HIGHER MODALITIES

MICHAEL SHULMAN

1. Higher modalities

We work in a "core" dependent type theory, such as that of Coq or Agda. Consider a collection of judgments of the following forms. (As always, unchanging ambient contexts are left implicit.)

$$\frac{A \ \mathsf{type}}{\bigcirc A \ \mathsf{type}} \qquad\qquad \frac{A \ \mathsf{type} \qquad a : A}{\mathsf{ret}_A(a) : \bigcirc A}$$

$$\frac{A \ \mathsf{type} \qquad (y : \bigcirc A) \vdash B \ \mathsf{type} \qquad (x : A) \vdash (b : \bigcirc B\,[\mathsf{ret}_A(x)/y]) \qquad \alpha : \bigcirc A}{(\mathsf{let}\ \mathsf{ret}(x) \coloneqq \alpha \ \mathsf{in}\ b) : \bigcirc B[\alpha/y]} \tag{1.1}$$

$$\frac{A \ \mathsf{type} \qquad (y : \bigcirc A) \vdash B \ \mathsf{type} \qquad (x : A) \vdash (b : \bigcirc B\,[\mathsf{ret}_A(x)/y]) \qquad a : A}{(\mathsf{let}\ \mathsf{ret}(x) \coloneqq \mathsf{ret}(a) \ \mathsf{in}\ b) \equiv b[a/x]} \tag{1.2}$$

The first two rules, plus the non-dependent version of the third, together look like a classical sort of monadic modality: we have $A \to \bigcirc A$ and

$$(A \to \bigcirc B) \to (\bigcirc A \to \bigcirc B).$$

However, here the modal operator $\bigcirc$ acts on *all* types, not just those with some claim to being called "propositions". This mandates the presence of a computation rule to match the introduction and elimination rules, and suggests the generalization to a dependent elimination rule.

However, these four rules are not quite enough for a well-behaved notion. With inductive and (presumably) higher inductive types, the dependent eliminator is sufficient to ensure homotopical initiality. (For instance, a W-type is a homotopically initial algebra for the corresponding endofunctor [AGS12], in that the space of maps from it to any other algebra is contractible.) But here, because the dependent eliminator only eliminates into types of the form $\bigcirc A$, the analogous argument fails. Thus, we need to strengthen the rules somehow.

Let us define $\mathsf{isModal}(A)$ to mean $\mathsf{isEquiv}(\mathsf{ret}_A)$; that is, that the map $A \to \bigcirc A$ is an equivalence. If $\mathsf{isModal}(A)$, we say that A is a **modal** type. The rule (1.1) implies that we can eliminate out of $\bigcirc A$ into any modal type (not just types of the form $\bigcirc B$, as asserted explicitly.) But we would also like to have that:

(i) For any type A, the type $\bigcirc A$ is modal, and
(ii) If B is modal, then precomposition with $\mathsf{ret}_A : A \to \bigcirc A$ yields an equivalence $(\bigcirc A \to B) \simeq (A \to B)$.

This would ensure that $\bigcirc$ is a *reflection* from types into modal types, or equivalently an *idempotent monad* on the category of types.

One approach is simply to assume (i) and (ii) as additional rules/axioms. Alternatively, it turns out that we can prove both of them if we assume

Date: October 24, 2012.

(iii) For any type A and any $\alpha_1, \alpha_2 : \bigcirc A$, the type $(\alpha_1 = \alpha_2)$ is modal.

Conversely, (iii) is provable from (i) and (ii). Take your pick of which assumptions seem better motivated. When these additional properties hold, I will call $\bigcirc$ a **(higher) modality**.

There are a number of possible variations:

- I've stated the computation rule (1.2) as a judgmental equality, but it could of course be a propositional one.
- We don't need to define $\mathsf{isModal}(A)$ as $\mathsf{isEquiv}(\mathsf{ret}_A)$; we could take "being modal" as an additional specified property of types. With suitable modifications to all the other rules, it then turns out that $\mathsf{isModal}(A)$ is provably equivalent to $\mathsf{isEquiv}(\mathsf{ret}_A)$.
- Note that (ii) implies the non-dependent version of the elimination rule (1.1) (with propositional computation). Conversely, (1.1) can be derived from (ii) and the other rules if we additionally assume that modal types are closed under dependent sums (which is, in turn, provable from the above presentation). If we have only (ii) but no dependent eliminator, we might speak of a **weak modality**.

For any (strong) modality, we can prove:

- The modal types are closed under products.
- They are also closed under *dependent* products: if $B(x)$ is modal for all $x : A$, then so is $\prod_{x:A} B(x)$, regardless of whether A is modal. In particular, $A \to B$ is modal if B is.
- The operation $\bigcirc$ preserves products.
- $\bigcirc$ preserves h-props. (I don't know whether it preserves all h-levels.)

Using these, we can define a "logic" of modal types in which conjunction, implication, and universal quantification are as usual, while disjunction and existential quantification are obtained by applying $\bigcirc$ to coproducts and dependent sums. Andrej has suggested that conversely, any "logic" in which implication and universal quantification are "natural" ought to come from a modality; I think probably this will require some "coherence" properties of the logic in order to get the stronger properties (i)–(iii).

Time for some examples:

Example 1.3. The identity is, of course, a modality.

Example 1.4. For any n, truncation to h-level n (defined as a higher inductive type) is a modality. In particular, this includes the h-prop reflection.

Example 1.5. Double negation $\bigcirc A = \neg\neg A = (A \to \bot) \to \bot$ is a modality, the "modality of classical logic". Note that $\neg\neg A$ is always an h-prop.

Example 1.6. $\bigcirc A \equiv \mathsf{unit}$ is a (trivial) modality. (In particular, a modality $\bigcirc$ need not preserve $\bot$.)

If $\bigcirc$ preserves pullbacks, we call it a **lex modality**. It is sufficient, for this, to ask that if $B : A \to \mathsf{Type}$ is a dependent type such that $\bigcirc A$ and $\bigcirc\left(\sum_{x:A} B(x)\right)$ are contractible, then each $\bigcirc B(x)$ is also contractible. In categorical semantics, lex modalities correspond to *subtoposes*.

Example 1.7. If U is an h-prop, then $\bigcirc A = (U \to A)$ is a modality. This is the "open subtopos" corresponding to U.

Example 1.8. Again if U is an h-prop, then let $\bigcirc A$ be the homotopy pushout of A and U over $A \times U$. This is the "closed subtopos" corresponding to U.

References

[AGS12] Steve Awodey, Nicola Gambino, and Kristina Sojakova. Inductive types in homotopy type theory. To appear in LICS 2012; arXiv:1201.3898, 2012. 1