Tests, Games, and Martin-Löf's Meaning Explanations for Intuitionistic Type Theory

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Intuitionistic logic and intuitionistic type theory

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1986 Martin-Löf, intensional intuitionistic type theory based on a logical framework (set-type distinction)

Intuitionistic Type Theory - a language for both mathematics and programming

- Full-scale framework for constructive mathematics in the style of Bishop ("ZF for intuitionism"). Others are e g
  - Myhill-Aczel constructive set theory,
  - Aczel-Feferman style type-free theories.
- A functional programming language with dependent types where all programs terminate (core of NuPRL, Coq, Agda, etc)

Type formers of intuitionistic type theory

To interpret predicate logic with identity:

$$\Pi x : A.B, \Sigma x : A.B, A + B, \mathrm{N}_0, \mathrm{N}_1, \mathrm{I}(A, a, b)$$

Other mathematical objects

$$\mathrm{N}, \mathrm{W}x : A.B, \mathrm{U}, \mathrm{U}_1, \mathrm{U}_2, \ldots$$

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- Extensions with general notion of inductive definition – important for programming.
- Extensions into the constructive higher infinite: super universes, universe hierarchies, Mahlo universes, autonomous Mahlo universes, general inductive-recursive definitions etc.

Meaning explanations extend as well.

What are Martin-Löf's meaning explanations?

Meaning explanations. Also called

> *direct semantics, intuitive semantics, standard semantics, syntactico-semantical approach*
>
> *"pre-mathematical" as opposed to "meta-mathematical":*

References:

- *Constructive Mathematics and Computer Programming*, LMPS 1979;
- *Intuitionistic Type Theory*, Bibliopolis, 1984;
- *Philosophical Implications of Type Theory*, Firenze lectures 1987.

Before 1979: normalization proofs, but no meaning explanations.

Natural numbers - computation to canonical form

Start with *untyped computation system* with notion of computation of *closed* expression to *canonical form* (whnf) $a \Rightarrow v$.

$$\mathrm{N} \Rightarrow \mathrm{N}$$

$$0 \Rightarrow 0 \qquad\qquad \mathrm{s}(a) \Rightarrow \mathrm{s}(a)$$

$$\frac{c \Rightarrow 0 \quad d \Rightarrow v}{\mathrm{R}(c,d,e) \Rightarrow v} \qquad\qquad \frac{c \Rightarrow \mathrm{s}(a) \quad e(a,\mathrm{R}(a,d,e)) \Rightarrow v}{\mathrm{R}(c,d,e) \Rightarrow v}$$

Natural numbers - meaning explanations

$$\frac{A \Rightarrow \mathrm{N}}{A\ type}$$

$$\frac{A \Rightarrow \mathrm{N} \quad A' \Rightarrow \mathrm{N}}{A = A'}$$

$$\frac{A \Rightarrow \mathrm{N} \quad a \Rightarrow 0}{a : A} \qquad \frac{A \Rightarrow \mathrm{N} \quad a \Rightarrow \mathrm{s}(b) \quad b : \mathrm{N}}{a : A}$$

$$\frac{A \Rightarrow \mathrm{N} \quad a \Rightarrow 0 \quad a' \Rightarrow 0}{a = a' : A} \qquad \frac{A \Rightarrow \mathrm{N} \quad a \Rightarrow \mathrm{s}(b) \quad a' \Rightarrow \mathrm{s}(b') \quad b = b' : \mathrm{N}}{a = a' : A}$$

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How to understand these rules, meta-mathematically (realizability) or pre-mathematically (meaning explanations)?

General pattern

$$\frac{A \Rightarrow \mathrm{C}(a_1, \ldots, a_m) \quad \cdots}{A \ type}$$

$$\frac{A \Rightarrow \mathrm{C}(a_1, \ldots, a_m) \quad A' \Rightarrow \mathrm{C}(a'_1, \ldots, a'_m) \quad \cdots}{A = A'}$$

where C is an m-place type constructor $(\mathrm{N}, \Pi, \Sigma, \mathrm{I}, \mathrm{U}$, etc), and

$$\frac{A \Rightarrow \mathrm{C}(a_1, \ldots, a_m) \quad a \Rightarrow \mathrm{c}(b_1, \ldots, b_n) \quad \cdots}{a : A}$$

$$\frac{A \Rightarrow \mathrm{C}(a_1, \ldots, a_m) \quad a \Rightarrow \mathrm{c}(b_1, \ldots, b_n) \quad a' \Rightarrow \mathrm{c}(b'_1, \ldots, b'_n) \quad \cdots}{a = a' : A}$$

where c is an n-place term constructor for the m-place type constructor C ($0, \mathrm{s}$ for N; λ for Π; $\mathrm{N}, \Pi, \ldots$ for U; etc).

Universe (ła Russell) - meaning explanations

$$\frac{A \Rightarrow \mathrm{U}}{A\ type}$$

$$\frac{A \Rightarrow \mathrm{U} \quad A' \Rightarrow \mathrm{U}}{A = A'}$$

$$\frac{A \Rightarrow \mathrm{U} \quad a \Rightarrow \mathrm{N}}{a : A} \qquad \frac{A \Rightarrow \mathrm{U} \quad a \Rightarrow \Pi x : B.C \quad B : \mathrm{U} \quad x : B \vdash C : \mathrm{U}}{a : A} \quad \cdots$$

$$\frac{A \Rightarrow \mathrm{U} \quad a \Rightarrow \mathrm{N} \quad a' \Rightarrow \mathrm{N}}{a = a' : A}$$

$$\frac{A \Rightarrow \mathrm{U} \quad a \Rightarrow \Pi x : B.C \quad a' \Rightarrow \Pi x : B'.C' \quad B = B' : \mathrm{U} \quad x : B \vdash C = C' : \mathrm{U}}{a = a' : A} \, .$$

(Remark: equality of types means "same elements and same equal elements" in Martin-Löf 1979)

The meaning of hypothetical judgments

$$x_1 : A_1, \ldots, x_n : A_n \vdash a : A$$

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$$a[a_1, \ldots, a_n / x_1, \ldots, x_n) : A[a_1, \ldots, a_n / x_1, \ldots, x_n]$$

provided

$$\begin{aligned} &a_1 : A_1, \\ &\vdots \\ &a_n : A_n[a_1, \ldots, a_{n-1} / x_1, \ldots, x_{n-1}], \end{aligned}$$

and, moreover,

$$a[a_1, \ldots, a_n / x_1, \ldots, x_n] = a[b_1, \ldots, b_n / x_1, \ldots, x_n] : A[a_1, \ldots, a_n / x_1, \ldots, x_n]$$

provided

$$\begin{aligned} &a_1 = b_1 : A_1, \\ &\vdots \end{aligned}$$

However ...

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The meaning of the identity type (Martin-Löf 1984, *Intuitionistic Type Theory*, p 32)

> *We now have to explain how to form canonical elements of* $\mathrm{I}(A, a, b)$*. The standard way to know that* $\mathrm{I}(A, a, b)$ *is true is to have* $a = b : A$*. Thus the introduction rule is simply: if* $a = b : A$*, then there is a canonical proof* r *of* $\mathrm{I}(A, a, b)$*. Here* r *does not depend on* a, b *or* A*; it does not matter what canonical element* $\mathrm{I}(A, a, b)$ *has when* $a = b : A$*, as long as it has one.*

Extensional intuitionistic type theory

We can then validate

I-*elimination*

$$\frac{c : \mathrm{I}(A, a, b)}{a = b : A}$$

I-*equality*

$$\frac{c : \mathrm{I}(A, a, b)}{c = \mathrm{r} : \mathrm{I}(A, a, b)}$$

which lead to:

- *extensional* intuitionistic type theory, with function extensionality
- non-normalizing terms (although $a : A$ implies that a and A have *canonical* form for *closed* terms).
- undecidable judgments, e g, it is not decidable whether $a : A$ even if we know that A *type*.

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Hence, Martin-Löf rejected the above rules of I-elimination and equality in 1986. (But they remained valid at Cornell.)

Extensional meaning explanations for intensional type theory?

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Possible attitudes:

- The meaning explanations provide *necessary*, but *not sufficient* criteria for validity of judgments. Judgments should also be *decidable* (a *global* condition).
- The judgments of extensional type theory are all valid, but we may sometimes prefer intensional type theory for pragmatic reasons.
- We need alternative *meaning explanations based on the normalization of open terms*, rather than just closed terms. Cf Martin-Löf 2009.

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> *Beware of bugs in the above code; I have only proved it correct, not tried it.*

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Judgments can be refuted and corroborated by running tests (cf Popper). The validity of a judgment should be *local*; it should not refer to the formal system of which it is a part.

Testing categorical judgments $a : A$

Compute the canonical form of A and a!

- If $A \Rightarrow \mathrm{N}$, then
  - if $a \Rightarrow 0$, then the test is successful.
  - if $a \Rightarrow \mathrm{s}(b)$, then test whether $b : \mathrm{N}$.
  - if $a \Rightarrow \mathrm{c}(b_1, \ldots, b_n)$ for some other constructor c (including λ), then the test fails.
- If $A \Rightarrow \mathrm{U}$
  - if $a \Rightarrow \mathrm{N}$, then the test is successful.
  - if $a \Rightarrow \Pi x : B.C$, then test whether $B : \mathrm{U}$ and $x : B \vdash C : \mathrm{U}$.

    $\vdots$
  - if $a \Rightarrow \mathrm{c}(b_1, \ldots, b_n)$ for some c which is not a constructor for small sets, then the test fails.

Testing hypothetical judgments

To test

$$x_1 : A_1, \ldots, x_n : A_n \vdash a : A$$

we need to *generate* arbitrary elements $a_1 : A_1, \ldots, a_n : A_n$ and test the categorical judgment

$$a[a_1, \ldots, a_n / x_1, \ldots, x_n] : A[a_1, \ldots, a_n / x_1, \ldots, x_n]$$

How to do this?

- No problem if $A_i \Rightarrow \mathrm{N}$. Generate $x_i := 0, \mathrm{s}(0), \ldots$.
- No problem if $A_i \Rightarrow \mathrm{I}(\mathrm{N}, m, n)$ for closed m, n. Generate r if $m = n : \mathrm{N}$.
- But what if $A_i \Rightarrow \mathrm{N} \rightarrow \mathrm{N}$?
- And what if $A_i \Rightarrow \mathrm{I}(\mathrm{N} \rightarrow \mathrm{N}, f, g)$?

The meaning of higher order functions

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- Use *game semantics*: lazily generate opponent *strategies* for f, e g, following Hyland and Ong's *arena games* with *innocent strategies*.
- A *refinement* of Martin-Löf's meaning explanations, where input generation plays a dual role to output computation. Technically, game semantics rather than realizability semantics.

Generating elements of identity types

Define meaning of identity type by induction on the type structure:

$$
\begin{array}{rcl}
\mathrm{I}(\mathrm{N}, m, n) & = & \mathrm{T}(m =_{\mathrm{N}} n) \; - \textit{compute recursive function} \\
\mathrm{I}(\Pi x : A.B, f, g) & = & \Pi x : A.\mathrm{I}(B, f(x), g(x)) \\
& \vdots &
\end{array}
$$

We can then define $\mathrm{r}_N, \mathrm{r}_{\Pi x:A.B}$, etc and validate the rules of I-elimination and I-equality in extensional type theory:

$$
\frac{c : \mathrm{I}(\mathrm{N}, m, n)}{m = n : \mathrm{N}} \qquad \frac{c : \mathrm{I}(\mathrm{N}, m, n)}{c = \mathrm{r}_{\mathrm{N}} : \mathrm{I}(\mathrm{N}, m, n)}
$$

$$
\frac{c : \mathrm{I}(\Pi x : A.B, f, g)}{f = g : \Pi x : A.B} \qquad \frac{c : \mathrm{I}(\Pi x : A.B, f, g)}{c = \mathrm{r}_{\Pi x:A.B} : \mathrm{I}(\Pi x : A.B, f, g)}
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Identity on universe?

What is $\mathrm{I}(\mathrm{U}, A, B)$? Freedom of choice.

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Extensional equality of codes. The elements of $\mathrm{I}(\mathrm{U}, A, B)$ are generated in the following way:

$$\mathrm{r}_{\mathrm{UN}} : \mathrm{I}(\mathrm{U}, \mathrm{N}, \mathrm{N}) \qquad \frac{c : \mathrm{I}(\mathrm{U}, A, A') \quad x : A \vdash d : \mathrm{I}(\mathrm{U}, B, B')}{\mathrm{r}_{\mathrm{U}\Pi}(c, d) : \mathrm{I}(\mathrm{U}, \Pi x : A.B, \Pi x : A'.B')} \qquad \ldots$$

Validates I-elimination and I-equality of extensional type theory for U.

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Validates I-elimination and I-equality of extensional type theory for U.

Isomorphism. Let

$$\mathrm{I}(\mathrm{U}, A, B) = A \cong_{\mathrm{U}} B$$

Validates univalence axiom for U, but not I-elimination (J) of intensional type theory for this type.

Groupoid model?

Groupoid model of intensional type theory with univalence for U in extensional type theory (conjecture)? Harper and Licata 2012, *Canonicity for 2-Dimensional Type Theory*

> *An alternative to our current work would be to parallel this approach, and define a groupoid interpretation into extensional type theory, and thereby inherit equality from the meta-language. The benefit of the approach we take here is that it provides a more direct description of the equational theory, presenting it directly in terms of the source language.*

Finite System T

Types

$$A \quad ::= \quad \mathrm{Bool} \mid A \to A$$

Terms

$$a \quad ::= \quad x \mid a\,a \mid \lambda x.a \mid \mathrm{tt} \mid \mathrm{ff} \mid \mathrm{if}\,a\,a\,a$$

Innocent head normal forms of type $A_1 \to \cdots \to A_m \to \mathrm{Bool}$

$$\lambda x_1 \cdots x_m.\mathrm{tt}$$
$$\lambda x_1 \cdots x_m.\mathrm{ff}$$
$$\lambda x_1 \cdots x_m.\mathrm{if}\,(x\,a_1\,\ldots\,a_n)\,a\,a'$$

generate one level of Curien's PCF Böhm trees.

Test of $a : \mathrm{Bool}$

Run the closed term a!

- $a \Rightarrow \mathrm{tt}$: the test succeeds.
- $a \Rightarrow \mathrm{ff}$: the test succeeds.
- $a \Rightarrow \lambda x.b$: the test fails.

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The arena for Bool:

$$\begin{tikzcd}
 & a \Rightarrow ? \arrow[dl, dash, bend right] \arrow[dr, dash, bend left] & \\
a \Rightarrow \mathrm{tt} & & a \Rightarrow \mathrm{ff}
\end{tikzcd}$$

Test of $a : \mathrm{Bool} \longrightarrow \mathrm{Bool}$

Test $x : \mathrm{Bool} \vdash a\,x : \mathrm{Bool}$!

- $a\,x \Rightarrow \mathrm{tt}$: the test succeeds.
- $a\,x \Rightarrow \mathrm{ff}$: the test succeeds.
- $a\,x \Rightarrow \mathrm{if}\, x\, b\, c$: generate head variable
  - either $x := \mathrm{tt}$, and test $x : \mathrm{Bool} \vdash b : \mathrm{Bool}$ or
  - or $x := \mathrm{ff}$, and test $x : \mathrm{Bool} \vdash c : \mathrm{Bool}$
- otherwise, the test fails

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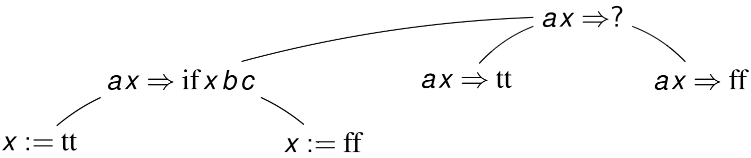
Arena for $\mathrm{Bool} \longrightarrow \mathrm{Bool}$:



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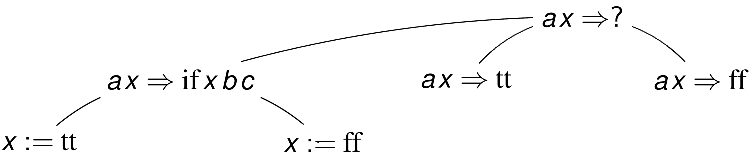

Correspondence with game semantics. Move in arbitrary innocent well-bracketed opponent strategy $x : \mathrm{Bool}$. Only head occurrence of x is instantiated at the first stage. Repetition of moves possible.

Test of $a : \mathrm{Bool} \to \mathrm{Bool}$

Test $\{\,\} : \mathrm{Bool} \vdash a\{\,\} : \mathrm{Bool}$!

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  - either $\{\,\} := \mathrm{tt}$, and test $\{\,\} : \mathrm{Bool} \vdash b : \mathrm{Bool}$ or
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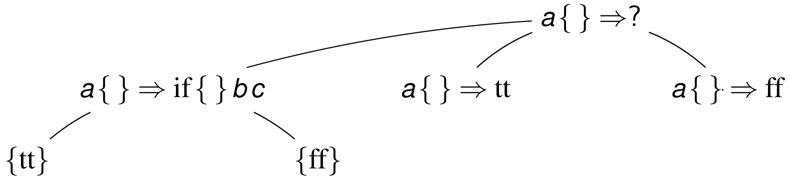
Arena for $\mathrm{Bool} \to \mathrm{Bool}$:



Correspondence with game semantics. Move in arbitrary innocent well-bracketed opponent strategy $\{\,\} : \mathrm{Bool}$. Only head occurrence of $\{\,\}$ is instantiated at the first stage. Repetition of moves possible.

Test of $a : (\mathrm{Bool} \to \mathrm{Bool}) \to \mathrm{Bool}$

Test $\{\ \} : \mathrm{Bool} \to \mathrm{Bool} \vdash a\{\ \} : \mathrm{Bool}$.

- $a\{\ \} \Rightarrow \mathrm{tt}$: the test succeeds.
- $a\{\ \} \Rightarrow \mathrm{ff}$: the test succeeds.
- $a\{\ \} \Rightarrow \mathrm{if}\,(\{\ \}\,b)\,c\,d$. See below.
- otherwise, the test fails.

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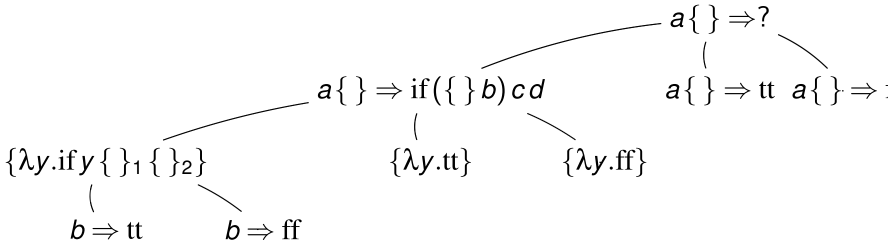


Playing $x\ (x\ \mathrm{tt})$ versus $\lambda y.\mathrm{if}\ y\ \mathrm{ff}\ \mathrm{tt}$

$$(\mathrm{Bool} \quad \to \quad \mathrm{Bool}) \quad \to \quad \mathrm{Bool}$$

$x\ (x\ \mathrm{tt}) \Rightarrow ?$

$x\ (x\ \mathrm{tt}) \Rightarrow \underline{x}\ (x\ \mathrm{tt})$

$x\ (x\ \mathrm{tt}) := ?$

$x\ \mathrm{tt} \Rightarrow \underline{x}\ \mathrm{tt}$

$x\ \mathrm{tt} := ?$

$\mathrm{tt} \Rightarrow \mathrm{tt}$

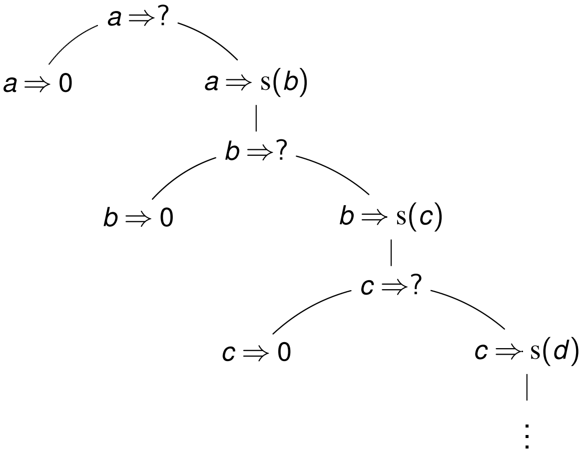
$x\ \mathrm{tt} := \mathrm{ff}$

$x\ \mathrm{tt} \Rightarrow \mathrm{ff}$

$x\ (x\ \mathrm{tt}) := \mathrm{tt}$

$x\ (x\ \mathrm{tt}) \Rightarrow \mathrm{tt}$

The arena for lazy N



Terms of intuitionistic type theory ($\mathrm{Bool}, \mathrm{N}, \Sigma, \Pi, \mathrm{U}$-fragment)

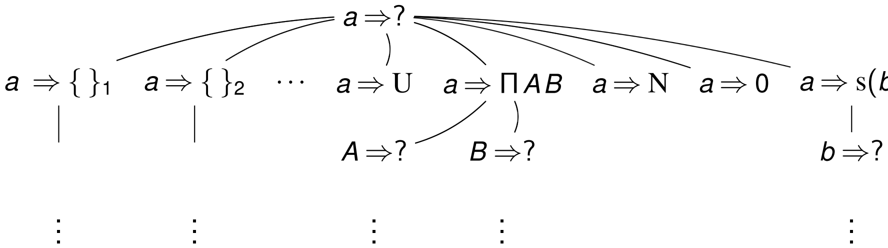
$$
\begin{array}{rcl}
a & ::= & x \mid a\,a \mid \lambda x.a \\
& & \mid \mathrm{tt} \mid \mathrm{ff} \mid 0 \mid \mathrm{s}\,a \mid (a, a) \\
& & \mid \mathrm{Bool} \mid \mathrm{N} \mid \Sigma\,a\,a \mid \Pi\,a\,a \mid \mathrm{U} \\
& & \mid \mathrm{if}\,a\,a\,a \mid \mathrm{natrec}\,a\,a\,a \mid \mathrm{fst}\,a \mid \mathrm{snd}\,a \mid \mathrm{urec}\,a\,a\,a\,a\,a
\end{array}
$$

Russell-style universe with universe recursion

Testing judgments as game playing

- computation to weak head normal form (player move)
- instantiation of variables (opponent move)
- matching constructors of terms with constructors of types (checking rules of the game)

The arena for Intuitionistic type theory



Transition system for testing

Configurations

$$\langle \mathcal{J}, \tau \rangle$$

where

- $\mathcal{J}$ is a judgment
- Formally τ is a *channel environment* - a partial association between channels (metavariables) and *atomic normal forms*
- a *channel* has a type and a context $\Gamma \vdash c : A$
- an *atomic normal form* records *one* move qua one level of unfolding of a Böhm tree corresponding to an opponent strategy.

Initial state of transition system

If $\mathcal{J}$ is a categorical (closed) judgment, then the initial state is

$$\langle \mathcal{J}, \emptyset \rangle$$

If $\mathcal{J}$ is a hypothetical judgment, then the initial channel environment τ_Γ is obtained by replacing all variables in the context Γ by channels of appropriate types:

$$\langle \mathcal{J}, \tau_\Gamma \rangle$$

Final states for test of typing judgment

Final states correspond to introduction rules without premises ("axioms")

$$\langle \mathrm{tt} : \mathrm{Bool}, \tau \rangle$$
$$\langle \mathrm{ff} : \mathrm{Bool}, \tau \rangle$$
$$\langle 0 : \mathrm{N}, \tau \rangle$$
$$\langle \mathrm{Bool} : \mathrm{U}, \tau \rangle$$
$$\langle \mathrm{N} : \mathrm{U}, \tau \rangle$$

Transitions corresponding to introduction rules

Transitions corresponding to introduction rules with premises

$$
\begin{array}{rcl}
\langle \mathrm{s}(a) : \mathrm{N}, \tau \rangle & \longrightarrow & \langle a : \mathrm{N}, \tau \rangle \\
\langle (a, b) : \Sigma A B, \tau \rangle & \longrightarrow & \langle a : A, \tau \rangle \\
\langle (a, b) : \Sigma A B, \tau \rangle & \longrightarrow & \langle b : B a, \tau \rangle \\
\langle \lambda x.b : \Pi A B, \tau \rangle & \longrightarrow & \langle b[x = c] : B c, \tau \cup \{\vdash c : A\} \rangle \\
\langle \Sigma A B : \mathrm{U}, \tau \rangle & \longrightarrow & \langle A : \mathrm{U}, \tau \rangle \\
\langle \Sigma A B : \mathrm{U}, \tau \rangle & \longrightarrow & \langle B c : \mathrm{U}, \tau \cup \{\vdash c : A\} \rangle \\
\langle \Pi A B : \mathrm{U}, \tau \rangle & \longrightarrow & \langle A : \mathrm{U}, \tau \rangle \\
\langle \Pi A B : \mathrm{U}, \tau \rangle & \longrightarrow & \langle B c : \mathrm{U}, \tau \cup \{\vdash c : A\} \rangle
\end{array}
$$

Transition from "generative evaluation"

Generative evaluation $\langle a, \tau_1 \rangle \Longrightarrow \langle v, \tau_2 \rangle$ combines computation of an expression a to canonical form v with instantiation of channels by extending the channel environment τ_1 to τ_2. We have the rule

$$\frac{\langle A, \tau_1 \rangle \Longrightarrow \langle V, \tau_2 \rangle \qquad \langle a, \tau_2 \rangle \Longrightarrow \langle v, \tau_2 \rangle}{\langle a : A, \tau_1 \rangle \longrightarrow \langle v : V, \tau_3 \rangle}$$

- v is a canonical term former: outermost form is term constructor
- V is a canonical type former: outermost form is type constructor

Pure computation to canonical form

Pure computation to canonical form

$$\frac{a \Rightarrow v}{\langle a, \tau \rangle \Longrightarrow \langle v, \tau \rangle}$$

Channel instantiation and generation in head contexts

Head contexts

$$HC[\,] ::= [\,] \mid [\,]\,a \mid \mathrm{if}\,[\,]\,a\,a \mid \mathrm{natrec}\,[\,]\,a\,a \mid \mathrm{urec}\,[\,]\,a\,a\,a\,a$$

Using old instantiation $c = a' : \tau$

$$\frac{a \Rightarrow HC[c] \qquad \langle HC[a'], \tau_1\rangle \Longrightarrow \langle v, \tau_2\rangle}{\langle a, \tau_1\rangle \Longrightarrow \langle v, \tau_2\rangle}$$

Creating new instantiation $c : A : \tau$ and $a' : ANF(\Gamma, V)$

$$\frac{a \Rightarrow HC[c] \qquad \langle A, \tau_1\rangle \Longrightarrow \langle V, \tau_2\rangle \qquad \langle HC[a'], \tau_2\rangle \Longrightarrow \langle v, \tau_3\rangle}{\langle a, \tau_1\rangle \Longrightarrow \langle v, \tau_3\rangle}$$

Atomic normal forms (opponent's moves)

- $\mathrm{tt}, \mathrm{ff} : \mathit{ANF}(\Gamma, \mathrm{Bool})$.
- $0 : \mathit{ANF}(\Gamma, \mathrm{N})$, $\mathrm{s}(c) : \mathit{ANF}(\Gamma, \mathrm{N})$, where $\Gamma \vdash c : \mathrm{N}$ is a new channel.
- $(c, d) : \mathit{ANF}(\Gamma, \Sigma\, A\, B)$ where $\Gamma \vdash c : A$ and $\Gamma \vdash d : B\, c$ are new channels.
- $\lambda x.c : \mathit{ANF}(\Gamma, \Pi\, A\, B)$, where $\Gamma, x : A \vdash c : B\, x$ is a new channel.
- $\mathrm{Bool}, \mathrm{N} : \mathit{ANF}(\Gamma, \mathrm{U})$, $\Sigma\, c\, d, \Pi\, c\, d : \mathit{ANF}(\Gamma, \mathrm{U})$ where $c : \Gamma \vdash \mathrm{U}$ and $\Gamma \vdash d\, c : \mathrm{U}$ are new channels.

Atomic normal forms (neutral case)

- $C_{\Gamma \vdash B}[x_i : A_i] : ANF(\Gamma, B)$, where $x_i : A_i : \Gamma$.

We use a new auxiliary atomic normal form $C_{\Gamma \vdash B}[a : A]$, which will expand to "neutral" atomic normal forms, when the the canonical form of the type A_i is determined.

Expansion of neutral terms for $\mathrm{Bool}, \mathrm{N}$, and U

$C_{\Gamma \vdash B}[a : A]$ is an atomic normal form which depends on A and a.

$$\frac{\langle A, \tau_1 \rangle \Rightarrow \langle \mathrm{Bool}, \tau_2 \rangle \quad \langle \mathrm{if}\, a\, c\, d, \tau_2 \cup \{\Gamma \vdash c : B, \Gamma \vdash d : B\} \rangle \Longrightarrow \langle v, \tau_3 \rangle}{\langle C_{\Gamma \vdash B}[a : A], \tau_1 \rangle \Longrightarrow \langle v, \tau_3 \rangle}$$

$$\frac{\langle A, \tau_1 \rangle \Rightarrow \langle \mathrm{N}, \tau_2 \rangle \quad \langle \mathrm{natcase}\, a\, c\, d, \tau_2 \cup \{\Gamma \vdash c : B, \Gamma \vdash d : \mathrm{N} \to B\} \rangle \Longrightarrow \langle v, \tau_3 \rangle}{\langle C_{\Gamma \vdash B}[a : A], \tau_1 \rangle \Longrightarrow \langle v, \tau_3 \rangle}$$

$$\frac{\langle A, \tau_1 \rangle \Rightarrow \langle \mathrm{U}, \tau_2 \rangle \quad \langle \mathrm{ucase}\, a\, c\, d\, e\, f, \tau_2 \cup \cdots \rangle \Longrightarrow \langle v, \tau_3 \rangle}{\langle C_{\Gamma \vdash B}[a : A], \tau_1 \rangle \Longrightarrow \langle v, \tau_3 \rangle}$$

Expansion of neutral terms for Σ and Π

Σ:

$$\frac{\langle A, \tau_1 \rangle \Rightarrow \langle \Sigma\, D\, E, \tau_2 \rangle \quad \langle C_{\Gamma \vdash B}[\mathrm{fst}\, a : D], \tau_2 \rangle \Longrightarrow \langle v, \tau_3 \rangle}{\langle C_{\Gamma \vdash B}[a : A], \tau_1 \rangle \Longrightarrow \langle v, \tau_3 \rangle}$$

$$\frac{\langle A, \tau_1 \rangle \Rightarrow \langle \Sigma\, D\, E, \tau_2 \rangle \quad \langle C_{\Gamma \vdash B}[\mathrm{snd}\, a : D\,(\mathrm{fst}\, a)], \tau_2 \rangle \Longrightarrow \langle v, \tau_3 \rangle}{\langle C_{\Gamma \vdash B}[a : A], \tau_1 \rangle \Longrightarrow \langle v, \tau_3 \rangle}$$

Π:

$$\frac{\langle A, \tau_1 \rangle \Rightarrow \langle \Pi\, D\, E, \tau_2 \rangle \quad \langle C_{\Gamma \vdash B}[a\, c : E\, c], \tau_2 \cup \{c : \Gamma \vdash D\} \rangle \Longrightarrow \langle v, \tau_3 \rangle}{\langle C_{\Gamma \vdash B}[a : A], \tau_1 \rangle \Longrightarrow \langle v, \tau_3 \rangle}$$

Summary

- The test manual determines the meaning of judgments, including equality judgments.
- Tests can corroborate or refute judgments.
- Tests with functional input leads us to games: input generation corresponds to playing opponent strategy.
- New interpretation of hypothetical judgments, type equality, and identity types.
- Rules of extensional type theory of Martin-Löf 1979 can be justified – work in progress to construct a game model of extensional type theory based on the test manual.