Univalent Foundations of Mathematics

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IAS Mathematics Members Seminar
November 2012

Introduction

A new connection has recently come to light between Logic and Topology, namely an interpretation of the constructive type theory of Martin-Löf into homotopy theory.

- Homotopy can be used to construct models of systems of constructive logic.
- Constructive type theory can be used as a formal calculus to reason about homotopy.
- The computational implementation of type theory allows computer verified proofs in homotopy theory, and elsewhere.
- The homotopical interpretation suggests some new logical constructions and axioms.

Univalent Foundations combines these into a new program for foundations of mathematics.
The *Univalence Axiom* is a new principle of reasoning.

Type theory

Martin-Löf constructive type theory consists of:

- **Types**: $X, Y, \ldots, A \times B,\ A \to B, \ldots$
- **Terms**: $x : A,\ b : B,\ \langle a, b \rangle,\ \lambda x.b(x), \ldots$
- **Dependent Types**: $x : A \vdash B(x)$
  - $\sum_{x:A} B(x)$
  - $\prod_{x:A} B(x)$
- **Equations** $s = t : A$

Formal calculus of typed terms and equations.
Presented as a deductive system by rules of inference.
Intended as a foundation for constructive mathematics.

Propositions as Types

The system has a dual interpretation:

- once as **mathematical** objects: types are “sets” and their terms are “elements”, which are being constructed,
- once as **logical** objects: types are “propositions” and their terms are “proofs”, which are being derived.

This is also known as the **Curry-Howard correspondence**:

0	1	$A + B$	$A \times B$	$A \to B$	$\sum_{x:A} B(x)$	$\prod_{x:A} B(x)$
$\bot$	$\top$	$A \vee B$	$A \wedge B$	$A \Rightarrow B$	$\exists_{x:A} B(x)$	$\forall_{x:A} B(x)$

Gives the system its **constructive character**.

Identity types

According to the logical interpretation we have:

- **propositional logic**: $A + B,\ A \times B,\ A \to B$,
- **predicate logic**: $B(x), C(x, y)$, with **quantifiers** $\prod$ and $\sum$.

So it's natural to add a primitive relation of **identity** between any terms of the same type:

$$x, y : A \vdash \mathtt{Id}_A(x, y)$$

This type represents the **logical** proposition "x is identical to y".

On the **mathematical** side, the identity type admits a newly discovered "geometric" interpretation.

Rules for identity types

The **introduction** rule says that $a : A$ is always identical to itself:

$$\mathtt{r}(a) : \mathtt{Id}_A(a, a)$$

The **elimination** rule is a form of Leibniz's law:

$$\frac{c : \mathtt{Id}_A(a, b) \qquad x : A \vdash d(x) : R\big(x, x, \mathtt{r}(x)\big)}{\mathtt{J}_d(a, b, c) : R(a, b, c)}$$

Schematically:

$$\text{“ } a = b \ \& \ R(x, x) \ \Rightarrow \ R(a, b) \text{ ”}$$

The homotopy interpretation

Suppose we have terms of ascending identity types:

$$
\begin{aligned}
a,\ b &: A \\
p,\ q &: \mathtt{Id}_A(a, b) \\
\alpha,\ \beta &: \mathtt{Id}_{\mathtt{Id}_A(a,b)}(p, q) \\
\ldots &: \mathtt{Id}_{\mathtt{Id}_{\mathtt{Id}_{\ldots}}}(\ldots)
\end{aligned}
$$

Consider the following interpretation:

$$
\begin{aligned}
\text{Types} &\quad\rightsquigarrow\quad \text{Spaces} \\
\text{Terms} &\quad\rightsquigarrow\quad \text{Maps} \\
a : A &\quad\rightsquigarrow\quad \text{Points } a : 1 \to A \\
p : \mathtt{Id}_A(a, b) &\quad\rightsquigarrow\quad \text{Paths } p : a \Rightarrow b \\
\alpha : \mathtt{Id}_{\mathtt{Id}_A(a,b)}(p, q) &\quad\rightsquigarrow\quad \text{Homotopies } \alpha : p \Rrightarrow q \\
&\vdots
\end{aligned}
$$

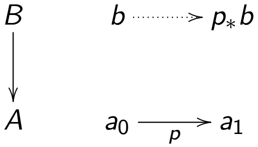
The homotopy interpretation: Type dependency

We still need to interpret dependent types $x : A \vdash B(x)$.
The identity rules imply the following:

$$\frac{p : \mathtt{Id}_A(a_0, a_1) \qquad b : B(a_0)}{p_* b : B(a_1)}$$

Logically, this just says "$a_0 = a_1 \ \& \ B(a_0) \Rightarrow B(a_1)$".

But topologically, it is a familiar **lifting property**:



This is the notion of a "fibration" of spaces.

The homotopy interpretation: Type dependency

Thus we continue the homotopy interpretation as follows:

$$\text{Dependent types} \quad x : A \vdash B(x) \quad \rightsquigarrow \quad \text{Fibrations} \quad \begin{array}{c} B \\ \downarrow \\ A \end{array}$$

The type $B(a)$ is the fiber of $B \to A$ over the point $a : A$

$$\begin{array}{ccc} B(a) & \longrightarrow & B \\ \downarrow & & \downarrow \\ 1 & \xrightarrow[a]{} & A. \end{array}$$

The homotopy interpretation: Identity types

To interpret the identity type $x, y : A \vdash \mathtt{Id}_A(x, y)$, we thus require a fibration over $A \times A$.
Take the space A^I of all paths in A:

$$\text{Identity type} \quad x, y : A \vdash \mathtt{Id}_A(x, y) \quad \rightsquigarrow \quad \text{Path space} \quad \begin{array}{c} A^I \\ \downarrow \\ A \times A \end{array}$$

The fiber $\mathtt{Id}_A(a, b)$ over a point $(a, b) \in A \times A$ is the space of paths from a to b in A.

$$\begin{array}{ccc} \mathtt{Id}_A(a, b) & \longrightarrow & A^I \\ \downarrow & & \downarrow \\ 1 & \xrightarrow[(a,b)]{} & A \times A. \end{array}$$

The homotopy interpretation: Identity types

The path space A^I classifies homotopies $\vartheta : f \Rightarrow g$ between maps $f, g : X \to A$,

$$\begin{array}{ccc} & & A^I \\ & \overset{\vartheta}{\nearrow} & \downarrow \\ X & \xrightarrow[(f,g)]{} & A \times A. \end{array}$$

So given any terms $x : X \vdash f, g : A$, an identity term

$$x : X \vdash \vartheta : \mathtt{Id}_A(f, g)$$

is interpreted as a **homotopy** between f and g.

The homotopy interpretation: Summary

1. There is a **topological interpretation** of the λ-calculus:

$$\begin{aligned} \text{types} &\rightsquigarrow \text{spaces} \\ \text{terms} &\rightsquigarrow \text{continuous functions} \\ &\dots \\ \text{computability} &\rightsquigarrow \text{continuity} \end{aligned}$$

2. Extend this to dependently typed λ-calculus with `Id`-types, using the **basic idea**:

   $p : \mathtt{Id}_X(a, b) \;\Leftrightarrow$

   $\qquad\qquad p$ is a path from point a to point b in the space X

   This forces dependent types to be fibrations, `Id`-types to be path spaces, and terms of `Id`-types to be homotopies.

The homotopy interpretation: First theorems

Instead of concrete spaces and homotopies, we use the axiomatic description provided by **Quillen model categories**.

- Gives a wide range of different models.
- Includes classical homotopy of spaces and simplicial sets.
- Allows the use of standard methods from categorical logic.

Theorem (Awodey & Warren 2006)

"Martin-Löf type theory has a **sound** *interpretation into abstract homotopy theory."*

Theorem (Gambino & Garner 2008)

"The homotopy interpretation of Martin-Löf type theory is also **complete***."*

1. The homotopy interpretation: Conclusion

Type theory provides a "logic of homotopy".

2. Homotopy type theory

How **expressive** is constructive type theory as a formal language for homotopy theory?

What facts, properties, and constructions from homotopy theory are logically expressible?

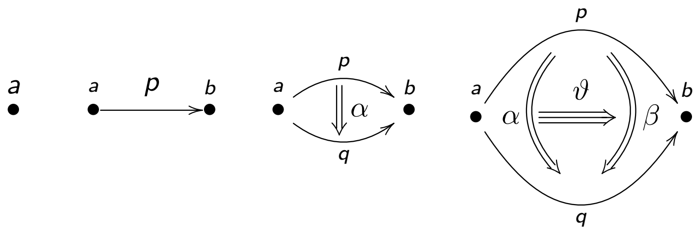
One example: the **fundamental group** and its higher-dimensional analogues are logical constructions.

Fundamental groupoids

Let's return to the system of identity terms of various orders:

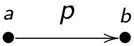
$$
\begin{aligned}
a,\ b &: A \\
p,\ q &: \mathtt{Id}_A(a,b) \\
\alpha,\ \beta &: \mathtt{Id}_{\mathtt{Id}_A(a,b)}(p,q) \\
\vartheta &: \mathtt{Id}_{\mathtt{Id}_{\mathtt{Id}_{\dots}}}(\alpha,\beta)
\end{aligned}
$$

These can be represented suggestively as follows:



Fundamental groupoids

As in topology, the terms of order 0 and 1, ("points" and "paths"),



bear the structure of a **groupoid**.

The laws of identity correspond to the **groupoid operations**:

$$
\begin{array}{rll}
r : \mathtt{Id}(a, a) & \text{reflexivity} & a \to a \\
s : \mathtt{Id}(a, b) \to \mathtt{Id}(b, a) & \text{symmetry} & a \leftrightarrows b \\
t : \mathtt{Id}(a, b) \times \mathtt{Id}(b, c) \to \mathtt{Id}(a, c) & \text{transitivity} & a \to b \to c
\end{array}
$$

This was first shown by Hofmann & Streicher (1998), who gave a model of intensional type theory using groupoids as types.

Fundamental groupoids

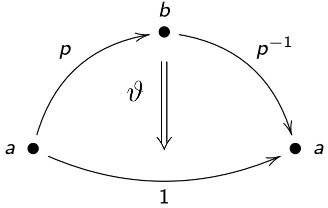
But also just as in topology, the **groupoid equations** of associativity, inverse, and unit:

$$p \cdot (q \cdot r) = (p \cdot q) \cdot r$$
$$p^{-1} \cdot p = 1 = p \cdot p^{-1}$$
$$1 \cdot p = p = p \cdot 1$$

do not hold strictly, but only "up to homotopy".

This means they are witnessed by terms of the next higher order:

$$\vartheta : \mathtt{Id}_{\mathtt{Id}}\left(p^{-1} \cdot p, 1\right)$$



Fundamental groupoids

The entire system of identity terms of all orders forms an infinite-dimensional graph, or "globular set":

$$A \leftleftarrows \mathtt{Id}_A \leftleftarrows \mathtt{Id}_{\mathtt{Id}_A} \leftleftarrows \mathtt{Id}_{\mathtt{Id}_{\mathtt{Id}_A}} \leftleftarrows \dots$$

It has the structure of a (weak), infinite-dimensional, groupoid, as already occurring homotopy theory:

Theorem (Lumsdaine, Garner & van den Berg, 2009)

The system of identity terms of all orders over any fixed type is a weak ∞-groupoid.

Every type has a **fundamental weak ∞-groupoid**.

Machine implementation

Now one can combine the following:

- the representation of homotopy theory in constructive type theory
- the well-developed implementations of type theory in computational proof assistants like Coq.

Allows computer verified proofs in homotopy theory and related fields, in addition to constructive mathematics.

This aspect is being very actively pursued right now in the Univalent Foundations Program; examples in Dan Licata's talk next week.

2. Homotopy type theory: Conclusion

- Type theory provides a logic of homotopy.
- Logical methods can in principle capture a lot of homotopy theory, e.g. the fundamental ∞-groupoid of a space is a *logical* construction.
- Some results are already being formalized: basic homotopy theory, elementary mathematics, and some new results in foundations.
- Some new logical concepts, constructions and axioms are also suggested by the homotopy interpretation.

3. Higher-dimensional inductive types

(Work in progress by Lumsdaine, Shulman & others.)

The natural numbers $\mathbb{N}$ are implemented in type theory as an inductively defined structure of type:

$$o : \mathbb{N}$$
$$s : \mathbb{N} \to \mathbb{N}$$

The **recursion property** is captured by an elimination rule:

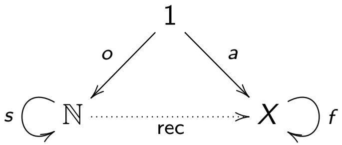
$$\frac{a : X \qquad f : X \to X}{\mathrm{rec}(a, f) : \mathbb{N} \to X}$$

such that:

$$\mathrm{rec}(a, f)(o) = a$$
$$\mathrm{rec}(a, f)(sn) = f(\mathrm{rec}(a, f)(n))$$

Higher-dimensional inductive types

This says just that $(\mathbb{N}, o, s)$ is the **free** structure of this type:



The map $\text{rec}(a, f) : \mathbb{N} \to X$ is unique with this property.

Higher-dimensional inductive types: The circle S^1

The topological circle $\mathbb{S} = S^1$ can also be given as an inductive type, now involving a higher-dimensional generator:

$$
\begin{aligned}
b &: \mathbb{S} \\
p &: b \rightsquigarrow b
\end{aligned}
$$

Here we have written $p : b \rightsquigarrow b$ for the "loop" $p : \mathtt{Id}_{\mathbb{S}}(b, b)$.

Higher-dimensional inductive types: The circle S^1

There is an associated recursion property, captured again by an elimination rule:

$$\frac{a : X \qquad q : a \rightsquigarrow a}{\mathsf{rec}(a, q) : \mathbb{S} \to X}$$

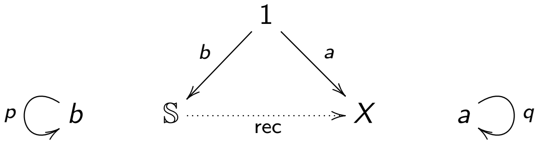
such that:

$$\mathsf{rec}(a, q)(b) = a$$
$$\mathsf{rec}(a, q)(p) = q$$

The map $\mathsf{rec}(a, q)$ acts on paths via the `Id`-elimination rule.

Higher-dimensional inductive types: The circle S^1

This says that $(\mathbb{S}, b, p)$ is the **free** structure of this (higher) type:



The map $\mathrm{rec}(a, q) : \mathbb{S} \to X$ is then unique up to homotopy.

Higher-dimensional inductive types: The circle S^1

Here is a sanity check:

Theorem (Shulmann 2011)

The type-theoretic circle $\mathbb{S}$ has the correct homotopy groups:

$$\pi_n(\mathbb{S}) = \begin{cases} \mathbb{Z}, & \text{if } n = 1, \\ 0, & \text{if } n \neq 1. \end{cases}$$

The proof is implemented in Coq. It combines classical homotopy theory with methods from constructive type theory, and uses Voevodsky's new Univalence Axiom.

Higher-dimensional inductive types: The interval I

The unit interval $\mathbb{I} = [0, 1]$ is also an inductive type, on the data:

$$0, 1 : \mathbb{I}$$
$$p : 0 \rightsquigarrow 1$$

Again writing $p : 0 \rightsquigarrow 1$ for the path $p : \mathtt{Id}_{\mathbb{I}}(0, 1)$.

Slogan: In topology, we start with the **interval** and use it to define the notion of a **path**.
In HoTT, we start with the notion of a **path**, and use it to define the **interval**.

3. Higher-dimensional inductive types: Conclusion

Many basic spaces and constructions can be introduced as HITs:

- higher spheres S^n and disks D^n,
- cylinders, tori, cell complexes, ...,
- suspensions ΣA and loop spaces ΩA,
- homotopy pullbacks, pushouts, etc.,
- truncations such as connected components $\pi_0(A)$,
- higher homotopy groups,
- Quillen model structure.

The formal system has **intrinsic geometric content**.

4. Univalence

The Univalent Foundations program is a new approach to the foundations of mathematics with both intrinsic geometric content and a computational implementation.

- Univalent Foundations applies not only to homotopy theory, but also subsumes **Set Theory** and **Formal Logic**.
- Voevodsky has proposed a new foundational axiom in this setting: the **Univalence Axiom**.
- It captures the informal mathematical practice of **identifying isomorphic objects**.
- It is formally **incompatible** with conventional foundations.
- It is formally **consistent** with homotopy type theory.

Isomorphism

In type theory, the notion of *type isomorphism* $A \cong B$ is definable as usual:

$$A \cong B \;\Leftrightarrow\; \textit{there are } f : A \to B \textit{ and } g : B \to A \\ \textit{such that } gfx = x \textit{ and } fgy = y.$$

Formally, there is a type of isomorphisms:

$$\mathrm{Iso}(A, B) \;:=\; \sum_{f:A\to B} \sum_{g:B\to A} \Big(\prod_{x:A} \mathtt{Id}_A(gfx, x) \times \prod_{y:B} \mathtt{Id}_B(fgy, y) \Big)$$

We say that $A \cong B$ if this type is “inhabited” by a closed term – which is then an isomorphism between A and B.

Isomorphism: Remarks

$$\mathrm{Iso}(A, B) \;:=\; \sum_{f:A\to B}\sum_{g:B\to A}\big(\prod_{x:A}\mathtt{Id}_A(gfx, x)\times\prod_{y:B}\mathtt{Id}_B(fgy, y)\big)$$

- It is convenient to add a "coherence" condition relating the *proofs* of $gfx = x$ and $fgy = y$.
- Under the homotopy interpretation, this then becomes the type of *homotopy equivalences*.
- The same notion also subsumes *logical equivalence*, *isomorphism of sets*, and *categorical equivalence*.

Invariance

One can show that in type theory, all *definable properties* $P(X)$ of types X *respect type isomorphism*, in the sense that the following inference holds:

$$\frac{A \cong B \qquad P(A)}{P(B)}$$

In this sense, *all definable properties are invariant*.

Now let us compare the indiscernability condition:

$$P(A) \Rightarrow P(B), \text{for all } P$$

with **identity of types** A and B.

Universes

An extension of type theory that allows reasoning about **identity of types** is a *universe* U, which then has an identity type:

$$\mathtt{Id}_U(A, B)$$

One can easily construct a "comparison map" of types:

$$\mathtt{Id}_U(A, B) \to \mathrm{Iso}(A, B).$$

So, of course, **identity implies isomorphism**.

The Univalence Axiom

Voevodsky's *Univalence Axiom* asserts that this comparison map is itself an isomorphism:

$$\mathtt{Id}_U(A, B) \xrightarrow{\sim} \mathrm{Iso}(A, B) \tag{UA}$$

So UA can be stated: *"Identity is isomorphic to isomorphism."*

The Univalence Axiom: Remarks

- Since UA is an iso, there is a map coming back:

$$\mathtt{Id}_U(A, B) \longleftarrow \mathrm{Iso}(A, B)$$

  So **isomorphic objects are identical**.
- The system with UA still has the **invariance property**:

$$\frac{A \cong B \qquad P(A)}{P(B)}$$

  In the presence of a universe, UA is equivalent to invariance.
- Incompatible with conventional foundations in set theory, but consistent with HoTT: Voevodsky has a model in SSets.
- **Conjecture** (Voevodsky): UA preserves the computational character of the total system.

Conclusion

Homotopy Type Theory *is a topological interpretation of constructive type theory that allows purely formal reasoning in homotopy theory.*

Univalent Foundations *is a new approach to the foundations of mathematics based on Homotopy Type Theory, with both intrinsic geometric content and a computational implementation.*

The Univalence Axiom *is a powerful new principle of reasoning that is incompatible with conventional foundations, and yet (conjecture!) computationally admissible.*

References and Further Information

General information:

www.HomotopyTypeTheory.org

Current state of the Univalent Foundations Program:

uf-ias-2012.wikispaces.com