Notes on type systems

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1 C-structures

1 C-structures as set level categories

The objects which we call C-structures are better know as "contextual categories". They where introduces by Cartmell in [3] and then described in more detail by Streicher (see [9, Def. 1.2, p.47]). It will be important for us to distinguish two notions of a category. What is understood by a category by most practicing mathematicians i.e. a category up to an equivalence, will be called, when an explicit distinction is needed, a category of h-level 3. A category as an algebraic object i.e. a category up to an isomorphism will be called a set-level category or category of h-level 2. A set-level category C is a pair of sets $Mor(C)$ and $Ob(C)$ with structure given by four maps

$$\partial_0, \partial_1 : Mor(C) \to Ob(C)$$

$$Id : Ob(C) \to Mor(C)$$

and

$$\circ : Mor(C)_{\partial_0} \times_{\partial_1} Mor(C) \to Mor(C)$$

which satisfy the well known conditions (note that we write composition of morphisms in the form $f \circ g$ where $f : Y \to X$ and $g : Z \to Y$).

A C-structure is a set-level category CC with additional structure of the form

1. a function $l : Ob(CC) \to \mathbf{N}$,
2. an object pt,
3. a map $ft : Ob(CC) \to Ob(CC)$,
4. for each $X \in Ob(CC)$ a morphism $p_X : X \to ft(X)$,
5. for each $X \in Ob(CC)$ such that $X \neq pt$ and each morphism $f : Y \to ft(X)$ an object f^*X and a morphism $q(f, X) : f^*X \to X$,

which satisfies the following conditions:

1. $l^{-1}(0) = \{pt\}$
2. for X such that $l(X) > 0$ one has $l(ft(X)) = l(X) - 1$
3. $ft(pt) = pt$
4. pt is a final object,
5. for $X \in Ob(CC)$ such that $X \neq pt$ and $f : Y \to ft(X)$ one has $ft(f^*X) = Y$ and the square

[2009.10.14.eq1]
$$\begin{array}{ccc} f^*X & \xrightarrow{q(f,X)} & X \\ {\scriptstyle p_{f^*X}}\downarrow & & \downarrow{\scriptstyle p_X} \\ Y & \xrightarrow{f} & ft(X) \end{array} \qquad (1)$$

is a pull-back square,

6. for $X \in Ob(CC)$ such that $X \neq pt$ one has $id^*_{ft(X)}(X) = X$ and $q(id_{ft(X)}, X) = id_X$,

7. for $X \in Ob(CC)$ such that $X \neq pt$, $f : Y \to ft(X)$ and $g : Z \to Y$ one has $(fg)^*(X) = g^*(f^*(X))$ and $q(fg, X) = q(f, X)q(g, f^*X)$.

Let $B_n(CC) = \{X \in Ob(CC) \,|\, l(X) = n\}$ and let $Mor_{n,m}(CC) = \{f : Mor(CC) | \partial_0(f) \in B_n \, and \, \partial_1(f) \in B_m\}$. One can reformulate the definition of a C-structure using $B_n(CC)$ and $Mor_{n,m}(CC)$ as the underlying sets together with the obvious analogs of maps and conditions the definition given above. In this reformulation there will be no use of $\neq$ and the only use of the existential qualifier will be as a part of "there exists a unique" condition. This shows that C-structures can be considered as models of a quasi-equational theory with sorts B_n, and $Mor_{n,m}$ and in particular all the results of [6] are applicable to C-structures.

We will also use the following notations:

1. $B(X) = \{Y \in Ob(CC) \,|\, ft(Y) = X \, and \, Y \neq pt\}$,

2. $\widetilde{Ob}(CC)$ is the set of pairs of the form (X, s) where $X \in Ob(CC)$, $X \neq pt$ and s is a section of the canonical morphism $p_X : X \to ft(X)$ i.e. a morphism $s : ft(X) \to X$ such that $p_X \circ s = Id_{ft(X)}$,

3. $\widetilde{B}_n = \{(X, s) \in \widetilde{Ob}(CC) \,|\, X \in B_n\}$ (note that $\widetilde{B}_0 = \emptyset$),

4. $\partial : \widetilde{B}_n \to B_n$ is the function defined by $\partial(X, s) = X$,

5. $\widetilde{B}(X) = \partial^{-1}(X)$ (note that $\widetilde{B}(pt) = \emptyset$).

2 C-substructures.

A C-substructure CC' of a C-structure CC is a subcategory of the underlying set-level category which is closed, in the obvious sense under the operations which define the C-structure on CC and such that the canonical squares which belong to CC' are pull-back squares in CC'. A C-substructure is called non-trivial if it contains at least one element other than pt. A C-substructure is itself a C-structure with respect to the induced structure. The following elementary result plays a key role in many constructions of type theory:

Proposition 2.1 *[***2009.10.15.prop1***] Let CC be a C-structure. Then for any family CC_α of C-substructures of CC, the intersection $CC' = \cap_\alpha CC_\alpha$ is a C-substructure.*

Proof: The only condition to check is that a canonical square which belongs to CC' is a pull-back square in CC'. This follows from the definition of pull-back squares and the fact that fiber products of sets commute with intersections of sets.

Corollary 2.2 *[***2009.10.15.cor1***] Let CC be a C-structure, C_0 a set of objects of CC and C_1 a set of morphisms of CC. Then there exists the smallest C-substructure $[C_1, C_0]$ which contains C_0 and C_1. It is called the C-substructure generated by C_0 and C_1.*

Lemma 2.3 *[***2009.10.15.l1***] Let CC be a C-structure and CC', CC'' be two C-substructures such that $Ob(CC') = Ob(CC'')$ (as subsets of $Ob(CC)$) and $\widetilde{Ob}(CC') = \widetilde{Ob}(CC'')$ (as subsets of $\widetilde{Ob}(CC)$). Then $CC' = CC''$.*

Proof: Let $f : Y \to X$ be a morphism in CC'. We want to show that it belongs to CC''. Proceed by induction on m where $X \in B_m$. For $m = 0$ the assertion is obvious. Suppose that $m > 0$. Since CC is a C-structure we have a commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow{s_f} & (p_X f)^* X & \xrightarrow{q(p_X f, X)} & X \\ \downarrow & & \downarrow p' & & \downarrow p \\ Y & \xrightarrow{=} & Y & \xrightarrow{p_X f} & ft(X) \end{array} \qquad (2)$$

[**2009.11.07.oldeq1**]

such that $f = q(p_X f, X)\, s_f$. Since the right hand side square is a canonical one, $((p_X f)^*\Gamma', s_f) \in \widetilde{Ob}(CC)$ and $ft(X) \in B_{m-1}$, the inductive assumption implies that $f \in CC''$.

Remark 2.4 In Lemma 2.3, it is sufficient to assume that $\widetilde{Ob}(CC') = \widetilde{Ob}(CC'')$. The condition $Ob(CC') = Ob(CC'')$ is then also satisfied. Indeed, let $X \in Ob(CC')$. Then $p_X^* X$ is the product $X \times X$ in CC. Consider the diagonal section $\Delta_X : X \to p_X^* X$ of $p_{p_X^*(X)}$. Since CC' is assumed to be a C-substructure we conclude that $\Delta_X \in \widetilde{Ob}(CC') = \widetilde{Ob}(CC'')$ and therefore $X \in Ob(CC'')$. It is however more convenient to think of C-substructures in terms of subsets of both Ob and $\widetilde{Ob}$.

Let CC be a C-structure. Let us say that a pair of subsets $C \subset Ob(CC)$, $\widetilde{C} \subset \widetilde{Ob}(CC)$ is saturated if there exists a C-substructure CC' such that $C = Ob(CC')$ and $\widetilde{C} = \widetilde{Ob}(CC')$. By Lemma 2.3 we have a bijection between C-substructures of CC and saturated pairs $(C, \widetilde{C})$.

Let us introduce the following notations. Let $X \in Ob(CC)$ and $i \geq 0$. Denote by $p_{X,i}$ the composition of the canonical projections $X \to ft(X) \to \ldots \to ft^i(X)$ such that $p_{X,0} = Id_X$ and $p_{X,1} = p_X$. For $f : Y \to ft^i(X)$ denote by $q(f, X, i) : f^*(X, i) \to X$ the morphism defined inductively by the rule

$$f^*(X, 0) = Y \qquad q(f, X, 0) = f,$$

$$f^*(X, i+1) = q(f, ft(X), i)^*(X) \qquad q(f, X, i+1) = q(q(f, ft(X), i), X).$$

In other words, $q(f, X, i)$ is the canonical pull-back of the morphism $f : Y \to ft^i(X)$ with respect to the sequence of canonical projections $X \to ft(X) \to \ldots \to ft^i(X)$.

Let $i \geq 1$, $f : Y \to ft^i(X)$ be a morphism and $s : ft(X) \to X$ an element of $\widetilde{Ob}(CC)$. Denote by $f^*(s, i)$ the element of $\widetilde{Ob}(CC)$ of the form $f^*(ft(X), i-1) \to f^*(X, i)$ which is the pull-back of s with respect to $q(f, ft(X), i-1)$.

Proposition 2.5 *[**2009.10.15.prop2**] A pair $(C, \widetilde{C})$ is saturated if and only if it satisfies the following conditions:*

1. *$pt \in C$,*
2. *if $X \in C$ then $ft(X) \in C$,*
3. *if $(s : ft(X) \to X) \in \widetilde{C}$ then $X \in C$,*
4. *if $(s : ft(X) \to X) \in \widetilde{C}$, $X' \in C$, $i \geq 1$ and $ft^i(X) = ft(X')$ then $q(p_{X'}, ft(X), i-1)^*(s) \in \widetilde{C}$,*
5. *if $(s_1 : ft(X) \to X) \in \widetilde{C}$, $i \geq 1$ and $(s_2 : ft^{i+1}(X) \to ft^i(X)) \in \tilde{C}$ then $q(s_2, ft(X), i-1)^*(s_1) \in \widetilde{C}$,*

6. *if $X \in C$ then the diagonal $s_{id_X} : X \to (p_X)^*(X)$ is in $\widetilde{C}$.*

Conditions (4) and (5) are illustrated by the following diagrams:

$$
\begin{array}{ccc}
p^*_{X'}(ft(X), i-1) & \xrightarrow{q(p_{X'}, ft(X), i-1)} & ft(X) \\
\downarrow{\scriptstyle q(p_{X'}, ft(X), i-1)^*(s)} & & \downarrow{\scriptstyle s} \\
p^*_{X'}(X, i) & \xrightarrow{q(p_{X'}, X, i)} & X \\
\downarrow & & \downarrow{\scriptstyle p_X} \\
p^*_{X'}(ft(X), i-1) & \xrightarrow{q(p_{X'}, ft(X), i-1)} & ft(X) \\
\downarrow & & \downarrow \\
\dots & & \dots \\
\downarrow & & \downarrow \\
X' & \xrightarrow{p_{X'}} & ft^i(X)
\end{array}
\qquad
\begin{array}{ccc}
s^*_2(ft(X), i-1) & \xrightarrow{q(s_2, ft(X), i-1)} & ft(X) \\
\downarrow{\scriptstyle q(s_2, ft(X), i-1)^*(s_1)} & & \downarrow{\scriptstyle s_1} \\
s^*_2(X, i) & \xrightarrow{q(s_2, X, i)} & X \\
\downarrow & & \downarrow{\scriptstyle p_X} \\
s^*_2(ft(X), i-1) & \xrightarrow{q(s_2, ft(X), i-1)} & ft(X) \\
\downarrow & & \downarrow \\
\dots & & \dots \\
\downarrow & & \downarrow \\
ft^{i+1}(X) & \xrightarrow{s_2} & ft^i(X)
\end{array}
$$

Proof: The "only if" part of the proposition is straightforward. Let us prove that for any $(C, \widetilde{C})$ satisfying the conditions of the proposition there exists a C-substructure CC' of CC such that $C = Ob(CC')$ and $\widetilde{C} = \widetilde{Ob}(CC')$.

For a morphism $f : Y \to X$ let $ft(f) = p_X f : Y \to ft(X)$. Any morphism $f : Y \to X$ in CC has a canonical representation of the form $Y \stackrel{s_f}{\to} X_f \stackrel{q_f}{\to} X$ where $X_f = ft(f)^*(X)$, $q_f = q(ft(f), X)$ and $s_f : Y \to X_f$ is the section of the canonical projection $X_f \to Y$ corresponding to f.

Define a candidate subcategory CC' setting $Ob(CC') = C$ and defining the set $Mor(CC')$ of morphisms of CC' inductively by the conditions:

1. $Y \to pt$ is in $Mor(CC')$ if and only if $Y \in C$,
2. $f : Y \to X$ is in $Mor(CC')$ if and only if $X \in Ob(C)$, $ft(f) \in Mor(CC')$ and $s_f \in \widetilde{C}$.

(note that the for $(f : Y \to X) \in Mor(CC')$ one has $Y \in C$ since $s_f : Y \to X_f$).

Let us show that if the condition of the proposition are satisfied then $(Ob(CC'), Mor(CC'))$ form a C-substructure of CC.

The subset $Ob(CC')$ contains pt and is closed under ft map by the first two conditions. The following lemma shows that $Mor(CC')$ contains identities and the compositions of canonical projections.

Lemma 2.6 *[***2009.10.16.l1***] Under the assumptions of the proposition, if $X \in C$ and $i \geq 0$ then $p_{X,i} : X \to ft^i(X)$ is in $Mor(CC')$.*

Proof: By definition of C-structures there exists n such that $ft^n(X) = pt$. Then $p_{X,n} \in Mor(CC')$ by the first constructor of $Mor(CC')$. By induction it remains to show that if $X \in C$ and $p_{X,i} \in Mor(CC')$ then $p_{X,i-1} \in Mor(CC')$. We have $ft(p_{X,i-1}) = p_{X,i}$ and $s_{p_{X,i-1}}$ is the pull-back of the diagonal $ft^{i-1}(X) \to (p_{ft^{i-1}(X)})^*(ft^{i-1}(X))$ with respect to the canonical morphism $X \to ft^{i-1}(X)$. The diagonal is in $\widetilde{C}$ by condition (6) and therefore $s_{p_{X,i-1}}$ is in $\widetilde{C}$ by repeated application of condition (4).

Lemma 2.7 *[2009.10.16.l3] Under the assumptions of the proposition, let $X \in C$, $(s : ft(X) \to X) \in \widetilde{C}$, $i \geq 0$, and $(f : Y \to ft^i(X)) \in Mor(CC')$. Then $q(f, ft(X), i-1)^*(s) : ft(f^*(X, i)) \to f^*(X, i)$ is in $Mor(CC')$.*

Proof: Suppose first that $ft^i(X) = pt$. Then $f = p_{Y,n}$ for some n and the statement of the lemma follows from repeated application of condition (4). Suppose that the lemma is proved for all morphisms to objects of length $j-1$ and let the length of $ft^i(X)$ be j. Consider the canonical decomposition $f = q_f s_f$. The morphism q_f is the canonical pull-back of $ft(f)$ and therefore the pull-back of s relative to q_f coincides with its pull-back relative to $ft(f)$ which is $\widetilde{C}$ by the inductive assumption. The pull-back of an element of $\widetilde{C}$ with respect to s_f is in $\widetilde{C}$ by condition (5).

Lemma 2.8 *[2009.10.16.l4] Under the assumptions of the proposition, let $g : Z \to Y$ and $f : Y \to X$ be in $Mor(CC')$. Then $fg \in Mor(CC')$.*

Proof: If $X = pt$ the the statement is obvious. Assume that it is proved for all f whose codomain is of length $< j$ and let X be of length j. We have $ft(fg) = ft(f)g$ and therefore $ft(fg) \in Mor(CC')$ by the inductive assumption. It remains to show that $s_{fg} \in \widetilde{C}$. We have the following diagram whose squares are canonical pull-back squares

$$\begin{array}{ccccc}
X_{fg} & \longrightarrow & X_f & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow p_X \\
Z & \xrightarrow{g} & Y & \xrightarrow{ft(f)} & ft(X)
\end{array}$$

which shows that $s_{fg} = g^*(s_f)$. Therefore, $s_{fg} \in Mor(CC')$ by Lemma 2.7.

Lemma 2.9 *[2009.10.16.l5] Under the assumptions of the proposition, let $X \in C$ and let $f : Y \to ft(X)$ be in $Mor(CC')$, then $f^*(X) \in C$ and $q(f, X) \in Mor(CC')$.*

Proof: Consider the diagram

$$\begin{array}{ccccc}
f^*(X) & \xrightarrow{q(f,X)} & X & & \\
s_{q(f,X)} \downarrow & & \downarrow s_{Id_X} & & \\
q(f,X)^*(X) & \longrightarrow & p_X^*(X) & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
f^*(X) & \xrightarrow{q(f,X)} & X & \longrightarrow & ft(X) \\
p_{f^*(X)} \downarrow & & \downarrow p_X & & \\
Y & \xrightarrow{f} & ft(X) & &
\end{array}$$

where the squares are canonical. By condition (6) we have $s_{Id} \in \widetilde{C}$. Therefore, by Lemma 2.7, we have $s_{q(f,X)} \in \widetilde{C}$. In particular, $q(f,X)^*(X) \in C$ and therefore $f^*(X) = ft(q(f,X)^*(X)) \in C$. The fact that $q(f,X) \in Mor(CC')$ follows from the fact that $s_{q(f,X)} \in \widetilde{C}$ and $ft(q(f,X)) = f \circ p_{f^*(X)}$ is in $Mor(CC')$ by previous lemmas.

Lemma 2.10 *[2009.10.16.l6] Under the assumptions of Lemma 2.9, the square*

$$\begin{array}{ccc} f^*(X) & \xrightarrow{q(f,X)} & X \\ {\scriptstyle p_{f^*(X)}}\downarrow & & \downarrow{\scriptstyle p_X} \\ Y & \xrightarrow{f} & ft(X) \end{array}$$

is a pull-back square in CC'.

Proof: We need to show that for a morphism $g : Z \to f^*(X)$ such that $p_{f^*(X)}g$ and $q(f, X)g$ are in $Mor(CC')$ one has $g \in Mor(CC')$. We have $ft(g) = p_{f^*(X)}g$, therefore by definition of $Mor(CC')$ it remains to check that $s_g \in \widetilde{C}$. The diagram

$$\begin{array}{ccccc} (f^*Y)_g & \longrightarrow & f^*Y & \xrightarrow{q(f,X)} & X \\ \downarrow & & \downarrow & & \downarrow \\ Z & \xrightarrow{ft(g)} & Y & \xrightarrow{f} & ft(X) \end{array}$$

shows that $s_g = s_{q(f,X)g}$ and therefore $s_g \in Mor(CC')$.

To finish the proof of the proposition it remains to show that $Ob(CC') = C$ and $\widetilde{Ob}(CC') = \widetilde{C}$. The first assertion is tautological. The second one follows immediately from the fact that for $(s : ft(X) \to X) \in \widetilde{Ob}(CC)$ one has $ft(s) = Id_{ft(X)}$ and $s_s = s$.

3 The sequent axiomatics of C-structures.

Proposition 2.5 suggests that a C-structure CC can be reconstructed from the sets $B_n = B_n(CC)$ and $\widetilde{B}_{n+1} = \widetilde{B}_{n+1}(CC)$, $n \geq 0$ together with the structures on these sets which correspond to the conditions of the proposition. Let us show that it is indeed the case.

In addition to the sets B_n and $\widetilde{B}_n$ and maps $ft : B_{n+1} \to B_n$ and $\partial : \widetilde{B}_{n+1} \to B_{n+1}$ let us consider the following maps given for all $m \geq n \geq 0$:

1. $T : (B_{n+1})_{ft} \times_{ft^{m+1-n}} (B_{m+1}) \to B_{m+2}$, which sends (Y, X) such that $ft(Y) = ft^{m+1-n}(X)$ to $p_Y^*(X, m+1-n)$,

2. $\widetilde{T} : (B_{n+1})_{ft} \times_{ft^{m+1-n}\partial} (\widetilde{B}_{m+1}) \to \widetilde{B}_{m+2}$, which sends (Y, s) such that $ft(Y) = ft^{m+1-n}\partial(s)$ to $p_Y^*(s, m+1-n)$,

3. $S : (\widetilde{B}_{n+1})_\partial \times_{ft^{m+1-n}} (B_{m+2}) \to B_{m+1}$, which sends (r, X) such that $\partial(r) = ft^{m+1-n}(X)$ to $r^*(X, m+1-n)$,

4. $\widetilde{S} : (\widetilde{B}_{n+1})_\partial \times_{ft^{m+1-n}\partial} (\widetilde{B}_{m+2}) \to \widetilde{B}_{m+1}$, which sends (r, s) such that $\partial(r) = ft^{m+1-n}\partial(s)$ to $r^*(s, m+1-n)$.

5. $\delta : B_{n+1} \to \widetilde{B}_{n+2}$ which sends X to the diagonal section of the projection $p_X^*X \to X$.

Note that we have:

1. for $Y \in B_{n+1}$, $X \in B_{m+1}$ such that $ft(Y) = ft^{m+1-n}(X)$ and $m \geq n \geq 0$ one has:

$$ft(T(Y,X)) = \begin{cases} T(Y, ft(X)) & \text{if } m > n \\ Y & \text{if } m = n \end{cases} \tag{3}$$

2. for $Y \in B_{n+1}$, $s \in \widetilde{B}_{m+1}$ such that $ft(Y) = ft^{m+1-n}\partial(s)$ and $m \geq n \geq 0$ one has:

$$\partial(\widetilde{T}(Y,s) = T(Y, \partial(s)) \tag{4}$$

3. for $r \in \widetilde{B}_{n+1}$, $X \in \widetilde{B}_{m+2}$ such that $\partial(r) = ft^{m+1-n}(X)$ and $m \geq n \geq 0$ one has:

$$ft(S(r,X)) = \begin{cases} S(r, ft(X)) & \text{if } m > n \\ ft(Y) & \text{if } m = n \end{cases} \tag{5}$$

4. for $r \in \widetilde{B}_{n+1}$, $s \in \widetilde{B}_{m+2}$ such that $\partial(r) = ft^{m+1-n}\partial(s)$ and $m \geq n \geq 0$ one has:

$$\partial(\widetilde{S}(r,s)) = S(r, \partial(s)) \tag{6}$$

5.
$$[\mathbf{2009.12.27.eq1}]\partial(\delta(X)) = T(X,X) \tag{7}$$

Let us denote by

$$T_j : (B_{n+j})_{ft^j} \times_{ft^{m+1-n}} (B_{m+1}) \to B_{m+1+j}$$

$$\widetilde{T}_j : (B_{n+j})_{ft^j} \times_{ft^{m+1-n}\partial} (\widetilde{B}_{m+1}) \to \widetilde{B}_{m+1+j}$$

the maps which are defined inductively by

$$T_j(Y,X) = \begin{cases} X & \text{if } j = 0 \\ T(Y, T_{j-1}(ft(Y), X)) & \text{if } j > 0 \end{cases} \qquad \widetilde{T}_j(Y,s) = \begin{cases} s & \text{if } j = 0 \\ \widetilde{T}(Y, \widetilde{T}_{j-1}(ft(Y), s)) & \text{if } j > 0 \end{cases} \tag{8}$$

Note that for any $i = 0, \ldots, j$ we have

$$T_j(Y,X) = T_i(Y, T_{j-i}(ft^i(Y), X))$$

and

$$\widetilde{T}_j(Y,s) = \widetilde{T}_i(Y, \widetilde{T}_{j-i}(ft^i(Y), s))$$

Lemma 3.1 *[**Tnft**] One has*

$$T_j(Y, ft(X)) = ft(T_j(Y,X))$$

Proof: For $n = 0$ the statement is obvious. For $n > 0$ we have by induction on j

$$T_j(Y, ft(X)) = T(Y, T_{j-1}(ft(Y), ft(X))) = T(Y, ft(T_{j-1}(ft(Y), X))) =$$

$$= ft(T(Y, T_{j-1}(ft(Y), X))) = ft(T_j(Y,X)).$$

Let $Y \in B_n$. Define by induction on $m \geq 0$ the following collection of data:

1. for any $X \in B_m$ a set $Mor_{n,m}(Y,X)$,

2. for any $i > 0$, $X \in B_{m+i}$ and $f \in Mor_{n,m}(Y, ft^i(X))$ - an element $f^*(X) \in B_{n+i}$

setting:

1. $Mor_{n,0}(Y, pt)$ is the one point set whose only element we denote by $p_{Y,n}$ and for $i > 0$ and $X \in B_i$ we set
$$p^*_{Y,n}(X) = T_n(Y, X)$$

2. for $m > 0$ one has:

   (a) for $X \in B_m$, $Mor_{n,m}(Y, X)$ is the set of pairs (rf, ftf) where $rf \in \widetilde{B}_{n+1}$, $ftf \in Mor_{n,m-1}(Y, ft(X))$ and $\partial(rf) = ftf^*(X)$,

   (b) for $i > 0$, $X \in B_{m+i}$ and $f = (rf, ftf) \in Mor_{n,m}(Y, ft^i(X))$ we set
$$f^*(X) = S(rf, ftf^*(X)).$$

To check that this construction is well defined we need to verify that $S(rf, ftf^*(X))$ is defined. We have $i > 0$, $X \in B_{m+i}$, $rf \in \widetilde{B}_{n+1}$ and $ftf \in Mor_{n,m-1}(Y, ft^{i+1}(X))$ and therefore $ftf^*(X) \in B_{n+i+1}$. It remains to check that $\partial(rf) = ft^i(ftf^*(X))$. By definition of $Mor_{n,m}$ we have $\partial(rf) = ftf^*(ft^i(X))$.

To verify that $ft^i(ftf^*(X)) = ftf^*(ft^i(X))$ it is sufficient to check that $ft(ftf^*(X')) = ftf^*(ft(X'))$ for $X' = ft^j(X)$ where $j = 0, \dots, i-1$. Then $X' =\in B_{m+i-j}$ and $ftf \in Mor_{n,m-1}(Y, ft^{i-j+1}(X'))$.

If $m = 1$ then $ftf = p_{Y,n}$ and we have
$$ft(p^*_{Y,n}(X')) = ft(T_n(Y, X')) = T_n(Y, ft(X')) = p^*_{Y,n}(ft(X')).$$
where the middle equality holds by Lemma 3.1.

If $m > 1$ then $ftf = (rf', ftf')$ where $rf' \in \widetilde{B}_{n+1}$, $ftf' \in Mor_{n,m-2}(Y, ft^{i+2}(X))$, $(ftf')^*(X') \in B_{n+i-j+2}$ and
$$ft(ftf^*(X')) = ft(S(rf', (ftf')^*(X'))) = S(rf', ft((ftf^*(X')))) =$$
$$S(rf', (ftf')^*(ft(X'))) = (ftf)^*(ft(X))$$
where the second equality holds by property (3) assumed above since $i > j$ and the third equality holds by the inductive assumption.

For $f \in Mor_{n,m}(Y, X)$ where $m > 0$ we define $r(f) \in \widetilde{B}_{n+1}$ and $ft(f) \in Mor_{n,m-1}(Y, ft(X))$ by the condition that $f = (r(f), ft(f))$.

Let $i > 0$, $f \in Mor_{n,m}(Y, ft^i(X))$ and $s \in \widetilde{B}_{m+i}$ where $\partial(s) = X$ define $f^*(s) \in \widetilde{B}_{n+i}$ as follows:

1. if $m = 0$ then $f^*(s) = p^*_{Y,n}(s) = \widetilde{T}_n(Y, s)$,

2. if $m > 0$ then $f^*(s) = (r(f), ft(f))^*(s) =$

Let now $g \in Mor_{n,m}(Z, Y)$, $f \in Mor_{m,k}(Y, X)$. Define the composition $f \circ g \in Mor_{n,k}(Z, X)$ as follows:

1. if $k=0$ then $f\circ g = p_{Z,n}$,
2. if $k>0$ then $f\circ g = (g^*(r(f)), ft(f)\circ g)$.

To show that our construction is well defined we need to verify that several conditions:

Let $f: Y\to X$ be a morphism such that $Y\in B_n$ and $X\in B_m$. Define a sequence $(s_1(f),\dots,s_m(f))$ of elements of $\widetilde{B}_{n+1}$ inductively by the rule

$$(s_1(f),\dots,s_m(f)) = (s_1(ft(f)),\dots,s_{m-1}(ft(f)),s_f) = (s_{ft^{m-1}(f)},\dots,s_{ft(f)},s_f)$$

where $ft(f)=p_X f$, s_f is defined by the diagram (2) and for $m=0$ we start with the empty sequence. This construction can be illustrated by the following diagram for $f:Y\to X$ where $X\in B_4$:

$$\begin{array}{ccccccccccc}
Y & \xrightarrow{s_4(f)} & Z_{4,3} & \longrightarrow & Z_{4,2} & \longrightarrow & Z_{4,1} & \longrightarrow & T_n(Y,X) & \longrightarrow & X \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & Y & \xrightarrow{s_3(f)} & Z_{3,2} & \longrightarrow & Z_{3,1} & \longrightarrow & T_n(Y,ft(X)) & \longrightarrow & ft(X) \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & & & Y & \xrightarrow{s_2(f)} & Z_{2,1} & \longrightarrow & T_n(Y,ft^2(X)) & \longrightarrow & ft^2(X) \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & Y & \xrightarrow{s_1(f)} & T_n(Y,ft^3(X)) & \longrightarrow & ft^3(X) \\
 & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & & & Y & \longrightarrow & pt
\end{array} \qquad (9)$$

which is completely determined by the condition that the squares are the canonical ones and the composition of morphisms in the i-th arrow from the top is $ft^i(f)$. For the objects Z_i^j we have:

$$\begin{array}{lll}
Z_{4,1} = S(s_1(f),T_n(Y,X)) & Z_{4,2} = S(s_2(f),Z_{4,1}) & Z_{4,3} = S(s_3(f),Z_{4,2}) \\
Z_{3,1} = S(s_1(f),T_n(Y,ft(X))) & Z_{3,2} = S(s_2(f),Z_{3,1}) & \\
Z_{2,1} = S(s_1(f),T_n(Y,ft^2(X))) & &
\end{array} \qquad (10)$$

A simple inductive argument similar to the one in the proof of Lemma 2.3 show that if $f,f':Y\to X$ are two morphisms such that $X\in B_m$ and $s_i(f)=s_i(f')$ for $i=1,\dots,m$ then $f=f'$. Therefore, we may consider the set $Mor(CC)$ of morphisms of CC as a subset in $\amalg_{n,m\geq 0} B_n\times B_m\times \widetilde{B}_{n+1}^m$.

Let us show how to describe this subset in terms of the operations introduced above.

Lemma 3.2 *[***2009.11.07.l1***] An element $(Y,X,s_1,\dots,s_m)$ of $B_n\times B_m\times\widetilde{B}_{n+1}^m$ corresponds to a morphism if and only if the element $(Y,ft(X),s_1,\dots,s_{m-1})$ corresponds to a morphism and $\partial(s_m)=Z_{m,m-1}$ where $Z_{m,i}$ is defined inductively by the rule:*

$$Z_{m,0}=T_n(Y,X)\qquad Z_{m,i+1}=S(s_{i+1},Z_{m,i})$$

Proof: Straightforward from the example considered above.

Let us show now how to identify the canonical morphisms $p_{X,i} : X \to ft^i(X)$ and in particular the identity morphisms.

Lemma 3.3 *[***2009.11.10.l1***] Let $X \in B_m$ and $0 \le i \le m$. Let $p_{X,i} : X \to ft^i(X)$ be the canonical morphism. Then one has:*

$$s_j(p_{X,i}) = \widetilde{T}_{m-j}(X, \delta_{ft^{m-j}(X)}) \qquad j = 1, \dots, m-i$$

Proof: Let us proceed by induction on $m-i$. For $i = m$ the assertion is trivial. Assume the lemma proved for $i+1$. Since $ft(p_{X,i}) = p_{X,i+1}$ we have $s_j(p_{X,i}) = s_j(p_{X,i+1})$ for $j = 1, \dots, m-i-1$. It remains to show that

$$[\mathbf{2009.11.10.eq1}]s_{m-i}(p_{X,i}) = \widetilde{T}_i(X, \delta_{ft^i(X)}) \qquad (11)$$

By definition $s_{m-i}(p_{X,i}) = s_{p_{X,i}}$ and (11) follows from the commutative diagram:

$$\begin{array}{ccccc}
X & \longrightarrow & ft^i(X) & & \\
{\scriptstyle s_p}\downarrow & & \downarrow{\scriptstyle \delta_{ft^i(X)}} & & \\
p^*_{X,i+1}(ft^i(X)) & \longrightarrow & p^*_{ft^i(X)}(ft^i(X)) & \longrightarrow & ft^i(X) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle p_{ft^i(X)}} \\
X & \longrightarrow & ft^i(X) & \longrightarrow & ft^{i+1}(X)
\end{array}$$

where $p = p_{X,i}$.

Lemma 3.4 *[***2009.11.10.l2***] Let $(X, s) \in \widetilde{B}_{m+1}$, $Y \in B_n$ and $f : Y \to ft(X)$. Define inductively $(f,i)^*(s) \in \widetilde{B}_{n+m+1-i}$ by the rule*

$$(f, 0)^*(s) = \widetilde{T}_n(Y, s)$$

$$(f, i+1)^*(s) = \widetilde{S}(s_{i+1}(f), (f, i)^*(s))$$

Then $f^*(s) = (f, m)^*(s)$.

Proof: It follows from the diagram:

$$\begin{array}{ccccccccccc}
Y & \xrightarrow{s_m(f)} & * & \longrightarrow & \dots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & ft(X)\\
{\scriptstyle f^*(s)}\downarrow & & \downarrow{\scriptstyle (f,m-1)^*(s)} & & & & \downarrow{\scriptstyle (f,1)^*(s)} & & \downarrow{\scriptstyle (f,0)^*(s)} & & \downarrow{\scriptstyle s}\\
* & \longrightarrow & * & \longrightarrow & \dots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & X\\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow\\
Y & \xrightarrow{s_m(f)} & * & \longrightarrow & \dots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & ft(X)\\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow\\
 & & Y & \xrightarrow{s_{m-1}(f)} & \dots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & ft^2(X)\\
 & & & & & & \downarrow & & \downarrow & & \downarrow\\
 & & & & & & \dots & & \dots & & \dots\\
 & & & & & & \downarrow & & \downarrow & & \downarrow\\
 & & & & & & Y & \xrightarrow{s_1(f)} & * & \longrightarrow & ft^{m-1}(X)\\
 & & & & & & & & \downarrow & & \downarrow\\
 & & & & & & & & Y & \longrightarrow & pt
\end{array}$$

Lemma 3.5 *Let $g : Z \to Y$, $f : Y \to X$ and $X \in B_m$. Then $s_i(fg) = g^* s_i(f)$.*

Proof: It follows immediately from the equations $s_{fg} = g^* s_f$ and $ft(fg) = ft(f)g$.

Lemma 3.6 *[**2009.11.10.l4**] Let $f : Y \to ft(X)$ be a morphism, $Y \in B_n$ and $X \in B_{m+1}$. Define $(f,i)^*(X)$ inductively by the rule:*

$$(f,0)^*(X) = T_n(Y,X)$$

$$(f,i+1)^*(X) = S(s_{i+1}(f), (f,i)^*(X))$$

Then $f^*(X) = (f,m)^*(X)$.

Proof: Similar to the proof of Lemma 3.4.

Lemma 3.7 *[**2009.11.10.l4**] Let $f : Y \to ft(X)$ be a morphism, $Y \in B_n$ and $X \in B_{m+1}$. Then*

$$s_i(q(f,X)) = \begin{cases} \widetilde{T}(f^*X, s_i(f)) & \text{if } i \le m \\ \\ \widetilde{T}(f^*X, \delta_X) & \text{if } i = m+1 \end{cases}$$

Proof: We have $s_i(q(f,X)) = s_{ft^{m+1-i}(q(f,X))}$. For $i \le m$ we have $ft^{m+1-i}(q(f,X)) = ft^{m-i}(f)p_{f^*X}$. Therefore,

$$s_{ft^{m+1-i}(q(f,X))} = s_{ft^{m-i}(f)p_{f^*X}} = p^*_{f^*X} s_{ft^{m-i}(f)} = \widetilde{T}(f^*X, s_i(f))$$

and for $i = m+1$ we have

$$s_i(q(f,X)) = s_{q(f,X)} = p^*_{f^*X}(\delta_X) = \widetilde{T}(f^*X, \delta_X).$$

The lemmas proved above show that a C-structure can be reconstructed from the pair of sets $B, \widetilde{B}$ connected by the maps ft, ∂, δ, T, $\widetilde{T}$, S and $\widetilde{S}$. While this way of encoding C-structures may be less convenient than their encoding as a pair of sets Ob and Mor connected by the maps ∂_0, ∂_1, c (composition), id, ft and $qpb : (f,X) \mapsto q(f,X)$, this fact has the following important corollary.

Proposition 3.8 *[**2009.11.10.prop1**] Let CC, CC' be two C-structures. Then there is a natural bijection between C-structure morphisms $F : CC \to CC'$ and pairs of maps $F_0 : Ob(CC) \to Ob(CC')$, $F_1 : \widetilde{Ob}(CC) \to \widetilde{Ob}(CC')$ which commute in the obvious sense with ft, ∂, T, $\widetilde{T}$, S, $\widetilde{S}$ and δ.*

Remark 3.9 Notes on the properties of the maps introduced above:

1. for $Y \in B_{\ge n+2}$, $S(\delta_{ft^{n+1}(Y)}, T(ft^{n+1}(Y), Y)) = Y$.

2. The maps S and T can be defined as $ft\,\partial\,\widetilde{S}\,\delta$ and $ft\,\partial\,\widetilde{T}\,\delta$ respectively.

2 Type systems

1 Systems of expressions

Free systems of expressions. Let M be a set and let $T(M)$ be the set of finite rooted trees whose vertices (including the root) are labeled by elements of M and such that for any vertex the set of edges leaving this vertex is ordered. Note that such ordered trees have no symmetries. We will use the following notations. For $T \in T(M)$ let $Vrtx(T)$ be the set of vertices of T and for $v \in Vrtx(T)$ let $lbl(v) = lbl(v)_T \in M$ be the label on v. We will sometimes write $v \in T$ instead of $v \in Vrtx(T)$. For $v \in Vrtx(T)$ let $[v] = [v]_T \in T(M)$ be the subtree in T which consists of v and all the vertices under v. Let $val(v)$ be the valency of v i.e. the number of edges leaving v and $ch_1(v), \ldots, ch_{val(v)}(v) \in Vrtx(T)$ be the "children" of v i.e. the end points of these edges. Let further $br_i(v) = [ch_i(v)]$ be the branches of $[v]$. We write $v \le w$ (resp. $v < w$) if $v \in [w]$ (resp. $v \in [w] - w$). We say that two vertices v and w are independent if $v \notin [w]$ and $w \notin [v]$.

For three sets A, B and $Cont$ let

$$AllExp(A,B;Con) = T(A \amalg B \amalg (Con \times (\amalg_{n\ge 0} B^n)))$$

Elements of $AllExp(A,B;Con)$ are called expressions over the alphabet Con (or with a set of constructors Con), free variables from A and bound variables from B.

An expression is called unambiguous if it satisfies the following conditions:

1. if $lbl(v) \in A \amalg B$ then $val(v) = 0$,

2. (a) if $v < v'$, $lbl(v) = (c; x_1, \ldots, x_n)$ and $lbl(v') = (c'; x'_1, \ldots, x'_{n'})$ then $\{x_1, \ldots, x_n\} \cap \{x'_1, \ldots, x'_{n'}\} = \emptyset$,

(b) if $lbl(v) = (c; x_1, \ldots, x_n)$ then $x_i \neq x_j$ for $i \neq j$,

3. if $lbl(v) = (c; x_1, \ldots, x_n)$ and $lbl(v') \in \{x_1, \ldots, x_n\}$ then $v' \in [v]$.

The first conditions says that a vertex labeled by a variable is a leaf. The second one is equivalent to saying that if the same variable is bound at two different vertices v, v' then these vertices are independent i.e. $[v] \cap [v'] = \emptyset$ and that a vertex can not bind the same variable twice. The third one says that all the leaves labeled by a bound variable lie under the vertex where it is boud. We let $UAExp(A, B; Con)$ denote the subset of unambiguous expressions in $AllExp(A, B; Con)$. Note that for for any $T \in UAExp(A, B; Con)$ and $v \in Vrtx(T)$ there is a subset $Ext(v) \subset B$ such that

$$[v] \in UAExp(A \amalg Ext(v), B \backslash Ext(v); Con)$$

Any triple of maps $f_{Con} : A \to A'$, $f_B : B \to B'$, $f_{Con} : Con \to Con'$ define a map

$$f_* = (f_A, f_B, f_{Con})_* : AllExp(A, B; Con) \to AllExp(A', B'; Con')$$

which changes labels in the obvious way. If f_B is injective then f_* maps unambiguous expressions to unambiguous ones.

An element T of $UAExp(A, B; Con)$ is said to be strictly unambiguous if for any $v \neq v'$ in $Vrtx(T)$ such that $lbl(v) = (c; x_1, \ldots, x_n)$ and $lbl(v') = (c'; x'_1, \ldots, x'_{n'})$ one has $\{x_1, , \ldots, x_n\} \cap \{x'_1, \ldots, x'_{n'}\} = \emptyset$ i.e. if the names of all bound variables are different. We let $SUAExp(A, B; Con)$ denote the subset of strictly unambiguous expressions in $UAExp(A, B; Con)$.

An element T of $UAExp(A, B; Con)$ is said to be α-equivalent to an element T' of $UAExp(A, B'; Con)$ if there is a set B'' , an element $T'' \in UAExp(A, B''; Con)$ and two maps $f : B'' \to B$, $f' : B'' \to B'$ such that $T = (Id, f, Id)_*(T'')$ and $T' = (Id, f', Id)_*(T'')$. The following lemma is straightforward:

Lemma 1.1 *[***2009.09.08.l1***] For any two sets A and Con one has:*

1. *α-equivalence is an equivalence relation,*

2. *for any set B and any element $T \in UAExp(A, B; Con)$ there exists an element $T' \in UAExp(A, \mathbf{N}; Con)$ such that $T \stackrel{\alpha}{\sim} T'$ and T' is strictly unambiguous,*

3. *fwo strictly unambiguous elements $T, T' \in UAExp(A, B; Con)$ are α-equivalent if and only if there exists a permutation $f : B \to B$ such that $(Id, f, Id)_*(T) = T'$.*

We let $Exp_\alpha(A; Con)$ denote the set of α-equivalence classes in $\amalg_B UAExp(A, B; Con)$. In view of Lemma 1.1 this set is well defined and can be also defined as the set of equivalence classes in $SUAExp(A, \mathbf{N}; Con)$ modulo the equivalence relation generated by the permutations on $\mathbf{N}$.

Note that for two α-equivalent expressions T_1, T_2 and a vertex $v \in V(T_1) = V(T_2)$ the expressions $[v]_{T_1}$ and $[v]_{T_2}$ need not be α-equivalent since some of the variables which are bound in T_1 may be free in $[v]$.

The maps $(f_A, f_B, f_{Con})_*$ respect α-equivalence. Therefore for any $f_A : A \to A'$ and $f_{Con} : Con \to Con'$ there is a well defined map

$$(f_A, f_{Con})_* : Exp_\alpha(A; Con) \to Exp(A'; Con')$$

which make $Exp_\alpha(-;-)$ into a covariant functors from pairs of sets to sets. In addition there is a well defined notion of substitution on $Exp_\alpha(-;Con)$ which may be considered as a collection of maps of the form:

$$Exp_\alpha(A;Con) \times (\prod_{a\in A} Exp_\alpha(X_a;Con)) \to Exp_\alpha(\amalg_{a\in A} X_a;Con)$$

given for all pairs $(A;\{X_a\}_{a\in A})$ where A is a set and $\{X_a\}_{a\in A}$ a family of sets parametrized by A. Alternatively, the substitution structure can be seen as a collection of maps

$$Exp_\alpha(Exp_\alpha(A;Con);Con) \to Exp_\alpha(A;Con)$$

given for all A and Con. These maps make the functor $Exp_\alpha(-;Con)$ into a monad (triple) on the category of sets which functorially depends on the set Con.

Example 1.2 [**lambda**] The mapping which sends a set X to the set of α-equivalence classes of terms of the untyped λ-calculus with free variables from X is a sub-triple of $Exp_\alpha(-;Con)$ where $Con = \{\lambda, ev\}$. Elements T of $UAExp(X, \mathbf{N}; \{\lambda, ev\})$ which belong to this sub-triple are characterized by the following "local" conditions:

1. for each $v \in T$, $lbl(v) \in X \amalg \mathbf{N} \amalg \{ev\} \amalg \{\lambda\} \times \mathbf{N}$
2. if $lbl(v) \in \{\lambda\} \times \mathbf{N}$ then $val(v) = 1$
3. if $lbl(v) = ev$ then $val(v) = 2$.

Example 1.3 /**propositional**/The mapping which sends a set X to the set of terms of the propositional calculus with free variables from X is a sub-triple of $Exp_\alpha(-;C_0)$ where $C_0 = \{\vee, \wedge, \urcorner, \Rightarrow\}$. Elements T of $UAExp(X, \mathbf{N}; C_0)$ which belong to this sub-triple are characterized by the following "local" conditions:

1. for all $v \in T$, $lbl(v) \in X \amalg C_0$
2. if $lbl(v) \in \{\vee, \wedge, \Rightarrow\}$ then $val(v) = 2$
3. if $lbl(v) =\urcorner$ then $val(v) = 1$.

Example 1.4 [**multisorted**] Consider first order logic with several sorts $GS = \{S_1, \ldots, S_n\}$. Let GP be the set of generating predicates and GF the set of generating functions. Let $C_1 = C_0 \amalg \{\forall, \exists\}$ and $C_2 = C_1 \amalg GP \amalg GF \amalg GS$. We can identify the α-equivalence classes of formulas of the first order language defined by GS and GF with free variables from a set X with a subset in $Exp_\alpha(X, \mathbf{N}; C_2)$. Vertices which are labeled by $(\forall; x)$ and $(\exists; x)$ have valency two. For such a vertex v, the first branch of $[v]$ is one vertex labeled by an element of GS giving the sort over which the quantification occurs and the second branch is the expression which is quantified. Now however, these subsets do not form a sub-triple of Exp_α since not all substitutions are allowed. By allowing all substitutions irrespectively of the sort we get (for each X) a subset in $Exp_\alpha(X;C_2)$ whose elements will be called pseudo-formulas.

The following operations on expressions are well defined up to the α-equivalence:

1. If $T_1, \ldots, T_m \in Exp_\alpha(A; Con)$, $a_1, \ldots, a_n$ are pair-wise different elements of A and $M \in Con$ we will write $(M, a_1, \ldots, a_n)(T_1, \ldots, T_m)$ for the expression whose root v is labeled by $(M, a_1, \ldots, a_n)$, $val(v) = n$ and $br_i(v) = T_i$.

2. For $T_1, T_2 \in Exp_\alpha(A; Con)$ and $v \in T_1$ we let $T_1(T_2/[v])$ be the expression obtained by replacing $[v]$ in T_1 with T_2' where T_2' is obtained from T_2 by the change of bound variables such that the bound variables of T_2' do not conflict with the variables of T_1.

3. For $T, R_1, \ldots, R_n \in Exp_\alpha(A; Con)$ and $y_1, \ldots, y_n \in A$ we let $T(R_1/y_1, \ldots, R_n/y_n)$ denote the expression obtained by changing R_i's by α-equivalent R_i' such that $bnd(R_i') \cap bnd(R_j)' = \emptyset$ for $i \neq j$, changing T to an α-equivalent T' such that $bnd(T') \cap (var(R_1') \cup \ldots \cup var(R_n')) = \emptyset$ and then replacing all the leaves of T' marked by y_i by R_i'.

In all the examples considered above, these operations correspond to the usual operations on formulas. The first operation can be used to directly associate expressions in our sense with the formulas. For example, the expression associated with the formula $\forall x : S.P(x, y)$ in a multi-sorted predicate calculus is $(\forall, x)(S, P(x, y))$ where as was mentioned above we use the same notation for an element of $A \amalg B \amalg (Con \times (\amalg_{n \geq 0} B^n))$ and the one vertex tree with the corresponding label.

Note: about representing elements of $AllExp(A, B; Con)$ by linear sequences of elements of $A \amalg B \amalg$??.

Reduction structures. Another component of the structure present in systems of expressions used in formal systems is the reduction relation. It is very important for our approach to type systems that the reduction relation is defined on all pseudo-formulas and is compatible with the substitution structure even when not all psedu-formulas are well formed formulas. In what follows we will consider, instead of a particular syntactic system, a pair $(S, \triangleright)$ where S is a continuous triple on the category of sets and $\triangleright$ is a reduction structure on S i.e. a collection of relations $\triangleright_X$ on $S(X)$ given for all finite sets X satisfying the following two conditions:

1. if $E \in S(\{x_1, \ldots, x_n\})$, $f_1, \ldots, f_n, f_i' \in S(\{y_1, \ldots, y_m\})$ and $f_i \triangleright_{\{y_1, \ldots, y_m\}} f_i'$ then
$$E(f_1/x_1, \ldots, f_i/x_i, \ldots f_n/x_n) \triangleright_{\{x_1, \ldots, x_n\}} E(f_1/x_1, \ldots, f_i'/x_i, \ldots f_n/x_n),$$

2. if $E, E' \in S(\{x_1, \ldots, x_n\})$, $f_1, \ldots, f_n \in S(\{y_1, \ldots, y_m\})$ and $E \triangleright_{\{x_1, \ldots, x_n\}} E'$ then
$$E(f_1/x_1, \ldots, f_n/x_n) \triangleright_{\{x_1, \ldots, x_n\}} E'(f_1/x_1, \ldots, f_n/x_n).$$

The following two results are obvious but important.

Proposition 1.5 *[***2009.10.17.prop1***] Let S be a continuous triple on Sets and $\triangleright_\alpha$ be a family of reduction structures on S. Then the intersection $\cap_\alpha \triangleright_\alpha : X \mapsto \cap_\alpha \triangleright_{\alpha, X}$ is a reduction structure on S.*

Corollary 1.6 *[***2009.10.17.cor1***] For any family (X_α, pre_α) of pairs of the form (X, pre) where X is a set and pre is a relation on $S(X)$ (i.e. a subset of $S(X) \times S(X)$) there exists the smallest reduction structure $\triangleright = \triangleright(X_\alpha, pre_\alpha)$ on S such that for each α and each $(f, g) \in pre_\alpha$ one has $f \triangleright g$.*

2 C-structures defined by a triple.

Let S be a continuous triple on $Sets$. Let $S - cor$ be the full subcategory of the Kleisli category of S whose objects are finite sets. Recall, that the set of morphisms from X to Y in $S - cor$ is the set of maps from X to $S(Y)$ i.e. $S(Y)^X$ (in other words, $S - cor$ is the category of free, finitely generated S-algebras). We will construct two C-structures $C(S)$ and $CC(S)$which are based on $(S - cor)^{op}$.

Examples:

1. If $S = Id$ i.e. $S(X) = X$ the $S - cor = FSets$ is the category of finite sets. It is easy to see that the category of finite sets is the free category with finite coproducts generated by one object. Therefore, $(FSets)^{op}$ can be thought of the free category with finite products generated by one object.

2. Let S be given by $S(X) = X \amalg A$ where A is a set. This corresponds to the system of expressions where all expressions are either variables or constants and the set of constants is A. The category $(S - cor)^{op}$ can be though of as the free category with finite products generated by an object U and the set A of morphisms $pt \to U$.

The categories of sets, finite sets or even the category of finite linearly ordered sets and their isomorphisms are all level 1 categories and so is the category $S - cor$. We can get a set-level model $C(S)$ for $(S - cor)^{op}$ by setting $Ob(C(S)) = \mathbf{N}$ and $Hom_{C(S)}(n, m) = S(\{1, \ldots, n\})^m$.

The category $C(S)$ extends to a C-structure which is defined as follows. The final object is 0. The map ft is given by

$$ft(n) = \begin{cases} 0 & \text{if } n = 0 \\ n - 1 & \text{if } n > 0 \end{cases}$$

The canonical projection $n \to n - 1$ is given by the sequence $(1, \ldots, n - 1)$. For $f = (f_1, \ldots, f_m) : n \to m$ the canonical square build on f and the canonical projection $m + 1 \to m$ is of the form

$$\begin{array}{ccc} n+1 & \xrightarrow{(f_1,\ldots,f_m,n+1)} & m+1 \\ \downarrow & & \downarrow \\ n & \xrightarrow{(f_1,\ldots,f_m)} & m \end{array}$$

Any morphism of triples $S \to S'$ defines a C-structure morphism $C(S) \to C(S')$. Non-trivial C-substructures of $C(S)$ are in one-to-one correspondence with continuous sub-triples of S.

Note: add notes that a continuous sub-triple of S is exactly the same as a subcategory in $S - cor$ which contains all (isomorphism classes of) objects. Intersection of two sub-triples is a sub-triple which allows us to speak of sub-triples (systems of expressions etc.) generated by a set of expressions. For the construction of type systems the category $S - cor$ is replaced by the C-structure $CC(S, X)$.

Note: that continuous triples on $Sets$ are the same as category structures on $\mathbf{N}$ which extend the a category structure of finite sets and where the addition remains to be coproduct.

Let now $CC(S)$ be the set-level category whose set of objects is $Ob(CC(S)) = \amalg_{n \geq 0} Ob_n$ where

$$Ob_n = S(\emptyset) \times \ldots \times S(\{1, \ldots, n - 1\})$$

and the set of morphisms is

$$mor(CC(S)) = \coprod_{n,m\geq 0} Ob_n \times Ob_m \times S(\{1,\ldots,n\})^m$$

with the obvious domain and codomain maps. The composition of morphisms is defined in the same way as in $C(S)$ such that the mapping $Ob(CC(S)) \to \mathbf{N}$ which sends all elements of Ob_n to n, is a functor. The associativity of compositions follows immediately from the associativity of compositions in $S - cor$.

Note that if $S(\emptyset) = \emptyset$ then $CC(S) = \emptyset$ and otherwise the functor $CC(S) \to (S - cor)^{op}$ is an equivalence, so that in the second case $C(S)$ and $CC(S)$ are indistinguishable as level 1 categories. However, as set level categories they are quite different.

The category $CC(S)$ is given a C-structure as follows. The final object is the only element of Ob_0, the map ft is defined by the rule

$$ft(T_1,\ldots,T_n) = (T_1,\ldots,T_{n-1}).$$

The canonical pull-back square defined by an object $(T_1,\ldots,T_{m+1})$ and a morphism $(f_1,\ldots,f_m) \in S(\{1,\ldots,n\})^m$ from $(R_1,\ldots,R_n)$ to $(T_1,\ldots,T_m)$ is of the form

[**2009.11.05.oldeq1**]

$$\begin{array}{ccc} (R_1,\ldots,R_n,T_{m+1}(f_1/1,\ldots,f_m/m)) & \xrightarrow{(f_1,\ldots,f_m,n+1)} & (T_1,\ldots,T_{m+1}) \\ \downarrow & & \downarrow \\ (R_1,\ldots,R_n) & \xrightarrow{(f_1,\ldots,f_m)} & (T_1,\ldots,T_m) \end{array} \tag{12}$$

Proposition 2.1 [**2009.10.01.prop2**] *With the maps defined above $CC(S)$ is a C-structure.*

Proof: Straightforward.

Note that the natural projection $CC(S) \to C(S)$ is a C-structure morphism. It's C-structure sections are in one-to-one correspondence with $S(\emptyset)$ such that $U \in S(\emptyset)$ corresponds to the section which takes the object n of $C(S)$ to the object $(U,\ldots,U)$ of $CC(S)$.

Any morphism of triples $S \to S'$ defines a C-structure morphism $CC(S) \to CC(S')$. C-substructures of $CC(S)$, which are discussed in more detail below, provide an important class of type systems over S.

There is another construction of a category from a continuous triple S which takes as an additional parameter a set Var which is called the set of variables. Let $F_n(Var)$ be the set of sequences of length n of pair-wise distinct elements of Var. Define the category $CC(S, Var)$ as follows. The set of objects of $CC(S, Var)$ is

$$Ob(CC(S,Var)) = \amalg_{n\geq 0} \amalg_{(x_1,\ldots,x_n)\in F_n(Var)} S(\emptyset) \times \ldots \times S(\{x_1,\ldots,x_{n-1}\})$$

For notational compatibility with the traditional type theory we will write the elements of $Ob(CC(S,X))$ as sequences of the form $x_1 : E_1,\ldots,x_n : E_n$. The set of morphisms is given by

$$Hom_{CC(S,,Var)}((x_1 : E_1,\ldots,x_n : E_n),(y_1 : T_1,\ldots,y_m : T_m)) = S(\{x_1,\ldots,x_n\})^m$$

The composition is defined in such a way that the projection

$$(x_1 : E_1, \ldots, x_n : E_n) \mapsto (E_1, E_2(1/x_1), \ldots, E_n(1/x_1, \ldots, n-1/x_{n-1}))$$

is a functor from $CC(S, X)$ to $CC(S)$. This functor is clearly an equivalence. There is an obvious final object and ft map on $CC(S, X)$. There is however a real problem in making it into a C-structure which is due to the following. Consider an object $(y_1 : T_1, \ldots, y_{m+1} : T_{m+1})$ and a morphism $(f_1, \ldots, f_m) : (x_1 : R_1, \ldots, x_n : R_n) \to (y_1 : T_1, \ldots, y_m : T_m)$. In order for the functor to $CC(S)$ to be a C-structure morphism the canonical square build on this pair should have the form

$$\begin{array}{ccc} (x_1 : R_1, \ldots, x_n : R_n, x_{n+1} : T_{m+1}(f_1/1, \ldots, f_m/m)) & \xrightarrow{(f_1,\ldots,f_m,n+1)} & (y_1 : T_1, \ldots, y_{m+1} : T_{m+1}) \\ \downarrow & & \downarrow \\ (x_1 : R_1, \ldots, x_n : R_n) & \xrightarrow{(f_1,\ldots,f_m)} & (y_1 : T_1, \ldots, y_m : T_m) \end{array}$$

where x_{n+1} is an element of X which is distinct from each of the elements $x_1, \ldots, x_n$. Moreover, we should choose x_{n+1} in such a way the the resulting construction satisfies the C-structure axioms for $(f_1, \ldots, f_m) = Id$ and for the compositions $(g_1, \ldots, g_n) \circ (f_1, \ldots, f_m)$. One can easily see that no such choice is possible for a finite set X. At the moment it is not clear to me whether or not such it is possible for an infinite X.

3 C-substructures of $CC(S)$.

Let TS be a C-substructure of $CC(S)$. By Lemma 2.3, TS is determined by the subsets $B = Ob(TS)$ and $\widetilde{B} = \widetilde{Ob}(TS)$ in $Ob(CC(S))$ and $\widetilde{Ob}(CC(S))$. By definition we have

$$Ob(CC(S)) = \coprod_{n \geq 0} \prod_{i=0}^{n-1} S(\{1, \ldots, i\})$$

An element of $\widetilde{Ob}(CC(S))$ is given by a pair (Γ, s) where $\Gamma \in Ob(CC(S))$ is an object and $s : ft(\Gamma) \to \Gamma$ is a section of the canonical morphism $p_\Gamma : \Gamma \to ft(\Gamma)$. It follows immediately from the definition of $CC(S)$ that for $\Gamma = (E_1, \ldots, E_{n+1})$, a morphism $(f_1, \ldots, f_{n+1}) \in S(\{1, \ldots, n\})^{n+1}$ from $ft(\Gamma)$ to Γ is a section of p_Γ if an only if $f_i = i$ for $i = 1, \ldots, n$. Therefore, any such section is determined by its last component f_{n+1} and mapping $((E_1, \ldots, E_{n+1}), (f_1, \ldots, f_{n+1}))$ to $(E_1, \ldots, E_n, E_{n+1}, f_{n+1})$ we get a bijection

$$[\mathbf{2009.10.15.eq2}]\widetilde{Ob}(CC(S)) \cong \coprod_{n \geq 0} (\prod_{i=0}^{n-1} S(\{1, \ldots, i\})) \times S(\{1, \ldots, n\})^2 \qquad (13)$$

For $\Gamma = (E_1, \ldots, E_n)$ we write $(\Gamma \rhd_{TS})$ if $(E_1, \ldots, E_n)$ is in B and $(\Gamma \vdash_{TS} t : T)$ if $(E_1, \ldots, E_n, T, t)$ is in $\widetilde{B}$. When no confusion is possible we will write $\vdash$ instead of $\vdash_{TS}$. We also write $l(\Gamma) = n$ and $ft(\Gamma) = (E_1, \ldots, E_{n-1})$.

The following result is an immediate corollary of Proposition 2.5.

Proposition 3.1 *[**2009.10.16.prop3**] Let S be a continuous triple on Sets. A pair of subsets*

$$B \subset \coprod_{n \geq 0} \prod_{i=0}^{n-1} S(\{1, \ldots, i\})$$

$$\widetilde{B} \subset \coprod_{n\geq 0} (\prod_{i=0}^{n-1} S(\{1,\dots,i\})) \times S(\{1,\dots,n\})^2$$

defines a C-substructure of $CC(S)$ if and only if the following conditions hold:

1. $(\rhd)$
2. $(\Gamma\rhd) \Rightarrow (ft(\Gamma)\rhd)$
3. $(\Gamma \vdash t : T) \Rightarrow (\Gamma, T\rhd)$
4. $(\Gamma_1, \Gamma_2, \vdash o : S) \wedge (\Gamma_1, T\rhd) \Rightarrow (\Gamma_1, T, s_{i+1}\Gamma' \vdash s_{i+1}o : s_{i+1}S)$ *where* $i = l(\Gamma_1)$
5. $(\Gamma_1, T, \Gamma_2 \vdash o : S) \wedge (\Gamma_1 \vdash r : T) \Rightarrow (\Gamma_1, d_{i+1}(\Gamma_2[r/i+1]) \vdash d_{i+1}(t[r/i+1]) : d_{i+1}(T[r/i+1]))$ *where* $i = l(\Gamma_1)$
6. $(\Gamma, T\rhd) \Rightarrow (\Gamma, T \vdash n+1 : T)$ *where* $n = l(\Gamma)$.

where for $E \in S(\{1,\dots,k\})$, $s_iE = E[i+1/i,\dots,k+1/k] \in S(\{1,\dots,k+1\}$ *and* $d_iE = E[i/i+1,\dots,k-1/k] \in S(\{1,\dots,k-1\}$

Note that conditions (4) and (5) together with condition (6) and condition (3) imply the following

4a $(\Gamma_1, \Gamma_2\rhd) \wedge (\Gamma_1, T\rhd) \Rightarrow (\Gamma_1, T, s_{i+1}\Gamma_2\rhd)$ where $i = l(\Gamma_1)$

5a $(\Gamma_1, T, \Gamma_2\rhd) \wedge (\Gamma_1 \vdash r : T) \Rightarrow (\Gamma_1, d_{i+1}(\Gamma_2[r/i+1])\rhd)$ where $i = l(\Gamma_1)$.

Note also that modulo condition (2), condition (1) is equivalent to the condition that $B \neq \emptyset$.

Remark 3.2 [**2010.08.07.rem1**] If one re-writes the conditions of Proposition 3.1 in the more familiar in type theory form where the variables introduced in the context are named rather than directly numbered one arrives at the following rules:

$$\frac{}{\rhd} \qquad \frac{x_1 : E_1,\dots,x_n : E_n\rhd}{x_1 : E_1,\dots,x_{n-1} : E_{n-1} \vdash} \qquad \frac{x_1 : E_1,\dots,x_n : E_n \rhd t : T}{x_1 : E_1,\dots,x_n : E_n \vdash}$$

$$\frac{x_1 : E_1,\dots,x_n : E_n \vdash t : T \quad x_1 : E_1,\dots,x_i : E_i, y : F\rhd}{x_1 : E_1,\dots,x_i : E_i, y : F, x_{i+1} : E_{i+1},\dots,x_n : E_n \vdash t : T}, \quad i = 0,\dots,n$$

$$\frac{x_1 : E_1,\dots,x_n : E_n \vdash t : T \quad x_1 : E_1,\dots,x_i : E_i \vdash r : E_{i+1}}{x_1 : E_1,\dots,x_i : E_i, x_{i+2} : E_{i+2}[r/x_{i+1}],\dots,x_n : E_n[r/x_{i+1}] \vdash t[r/x_{i+1}] : T[r/x_{i+1}]}, \quad i = 0,\dots,n-1$$

$$\frac{x_1 : E_1,\dots,x_n : E_n\rhd}{x_1 : E_1,\dots,x_n : E_n \vdash x_n : E_n}$$

which are similar to (and probably equivalent) the "basic rules of DTT" given in [4, p.585]. The advantage of the rules given here is that they are precisely the ones which are necessary and sufficient for a given collection of contexts and judgements to define a C-structure.

Lemma 3.3 *[**2009.11.05.l1**] Let S, B, $\widetilde{B}$ be as above and let $(E_1, \ldots, E_n), (T_1, \ldots, T_m) \in B$ and $(f_1, \ldots, f_m) \in S(\{1, \ldots, n\})^m$. Then*

$$(f_1, \ldots, f_m) \in Hom_{TS}((E_1, \ldots, E_n), (T_1, \ldots, T_m))$$

if and only if $(f_1, \ldots, f_{m-1}) \in Hom_{TS}((E_1, \ldots, E_n), (T_1, \ldots, T_{m-1}))$ and

$$(E_1, \ldots, E_n, T_m(f_1/1, \ldots, f_{m-1}/m-1), f_m) \in \widetilde{B}$$

Proof: Straightforward using the fact that the canonical pull-back squares in $CC(S)$ are given by (12).

4 Type systems over S.

Definition 4.1 *[**typesystem**] Let S be a continuous triple on Sets. A type system over S is a collection of data of the form:*

$$B \subset \coprod_{n\geq 0} \prod_{i=0}^{n-1} S(\{1, \ldots, i\})$$

$$Beq \subset \coprod_{n\geq 0} (\prod_{i=0}^{n-1} S(\{1, \ldots, i\})) \times S(\{1, \ldots, n\})^2$$

$$\widetilde{B} \subset \coprod_{n\geq 0} (\prod_{i=0}^{n-1} S(\{1, \ldots, i\})) \times S(\{1, \ldots, n\})^2$$

$$\widetilde{Beq} \subset \coprod_{n\geq 0} (\prod_{i=0}^{n-1} S(\{1, \ldots, i\})) \times S(\{1, \ldots, n\})^3$$

For $\Gamma = (T_1, \ldots, T_n) \in \coprod_{n\geq 0} \prod_{i=0}^{n-1} S(\{1, \ldots, i\})$ and $S_1, S_2 \in S(\{1, \ldots, i\})$ we write $(\Gamma \vdash S_1 = S_2)$ to signify that $(T_1, \ldots, T_n, S_1, S_2) \in Beq$. Similarly for $S, o, o' \in S(\{1, \ldots, n\})$ we write $(\Gamma \vdash o = o' : S)$ to signify that $(T_1, \ldots, T_n, S, o, o') \in \widetilde{Beq}$. These data should satisfy the following conditions:

1. *Conditions (1)-(6) on B and $\widetilde{B}$ from Proposition 3.1 (referred to below as conditions (1.1)-(1.6) from Definition 4.1).*

2.
$$\begin{array}{ll} (a) & (\Gamma \vdash T = T') \Rightarrow (\Gamma, T\rhd) \\ (b) & (\Gamma, T\rhd) \Rightarrow (\Gamma \vdash T = T) \\ (c) & (\Gamma \vdash T = T') \Rightarrow (\Gamma \vdash T' = T) \\ (d) & (\Gamma \vdash T = T') \wedge (\Gamma \vdash T' = T'') \Rightarrow (\Gamma \vdash T = T'') \end{array}$$

3.
$$\begin{array}{ll} (a) & (\Gamma \vdash o = o' : T) \Rightarrow (\Gamma \vdash o : T) \\ (b) & (\Gamma \vdash o : T) \Rightarrow (\Gamma \vdash o = o : T) \\ (c) & (\Gamma \vdash o = o' : T) \Rightarrow (\Gamma \vdash o' = o : T) \\ (d) & (\Gamma \vdash o = o' : T) \wedge (\Gamma \vdash o' = o'' : T) \Rightarrow (\Gamma \vdash o = o'' : T) \end{array}$$

4.

$$\begin{array}{ll} (a) & (\Gamma_1 \vdash T = T') \wedge (\Gamma_1, T, \Gamma_2 \vdash S = S') \Rightarrow (\Gamma_1, T', \Gamma_2 \vdash S = S') \\ (b) & (\Gamma_1 \vdash T = T') \wedge (\Gamma_1, T, \Gamma_2 \vdash o = o' : S) \Rightarrow (\Gamma_1, T', \Gamma_2' \vdash o = o' : S) \\ (c) & (\Gamma \vdash S = S') \wedge (\Gamma \vdash o = o' : S) \Rightarrow (\Gamma \vdash o = o' : S') \end{array}$$

5.

$$\begin{array}{lll} (a) & (\Gamma_1, T \rhd) \wedge (\Gamma_1, \Gamma_2 \vdash S = S') \Rightarrow (\Gamma_1, T, s_{i+1}\Gamma_2 \vdash s_{i+1}S = s_{i+1}S') & i = l(\Gamma) \\ (b) & (\Gamma_1, T \rhd) \wedge (\Gamma_1, \Gamma_2 \vdash o = o' : S) \Rightarrow (\Gamma_1, T, s_{i+1}\Gamma_2 \vdash s_{i+1}o = s_{i+1}o' : s_{i+1}S) & i = l(\Gamma) \end{array}$$

6.

$$\begin{array}{ll} (a) \quad (\Gamma_1, T, \Gamma_2 \vdash S = S') \wedge (\Gamma_1 \vdash r : T) \Rightarrow & \\ (\Gamma_1, d_{i+1}(\Gamma_2[r/i+1]) \vdash d_{i+1}(S[r/i+1]) = d_{i+1}(S'[r/i+1])) & i = l(\Gamma_1) \\ (b) \quad (\Gamma_1, T, \Gamma_2 \vdash o = o' : S) \wedge (\Gamma_1 \vdash r : T) \Rightarrow & \\ (\Gamma_1, d_{i+1}(\Gamma_2[r/i+1]) \vdash d_{i+1}(o[r/i+1]) = d_{i+1}(o'[r/i+1]) : d_{i+1}(S[r/i+1])) & i = l(\Gamma_1) \end{array}$$

7.

$$\begin{array}{ll} (a) \quad (\Gamma_1, T, \Gamma_2, S \rhd) \wedge (\Gamma_1 \vdash r = r' : T) \Rightarrow & \\ (\Gamma_1, d_{i+1}(\Gamma_2[r/i+1]) \vdash d_{i+1}(S[r/i+1]) = d_{i+1}(S[r'/i+1])) & i = l(\Gamma_1) \\ (b) \quad (\Gamma_1, T, \Gamma_2 \vdash o : S) \wedge (\Gamma_1 \vdash r = r' : T) \Rightarrow & \\ (\Gamma_1, d_{i+1}(\Gamma_2[r/i+1]) \vdash d_{i+1}(o[r/i+1]) = d_{i+1}(o[r'/i+1]) : d_{i+1}(S[r/i+1])) & i = l(\Gamma_1) \end{array}$$

Definition 4.2 *[simandsimeq] Given S, B, Beq, $\widetilde{B}$ and $\widetilde{Beq}$ as above and assuming that conditions (1.2) and (1.3) hold, define relations $\sim_n$ on B_n and $\simeq_n$ on $\widetilde{B}_n$ as follows:*

1. *for $\Gamma = (T_1, \ldots, T_n)$, $\Gamma' = (T_1', \ldots, T_n')$ in B_n we set $\Gamma \sim_n \Gamma'$ iff $ft(\Gamma) \sim_{n-1} ft(\Gamma')$ and $T_1, \ldots, T_{n-1} \vdash T_n = T_n'$,*
2. *for $(\Gamma \vdash o : S)$, $(\Gamma' \vdash o' : S')$ in $\widetilde{B}_n$ we set $(\Gamma \vdash o : S) \simeq_n (\Gamma' \vdash o' : S')$ iff $(\Gamma, S) \sim_n (\Gamma', S')$ and $(\Gamma \vdash o = o' : S)$.*

Lemma 4.3 *[iseqrelsiml1] Let S, B, Beq, $\widetilde{B}$ and $\widetilde{Beq}$ be as above. Then for all $n \geq 0$, one has:*

1. *If conditions (1.2), (4a) of Definition 4.1 holds then $(\Gamma \vdash S = S') \wedge (\Gamma \sim_n \Gamma') \Rightarrow (\Gamma' \vdash S = S')$.*
2. *If conditions (1.2), (1.3), (4a), (4b), (4c) hold then $(\Gamma \vdash o = o' : S) \wedge (\Gamma, S \sim_{n+1} \Gamma', S') \Rightarrow (\Gamma' \vdash o = o' : S')$.*

Proof: (1) For $n = 0$ the assertion is obvious. Therefore by induction we may assume that $(\Gamma \vdash S = S') \wedge (\Gamma \sim_i \Gamma') \Rightarrow (\Gamma' \vdash S = S')$ for all $i < n$ and all appropriate Γ,Γ', S and S' and that $(T_1, \ldots, T_n \vdash S = S') \wedge (T_1, \ldots, T_n \sim_n T_1', \ldots, T_n')$ holds and we need to show that $(T_1', \ldots, T_n' \vdash S = S')$ holds. Let us show by induction on j that $(T_1', \ldots, T_j', T_{j+1}, \ldots, T_n \vdash S = S')$ for all $j = 0, \ldots, n$. For $j = 0$ it is a part of our assumptions. By induction we may assume that $(T_1', \ldots, T_j', T_{j+1}, \ldots, T_n \vdash S = S')$. By definition of $\sim_n$ we have $(T_1, \ldots, T_j \vdash T_{j+1} = T_{j+1}')$. By the inductive assumption we have $(T_1', \ldots, T_j' \vdash T_{j+1} = T_{j+1}')$. Applying (4a) with $\Gamma_1 = (T_1', \ldots T_j')$, $T = T_{j+1}$, $T' = T_{j+1}'$ and $\Gamma_2 = (T_{j+2}, \ldots, T_n)$ we conclude that $(T_1', \ldots, T_{j+1}', T_{j+2}, \ldots, T_n \vdash S = S')$.

(2) By the first part of the lemma we have $\Gamma' \vdash S = S'$. Therefore by (4c) it is sufficient to show that $(\Gamma \vdash o = o' : S) \wedge (\Gamma \sim_n \Gamma') \Rightarrow (\Gamma' \vdash o = o' : S)$. The proof of this fact is similar to the proof of the first part of the lemma using (4b) instead of (4a).

Proposition 4.4 *[iseqrelsim] Let S, B, Beq, $\widetilde{B}$ and $\widetilde{Beq}$ be as above. Then one has:*

1. *Assume that conditions (1.2), (2b), (2c), (2d) and (4a) of Definition 4.1 hold. Then $\sim_n$ is an equivalence relation for each $n \geq 0$.*
2. *Assume that conditions of the previous part of the proposition as well as conditions (1.3), (3b), (3c), (3d), (4b) and (4c) hold. Then $\simeq_n$ is an equivalence relation for each $n \geq 0$.*

Proof: (1) Reflexivity follows directly from (1.2) and (2b). The relation $\sim_0$ is symmetric by (2c). Let $(\Gamma, T) \sim_{n+1} (\Gamma', T')$. By induction we may assume that $\Gamma' \sim_n \Gamma$. By Lemma 4.3(a) we have $(\Gamma' \vdash T = T')$ and by (2c) we have $(\Gamma' \vdash T' = T)$. We conclude that $(\Gamma', T') \sim_{n+1} (\Gamma, T)$ i.e. that $\sim_{n+1}$ is symmetric. The proof of transitivity is by a similar induction.

(2) Reflexivity follows directly from reflexivity of $\sim_*$, (1.3) and (3b). Symmetry and transitivity are also easy using Lemma 4.3.

From this point on we assume that all conditions of Definition 4.1 hold. Let $B'_n = B_n/\sim_n$ and $\widetilde{B}'_n = \widetilde{B}_n/\simeq_n$. It follows immediately from our definitions that the functions $ft : B_{n+1} \rightarrow B_n$ and $\partial : \widetilde{B}_n \rightarrow B_n$ define functions $ft' : B'_{n+1} \rightarrow B'_n$ and $\partial' : \widetilde{B}'_n \rightarrow B'_n$.

Lemma 4.5 *[surjl1] Under the above assumptions the following maps are surjective for all $m \geq n \geq 0$:*

$$\pi_{T,m,n} : (B_{n+1})_{ft} \times_{ft^{m+1-n}} (B_{m+1}) \rightarrow (B'_{n+1})_{ft'} \times_{(ft')^{m+1-n}} (B'_{m+1})$$
$$\pi_{\widetilde{T},m,n} : (B_{n+1})_{ft} \times_{ft^{m+1-n}\partial} (\widetilde{B}_{m+1}) \rightarrow (B'_{n+1})_{ft'} \times_{(ft')^{m+1-n}\partial'} (\widetilde{B}'_{m+1})$$
$$\pi_{S,m,n} : (\widetilde{B}_{n+1})_{\partial} \times_{ft^{m+1-n}} (B_{m+2}) \rightarrow (\widetilde{B}'_{n+1})_{\partial''} \times_{(ft')^{m+1-n}} (B'_{m+2})$$
$$\pi_{\widetilde{S},m,n} : (\widetilde{B}_{n+1})_{\partial} \times_{ft^{m+1-n}\partial} (\widetilde{B}_{m+2}) \rightarrow (\widetilde{B}'_{n+1})_{\partial'} \times_{(ft')^{m+1-n}\partial'} (B'_{m+2})$$

Proof: We will show that the projections

$$(B_{n+1})_{ft} \times_{ft^{m+1-n}} (B_{m+1}) \rightarrow (B'_{n+1})_{ft'} \times_{(ft')^{m+1-n}} (B_{m+1})$$
$$(B_{n+1})_{ft} \times_{ft^{m+1-n}\partial} (\widetilde{B}_{m+1}) \rightarrow (B'_{n+1})_{ft'} \times_{(ft')^{m+1-n}\partial'} (\widetilde{B}_{m+1})$$
$$(\widetilde{B}_{n+1})_{\partial} \times_{ft^{m+1-n}} (B_{m+2}) \rightarrow (\widetilde{B}'_{n+1})_{\partial'} \times_{(ft')^{m+1-n}} (B_{m+2})$$
$$(\widetilde{B}_{n+1})_{\partial} \times_{ft^{m+1-n}\partial} (\widetilde{B}_{m+2}) \rightarrow (\widetilde{B}'_{n+1})_{\partial'} \times_{(ft')^{m+1-n}\partial'} (B_{m+2})$$

are already surjective.

(1) We need to show that for $(\Gamma_1, T\rhd)$, $(\Gamma'_1, \Gamma_2\rhd)$ where $\Gamma_1 \sim_n \Gamma'_1$ there exists $(\Gamma'_1, T'\rhd)$ such that $(\Gamma_1, T) \sim_n (\Gamma'_1, T')$. It is sufficient to take $T = T'$. Indeed by (2b) we have $\Gamma \vdash T = T$, by Lemma 4.3(2) we conclude that $\Gamma' \vdash T = T$ and by (1a) that $\Gamma', T\rhd$.

(2) Same proof as for (1).

(3) We need to show that for $(\Gamma_1 \vdash o : S)$, $(\Gamma'_1, S', \Gamma_2\rhd)$ where $(\Gamma, S) \sim_{n+1} (\Gamma', S')$ there exists $(\Gamma'_1 \vdash o' : S')$ such that $(\Gamma'_1 \vdash o' : S') \simeq_{n+1} (\Gamma_1 \vdash o : S)$. It is sufficient to take $o' = o$. Indeed, by (3b) we have $(\Gamma_1 \vdash o = o : S)$, by Lemma 4.3(2) we conclude that $(\Gamma'_1 \vdash o = o : S')$ and by (2a) that $(\Gamma'_1 \vdash o : S')$.

(4). Same proof as for (3).

Lemma 4.6 *[**TSetc**] Under the above assumptions the maps $T, \widetilde{T}, S, \widetilde{S}$ and δ which form the C-structure on $(B, \widetilde{B})$ define unique maps*

$$T' : (B'_{n+1})_{ft'} \times_{(ft')^{m-n}} (B'_m) \to B'_{m+1}$$

$$\widetilde{T}' : (B'_{n+1})_{ft'} \times_{(ft')^{m+1-n}\partial'} (\widetilde{B}'_{m+1}) \to \widetilde{B}'_{m+2}$$

$$S' : (\widetilde{B}'_{n+1})_{\partial'} \times_{(ft')^{m-n}} (B'_{m+1}) \to B'_m$$

$$\widetilde{S}' : (\widetilde{B}'_{n+1})_{\partial'} \times_{(ft')^{m+1-n}\partial'} (\widetilde{B}'_{m+2}) \to \widetilde{B}'_{m+1}$$

$$\delta' : B'_{n+1} \to \widetilde{B}'_{n+2}$$

Proof: Uniqueness follows immediately from Lemma 4.5. Let us show existence.

(1) Given $(\Gamma_1, T\rhd) \sim_{n+1} (\Gamma'_1, T'\rhd)$ and $(\Gamma_1, \Gamma_2\rhd) \sim_m (\Gamma'_1, \Gamma'_2\rhd)$ we have to show that

$$(\Gamma_1, T, s_{n+1}\Gamma_2) \sim_{m+1} (\Gamma'_1, T', s_{n+1}\Gamma'_2).$$

Proceed by induction on $m-n = l(\Gamma_2)$. For $l(\Gamma_2) = 0$ the assertion is obvious. Let $(\Gamma_1, T\rhd) \sim_{n+1} (\Gamma'_1, T'\rhd)$ and $(\Gamma_1, \Gamma_2, S\rhd) \sim_m (\Gamma'_1, \Gamma'_2, S'\rhd)$. The later condition is equivalent to $(\Gamma_1, \Gamma_2\rhd) \sim_m (\Gamma'_1, \Gamma'_2\rhd)$ and $(\Gamma_1, \Gamma_2 \vdash S = S')$. By the inductive assumption we have $(\Gamma_1, T, s_{n+1}\Gamma_2) \sim_{m+1} (\Gamma'_1, T', s_{n+1}\Gamma'_2)$. By (5a) we conclude that $(\Gamma_1, T, s_{n+1}\Gamma_2 \vdash s_{n+1}S = s_{n+1}S')$. Therefore by definition of $\sim_{m+1}$ we have $(\Gamma_1, T, s_{n+1}\Gamma_2, s_{n+1}S) \sim_{m+1} (\Gamma'_1, T', s_{n+1}\Gamma'_2, s_{n+1}S')$.

(2) Given $(\Gamma_1, T\rhd) \sim_{n+1} (\Gamma'_1, T'\rhd)$ and $(\Gamma_1, \Gamma_2 \vdash o : S) \simeq_{m+1} (\Gamma'_1, \Gamma'_2 \vdash o' : S')$ we have to show that $(\Gamma_1, T, s_{n+1}\Gamma_2 \vdash s_{n+1}o : s_{n+1}S) \simeq_{m+2} (\Gamma'_1, T', s_{n+1}\Gamma'_2 \vdash s_{n+1}o' : s_{n+1}S')$. We have $(\Gamma_1, \Gamma_2, S) \sim_{m+1} (\Gamma'_1, \Gamma'_2, S')$ and $(\Gamma_1, \Gamma_2 \vdash o = o' : S)$. By (5b) we get $(\Gamma_1, T, s_{n+1}\Gamma_2 \vdash s_{n+1}o = s_{n+1}o' : s_{n+1}S)$. By (1) of this lemma we get $(\Gamma_1, T, s_{n+1}\Gamma_2, s_{n+1}S) \sim_{m+2} (\Gamma'_1, T', s_{n+1}\Gamma'_2, s_{n+1}S')$ and therefore by definition of $\simeq$ we get $(\Gamma_1, T, s_{n+1}\Gamma_2 \vdash s_{n+1}o : s_{n+1}S) \simeq_{m+2} (\Gamma'_1, T', s_{n+1}\Gamma'_2 \vdash s_{n+1}o' : s_{n+1}S')$.

(3) Given $(\Gamma_1 \vdash r : T) \simeq_{n+1} (\Gamma'_1 \vdash r' : T')$ and $(\Gamma_1, T, \Gamma_2\rhd) \sim_{m+1} (\Gamma'_1, T', \Gamma'_2\rhd)$ we have to show that

$$(\Gamma_1, d_{n+1}(\Gamma_2[r/n+1])) \sim_m (\Gamma'_1, d_{n+1}(\Gamma'_2[r'/n+1])).$$

Proceed by induction on $m-n = l(\Gamma_2)$. For $l(\Gamma_2) = 0$ the assertion follows directly from the definitions. Let $(\Gamma_1 \vdash r : T) \simeq_{n+1} (\Gamma'_1 \vdash r' : T')$ and $(\Gamma_1, T, \Gamma_2, S\rhd) \sim_m (\Gamma'_1, T', \Gamma'_2, S'\rhd)$. The later condition is equivalent to $(\Gamma_1, T, \Gamma_2\rhd) \sim_m (\Gamma'_1, T', \Gamma'_2\rhd)$ and $(\Gamma_1, T, \Gamma_2 \vdash S = S')$. By the inductive assumption we have $(\Gamma_1, d_{n+1}(\Gamma_2[r/n+1])) \sim_m (\Gamma'_1, d_{n+1}(\Gamma'_2[r'/n+1]))$. It remains to show that $(\Gamma_1, d_{n+1}(\Gamma_2[r/n+1]) \vdash d_{n+1}(S[r/n+1]) = d_{n+1}(S'[r'/n+1]))$. By (2d) it is sufficient to show that $(\Gamma_1, d_{n+1}(\Gamma_2[r/n+1]) \vdash d_{n+1}(S[r/n+1]) = d_{n+1}(S'[r/n+1]))$ and $(\Gamma_1, d_{n+1}(\Gamma_2[r/n+1]) \vdash d_{n+1}(S'[r/n+1]) = d_{n+1}(S'[r'/n+1]))$. The first relation follows directly from (6a). To prove the second one it is sufficient by (7a) to show that $(\Gamma_1, T, \Gamma_2, S'\rhd)$ which follows from our assumption through (2c) and (2a).

(4) Given $(\Gamma_1 \vdash r : T) \simeq_{n+1} (\Gamma'_1 \vdash r' : T')$ and $(\Gamma_1, T, \Gamma_2 \vdash o : S) \simeq_{m+2} (\Gamma'_1, T', \Gamma'_2 \vdash o' : S')$ we have to show that

$$(\Gamma_1, d_{n+1}(\Gamma_2[r/n+1]) \vdash d_{n+1}(o[r/n+1]) : d_{n+1}(S[r/n+1])) \simeq_{m+1}$$

$$(\Gamma'_1, d_{n+1}(\Gamma'_2[r'/n+1]) \vdash d_{n+1}(o'[r'/n+1]) : d_{n+1}(S'[r'/n+1])).$$

or equivalently that $(\Gamma_1, d_{n+1}(\Gamma_2[r/n+1]), d_{n+1}(S[r/n+1])) \sim_{m+1} (\Gamma'_1, d_{n+1}(\Gamma'_2[r'/n+1]), d_{n+1}(S'[r'/n+1]))$ and $(\Gamma_1, d_{n+1}(\Gamma_2[r/n+1]) \vdash d_{n+1}(o[r/n+1]) = d_{n+1}(o'[r'/n+1]) : d_{n+1}(S[r/n+1]))$. The first statement follows from part (3) of the lemma. To prove the second statement it is sufficient by (3d) to show that $(\Gamma_1, d_{n+1}(\Gamma_2[r/n+1]) \vdash d_{n+1}(o[r/n+1]) = d_{n+1}(o'[r/n+1]) : d_{n+1}(S[r/n+1]))$ and $(\Gamma_1, d_{n+1}(\Gamma_2[r/n+1]) \vdash d_{n+1}(o'[r/n+1]) = d_{n+1}(o'[r'/n+1]) : d_{n+1}(S[r/n+1]))$. The first assertion follows directly from (6b). To prove the second one it is sufficient in view of (7b) to show that $(\Gamma_1, T, \Gamma_2 \vdash o' : S)$ which follows conditions (3c) and (3a).

(5) Given $(\Gamma, T) \sim_{n+1} (\Gamma', T')$ we need to show that $(\Gamma, T \vdash (n+1) : T) \simeq_{n+2} (\Gamma', T' \vdash (n+1) : T')$ or equivalently that $(\Gamma, T, T) \sim_{n+2} (\Gamma, T', T')$ and $(\Gamma, T \vdash (n+1) = (n+1) : T)$. The second part follows from (3b). To prove the first part we need to show that $(\Gamma, T \vdash T = T')$. This follows from our assumption by (5a).

Definition 4.7 *[**2009.11.4.def1**] Let S, $\triangleright$ and TS be as above. Let further $(\mathcal{C}, p)$ be a category with a universe structure. A closed model of TS with values in $(\mathcal{C}, p)$ is a C-structure morphism*

$$M : TS \to CC(\mathcal{C}, p)$$

which is compatible with $\triangleright$ i.e. such that the following conditions hold:

1. *if $(E_1, \dots, E_n) \in Ob(TS)$, $i = 1, \dots, n$ and $E'_i \in S(\{x_1, \dots, x_{i-1}\})$ is such that $E_i \triangleright E'_i$ then $M(E_1, \dots, E_n) = M(E_1, \dots, E'_i, \dots, E_n)$,*
2. *if $(f_1, \dots, f_m) \in Hom_{TS}((E_1, \dots, E_n), (T_1, \dots, T_m))$, $i = 1, \dots, m$ and $f'_i \in S(\{1, \dots, n\})$ is such that $f_i \triangleright f'_i$ then*
$$M((f_1, \dots, f_m); (E_1, \dots, E_n); (T_1, \dots, T_m)) = M((f_1, \dots, f'_i, \dots, f_m); (E_1, \dots, E_n); (T_1, \dots, T_m)$$
3. *if $(E_1, \dots, E_n), (T_1, \dots, T_m) \in Ob(TS)$, $(f_1, \dots, f_m) \in S(\{1, \dots, n\})^m$, $i = 1, \dots, n$ and $E'_i \in S(\{1, \dots, i-1\})$ is such that $E_i \triangleright E'_i$ then*
$$M((f_1, \dots, f_m); (E_1, \dots, E_n); (T_1, \dots, T_m)) = M((f_1, \dots, f_m); (E_1, \dots, E'_i, \dots, E_n); (T_1, \dots, T_m))$$
4. *if $(E_1, \dots, E_n), (T_1, \dots, T_m) \in Ob(TS)$, $(f_1, \dots, f_m) \in S(\{1, \dots, n\})^m$, $i = 1, \dots, m$ and $T'_i \in S(\{1, \dots, i-1\})$ is such that $T_i \triangleright T'_i$ then*
$$M((f_1, \dots, f_m); (E_1, \dots, E_n); (T_1, \dots, T_m)) = M((f_1, \dots, f_m); (E_1, \dots, E_n); (T_1, \dots, T'_i, \dots, T_m))$$

... are called the subset of type sequents and the subset of term sequents of a type system. By Lemma 2.3 they uniquely determine the type system.

Elements of Seq_0 are called contexts and elements of Seq_1 are called judgements. Proposition 3.1 shows that for any type system TS and any $(E_1, \dots, E_n, t, T)$ in $Seq_1(TS)$ the sequence $(E_1, \dots, E_n)$ is in Seq_0 i.e. the first part of a judgement should be a valid context.

One also often uses the notation $E_1, E_2, \dots, E_n \vdash T : Type$ which is equivalent to $E_1, E_2, \dots, E_n, T \rhd$. The meaning assigned to these subsets is as follows:

1. $E_1, E_2, \dots, E_n \rhd$ means that E_1 is a well formed closed type expression and for $i > 1$, $E_i(1, \dots, i-1)$ is a well formed type expression in the context where variables $1, \dots, i-1$ have types $E_1, \dots, E_{i-1}$ respectively,
2. $E_1, E_2, \dots, E_n \vdash t : T$ means that $E_1, E_2, \dots, E_n, T \rhd$ and in the context where variables $1, \dots, n$ are of the types $E_1, \dots E_n$ respectively, $t(1, \dots, n)$ is a well formed term expression of type $T(1, \dots, n)$.

3 C-structures defined by universes in 1-categories

C-structures $CC(\mathcal{C}, p)$.

Definition 0.8 *[**2009.11.1.def1**] Let $\mathcal{C}$ be a (level 1) category. A universe on $\mathcal{C}$ is a morphism $p : \widetilde{U} \to U$ together with a mapping which assigns to any morphism $f : X \to U$ in $\mathcal{C}$ a pull-back square*

$$\begin{array}{ccc} (X,f) & \xrightarrow{Q(f)} & \widetilde{U} \\ {\scriptstyle p_{(X,f)}}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & U \end{array}$$

In what follows we will write $(X, f_1, \ldots, f_n)$ for $(\ldots((X, f_1), f_2)\ldots, f_n)$.

Let $\mathcal{C}$ be a 1-category, p a universe on $\mathcal{C}$ and pt a final object of $\mathcal{C}$. For such a triple define a C-structure $CC = CC(\mathcal{C}, p)$ as follows. Objects of CC are sequences of the form $(F_1, \ldots, F_n)$ where $F_1 \in Hom_{\mathcal{C}}(pt, U)$ and $F_{i+1} \in Hom_{\mathcal{C}}((pt, F_1, \ldots, F_i), U)$. Morphisms from $(G_1, \ldots, G_n)$ to $(F_1, \ldots, F_m)$ are given by

$$Hom_{CC}((G_1, \ldots, G_n), (F_1, \ldots, F_m)) = Hom_{\mathcal{C}}((pt, G_1, \ldots, G_n), (pt, F_1, \ldots, F_m))$$

and units and compositions are defined as units and compositions in $\mathcal{C}$ such that the mapping $(F_1, \ldots, F_n) \to (pt, F_1, \ldots, F_n)$ is a full embedding of the underlying category of CC to $\mathcal{C}$. The image of this embedding consists of objects X for which the canonical morphism $X \to pt$ is a composition of morphisms which are (canonical) pull-backs of p. We will denote this embedding by int.

The final object of CC is the empty sequence $()$. The map ft sends $(F_1, \ldots, F_n)$ to $(F_1, \ldots, F_{n-1})$. The canonical morphism $p_{(F_1,\ldots,F_n)}$ is the projection

$$p_{((pt,F_1,\ldots,F_{n-1}),F_n)} : ((pt, F_1, \ldots, F_{n-1}), F_n) \to (pt, F_1, \ldots, F_{n-1})$$

For an object $(F_1, \ldots, F_{m+1})$ and a morphism $f : (G_1, \ldots, G_n) \to (F_1, \ldots, F_m)$ the canonical pull-back square is of the form

[**2009.10.26.eq3**]
$$\begin{array}{ccc} (G_1, \ldots, G_n, F_{m+1}f) & \xrightarrow{q(f)} & (F_1, \ldots, F_{m+1}) \\ {\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle p_F} \\ (G_1, \ldots, G_n) & \xrightarrow{f} & (F_1, \ldots, F_m) \end{array} \qquad (14)$$

where $int(p_F) = p((pt, F_1, \ldots, F_{n-1}), F_n)$, $int(p_G) = p((pt, G_1, \ldots, G_{n-1}), F_{m+1} \circ f)$ and $q(f)$ is the morphism such that $p_F q(f) = f p_G$ and $Q(F_{m+1}) int(q(f)) = Q(F_{m+1} f)$. The unity and composition axioms for the canonical squares follow immediately from the unity and associativity axioms for compositions of morphisms in C.

Let $(\mathcal{C}, p, pt)$ and $(\mathcal{C}', p', pt')$ be two sets of data as above. Let $\Phi : \mathcal{C} \to \mathcal{C}'$ be a functor which takes distinguished squares in $\mathcal{C}$ to pull-back squares in $\mathcal{C}'$ and such that $\Phi(pt) \to pt'$ is an isomorphism, let further $\phi : \Phi(U) \to U'$, $\widetilde{\phi} : \Phi(\widetilde{U}) \to \widetilde{U}'$ be two morphisms such that

$$\begin{array}{ccc} \Phi(\widetilde{U}) & \xrightarrow{\widetilde{\phi}} & \widetilde{U}' \\ {\scriptstyle \Phi(p)}\downarrow & & \downarrow{\scriptstyle p'} \\ \Phi(U) & \xrightarrow{\phi} & U' \end{array}$$

is a pull-back square. Denote by ψ the isomorphism $\psi : pt' \to \Phi(pt)$.

Define a functor $H = H(\Phi, \phi, \widetilde{\phi})$ from $CC(\mathcal{C}, p)$ to $CC(\mathcal{C}', p')$ as follows. We define by induction on n objects $H(F_1, \dots, F_n) \in CC(\mathcal{C}', p')$ and isomorphisms

$$\psi_{(F_1,\dots,F_n)} : int'(H(F_1, \dots, F_n)) \to \Phi(int(F_1, \dots F_n))$$

where int and int' are the canonical functors $CC(\mathcal{C}, p) \to \mathcal{C}$ and $CC(\mathcal{C}', p') \to \mathcal{C}'$ respectively.

For $n = 0$ we set $H(pt) = pt$ and $\psi_{()} = \psi$. For $n > 0$ let

$$(F'_1, \dots, F'_{n-1}) = H(F_1, \dots, F_{n-1})$$

and let $F_n : int(F_1, \dots, F_{n-1}) \to U$. Define F'_n as the composition

$$[\mathbf{2009.10.26.eq5}]F'_n : int'(F'_1, \dots, F'_{n-1}) \stackrel{\psi_{(F_1,\dots,F_{n-1})}}{\longrightarrow} \Phi(int(F_1, \dots, F_{n-1})) \stackrel{\Phi(F_n)}{\to} \Phi(U) \stackrel{\phi}{\to} U' \quad (15)$$

and let $H(F_1, \dots, F_n) = (F'_1, \dots, F'_{n-1}, F'_n)$. Then

$$int'(H(F_1, \dots, F_n)) = (int'(H(F_1, \dots, F_n)), F'_n)$$

To define

$$\psi_{(F_1,\dots,F_n)} : int'(H(F_1, \dots, F_n)) \to \Phi(int(F_1, \dots, F_n))$$

observe that by our conditions on $\phi, \widetilde{\phi}$ and Φ the squares of the diagram

$$\begin{array}{ccccc} \Phi(int(F_1, \dots, F_n)) & \xrightarrow{\Phi(Q(F_n))} & \Phi(\widetilde{U}) & \longrightarrow & \widetilde{U}' \\ \downarrow & & \downarrow & & \downarrow \\ \Phi(int(F_1, \dots, F_{n-1})) & \xrightarrow{\Phi(F_n)} & \Phi(U) & \xrightarrow{\phi} & U' \end{array}$$

are pull-back. Therefore there is a unique morphism $\psi_{(F_1,\dots,F_n)}$ such that the diagram

$$[\mathbf{2009.10.26.eq2}] \quad \begin{array}{ccccc} int'(H(F_1, \dots, F_n)) & \xrightarrow{\psi_{(F_1,\dots,F_n)}} & \Phi(int(F_1, \dots, F_n)) & \xrightarrow{\widetilde{\phi}\Phi(Q(F_n))} & \widetilde{U}' \\ \downarrow & & \downarrow & & \downarrow \\ int'(H(F_1, \dots, F_{n-1})) & \xrightarrow{\psi_{(F_1,\dots,F_{n-1})}} & \Phi(int(F_1, \dots, F_{n-1})) & \xrightarrow{\phi\Phi(F_n)} & U' \end{array} \quad (16)$$

commutes and

$$[\mathbf{2009.10.26.eq7}]\widetilde{\phi}\Phi(Q(F_n))\psi_{(F_1,\dots,F_n)} = Q(\phi\Phi(F_n)\psi_{(F_1,\dots,F_{n-1})}) \quad (17)$$

and this morphism is an isomorphism.

To define H on morphism we use the fact that morphisms $\psi_{(F_1,\dots,F_n)}$ are isomorphisms and for $f : (F_1, \dots, F_n) \to (G_1, \dots, G_m)$ we set

$$[\mathbf{2009.10.26.eq6}]H(f) = \psi^{-1}_{(G_1,\dots,G_m)}\Phi(f)\psi_{(F_1,\dots,F_n)} \quad (18)$$

The fact that this construction gives a functor i.e. satisfies the unity and composition axioms is straightforward.

It remains to verify that this morphism respects the rest of the C-structure. It is clear that it respects the length function and the ft maps. The fact that it takes the canonical projections to canonical projections is equivalent to the commutativity of the left hand side square in (16).

Consider a canonical square of the form (14). Its image is a square of the form

$$[\mathbf{2009.10.26.eq4}] \quad \begin{array}{ccc} (G'_1,\ldots,G'_n,G'_{n+1}) & \xrightarrow{H(q(f))} & (F'_1,\ldots,F'_{m+1}) \\ {\scriptstyle H(p_G)}\downarrow & & \downarrow{\scriptstyle H(p_F)} \\ (G'_1,\ldots,G'_n) & \xrightarrow{H(f)} & (F'_1,\ldots,F'_m) \end{array} \qquad (19)$$

We already know that the vertical arrows are canonical projections. Therefore, in order to prove that (19) is a canonical square in $CC(\mathcal{C}',p')$ we have to show that $G'_{n+1} = F'_{m+1} int(H(f))$ and

$$[\mathbf{2009.10.26.eq8}] Q(F'_{m+1}) int(H(q(f))) = Q(F'_{m+1} int(H(f))) \qquad (20)$$

By (15) we have

$$G'_{n+1} = \phi\Phi(F_{m+1}f)\psi_{(G_1,\ldots,G_n)}$$
$$F'_{m+1} = \phi\Phi(F_{m+1})\psi_{(F_1,\ldots,F_m)}$$

and by (18)

$$int(H(f)) = \psi^{-1}_{(F_1,\ldots,F_m)}\Phi(f)\psi_{(G_1,\ldots,G_n)}$$
$$int(H(q(f))) = \psi^{-1}_{(F_1,\ldots,F_{m+1})}\Phi(q(f))\psi_{(G_1,\ldots,G_n,F_{m+1}f)}$$

Therefore the relation $G'_{n+1} = F'_{m+1} int(H(f))$ follows immediately and the relation (20) follows by application of (17).

Our construction of H shows that if Φ is a full embedding and ϕ and $\widetilde{\phi}$ are isomorphisms then H is an isomorphism of C-structures. This implies in particular that considered up to a canonical isomorphism $CC(\mathcal{C},p)$ depends only on the equivalence class of the pair $(\mathcal{C},p)$ i.e. that our construction maps pairs $(\mathcal{C},p)$ which are of h-level 3 to C-structures which are at the set level.

Let us describe now an inverse construction which shows that any C-structure is isomorphic to a C-structure of the form $CC(\mathcal{C},p)$. Let CC be a C-structure. Denote by $PreShv(CC)$ the 1-category of contravariant functors from the category underlying CC to $Sets$.

Let Ty be the functor which takes an object $\Gamma \in CC$ to the set

$$Ty(\Gamma) = \{\Gamma' \in CC \mid ft(\Gamma') = \Gamma\}$$

and a morphism $f : \Delta \to \Gamma$ to the map $\Gamma' \mapsto f^*\Gamma'$. It is a functor due to the composition and unity axioms for f^*. Let Tm be the functor which takes an object Γ to the set

$$Tm(\Gamma) = \{s \in \widetilde{CC} \mid ft\,\partial(s) = \Gamma\}$$

and a morphism $f : \Delta \to \Gamma$ to the map $s \mapsto f^*(s)$. Let further $p : Tm \to Ty$ be the morphism which takes s to $\partial(s)$. It is well defined as a morphisms of families of sets and forms a morphism of presheaves since $\partial(f^*(s)) = f^*(\partial(s))$.

Proposition 0.9 *[***2009.12.28.prop1***] For any C-structure CC there is a natural isomorphism*

$$CC = CC(PreShv(CC), p)$$

Proof: We start with the key lemma. (In what follows we identify objects of CC with the corresponding representable presheaves and, for a presheaf F and an object Γ, we identify morphisms $\Gamma \to F$ in $PreShv(CC)$ with $F(\Gamma)$).

Lemma 0.10 *[**2009.12.28.l1**] Let $\Gamma' \in Ob(CC)$ and let $\Gamma = ft(\Gamma')$. Then the square*

$$\begin{array}{ccc} \Gamma' & \xrightarrow{\delta_{\Gamma'}} & Tm \\ {\scriptstyle p_{\Gamma'}}\downarrow & & \downarrow{\scriptstyle p} \\ \Gamma & \xrightarrow{\Gamma'} & Ty \end{array}$$

is a pull-back square.

Proof: We have to show that for any $\Delta \in CC$ the obvious map

$$[\mathbf{2009.12.28.eq2}] Hom(\Delta, \Gamma') \to Hom(\Delta, \Gamma) \times_{Ty(\Delta)} Tm(\Delta) \tag{21}$$

is a bijection. Let $f_1, f_2 : \Delta \to \Gamma'$ be two morphisms such that their images under (21) coincide i.e. such that $p_{\Gamma'} f_1 = p_{\Gamma'} f_2$ and $f_1^*(\delta_{\Gamma'}) = f_2^*(\delta'_{\Gamma})$. These two conditions are equivalent to saying, in the notation introduced above, that $ft(f_1) = ft(f_2)$ and $s_{f_1} = s_{f_2}$. This implies that $f_1 = f_2$ i.e. that (21) is injective. Let $f : \Delta \to \Gamma$ be a morphism and $s \in Tm(\Delta)$ a section such that $ft(\partial(s)) = f^*(\Gamma')$. Then the composition $q(f, \Gamma')s$ is a morphism $f' : \Delta \to \Gamma'$ such that $p_{\Gamma'} f' = f$. We also have

$$(f')^*(\delta_{\Gamma'}) = s^* q(f, \Gamma')^*(\delta_{\Gamma'}) = s$$

which proves that (21) is surjective.

To construct the required isomorphism we now choose a universe structure on p such that the pull-back squares associated with morphisms from representable objects are squares (21). The isomorphism is now obvious.

Definition 0.11 *[**2009.12.27.def1**] Let CC be a C-structure. A closed model of CC is a collection of data of the following form:*

1. *A 1-category $\mathcal{C}$,*
2. *a universe $p : \widetilde{U} \to U$ in $\mathcal{C}$ and a final object pt of $\mathcal{C}$,*
3. *a C-structure morphism $CC \to CC(\mathcal{C}, p)$.*

The following proposition shows that any "model" of a C-structure can be viewed as a closed model.

Proposition 0.12 *[**2009.12.27.prop1**] Let $\mathcal{C}$ be a 1-category, CC be a C-structure and $M : CC \to \mathcal{C}$ a functor such that $M(pt_{CC})$ is a final object of $\mathcal{C}$ and M maps distinguished squares of CC to pull-back squares of $\mathcal{C}$. Then there exists a universe $p_M : \widetilde{U}_M \to U_M$ in $PreShv(\mathcal{C})$ and a C-structure morphism $M' : CC \to CC(PreShv(\mathcal{C}), p_M)$ such that the square*

$$\begin{array}{ccc} CC & \xrightarrow{M} & \mathcal{C} \\ \downarrow{\scriptstyle M'} & & \downarrow \\ CC(PreShv(\mathcal{C}), p_M) & \xrightarrow{int} & PreShv(C) \end{array}$$

where the right hand side vertical arrow is the Yoneda embedding, commutes up to a canonical isomorphism.

Proof: We will write $p : \widetilde{U} \to U$ instead of $p_M : \widetilde{U}_M \to U_M$. Set

$$\widetilde{U} = \amalg_{\Gamma \in CC_{>0}} M(\Gamma) \qquad U = \amalg_{\Gamma \in CC_{>0}} M(ft(\Gamma)) \qquad p = \amalg_{\Gamma \in CC_{>0}} M(p_\Gamma)$$

Let pt be final object of $PreShv(\mathcal{C})$. Set $M'(pt_{CC}) = pt$.

....

Π-universes in lcc categories. Recall that a (level 1) category $\mathcal{C}$ is called a lcc (locally Cartesian closed) category if it has fiber products and all the over-categories $\mathcal{C}/X$ have internal Hom-objects.

Definition 0.13 *[**2009.10.27.def1**] Let $\mathcal{C}$ be an lcc category and let $p_i : \widetilde{U}_i \to U_i$, $i = 1, 2, 3$ be three morphisms in $\mathcal{C}$. A Π-structure on (p_1, p_2, p_3) is a Cartesian square of the form*

$$[\mathbf{Pisq1}] \qquad \begin{array}{ccc} \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times \widetilde{U_2}) & \xrightarrow{\widetilde{P}} & \widetilde{U}_3 \\ {\scriptstyle p_2'}\downarrow & & \downarrow{\scriptstyle p_3} \\ \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) & \xrightarrow{P} & U_3 \end{array} \qquad (22)$$

such that p_2' is the natural morphism defined by p_2. A Π-structure on $p : \widetilde{U} \to U$ is a Π-structure on (p, p, p).

Remark 0.14 A Π-structure on (p_1, p_2, p_3) corresponds to the rule

$$\frac{\Gamma, X : U_1, f : X \to U_2 \rhd}{\Gamma, X : U_1, f : X \to U_2 \vdash \prod x : X.ev(f, x) : U_3}$$

Let $\mathcal{C}$ be as above, $p : \widetilde{U} \to U$ and let $(\widetilde{P}, P)$ be a Π-structure on (p, p, p). Let us construct a structure of Π-C-structure on $CC = CC(\mathcal{C}, p)$.

We start by recalling some level 1 constructions in $\mathcal{C}$.

Lemma 0.15 *[**2009.11.24.l5**] Consider a pair of pull back squares*

$$[\mathbf{2009.11.24.eq3}] \qquad \begin{array}{ccc} I_2 & \xrightarrow{\widetilde{F}_1} & \widetilde{U}_1 \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ I_1 & \xrightarrow{F_1} & U_1 \end{array} \qquad \begin{array}{ccc} I_3 & \xrightarrow{\widetilde{F}_2} & \widetilde{U}_2 \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle p_2} \\ I_2 & \xrightarrow{F_2} & U_2 \end{array} \qquad (23)$$

Then there exists a unique morphism $f_{F_1,F_2} : I_1 \to \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2)$ such that its composition with the natural morphism to U_1 is F_1 and the composition of its adjoint

$$ev \circ (f_{F_1,F_2} \times_{U_1} \widetilde{U}_1) : I_2 = I_1 \times_{U_1} \widetilde{U}_1 \to U_1 \times U_2$$

with the projection to U_2 is F_2.

Proof: Follows immediately from the definition of internal Hom-objects.

Lemma 0.16 *[**2009.11.24.l3**] In the notation of Lemma 0.15 let*

$$\begin{array}{ccc} J_2 & \xrightarrow{\phi_2} & I_2 \\ \downarrow & & \downarrow{\scriptstyle q_1} \\ J_1 & \xrightarrow{\phi_1} & I_1 \end{array} \qquad \begin{array}{ccc} J_3 & \xrightarrow{\phi_3} & I_3 \\ \downarrow & & \downarrow{\scriptstyle q_2} \\ J_2 & \xrightarrow{\phi_2} & I_2 \end{array}$$

be two pull-back squares. Then $f_{F_1\phi_1,F_2\phi_2} = f_{F_1,F_2} \circ \phi_1$.

Proof: Straightforward.

Let $p_1 : \widetilde{U}_1 \to U_1$, $p_2 : \widetilde{U}_2 \to U_2$ be a pair of morphisms in an lcc $\mathcal{C}$. Consider a pull-back square of the form

[**2009.11.24.eq4**]
$$\begin{array}{ccc} Fam_2(p_1,p_2) & \longrightarrow & \widetilde{U}_2 \\ {\scriptstyle p_{12}}\downarrow & & \downarrow{\scriptstyle p_2} \\ \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) \times_{U_1} \widetilde{U_1} & \xrightarrow{pr\circ ev} & U_2 \end{array} \qquad (24)$$

where

$$ev : \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) \times_{U_1} \widetilde{U_1} \to U_1 \times U_2$$

is the canonical morphism.

Then for any two pull-back squares as in Lemma 0.15, the morphism f_{F_1,F_2} defines factorizations of the pull-back squares (23) of the form

$$\begin{array}{ccccc} I_2 & \xrightarrow{f_{F_1,F_2}\times_{U_1}\widetilde{U}_1} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) \times_{U_1} \widetilde{U_1} & \xrightarrow{pr} & \widetilde{U}_1 \\ {\scriptstyle q_1}\downarrow & & \downarrow & & \downarrow{\scriptstyle p_1} \\ I_1 & \xrightarrow{f_{F_1,F_2}} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) & \longrightarrow & U_1 \end{array}$$

and

$$\begin{array}{ccccc} I_3 & \longrightarrow & Fam_2(p_1,p_2) & \longrightarrow & \widetilde{U}_2 \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle p_{12}} & & \downarrow{\scriptstyle p_2} \\ I_2 & \xrightarrow{f_{F_1,F_2}\times_{U_1}\widetilde{U}_1} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) \times_{U_1} \widetilde{U_1} & \xrightarrow{pr\circ ev} & U_2 \end{array}$$

respectively and joining the left hand side squares of these diagrams we get a diagram with pull-back squares of the form

$$\begin{array}{ccc} I_3 & \longrightarrow & Fam_2(p_1,p_2) \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle p_{12}} \\ I_2 & \xrightarrow{f_{F_1,F_2}\times_{U_1}\widetilde{U}_1} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) \times_{U_1} \widetilde{U_1} \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle pr} \\ I_1 & \xrightarrow{f_{F_1,F_2}} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) \end{array}$$

Let

$$g : \underline{Hom}_{U_1}(\widetilde{U_1}, U_1 \times \widetilde{U}_2) \times_{U_1} \widetilde{U_1} \to Fam_2(p_1, p_2)$$

be the morphism over $\underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) \times_{U_1} \widetilde{U_1}$ whose composition with the projection $Fam_2(p_1, p_2) \to \widetilde{U}_2$ equals $pr \circ \widetilde{ev}$ where

$$\widetilde{ev} : \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times \widetilde{U}_2) \times_{U_1} \widetilde{U_1} \to U_1 \times \widetilde{U}_2$$

is the canonical morphism.

Lemma 0.17 *[**2009.11.24.l2**] The pair*

$$(\underline{Hom}_{U_1}(\widetilde{U_1}, U_1 \times \widetilde{U}_2) \to \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2), g)$$

is universal for (p_{12}, pr).

Proof: For a given $w : Z \to \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2)$, a morphism $Z \to \underline{Hom}_{U_1}(\widetilde{U_1}, U_1 \times \widetilde{U}_2)$ over $\underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2)$ is the same as a morphism $Z \times_{U_1} \widetilde{U}_1 \to \widetilde{U}_2$ such that the adjoint of its composition with $p_2 : \widetilde{U}_2 \to U_2$ is w.

A morphism from Z to the universal pair for p_{12} over $\underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2)$ is a morphism $Z \times_{U_1} \widetilde{U_1} \to \widetilde{U}_2$ whose composition with p_2 is $(pr \circ ev) \circ (w \times_{U_1} Id_{\widetilde{U_1}})$ which coincides with the condition that the composition of its adjoint with p_2 is w. This can be also seen from the diagram

$$\begin{array}{ccccc}
 & & Fam_2(p_1, p_2) & \longrightarrow & \widetilde{U}_2 \\
 & & \downarrow{\scriptstyle p_{12}} & & \downarrow{\scriptstyle p_2} \\
\underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times \widetilde{U}_2) \times_{U_1} \widetilde{U}_1 & \longrightarrow & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) \times_{U_1} \widetilde{U}_1 & \xrightarrow{pr \circ ev} & U_2 \\
\downarrow & & \downarrow{\scriptstyle pr} & & \\
\underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times \widetilde{U}_2) & \longrightarrow & \underline{Hom}_{U_1}(\widetilde{U_1}, U_1 \times U_2) & &
\end{array}$$

Lemma 0.18 *[**2009.11.24.l4**] For two pull back squares as in (23), consider a pull-back square of the form*

$$\begin{array}{ccc}
R(F_1, F_2) & \longrightarrow & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times \widetilde{U}_2) \\
\downarrow & & \downarrow \\
I_1 & \xrightarrow{f_{F_1, F_2}} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2)
\end{array}$$

and the morphism

$$g_{F_1, F_2} : R(F_1, F_2) \times_{I_1} I_2 \to I_3$$

whose composition with the morphism $I_3 \to \widetilde{U}_2$ coincides with the composition

$$R(F_1, F_2) \times_{I_1} I_2 = R(F_1, F_2) \times_{U_1} \widetilde{U}_1 \to \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times \widetilde{U}_2) \times_{U_1} \widetilde{U}_1 \stackrel{pr \circ ev}{\to} \widetilde{U}_2$$

Then $(R(F_1, F_2), g_{F_1, F_2})$ is a universal pair for (q_1, q_2).

Proof: It follows from Lemma 0.17 and the fact that in a lcc a pull-back of a universal pair is a universal pair.

Let us now construct a Π-C-structure on $CC = CC(\mathcal{C}, p)$. Let $n \geq 2$ and $(F_1, \ldots, F_n) \in CC$. Denote $(pt, F_1, \ldots, F_{n-2})$ by I. Then we have two morphisms $F_{n-1} : I \to U$ and $F_n : (I, F_{n-1}) \to U$.

Applying Lemma 0.15 to the corresponding pull-back squares we get a morphism

$$f_{F_{n-1},F_n} : I \to \underline{Hom}_U(\widetilde{U}, U \times U)$$

Set $\Pi(F_1, \ldots, F_n) = (I, P \circ f_{F_{n-1},F_n}) = (F_1, \ldots, F_{n-2}, P \circ f_{F_{n-1},F_n})$. Since the square (22) is a pull-back square there is a unique morphism $\Pi(F_1, \ldots, F_n) \to \underline{Hom}_U(\widetilde{U}, U \times \widetilde{U})$ such that the diagram

$$\begin{array}{ccccc}
\Pi(F_1, \ldots, F_n) & \longrightarrow & \underline{Hom}_U(\widetilde{U}, U \times \widetilde{U}) & \xrightarrow{\widetilde{P}} & \widetilde{U} \\
\downarrow & & \downarrow & & \downarrow \\
I & \xrightarrow{f_{F_{n-1},F_n}} & \underline{Hom}_U(\widetilde{U}, U \times U) & \xrightarrow{P} & U
\end{array}$$

commutes and the composition of the two upper arrows is $Q(f_{F_{n-1},F_n})$. The left hand side square in this diagram is automatically a pull-back square. Applying to this square Lemma 0.18 we obtain a morphism

$$eval_{(F_1,\ldots,F_n)} : (I, F_{n-1}, (P \circ f_{F_{n-1},F_n}) \circ pr) \to (I, F_{n-1}, F_n)$$

over (I, F_{n-1}) (where $pr : (I, F_{n-1}) \to I$ is the projection).

The fact that this construction satisfies the first condition of Definition 1.2 follows from Lemma 0.16. The fact that it satisfies the second condition of this definition follows from Lemma 0.18.

Σ-universes in lcc categories.

Definition 0.19 *[***2009.10.27.def2***] Let $\mathcal{C}$ be an lcc category and $p_i : \widetilde{U}_i \to U_i$, $i = 1, 2, 3$ be three morphisms in $\mathcal{C}$. A Σ-structure on (p_1, p_2, p_3) is a diagram of the form*

$$\begin{array}{ccccc}
\widetilde{U}_2 & \longleftarrow & Fam_\bullet(U_1, U_2) & \longrightarrow & \widetilde{U}_3 \\
{\scriptstyle p_2}\downarrow & & \downarrow & & \\
U_2 & \xleftarrow{pr_{U_2} eval} & \widetilde{U_1} \times_{U_1} \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) & & \downarrow{\scriptstyle p_3} \\
 & & \downarrow{\scriptstyle pr} & & \\
 & & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) & \xrightarrow{\Sigma} & U_3
\end{array}$$

such that p_2' is the natural morphism defined by p_2, eval is the canonical evaluation morphism and both the square and the vertical rectangle are Cartesian. A Σ-structure on $p : \widetilde{U} \to U$ is a Σ-structure on (p, p, p).

A Σ-structure on (p_1, p_2, p_3) corresponds to the rule

$$\frac{\Gamma, X : U_1, f : X \to U_2 \rhd}{\Gamma, X : U_1, f : X \to U_2 \vdash \sum x : X.ev(f, x) : U_3}$$

Definition 0.20 *[2009.11.2.def1] Let $\mathcal{C}$ be an lcc category and $p : \widetilde{U} \to U$ be a morphism. A Prop-structure on p is a collection of data of the following form:*

1. *two pull-back squares*

$$\begin{array}{ccc} P & \longrightarrow & \widetilde{U} \\ \downarrow & & \downarrow p \\ pt & \longrightarrow & U \end{array} \qquad \begin{array}{ccc} \widetilde{P} & \longrightarrow & \widetilde{U} \\ p_0 \downarrow & & \downarrow p \\ P & \longrightarrow & U \end{array}$$

2. *a Π-structure on (p, p_0, p_0).*

A $Prop$-structure on p corresponds to the rules:

$$\frac{}{x : P\rhd} \qquad \frac{}{x : P, y : x\rhd} \qquad \frac{\Gamma, f : X \to P\rhd}{\Gamma, f : X \to P \vdash \prod x : X.ev(f, x) : P}$$

4 Universes in the category of simplicial sets

1 Well-ordered morphisms of simplicial sets

Let X, Y be simplicilal sets. A well-ordered morphism $p : Y \to X$ is a pair which consists of a morphism $Y \to X$ (also denoted by p) and of a function which assigns to each $n \geq 0$ and each $\sigma \in X_n$ a well-ordering on $p^{-1}(\sigma) \subset Y_n$.

Note that there is a unique well-ordering on any isomorphism but, for example, the morphism $pt \amalg pt \to pt$ has uncountably many well-orderings since $pt_n = pt$ for all n and we require no compatibility conditions for well orderings of the fibers over different simplexes of the target.

If $p : Y \to X$, $p' : Y' \to X$ are two well-ordered morphisms then we define a standard isomorphism from Y to Y' over X as an isomorphism over X such that for each $n \geq 0$ and each $\sigma \in X_n$ the bijection $p^{-1}(\sigma) \to (p')^{-1}(\sigma)$ is order-preserving. Since there is at most one order-preserving bijection between two well-ordered sets, there is at most one standard isomorphism between two well-ordered simplicial sets over X.

Let $WOM(X, < \alpha)$ be the set of standard isomorphism classes of well-ordered simplicial sets $p : Y \to X$ over X such that for each $n \geq 0$ and each $\sigma \in X_n$ the fiber $p^{-1}(\sigma)$ has cardinality $< \alpha$. For any $f : X' \to X$ the pull-back $p' : Y' = X' \times_X Y \to X'$ of a well-ordered morphism has a natural well-ordering which makes $WOM(X, < \alpha)$ into a functor from $\Delta^{op}Sets$ to $Sets$.

Consider $WOM(\Delta^n, < \alpha)$. These sets depend on Δ^n functorially and therefore define a simplicial set $WOM(< \alpha)$. Let $\widetilde{WOM}(\Delta^n, < \alpha)$ be the set of pairs $p : Y \to \Delta^n$, $s \in Y_n$ where $p \in WOM(\Delta^n, < \alpha)$ and $s \in p^{-1}(\sigma_n)$ where σ_n is the non-degenerate n-simplex of Δ^n. These sets also depend on Δ^n functorially and define a simplicial set $\widetilde{WOM}(< \alpha)$.

Since $p^{-1}(\sigma)$ carries a well-ordering the natural projection $\widetilde{WOM}(< \alpha) \to WOM(< \alpha)$ carries a natural well-ordering.

Proposition 1.1 *[2009.12.10.pr1] The morphism $\widetilde{WOM}(< \alpha) \to WOM(< \alpha)$ is a universal well-ordered morphism with fibers of cardinality $< \alpha$. In particular, $WOM(< \alpha)$ represents the functor $WOM(-, < \alpha)$.*

Proof: Straightforward.

Note that $WOM(<\alpha)$ is obviously a contractible Kan simplicial set for any $\alpha > 0$.

Let us consider now the sub-object $WOF(<\alpha)$ of $WOM(<\alpha)$ which classifies well-ordered Kan fibrations whose fibers have cardinality $<\alpha$ and let $\widetilde{WOF}(<\alpha) \to WOF(<\alpha)$ be the corresponding universal fibration.

The idea of the proof of the following result and in general the idea to use minimal fibrations is due to A. Bousfield and reached me through Peter May and Rick Jardine.

Proposition 1.2 *[**2009.12.8.prop1**] Let α be an infinite cardinal. Then the simplicial set $WOF(<\alpha)$ is Kan.*

Proof: One can easily see that it is sufficient to show that for any horn inclusion $\Lambda^n_k \to \Delta^n$ and any Kan fibration $p : B \to \Lambda^n_k$ there exists a pull-back square of the form

[**2009.12.8.eq1**]
$$\begin{array}{ccc} B & \longrightarrow & C \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \Lambda^n_k & \longrightarrow & \Delta^n_k \end{array} \qquad (25)$$

where q is a Kan fibration whose fibers have cardinality $<\alpha$. By Quillen's Lemma ([8]) there is a factorization of p of the form $B \stackrel{p'}{\to} B' \stackrel{p''}{\to} \Lambda^n_k$ where p' is a trivial fibration and p'' is a minimal fibration. Since trivial fibrations are surjective, both p' and p'' have fibers of cardinality $<\alpha$. By [5, Cor. 11.7, p.45] the fibration p'' is isomorphic to a fibration $F \times \Lambda^n_k \to \Lambda^n_k$ where F is a Kan simplicial set. Together with Lemma 1.4 it shows that there is a diagram of the form

$$\begin{array}{ccc} B & \longrightarrow & C \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle q'} \\ F \times \Lambda^n_k & \longrightarrow & F \times \Delta^n \\ \downarrow & & \downarrow \\ \Lambda^n_k & \longrightarrow & \Delta^n \end{array}$$

with pull-back squares such that q' is a trivial fibration with fibers of cardinality $<\alpha$. The external square of this diagram has the required form (25).

Lemma 1.3 *[**2009.12.11.l1**] Let α be an infinite cardinal. Let $p : Y \to X$ be a map of simplicial sets such that for each $n \geq 0$, $x \in X_n$ one has $|p^{-1}(x) \cap Y^{nd}_n| < \alpha$ where Y^{nd}_n is the subset of non-degenate simplexes in Y_n. Then for each $n \geq 0$, $\sigma \in X_n$ one has $|p^{-1}(x)| < \alpha$.*

Proof: Since for any surjection s the map $s^* : X_m \to X_n$ is an inclusion and there are only finitely many surjections of the form $[n] \to [m]$ (where $[n] = \{0, \dots, n\}$) there exists only finitely many pair-wise distinct pairs (x_i, s_i) where $x_1, \dots, x_d \in X_{m_i}$ and $s : [n] \to [m_i]$ is a surjection, such that $s^*_i(x_i) = x$.

Consider the map

[**2009.12.11.eq1**] $$\amalg_i\, s^*_i : \amalg_i (p^{-1}(x_i) \cap Y^{nd}_{m_i}) \to p^{-1}(x) \qquad (26)$$

If $y \in p^{-1}(x)$ then there exists $s : [n] \to [m]$ and $y' \in Y_m^{nd}$ such that $s^*(y') = y$. Then $s^*p(y') = p\, s^*(y') = x$ and therefore $s = s_i$ for $i = 1, \dots, d$. We conclude that the map (26) is surjective and therefore $|p^{-1}(x)| < \alpha$.

Lemma 1.4 *[**2009.12.8.l4**] Let $\alpha > \aleph_0$ be an cardinal. Let $j : A \to X$ be a cofibration (monomorphism) and $p : B \to A$ be a trivial Kan fibration with fibers of cardinality $< \alpha$. Then there exists a pull-back square of the form*

$$\begin{array}{ccc} B & \longrightarrow & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{j} & X \end{array} \qquad (27)$$

[**2009.12.8.eq2**]

such that q is a trivial Kan fibration with fibers of cardinality $< \alpha$.

Proof: Define inductively squares

$$\begin{array}{ccc} B & \longrightarrow & B_i \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_i} \\ A & \xrightarrow{j} & X \end{array} \qquad (28)$$

[**2009.12.8.eq3**]

setting $p_0 = p$ and defining B_{i+1} by the push-out square of the form

$$\begin{array}{ccc} \amalg_n \amalg_{Q_{n,i}} \partial\Delta^n & \longrightarrow & B_i \\ \downarrow & & \downarrow \\ \amalg_n \amalg_{Q_{n,i}} \Delta^n & \longrightarrow & B_{i+1} \end{array} \qquad (29)$$

[**2009.12.11.eq2**]

where $Q_{n,i}$ is the set of commutative squares of the form

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & B_i \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & X \end{array}$$

such that its base simplex i.e. the simplex corresponding to the map $\Delta^n \to X$ does not belong to A.

Since for such a map f one has $f^{-1}(A) \subset \partial\Delta^n$ the squares (28) are pull-back squares. Define Y as $colim\, B_i$. Then one verifies easily that (27) is a pull-back square and q is a fibration. Let us show that the fibares of q have cardinality $< \alpha$. Since $\alpha > \aleph_0$ it is sufficient to show that, assuming that the fibers of $B_i \to X$ are of cardinality $< \alpha$, the fibers of B_{i+1} are. The squares (29) show that for each n and $x \in X_n$ the fiber $p_{i+1}^{-1}(x) \cap (B_{i+1})_n^{nd}$ is of the form $(p_i^{-1}(x) \cap (B_i)_n^{nd}) \amalg Q(n,i;x)$ where $Q(n,i;x)$ is the subset in $Q(n,i)$ which consists of squares whose base simplex $\Delta^n \to X$ is x. It remains to observe that the number of such squares is $< \alpha^{n+1}$ and to apply Lemma 1.3.

The category $\Delta^{op}Sets$ is a topos and in particular an lcc. The relative internal Hom-objects in $\Delta^{op}Sets$ can be explicitly described as follows.

Lemma 1.5 *[**2009.12.8.l5**] Let $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ be morphisms of simplicial sets. Consider the simplicial set $H(p_1, p_2)$ whose set of n-simplexes is the set of pairs of the form $(f, \widetilde{f})$ where $f : \Delta^n \to B$ and $\widetilde{f} : f^*(p_1) \to p_2$ is a morphism over B.*

Let $H(p_1, p_2) \to B$ be the morphism $ev : (f, \widetilde{f}) \mapsto f$ and let $H(p_1, p_2) \times_B E_1 \to E_2$ be the morphism which sends $((f, \widetilde{f}), \sigma)$ to $\widetilde{f}(\sigma)$. Then $(H(p_1, p_2), ev)$ is an internal Hom-object from E_1 to E_2 over B.

Lemma 1.6 *[**2009.12.8.l6**] Let $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ be Kan fibrations. Then $H(p_1, p_2) \to B$ is a Kan fibration.*

Proof: It follows immediately from definitions and the fact that for a fibration $p_1 : E_1 \to B$ and an anodyne morphism $A \to X$ over B, the morphism $A \times_B E_1 \to X \times_B E_1$ is anodyne.

Lemma 1.7 *[**2009.12.9.l1**] Let $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ be Kan fibrations and $f : E_1 \to E_2$ a morphism over B which is a weak equivalence. Then for any $g : B' \to B$ the pull-back $f' : B' \times_B E_1 \to B' \times_B E_2$ is a weak equivalence.*

Proof: Using the factorization of f into a trivial cofibration and a trivial fibration and the fact that the pull-back of a trivial fibration is a trivial fibration we may assume that f is a trivial cofibration. A trivial cofibration between two fibrant objects (in the category over B) is a homotopy equivalence and the pull-back of a homotopy equivalence is a homotopy equivalence.

Lemma 1.8 *[**2009.12.9.l3**] Let $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ be Kan fibrations and $f : E_1 \to E_2$ a morphism over B. Suppose that for any $n \geq 0$ and any simplex $\sigma : \Delta^n \to B$ the pull-back $f_\sigma : \Delta^n \times_B E_1 \to \Delta^n \times_B E_2$ is a weak equivalence. Then f is a weak equivalence.*

Proof: Replacing p_1, p_2 by minimal fibrations we may assume that p_1, p_2 are minimal. Then our condition implies that f_σ is an isomorphism for each σ and therefore is an isomorphism globally.

Let p_1, p_2 be Kan fibrations as above. Consider the internal Hom-object $H(p_1, p_2)$. A morphism $f : A \to H(p_1, p_2)$ defines a morphism $pr(f) : A \to B$ and a morphism $fib(f) : A \times_B E_1 \to A \times_B E_2$. Let $Eq(p_1, p_2)_n$ be the subset of simplexes $\sigma : \Delta^n \to H(p_1, p_2)$ such that $fib(\sigma)$ is a weak equivalence. Lemma 1.7 implies that these subsets form a simplicial subset in $H(p_1, p_2)$ which we denote by $Eq(p_1, p_2)$ or $Eq_B(p_1, p_2)$.

Lemma 1.9 *[**2009.12.9.l2**] Let p_1, p_2 be Kan fibrations as above and $f : A \to H(p_1, p_2)$ a morphism. The $fib(f)$ is a weak equivalence if and only if $Im(f) \subset Eq(p_1, p_2)$.*

Proof: Straightforward using Lemmas 1.7 and 1.8.

Lemma 1.10 *[**2009.12.9.l4**] Let p_1, p_2 be Kan fibrations as above, $f : E_1 \to E_2$ a morphism over B and $b \in B$. Assume that B is connected and that $p_1^{-1}(b) \to p_2^{-1}(b)$ is a weak equivalence. Then f is a weak equivalence.*

Proof: In view of Lemma 1.8 we may assume that $B = \Delta^n$. Since the pull-back of a weak equivalence along a fibration is a weak equivalence and $b : \Delta^0 \to \Delta^n$ is a weak equivalence, we conclude that $p_1^{-1}(b) \to E_1$ and $p_2^{-1}(b) \to E_2$ are weak equivalences. Therefore, if $f_b : p_1^{-1}(b) \to p_2^{-1}(b)$ is a weak equivalence then so is f.

Lemma 1.11 *[2009.12.9.l5] Let p_1, p_2 be Kan fibrations as above. Then $Eq(p_1, p_2)$ is a union of connected components of $H(p_1, p_2)$ i.e. if (A, a) is a connected pointed simplicial set and $f : A \to H(p_1, p_2)$ a morphism such that $f(a) \in Eq(p_1, p_2)$ then $Im(f) \subset Eq(p_1, p_2)$.*

Proof: Follows immediately from Lemma 1.10.

Let $p : E \to B$ be a fibration. Let $p_1 : E \times B \to B \times B$ and $p_2 : B \times E \to B \times B$ be the obvious projections. Consider the space $H(p_1, p_2)$ over $B \times B$. The natural isomorphism $p_1^{-1}(\Delta(B)) = p_2^{-1}(\Delta(B))$ where Δ is the diagonal defines a morphism $B \to H(p_1, p_2)$ over $B \times B$ which, by Lemma 1.9, takes values in $Eq(p_1, p_2)$. Let us denote this morphism by $mm_p : B \to Eq(p_1, p_2)$.

Definition 1.12 *[2009.12.9.def1] A Kan fibration $p : E \to B$ is called univalent if the morphism $mm_p : B \to Eq(p_1, p_2)$ defined above is a weak equivalence.*

Theorem 1.13 *[2009.12.9.th1] The Kan fibration*

$$p_{fib} : \widetilde{WOF}(< \alpha) \to WOF(\alpha)$$

is univalent.

Proof: Let $E = \widetilde{WOF}(< \alpha)$ and $B = WOF(< \alpha)$. Let $P_1 : E \times B \to B \times B$, $P_2 : B \times E \to B \times B$ be the projections. Proposition 1.1 implies easily that the space $H(P_1, P_2)$ represents the functor which sends X into the set of (standard isomorphism classes of) triples of the form $p_1 : Y_1 \to X$, $p_2 : Y_2 \to X$, $f : Y_1 \to Y_2$ where p_1, p_2 are well ordered Kan fibrations with fibers of cardinality $< \alpha$ and f is a morphism over X. The subspace $Eq(P_1, P_2)$ classifies triples such that f is a weak equivalence.

Consider now the morphism $r : B \to Eq(P_1, P_2) \to B \times B \stackrel{pr_2}{\to} B$. To prove the theorem it is sufficient to show that the composition $Eq(P_1, P_2) \to B \to Eq(P_1, P_2)$ is homotopic to the identity. This composition represents the functor morphism which sends (p_1, p_2, f) to (p_2, p_2, id).

Applying Lemma 1.14 to the universal equivalence of fibrations over $Eq(P_1, P_2)$ and using the axiom of choice we construct the required homotopy.

Lemma 1.14 *[2009.12.11.l3] Let $p_1 : Y_1 \to X$, $p_2 : Y_2 \to X$ be two Kan fibrations and $f : Y_1 \to Y_2$ be a morphism over X which is a weak equivalence. Then there exists a fibration $q : Z \to X \times \Delta^1$ and a morphism $F : Z \to Y_2 \times \Delta^1$ over X such that the fiber of F over $X \times \{0\}$ is isomorphic to f and the fiber over $X \times \{1\}$ is isomorphic to Id_{Y_2}.*

In addition if $\alpha > \aleph_0$ is a cardinal and the fibers of p_1 and p_2 have cardinality $< \alpha$ then we can choose q such that its fibers have cardinality $< \alpha$.

Proof: Let $Y_1 \stackrel{p_1'}{\to} Y_1' \stackrel{p_1''}{\to} X$, $Y_2 \stackrel{p_2'}{\to} Y_2' \stackrel{p_2''}{\to} X$ be factorizations of p_1 and p_2 such that p_i' is a trivial fibration and p_i'' a minimal fibration which exist by [8]. If s_1 is a section of p_1' (which exist since all simplicial sets are cofibrant) then $p_2' f\, s$ is a weak equivalence between two minimal fibrations over X and therefore an isomorphism. Let us denote it by $f' : Y_1' \to Y_2'$.

Applying Lemma 1.4 to the trivial fibration $p'_1 \amalg p'_2 : Y_1 \amalg Y_2 \to Y'_1 \amalg Y'_2$ and monomorphism $j = (i_0\, f' \amalg i_1) : Y'_1 \amalg Y'_2 \to Y'_2 \times \Delta^1$ we obtain a pull-back square of the form

$$\begin{array}{ccc} Y_1 \amalg Y_2 & \xrightarrow{k} & Z \\ {\scriptstyle p'_1 \amalg p'_2}\downarrow & & \downarrow{\scriptstyle q} \\ Y'_1 \amalg Y'_2 & \xrightarrow{j} & Y'_2 \times \Delta^1 \end{array}$$

Consider now the square

$$\begin{array}{ccc} Y_1 \amalg Y_2 & \xrightarrow{i_0\, f \amalg i_1} & Y_2 \times \Delta^1 \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle p'_2 \times Id} \\ Z & \xrightarrow[q]{} & Y'_2 \times \Delta^1 \end{array}$$

Let us show that it commutes. (The following argument was supplied by Thomas Streicher). It clearly commutes on the Y_2 summand. On the Y_1 summand the corresponding maps are (up to inclusions into $Y'_2 \times \Delta^1$) of the form $f' \circ p'_1$ and $p'_2 \circ f$. Note that a priory it is not clear that $f' \circ p'_1 = p'_2 \circ f$. However these two maps are homotopic since in the homotopy category over X, the morphism p'_1 and therefore its section s_1 are isomorphisms and therefore $s_1 \circ p'_1$ is homotopic to the identity. On the other hand Y'_2 is a minimal fibration over X and any two morphisms with values in this simplicial set which are homotopic and coincide after projection to X are equal.

Since k is a cofibration (monomorphism) and $p'_2 \times Id$ is a trivial fibration, there is a morphism $F : Z \to Y_2 \times \Delta^1$ which splits this square into two commutative triangles. One verifies easily that the pair (Z, F) satisfies the conditions of the lemma.

Let $p' : E \to B$, $p : \widetilde{U} \to U$ be two Kan fibrations. For a simplicial set X denote by $HInd(p', p)(X)$ the set of pairs $(\widetilde{f}, f)$ where $\widetilde{f} : E \times X \to \widetilde{U}$, $f : B \times X \to \widetilde{U}$ are morphisms such that the square

[2009.12.23.eq1]
$$\begin{array}{ccc} E \times X & \xrightarrow{\widetilde{f}} & \widetilde{U} \\ {\scriptstyle p' \times Id_X}\downarrow & & \downarrow{\scriptstyle p} \\ B \times X & \xrightarrow{f} & U \end{array} \tag{30}$$

is a homotopy pull-back square i.e. such that $p \circ \widetilde{f} = f \circ (p' \times Id_X)$ and the obvious morphism $E \times X \to (B \times X) \times_U \widetilde{U}$ is a weak equivalence. Since p' and p are fibrations, the composition of a homotopy pull-back square of the form (30) with a pull-back square

$$\begin{array}{ccc} E \times X' & \longrightarrow & E \times X \\ \downarrow & & \downarrow \\ B \times X' & \longrightarrow & B \times X \end{array}$$

defined by any morphism $f : X' \to X$, is a homotopy pull-back square. Therefore $HInd(p', p)(-)$ is a contravariant functor on $\Delta^{op}Sets$ and Lemma 1.8 implies easily that it is represented by the simplicial set $HInd(p', p)$ whose set of n-simplexes is $HInd(p', p)(\Delta^n)$.

Proposition 1.15 *[**2009.12.23.prop1**] A Kan fibration $p : \widetilde{U} \to U$ such that U is a Kan simplicial set is univalent if and only if for any Kan fibration $p' : E \to B$, $(HInd(p', p) \neq \emptyset) \Rightarrow$ $(HInd(p', p)$ is contractible).*

Proof: Let $p' : E \to B$ be a Kan fibration such that $HInd(p', p) \neq \emptyset$ i.e. such that there exists a pull-back square of the form

$$\begin{array}{ccc} E & \xrightarrow[\widetilde{f}]{} & \widetilde{U} \\ {\scriptstyle p}\downarrow & & \downarrow \\ B & \xrightarrow[f]{} & U \end{array}$$

Let X be a simplicial set. Then a morphism $X \to HInd(p', p)$ is given by a pair of a morphism $f_X : B \times X \to U$ and a weak equivalence $E \times X \to (B \times X)_{f_X} \times_p \widetilde{U}$ over $B \times X$. The morphism $E \times X \to B \times X$ is canonically isomorphic to the projection $(B \times X)_{f \circ pr_B} \times_p \widetilde{U} \to B \times X$. Therefore, morphisms $X \to HInd(p', p)$ correspond to morphisms $B \times X \to Eq(p \times Id_U, Id_U \times p)$ whose composition with $Eq(p \times Id_U, Id_U \times p) \stackrel{p_{Eq}}{\to} U \times U \stackrel{pr_2}{\to} U$ equals $f \circ pr_B : B \times X \to B \stackrel{f}{\to} U$.

Since U is assumed to be a Kan simplicial set the morphism $pr_2 \circ p_{Eq}$ is a Kan fibration. If p is univalent it is a trivial Kan fibration and from the previous description of $HInr(p', p)$ we conclude that for any cofibration $X \subset Y$ and a morphism $F = (\widetilde{f}, f) : X \to HInt(p', p)$ there exists an extension of F to Y i.e. that $HInt(p', p) \to pt$ is a trivial Kan fibration.

To prove the other implication consider the case when $B = pt$. Then our considerations show that $HInd(p', p)$ is isomorphic to the fiber of $pr_2 \circ p_{Eq}$ over $f(pt) \in U_0$. Since any Kan fibrations with contractible fibers is a trivial Kan fibration we conclude that the required implication holds.

2 Well-ordered simplicial sets

We consider a triple (ST, ST', M) where ST, ST' are ZF-like set-theories and M is a model of ST and ST'. These data defines "the set of all ST sets" as an ST'-set. Similarly, these data provides an unambiguous definition for objects such as "the set of isomorphism classes of simplicial sets" etc.

Our first step is to choose a convenient set-level model of the 1-category of simplicial sets.

Definition 2.1 *[***2009.12.8.def1***] A well-ordered simplicial set is a simplicial set $(X_n)_{n \geq 0}$ together with well orderings $\prec$ on each of X_n.*

Note that the well orderings on X_n are note assumed to be compatible with the boundary or degeneracy maps. By a morphism between two well-ordered simplicial sets we will mean a morphism between the corresponding simplicial sets without any regard for orderings. A morphism which preserves well-orderings on each of X_n will be called a standard morphism.

The standard facts about well-ordered sets imply that there is at most one standard isomorphism between any two well-ordered simplicial sets. Therefore, we may consider a set level model C of $\Delta^{op}Sets$ where $Ob(C)$ is the set of standard isomorphism classes of well-ordered simplicial sets and $Mor(X, Y)$ is the set of all morphisms from X to Y. The uniqueness of standard isomorphisms implies that the composition of morphisms is well defined.

For well-ordered simplicial sets X, Y denote by $X \times Y$ the well-ordered simplicial set whose terms $X_n \times Y_n$ are well-ordered with respect to the lexicographical ordering such that the projection to X (but not to Y) is a standard morphism.

For $f : X' \to X$ and $p : Y \to X$ define the standard pull-back square

$$\begin{array}{ccc} f^*(Y,p) & \xrightarrow{q(f,Y,p)} & Y \\ {\scriptstyle p_f}\downarrow & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{f'} & X \end{array}$$

setting $f^*(Y,p)$ to be the subset in $X' \times Y$ defined by the usual equations with the induced well-ordering.

One verifies easily the following results.

Lemma 2.2 *[***2009.12.8.l1***] For any p the morphism p_f is standard and $p : Y \to X$ is standard if and only if $Id_X^*(p) = p$.*

Lemma 2.3 *[***2009.12.8.l2***] For any $p : Y \to X$ and $g : X'' \to X'$, $f : X' \to X$ one has $(fg)^*(p) = g^*f^*(p)$, $q(fg, Y, p) = q(g, f^*(p), p_f)q(f, Y, p)$ and $p_{fg} = (p_f)_g$.*

Note that $q(f, Y, p)$ need not be standard even if both p and f are standard (consider e.g. the case when $X = pt$).

In what follows we choose a well-ordering on the sets Δ_i^n and consider the standard simplexes as objects of C with respect to this ordering.

5 Type theoretic constructs in terms of C-structures

1 Π-C-structures

The notion of a Π-C-structure is equivalent to the notion of a contextual category with products of families of types from [9]. We use the name Π-C-structures to emphasize the fact that we are dealing here with an additional structure on a C-structure rather than with a property of such an object.

Let us recall first the following definition.

Definition 1.1 *[***2009.11.24.def2***] Let $\mathcal{C}$ be a 1-category. Let $g : Z \to Y$, $f : Y \to X$ be a pair of morphisms such that for any $U \to X$ a fiber product $U \times_X Y$ exists. A pair*

$$(w : W \to X, h : W \times_X Y \to Z)$$

such that $g \circ h = pr$ is called a universal pair for (f, g) if for any $U \to X$ the map

$$Hom_X(U, W) \to Hom_Y(U \times_X Y, Z)$$

of the form $u \mapsto h \circ (u \times Id_Y)$ is a bijection.

If a universal pair exists then it is easily seen to be unique up to a canonical isomorphism. We denote such a pair by $(\Pi(g, f), e_{g,f} : \Pi(g, f) \times_X Y \to Z)$. Note that if $f' : Y \to X$ and $pr : Y' \times_X Y \to Y$ is the projection then

$$(\Pi(pr, f), pr' \circ e_{pr,f} : \Pi(g, f) \times_X Y \to Y') = (\underline{Hom}_X(Y, Y'), ev : \underline{Hom}_X(Y, Y') \times_X Y \to Y')$$

so that relative internal Hom-objects are particular cases of universal pairs.

Definition 1.2 *[2009.11.24.def1] A* Π*-C-structure is a C-structure* CC *together with additional data of the form*

1. *for each* $Y \in Ob(CC)_{\geq 2}$ *an object* $\Pi(Y) \in Ob(CC)$ *such that* $ft(\Pi(Y)) = ft^2(Y)$*,*
2. *for each* $Y \in Ob(CC)_{\geq 2}$ *a morphism* $eval : T(ft(Y), \Pi(Y)) = p^*_{ft(Y)}(\Pi(Y)) \to Y$ *over* $ft(Y)$*,*

such that

(i) for any $f : Z \to ft^2(Y)$ *one has* $f^*(\Pi(Y)) = \Pi(f^*(Y))$ *and* $f^*(eval_Y) = eval_{f^*(Y)}$*,*

(ii) $(\Pi(Y), eval_Y))$ *is a universal pair for* $(p_Y, p_{ft(Y)})$*.*

Let us now prove that this definition can be re-written in a less compact but purely equational form. As before let us write B_n for $Ob(CC)_n$, $\widetilde{B}_n$ for $\widetilde{Ob}(CC)_n$ etc.

The C-structure is completely determined by the sets $B_n, \widetilde{B}_{n+1}$, $n \geq 0$ and maps $\partial : \widetilde{B}_{n+1} \to B_{n+1}$, $ft : B_{n+1} \to B_n$, $\delta : B_n \to \widetilde{B}_{n+1}$ and the maps $T_{n+1}, \widetilde{T}_{n+1}, S_{n+1}, \widetilde{S}_{n+1}$ considered above.

Suppose now that we are given a Π-C-structure. Then we have maps

1. $\Pi : B_{n+2} \to B_{n+1}$, $n \geq 0$,
2. $\lambda : \widetilde{B}_{n+2} \to \widetilde{B}_{n+1}$, $n \geq 0$,
3. $ev : (\widetilde{B}_{n+1})_{\partial} \times_{ft} (B_{n+2})_{\Pi} \times_{\partial} (\widetilde{B}_{n+1}) \to \widetilde{B}_{n+1}$, $n \geq 0$

as follows. The map Π is the map from Definition 1.2. Since $(\Pi(Y), eval_Y)$ is a universal pair for $(p_Y, p_{ft(Y)})$ the mapping

$$\phi_Y : \{f \in \widetilde{B}_{n+1} \,|\, \partial(f) = \Pi(Y)\} \to \{s \in \widetilde{B}_{n+2} \,|\, \partial(s) = Y\}$$

given by the formula

$$\phi_Y(f) = eval_Y \circ \widetilde{T}(ft(Y), f)$$

is a bijection. One defines λ_Y as the inverse to this bijection.

The map ev sends a triple (r, Y, f) such that $\partial(r) = ft(Y)$ and $\partial(f) = \Pi(Y)$ to

$$ev(r, Y, f) = \widetilde{S}(r, eval \circ \widetilde{T}(ft(Y), f))$$

as partially illustrated by the following diagram:

$$\begin{array}{ccccc}
 & & Y & \longleftarrow & S(r,Y) \\
 & & {\scriptstyle p_Y}\downarrow & & \downarrow \\
p^*_{ft(Y)}(\Pi(Y)) & \longrightarrow & ft(Y) & \underset{r}{\longleftarrow} & ft^2(Y) \\
\downarrow & & \downarrow & & \\
\Pi(Y) & \xrightarrow{p_{\Pi(Y)}} & ft^2(Y) & &
\end{array}$$

Lemma 1.3 *[**2009.11.30.l1**] Let $n \geq i \geq 0$, $Y \in B_{n+2}$, $g : Z \to ft^{i+2}(Y)$ and $f \in \widetilde{B}(\Pi(Y))$. Then one has*

$$g^*(\phi_Y(f), i+2) = \phi_{g^*(Y,i+2)}(g^*(f, i+1))$$

Proof: Let $h_1 = q(g, ft(Y), i+1)$, $h_2 = q(g, ft^(Y), i+2)$. Then one has

$$g^*(\phi_Y(f), i+2) = h_1^*(\phi_Y(f)) = h_1^*(eval_Y \circ \widetilde{T}(ft(Y), f)) = h_1^*(eval_Y) \circ h_1^*(\widetilde{T}(ft(Y), f))$$
$$= eval_{h_1^*(Y)} p^*_{g^*(ft(Y),i+1)}(h_2^*(f)) = \phi_{h_1^*(Y)}(h_2^*(f)) = \phi_{g^*(Y,i+2)}(g^*(f, i+1)).$$

As an immediate corollary of Lemma 1.3 we have:

Lemma 1.4 *[**2009.11.30.l2**] Let $n \geq i \geq 0$, $Y \in B_{n+2}$, $g : Z \to ft^{i+2}(Y)$ and $r \in \widetilde{B}(Y)$. Then one has*

$$g^*(\lambda(r), i+1) = \lambda(g^*(r, i+2)).$$

Lemma 1.5 *[**2009.11.30.l3**] Let $n \geq i \geq 0$, $Y \in B_{n+2}$, $g : Z \to ft^{i+2}(Y)$, $r \in \widetilde{B}(ft(Y))$ and $f \in \widetilde{B}(\Pi(Y))$. Then one has*

$$g^*(ev(r, Y, f), i+1) = ev(g^*(r, i+2), g^*(Y, i+2), g^*(f, i+1))$$

Proof: Let $h_1 = q(g, ft(Y), i+1)$, $h_2 = q(g, ft^(Y), i+2)$. Then one has:

$$g^*(ev(r, Y, f), i+1) = h_2^*(\widetilde{S}(r, eval \circ \widetilde{T}(ft(Y), f))) = h_2^*(r^*(eval \circ \widetilde{T}(ft(Y), f))) =$$
$$= (h_2^*(r))^* h_1^*(eval \circ \widetilde{T}(ft(Y), f))) = (h_2^*(r))^*(h_1^*(eval) \circ h_1^* p^*_{ft(Y)}(f)) =$$
$$= (g^*(r, i+2))^*(eval \circ p^*_{g^*(ft(Y),i+1)}(h_2^*(f))) = ev(g^*(r, i+2), g^*(Y, i+2), g^*(f, i+1)).$$

Proposition 1.6 *[**2009.11.29.prop1**] Let $CC = (B_n, \widetilde{B}_n, ft, \partial, \delta)$ be a C-structure. Let further $(\Pi, eval)$ be a Π-structure on CC. Then the maps Π, λ, ev defined by this structure satisfy the following conditions:*

1. *for $Y \in B_{n+2}$ one has*
   (a) *$ft\,\Pi(Y) = ft^2(Y)$,*
   (b) *for $n+1 \geq i \geq 1$, $Z \in B_{n+2-i}$ such that $ft(Z) = ft^{i+1}(Y)$, $T(Z, \Pi(Y)) = \Pi(T(Z, Y))$,*
   (c) *for $n+1 \geq i \geq 1$, $t \in \widetilde{B}_{n+1-i}$ such that $\partial(t) = ft^{i+1}(Y)$, $S(t, \Pi(Y)) = \Pi(S(t, Y))$,*
2. *for $s \in \widetilde{B}_{n+2}$ one has*
   (a) *$\partial\,\lambda(s) = \Pi\,\partial(s)$,*
   (b) *for $n+1 \geq i \geq 1$, $Z \in B_{n+2-i}$ such that $ft(Z) = ft^{i+1}\,\partial(s)$, $\widetilde{T}(Z, \lambda(s)) = \lambda(\widetilde{T}(Z, s))$,*
   (c) *for $n+1 \geq i \geq 1$, $t \in \widetilde{B}_{n+1-i}$ such that $\partial(t) = ft^{i+1}\,\partial(s)$, $\widetilde{S}(t, \lambda(s)) = \lambda(\widetilde{S}(t, s))$,*
3. *for $r \in \widetilde{B}_{n+1}$, $Y \in B_{n+2}$ and $f \in \widetilde{B}_{n+1}$ such that $\partial(r) = ft(Y)$ and $\partial(f) = \Pi(Y)$ one has*
   (a) *$\partial(ev(r, Y, f)) = S(r, Y)$,*

(b) for $n+1 \geq i \geq 1$, $Z \in B_{n+2-i}$ such that $ft(Z) = ft^{i+1}(Y)$,

$$\widetilde{T}(Z, ev(r, Y, f)) = ev(\widetilde{T}(Z, r), T(Z, Y), \widetilde{T}(Z, f)),$$

(c) for $n+1 \geq i \geq 1$, $t \in \widetilde{B}_{n+1-i}$ such that $\partial(t) = ft^{i+1}(Y)$,

$$\widetilde{S}(t, ev(r, Y, f)) = ev(\widetilde{S}(t, r), S(t, Y), \widetilde{S}(t, f)),$$

4. *for $r \in \widetilde{B}_{n+1}$, $s \in \widetilde{B}_{n+2}$ such that $ft(\partial(s)) = \partial(r)$*

$$ev(r, \partial\, s, \lambda(s)) = \widetilde{S}(r, s)$$

(β-reduction),

5. *for $Y \in B_{n+2}$, $f \in \widetilde{B}_{n+1}$ such that $\partial(f) = \Pi(Y)$,*

$$[\mathbf{2009.11.30.oldeq1}]\lambda(ev(\delta_{ft(Y)}, T(ft(Y), Y), \widetilde{T}(ft(Y), f))) = f \qquad (31)$$

(η-reduction).

Proof: (1a) Follows from Definition 1.2(1). (1b) Follows from Definition 1.2(i) applied to $f = q(p_Z, ft^2(Y), i-1)$. (1c) Follows from Definition 1.2(i) applied to $f = q(t, ft^2(Y), i-1)$.

(2a) Follows from the definition of λ. (2b) Follows from Lemma 1.4 applied to p_Z. (2c) Follows from Lemma 1.4 applied to t.

(3a) Follows from the definition of ev. (3b) Follows from Lemma 1.5 applied to p_Z. (3c) Follows from Lemma 1.5 applied to t.

(4) One has

$$ev(r, \partial\, s, \lambda(s)) = r^*(eval \circ (p^*_{ft(Y)}(\lambda(s)))) = r^*(\phi_Y(s)) = r^*(s) = \widetilde{S}(r, s).$$

(5) Let $T_1 = T(ft(Y), ft(Y))$ and $T_2 = T(ft(Y), Y)$. Then

$$ev(\delta_{ft(Y)}, T(ft(Y), Y), \widetilde{T}(ft(Y), f)) = \delta^*_{ft(Y)}(eval_{T_2} \circ p^*_{T_1}(p^*_{ft(Y)}(f))) =$$

$$= \delta^*_{ft(Y)}(eval_{T_2}) \circ \delta^*_{ft(Y)} p^*_{T_1} p^*_{ft(Y)}(f) = eval_{\delta^*_{ft(Y)}(T_2)} \circ p^*_{ft(Y)}(f) = eval_Y \circ p^*_{ft(Y)}(f) = \phi_Y(f)$$

which implies (31) by definition of λ.

The converse to Proposition 1.6 holds as well. Let $CC = (B_n, \widetilde{B}_n, ft, \partial, \delta)$ be a C-structure and let

1. $\Pi : B_{n+2} \to B_{n+1}$, $n \geq 0$,
2. $\lambda : \widetilde{B}_{n+2} \to \widetilde{B}_{n+1}$, $n \geq 0$,
3. $ev : (\widetilde{B}_{n+1})_\partial \times_{ft} (B_{n+2})_\Pi \times_\partial (\widetilde{B}_{n+1}) \to \widetilde{B}_{n+1}$, $n \geq 0$

be maps satisfying the conclusion of Proposition 1.6. For each $Y \in \widetilde{B}_{n+2}$ define a morphism

$$eval_Y : T(ft(Y), \Pi(Y)) \to Y$$

by the formula

$$eval_Y = q(p_Z, Y) \circ ev(p^*_Z(\delta_{ft(Y)}), T_2(Z, Y), \delta_Z)$$

where $Z = p^*_{ft(Y)}(\Pi(Y))$.

Proposition 1.7 *[2009.11.30.prop2] Under the assumption made above the morphisms $eval_Y$ are well defined and $(\Pi, eval)$ is a Π-structure on CC.*

Proof: Let us show that eva_Y is well defined. This requires us to check the following conditions:

1. $ft^2(Y) = ft(\Pi(Y))$, therefore Z is defined,
2. $ft(Z) = ft\partial(\delta_{ft(Y)})$ since $ft(Z) = ft(Y)$, therefore $p_Z^*(\delta_{ft(Y)})$ is defined,
3. $ft^2(Z) = ft^2(Y)$, therefore $T_2(Z,Y)$ is defined,
4. $\partial(p_Z^*(\delta_{ft(Y)}))) = p_Z^* p_{ft(Y)}^*(ft(Y))$, $ft(T_2(Z,Y)) = T_2(Z, ft(Y)) = p_Z^* p_{ft(Y)}^*(ft(Y))$,
5. $\partial(\delta_Z) = p_Z^*(Z) = p_Z^* p_{ft(Y)}^*(\Pi(Y)) = \Pi_{T_2(Z,Y)}$, therefore $ev = ev(p_Z^*(\delta_{ft(Y)}), T_2(Z,Y), \delta_Z)$ is defined,
6. $$\partial(ev) = (p_Z^*(\delta_{ft(Y)}))^*(T_2(Z,Y)) = (p_Z^*(\delta_{ft(Y)}))^* T(Z, T(ft(Y), Y)) =$$
$$= (p_Z^*(\delta_{ft(Y)}))^*(p_Z)^*((p_{ft(Y)})^*(Y,2),2) = (p_Z^*(\delta_{ft(Y)}))^* q(p_Z, p_Y^*(ft(Y)))^*(p_{ft(Y)})^*(Y,2) =$$
$$= (p_Z^*(\delta_{ft(Y)}))^* q(p_Z, p_Y^*(ft(Y)))^* q(p_{ft(Y)}, ft(Y))^*(Y) =$$
$$= (q(p_{ft(Y)}, ft(Y)) q(p_Z, p_Y^*(ft(Y))) p_Z^*(\delta_{ft(Y)}))^*(Y) = p_Z^*(Y)$$
and $q(p_Z, Y) : p_Z^*(Y) \to Y$. Therefore $eval_Y$ is defined and is a morphism from Z to Y as required by Definition 1.2(2).

We leave the verification of the conditions (i) of (ii) of Definition 1.2 for the later, more mechanized version of this paper.

2 Impredicative Π-universe structures.

Definition 2.1 *[2009.12.04.def1] Let $CC = (B, \widetilde{B}, \ldots, \Pi, \ldots)$ be Π-C-structure. An impredicative Π-universe structure on CC is a collection of data of the form*

1. *an object $\widetilde{\Omega} \in B_2$,*
2. *for any $n \geq 0$, $Y \in B_{n+1}$, $g : Y \to ft(\widetilde{\Omega})$ a morphism $\pi_\Omega(g) : ft(Y) \to ft(\widetilde{\Omega})$,*

such that the following conditions hold

(i) for any g as above $\pi_\Omega(g)^*(\widetilde{\Omega}) = \pi(g^*(\widetilde{\Omega}))$,

(ii) for any g as above and $h : Z \to ft(Y)$ one has

$$\pi_\Omega(g) \circ h = \pi_\Omega(g \circ q(h, Y))$$

The sequent presentation of an impredicative Π-structure looks as follows. Given an impredicative Π-universe $(\widetilde{\Omega}, \pi_\Omega)$ denote by Ω the object $ft(\widetilde{\Omega})$. Note that for any $Y \in B_n$ and $Z \in B_1$ the mapping which sends $s \in \widetilde{B}_{n+1}(T_n(Y, Z))$ to $s \circ q(p_{Y,n}, Z)$ defines a bijection $\phi_Y : \widetilde{B}_{n+1}(T_n(Y, Z)) \to Hom_{CC}(Y, Z)$.

For any $n \geq 0$, $Y \in B_{n+1}$, $s \in \widetilde{B}_{n+2}(T_{n+1}(Y, \Omega))$ define $\Pi_\Omega(s) \in \widetilde{B}_{n+1}(T_n(ft(Y), \Omega))$ by the formula

$$\Pi_\Omega(s) = \phi^{-1}_{ft(Y)}(\pi_\Omega(\phi_Y(g))).$$

One verifies immediately that the conditions of Definition 2.1 imply that

1. $S(\Pi_\Omega(s), T_n(ft(Y), \widetilde{\Omega})) = \Pi(S(s, T_{n+1}(Y, \widetilde{\Omega})))$,
2. for $n+1 \geq i \geq 1$, $Z \in B_{n+2-i}$ such that $ft(Z) = ft^i(Y)$, $\widetilde{T}(Z, \Pi_\Omega(s)) = \Pi_\Omega(\widetilde{T}(Z, s))$,
3. for $n+1 \geq i \geq 1$, $t \in \widetilde{B}_{n+1-i}$ such that $\partial(t) = ft^i(Y)$, $\widetilde{S}(t, \Pi_\Omega(s)) = \Pi_\Omega(\widetilde{S}(t, s))$.

Conversely one has:

Proposition 2.2 *[***2009.12.4.prop1***] Let $CC = (B, \widetilde{B}, \ldots, \Pi, \ldots)$ be Π-C-structure. Let $\widetilde{\Omega} \in B_2$, $\Omega = ft(\widetilde{\Omega})$ and*

$$\Pi_\Omega : (B_{n+1})_{T_{n+1}(-,\Omega)} \times_\partial (\widetilde{B}_{n+2}) \to \widetilde{B}_{n+1}$$

be maps satisfying conditions (1),(2),(3) listed above. Then they correspond to a unique impredicative Π-structure on CC.

3 Predicative Π-universe structures.

Definition 3.1 *[***2009.12.1def4***] Let $CC = (B, \widetilde{B}, \ldots, \Pi, \ldots)$ be Π-C-structure. A predicative Π-universe structure on CC is a collection of data of the form*

1. *an object $\widetilde{\Omega} \in B_2$,*
2. *for any $f : X \to ft(\widetilde{\Omega})$, $g : f^*(\widetilde{\Omega}) \to ft(\Omega)$ a morphism $\Pi_\Omega(f, g) : X \to ft(\widetilde{\Omega})$,*

such that the following conditions hold

(i) *for any f, g as above $\Pi_\Omega(f, g)^*(\widetilde{\Omega}) = \Pi(g^*(\widetilde{\Omega}))$,*

(ii) *for any f, g as above and $h : Z \to X$ one has*

$$\Pi_\Omega(f, g) \circ h = \Pi_\Omega(f \circ h, g \circ q(h, f^*(\widetilde{\Omega})))$$

Note that any impredicative universe structure defines a predicative universe structure by the formula $\Pi(f, g) = \Pi(g)$.

The sequent representation of a predicative Π-universe structure looks as follows.

Proposition 3.2 *[***2009.12.4.prop2***] Let $CC = (B, \widetilde{B}, \ldots, \Pi, \ldots)$ be Π-C-structure. Any predicative Π-universe structure on CC is uniquely determined by a collection of data of the form*

1. *an object $\widetilde{\Omega} \in B_2$ (we will write Ω for $ft(\widetilde{\Omega})$),*
2. *a morphism $\Pi_\Omega : \Pi(T_2(\widetilde{\Omega}, ft(\widetilde{\Omega}))) \to ft(\widetilde{\Omega})$,*

which satisfies the following conditions.

6 The system of Coq

The goal of these notes is to collect the material needed to prove that the system of inductive constructors and reductions supported by the Coq proof assistant is compatible with the univalent interpretation of type theory. For information about inductive definitions in Coq see [10], [2][p.77] (good notations) and [7].

1 A type system CIC0

We start with a subset of current Coq system with a simplified universe structure which we call CIC0. Namely, we will only allow for two universes **Prop** and **Type0** $= Type(0)$ with **Prop** $\subset$ **Type0** and **Prop** : **Type0** but without **Type0** : **Type0**. We only allow universal quantification over expressions which type to **Type0** and our **Prop** is impredicative i.e. the product of any family of members of **Prop** again types to **Prop**. We will also use slightly different syntax in our description. We will use $\prod$ and λ and most notably we will make the evaluation explicit and having three arguments the first of which is the domain of the function to be evaluated. This allows one to have unambiguous η-reduction.

For simplicity of notation we will use named free variables (instead of the free variables being always named by natural numbers as in Proposition 3.1). We will also use "vector notation" writing $\vec{x} : \vec{E}$ for $x_1 : E_1, \ldots, x_n : E_n$ etc. and write Γ to denote any valid context. To distinguish the names of variables and constructors (including constants) from the symbols which denote expressions we will use bold face for the former (except for $\prod$, λ etc. where no ambiguity is possible). Since writing out the three-argument evaluation expressions would make the text very hard to read we will often write $\mathbf{ev}(f, \vec{x})$ instead of $\mathbf{ev}(X_1, \ldots, \mathbf{ev}(X_i, \ldots, \mathbf{ev}(X_1, f, x_1), \ldots, x_i), \ldots, x_n)$ where X_i is the type of x_i.

Remark 1.1 [**2010.08.08.rem1**] Note that if one does not include the substitution rules of Proposition 3.1 explitly, the context and judgement formation rules should be such that the variables introduced by a context "above the line" can appear in the expressions "below the line" either among the context variables or among the bounded variables of the expressions. If a variable introduced in a context "above the line" appears in an expression below the line as a free variable such a rule will be unstable under the substitution of this variable by an expression.

The basic context and judgement formation rules in CIC0 are as follows.

Basic rules

$$\frac{\mathbf{x_1} : E_1, \ldots, \mathbf{x_n} : E_n \rhd}{\mathbf{x_1} : E_1, \ldots, \mathbf{x_n} : E_n \vdash \mathbf{x_i} : E_i} \quad i = 1, \ldots, n$$

The universe structure

$$\frac{\Gamma \rhd}{\Gamma, \mathbf{T} : \mathbf{Type0} \rhd} \qquad \frac{\Gamma \vdash T : \mathbf{Type0}}{\Gamma, \mathbf{t} : T \rhd} \qquad \frac{\Gamma \rhd}{\Gamma, \mathbf{P} : \mathbf{Prop} \rhd} \qquad \frac{\Gamma \vdash P : \mathbf{Prop}}{\Gamma, \mathbf{p} : P \vdash}$$

$$\frac{\Gamma \rhd}{\Gamma \vdash \mathbf{Prop} : \mathbf{Type0}} \qquad \frac{\Gamma \vdash P : \mathbf{Prop}}{\Gamma \vdash P : \mathbf{Type0}}$$

Underlying λ-calculus

$$\frac{\Gamma, \mathbf{t1} : T1, \mathbf{t2} : T2 \rhd}{\Gamma, \mathbf{t3} : \prod \mathbf{t1} : T1.T2 \rhd}$$

$$\frac{\Gamma \vdash T1 : \mathbf{Type0} \quad \Gamma, \mathbf{t1} : T1 \vdash T2 : \mathbf{Type0}}{\Gamma \vdash \prod \mathbf{t1} : T1.T2 : \mathbf{Type0}}$$

$$\frac{\Gamma, \mathbf{t1} : T1 \vdash T2 : \mathbf{Prop}}{\Gamma \vdash \prod \mathbf{t1} : T1.T2 : \mathbf{Prop}}$$

$$\frac{\Gamma, \mathbf{t1} : T1 \vdash t2 : T2}{\Gamma \vdash \lambda \, \mathbf{t1} : T1.t2 : \prod \mathbf{t1} : T1.T2} \qquad \frac{\Gamma \vdash f : \prod \mathbf{t1} : T1.T2 \quad \Gamma \vdash t : T1}{\Gamma \vdash \mathbf{ev}(T1, f, t) : T2[t/\mathbf{t1}]}$$

$$\beta - reduction: \ \mathbf{ev}(T1, \lambda \, \mathbf{t1} : T1.t2, t) \searrow t2[t/\mathbf{t1}] \qquad \eta - reduction: \ \lambda \, \mathbf{t1} : T1.\mathbf{ev}(T1, f, \mathbf{t1}) \searrow f$$

Note that the β- and η-reductions are defined on the level of the system of expressions.

Inductive types (Ia)

$$\frac{\Gamma \rhd}{\Gamma \vdash \emptyset : \mathbf{Prop}}$$

$$\frac{\Gamma, \mathbf{t} : T \rhd \quad \Gamma \vdash x : \emptyset}{\Gamma \vdash \epsilon(x, T) : T} \qquad (resp. \ \frac{\Gamma \vdash T : Type0 \quad \Gamma \vdash x : \emptyset}{\Gamma \vdash \epsilon(x, T) : T} \ )$$

$$\frac{\Gamma, \mathbf{t1} : T1 \rhd \quad \Gamma, \mathbf{t2} : T2 \rhd}{\Gamma, \mathbf{t} : \amalg(T1, T2) \rhd} \qquad (resp. \ \frac{\Gamma \vdash T1 : Type0 \quad \Gamma \vdash T2 : Type0}{\Gamma \vdash \amalg(T1, T2) : Type0} \ )$$

Inductive types (II)

What follows is just an explicit form of a general inductive construction of the CIC with additional restrictions on the "sizes" of the relevant type expressions and with the condition that in the constructors all the non-recursive components are grouped together and placed in front of the "recursive" ones. The input ("above the line") data for a general inductive definition in a context $\Gamma \rhd$ looks as follows:

1. integers: $na \geq 0$ (the number of "pseudo-parametrs") and $nc \geq 0$ (the number of constructors),

2. for each $i = 1, \ldots, nc$ and integer $nd^{(i)} \geq 0$,

3. a valid context $(\Gamma, \mathbf{a_1} : A_1, \ldots, \mathbf{a_{na}} : A_{na} \rhd)$

4. for each $i = 1, \ldots, nc$ a valid context of the form $(\Gamma, \vec{\mathbf{b}}^{(i)} : \vec{B}^i \rhd)$, such that each $B_j^{(i)}$ types to **Type0** in the context where it is defined,

5. for each $i = 1, \ldots, nc$ and each $j = 1, \ldots, na$ a valid judgement of the form

$$(\Gamma, \vec{\mathbf{b}}^{(i)} : B^{(i)} \vdash m_j^{(i)} : A_j[m_1^{(i)}/\mathbf{a}_1, \ldots, m_{j-1}^{(i)}/\mathbf{a}_{j-1}]),$$

6. for each $i = 1, \ldots, nc$ and each $j = 1, \ldots, nd^{(i)}$ a valid context of the form $(\Gamma, \vec{\mathbf{b}}^{(i)} : \vec{B}^{(i)}, \vec{\mathbf{d}}^{(i),j} : \vec{D}^{(i),j} \rhd)$, such that each expression $D_k^{(i),j}$ types to **Type0** in the context where it is defined,

7. for each $i = 1, \ldots, nc$, $j = 1, \ldots, nd^{(i)}$ and $k = 1, \ldots, na$ a valid judgement of the form

$$(\Gamma, \vec{\mathbf{b}}^{(i)} : B^{(i)}, \vec{\mathbf{d}}^{(i),j} : \vec{D}^{(i),j} \vdash q_k^{(i),j} : A_j[q_1^{(i),j}/\mathbf{a}_1, \ldots, q_{k-1}^{(i),j}/\mathbf{a}_{k-1}])$$

For simplicity of notation we will write simply A, B, D, q and m for $\vec{\mathbf{a}} : \vec{A}$, $(\vec{\mathbf{b}}^{(1)} : \vec{B}^{(1)}), \ldots, (\vec{\mathbf{b}}^{(nc)} : \vec{B}^{(nc)})$ etc. To further simplify the matter we will write BID ("Basic Input Data") for the sequence $(na; nc; nd^{(1)}, \ldots, nd^{(nc)}; A; B; D; q; m)$ which has to be included in the notation for every object generated by the inductive construction.

The output consists of the following objects:

1. A valid judgement of the form $(\Gamma \vdash \mathbf{IT}(BID) : \prod \vec{\mathbf{a}} : \vec{A}.\mathbf{Type0})$. Note that the expression $\mathbf{IT}(BID)$ bounds the variables $\mathbf{a}_1, \ldots, \mathbf{a}_{na}$, $\vec{\mathbf{b}}^{(i)}$ and $\vec{\mathbf{d}}^{(i),j}$.

2. For each $i = 1, \ldots, nc$ a valid judgement of the form

$$\Gamma \vdash \mathbf{c}(i, BID) : \prod \vec{\mathbf{b}}^{(i)} : \vec{B}^{(i)},$$
$$\prod \mathbf{z}_1^{(i)} : (\prod \vec{\mathbf{d}}^{(i),1} : \vec{D}^{(i),1}, \mathbf{ev}(\mathbf{IT}(BID), \vec{q}^{(i),1})),$$
$$\ldots,$$
$$\prod \mathbf{z}^{(i),nd^{(i)}} : (\prod \vec{\mathbf{d}}^{(i),nd^{(i)}} : \vec{D}^{(i),nd^{(i)}}, \mathbf{ev}(\mathbf{IT}(BID), \vec{q}^{(i),nd^{(i)}})),$$
$$\mathbf{ev}(\mathbf{IT}(BID), \vec{m}^{(i)})$$

where each $\mathbf{c}(i, BID)$ again bounds all of the variables $\mathbf{a}_1, \ldots, \mathbf{a}_{na}$, $\vec{\mathbf{b}}^{(i)}$ and $\vec{\mathbf{d}}^{(i),j}$.

3. A valid judgement of the form

$$\Gamma \vdash \mathbf{rect}(\vec{\mathbf{ar}}, \mathbf{r}, BID) : \mathbf{Trect}(\vec{\mathbf{ar}}, \mathbf{r}, BID)$$

where

$$Trect(\vec{\mathbf{ar}}, \mathbf{r}, BID) :=$$
$$= \prod P : \left(\prod \vec{\mathbf{ar}} : \vec{A}, \mathbf{ev}(\mathbf{IT}(BID), \vec{\mathbf{ar}}) \to Type0\right),$$

$$(RC^{(1)}->\ldots->RC^{(nc)}->\prod \mathbf{\vec{ar}}:\vec{A},\prod \mathbf{r}:\mathbf{ev}(\mathbf{IT}(BID),\mathbf{\vec{ar}}),\mathbf{ev}(\mathbf{ev}(P,\mathbf{\vec{ar}}),\mathbf{r}))$$

and

$$RC^{(i)}=\prod \vec{\mathbf{b}}^{(i)}:\vec{B}^{(i)},$$
$$\prod \mathbf{z}_1^{(i)}:(\prod \vec{\mathbf{d}}^{(i),1}:\vec{D}^{(i),1},\mathbf{ev}(\mathbf{IT}(BID),\vec{q}^{(i),1})),$$
$$\prod \mathbf{y}_1^{(i)}:(\prod \vec{\mathbf{d}}^{(i),1}:\vec{D}^{(i),1},\mathbf{ev}(\mathbf{ev}(P,\vec{q}^{(i),1}),\mathbf{ev}(\mathbf{z}_1^{(i)},\vec{\mathbf{d}}^{(i),1}))),$$
$$\ldots$$
$$\prod \mathbf{z}_{nd^{(i)}}^{(i)}:(\prod \vec{\mathbf{d}}^{(i),nd^{(i)}}:\vec{D}^{(i),nd^{(i)}},\mathbf{ev}(\mathbf{IT}(BID),\vec{q}^{(i),nd^{(i)}})),$$
$$\prod \mathbf{y}_{nd^{(i)}}^{(i)}:(\prod \vec{\mathbf{d}}^{(i),nd^{(i)}}:\vec{D}^{(i),nd^{(i)}},\mathbf{ev}(\mathbf{ev}(P,\vec{q}^{(i),nd^{(i)}}),\mathbf{ev}(\mathbf{z}_{nd^{(i)}}^{(i)},\vec{\mathbf{d}}^{(i),nd^{(i)}}))),$$
$$\mathbf{ev}(\mathbf{ev}(P,\vec{m}^{(i)}),\mathbf{ev}(\mathbf{ev}(\mathbf{c}(i,BID),\vec{b}^{(i)}),\vec{\mathbf{z}}^{(i)})).$$

4. The constructions **IT**, **c** and **rect** satisfy the following reduction rule which is called ι-reduction:

Note that with the previous definition (which we will call P-definition) one can not use inductive elimination to define functions from inductive types to "large types". E.g. given a type expression T there is no way to define a function $f:nat->Type0$ such that $f\,n=T^n$.

Alternatively we can define the Q-form of inductive constructions as follows. The input data is the same. The output data is of the form:

1. A valid judgement of the form $(\Gamma\vdash\mathbf{IT}(BID):\prod\vec{\mathbf{a}}:\vec{A}.\mathbf{Type0})$.
2. For each $i=1,\ldots,nc$ a valid judgement of the form

$$\Gamma\vdash\mathbf{c}(i,BID):\prod \vec{\mathbf{b}}^{(i)}:\vec{B}^{(i)},$$
$$\prod \mathbf{z}_1^{(i)}:(\prod \vec{\mathbf{d}}^{(i),1}:\vec{D}^{(i),1},\ \mathbf{ev}(\mathbf{IT}(BID),\vec{q}^{(i),1})),$$
$$\ldots,$$
$$\prod \mathbf{z}_{nd^{(i)}}^{(i)}:(\prod \vec{\mathbf{d}}^{(i),nd^{(i)}}:\vec{D}^{(i),nd^{(i)}},\ \mathbf{ev}(\mathbf{IT}(BID),\vec{q}^{(i),nd^{(i)}})),$$
$$\mathbf{ev}(\mathbf{IT}(BID),\vec{m}^{(i)})$$

3. For any valid context of the form

$$(\Gamma,\mathbf{\vec{ar}}:\vec{A},\mathbf{r}:\mathbf{ev}(\mathbf{IT}(BID),\mathbf{\vec{ar}}),\mathbf{x}:Q\rhd)$$

a valid judgement of the form

$$\Gamma\vdash\mathbf{rect}(\mathbf{\vec{ar}},\mathbf{r},Q,BID):\mathbf{TrectQ}(\mathbf{\vec{ar}},\mathbf{r},Q,BID)$$

where

$$\mathbf{TrectQ}(\mathbf{\vec{ar}},\mathbf{r},Q,BID):=$$
$$RCQ^{(1)}->\ldots->RCQ^{(nc)}->\prod \mathbf{\vec{ar}}:\vec{A},\prod \mathbf{r}:\mathbf{ev}(\mathbf{IT}(BID),\mathbf{\vec{ar}}),Q$$

and

$$RCQ^{(i)} = \prod \vec{\mathbf{b}}^{(i)} : \vec{B}^{(i)},$$

$$\prod \mathbf{z}_1^{(i)} : (\prod \vec{\mathbf{d}}^{(i),1} : \vec{D}^{(i),1}, \mathbf{ev}(\mathbf{IT}(BID), \vec{q}^{(i),1})),$$

$$\prod \mathbf{y}_1^{(i)} : (\prod \vec{\mathbf{d}}^{(i),1} : \vec{D}^{(i),1}, Q[\vec{q}^{(i),1}/\vec{\mathbf{ar}}, \mathbf{ev}(\mathbf{z}_1^{(i)}, \vec{\mathbf{d}}^{(i),1})/\mathbf{r}]),$$

$$\ldots$$

$$\prod \mathbf{z}_{nd^{(i)}}^{(i)} : (\prod \vec{\mathbf{d}}^{(i),nd^{(i)}} : \vec{D}^{(i),nd^{(i)}}, \mathbf{ev}(\mathbf{IT}(BID), \vec{q}^{(i),nd^{(i)}})),$$

$$\prod \mathbf{y}_{nd^{(i)}}^{(i)} : (\prod \vec{\mathbf{d}}^{(i),nd^{(i)}} : \vec{D}^{(i),nd^{(i)}}, Q[\vec{q}^{(i),nd^{(i)}}/\vec{\mathbf{ar}}, \mathbf{ev}(\mathbf{z}_{nd^{(i)}}^{(i)}, \vec{\mathbf{d}}^{(i),nd^{(i)}})/\mathbf{r}]),$$

$$Q[\vec{m}^{(i)}/\vec{\mathbf{ar}}, \mathbf{ev}(\mathbf{ev}(\mathbf{c}(i, BID), \vec{b}^{(i)}), \vec{\mathbf{z}}^{(i)})/\mathbf{r}].$$

4. The constructions **IT**, **c** and **rect** satisfy the following ι-reduction rule:

Finally, there is the following R-form of inductive constructions which is stronger than the previous two. The input data is the same but without the restriction on B and D to be small. The output data is of the form:

1. A valid context of the form $(\Gamma, \vec{\mathbf{ar}} : \vec{A}, \mathbf{r} : \mathbf{ITR}(\vec{\mathbf{ar}}, BID)\triangleright)$,
2. For each $i = 1, \ldots, nc$ a valid judgement of the form

$$\Gamma \vdash \mathbf{c}(i, BID) : \prod \vec{\mathbf{b}}^{(i)} : \vec{B}^{(i)},$$

$$\prod \mathbf{z}_1^{(i)} : (\prod \vec{\mathbf{d}}^{(i),1} : \vec{D}^{(i),1}, \mathbf{ITR}(BID)[\vec{q}^{(i),1}/\vec{\mathbf{ar}}]),$$

$$\ldots,$$

$$\prod \mathbf{z}_{nd^{(i)}}^{(i)} : (\prod \vec{\mathbf{d}}^{(i),nd^{(i)}} : \vec{D}^{(i),nd^{(i)}}, \mathbf{ITR}(BID)[\vec{q}^{(i),nd^{(i)}}/\vec{\mathbf{ar}}]),$$

$$\mathbf{ITR}(BID)[\vec{m}^{(i)}/\vec{\mathbf{ar}}]$$

3. For any valid context of the form

$$(\Gamma, \vec{\mathbf{ar}} : \vec{A}, \mathbf{r} : \mathbf{ITR}(BID), \mathbf{x} : Q\triangleright)$$

   a valid judgement of the form

$$\Gamma \vdash \mathbf{rect}(\vec{\mathbf{ar}}, \mathbf{r}, Q, BID) : \mathbf{TrectR}(\vec{\mathbf{ar}}, \mathbf{r}, Q, BID)$$

   where

$$TrectR(\vec{\mathbf{ar}}, \mathbf{r}, Q, BID) :=$$

$$RCQ^{(1)} -> \ldots -> RCQ^{(nc)} -> \prod \vec{\mathbf{ar}} : \vec{A}, \prod \mathbf{r} : \mathbf{ITR}(BID), Q$$

   and

$$RCQ^{(i)} = \prod \vec{\mathbf{b}}^{(i)} : \vec{B}^{(i)},$$

$$\prod \mathbf{z}_1^{(i)} : (\prod \vec{\mathbf{d}}^{(i),1} : \vec{D}^{(i),1}, \mathbf{ITR}(BID)[\vec{q}^{(i),1}/\vec{\mathbf{ar}}]),$$

$$\prod \mathbf{y}_1^{(i)} : (\prod \vec{\mathbf{d}}^{(i),1} : \vec{D}^{(i),1}, Q[\vec{q}^{(i),1}/\vec{\mathbf{ar}}, \mathbf{ev}(\mathbf{z}_1^{(i)}, \vec{\mathbf{d}}^{(i),1})/\mathbf{r}]),$$

$$\dots$$

$$\prod \mathbf{z}^{(i)}_{nd^{(i)}} : (\prod \vec{\mathbf{d}}^{(i),nd^{(i)}} : \vec{D}^{(i),nd^{(i)}}, \mathbf{ITR}(BID)[\vec{q}^{(i),nd^{(i)}}/\vec{\mathbf{ar}}]),$$

$$\prod \mathbf{y}^{(i)}_{nd^{(i)}} : (\prod \vec{\mathbf{d}}^{(i),nd^{(i)}} : \vec{D}^{(i),nd^{(i)}}, Q[\vec{q}^{(i),nd^{(i)}}/\vec{\mathbf{ar}}, \mathbf{ev}(\mathbf{z}^{(i)}_{nd^{(i)}}, \vec{\mathbf{d}}^{(i),nd^{(i)}})/\mathbf{r}]),$$

$$Q[\vec{m}^{(i)}/\vec{\mathbf{ar}}, \mathbf{ev}(\mathbf{ev}(\mathbf{c}(i, BID), \vec{b}^{(i)}), \vec{\mathbf{z}}^{(i)})/\mathbf{r}].$$

4. The constructions **ITR**, **c** and **rect** satisfy the following ι-reduction rule:

In the case of R-form we also have an extra rule which says that if B and D are small (type to $Type0$) then $(\Gamma, \vec{\mathbf{ar}} : \vec{A} \vdash \mathbf{ITR}(\vec{\mathbf{ar}}, BID) : Type0)$ is a valid judgement.

The Coq syntax for an inductive definition with these input data and values in a sort s (where $s = Prop, Set$ or $Type$) would look as follows:

Inductive $X : forall\ \vec{\mathbf{a}} : \vec{A}, s := c^{(1)} : C^{(1)} \mid \dots \mid c^{(nc)} : C^{(nc)}$.

Where

$$C^{(i)} := forall\ \vec{\mathbf{b}}^{(i)} : \vec{B}^{(i)},$$

$$forall\ \mathbf{z}^{(i)}_1 : (forall\ \vec{\mathbf{d}}^{(i)}_1 : \vec{D}^{(i)}_1,\ X\ \vec{q}^{(i)}_1), \dots, forall\ \mathbf{z}^{(i)}_{nd^{(i)}} : (forall\ \vec{\mathbf{d}}^{(i)}_{nd^{(i)}} : \vec{D}^{(i)}_{nd^{(i)}},\ X\ \vec{q}^{(i)}_{nd^{(i)}}),$$

$$X\ \vec{m}^{(i)}.$$

We will use below the notation $C^{(i)}$ for the direct analog of this expression in $CIC0$ as well. To write this analog explicitly one has to replace all "$forall\ w : W$," with "$\prod w : W$." and write our three-arguments $\mathbf{ev}(\dots,\dots,\dots)$ wherever applications occur. For example, $X\,\vec{m}^{(i)}$ will look as follows:

$$\mathbf{ev}(X, \vec{m}^{(i)}) = \mathbf{ev}(A_{na}[m^{(i)}_1/\mathbf{a}_1, \dots, m^{(i)}_{na-1}/\mathbf{a}_{na-1}], \dots, \mathbf{ev}(A_j[m^{(i)}_1/\mathbf{a}_1, \dots, m^{(i)}_{j-1}/\mathbf{a}_{j-1}], \dots$$

$$\dots, \mathbf{ev}(A_1, X, m^{(i)}_1), \dots, m^{(i)}_j), \dots m^{(i)}_{na})$$

Example 1.2 The following list gives the form of our BID for some of the inductive constructions which are often used in Coq.

1. To define natural numbers nat one takes $na = 0$, $nc = 2$, $nd^{(1)} = 0$, $nd^{(2)} = 1$, $\vec{B}^{(1)} = ()$ (empty sequence), $\vec{B}^{(2)} = ()$, $\vec{D}^{(2)}_1 = ()$.

2. To define binary trees one takes $na = 0$, $nc = 2$, $nd^{(1)} = 0$, $nd^{(2)} = 2$, $\vec{B}^{(1)} = ()$, $\vec{B}^{(2)} = ()$, $\vec{D}^{(2)}_1 = ()$, $\vec{D}^{(2)}_2 = ()$.

3. Given $(\Gamma \vdash T : \mathbf{Type0})$ one defines the equality types for T using the input data of the form $na = 2$, $nc = 1$, $nd^{(1)} = 0$, $\vec{A} = (\mathbf{t1} : T, \mathbf{t2} : T)$, $\vec{B}^{(1)} = (\mathbf{t} : T)$, $\vec{m}^{(1)} = (\mathbf{t}, \mathbf{t})$.

4. Given $(\Gamma \vdash T1 : \mathbf{Type0})$ and $(\Gamma, \mathbf{t1} : T1 \vdash T2 : \mathbf{Type0})$ one defines the dependent sum (in the standard notation $\sum \mathbf{t1} : T1.T2$) using the input data $na = 0$, $nc = 1$, $nd^{(1)} = 0$, $\vec{B}^{(1)} = (\mathbf{t1} : T1, \mathbf{t2} : T2)$.

5. empty, unit, unions.

2 Representing inductive definitions in Coq as combinations of elementary ones

In Coq an inductive definition with parameters is just a combination of an inductive definition without parameters in a wider context with the dependent product and abstraction. For a detailed translation see Appendix A. In what follows we will discuss only inductive definitions of the basic Calculus of Inductive Constructions as described in [10], [7] and [2, pp.77-80].

Recall, that in the notation of CIC one writes $(a : A)$ for what in Coq is $forall\, a : A$, and in the standard type-theoretic notation $\prod a : A$. and $[a : A]$ for what in Coq is $fun\, a : A =>$ and in the standard type-theoretic notation $\lambda\, a : A.$.

A general expression for an inductive type in a context Γ is of the form $I(X : \vec{A}\, s)\{C^{(1)}, \ldots, C^{(nc)}\}$ where $\vec{A}\, s$ is an "arity" i.e. a valid type expression in Γ of the form

$$(\vec{a} : \vec{A})s = (a_1 : A_1) \ldots (a_{na} : A_{na})s$$

where s is a sort and each $C^{(k)}$ is a type expression defined in the context $\Gamma, X : \vec{A}\, s$ which has the form of a "type of constructor expression" (cf. [2, p.77]).

In Coq inductive type of the form $I(X : \vec{A}\, s)\{C^{(1)}, \ldots, C^{(nc)}\}$ is introduced using the following syntax:

Inductive X: forall a_1:A_1, forall a_2:A_2, ... ,forall a_na:A_na, s := cc_1 : C_1 | ... | cc_nc : C_nc .

Lemma 2.1 *[**2010.1.19.l2**] Any type of constructor expression C in variable X of type $\vec{A}\, s$ can be written in a unique way in the form:*

$$C = (z_{1,1} : (\vec{d} : \vec{D}_{1,1})\, X\, \vec{q}_{1,1}) \ldots (z_{1,nd_1} : (\vec{d} : \vec{D}_{1,nd_1})\, X\, \vec{q}_{1,nd_1})(b_1 : B_1)$$

$$(z_{2,1} : (\vec{d} : \vec{D}_{2,1})\, X\, \vec{q}_{2,1}) \ldots (z_{2,nd_2} : (\vec{d} : \vec{D}_{2,nd_2})\, X\, \vec{q}_{2,nd_2})(b_2 : B_2)$$

$$\ldots\ldots\ldots$$

$$(b_{nb} : B_{nb})(z_{nb,1} : (\vec{d} : \vec{D}_{nb,1})\, X\, \vec{q}_{nb,1}) \ldots (z_{nb,nd_{nb}} : (\vec{d} : \vec{D}_{nb,nd_{nb}})\, X\, \vec{q}_{nb,nd_{nb}})\, X\, \vec{m}$$

where B_i, $\vec{D}_{i,j}$, $\vec{q}_{i,j}$ and $\vec{m}$ do not depend on X i.e. can be defined in a context which does not contain X.

Since $z_{k,l}$ can only appear in a context which contains X we get the following observation:

Lemma 2.2 *[**2010.1.14.l1**] In a type of constructor expression of the form given above $\vec{D}_{i,j}$, $\vec{q}_{i,j}$, $\vec{m}$ and B_i do not depend on $z_{k,l}$.*

The dependencies between different sub-terms of C can be visualized by the following diagram:

[2010.1.14.eq1]

$$\begin{array}{ccccc} & (\Gamma, B_1, \ldots, B_i, \vec{D}_{i,1}), \ldots, (\Gamma, B_1, \ldots, B_i, \vec{D}_{i,nd_i}) & & & (\Gamma, \vec{D}_{1,1}), \ldots, (\Gamma, \vec{D}_{1,nd_1}) \\ & \downarrow & & & \downarrow \\ \ldots \longrightarrow & (\Gamma, B_1, \ldots, B_i) & \longrightarrow \ldots \longrightarrow & & (\Gamma) \end{array} \quad (32)$$

The sequences $\vec{q}_{i,j}$ correspond to morphisms $(\Gamma, B_1, \ldots, B_i, \vec{D}_{i,j}) \to (\Gamma, \vec{A})$ over Γ and the sequence $\vec{m}$ to a morphism $(\Gamma, \vec{B}) \to (\Gamma, \vec{a} : \vec{A})$ over Γ.

Due to the structure of the dependencies diagram (32), the expression

$$[\mathbf{2010.1.21.eq2}]C' = (\vec{b} : \vec{B})(z_{1,1} : (\vec{d} : \vec{D}_{1,1})\, X\, \vec{q}_{1,1}) \ldots (z_{nb,nd_{nb}} : (\vec{d} : \vec{D}_{nb,nd_{nb}})\, X\, \vec{q}_{nb,nd_{nb}})\, X\, \vec{m} \tag{33}$$

is well formed in $(\Gamma, X : \vec{A}\, s)$ and there is a canonical bijection between the terms of type C and terms of type C'.

The inductive machinery of Coq defines for any collection of contexts of the form

$$[\mathbf{2010.1.21.eq1}](\Gamma, \vec{a} : \vec{A}, s), \qquad (\Gamma, X : \vec{A}\, s, C^{(1)} : s''), \ldots (\Gamma, X : \vec{A}\, s, C^{(nc)} : s'') \tag{34}$$

where $C^{(k)}$ are of the form described above, a term $I = I(X : \vec{A}\, s)\{C^{(1)}, \ldots, C^{(nc)}\}$ of type $(\vec{a} : \vec{A})s$ in Γ, a sequence of terms $c^{(k)}$ of types $C^{(k)}[I/X]$ for $k = 1, \ldots, nc$ in Γ and , for each "admissible" sort s' a "recursor" term I_{rec} (again in Γ). The type of this term is

$$[\mathbf{2010.1.17.eq1}]I_{rec} : (Q : (\vec{a} : \vec{A})(r : I\, \vec{a})s')(f_1 : \Delta\{I/X, C^{(1)}, Q, c^{(1)}/c\}) \ldots$$
$$\ldots (f_{nc} : \Delta\{I/X, C^{(nc)}, Q, c^{(nc)}/c\})(\vec{a} : \vec{A})(r : I\, \vec{a})Q\, \vec{a}\, r. \tag{35}$$

where for C of the form given above and defined for a variable X, $\Delta\{X, C, Q, c\}$ is the type expression in the context $(\Gamma, X : (\vec{a} : \vec{A})s, Q : (\vec{a} : \vec{A})(r : X\, \vec{a})s', c : C)$ of the form:

$$(z_{1,1} : (\vec{d} : \vec{D}_{1,1})X\, \vec{q}_{1,1})(y_{1,1} : (\vec{d} : \vec{D}_{1,1})Q\, \vec{q}_{1,1}(z_{1,1}\, \vec{d})) \ldots (z_{1,nd_1} : (\vec{d} : \vec{D}_{1,nd_1})X\ \vec{q}_{1,nd_1})$$
$$(y_{1,nd_1} : (\vec{d} : \vec{D}_{1,nd_1})Q\, \vec{q}_{1,nd_1}(z_{1,nd_1}\, \vec{d}))(b_1 : B_1)$$
$$(z_{2,1} : (\vec{d} : \vec{D}_{2,1})X\ \vec{q}_{2,1})(y_{2,1} : (\vec{d} : \vec{D}_{2,1})Q\, \vec{q}_{2,1}(z_{2,1}\, \vec{d})) \ldots (z_{2,nd_2} : (\vec{d} : \vec{D}_{2,nd_2})X\ \vec{q}_{2,nd_2})$$
$$(y_{2,nd_2} : (\vec{d} : \vec{D}_{2,nd_2})Q\, \vec{q}_{2,nd_2}(z_{2,nd_2}\, \vec{d}))(b_2 : B_2)$$
$$\ldots$$
$$(b_{nb} : B_{nb})(z_{nb,1} : (\vec{d} : \vec{D}_{nb,1})X\ \vec{q}_{nb,1})(y_{nb,1} : (\vec{d} : \vec{D}_{nb,1})Q\, \vec{q}_{nb,1}(z_{nb,1}\, \vec{d})) \ldots$$
$$(z_{nb,nd_{nb}} : (\vec{d} : \vec{D}_{nb,nd_{nb}})X\ \vec{q}_{nb,nd_{nb}})(y_{nb,nd_{nb}} : (\vec{d} : \vec{D}_{nb,nd_{nb}})Q\, \vec{q}_{nb,nd_{nb}}(z_{nb,nd_{nb}}\, \vec{d}))$$
$$Q\, \vec{m}\, (c\, z_{1,1}\, \ldots\, z_{1,nd_1}\, b_1\, z_{2,1}\, \ldots\, z_{2,nd_2}\, b_2\, \ldots\, b_{nb}\, z_{nb,1}\, \ldots\, z_{nb,nd_{nd}})$$

([2, p.78] uses the notation $\Delta\{I, X, C_k, Q, c\}$ for $\Delta\{I/X, C^{(k)}, Q, c\}$). Because of the structure of the dependency diagram (32) the expression

$$\Delta'\{X, C, Q, c\} = (\vec{b} : \vec{B})(z_{1,1} : (\vec{d} : D_{1,1})\, X\, \vec{q}_{1,1})(y_{1,1} : (\vec{d} : \vec{D}_{1,1})Q\, \vec{q}_{1,1}(z_{1,1}\, \vec{d})) \ldots$$
$$\ldots (z_{nb,nd_{nb}} : (\vec{d} : D_{nb,nd_{nb}})\, X\, \vec{q}_{nb,nd_{nb}})(y_{nb,nd_{nb}} : (\vec{d} : \vec{D}_{nb,nd_{nb}})Q\, \vec{q}_{nb,nd_{nb}}(z_{nb,nd_{nb}}\, \vec{d}))$$
$$(Q\, \vec{m}\, (c'\, \vec{b}\, z_{1,1}\, \ldots\, z_{nb,nd_{nb}}))$$

where c' is the term of C' corresponding to c, is a well defined type expression and its terms are in a canonical bijection with the terms of $\Delta\{X, C, Q, c\}$. For a term f of $\Delta\{X, C, Q, c\}$ we will denote by f' the corresponding term of $\Delta'\{X, C, Q, c\}$ (note that our $\Delta'\{\ldots\}$ is not to be confused with $\Delta'[\ldots]$ used in [2, p.79]). We will also write I'_{rec} for the term of the type

$$(Q : (\vec{a} : \vec{A})(r : I\, \vec{a})s')(f'_1 : \Delta'\{I/X, C^{(1)}, Q, c^{(1)}/c\}) \ldots$$
$$\ldots (f'_{nc} : \Delta'\{I/X, C^{(nc)}, Q, c^{(nc)}/c\})(\vec{a} : \vec{A})(r : I\, \vec{a})Q\, \vec{a}\, r.$$

corresponding to I_{rec}.

The data produced by an inductive definition satisfies the ι-reduction(s). For an inductive definition of the form given above and expressed in the form of (definitional) equalities in the context

$$(\Gamma, Q : (\vec{a} : \vec{A})(r : I\,\vec{a})s',$$

$$f'_1 : \Delta'\{I/X, C^{(1)}, Q, c^{(1)}/c\}, \ldots, f'_{nc} : \Delta'\{I/X, C^{(nc)}, Q, c^{(nc)}/c\},$$
$$\vec{b} : \vec{B}, z_{1,1} : (\vec{d} : \vec{D}_{1,1})\, I\, \vec{q}_{1,1}, \ldots, z_{nb,nd_{nb}} : (\vec{d} : \vec{D}_{nb,nd_{nb}})\, I\ \vec{q}_{nb,nd_{nb}})$$

(i.e. using the ordering of variables b_i and $z_{i,j}$ corresponding to C' and Δ') they take the following form (for $i = 1, \ldots, nc$):

$$[\mathbf{2010.1.19.eq1}] I'_{rec}\ Q\ \vec{f'}\ \vec{m}\ (c'^{(i)}\ \ \vec{b}\ \vec{z}) =$$
$$= f'_i\, \vec{b}\, z_{0,1}\, ([\vec{d} : \vec{D}_{0,1}]\, (I'_{rec}\ \ Q\, \vec{f'}\, \vec{q}_{0,1}\, (z_{0,1}\vec{d})))\ldots$$
$$\ldots z_{nb,nd_{nb}}\, ([\vec{d} : \vec{D}_{nb,nd_{nb}}]\, (I'_{rec}\ Q\, \vec{f'}\, \vec{q}_{nb,nd_{nb}}\, (z_{nb,nd_{nb}}\, \vec{d}))). \tag{36}$$

Let us consider inductive constructions in Coq of the following particular forms (following the syntax of Coq we write Type for any sort, note that several occurrences of Type in the same expression may actually refer to different sorts):

Inductive unit : Type := tt:unit.

Inductive Sum (T:Type) (Pf:T-> Type) : Type := pair: (forall t:T, forall x: Pf t, Sum T Pf).

Sum_rect: forall (T : Type) (Pf : T -> Type) (P : Sum T Pf -> Type), (forall (t : T) (x : Pf t), P (pair T Pf t x)) -> forall s : Sum T Pf, P s

Inductive emptytype : Type := .

Inductive Union (T1:Type) (T2:Type) : Type := ii1: T1 -> Union T1 T2 | ii2: T2 -> Union T1 T2.

Union_rect: forall (T1 T2 : Type) (P : Union T1 T2 -> Type), (forall t : T1, P (i1 T1 T2 t)) -> (forall t : T2, P (i2 T1 T2 t)) -> forall u : Union T1 T2, P u

Inductive Eq (T:Type): T -> T-> Type := ideq: forall t:T, eq T t t .

Eq_rect: forall (T : Type) (P : forall t t0 : T, Eq T t t0 -> Type), (forall t : T, P t t (ideq T t)) -> forall (y y0 : T) (m : Eq T y y0), P y y0 m

Inductive IC(A:Type)(B:Type)(Df:B->Type)(q:forall b:B, forall d: Df b, A)(m:forall b:B, A) : A -> Type :=
cic: forall b:B, ((forall d: Df b, IC A B Df q m (q b d)) -> IC A B Df q m (m b)).

IC_rect: forall (A B : Type) (Df : B -> Type) (q : forall b : B, Df b -> A) (m : B -> A) (P : forall a : A, IC A B Df q m a -> Type), (forall (b : B) (i : forall d : Df b, IC A B Df q m (q b d)), (forall d : Df b, P (q b d) (i d)) -> P (m b) (cic A B Df q m b i)) -> forall (y : A) (i : IC A B Df q m y), P y i

Inductive IP0(B:Type)(Df:B->Type) : Type :=
cip0: forall b:B, ((forall d: Df b, IP0 B Df) -> IP0 B Df).

Inductive IP(A:Type)(Bf:forall a:A,Type)(Df:forall a:A, forall b: Bf a, Type)(q: forall a:A, forall b: Bf a, forall d: Df a b, A) (a:A): Type :=
cip: forall b: Bf a, forall f: (forall d: Df a b, (IP A Bf Df q (q a b d))), (IP A Bf Df q a).

IP_rect: forall (A : Type) (Bf : A -> Type) (Df : forall a : A, Bf a -> Type) (q : forall (a : A) (b : Bf a), Df a b -> A) (P : forall a : A, IP A Bf Df q a -> Type), (forall (a : A) (b : Bf a) (f : forall d : Df a b, IP A Bf Df q (q a b d)), (forall d : Df a b, P (q a b d) (f d)) -> P a (cip A Bf Df q a b f)) -> forall (a : A) (i : IP A Bf Df q a), P a i

Let us show how to construct an interpretation of Coq in itself which transforms any inductive definition into a sequence of definitions of forms unit, Sum, emptytype, Union, Eq and IP. Definitions of the form IC will be used for an intermediate step in the construction.

Given any context of the form $\Gamma, \vec{x} : \vec{T}$ where $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{T} = T_1, \ldots, T_n$ we can form a new context $\Gamma, x : \Sigma(\vec{x} : \vec{T})$ where $\Sigma(\vec{x} : \vec{T})$ is defined by repeated application of the construction *Sum* such that for $n = 1$ we have $\Sigma(x : T) = T$ and for $n > 1$,

$$\Sigma(\vec{x} : \vec{T}) = Sum\, T_1\, (fun\, x_1 : T_1 => \Sigma((x_2, \ldots, x_n) : (T_2, \ldots, T_n)))$$

For $n = 0$ we set $\Sigma(\vec{x} : \vec{T}) = unit$.

Similarly given a sequence of contexts of the form $\Gamma, x_i : T_i$ where $i = 1, \ldots, n$ we can define a context $\Gamma, x : \amalg_i T_i$ where for $n = 1$ we have $\amalg T = T$ and for $n > 1$,

$$\amalg_i T_i = Union\, T_1\, (\amalg_i T_{i+1})$$

For $n = 0$ we set $\amalg_i T_i = emptytype$.

Given any arity $\vec{A}\, s$ of the form $(\vec{a} : \vec{A})s$ define A as $\Sigma\, (\vec{a} : \vec{A})$ and $A\, s$ as $(a : A)s$. Given any C of the form given above let us define the following:

$$B = \Sigma(\vec{b} : \vec{B})$$

(in the context Γ),

$$D'_{i,j} = \Sigma(\vec{d} : \vec{D}'_{i,j})$$

in the context $\Gamma, b : B$ where $\vec{D}'_{i,j}$ are obtained from $\vec{D}_{i,j}$ by replacing $b_1 : B_1, \ldots, b_i : B_i$ with the corresponding projections of $b : B$ and

$$D' = \amalg_{i,j} D'_{i,j}$$

also in the context $\Gamma, b : B$. The sequences $\vec{q}_{i,j}$ define a function $qf : (\Gamma, b : B, d' : D') \to (\Gamma, a : A)$ over Γ and the sequence $\vec{m}$ defines a function $mf : (\Gamma, b : B) \to (\Gamma, a : A)$ over Γ.

Suppose now that we have a sequence of type of constructor expressions $C^{(1)}, \ldots, C^{(nc)}$ in a variable X of arity $\vec{A}\, s$. Denote by $B^{(k)}, D'^{(k)}, qf^{(k)}, mf^{(k)}$ where $k = 1, \ldots, nc$ the objects defined above which correspond to the expression $C^{(k)}$. Let us do the groupings again. Set $BB = \amalg_k B^{(k)}$. Then there are functions of the form $Df^{(k)} : BB \to Type$ such that for $b : B_j$, $j \neq k$ one has $Df^{(k)} i_j b = \emptyset$ and for $b : B_k$ one has $Df^{(k)} i_k b = D'^{(k)}$. Set $DD = \amalg_k Df^{(k)} b$ such that we have a valid context $(\Gamma, bb : BB, dd : DD)$. The morphisms $qf^{(k)}$ and $mf^{(k)}$ define now morphisms $qf : (\Gamma, bb : BB, dd : DD) \to (\Gamma, a : A)$ and $mf : (\Gamma, bb : BB) \to (\Gamma, a : A)$ over Γ which we represent by terms $\Gamma \vdash q : (bb : BB)(dd : DD)A$ and $\Gamma \vdash m : BB \to A$.

This construction provides for any inductive definition $I(X : \vec{A}\, s)\{\vec{C}\}$ of the form permitted in Coq a set-up consisting of valid contexts and sequents of the form:

$$[\mathbf{2010.1.18.eq2}](\Gamma, a : A), \qquad (\Gamma, bb : BB, dd : DD)$$
$$(\Gamma \vdash q : (bb : BB)(dd : DD)A), \qquad (\Gamma \vdash m : BB \to A) \tag{37}$$

where BB, DD, q and m are expressions which use only the dependent sum and disjoint union constructions and the original expressions $\vec{A}$ and $\vec{C}$. If $q' : \sum((bb, dd) : (BB, DD)) -> A$ is adjoint to q then our data can be shows in the form of a diagram:

$$\begin{array}{ccc} \sum((bb,dd):(BB,DD)) & \xrightarrow{q'} & A \\ {\scriptstyle pr_1}\downarrow & & \\ BB & & \\ {\scriptstyle m}\downarrow & & \\ A & & \end{array}$$

When $na = 0$ i.e. $A = unit$ we can ignore q and m and we obtain the diagram

$$\sum((bb, dd) : (BB, DD)) \stackrel{pr1}{\to} BB$$

Proposition 2.3 *[**2010.1.18.prop1**] Let $I(X : \vec{A}\, s)\{\vec{C}\}$ be a valid inductive definition in Coq in a context Γ.*

If $na > 0$, consider the type $I = IC\, A\, BB\, (fun\, bb : BB => DD)\, q\, m$ where A, BB, DD, q and m are defined based on $\vec{A}\, s$ and $\vec{C}$ as explained above. Then there is a term expression based on $IC_{rect}\, A\, BB\, (fun\, bb : BB => DD)\, q\, m$ of type (35) in Γ and this expression satisfies the same reduction rules as (35).

If $na = 0$, consider the type $I = IP0\, BB\, (fun\, bb : BB => DD)$. Then there is a term expression based on $IP0_{rect}\, BB\, (fun\, bb : BB => DD)$ of type (35) in Γ and this expression satisfies the same reduction rules as (35).

Example 2.4 *[**2010.8.4.ex1**]*Let us consider the construction described above in the case of the standard definition of natural numbers:

Inductive nat : Type := O : nat | S : nat -> nat.

We have $s = Type$, $A = unit$, $nc = 2$,

$$C^{(1)} = nat \quad B^{(1)} = unit \quad D^{(1)} = emptytype$$

$$C^{(2)} = (z^{(2)}_{0,1} : nat)\, nat \quad B^{(2)} = unit \quad D^{(2)} = unit$$

When we group $C^{(1)}$ and $C^{(2)}$ together we get:

$$BB = Union\, unit\, unit$$

The explicit forms for $Df1 = Df^{(1)}$ and $Df2 = Df^{(2)}$ in Coq are:

```
Fixpoint Df1 (bb:BB) Type :=
match bb:BB with
ii1 tt1 => emptytype |
ii2 tt2 => emptytype
end.
```

```
Fixpoint Df2 (bb:BB) Type :=
match bb:BB with
ii1 tt1 => unit |
ii2 tt2 => emptytype
end.
```

and DD (in the context (Γ, bb:B)) is of the form Union (Df1 bb) (Df2 bb). Therefore, our construction would replace the usual definition of nat by

Definition nat':= IP0 BB (fun bb:BB => Union (Df1 bb) (Df2 bb)).

which is equivalent to the one of the form

nat' : Type := (Union nat' unit) -> nat'.

Proposition 2.3 can be informally summarized by saying that the procedure described above allows one to express any inductive definition of Coq as a combination of a number of dependent sums, dependent products and a single inductive definition of the form IC.

We will show now how to transform, using eq, a definition of the form IC into an equivalent (in the sense clarified by Proposition 2.5 below) inductive definition of the form IP.

Consider a set-up of the form (37). In the context $\Gamma, a : A$ define $B' = \Sigma\, b : B, eq\, A\, a\, (m\, b)$. Consider the term sequent

$$(\Gamma, a : A, b' : B', d' : D[(pr1\, b')/b] \vdash q\, (pr1\, b')\, d' : A)$$

This sequent is of the form which can serve as an input for the construction of IP. Set

$$I = IP\ A\ Bf'\ Df'\ q'$$

where $Bf' = (fun\, a => B')$, $Df' = (fun\, a => (fun\, b' => D[(pr1\, b')/b]))$ and $q' = (fun\, a => (fun\, b' => (fun\, d' => a\, (pr1\, b')\, d')))$.

Proposition 2.5 *[**2010.1.25.prop1**] In the notations introduced above, there exists a term expression in Γ based on $IP_{rect}\ A\ Bf'\ Df'\ q'$ which has the same type as $IC_{rect}\ A\ B\ (fun\, b : B => D)\ q\ m$ and satisfies the same ι-reduction property.*

3 Interpretations of inductive definitions in lccc's

To get started let is consider an interpretation [-] of the calculus of construction (with some universes) in a lccc $\mathcal{C}$ which is compatible with dependent products.

We start by interpreting dependent sums (which can be seen in Coq as inductive definitions with one strictly positive constructor) in the usual way. After they are interpreted we can always replace an expression of the form $(x_1 : T_1)(x_2 : T_2)....(x_n : T_n)T_{n+1}$ by $(x : (\sum x_1 :$

$T_1, \ldots, \sum x_{n-1} T_{n-1}, T_n))\, T_{n+1}$ i.e. to replace all sequences of dependent products by a single dependent product parametrized by a dependent sum.

We also interpret inductive definitions with several constructors of the form $T_i \to I$ where T_i's do not depend on I as disjoint unions.

Using this reduction and also identifications of *Prop* and *Set* with subtypes of *Type* we may assume that a general inductive definition is of the form $X = Ind(X : A \to Type)\{C_1, \ldots, C_n\}$ where $C_i = (z_i : B_i)\, X\, m_i$, X is not present in m_i and B_i is a dependent sum of the form

$$[\mathbf{2009.1.8.eq1}] B_i = \sum z_{i,0} : B_{i,0}, \sum z_{i,1} : B_{i,1}, \sum \ldots, B_{i,n_i} \tag{38}$$

where $B_{i,j}$ does not depend on X or is of the form $(y_{i,j} : D_{i,j})\, X\, q_{i,j}$ where $D_{i,j}$ and $q_{i,j}$ do not depend on X.

Presumably, these conditions imply in particular that if $B_{i,j} = (y_{i,j} : D_{i,j})\, X\, q_{i,j}$ and $k > j$ then $B_{i,k}$ does not depend on $z_{i,j}$. Therefore, we can collect all the terms $B_{i,j}$ which do not depend on X into $B_{i,0}$ and assume that for $j > 0$ the term $B_{i,j}$ is of the form $(y_{i,j} : D_{i,j})\, X\, q_{i,j}$. Note that these terms do not depend on each other for various $j_1, j_2 > 0$.

Suppose we have already interpreted everything "before" this definition. By passing to the slice category we may assume that we work in an empty context. Then $[B_{i,0}]$ is an object of $\mathcal{C}$ and each of the $[D_{i,j}]$ is a family of objects over $[B_{i,0}]$. Let $E_{i,j} = \sum z_{i,0} : B_{i,0}, D_{i,j}$. Then $[p_{i,j}] : [E_{i,j}] \to [B_{i,0}]$ is the map whose fibers give this family and $[q_{i,j}]$ can be seen as a morphism $[q_{i,j}] : [E_{i,j}] \to A$.

Let $Tot(X) = \sum a : A, Xa$. Then $[Tot(X)]$ is an object over $[A]$ and $=$ the object $[B_i]$ as an object over $[B_{i,0}]$ is $\prod_j [p_{i,j}]_* [q_{i,j}]^* ([Tot(X)]/[A])$. By taking disjoint union of the types $E_{i,j}$ for $j > 0$ we may collect them into one type E_i and the maps $p_{i,j}$ and $q_{i,j}$ into two maps $p_i : E_i \to B_{i,0}$, $q_i : E_i \to A$.

Since m_i does not depend on X it means in particular that, as a function $B_i \to A$ it only depends on $z_{i,0}$ i.e. that m_i is a function $B_{i,0} \to A$. Summing things up we find that each constructor C_i defines three morphisms $p_i : E_i \to B_{i,0}$, $q_i : E_i \to A$ and $m_i : B_{i,0} \to A$ of which p_i is a "display map" (i.e. the canonical morphism $E_i \to ft(E_i)$):

$$\begin{array}{ccc} & E_i & \xrightarrow{q_i} & A \\ & \downarrow{\scriptstyle p_i} & & \\ A & \xleftarrow[m_i]{} & B_{i,0} & \end{array}$$

and for $X : A \to Type$, a term c_i of type $C_i(X)$ is a morphism $(p_i)_*(q_i)^*(Tot(X)/A) \to (m_i)^*(Tot(X)/A)$.

We conclude that an interpretation of such a term is a morphism

$$[p_i]_* [q_i]^* ([Tot(X)]/[A]) \to [m_i]^* ([Tot(X)]/[A])$$

or equivalently by adjunction a morphism of the form

$$[m_i]_\# [p_i]_* [q_i]^* ([Tot(X)]) \to [Tot(X)]$$

over $[A]$. We can further collect these morphisms together for different i setting

$$E = \amalg_i E_i \quad B = \amalg_i B_{i,0} \quad q = \amalg_i q_i \quad p = \amalg_i p_i \quad m = \amalg_i m_i$$

and define a functor $F(X') = [m]_\# [p]_* [q]^* (X')$ from $\mathcal{C}/[A]$ to itself.

An inductive definition introduces the following data:

1. an object I such that $ft(I) = A$,
2. a morphism $c_X : p_* q^*(I) \to m^*(I)$ over A,
3. a mapping which assigns to any pair (P, c_P) where P is an object such that $ft(P) = I$ and $c_P : p_* q^*(I) \to m^*(P)$, a section $s : I \to P$ of p_P such that $m^*(s)\, c_X = c_P\, (p_* q^*(s))$.

It is not difficult to show now that an initial algebra for F provides an interpretation for I with all of its recursors. However, in the case of the univalent models this is not a satisfactory interpretation since for a fibration $X' \to [A]$ the morphism $F(X') \to [A]$ needs not be a fibration unless $[m]$ happened to be a fibration.

In order to overcome this difficulty one re-writes any inductive definition as a combination of dependent sums, disjoint unions, equivalence types and IP constructions as explained above. A univalent (compatible with the equivalence axiom) interpretation of dependent sums, disjoint unions and equivalence types is known. We obtain a univalent interpretation of IP types using [1, Th. 5.6] since the IP construction can be interpreted as the initial algebra of a functor which takes Kan fibrations to Kan fibrations and which depends on its parameters in a way which respects weak equivalences.

References

[1] Michael Abbott, Thorsten Altenkirch, and Neil Ghani. Categories of containers. In *In Proceedings of Foundations of Software Science and Computation Structures*, 2003.

[2] Frédéric Blanqui. Type theory and rewritting. In *Ph. D. Thesis*, pages 1–139. Orsay, 2001.

[3] John Cartmell. Generalised algebraic theories and contextual categories. *Ann. Pure Appl. Logic*, 32(3):209–243, 1986.

[4] Bart Jacobs. *Categorical logic and type theory*, volume 141 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1999.

[5] Peter May. *Simplicial objects in algebraic topology*. Van Nostrand, 1968.

[6] E. Palmgren and S. J. Vickers. Partial horn logic and Cartesian categories. *Ann. Pure Appl. Logic*, 145(3):314–353, 2007.

[7] Christine Paulin-Mohring. Inductive definitions in the system Coq; rules and properties. In *Typed lambda calculi and applications (Utrecht, 1993)*, volume 664 of *Lecture Notes in Comput. Sci.*, pages 328–345. Springer, Berlin, 1993.

[8] Daniel G. Quillen. The geometric realization of a Kan fibration is a Serre fibration. *Proc. Amer. Math. Soc.*, 19:1499–1500, 1968.

[9] Thomas Streicher. *Semantics of type theory*. Progress in Theoretical Computer Science. Birkhäuser Boston Inc., Boston, MA, 1991. Correctness, completeness and independence results, With a foreword by Martin Wirsing.

[10] B. Werner. Une thorie des constructions inductives. In *Thse de Doctorat*, 1994.