Presheaf model for simplicial sets

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Abstract

By means of a countermodel we show that the homotopy equivalence of fibers of a Kan fibration cannot be proved constructively.

A simplicial set (= functor $\Delta^{op} \to \mathsf{Set}$) is given by a collection of sets X_n with maps $u \longmapsto uf,\ X_n \to X_m$ for all monotone $f : [m] \to [n]$. This definition makes sense in any presheaf model. We shall work with a truncated version and have only two sets X_0, X_1 with maps $d_0, d_1 : X_1 \to X_0$ and $s : X_0 \to X_1$ such that $d_0(s(x)) = d_1(s(x)) = x$ for all $x \in X_0$. We write $e : a \to a'$ if e is in X_1 and $d_0(e) = a'$ and $d_1(e) = a$. This truncation simplifies the presentation and actually provides a stronger counterexample, which will be further strengthened in that we start from a Kan fibration with explicit filling operators.

We call a truncated simplicial set C_1, C_0, d_0, d_1, s an explicit Kan graph, or *Kan graph* for short, if it has the following filling operation: for all a, b, c in C_0 and $e : a \to b$ and $f : a \to c$ there exists $g : b \to c$. This element g is supposed to be given as a function, the *filler*, of a, b, c, e, f. Note the symmetry in e, b and f, c. Also the fillers in conditions (3) and (4) below are meant to be explicit.

Definition 0.1 Any Kan fibration $E \to \Delta^1$ with explicit filling operators, can be described in the truncated version by the following data:

1. Two Kan graphs A_0, A_1 and B_0, B_1 (A is the fiber over 0 and B the fiber over 1), with their respective maps $d_i : A_1 \to A_0$ and $d_i : B_1 \to B_0$ and $s : A_0 \to A_1$ and $s : B_0 \to B_1$ (no confusion will arise from using the same notation).
2. A set G and two maps $d_0 : G \to A_0$ and $d_1 : G \to B_0$. Again we write $e : a \to b$ if e is in G such that $d_0(e) = a$ and $d_1(e) = b$.
3. The following filling conditions: for all a in A_0 there exist b in B_0 and $e : a \to b$ in G and for all b in B_0 there exists a a in A_0 and $e : a \to b$. Thus G represents the liftings of $01 \in \Delta^1[1]$ to the (truncated) fibers A and B.
4. The following (tricky) filling conditions: for all a in A_0, b in B_0, c in $A_0 + B_0$ and $e : a \to b$ in G and $f : a \to c$ in $A_1 + G$, there exists $g : c \to b$ in $G + B_1$; for all a in A_0, b in B_0, c in $A_0 + B_0$ and $e : a \to b$ in G and $f : c \to b$ in $G + B_1$, there exists $g : a \to c$ in $A_1 + G$. (Very tricky indeed, the Kan graph property of A and B follows from these!)

Now that we have expressed in an explicit way what is a truncated Kan fibration over Δ^1, we formulate the homotopy of the fibers in terms of the data above.

Proposition 0.2 *Given a truncated Kan fibration as in Definition 0.1, there exist* $f_0 : A_0 \to B_0,\ g_0 : B_0 \to A_0$ *and* $f_1 : A_1 \to B_1,\ g_1 : B_1 \to A_1$ *such that:*

1. *for all a in A_0 there exists $u : a \to g_0(f_0(a))$ in A_1*
2. *for all b in B_0 there exists $v : b \to f_0(g_0(b))$ in B_1*
3. *we have $f_0(d_i(u)) = d_i(f_1(u))$ for all u in A_1 $(i = 0, 1)$*
4. *we have $g_0(d_i(v)) = d_i(g_1(u))$ for all v in B_1 $(i = 0, 1)$*
5. *(crucial condition) we have $f_1(s(a)) = s(f_0(a))$ for all $a \in A_0$ and $g_1(s(b)) = s(g_0(b))$ for all $b \in B_0$*

Proof. (Classical) We use condition (3) in Definition 0.1 on G to define f_0 and g_0 such that for all a in A_0 there exists $u : a \to f_0(a)$ in G and for all b in B_0 there exists $v : g_0(b) \to b$ in G.

Before we continue defining f_1 and g_1, let us verify clauses (1) and (2). Let $a \in A_0$ and consider $u : a \to f_0(a)$ and $v : g_0(f_0(a)) \to f_0(a)$ in G. By condition (4) in Definition 0.1 we get (1). In a similar way (2) can be verified.

To define f_1, let u be in A_1. We distinguish between u degenerate or not. If u is degenerate, i.e., equal to $s(a)$ for some a in A_0 (alternatively: $u = s(d_0(u))$), define

$$f_1(u) \;:=\; s(f_0(a))$$

Otherwise, consider $u : a \to a'$. By condition (4) in Definition 0.1, taking $b = f_0(a)$ and $c = a'$, we can find an edge $w : a' \to f_0(a)$ in G, and then using condition (4) in Definition 0.1 a second time, we can find an edge $f_0(a) \to f_0(a')$, which satisfies (3). In a similar way one defines g_1 satisfying (4). Both f_1 and g_1 satisfy (5) per construction. □

Notice the use of case distinction on u in A_1 (v in B_1) being degenerate or not. There is actually an alternative proof where we use decidability of equality on A_0 (and B_0): if $d_0(u) = d_1(u)$ ($d_0(v) = d_1(v)$) then $f_1(u) = s(d_0(u))$ ($g_1(v) = s(d_0(v))$).

The next result is that some use of classical logic is essential in this argument, by an appeal to the soundness of Kripke semantics for intuitionistic logic.

Proposition 0.3 *The previous proposition does not hold in the Kripke model over the poset* $0 \leqslant 1 \leqslant 2$.

Proof. The intuition is that a set X in the model evolves over time as $X(0) \to X(1) \to X(2)$. We can interpret the transition map $X(i) \to X(j)$ as adding new elements or equating elements. The following table shows A_0, A_1, B_0, B_1, G *changing* over time. (As presheaves, time should be reversed.)

Day	A_0	A_1	G	B_1	B_0
0	$\{a, a'\}$	$\{s(a), s(a')\}$	$\{w : a{\to}b, w' : a'{\to}b'\}$	$\{s(b), s(b'), z : b{\to}b, z' : b'{\to}b'\}$	$\{b, b'\}$
1		$+\{u : a{\to}a', u' : a'{\to}a\}$	$+\{x : a{\to}b', x' : a'{\to}b\}$	$+\{v : b{\to}b', v' : b'{\to}b\}$	
2	$\{a{=}a'\}$	$\{u{=}u'{=}s(a){=}s(a')\}$	$\{x{=}x'{=}w{=}w'\}$	$\{z{=}v{=}v'{=}z'\ ,\ s(b){=}s(b')\}$	$\{b{=}b'\}$

Table 1: Three days in the life of A_0, A_1, G, B_1, B_0 (only what *changes*)

In words, the table shows how edges are added from day 0 to day 1. From day 1 to day 2, A_0 collapses to one point with all edges degenerated; also B_0 collapses to one point, but the edges $z, v, z'v'$ collaps into one *non-degenerated* self-loop; G collapses to one edge.

All preconditions are now satisfied in the Kripke sense, but there is no way to define f_0, f_1, g_0, g_1 satisfying the required properties. Indeed, the function $f_0(0)$ has to be $a \longmapsto b,\ a' \longmapsto b'$ or $a \longmapsto b',\ a' \longmapsto b$. In both cases, we have to have $f_1(1)$ sending u to v or v'. But then there is a problem for defining $f_1(2)$ which has to send $s(a)$ both to $s(b)$ and to $v = v'$, see the diagram below. □

We thank Peter Lumsdaine and Mike Shulman for the edges z, z' that were initially missing.