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Profinite Algebraic Homotopy (shortened version)

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Introduction

The work for this book grew out of an attempt in the early 1980s to apply the then newly emerging theory of crossed modules, crossed complexes, and their relationship with combinatorial group theory, cohomology and homology of groups and, more generally, 'combinatorial homotopy theory' in the sense of Whitehead's [164], to problems in the algebraic homotopy of pro-finite spaces and related profinite groups. Profinite groups arise naturally as Galois groups and Serre had written extensively on their theory and applications. Grothendieck, [78], had followed him in stressing the analogy between the Galois group and algebraic geometric versions of the fundamental group of a scheme. These latter objects were also naturally profinite groups. Artin and Mazur, [6], had identified that the étale homotopy type of a scheme was closely linked to processes of profinite completion and these had been taken up and exploited, initially by Sullivan, and then by a whole host of homotopy theorists, culminating in the lecture notes of Bousfield and Kan, [19], in 1972.

Again in the early 1980s, I had the privilege to be a participant in a fascinating exchange of ideas between Grothendieck at Montpellier and Ronnie Brown and myself at Bangor. The subject of the correspondence was the 'Pursuit of Stacks' and the use of higher homotopy groupoids to provide algebraic models for homotopy types. The formal structure revealed in the categorical treatment of the fundamental group of a scheme, [78], defining it in terms of the classification of finite coverings, had, several years earlier, seemed to me possibly to lead to a generalisation replacing the ordinary category theory, used as a tool in the theory, by 'lax' versions and eventually by 'homotopy coherent' constructions. In the initial stage this would require profinite groupoids as used in Magid's Galois theory of commutative rings, [116], and eventually, if one was to get higher order coverings, working profinite models for profinite homotopy types both of spaces and perhaps of schemes. This was far more than I could dream of doing, but it suggested some good problems to try out the ideas at a low level. Then I found that my ideas were not as daft as they might seem. Grothendieck showed us letters that he had written some years earlier in which he outlined an attack on just this. His conjectural theory considered the action of the fundamental group of a space or scheme on the fibre of a covering space as being just the first of an infinite ladder of 'homotopy coherent actions' of homotopy n-types of spaces on 'fibres' which were models for homotopy $(n-1)$-types. The draft plan of attack had to be (i) understand homotopy coherence as a generalisation of the bicategorical methods as developed, for instance, by Benabou, (ii) understand algebraic models for n-types, and possibly (iii) understand the extent to which the available theory of algebraic models of n-types (at that time mainly the MacLane-Whitehead result on 2-types, [114], and the theory of crossed complexes / homotopy systems in [164]) could be extended in a meaningful way to the profinite case so as to be applicable in algebraic geometry. The evidence that some such theory might be possible was strong, at least in the topological (non-profinite) case. There were results on representation of cohomology classes in terms of crossed n-fold extensions, with indications of possible extensions to non-Abelian cohomology, and interpretations of combinatorial data in the theory of presentations of groups that suggested possible links there, both with classification of extensions and directly with cohomology. Above all, Whitehead's two 'Combinatorial Homotopy' papers and in particular the second one, [164], had been an attempt to extend the insights of the combinatorial group theory developed by Reidemeister in the 1930s, to exactly a combinatorial treatment of homotopy types and thus of their possible algebraic models, in other words to develop an '*algebraic homotopy*' theory. Whitehead's vision of such a theory is discussed more fully in the introduction to Chapter 2, and in particular, at the start of section 2.1. It basically aimed to reduce the classification of 'nice' space to that of algebraic models for their homotopy types, up to some algebraic notion of homotopy.

In 1982, a student, Fahmi Korkes, had started working with me on the generalisation of the basic constructions of profinite group theory to the theory of crossed modules and crossed complexes. This was, in part, designed to test if results would generalise as mentioned above, but was largely parallel to the 'Grothendieck programme' which was both much too large for a PhD project and also only becoming apparent at that time. Brown and Huebschmann, [29], had shed new light on old ideas in combinatorial group theory and, in particular, on 'identities among relations'. The methods used both in combinatorial group theory and in cohomology of groups, made frequent use of Eilenberg-MacLane spaces, $K(G,1)$, and of construction based on 'the' classifying space, BG, of a group G. The construction, and properties of these not only gave information on the presentations of the group but also, within algebraic topology, on the universal G-bundles. If G was a profinite group, however, the constructions did *not* take into account the extra topological aspect of G, so Fahmi's project was to try to get around this by using 'profinite' crossed modules and 'profinite' crossed complexes to provide profinite algebraic analogues of all the main topological constructions of basic combinatorial group theory and cohomology and then, to some extent, to evaluate them to see if they did what had been hoped for. He did an excellent job. His thesis, [100], showed that most of the crossed gadgetry went through without problem, in fact a surprising amount went through with little extra work, only occasionally needing hard new arguments. He also realised that the treatments, available at that time, of cohomology were unduly restrictive, and unnatural, when it came to choice of coefficients, and that they were not adapted to the task he had been set.

A referee of one of the joint papers we wrote after his thesis suggested that the material would fit better in a monograph as some of the results were reasonably easy extensions of published material, but that the original material was not well known amongst the users of profinite groups. We therefore started thinking of a monograph of this form. The main problems however were non-mathematical ones. They were caused by 'events' in the 'outside world' and, in particular wars in the Middle East. There were other problems to writing a short monograph. There was no adequate coverage of the background material in any small set of sources. It was available in journal papers dotted about the literature, so we could not merely quote the existing non-profinite theory. It would have to be developed in the monograph itself.

This monograph finally attempts to do that. It has been delayed by 'events' as I mentioned before. It aims to provide some of the necessary machinery for developing an adequate, well structured theory of cohomology of profinite groups, a combinatorial group theory for profinite groups and some of the relations with the n-stacks programme of Grothendieck. On the way it provides the first treatment, in monograph form, of much of the *crossed algebraic homotopy* or *crossed homotopical algebra* developed in the last 20 years.

Of course, in the time since this monograph was started, other sources relevant to the area have appeared. On the general area of Galois theory, there is the excellent book by Borceux and Janelidze, [18]; much has been written on profinite groups as such, see for instance, Dixon *et al*, [46], and some of the inadequacies that we had to work with, e.g. the unnatural restriction on coefficients for cohomology of pro-finite groups, no longer apply, cf. Ribes and Zalesskii, [148]. The ordinary theory of algebraic homotopy including some discussion of crossed structures has been extensively developed by Baues in a serious of books and articles, for instance, [9–11], but the full crossed 'menagery' is still not adequately represented. The theory of stacks has also progressed considerable, - but the full 'Grothendieck programme' has still not been completed, although much work has gone into it. The original needs for this monograph have thus changed in detail but are still there. Some of the ideas, problems and results that it reveals or reviews are still not well understood as far as that programme is concerned.

The theme of profinite n-types is represented in some form in most of the chapters. Another theme is cohomology of profinite groups, and there are strong links between these two themes. We give quite a detailed description of profinite group cohomology as not only are some of the results needed for a thorough treatment of profinite n-types, but also the 'crossed' approach to the representation of such n-types has applications in both the cohomology and the homology of groups. The treatment we have given is thus not limited just to those parts of the theory needed directly for the profinite algebraic homotopy as such.

The original work on the first few chapters is based on [100]. More examples and some more results have been added by me. Later sections have been added that are relevant to the wider and deeper understanding of profinite homotopy types, the structure of profinite simplicial groups and thus, indirectly, to problems relating to Grothendieck's *Pursuit of Stacks* in its profinite form.

Originally I had hoped to write this jointly with Korkes, but this had proved impossible due to world events and I have finally decided to complete the monograph myself. It, of course, has benefitted enormously from the initial firm foundation given by his work for his PhD and I must add the usual disclaimer about all the errors being my own!

1

Algebraic Preliminaries

In these first two chapters we will need to recall various more or less well known facts on profinite groups, simplicial groups, homotopy theory and completions. To avoid that this becomes as large as the rest of the book put together and as most of the proofs are relatively easily available in 'the literature', we will present merely an outline of much of the more routine theory giving references that are more than adequate for the details. Much of this material can be 'skimmed' on first reading, then studied more deeply when called on later on in the book.

1.1 Pro-objects

We will need to be able to talk of profinite groups, profinite spaces, and related topics in at least two different ways. In one the view of, say, a profinite group is as a topological group with particular properties on the quotients by its closed normal subgroups. This is often a natural way in which the object emerges from other considerations, say as the Galois group of an extension or the fundamental group of a scheme in the étale topology. It is not the only view possible and sometimes the profinite group is exactly that a pro-finite group, that is, a projective system of finite groups. Both views are useful and are equivalent to each other. For applications in topology and algebraic geometry, it is often the second one that gives the clearer insight so we will start with that, introducing the general notion of a pro-object in an arbitrary category, $\mathcal{C}$.

Definition: A small category $\mathcal{I}$ is said to be *filtering* if:
(i) for each pair of objects i, j of $\mathcal{I}$, there is an object k and morphisms $k \to i$, $k \to j$,
and
(ii) for each parallel pair of morphisms $i \rightrightarrows j$ between two objects i and j

of $\mathcal{I}$, there is an object k and a morphism $k \to i$ such that the two composites $k \longrightarrow i \rightrightarrows j$ are equal.

Let $\mathcal{C}$ be a category. A *pro-object* in $\mathcal{C}$ is specified by a small filtering category $\mathcal{I}$ and a functor $X : \mathcal{I} \to \mathcal{C}$. A pro-object in $\mathcal{C}$ may also be called a *projective system in* $\mathcal{C}$ or a *filtering diagram of objects of* $\mathcal{C}$.

If $X : \mathcal{I} \to \mathcal{C}$ and $Y : \mathcal{J} \to \mathcal{C}$ are two pro-objects in $\mathcal{C}$, then a morphism from X to Y is an element of the set $lim_j colim_i \mathcal{C}(X(i), Y(j))$. Pro-objects in $\mathcal{C}$ and their morphisms form a category denoted here by $Pro{-}\mathcal{C}$.

Remarks: (i) Luckily in the context in which we will be working, it is usually possible to avoid this somewhat complicated and somewhat 'stark' description of morphisms of pro-objects. It can be 'deconstructed' to give a description of such a morphism in terms of interrelating morphisms, $f_{ij} : X(i) \to Y(j)$, but for the most part we will have finiteness conditions on the objects $X(i)$ and $Y(j)$ which will allow us to use other means as well.

(ii) This description of $Pro{-}\mathcal{C}$ is explored in detail in several texts notably [5, 75], Artin and Mazur's *Étale homotopy theory* lecture notes, [6] and various other books on that area and the related area of *Shape Theory*, see, for instance, Cordier and Porter, [39]. It is not necessary to understand, from the start, the lim-colim definition of the morphism sets, so the reader should not be 'phased' by it, in fact, we introduce lim below, however some categorical knowledge will be needed and assumed from time to time.

(iii) The definition of pro-object given here can be simplified somewhat by replacing the condition that the domain 'indexing categories' for pro-objects be small filtering categories by demanding rather that they be 'directed sets', or more exactly small categories associated to directed sets. In such categories there is at most one morphism between any two objects so the second condition on filtering is redundant and thinking of the existence of $i \to j$ as being an indicator of a relationship $i \geq j$ condition (i) is the usual 'directed set' condition, see below. Given any pro-object $X : \mathcal{I} \to \mathcal{C}$ in the general sense, one can construct a directed set $\overline{\mathcal{I}}$ and a functor $\overline{X} : \overline{\mathcal{I}} \to \mathcal{C}$ such that X and $\overline{X}$ are isomorphic in $Pro{-}\mathcal{C}$. Thus for all intents and purposes we can replace X by $\overline{X}$.

It is useful to have a criterion that will imply that two pro-objects are isomorphic. In ordinary calculus, one is useful to the idea that a convergent sequence remains convergent if one deletes elements so that for any deleted element the remains one with higher index. A similar idea works with categorical limits and also with pro-objects. Suppose that $X : \mathcal{I} \to \mathcal{C}$ is a pro-object in $\mathcal{C}$ and we have a subcategory $\mathcal{J}$, we can restrict X to J to get a new pro-object that we will call $X_{\mathcal{J}}$ for the moment. We need conditions that will imply that $X_{\mathcal{J}}$ is isomorphic to X, itself.

Definition: The category $\mathcal{J}$ is *cofinal* or *initial* in $\mathcal{I}$ if given any $i \in \mathcal{I}$, there is a $j \in \mathcal{J}$ with a morphism $j \to i$.

The following is well known:

Proposition 1. *If $\mathcal{J}$ is a cofinal subcategory of $\mathcal{I}$, then for any pro-object X indexed by $\mathcal{I}$, $X \cong X_{\mathcal{J}}$.* ∎

There is a notion of *final functor* between filtering categories which extends the above. We will not be needing it, but discussion of this can be found in many sources on category theory.

1.2 Profinite topological spaces and profinite spaces

For much of the time we can replace pro-objects in a category $\mathcal{C}$ by objects of $\mathcal{C}$, which are 'topologised', at least in the presence of some finiteness assumptions. The first instance of this is when $\mathcal{C}$ is the category of finite sets, and the resulting objects are profinite spaces. Although, as mentioned above, we will need sometimes to assume a certain level of categorical knowledge, it is convenient to recall here the usual and elementary definition of inverse systems and inverse limits. The former of these is just a special case of the definition of pro-object that we have just seen:

Definition: Let $\mathcal{C}$ be a category and let $\mathcal{I}$ be a directed set with respect to a relation $\leq$, (i.e., $\leq$ is reflexive and transitive and to every pair $i_1, i_2 \in \mathcal{I}$, there is an $i \in \mathcal{I}$ such that $i \geq i_1$ and $i \geq i_2$). For each $i \in \mathcal{I}$, suppose that we have an object $S_i \in \mathcal{C}$ such that for each $i \leq j$, we have a morphism $\alpha_i^j : S_j \to S_i$ in $\mathcal{C}$ satisfying the following:
(i) each $\alpha_i^i : S_i \to S_i$ is the identity,
and
(ii) if there is a k with $i \leq j \leq k$ then $\alpha_i^k = \alpha_i^j \alpha_j^k$.

Such a system will be called an *inverse system* or *projective system* in $\mathcal{C}$. It is, of course, just a special case of the definition of pro-object in $\mathcal{C}$ given earlier, but it has the advantage of giving precisely what needs to be checked when specifying such a system.

Definition: Let S be an object in $\mathcal{C}$ and let $\{S_i, \alpha_i^j\}$ be an inverse system in $\mathcal{C}$. If for each $i \in \mathcal{I}$, we have a morphism $\alpha_i : S \to S_i$ in $\mathcal{C}$ such that, whenever $i \leq j$, we have $\alpha_i^j \alpha_j = \alpha_i$, then we shall call $\{S, \alpha_i\}$ a *cone* over $\{S_i, \alpha_i^j\}$. If $\{S, \alpha_i\}$ is universal amongst cones over $\{S_i, \alpha_i^j\}$, so if for every cone $\{X, \lambda_i\}$ over $\{S_i, \alpha_i^j\}$, there is a unique morphism $\phi : \{X, \lambda_i\} \to \{S, \alpha_i\}$ of such cones (i.e., a unique morphism $\Phi : X \to S$ in $\mathcal{S}$ such that $\lambda_i = \alpha_i \phi$, for each i in $\mathcal{I}$), then we say $\{S, \alpha_i\}$ is the *inverse limit* (or *projective limit* or sometimes simply *limit*) of $\{S_i, \alpha_i^j\}$ in $\mathcal{C}$ and denote it

$$S = Lim\, S_i.$$

Example: The usual construction of inverse limits in the category of Sets is as follows:

Given $\{S_i, \alpha_i^j\}$, form the product $\prod_k S_k$ and then let

$$S = \{(s_k) \mid s_k \in S_k \text{ and if } i \leq j, s_i = \alpha_i^j s_j\}.$$

Taking α_i to be the projection onto the i^{th} factor, it is easily checked that $\{S, \alpha_i\}$ satisfies the universal property above and so 'is' the inverse limit of $\{S_i, \alpha_i^j\}$. Of course, as usual with categorical definitions involving universal properties, an inverse limit is determined only up to isomorphism. In the categories of sets, groups, rings and topological spaces, inverse limits always exist.

Definition: A *profinite space* is an inverse (projective) limit of finite discrete spaces in the category of topological spaces. The category of profinite spaces and continuous maps will be denoted $Prof$.

Remark: Later we will have occasion to consider simplicial profinite spaces. These can be considered as inverse limits of finite simplicial sets. Due to the use of the term 'spaces' as synonymous with 'simplicial set' by quite a large number of workers in that field, some researchers refer to simplicial profinite spaces as 'profinite spaces', and this can cause some confusion. Here we will try to make the distinction between these two different meanings of the term, although it is usually clear when referring to a source which meaning is being attached to it.

The following is well known and a proof can be found in, for instance, [18] or [116]:

Proposition 2. *Let X be a topological space. Then X is a profinite space if and only if X is compact Hausdorff and totally disconnected.* ■

Definition: Let X be a space. A *profinite completion* of X is a profinite space $\hat{X}$ together with a continuous map $\eta_X : X \to \hat{X}$ such that if given any profinite space Y and a continuous map $g : X \to Y$, there is a unique continuous map $\psi : \hat{X} \to Y$ with $\psi\eta_X = g$.

In what follows we will give two constructions of a profinite completion $\hat{X}$ of a space X.

Let $\Omega(X)$ be the family of all equivalence relations R defined on X with the property that X/R is finite and discrete in the quotient topology. Then we take

$$\hat{X} = Lim_{R\in\Omega(X)}(X/R).$$

This is easily shown to be a profinite completion of X.

Another interesting way of constructing a profinite completion is to take $Bool(X)$ to be the Boolean algebra of all continuous $\mathbb{Z}_2$-valued functions on X, where $\mathbb{Z}_2 = \{0, 1\}$ is given the discrete topology or equivalently as being the algebra of closed-open subsets of X. Then we set $\hat{X} = Max(Bool(X))$, the maximal ideal space of $Bool(X)$ and there is a Gelfand map,

$$\eta_X : X \to \hat{X}.$$

This exploits Stone duality between Boolean algebras and profinite spaces. Because of this dualtiy, profinite spaces are called *Boolean spaces* by some authors. Taking this further leads to locale theory and prodiscrete locales. We will not be exploring that theory here, but note that it may be a useful generalisation of the situations, we will be discussing.

Proposition 3. *Let X be a profinite space, and let X_1 be a dense subspace of X. If $\hat{X}_1$ is the profinite completion of X_1, then there is a map*

$$\hat{q} : \hat{X}_1 \to X$$

that is surjective.

Proof: Take $Max(Bool(X))$, as above and similarly for X_1. The inclusion of X_1 into X induces a map between these spaces, but of course, $Max(Bool(X))$ and X are naturally homeomorphic. It is then relatively simple to use the density of X_1 in X to prove that the induced map $Bool(X) \to Bool(X_1)$ is one-one and hence that it induces a surjective map on applying Max. ■

1.3 Profinite and pro-$\mathcal{C}$ groups.

The category of profinite groups forms a natural extension of the category of finite groups, but it carries a much richer structure as it allows for the formation of free objects, and coproducts as well as the projective limits that one might expect, given that the objects of this extension are projective limits of finite groups. Here we will have room only to sketch some of the basic theory and to prove some important but fairly elementary results. For more on this elementary and by now fairly classical theory, we refer the reader to the basic sources such as Serre, [152], Ribes, [147], Schatz, [150] and also more recent texts such as that by Dixon, de Sautoy, Mann and Segal, [46] and Ribes and Zalesskii, [148].

Definition: A *profinite group* is an inverse limit of a system of finite groups.

The category of profinite groups and continuous homomorphisms between them will be denoted by $Prof.Grps$.

Theorem 1. *The following conditions are equivalent:*

(i) G is a profinite group;
(ii) G is a compact Hausdorff group in which the family of open normal subgroups forms a fundamental system of neighbourhoods at the identity;
(iii) G is a compact, Hausdorff and totally disconnected topological group. ■

Examples (i) If G is an abstract group, we can define a topology on G by taking, as a system of neighbourhoods of the identity, the normal subgroups, U, of finite index in G. Associated with this system of normal subgroups of

G, we have an inverse system of finite groups whose limit will be denoted $\hat{G}$ which will be called the *profinite completion* of G. Thus

$$\hat{G} = Lim_{U \in \Omega(G)} G/U,$$

where $\Omega(G)$ is the directed set of normal subgroups of finite index in G.

(ii) Let $\mathbb{N}$ be the natural numbers, partially ordered by division, so that for $m, n \in \mathbb{N}$, $n \leq m$ means $n \mid m$, and let $\phi_n^m : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be the natural projection. Then one has a profinite group

$$\hat{\mathbb{Z}} = Lim_m \mathbb{Z}/m\mathbb{Z}.$$

This is the profinite completion of the additive group, $\mathbb{Z}$, of integers.

(iii) Let $\mathbb{N}$ be the natural numbers and suppose p is a prime number. If $m, n \in \mathbb{N}$, and $m \leq n$ in the ordinary sense, define $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$, to be the natural projection. The limit

$$\mathbb{Z}_p = Lim_n \mathbb{Z}/p^n\mathbb{Z}$$

is a profinite group and coincides with the additive group of p-adic integers.

We will often want to restrict attention to subcategories of the category of profinite groups. For instance the last example above is an example of a 'pro-p group'. More generally, let $\mathcal{C}$ be any class of finite groups that is closed under the formation of subgroups, homomorphic images and extensions.

Definition: A *pro-$\mathcal{C}$ group* is an inverse limit of an inverse system of groups in the class $\mathcal{C}$.

The subcategory of $Prof.Grps$ consisting of the pro-$\mathcal{C}$ groups and the continuous homomorphisms between them will be denoted $Pro - \mathcal{C}$.

This notation now has two deinitions but as the corresponding categories are equivalent this causes no problem.

If G is a pro-$\mathcal{C}$ group, then an open normal subgroup U of G might be called $\mathcal{C}$-cofinite as G/U is a group in $\mathcal{C}$. The term will be extended to any group G, so that a normal subgroup N of G will be called *$\mathcal{C}$-cofinite* if G/N is in $\mathcal{C}$.

In particular, when $\mathcal{C}$ is the class of all groups of order a power of p, then we simply say 'pro-p' rather than specifying $\mathcal{C}$ and saying pro-$\mathcal{C}$. These pro-p groups are important in many areas of mathematics and we refer the reader to, for instance, [46], for an in-depth treatment of their theory. We note that a profinite group is a pro-p group if and only if every open normal subgroup has index some power of p. More generally, if ℓ is a family of prime numbers, we can form the class of all finite groups whose orders are products of primes from ℓ. The corresponding limit groups are then called *pro-ℓ groups*, for short.

The categories of form $Pro - \mathcal{C}$ form varieties in $Prof.Grps$. Recall that a *variety* in any algebraic context meands a subcategory of 'algebras' closed under products, subobjects and quotients. We note the condition on $\mathcal{C}$ implies the closure of $\mathcal{C}$ under finite products, so $\mathcal{C}$ is what is called a *pseudovariety*, see later. The category $Prof.Grps$ is monadic over the category of Spaces. This means that free objects exist in all the $Pro-\mathcal{C}$. A good reference for this is Gildenhuys and Kennison, [69]. We will need a good knowledge of these free constructions so will discuss them more fully shortly.

We will often give definitions in detail only for profinite groups themselves, but there are nearly always variants for when the class of finite groups is replaced, either by a general $\mathcal{C}$, or, sometimes, for the class of 'p-groups', as certain ideas are especially well structured in that case.

Returning to general profinite groups, we say that a subset S of a profinite group G *generates* G if the (abstract) subgroup generated by S is dense in G. We call G *finitely generated* if G has a finite subset which generates it.

Next we list some elementary facts about profinite groups:

(A) A closed subgroup H of a profinite group G is profinite.
(B) Any quotient group G/H of a profinite group, G, by a closed normal subgroup H, is profinite.
(C) The product of a family of profinite groups is profinite.
(D) The inverse limit of a system of profinite groups is profinite.

We can elaborate on this last fact.

Proposition 4. *Let Lim_I denote the inverse limit functor from the category of inverse systems of profinite groups, over a fixed directed set I, then Lim_I is an exact functor (i.e., it preserves exact sequences).* ■

There is a result that will be used time and again in this book. Its central importance is manifested by its position in Serre's book, [152]. It is his Proposition 1 on page 2. Given its importance we will give a sketch of its proof even though it is very well known and can be found in other sources as well, see Schatz, [150], for instance.

We will need a number of lemmas.

Lemma 1. *(i) If H is a closed subgroup of a profinite group G, then the space of cosets G/H is a profinite space.*
(ii) Let G be a compact group and $\{S_i\}$ a decreasing filtered family of closed subgroups of G. Writing $S = \bigcap S_i$, the natural mapping

$$G/S \to Lim\ G/S_i$$

is a homeomorphism.

Proof: Part (i) is fairly routine. For part (ii) note that the mapping is injective with dense image and as G/S is compact, the result follows. ■

The 'important result' is the existence of continuous sections for epimorphisms and of continuous transversals for subgroups. In the case of abstract groups where continuity does not play a part, there is no problem. You use the Axiom of Choice. For profinite groups, more work is needed although of course the Axiom of Choice will still be needed. Serre in fact proves a more general result:

Proposition 5. *(Serre, [152], p.2) Let H and K be two closed subgroups of a profinite group G with $K \subset H$. There is a continuous section*

$$s : G/H \to G/K$$

to the natural projection from G/K to G/H.

The proof will need a further lemma.

Lemma 2. *(i) There is a section in the above setting if H/K is finite.*
(ii) If H and K are normal in G, the extension

$$1 \to H/K \to G/K \to G/H \to 1$$

is trivial on an open subgroup of G/H.

Proof: Let U be an open normal subgroup of G such that $U \cap H \subset K$. The restriction of the projection from G/K to G/H to the image of U in G/K is then injective (and is a homomorphism when H and K are normal, which proves (ii)). It then has a section on the image of U in G/H, but that is an open set and translates of it cover G/H. Using those translates we construct a section on the whole of G/H. ∎

Proof of Proposition 5: Let $\mathcal{X}$ be the set of pairs (S, s) with S a closed subgroup of G with $K \subseteq S \subseteq H$ and where s is a continuous section $s : G/H \to G/S$. The idea is to show $\mathcal{X}$ has a maximal element with $S = K$ by using Zorn's lemma. We order $\mathcal{X}$ in the obvious way. Lemma 1 (ii) shows that $\mathcal{X}$ is an inductive set, (every totally order subset has a maximal element) and then, using Lemma 2, we obtain that if $\mathcal{X}$ has a maximal element then it must have $S = K$. As planned, invoking Zorn's lemma completes the proof. ∎

Corollary 1. *(Continuous sections exist.) Any epimorphism of profinite groups has a continuous section.*

Proof: Let $\theta : G_1 \to G_2$ be an epimorphism of profinite groups, then, of course, we may assume $G_2 = G_1/H$ for $H = Ker\ \theta$ and the result follows from the case $K = 1$ of the above proposition. ∎

Any section of a more general $G \to G/H$, where H is a closed subgroup, but is not necessarily normal, will give a *continuous transversal* for H in G, i.e., a continuous choice of coset labels. Of course, one can normalise any such continuous transversal, t, so that $t(1H) = 1_G$. It will be useful to have this as a formal result here so:

Corollary 2. *(Continuous transversals exist.) Given any closed subgroup H of a profinite group G, there is a continuous transversal $t : G/H \to G$ such that $t(1H) = 1_G$.* ■

Of course, 'sections' need not be 'splittings'. When a section is a homomorphism, then it is usually called a *splitting* and the epimorphism is called a *split epimorphism*. In the abstract case, split epimorphisms of groups correspond to the projections of semidirect products. The profinite analogue of this goes through without problem, but as we will need profinite semidirect products rather a lot, we will need to set up notation and terminology with some care.

Suppose $\pi : H \to G$ is a split epimorphism of profinite groups with $K = Ker\,\pi$, and $s : G \to H$, a chosen continuous splitting, then G acts continuously on K (on the left for convenience) by noting that if $\iota : K \to H$ is the inclusion, $g \in G$ and $k \in K$, then $s(g).\iota(k).s(g)^{-1}$ is again in K and so gives an element ${}^gk \in K$ such that

$$\iota({}^gk) = s(g).\iota(k).s(g)^{-1}.$$

This gives a continuous action:

$$G \times K \to K$$

$$(g, k) \longmapsto {}^gk.$$

As automorphism groups of profinite groups need not be themselves profinite, it is usually easier to define continuous actions in this direct way as a continuous map from a product satisfying the evident properties rather than as a continuous map to some topological group of continuous automorphisms.

Given any G, and K and such a continuous action, we can form a group $K \rtimes G$ with $K \times G$ as underlying space and the usual multiplication

$$(k_1, g_1)(k_2, g_2) = (k_1{}^{g_1}k_2, g_1g_2).$$

This is continuous and makes $K \rtimes G$ into a profinite group. The projection $\pi : K \rtimes G \to G$ is given by (k, g) goes to g. It is continuous, and a split epimorphism with splitiing given by $s(g) = (1_K, g)$. Of course, $K \cong Ker\,\pi$ and if we started with a profinite group H as above, and a split epimorphism, π, we retrieve them, up to isomorphism, since $H \cong K \rtimes G$, etc. compatibly with the split projections.

As was said above, all this is well known, but we will need the notation thus set up.

1.4 Free profinite groups

For the basic definitions and results in this section we refer the reader to Serre, [152], Ribes, [147], or Shatz, [150].

Definitions: (i) Let S be a set and G a profinite group. We say that a map $u : S \to G$ is *convergent to* $1 \in G$ if every open normal subgroup of G contains all but a finite number of the $u(s)$ for $s \in S$.

(ii) Again let S be a set. A *free profinite group* on S is a profinite group $F(S)$ together with a map $u : S \to F(S)$ convergent to $1 \in F(S)$ such that given any profinite group G and a map $v : S \to G$ convergent to $1 \in G$, there exists a unique homomorphism $\phi : F(S) \to G$ such that $\phi u = v$.

Now given a set S, let $F_d(S)$ be the (discrete) free group on the set S with the natural mapping $u : S \to F_d(S)$ being 'inclusion of generators'. Let $\Omega(F_d)$ be the set of all normal subgroups U of $F_d(S)$ such that

(i) $F_d(S)/U$ is finite,

and

(ii) U contains all but a finite number of the $u(s)$, $s \in S$.

Let
$$F(S) = Lim_{U\in\Omega(F_d)} F_d(S)/U.$$
We claim that $F(S)$ is a free profinite group on S in the above sense. This is well known, a version was already given in Serre's notes, [152], so we will sketch the argument only.

Suppose that G is a profinite group and $v : S \to G$ is convergent to 1 in G. As $F_d(S)$ is the free group on S, v extends to a unique homomorphism $\overline{v} : F_d(S) \to G$ satisfying $\overline{v}u = v$.

Suppose that U is an open normal subgroup of G, then we claim $\overline{v}^{-1}(U)$ is in $\Omega(F_d)$. Certainly it is the kernel of the composite
$$F_d(S) \to G \to G/U$$
and G/U is finite, so (i) is satisfied and we only need to check (ii), which however follows since v itself converges to 1.

It is now routine to check that $\overline{v}$ induces a unique continuous $\phi : F(S) \to G$ such that $\phi u = v$.

If S is finite, condition (ii) is trivially satisfied and $F(S)$ is just $F_d(S)$. To remind ourselves of the profiniteness of $F(S)$, we sometimes may write it also as $\hat{F}(S)$.

For an infinite S, the construction given above gives $u : S \to \hat{F}(S)$ and so $\hat{u} : \hat{S} \to \hat{F}(S)$, with S being considered as a discrete topological space, and $\hat{S}$, of course, being its profinite completion. This map $\hat{u}$ is then continuous. This leads to a definition that will often be needed later.

Definition: Let X be a profinite space. A *free profinite group on* X is a profinite group $F(X)$ together with a continuous map $f : X \to F(X)$ satisfying the universal property:

if $h : X \to G$ is any continuous map from X to a profinite group, G, then there exists a unique continuous homomorphism $\phi : F(X) \to G$ such that $\phi f = h$.

A proof of the existence of free profinite groups on all profinite spaces can be manufactured by writing the profinite space as limit of its finite quotients and then using the above constructions to construct an inverse system of free profinite groups whose limit is the desired construct. Alternatively an elegant proof can be found in Magid's paper, [115], Proposition 7.

Many results on free groups generalise directly to free profinite groups, due to the above universal property in part, but not all do so. For instance the Nielsen-Schreier theorem states that any subgroup of a free group is free, but any profinite group is compact and Hausdorff as a topological group, so open subgroups of a free profinite group should not even be expected to be profinite for fairly trivial reasons. Worse than this however is true. It is not the case even that closed subgroups of free profinite groups are necessarily free, see, for instance, Lubotsky and Van den Dries, [109]. Because of this it is necessary to take some care with generalisation of the theory of presentations to give profinite presentations of profinite groups. Working with a class $\mathcal{C}$ and hence with free pro-$\mathcal{C}$ groups, again one gets variations depending on what $\mathcal{C}$ is. Some of these will be used later on and we will then give a more detailed description of their behaviour.

Definition: Let $F(X)$ be a free profinite group on a profinite space, X and let $N(R)$ be the closed normal closure of a closed subspace, R, of $F(X)$. If G is a profinite group continuously isomorphic to the quotient group, $F(X)/N(R)$ then $(X : R)$ is called a *profinite presentation* of G.

If $N(R)$ is itself free as a profinite group, then we will say that $(X : R)$ is a *free profinite presentation* of G.

Remarks: (i) Any profinite presentation of a finite group can be replaced by a free one.

(ii) There are pro-p presentations of pro-p groups and more general pro-$\mathcal{C}$ ones of the groups from these varieties. We will use these without further mention when necessary.

One final result on free profinite groups is the following that is sometimes useful :

Proposition 6. *Let, as before, $F_d(S)$ be the free group on a finite set, S, and let $\hat{F}(S)$ be a free profinite group on S. If $\phi : F_d(S) \to \hat{F}(S)$ is the natural continuous morphism, then $\phi(F_d(S))$ is dense in $\hat{F}(S)$.* ■

The proof is easy and is omitted.

1.5 Group and groupoid objects in a category.

If $\mathcal{D}$ is an arbitrary category having finite products, then one can formulate what it means for an object M to have a group structure. Letting T be the terminal object of $\mathcal{D}$, we have to specify

1. a multiplication map
$$\mu : M \times M \to M;$$
2. a map
$$e : T \to M,$$
thought of as 'picking out the multiplicative identity' and
3. an inversion map
$$(\ \_\ )^{-1} : M \to M.$$

(In $Sets$, T is a singleton set, so e is determined by an element of M.)

This structure has to satisfy certain conditions. These can be specified in terms of commutativity conditions on various diagrams, e.g., for instance saying that

$$\begin{tikzcd} M \cong T \times M \arrow[r, "e \times M"] \arrow[dr, "="'] & M \times M \arrow[d, "\mu"] \\ & M \end{tikzcd}$$

commute, just says "$\mu(e,m) = m$", or, more conventionally, $e.m = m$, i.e., that e is a left identity for multiplication. (These diagrams can be found in many books on category theory and so will be omitted here.) Group objects in $\mathcal{D}$ form a category, $Gr(\mathcal{D})$. An alternative view is that the functor $\mathcal{D}(\_\,, M)$ from $\mathcal{D}^{op}$ to $Sets$ will be naturally group valued with multiplication given by the natural transformation

$$\mathcal{D}(\_\,, \mu) : \mathcal{D}(\_\,, M) \times \mathcal{D}(\_\,, M) \cong \mathcal{D}(\_\,, M \times M) \to \mathcal{D}(\_\,, M)$$

and the identity and inverse similarly induced from e and $(\ \_\ )^{-1}$ using, for the first, the fact that $\mathcal{D}(\_\,, T)$ is a singleton set. It is also clear what an Abelian group object in $\mathcal{D}$ should be. The category of Abelian group objects in $\mathcal{D}$ will be denoted by $Ab(\mathcal{D})$.

For example, a group object in the category of profinite spaces is exactly a profinite group. Later we will need to consider other case of this idea, and so give another type of example. A group object in the category of groups itself is just an Abelian group. We leave as an exercise a proof of this using the well known Eckmann-Hilton argument. An interesting observation is that the structure of internal group objects in the category of groupoids is not so simple. These 'group-groupoids' can equivalently be definied using the cat^1-group formulation that we will considering in section 5.1.1.

Such group-groupoids can also be considered as groupoid objects in the category of groups. We will be needing other examples of such 'internal groupoids' such as profinite groupoids, which are groupoids in the category of profinite spaces, so we will introduce them in general. We first need some subsidiary definitions, that will be useful in their own right.

Definition: Let $\mathcal{D}$, again, be a category. A *directed graph object* in $\mathcal{D}$ consists of an object, A, of $\mathcal{D}$ called the *object of arrows* (or *directed edges*) and an object, V, of $\mathcal{D}$ called the *object of vertices* together with two morphisms

$$s, t : A \to V$$

of $\mathcal{D}$, which are called the *source* and *target* morphisms.

The directed graph object is *reflexive* if, in addtion, there is a morphism

$$i : V \to A$$

such that $si = ti = id_V : V \to V$.

This morphism is thought of as assigning an 'identity' arrow to each vertex and as such arrows are loops their source and targets are equal. Of course there is no composition in our setting as yet so 'identity' is just a convenient label. We thus have a diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "s"] \arrow[r, shift left=0.5, "t"'] & V \arrow[l, shift left=1.5, "i"] \end{tikzcd}$$

with $si = ti = id_V$.

Example: If $\mathcal{D}$ has a terminal object, T and V is that object, then as the 'hom-set' $\mathcal{D}(A, V)$ is a singleton set, we must have $s = t$ and i makes A into a *pointed object*. For instance, any group object in $\mathcal{D}$ gives a pointed object by forgetting the multiplication map, μ.

For the next definition, leading up to that of an internal category, we need $\mathcal{D}$ to have finite limits so we can form a pullback.

Definition: Let $\mathcal{D}$ be a category with finite limits. Given a reflexive directed graph in $\mathcal{D}$, the object C_2 given by the pullback

$$\begin{tikzcd} C_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & A \arrow[d, "s"] \\ A \arrow[r, "t"'] & V \end{tikzcd}$$

is called the *object of composable pairs of arrows*.

We may write $C_2 = A\ {}_t\!\times_s A$, or more inexactly $A \times_V A$ for this pullback over V. In the case $\mathcal{D} = Sets$, C_2 is constructed as a subset of $A \times A$ consisting of those pairs of arrows, (a_1, a_2), where $t(a_1) = s(a_2)$, which explains the terminology. This object is the key to defining an internal category in $\mathcal{D}$, as, in a category, composition is 'partial', i.e., not necessarily defined for all pairs of arrows, just for those that match up their target and source, i.e., which are composable.

We now change notation slightly and will replace V by C_0, and A by C_1. This is to emphasise the various levels in an internal category C, so C_0 is the object of objects, C_1 is the object of arrows and C_2 is the object of composable arrows.

Definition: An *internal category* C in $\mathcal{D}$ consists of data (displayed in the diagram

$$\begin{tikzcd} C_2 \arrow[r, shift left=2, "p_1"] \arrow[r, "m"] \arrow[r, shift right=2, "p_2"'] & C_1 \arrow[r, shift left=2, "s"] \arrow[r, "t"'] & C_0 \arrow[l, shift left=3, "i"] \end{tikzcd}$$

with C_2 the object of composable pairs of arrows of the reflexive directed graph given by (C_1, C_0, s, t, i):
C_0 is the *object of objects* of C;
C_1 is the *object of arrows* of C;
s is the *domain* or *source* morphism;
t is the *codomain* or *target* morphism;
i is the *'identity'* morphism;
and
m is the *composition* or *multiplication* morphism.
These are to satisfy various well known axioms such as

$$sm = sp_1, \quad tm = tp_1$$

i.e., the domain of the composite of two arrows is the domain of the first, etc., and other axioms usually displayed diagrammatically, such as that giving associativity of composition:

First form the limit, C_3, of the diagram

$$\begin{tikzcd} C_3 \arrow[rr, "p_3"] \arrow[dr, "p_2"] \arrow[dd, "p_1"'] & & C_1 \arrow[d, "s"] \\ & C_1 \arrow[r, "t"] \arrow[d, "s"] & C_0 \\ C_1 \arrow[r, "t"'] & C_0 & \end{tikzcd}$$

As we have $sm = sp_1$, etc., the projections define two induced maps from C_3 to $C_2 \,{}_t\!\times_s C_1$ and to $C_1 \,{}_t\!\times_s C_2$ and, composing these with $m \times id$ and with $id \times m$ gives two maps to C_2. The associativity axiom says. of course, that

$$m(m \times id) = m(id \times m).$$

A full set of diagrammatic axioms can be found in Borceux's 'Handbook', [15–17], or Borceux and Janelidze, [18], section 7.1. There is an obvious notion of morphism or internal functor between internal categories in $\mathcal{D}$ and the resulting category will be denoted by $Cat(\mathcal{D})$.

An internal category C in $\mathcal{D}$ will be an *internal groupoid* if, in addition, there is an inversion morphism, $r : C_1 \to C_1$ satisfying $sr = t$, $r^2 = id$ and the composites

$$C_1 \xrightarrow{\Delta} C_1 \times C_1 \xrightarrow{r\times id} C_1 {}_t\times_s C_1 \xrightarrow{m} C_1$$

and

$$C_1 \xrightarrow{t} C_0 \xrightarrow{i} C_1,$$

are equal.

As we would in a *Set* based situation, we will write a^{-1} for $r(a)$, then this last condition expresses that the composite of a^{-1} with a is the identity on the target of a. Again we refer to Borceux and Janelidze, [18], for a detailed discussion.

The category of internal groupoids in $\mathcal{D}$ will be denoted $Grpd(\mathcal{D})$.

Examples: 1. In $\mathcal{D} = Sets$, here an internal category is just a small category and an internal groupoid the type of structure we have called a groupoid (as most of the examples are small by nature).

2. For $\mathcal{D} = Prof$, an internal groupoid is what Magid, [115], for example, has called a *profinite groupoid*. '*A profinite groupoid is a groupoid whose sets of objects and morphisms are profinite (spaces) and the functions assigning range and domain to morphisms and identity morphisms to objects are continuous*,' ([115], p.502). If the space of objects is a singleton, it is just a profinite group.

3. For $\mathcal{D} = Grps$, the category of groups, then an internal category is automatically an internal groupoid and is an example of the structure sometimes called a '*group-groupoid*' or also a '*(strict) 2-group*'. The category $Cat(Grps)$ is thus the same as $Grpd(Grps)$, and, in fact, is equivalent to the categories of cat^1-groups and crossed modules that we will meet shortly. A discussion of these equivalences and especially the first of them, is given in Brown-Spencer, [32]).

An important identity known as the 'interchange law' helps, perhaps, to explain the structure of an internal category in this context. If we have an internal category, C, within *Grps*, then the object of composable pairs C_2 is a group and the multiplication $m : C_2 \to C_1$ is a group homomorphism. Writing $m(c, c')$ in infix notation as $c\#_1 c'$ and writing $\#_0$ for the multiplication in the group C_1, the statement that m is a homomorphism is equivalent to stating that, for all $a, b, c, d \in C_1$,

$$(a\#_0 b)\#_1(c\#_0 d) = (a\#_1 c)\#_0(b\#_1 d),$$

whenever either side is defined. This is known as the *interchange law*. It is an interesting exercise to show that this implies that the subgroup $[Ker\, s, Ker\, t]$ of C_1 is trivial and conversely if this subgroup is trivial for some internal reflexive directed graph in *Grps*, then there is an internal composition naturally defined on that graph making it into an internal category. This $[Ker\, s, Ker\, t] = 1$ condition is the key to the cat^1-group definition of Loday, [106], that we will be examining in some detail starting in section 5.1.1.

Remark: For certain purposes 'finiteness' is not so natural a condition to impose and unfortunately, topological inverse limits outside the setting of compact spaces are less well behaved. A suitable replacement is to replace 'spaces' by 'locales'. Intuitively locales are open set lattices associated to spaces, but there is no necessity for a locale to be 'spatial' in that way. Taking 'prodiscrete localic groupoids' and thus, essentially, limits within tge category of internal groupoids in the category of locales, gets around many of the difficulties when finiteness in not 'natural" or is not 'available'. This is discussed in Borceux and Janelidze, [18]. The context there is non-Galoisian Galois theory and the Joyal-Tierney Galois theory of Grothendieck toposes, see their references, but localic groupoids have been applied in other settings even nearer to our interests than that, and they may warrant further research along parallel lines to the profinite, and hence spatial, case that we will be considering.

We end this discussion with an important example of particular type of profinite groupoid, namely the action groupoid of a group action, when the group and the space on which it acts are both profinite. (Again this notion can be internalised to suitable classes of categories, but we will not formally do this.) We start by looking at the classical discrete case, then the profinite version is simple to give.

Definition: Given a group G and a set X, a *left action* of G on X is a function

$$G \times X \to X$$

$$(g, x) \mapsto g \cdot x,$$

(or sometimes ${}^g x$, if this is more appropriate), such that (i) if $g_1, g_2 \in G$ and $x \in X$,

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x,$$

and (ii) $1_G \cdot x = x$ for $x \in X$. (Of course, 1_G, as usual, denotes the identity element of G.)

The *action groupoid of such a group action* is the groupoid having X as its object set and in which the set of morphisms is $G \times X$ with source given by the projection onto X and the target by the action, so the arrow (g, x) goes from x to $g \cdot x$. The identity function sends x to $(1_G, x)$.

The profinite case is obtained by requiring G to be a profinite group, X to be a profinite space and the action to be continuous. The resulting action groupoid is naturally a profinite groupoid, which will naturally be called the *profinite action groupoid* of the *profinite action*.

1.6 Enriched categories - just a taster

There is a variant notion of profinite groupoid in which the collection of objects need not be a profinite space, this being merely demanded of each 'hom-set' $G(X, Y)$. One of the best examples of this is the profinite groupoid $\widehat{Braid}$ introduced by Drinfel'd, [47], (cf. Jarvis, [93], for an introduction). This combines various ideas that will be useful later, but will not be needed in great detail

The example centres on the idea of a braided monoidal category. Categories such as categories of Abelian groups, modules, etc., have a tensor product that allows one to multiply objects or morphisms together in such a way that the result behaves a bit like an internal monoid in the 'category of categories'. A similar structure occurs if a category has finite products. The product of two objects defines the multiplication. Of course, products are only determined 'up to isomorphism' so associativity and identities are not satisfied 'on the nose'. Abstracting this gives the notion of a monoidal category, i.e., a category $\mathcal{C}$, with a multiplication $\otimes$ defined on it, so one can form $A \otimes B$ or $f \otimes g$ for objects A, B or morphisms f, g. There is a unit object I, an associativity isomorphism, and so on, and the whole structure has to satisfy some axioms that are fairly obvious as they are lax versions of the monoid axioms. For instance the associativity isomorphism is a natural isomorphism,

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C),$$

and has to satisfy a pentagon axiom related to the different ways of going from $((A \otimes B) \otimes C) \otimes D$ to $A \otimes (B \otimes (C \otimes D))$. These axioms can be found in many books on category theory so will not be given here. The result is called a *monoidal category.* So a monoidal category is a bit like a monoid in the category of categories, except that no requirement is made for it to be a small category.

A monoidal category is *symmetric* if there are natural isomorphisms $\tau_{A,B} : A \otimes B \cong B \otimes A$ (that is, a sort of commutativity isomorphism), which are to be compatible with the other structural isomorphisms and are such that $\tau_{B,A}\tau_{A,B} = Id_{A \otimes B}$. (We will be needing symmetric monoidal categories briefly in a short while.) Again the axioms can be found in MacLane, [113], or Borceux's second volume, [16], so we will not give them here.

We can form a symmetric monoidal category on a single generating object by building a category $Symm$ having the non-negative integers as objects and with

$$Symm(m, n) = \begin{cases} \emptyset & \text{if } m \neq n \\ S_n & \text{if } m = n \end{cases}$$

so it is the disjoint union of the various symmetric groups, S_n, indexed by the non-negative integers. The 'tensor product' is given by addition on the objects.

It often occurs that, although there are natural isomorphisms $\tau_{A,B} : A \otimes B \cong B \otimes A$, the second condition, that is, $\tau_{B,A}\tau_{A,B} = Id_{A \otimes B}$, need not be satisfied, although the first is. The result is then called a *braided monoidal category*. Again there is a beautifully simple description of the free singly generatoed braided monoidal category. It is $Braid$ with object set, again, the non-negative integers and

$$Braid(m, n) = \begin{cases} \emptyset & \text{if } m \neq n \\ Br_n & \text{if } m = n \end{cases}$$

where Br_n is the braid group on n strands. We will look at presentations of the braid groups in some detail later on. Drinfel'd's use was of a profinite version of this, $\widehat{Braid}$, in which each braid group is profinitely completed, but the set of objects is not changed.

This is typical of a slightly different form of adding structure such as profiniteness to a (small) category and we will meet this in various contexts in later chapters.

The idea of an internal category was based on the 'category as graph plus composition' paradigm, but this is not the only one possible. We can, and often do, view a category, $\mathcal{C}$, having X as its collection of objects, as a doubly indexed collection of sets $\{\mathcal{C}(A,B)|A,B \in X\}$, together with composition maps

$$C^B_{A,C} : \mathcal{C}(A,B) \times \mathcal{C}(B,C) \to \mathcal{C}(A,C),$$

and identity element that we will think of as being maps $1_A : \{*\} \to \mathcal{C}(A,A)$, satisfying the usual well known axioms. In an enriched category, we have some background category $\mathcal{B}$, and a collection X of 'objects'. For each pair $A,B \in X$, we have an object $\mathcal{C}(A,B)$ of $\mathcal{B}$, a sort of $\mathcal{B}$-enriched hom-set'. In addition to make things work we will need $\mathcal{B}$ to be a symmetric monoidal category, so we can form

$$\mathcal{C}(A,B) \otimes \mathcal{C}(B,C)$$

for objects $A,B,C \in X$. We also want identities, $1_A : I \to \mathcal{C}(A,A)$, where I is the unit of the monoidal structure. This enriched category structure is to satisfy some axioms and again we will omit these. They are the careful result of adapting the axioms for a category (associativity and identities) to a setting in which, for instance, $(\mathcal{C}(A,B) \otimes \mathcal{C}(B,C)) \otimes \mathcal{C}(C,D)$ is not equal to $\mathcal{C}(A,B) \otimes (\mathcal{C}(B,C) \otimes \mathcal{C}(C,D))$, merely isomorphic to it in a controlled and coherent way.

Examples: (i) Simplicially enriched categories and groupoids. We will meet these in earnest in the next chapter. Many categories involved in algebraic homotopy have extra structure that means that each $\mathcal{C}(X,Y)$ has the structure of a simplicial set. These are mainly large examples of simplicially enriched categories, but any homotopy type can be represented by a (small) simplicially enriched groupoid.

(ii) The profinite completion, $\widehat{Braid}$, of Drinfel'd's approach to Grothendieck-Teichmuller theory, is a profinitely enriched category. This corresponds to $\mathcal{B} = Prof$.

(iii) Taking $\mathcal{B}$ to be one of the many categories of modules over a commutative ring, k gives another collection of very useful enriched categories, the k-additive categories. For algebraic homotopy theory, categories enriched over various categories of chain complexes, or groupoids, (cf., Gabriel and Zisman, [66]), or more generally crossed complexes (see later) are also useful.

1.7 Profinite completions for groupoids

There are two ways of viewing groupoids, as we have seen. The 'finite' objects in the two cases are different, so there are two related, but distinct, forms of profinite completion for groupoids. Both are useful.

a) Internal groupoids in the category of finite sets form a subcategory of the category of groupoids. this is a proreflexive subcategory, so the inclusion has a left proadjoint, which will be a profinite completion functor. Put more simply, if $Grpd(FinSets)$ is the category of such finite groupoids, with objects for which both the object set and arrow set are finite, then although the inclusion of this into $Grpd = Grpd(Sets)$ does not have an adjoint, extending this to

$$Pro.Grpd(FinSets) \to Pro.Grpd,$$

this extended functor does have a left adjoint (and so a *proadjoint*, cf., for example, [39], p. 48-53) for the original inclusion. This gives an *internal profinite completion*. A pro-p version of this was introduced by Morel in [124].

b) Groupoids enriched over finite sets have each $G(x, y)$ a finite set. These again form a proreflexive subcategory of the category of groupoids and there is a corresponding *enriched profinite completion*. On the subcategory of groupoids with finitely many objects, the two constructions coincide.

In this enriched setting the profinite completion will not 'complete' the object set. For an extreme example, any set X yields a finitely enriched groupoid with each $G(x, x) = \{1_x\}$ and $G(x, y)$ with $x \neq y$ being empty. It is a 'discrete' groupoid in the terminology of groupoid theory. As it is finitely enriched, its enriched profinite completion is itself. Its internal profinite completion is the profinite completion of X, i.e., a profinite space considered as the profinite groupoid $(\hat{X}, \hat{X}, s = t = id_{\hat{X}})$. The Drinfel'd construction is an enriched completion and the enriched completion construction has also been used by Baker, [8], in unpublished work.

We will not be needing to use these completions that much in detail as we will usually be concentrating on the group, rather than the groupoid, based theory for simplicity, however there are some techniques for manipulating groupoids that are not that well known, but are very useful, and are well illustrated in the study of these two forms of profinite completion. We therefore will devote some time to their development.

An extremely useful construction for groupoids is a '*change of objects*' functor. This is described in full detail in Higgin's, [82], and can also be found in Brown, [23]. Suppose G and H are groupoids with object sets $X = Ob(G)$ and $Y = Ob(H)$, respectively. If $f : X \to Y$ is a function, then we can form a groupoid $f^*(H)$ with objects the elements of X by

$$f^*(H)(x, x') = H(f(x), f(x')).$$

Let $Grpd_f(G, H)$ denote the set of groupoid morphisms from G to Hhaving f as their object assignment functions and $Grpd/X$ the category of groupoids with objects set, X and in which morphisms are the identity on objects, then

$$Grpd_f(G,H) \cong Grpd/X(G, f^*(H)),$$

as is easily checked.

More interestingly, $Grpd_f(G,H) \cong Grpd/Y(f_*(G),H)$, for an 'induced groupoid' $f_*(G)$ on the object set, Y. We will not give the formal construction in terms of words (for which see Brown, [23]), but an idea of what is involved can be obtained by considering the example in which G is the 'interval' groupoid $\mathcal{I}$ with object set $X = \{0,1\}$ and with each $\mathcal{I}(x,x')$, a singleton set, and consider $Y = \{*\}$, a single object. The unique mapping, f, from X to Y induces a groupoid on Y, which is the infinite cyclic group, C_∞ (or if you prefer, $\mathbb{Z}$), generated by the image of the element $i : 0 \to 1$ in $\mathcal{I}$. This example shows that although G may be a finite groupoid, the induced groupoid, $f_*(G)$ can be infinite, since as F is not injective, new composites of the old arrows can be formed when objects are 'glued together' in Y.

The 'induction' process induces a functor

$$f_* : Grpd/X \to Grpd/Y,$$

but this functor does not preserve the subcategories of finitely enriched objects. There is a profinite analogue

$$f_*^{Prof} : Prof.Grpd/X \to Prof.Grpd/Y,$$

where one either needs X and Y finite, or to be talking about profinitely enriched groupoids. (We leave the detailed exploration of this for the reader.) In our example above, $f_*^{Prof}(\mathcal{I})$ will be the profinite completion of the infinite cyclic group. This is significant geometrically since it models, classically, the covering spaces of the circle and, for the profinite case, the inverse system of finite covering spaces of the circle. The theory of covering groupoids, as given in Brown, [23], clearly has a profinite analogue.

It has been claimed, above, that $Grpd(Fin) \to Grpd$ has a proadjoint. This is clearly given by taking the limit of all finite quotients of a groupoid, G, but there is one problem. If we denote this by $\tilde{G}$, what do the object and arrow spaces of $\tilde{G}$ look like? The source and target maps tie them together, so we seem not to be free to profinitely complete each part separately without potentially destroying other structure such as the composition. However we can obtain some information on this process:

Proposition 7. *For any groupoid, G, we have a natural isomorphism*

$$Ob(\tilde{G}) \cong \widehat{Ob(G)},$$

the profinite completion of the object set of G.

Proof: We will write (G, X, s, t, i), or similar for a groupoid, if the composition is 'understood'.

There are 'object set' functors O and O_P from $Grpd$ to $Sets$ and from $Prof.Grpd$ to $Prof$, respectively, giving the object of objects in each case, $O(G, X, s, t, i) = X$, etc. These functors have both left and right adjoints, of which we will use the right adjoints here. The right adjoint of O is given by $R(X) = (X \times X, X, p_1, p_2, diag)$, in what is hoped is an evident notation. The right adjoint for O_P is, of course, 'the same'.

We also have the forgetful functors from $Prof$ to $Sets$, and from $Prof.Grpd$ to $Grpd$. These will be denoted U and U_{Grpd} respectively. These have left adjoint, the respective profinite completions, and we will use the same notation as before, with hats and tildes. We note that UO_P and OU_{Grpd} are equal as are $U_{Grpd}R_P$ and RU, since a product profinite space has underlying set the product of the underlying sets.

Next we consider a profinitely completed groupoid $\hat{G}$ and a profinite space Y. We have the following natural isomorphisms:

$$\begin{aligned} Prof(O_P(\tilde{G}), Y) &\cong Prof.Grpd(\hat{G}, R_P(Y)) \\ &\cong Grpd(G, U_{Grpd}(R_P(Y))) \\ &\cong Grpd(G, RU(Y)) \\ &\cong Sets(O(G), U(Y)) \\ &\cong Prof(\widehat{O(G)}, Y), \end{aligned}$$

which gives the result, as the two objects in question represent the same functor. ■

The defining property of a profinite completion in any context is its universal property. In our situation, this is : if G is a groupoid and H is a finite groupoid, then

$$Grpd(G, H) \cong Prof.Grpd(\tilde{G}, H).$$

Here, of course, we have tacitly omitted any notation to distinguish between H as a finite groupoid in $Grpd$ and H as a discrete profinite groupoid. Now suppose $f : G \to H$ is a morphism of groupoids, and we denote the corresponding functions on objects and arrows by f_0 and f_1 respectively:

$$\begin{tikzcd} G_1 \arrow[r, "f_1"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & H_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ G_0 \arrow[r, "f_0"'] & H_0 \end{tikzcd}$$

We have that f factors via $f_{0*}(G)$:

$$G \xrightarrow{\eta} f_{0*}(G) \xrightarrow{f'} H,$$

where η is over f_0, so $\eta_0 = f_0$, whilst f' is a morphism in $Grpd/H_0$. We can restrict attention to the cases in which f' is an epimorphism or quotient. We thus need a description of quotient morphisms of groupoids, but this is readily available in Higgin's monograph, [82] or in [23] and can be given in terms of normal subgroupoids, in general, but as our quotient morphism is the identity on objects the particular type of normal subgroupoid that we need will be very simple.

Definition: Let G be a groupoid. A subgroupoid N of G is called a *normal subgroupoid* if it satisfies:
(i) $Ob(N) = Ob(G)$,
and
(ii) for any $g : x_1 \to x_2$ in G, $gN\{x_2\}g^{-1} \subseteq N\{x_1\}$.
(Here we have adopted the convention that $N\{x\}$ denotes the *vertex group* of N at x, also called the *automorphism group*, thus $N(x) = N(x, x)$.)

Given any normal subgroupoid N of G, we can form a quotient groupoid G/N with arrows equivalence classes under $g_1 \sim_N g_2$ if there are $m, n \in N$ such that $g_2 = m.g_1.n$. The proof that this does give a groupoid is fairly routine. There is a quotient map from G to G/N and, of course, its kernel is N. We have $N = Kerf'$ is a normal subgroupoid of $f_{0*}(G)$. The following should now be clear.

Proposition 8. *a) If G is a groupoid with G_0 finite, then $\tilde{G}$ the limit of the inverse system of quotients by all cofinite normal subgroupoids of G.*
b) In general, for G a groupoid, there is for each finite quotient of G_0, say $f_0 : G_0 \to X$, a profinite groupoid $\widehat{f_{0*}(G)}$ constructed as in a) and $\tilde{G} = Lim_f \widehat{f_{0*}(G)}$. ∎

We thus have a reasonably good description of the 'internal' profinite completion of a groupoid. The discussion also suggests a way of describing the 'enriched' profinite completion.

Suppose $f : G \to H$ and H is a finitely enriched groupoid, i.e., each $H(a, b)$ is a finite set, then the same is true of $f_0^*(H)$ and, dividing out by $Ker(f' : G \to f_0^*(H))$ yields an image of f in $f_0^*(H)$. Taking the inverse limit of all these quotients will give the enriched completion functor. (In fact, the normal subgroupoids can again be chosen to be (cofinally) of the form of a disjoint union of $N(x)$, the only requirement being that G/N be finitely enriched.)

1.8 Free profinite groupoids

We will occasionally need not just free profinite or pro-$\mathcal{C}$ groups, but their many object analogues, i.e., free profinite groupoids. The existence of free profinite categories and groupoids on a profinite directed graph has been investigated by Almeida and Weil, [2].

First some terminology: by a *profinite graph*, we will mean an inverse limit of finite graphs. The graphs will be directed, but need not be reflexive.

Of course, a profinite category is likewise an inverse limit of finite categories (i.e., internal categories in $FinSets$). In more generality, to replace the class $\mathcal{C}$ of 'pro-$\mathcal{C}$' theory, Almeida and Weil use a pseudovariety $\mathcal{V}$.

Definition: A *pseudovariety, $\mathcal{V}$, of categories* is a class of finite categories (in the above 'internal' sense), which is closed under taking 'divisors' and finite products. Here a category C is a *divisor* of D if there is another category E and functors

$$\begin{tikzcd} E \arrow[r, "\text{faithful}"] \arrow[d, "\text{subquotient}"'] & D \\ C & \end{tikzcd}$$

generalising the situation for 'subquotients' of groups. (By 'subquotients' here we mean that the morphism is surjective on 'homs', but injective on objects, i,e, $\phi_{e,e'} : E(e,e') \to C(\phi e, \phi e')$ is always onto, but $\phi : Ob(E) \to Ob(C)$ is injective.) A category or groupoid which is an inverse limit of categories, resp., groupoids, in such a $\mathcal{V}$ is called a *pro-$\mathcal{V}$ category*, resp. *pro-$\mathcal{V}$ groupoid*.

The fact that $\mathcal{V}$ is a pseudovariety means that it is closed under pullbacks and so the category of morphisms from a given groupoid G to objects in $\mathcal{V}$ is filtering and the resulting limit is the *pro-$\mathcal{V}$ completion* of G.

The existence of free pro-$\mathcal{V}$ groupoids on a given profinite graph can be proved along the following standard line. We will however also give a separate proof, as although the intuitive idea for the standard one is a good one, there is a hidden subtle technicality that is better to avoid.

First a *profinite graph*, A, is an inverse limit of finite graphs within the category of all (directed) graphs, given the inverse limit topology in the obvious way. A profinite groupoid G is *A-generated* by a continuous morphism $\phi_G : A \to G$ of profinite graphs, if the smallest closed subcategory of G containing $\phi_G(A)$ is G itself.

For a given pseudovariety, $\mathcal{V}$, the A-generated members of $\mathcal{V}$ form an inverse system and we can take its limit to get the *free pro-$\mathcal{V}$ groupoid*, $F_{\mathcal{V}}(A)$, on A. The problem is to check in detail that the system is not too large. (Readers with a good knowledge of categorical arguments will realise what has to be done, namely to verify a solution set condition.) Because of this we will adopt a slightly different approach, which implicitly checks the solution set condition along the way and also provides a more explicit description of $F_{\mathcal{V}}(A)$.

A general profinite graph A can be written as $A = Lim\, A_i$ for some system $\{A_i : i \in I\}$ of finite graphs. For each A_i, we can form $F_V(A_i)$ as the pro-$\mathcal{V}$ completion of the free category on the graph A_i. The two universal properties, of free category and completions, combine to give that this is the free pro-$\mathcal{V}$ category on the finite graph A_i. Using the functoriality of this construction, we form an inverse system $\{F_{\mathcal{V}}(A_i) : i \in I\}$ with the induced 'bonding' morphisms. We now take the limit of this which we will denote $F_{\mathcal{V}}(A)$. This will be a pro-$\mathcal{V}$ groupoid, but we have to check the universal property.

Suppose G is a groupoid in $\mathcal{V}$ and $\phi : A \to G$ a continuous morphism of profinite graphs. As G is finite, ϕ factors through one of the A_is, say by $\phi_i : A_i \to G$, and this extends uniquely to a morphism from $F_{\mathcal{V}}(A_i)$ to G. As this will be the case, compatibly, for cofinally many A_i, we get a well defined and unique morphism $\overline{\phi} : F_{\mathcal{V}}(A) \to G$ extending ϕ. We have:

Proposition 9. *There is a free pro-$\mathcal{V}$ groupoid generated by any profinite graph.* ■

Remark: It is worth noting that just as (group) varieties correspond to defining words, pseudovarieties correspond to certain 'pseudoidentities' and one needs the theory of free pro-$\mathcal{V}$ categories to obtain a full description; see Almeida and Weil, [2]. A similar idea will play a role later when we give sets of defining equations for variety-like classes of (profinite) homotopy types.

1.9 Pseudocompact Algebras and Modules.

The natural setting for replacing the role of the group algebra of a group, G, or for that of G-modules, when extending structure to a profinite or pro-$\mathcal{C}$ setting would seem to be that of pseudocompact rings and modules. Other choices are used, for instance, Boggi, [14], develops a theory of continuous cohomology for profinite groups using complete totally disconnected R-modules for R a topologically compact unitary ring, but pseudocompact rings and modules, as we will see, do a good job and neatly combine the topological and proobject aspects that we tend to use in a combined and complementary way throughout this book. Pseudocompact modules also seem to arise naturally in the profinite 'crossed' algebraic contexts that we will be introducing, so they are the natural setting for us.

Our main source for the material in this section is Brumer's paper, [34], with some knowledge of Gabriel, [65], being helpful for some generalities on linearly compact rings. For one or two of the results we also use Gildenhuys and MacKay, [71]. As usual as the material is essentially 'well known' here, proofs will sometimes be omitted or sketched.

Definition: A complete Hausdorff topological ring, A, is said to be a *pseudocompact ring* if it admits a system of open neighbourhoods of 0 consisting of two sided ideals, I, for which A/I is an Artin ring, i.e., has the descending chain condition on its two sided ideals.

Of course, here we have a 'ring' not an 'algebra'. If instead of working over $\mathbb{Z}$ and thus essentially over its profinite completion, $\hat{\mathbb{Z}}$, we had used a fixed commutative (pseudocompact) ring k, then we would get the notion of a *pseudocompact algebra*, A.

The example of a pseudocompact ring that we will use continually is the completed group algebra of a profinite group, which we will introduce shortly.

Definition: Let A be a pseudocompact ring. A complete Hausdorff topological A-module M is said to be a *pseudocompact A-module* if it has a system of open neighbourhoods of 0 consisting of submodules N of M for which M/N has finite length.

The category of pseudocompact A-modules and continuous homomorphisms between them will be denoted $Pc.A - Mod$.

If $\phi : A \to B$ is a continuous homomorphism of pseudocompact algebras and $\theta : M \to N$ is a continuous homomorphism of topological Abelian groups such that M is a pseudocompact A-module, and N is similarly one for B, then θ is said to be *compatible* with ϕ, or is a *morphism over* ϕ, if, for all $a \in A$, $m \in M$, $\theta(a.m) = \phi(a).\theta(m)$.

We form a category, $Pc.Mod$, with such pairs (A, M), with A a pseudocompact algebra and M a pseudocompact A-module, as objects and with compatible pairs (ϕ, θ) as the morphisms. This category of pseudocompact modules over varying algebras has the usual type of properties, corresponding to change of rings, such as being fibred and op-fibred over the category of pseudocompact algebras, but we will not be using that structure at that level of abstraction. We will however need the essential induced and restricition constructions on modules which is behind that abstract structure.

1.10 The completed group algebra of a profinite group, *G*.

Let $\hat{\mathbb{Z}}$ be the profinite completion of the ring of integers, $\mathbb{Z}$, then $\hat{\mathbb{Z}}$ is itself a pseudocompact ring as it is the inverse limit of its *finite* quotients. Now let G be a profinite group.

Definition: The *complete group algebra,* $\hat{\mathbb{Z}}[\![G]\!]$, of G over $\hat{\mathbb{Z}}$ is the inverse limit of the ordinary group algebras, $\hat{\mathbb{Z}}[G/U]$ of the finite quotients, G/U for $U \in \Omega(G)$ over $\hat{\mathbb{Z}}$;

$$\hat{\mathbb{Z}}[\![G]\!] = Lim_{U \in \Omega(G)} \hat{\mathbb{Z}}[G/U].$$

This is a pseudocompact ring. It can also be written as an inverse limit of finite group algebras over the finite quotient rings of $\mathbb{Z}$. There are pro-$\mathcal{C}$ variants, which will be denoted, $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$ in general, and $\hat{\mathbb{Z}}_p[\![G]\!]$, in the pro-$p$ case. In general given an arbitrary pseudocompact ring A, we can form $A[\![G]\!]$ in a similar way, based on the $A[G/U]$. In particular, we may want to use this with $A = \hat{\mathbb{Z}}_{\mathcal{C}}$, the pro-$\mathcal{C}$ completion of $\mathbb{Z}$ and for G, a pro-$\mathcal{C}$ group, so that all the G/U will be in $\mathcal{C}$.

There is an obvious notion of pseudocompact $\hat{\mathbb{Z}}[\![G]\!]$-modules and we get a category that we will denote by $Pc.G - Mod$.

Example: The Magnus algebra. Let p be a prime and $\hat{\mathbb{Z}}_p$, as before, be the pseudocompact ring of p-adic integers. For any positive integer, n, let $F(n)$ denote the free pro-p group on n generators, $x_1, \ldots, x_n$.

Consider the algebra, $A(n)$, of formal power series in non-commuting indeterminates, $t_1, \ldots, t_n$ with coefficients in $\hat{\mathbb{Z}}_p$. This algebra, $A(n)$, with a topology given by convergence of coefficients, is compact. The multiplicative group U of elements of $A(n)$, which have constant term equal to 1 forms a pro-p group, and contains all the elements $1+t_i$. Associating x_i to $1+t_i$ gives an isomorphism between $F(n)$ and U, which extends to one between $\hat{\mathbb{Z}}_p[\![F(n)]\!]$ and $A(n)$, see Serre, [152], or Lazard, [102]. The algebra, $A(n)$, is sometimes called the *Magnus algebra on n indeterminates*, and it, and its associated Lie algebras have been used extensively in the study of pro-p groups.

1.11 Some generalities on Pseudocompact modules

Throughout let A be a fixed pseudocompact ring.

Proposition 10. *The category $Pc.A-Mod$ is an Abelian category with exact projective limits.* ■

Most of this is well known or even ‘routine’. The exactness of projective limits follows the same lines as well known and classical results on the exactness of projective limits on systems of linearly compact modules.

The section lemma (Proposition 5) has an analogue for pseudocompact modules:

Proposition 11. *Let $f : M \to N$ be an epimorphism of pseudocompact A-modules, then there is a continuous section $s : N \to M$ such that $fs(n) = n$ for all $n \in N$.* ■

The proof is easy to adapt from that earlier ‘section lemma’.

One use of this is in the proof of the important:

Theorem 2. *The category $Pc.A - Mod$ has enough projectives.* ■

The proof, given in Brumer, [34], proceeds by proving that the free pseudocompact A-module of a set, S, exists. We will need that, but it is better to go for a more general result. First let us give an obvious definition.

Definition: Let X be a profinite space. A *free pseudocompact A-module on X* is a pair $(A^{(X)}, h)$, where $A^{(X)}$ is a pseudocompact A-module, and $h : X \to A^{(X)}$ is a continuous function such that if M is any pseudocompact A-module and $g : X \to M$ is continuous, then there is a unique continuous homomorphism $f : A^{(X)} \to M$ such that $fh = g$.

Proposition 12. *The free pseudocompact A-module, $(A^{(X)}, h)$, exists for any profinite space X.*

The proof given above for the existence of free profinite groups on profinite spaces can be imitated here, (cf., Haran, [81]), however we will give an alternative more direct form, adapted from Magid's proof, [115], of the profinite group existence theorem.

Proof: Let $A(X)$ denote the free A-module on the underlying set of X. Let $\mathcal{N}$ be the set of all submodules N of $A(X)$ such that (i) $A(X)/N$ has finite length and (ii) for any $a \in A(X)$, $(a+N) \cap X$ is an open-closed subset of X. Partial order $\mathcal{N}$ by inverse inclusion. Let $A^{(X)} = Lim\{A(X)/N | N \in \mathcal{N}\}$. The compatible maps, for $N \in \mathcal{N}$, $X \to A(X) \to A(X)/N$ induce a continuous map $h : X \to A^{(X)}$. Next suppose M is any pseudocompact A-module and $g : X \to M$ is continuous, and that $M = Lim\, M_i$ with M_i of finite length. By the universal property of free modules, the composite $X \to M \to M_i$ factors through $A(X)$ as $X \to A(X) \to M_i$. Set $K = Ker(A(X) \to M_i)$, then $A(X)/K$ has finite length and for any $a \in A(X)$, $(a+K) \cap X$ is an open-closed subset of X. The rest is now fairly routine. ∎

Of course, since we have enough projective pseudocompact modules, we can form projective resolutions of any pseudocompact module and this releases a large part of homological algebra for our use. Generally we will assume an elementary knowledge of standard homological algebra.

Later we will be needing the torsion groups, Tor_n^G, and also will be generalising tensor products of modules to tensor products of groups, both in the discrete and profinite settings. We will thus need a 'completed tensor product' of pseudocompact modules, not only for comparison but as a means to extract information from the new constructions.

Let A, as usual, be a pseudocompact algebra over a commutative pseudocompact ring k, M a right and N a left pseudocompact A-module.

Definition: The *completed tensor product* of M and N is a pseudocompact k-module, $M\hat{\otimes}_A N$, and a continuous A-bilinear morphism $\alpha : M \times N \to M\hat{\otimes}_A N$ with the following universal property:

given any continuous A-bilinear morphism, $\beta : M \times N \to C$, where C is a pseudocompact k-module, (so $\beta(ma, n) = \beta(m, an)$ for all $a \in A, m \in M, n \in N$), there is a unique continuous k-module morphism

$$g : M\hat{\otimes}_A N \to C$$

such that $g\alpha = \beta$.

Proposition 13. *(Brumer, [34], p.446) For any pair M, N, as above, a completed tensor product $M\hat{\otimes}_A N$ exists.*

Proof: We set $M\hat{\otimes}_A N = Lim\,(M/U \otimes_A N/V)$, where the U (resp. the V), are open submodules of M (resp. N). Since M/U and N/V are k-modules of finite length, so is their tensor and so $M\hat{\otimes}_A N$ is pseudocompact as a k-module.

The natural k-bilinear morphisms from $M \times N$ to $M/U \otimes_A N/V$ induce the desired bilinear $\alpha : M \times N \to M\hat{\otimes}_A N$, upon passing to the limit. ■

The usual results on $\otimes$ generalise without difficulty to the completed tensor. The category of two sided pseudocompact A-modules with $\hat{\otimes}_A$ forms a symmetric monoidal category, and using projective resolutions of either M or N, we can define the left derived functors, $Tor_n^A(M,N)$, of the functor $T(M,N) = M\hat{\otimes}_A N$.

We will need this Tor several times, but of even more importance will be the use of the change of rings functors, that is, the construction of restricted and induced modules and their variants.

Change of rings: Suppose $f : A \to B$ is a continuous morphism of pseudocompact k-algebras.

Definition: a) Suppose N is a pseudocompact B-module. We denote by $f^*(N)$, the pseudocompact A-module with underlying compact Abelian group, N, and with continuous A-action given by $a.n = f(a).n$. We say $f^*(N)$ is obtained by restricting along f.

b) Suppose M is a pseudocompact A-module, then $f_*(M)$ denotes the pseudocompact B-module given by $f_*(M) = B\hat{\otimes}_A M$, analogously to the classical discrete case.

These constructions are examples of a type we will see many times, and have, in fact, already met with the change of object set / induction construction for groupoids in section 1.7, so we will make one or two observations.

Let $Hom_f(M,N)$ denote the set of continuous k-module morphisms, ϕ, from a pseudocompact A-module, M to a pseudocompact B-module N, satisfying : for all $a \in A$, $\phi(a.m) = f(a).\phi(m)$.

Proposition 14. *There are natural isomorphisms*

$$Pc.A - Mod(M, f^*(N)) \cong Hom_f(M,N) \cong Pc.B - Mod(f_*(M), N).$$

■

The proof is well known and routine.

One can consider a category $Pc.Mod$ of pseudocompact modules over all pseudocompact algebras, and thus consisting of pairs (A,M) with A a pseudocompact k-algebra and M a pseudocompact A-module. Morphisms from (A,M) to (B,M) consist of pairs (f,ϕ), as above, so $\phi(a.m) = f(a).\phi(m)$. The above results show, after a little more work, that the obvious functor from $Pc.Mod$ to $Pc.k{-}Alg$ is a (bi)fibration of categories. The features that we will use of this are mirrored in the other 'restriction - induction' contexts that we will encounter.

Finally we should note that if A is a pro-$\mathcal{C}$ Abelian group, and G is a pro-$\mathcal{C}$ group which acts continuously on A, then A is naturally a pseudocompact $\hat{\mathbb{Z}}_p[\![G]\!]$-module, and conversely, see Gildenheys and Mackay, [71], for instance.

2

Algebraic homotopy theory: preliminaries.

The title of this monograph involves the words ‘algebraic homotopy’, but what is this? In this chapter we will try to give some of the tools of this subject, but also to sketch some sort of overview of how it relates to other terms such as ‘homotopical algebra’ and ‘abstract homotopy theory’. An overview of the area has been published, [141], showing how its relation to abstract homotopy theory and to the ‘Grothendieck programme’. It is also involved centrally with the modelling of homotopy types and this aspect is well introduced by Baues’ contribution to the Handbook of Algebraic Topology, [11]. Of course, Baues has written extensively on the area, see, for instance, [9], and [10]. We will, later on, use various ‘crossed’ objects, ‘crossed modules’, ‘crossed squares’, ‘crossed complexes’ etc. and will, in general, introduce these thoroughly, however as some of that material is in the forthcoming book by Brown, Higgins and Sivera, [28], sometimes the easier verifications will be ‘left to the reader’. As usual in these preliminary chapters, we will sketch results rather than always providing proofs, and may occasionally refer forward to a detailed treatment in later chapters.

2.1 Algebraic Homotopy and Algebraic Models for Homotopy Types

We start by quoting J.H.C.Whitehead, who can be thought of as the founding father of Algebraic Homotopy, cf. [166].

> The ultimate aim of algebraic homotopy is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that ‘analytic’ is equivalent to ‘pure’ projective geometry.
>
> J.H.C.Whitehead, [166], (quoted in Baues, [9]).

A statement of the aims of ‘algebraic homotopy’ might thus include the following homotopy classification problem (from the same source, J.H.C.Whitehead, [166]):

Classify the homotopy types of polyhedra, X, Y, $\ldots$, by algebraic data.
Compute the set of homotopy classes of maps, $[X, Y]$, in terms of the classifying data for X, Y.

These aims, but with the enlargement of the class of objects of study to include many other types of 'spaces', are still valid for the standard form of algebraic homotopy. One looks for a category of algebraic objects whose structure accurately mirrors aspects of the homotopy category of, in this classical case, polyhedra or CW-complexes. Slightly more exactly, one searches for a nice "algebraic" category $\mathcal{A}$ together with a functor or functors

$$F : \mathit{Spaces} \to \mathcal{A}$$

and an algebraically defined notion of homotopy, $\simeq$, in $\mathcal{A}$ such that
a) if $X \simeq Y$ in *Spaces*, then $F(X) \simeq F(Y)$ in $\mathcal{A}$;
b) if $f \simeq g$ in *Spaces*, then $F(f) \simeq F(g)$ in $\mathcal{A}$,
and F induces an equivalence of homotopy categories

$$Ho(\mathit{Spaces}) \simeq Ho(\mathcal{A}).$$

Here, classically, *Spaces* is a category of topological spaces such as polyhedra or CW-complexes, but the aims are equally valid for objects in other related categories such as those of simplicial sets, their profinite completions, or simplicial groups, which are already partially 'combinatorised' or 'algebraised'.

The title of Whitehead's paper, [164], was 'Combinatorial homotopy II' and he was inspired in part by the example of combinatorial group theory as developed by Reidemeister and others in the 1930s. The algebra he introduced, and which in part we will be using, was often applicable both in homotopy and in group theoretic contexts. From there to group cohomology, and homological algebra is only a short distance and we will explore some of these links in the profinite case.

The 'combinatorial' aspect lead to an idea for building both the 'algebras' and the 'modelising functor'.

Ideal Scenarios: i) If we know how a space X is constructed from simpler objects (e.g. from 'cells' or 'simplices') and if we know F on these simple objects, then we can 'calculate' $F(X)$ completely, (e.g. not just up to extensions of groups or its analogue in $\mathcal{A}$). For this to be the case, we could do with a result of the form of the *van Kampen Theorem* which allows one to build the homotopy information by decomposing the space:

Recall (cf., for instance, Brown, [23]), the van Kampen theorem says that if $X = A \cup B$, A, B are open, then

$$\begin{tikzcd} \Pi_1(A \cap B) \arrow[r] \arrow[d] & \Pi_1(A) \arrow[d] \\ \Pi_1(B) \arrow[r] & \Pi_1(X) \end{tikzcd}$$

is a pushout of groups (or groupoids).

(For the group version, one needs $A \cap B$ arcwise connected; for the groupoid version $A \cap B$ is a union of arcwise connected components and one bases the groupoids on at least one point in each component, cf. Brown, [23]).)

Here we will work simplicially, often deriving our algebraic models from simplicial groups. In that setting the role of the van Kampen Theorem is replaced by combinatorial and algebraic arguments.

ii) It is to be hoped that the 'algebra' in $\mathcal{A}$ reflects the 'geometry' in the spaces. This is evident in the fundamental groupoid where the algebraic composition of path classes is defined via a geometric construction.

iii) Ideally the detailed 'homotopy structure' of Top, $Poly$, or our current favourite category of 'spaces' will be reflected in $\mathcal{A}$, ..., but this leads us to a basic question: what does 'detailed homotopy structure' mean? In fact: *What should be meant by 'homotopy'?*

These 'ideal scenarios' are highly unrealistic. There are however various ideas that may reduce the problem to more manageable proportions.

We could:

(a) restrict at least some of the spaces being considered using geometric properties, e.g. having dimension $\leq n$ (cf. Whitehead, [165] or more recently Baues, [9–11]);

or

(b) find a model which models fully only certain homotopy types (typically those having some condition such as $\pi_i(X) = 0$ if $i > n$);

or

(c) find a model that classifies *all* spaces and maps, but up to a weaker relation than homotopy, (e.g. up to n-equivalence, cf. Whitehead [164], but beware the definition of n-equivalence will be slightly different in more recent work).

The specific examples of these strategies are, of course, not the only ones possible, but they have the merit of being linked in the idea of n-type, (Fox, [59], Whitehead, [164], Loday, [106], etc.). The idea is that n-equivalence measures information detectable with maps coming from polyhedra of dimension $\leq n$, so the $\pi_i(X)$ for $i > n$ do not have as much significance for this notion of equivalence. Each n-equivalence class of spaces ($= n$-type) has a representative X with $\pi_i(X) = 0$ for $i > n$, so here the three ideas are strongly linked.

Examples: (For simplicity, assume that X is a connected CW-complex or polyhedron.)

$n = 1$: the fundamental group $\pi_1(X)$ or groupoid $\Pi_1(X)$, completely models the 1-type of X, classifies maps from 1-dimensional complexes into X and also classifies covering spaces of X.

Before we try to go to higher values of n, we need some more notation and terminology.

We write X^n for the union of the i-cells for $i \leq n$. 'Recall' that if (X, A) is a pair of spaces, with $A \subset X$, $x_0 \in A$, then $\pi_n(X, A, x_0)$ is the n^{th} relative homotopy group of (X, A). It consists of homotopy classes of maps from an n-cube I^n into X that map all but one face of I^n to x_0 and the remaining face into A. The detailed description will not be needed here, but can be found in most books on homotopy theory. Restricting the maps to the last face gives a homomorphism

$$\partial : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0).$$

We can now handle the case $n = 2$:

MacLane and Whitehead, [114], showed that the algebraic structure of

$$\partial : \pi_2(X, X^1, x_0) \to \pi_1(X^1, x_0)$$

models the 2-type of X (Their 3-type is our 2-type - the terminology has changed in the years since their work was published.)

The structure referred to is that of a crossed module (see below, section 3.1.1). We note that
(i) $Ker\,\partial \cong \pi_2(X)$;
(ii) $Im\,\partial \lhd \pi_1(X^1)$
and
(iii) $Coker\,\partial \cong \pi_1(X)$,
so the usual invariants π_1 and π_2 can be found from this data.

2.2 What is a Homotopy Theory?

To set up an algebraic homotopy theory, we first need briefly to ask what is a homotopy theory and to give some examples that will be useful later on. We will not give details. In general the viewpoint adopted will be a mix of that from the overview article, [141], the monograph by Kamps and Porter, [95], and the ideas of Baues, [9–11].

A brief list of contexts for a homotopy theory might include Spaces, Groupoids, Simplicial Sets, Simplicial Objects in other categories, Chain Complexes and Small Categories.

In more detail:

Spaces: There is perhaps no real need for explanation here, but there are various points worth making that can serve as an introduction to some of the ideas that come later. We consider a 'suitable' category, Top, of topological spaces and continuous maps. Homotopy between maps, denoted $\simeq$, is defined by maps from cylinders, $X \times I$, or alternatively to cocylinders, Y^I, in the usual way. (The cocylinder Y^I is the space of paths in Y and may not exist in some categories of spaces that we might want to use.)

Following the usual convention, we put $[X, Y] = Top(X, Y)/\simeq$. The category $Top/\simeq$ has spaces as objects but these $[X, Y]$ as sets of morphisms.

Homotopy types correspond to isomorphism classes in $Top/\simeq$. The maps in Top which give isomorphisms in $Top/\simeq$ are called *homotopy equivalences.*

There are special classes of maps called *cofibrations* and *fibrations.* These are defined by the *homotopy extension* and *homotopy lifting properties*, respectively. For example if $f : X \to Y$ is a cofibration then given any homotopy $H : X \times I \to Z$ and any map $g : Y \to Z$ such that $H|X \times \{0\} = gf$, there is a homotopy $K : Y \times I \to Z$ extending H and starting with g, i.e., $K(f \times I) = H$ and $K|Y \times \{0\} = g$. We may also say that f satisfies the *homotopy extension property.*

Our final points on Top relate to pointed spaces and homotopy groups. The structures here are well known but will serve to introduce the terminology 'weak equivalence'. A *pointed space* is a pair (X, x_0), where $x_0 \in X$. 'Pointed maps' of pointed spaces, $f : (X, x_0) \to (Y, y_0)$, send x_0 to y_0 and give a category Top_0. Homotopies between such 'pointed' maps are assumed to be constant on these base points. For $n = 0$, S^0 is the 0-sphere, that is, a two point space $\{-1, 1\}$ and we write $\pi_0(X, x_0) = [(S^0, 1), (X, x_0)]$. It is the set of arcwise connected components of X, pointed at the component corresponding to x_0. This notation π_0 is also used in the unpointed case. In general we write $\pi_n(X, x_0) = [(S^n, 1), (X, x_0)]$. For $n \geq 1$, $\pi_n(X, x_0)$ has a natural group structure and for $n \geq 2$, this structure is Abelian. A map $f : X \to Y$ of spaces is called a *weak equivalence* if $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection and for all $n \geq 1$ and all possible basepoints $x_0 \in X$, $\pi_n(f) : \pi_n(X) \to \pi_n(Y)$ is an isomorphism. Any homotopy equivalence is a weak equivalence but the converse does not hold in general. For 'locally nice' spaces such as polyhedra and more generally for CW-complexes, the two concepts coincide and any weak equivalence between such spaces has a homotopy inverse.

Groupoids: We have already looked at these. They are small categories in which all morphisms are invertible and groups correspond to the special case in which there is only one object. As examples, there are the fundamental groupoids of spaces, see, for instance, [23].

The groupoid I consists of two objects 0 and 1, their identities, and two morphisms $\iota : 0 \to 1$, $\iota^{-1} : 1 \to 0$. We met it previously in the section on profinite groupoids. Homotopy of groupoids can be defined by a cylinder $G \times I$ or by a 'cocylinder' H^I, since, if G and H are groupoids, there is a natural isomorphism

$$Grpd(G \times I, H) \cong Grpd(G, H^I).$$

This 'cocylinder' is just the category of functors from I to H. It is easily seen to be a groupoid.

Simplicial Sets: These form an extremely useful category in which to do homotopy theory and we will use them and related structures a lot later on. The basic theory can be found in the first half of the survey article by Curtis, [42], for fuller treatments, see May, [117], or Gabriel-Zisman, [66], and for a modern viewpoint, the book by Goerss and Jardine, [73].

Let $[n] = \{0 < 1 < \ldots < n\}$, considered as an ordered set or as a small category. Looked at for small values of n, it is clear why it is considered as a categorical simplex.

$$e.g. \qquad [2] = \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

Writing Cat for the category of small categories and Δ for the full subcategory of Cat determined by the objects, $[n]$ for $n \geq 0$, a *simplicial set* is a functor $K : \Delta^{op} \to Sets$ and $\mathcal{S} = Simp.Sets$ is just the notation for $Sets^{\Delta^{op}}$, the category of contravariant functors from Δ to $Sets$ and all natural transformations between them. A simplicial set K is often written diagrammatically as

$$\cdots\cdots \; K_2 \; K_1 \; K_0.$$

The maps $d_i : K_n \to K_{n-1}$, $0 \leq i \leq n$ are called the *face maps*, the maps $s_i : K_n \to K_{n+1}$, $0 \leq i \leq n$ are called the *degeneracies*. The d_i and s_i satisfy the "simplicial identities".

$$d_i d_j = d_{j-1} d_i \qquad \text{if } \ i < j,$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } \quad i < j, \\ id & \text{if } i = j \quad \text{ or } j+1, \\ s_j d_{i-1} & \text{if } i > j+1, \end{cases}$$

$$s_i s_j = s_j s_{i-1} \qquad \text{if } \quad i > j.$$

As a simple example of a simplicial set, we can take $\Delta[n] = \Delta(-, [n])$, the 'simplicial' n-simplex. This is generated by the identity map on $[n]$, which we denote by ι_n. Given any simplicial set, K and $x \in K_q$, there is a unique simplicial map, $\mathbf{x} : \Delta[q] \to K$, defined by $\mathbf{x}(\iota_q) = x$. Because of this, many arguments in simplicial set theory depend on the structure of examples such as $\Delta[n]$, $\Delta[p] \times \Delta[q]$, and so on.

The link with spaces comes via the topological originals of these 'simplicial' n-simplices. Let Δ^n be the topological n-simplex, represented by

$$\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1\}.$$

There are maps $\delta_i : \Delta^n \to \Delta^{n+1}$ given by

$$\delta_i(t_0, \ldots, t_n) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n),$$

for $0 \leq i \leq n$, which insert Δ^n as the i^{th} face of Δ^{n+1}. Similarly let $\sigma_i : \Delta^n \to \Delta^{n-1}$ be the map

$$\sigma_i(t_0, \dots, t_n) = (t_0, \dots, t_{i-1} + t_i, \dots, t_n).$$

Let X be a space and set $Sing(X)_n = Top(\Delta^n, X)$. The compositions with the δ_i and σ_i yield face and degeneracy maps that make $Sing(X)$ a simplicial set, the *singular complex* of X.

The singular complex construction gives a functor

$$Sing : Top \to S.$$

This has a left adjoint $|\quad| : S \to Top$ called *geometric realisation*:

$$Top(|K|, X) \simeq S(K, Sing(X))$$

and the natural map

$$|Sing(X)| \to X$$

is a weak equivalence.

It is sometimes the case that within a category of spaces, Top, one can form mapping spaces Y^X, but in any case provided that for each n, Δ^n is in our chosen category, Top, and for all X in Top, $X \times \Delta^n$ is there as well, we can form something that plays the rôle of $Sing(Y^X)$ namely $\underline{Top}(X, Y)$, where $\underline{Top}(X, Y)_n = Top(X \times \Delta^n, Y)$. Similarly within $\mathcal{S}$ itself, we can form 'simplicial mapping spaces', $\underline{\mathcal{S}}(K, L)_n = \mathcal{S}(K \times \Delta[n], L))$. These extra structures are examples of simplicial enrichment of the basic categories Top and $\mathcal{S}$. We will briefly consider the relevance of this type of structure later.

We can apply the same sort of construction to small categories. We saw that $[n] = \{0 < 1 < \dots < n\}$ can be considered either as an ordered set or a small category. If C is a small category or a groupoid, we can form a simplicial set, $Ner(C)$, defined by $Ner(C)_n = Cat([n], C)$, with the obvious face and degeneracy maps induced by composition with the analogues of the δ_i and σ_i. The simplicial set, $Ner(C)$, is called the *nerve of the category* C. An n-simplex in $Ner(C)$ is a sequence of n composable arrows in C. We will explore the structure of such nerves later on.

If we are to use simplicial sets adequately in 'algebraic homotopy', then we clearly need an 'adequate' notion of homotopy. Given two simplicial sets K and L, we can form their product $K \times L$ by setting $(K \times L)_q = K_q \times L_q$, and, with face and degeneracy maps applied 'componentwise', i.e. $d_i(x, y) = (d_i x, d_i y)$, $s_j(x, y) = (s_j x, s_j y)$. This does give a (categorical) product within the category $\mathcal{S}$.

Because of this, the obvious way to define a homotopy between two simplicial maps $f_0, f_1 : K \to L$ is to use the 'cylinder' $K \times I = K \times \Delta[1]$ First let $e_0(K) : K \to K \times I$ send x to $(x, 0$, and e_1 similarly send x to $(x, 1)$. The picture should be clear, but a bit more precision will make life easier later on. The simplicial 'unit interval' $I = \Delta[1]$ is generated by a 1-simplex $\iota_1 \in I_1 = \Delta[1]_1 = \Delta(1, 1)$. There are two 0-simplices, which we denote $0 = d_1\iota_1$ and $1 = d_0\iota_1$. These generate degenerate q-simplices $s_0^{(q)}0$

and $s_0^{(q)}1$, which we will usually denote by 0 and 1 for simplicity. The maps e_0 and e_1 are thus more properly written $e_i(x) = (x, s_0^{(q)}i)$ if $x \in K_q$ and for $i = 0, 1$.

Definition: Given two maps $f_0, f_1 : K \to L$, a *homotopy* between them is a simplicial map

$$h : K \times I \to L$$

such that $f_i = he_i$, for $i - 0, 1$. The two maps f_0 and f_1 are then said to be *homotopic*, written $f_0 \simeq f_1$.

Unfortunately in this fairly elementary and unstructured setting homotopy is not an equivalence relation on the set, $\mathcal{S}(K, L)$. We refer the reader to standard texts and articles on simplicial sets for a fuller discussion of this.

The non-degenerate simplices of $K \times I$ include some which are componentwise degenerate: if $x \in K_q$, and $t \in I_q$, then $s_i x, s_j t)$ will in general be non-degenrate. Since x is the image of $\iota_q \in \Delta[q]_q$, we can determine a homotopy h by its values when composed with each $\mathbf{x} \times Id_I : \Delta[q] \times \Delta[1] \to K \times \Delta[1]$. Any $(q+1)$-simplex of $\Delta[q] \times \Delta[1]$ has the form of a path from $(0, 0)$ to $q, 1)$ through a $q \times 1$ lattice, which follows the bottom until it goes vertically one step, then continues to the end at $(q, 1)$: $\tau_i^q = (0, 1, 2, \ldots, i, i, i+1, \ldots, q) \times (0, 0, 0, \ldots, 0, 1, 1, \ldots, 1)$. The picture

$$\begin{array}{ccccccccc}
(0,1) & \text{———} & (1,1) & \text{———} & \cdots \text{———} & (q-1,1) & \text{———} & (q,1) \\
| & & | & & & | & & | \\
(0,0) & \text{———} & (1,0) & \text{———} & \cdots \text{———} & (q-1,0) & \text{———} & (q,0)
\end{array}$$

may help. For example, $(0, 0) - (1, 0) - (1, 1) - (2, 1) - (3, 1)$ gives a 4-simplex in $\Delta[3] \times \Delta[1]$.

Thus to specify h we need to specify the mappings $h_i^q + K_q \to L_{q+1}$, where $h_i^q(x) = h(\mathbf{x} \times id)(\tau_i^q)$, for each x and $i = 0, 1, \ldots, n$. Of course, these maps have to satisfy various relations. This gives an alternative definition of homotopy:

A homotopy h from f_0 to f_1 is a system of mappings

$$((h_i)_{0 \le i \le q} : K_q \to L_{q+1})_{q \ge 0},$$

which satisfy the following set of homotopy identities

1. $d_0 h_0 = f_1, \quad d_{q+1} h_q = f_0$;
2. $d_i h_j = \begin{cases} h_{j-1} d_i & i < j, \\ h_j d_{i-1} & i > j+1, \end{cases}$
   and $d_{j+1} h_{j+1} = d_{j+1} h_j$;
3. $s_i h_j = \begin{cases} h_{j+1} s_i & i \le j, \\ h_j s_{i-1} & i > j. \end{cases}$

We can see that this form of the definition allows us an immediate generalisation of the notion of homotopy to categories of simplicial objects in other settings, as, for instance, to get a workable definition of homotopy between morphisms of 'simplicial groups', (see below), we merely have to add the condition that the various h_is are group homomorphisms.

Before we make an attempt to answer the question posed at the outset of this section, we need to look at a class of other example categories in more detail.

2.3 Simplicial Groups

2.3.1 Simplicial Objects in Categories other than *Sets*.

If $\mathcal{A}$ is any category, we can form $Simp.\mathcal{A} = \mathcal{A}^{\Delta^{op}}$. (Sometimes we will use a variant notation: $Simp(\mathcal{A})$, as occasionally the first notation may be ambiguous.)

These categories often have a good notion of homotopy as briedfly discussed above; see also the discussion of simplicially enriched categories in [95]. Of particular use are:

(i) *Simp.Ab*, the category of simplicial Abelian groups. This is equivalent to the category of chain complexes by the Dold-Kan theorem, see later, section **??**, for a version of this.

(ii) *Simp.Grps*, the category of simplicial groups. (This 'models' all connected homotopy types, by Kan, [97] (cf., Curtis, [42]).) There are adjoint functors $G : S_{conn} \to Simp.Grps$, $\overline{W} : Simp.Grps \to S_{conn}$, with the two natural maps $G\overline{W} \to Id$ and $Id \to \overline{W}G$ being weak equivalences. Later we will examine results on simplicial groups by Carrasco, [35], that generalise the Dold-Kan theorem to the non-Abelian case, (cf., Carrasco and Cegarra, [36]).

(iii) '*Simp.Grpd*': in 1984 Dwyer and Kan, [49], (and also Joyal and Tierney, and Duskin and van Osdol, cf., Nan Tie, [130, 131]) noted how to generalise the $(G, \overline{W})$ adjoint pair to handle all simplicial sets, not just the connected ones. (Beware there are several important printing errors in the paper [49].) For this they used a special type of simplicial groupoid. Although the term used in [49] was exactly that, 'simplicial groupoid', this is really a misnomer and may give the wrong impression as not all simplicial objects in the category of groupoids are used. A probably better term would be 'simplicially enriched groupoid', as we will explain slightly later, see section 2.6.1. We will denote this category by $\mathcal{S}-Grpds$. This category 'models' all homotopy types using a mix of algebra and combinatorial structure. We will shortly describe both G and $\overline{W}$ in some detail.

(iv) *BiSimp*: We can take $\mathcal{A} = \mathcal{S}$ and get simplicial objects in the category of simplicial sets, i.e. bisimplicial sets, or, with $\mathcal{A} = Simp.Grps$, we get $BiSimp(Grps)$, the category of bisimplicial groups. These are less scarry than one might think. The usual sort of manipulation for functions (or here functors) of two variables identifies their objects as being of the form $X : \Delta^{op} \times \Delta^{op} \to \mathcal{A}$, and so their study depends on the combinatorics of the objects of $\Delta \times \Delta$. We will only be using them once or twice right at the end of the book. There are, of course, n-simplicial sets, etc., being the n-variable version of these.

(v) *Nerves of internal categories:* Suppose that $\mathcal{D}$ is a category with finite limits and C is an internal category in $\mathcal{D}$. In our earlier discussion in section 1.5, we had the diagram

$$\begin{tikzcd} C_2 \arrow[r, shift left=2, "p_1"] \arrow[r, "m"] \arrow[r, shift right=2, "p_2"'] & C_1 \arrow[r, shift left=2, "s"] \arrow[r, "t"'] & C_0 \arrow[l, shift left=2, "i"] \end{tikzcd} .$$

We also encountered an object C_3 in discussing the associativity of composition. If one interprets this for $\mathcal{D} = Sets$, it becomes clear that this diagram is part of the diagram specifying the nerve of the small category, C, with C_0 the set of objects, C_1 that of morphisms, C_2 that of composable pairs and so on. We have not specified the two degeneracies from C_1 to C_2 in the diagram, but this is merely because we left the details of the rules governing identities out of our discussion. Of course, this does not depend on working in *Sets* and we could work in an arbitrary $\mathcal{D}$, with an arbitrary internal category C therein, having C_0 as object of objects, and C_1 as object of arrows. We could then build a simplicial object in $\mathcal{D}$ as follows: take

$$C_n = \underbrace{C_1 \times_{C_0} C_1 \times_{C_0} C_1 \times_{C_0} \ldots \times_{C_0} C_1}_{n},$$

define face and degeneracies by the same sort of rules as in the set based nerve, that is, in dimension n, d_0 and d_n each leave out an end, whilst the d_i use the multiplication in the category to get a composite of two adjacent 'arrows', and the degeneracies are 'insertion of identities'. (Working out how to do these morphisms in terms of diagrams is quite fun!) We thus get a simplicial object in $\mathcal{D}$ called the *nerve of the internal category, C*, We will use this in several situations later in a key way. In particular we will use the case $\mathcal{D} = Grps$.

2.3.2 Homotopies for simplicial groups

If we have two simplicial morphisms $f_0, f_1 : G \to H$ between simplicial groups, it is easy to adapt the second, alternative, definition of homotopy to this richer setting merely by requiring that the $h_i : G_q \to H_{q+1}$ all be homomorphisms, in addition to satisfying the simplicial identities of that definition. This gives a good working definition which will adapt, without difficulty, to profinite simplicial groups or the 'simplicial groupoids' that we mentioned slightly earlier.

There is another way of approaching such homotopies that can also be useful. Suppose G is a simplicial group and K ne a simplicial set. For convenience we will assume each K_q is finite. We form a new simplicial group $G\overline{\otimes}K$ by setting

$$(G\overline{\otimes}K)_q = \coprod_{K_q} G_q,$$

the K_q-fold copower of G_q. The elements of $(G\overline{\otimes}K)_q$ are words in symbols $g \otimes x$, for $g \in G_q$ and $x \in K_q$ and the only relations are of the form

$$(g_1 \otimes x)(g_2 \otimes x) = (g_1 g_2 \otimes x),$$

for $g_1, g_2 \in G_q$. The face maps are given by $d_i(g \otimes x) = d_i(g) \otimes d_i(x)$ and the degeneracies by $s_j(g \otimes x) = s_j(g) \otimes s_j(x)$. We leave the reader to show that a homotopy h as above, yields a simplicial group morphism $h : G\overline{\otimes}\Delta[1] \to H$ satisfying the obvious conditions, and conversely, from such a simplicial morphism, we would get a family of maps (h_i), as before. In other words this construction acts as a cylinder on G.

2.3.3 The Moore complex and the homotopy groups of a simplicial group

Given a simplicial group G, *the Moore complex*, (NG, ∂), *of* G is the chain complex defined by

$$NG_n = \bigcap_{i=1}^{n} Ker\, d_i^n$$

with $\partial_n : NG_n \to NG_{n-1}$ induced from d_0^n by restriction.

The n^{th} *homotopy group*, π_n(G), of G is the n^{th} homology of the Moore complex of G, i.e.,

$$\begin{aligned}\pi_n(G) &\cong H_n(NG, \partial),\\ &= \big(\bigcap_{i=0}^{n} Ker\, d_i^n\big)/d_{n+1}^{n+1}\big(\bigcap_{i=0}^{n} Ker\, d_i^{n+1}\big).\end{aligned}$$

The interpretation of NG and $\pi_n(G)$ is as follows:

for $n = 1,\ g \in NG_1$,

$$\partial g \bullet \xrightarrow{\ g\ } \bullet 1$$

and $g \in NG_2$ looks like

$$\begin{tikzcd} & \bullet \arrow[dr, "1"] \arrow[d, phantom, "g"] & \\ \bullet \arrow[ur, "\partial g"] \arrow[rr, "1"'] & & \bullet \end{tikzcd}$$

and so on.

We note that $g \in$ NG$_2$ is in $Ker\, \partial$ if it looks like

$$\begin{tikzcd} & \bullet \arrow[dr, "1"] & \\ \bullet \arrow[ur, "1"] \arrow[rr, "1"'] & g & \bullet \end{tikzcd}$$

whilst it will give the trivial element of $\pi_2(G)$ if there is a 3-simplex x with g on its third face and all other faces identity.

This simple interpretation of the elements of NG and $\pi_n(G)$ will 'pay off' later by aiding interpretation of some of the elements in other situations.

Definition: A simplicial group, G, is *augmented* by specifying a constant simplicial group $K(G_{-1}, 0)$ and a surjective group homomorphism, $f = d_0^0 : G_0 \to G_{-1}$ with $fd_0^1 = fd_1^1 : G_1 \to G_{-1}$. An augmentation of the simplicial group G is then a map

$$G \longrightarrow K(G_{-1}, 0),$$

or more simply $f : G_0 \longrightarrow G_{-1}$. An augmented simplicial group, (G, f), is *acyclic* if the corresponding complex is acyclic, i.e., $H_n(NG) \cong 1$ for $n > 0$ and $H_0(NG) \cong G_{-1}$.

2.4 Kan complexes and Kan fibrations.

Within the category of simplicial sets, there is an important subcategory determined by those objects that satisfy the Kan condition, that is the *Kan complexes*.

As before we set $\Delta[n] = \Delta(-, [n]) \in S$, then, for each i, $0 \leq i \leq n$, we can form, within $\Delta[n]$, a subsimplicial set, $\Lambda^i[n]$, called the (n, i)-*horn* or (n, i)-*box* by discarding the top dimensional n-simplex (given by the identity map on $[n]$) and its i^{th} face. We must also discard all the degeneracies of those simplices.

By an (n, i)-*horn* or *box* in a simplicial set K, we mean a simplicial map $f : \Lambda^i[n] \to K$. Such a simplicial map corresponds intuitively to a family of n simplices of dimension $(n-1)$, fitting together to form a 'funnel' or 'empty horn' shaped subcomplex within K. Of course, the simplicial map may send different simplices in $\Lambda^i[n]$ to the same image, but that is no bother. The idea is that a Kan fibration of simplicial sets is a map in which the horns in the domain can be 'filled' if their images in the codomain can be. More formally:

Definition: A map $p : E \to B$ is a *Kan fibration* if, for any n, i as above, given any (n, i)-horn in E, specified by a map $f_1 : \Lambda^i[n] \to E$, together with an n-simplex, $f_0 : \Delta[n] \to B$, such that

$$\begin{tikzcd} \Lambda^i[n] \arrow[r, "f_1"] \arrow[d, "inc"'] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "f_0"'] & B \end{tikzcd}$$

commutes, then there is an $f : \Delta[n] \to E$ such that $pf = f_0$ and $f.inc = f_1$, i.e., f lifts f_0 and extends f_1.

Definition: A simplicial set, K, is a *Kan complex* if the unique map $K \to \Delta[0]$ is a Kan fibration. This is equivalent to saying that every horn in K has a filler, i.e., any $f_1 : \Lambda^i[n] \to Y$ extends to an $f : \Delta[n] \to Y$.

Singular complexes, $Sing(X)$, and the simplicial mapping spaces, $\underline{Top}(X, Y)$, are always Kan complexes. The nerve of a category, C, is a Kan complex if and only if the category is a groupoid. If G is a simplicial group, then its underlying simplicial set is a Kan complex. Moreover, and this is of importance later on, given a box in G, there is an algorithm for filling it using products of degeneracy elements.

A form of this algorithm is given below and it is discussed in Kamps and Porter, [95], where the reason why it works is explored. More generally if $f : G \to H$ is an epimorphism of simplicial groups, then the underlying map of simplicial sets is a Kan fibration. This will have significant implications later on as it is natural to choose the identity element in a simplicial group as the base point of its underlying simplicial set, so the fibre of a fibration corresponds precisely to the kernel of the epimorphism.

The following description of the algorithm is adapted from May's monograph, [117], page 67.

Proposition 15. *Let G be a simplicial group, then every box has a filler.*

Proof: Let $(y_0, \ldots, y_{k-1}, -, y_{k+1}, \ldots, y_n$ give a box in G_{n-1}, so the y_is are $(n-1)$ simplices that fit together as if they were all but one, the k^{th} one, of the faces of an n-simplex. There are three cases:

(i) $k = 0$: Let $w_n = s_{n-1}y_n$ and then $w_i = w_{i+1}(s_{i-1}d_iw_{i+1})^{-1}s_{i-1}y_i$ for $i = n, \ldots, 1$. The w_1 satisfies $d_iw_1 = y_i$, $i \neq 0$;
(ii) $0 < k < n$: Let $w_0 = s_0y_0$ and $w_i = w_{i-1}(s_id_iw_{i-1})^{-1}s_iy_i$ for $i = 0, \ldots, k-1$, then take $w_n = w_{k-1}(s_{n-1}d_nw_{k-1})^{-1}s_{n-1}y_n$, and finally a downwards induction given by $w_i = w_{i+1}(s_{i-1}d_iw_{i+1})^{-1}s_{i-1}y_i$, for $i = n, \ldots, k+1$, then w_{k+1} gives $d_iw_{k+1} = y_i$ for $i \neq k$;
(iii) the third case, $k = n$ uses $w_0 = s_0y_0$ and $w_i = w_{i-1}(s_id_iw_{i-1})^{-1}s_iy_i$ for $i = 0, \ldots, n-1$, then w_{n-1} satisfies $d_iw_{n-1} = y_i$, $i \neq n$. ■

2.5 Again: ‘ What is a homotopy theory?’

In each of these settings, and in many more, one has a notion of equivalence (weak equivalence, quasi-isomorphism or homotopy equivalence depending on the context). Suppose C is a category with a collection, Σ, of maps called *weak equivalences* (no properties are thought of as being attached to the name, for the moment it is just a name). We can form a new category $Ho(C) = C(\Sigma^{-1})$ by ‘formally inverting’ the morphisms in Σ. We do this by taking for each $f \in \Sigma$, say $f \in C(X, Y)$, a new symbol f^{-1} and we add it into $C(Y, X)$. We then form composite words in the old arrows together with all these new ‘inverses’ and if we ever see a pair, ff^{-1} or $f^{-1}f$, we cancel it out, (see Gabriel and Zisman, [66], for a proper description of this process). The general construction of this category $C(\Sigma^{-1})$, is sometimes called the *category of fractions of C with respect to Σ. we also say it is obtained by* formally inverting *the arrows in Σ, hence the notation.*) The resulting category comes with a functor $\gamma : C \to Ho(C)$ with the nice universal property that if $\alpha : C \to D$ is any functor such that for all $f \in \Sigma$, $\alpha(f)$ is an isomorphism in D, then α factors uniquely through γ, i.e., there is a unique $\overline{\alpha} : Ho(C) \to D$ such that

$$\begin{tikzcd} C \arrow[rr, "\alpha"] \arrow[dr, "\gamma"'] & & D \\ & Ho(C) \arrow[ur, "\overline{\alpha}"'] & \end{tikzcd}$$

commutes. Here this category will be called the *homotopy category* of C (and often we will miss out mention of Σ), adopting special notation for special cases.

We note that using this construction, one can prove, for instance, that $Ho(Top) \simeq Ho(\mathcal{S}-Grpd)$ for a suitable definition of weak equivalence in $\mathcal{S}-Grpd$ (cf. Dwyer and Kan [49]), and that is a very significant step towards a possible algebraic homotopy theory as Whitehead proposed.

Common structure in the examples? There are various interacting structures and therein lies the problem in deciding exactly what is an ‘abstract homotopy theory’. We note various attempts to encode at least part of that structure.

a) **Quillen**: [142–144] This is one of the most widespread of the structures, so has often been considered as the basic abstract homotopy theory to use. It considers a category C with infinite limits and colimits and three classes of morphisms called *weak equivalences*, *fibrations* and *cofibrations*, whose behaviour, and, in particular, whose interaction, is governed by various axioms. (We do not give them here as they can be found in many sources in the literature.) The origins of the work may be found in deformation theory and the need for a cohomology of commutative algebras (cf. Quillen, [144]). Once developed, Quillen used it highly successfully in [143] to produce new results on rational homotopy theory.

One may criticise this approach from various viewpoints, whilst still acknowledging its great importance. For instance, the weak equivalences, etc. are given right from the start and no guidance is given how these classes might arise, or once ‘given’ might be interpreted. Thus weak equivalences are often given by ‘external’ information (e.g., $f : K \to L$ in $\mathcal{S}$ is defined to be a weak equivalence if $|f|$ is a weak equivalence in Top). This may not help one when interpreting the notions geometrically. Linked to this is the fact that several important ideas such as that of fundamental groupoid, homotopy limits and even homotopy equivalence are ‘out of place’ in the original theory and later variants have amended the basic definition to ‘correct’ these deficiencies.

b) **Kan**: [96], and **Kamps**: see [95] and the references therein to both Kan and Kamps. Here the ‘primitive’ idea is that of abstracting the structure of the functor ‘X goes to $X \times [0, 1]$’ used as the basis for topological homotopy theory. Dually one can use ‘X goes to X^I’ when that exists.

Let $\mathcal{C}$ be a category. A *cylinder functor* on $\mathcal{C}$ is a functor $I : \mathcal{C} \to \mathcal{C}$ together with natural transformations $e_0, e_1 : Id_{\mathcal{C}} \to I$ and $p : I \to Id_C$ with $pe_0 = pe_1 = Id$. This defines a notion of homotopy in an obvious way, and hence a relation on the sets, $\mathcal{C}(X, Y)$, etc. This relation need <u>not</u> be an equivalence relation, but one can still form $\mathcal{C}(\Sigma^{-1})$ for Σ the class of homotopy equivalences.

This theory has been more than adequately described in Kamps and Porter [95] and so will not be given again here. Under various condition on the cylinder, it allows for the definition of classes of homotopy equivalences and cofibrations that satisfy a weakened form of Quillen’s axioms that are due to K. Brown, [20]. Baues has a similar theory, as follows:

c) **Baues**, [9]: Baues uses interacting structures, one of Quillen type (or rather of K. Brown’s version of half of Quillen’s theory) and the other of cylinder functor type. The two structures are called *cofibration categories* and I-categories.

Cofibration category; ($\mathcal{C}$, *cof*, *w.e.*): i.e., a category $\mathcal{C}$ with two classes of morphisms, *cof* of *cofibrations* and *w.e.* of *weak equivalences*, satisfying four axioms, see Baues, [9].

I-category ($\mathcal{C}$, cof, I, $\emptyset$): Here $\mathcal{C}$ is a category, ‘cof’ is a class of ‘cofibrations’, $\emptyset$ is the initial object of $\mathcal{C}$ and I is a ‘cylinder functor’.

These are required to satisfy some axioms, again given in [9]. The axioms are generally intuitive and are easy to use, giving, after a reasonable amount of work:

Theorem 3. *(Baues, [9])If ($\mathcal{C}$, cof, I, $\emptyset$) is an I-category and we let w.e. be the class of homotopy equivalences with repect to I, then ($\mathcal{C}$, cof, w.e.) is a cofibration category.*

Baues develops a large segment of homotopy theory in this setting and gives a large number of examples. He then goes on to get deep new results on the homotopy theory of spaces via this abstract homotopy approach, (see also Baues, [10, 11]).

2.6 Simplicial background

In addition to the introductory material above, we will need some more background on simplicial thoery. This also gives a good opportunity to set up notation and terminology. Additional material on simplicial theory, i.e., beyond the 'classical' sources of Curtis, [42], May, [117], and Gabriel and Zisman, [66], can be found in books such as that by Goerss and Jardine, [73].

2.6.1 Simplicially enriched categories and groupoids.

First a word about 'enrichment'. We have met internal categories and groupoid above, and also group objects in a category. Simplicial groups are group objects in $\mathcal{S}$, for instance. They have another aspect namely 'simplicially enriched groupoids with one object'.

As was mentioned in the previous chapter, in many situations we have a category whose 'hom-sets' have additional structure. For instance, in a category of modules, the set of morphisms between modules all have the structure of an Abelian group. We say the category of modules is 'enriched' over the category of Abelian groups. Similarly we saw earlier, section 2.2, that both Top and $\mathcal{S}$ itself were 'simplicially enriched'. We will briefly revisit this next.

Categories with simplicial 'hom-sets': We assume that we have a category $\mathcal{A}$, whose objects will be denoted by lower case letter, x, y, z, $\dots$, at least in the generic case, and for each pair of such objects, (x, y), a simplicial set $\mathcal{A}(x, y)$ is given; for each triple x, y, z of objects of $\mathcal{A}$, we have a simplicial map, called *composition*

$$\mathcal{A}(x, y) \times \mathcal{A}(y, z) \longrightarrow \mathcal{A}(x, z);$$

and for each object x a map

$$\Delta[0] \to \mathcal{A}(x, x)$$

that 'names' or 'picks out' the 'identity arrow' in the set of 0-simplices of $\mathcal{A}(x, x)$. This data is to satisfy the obvious axioms, associativity and identity, suitably adapted to this situation. Such a set up will be called a *simplicially enriched category* or more simply *an $\mathcal{S}$-category*. Enriched category theory is a well established branch of category theory. It has many useful tools and not all of them have yet been exploited for the particular case of $\mathcal{S}$-categories and its applications in homotopy theory. Here are the two main standard examples again plus some more:

Examples (i) $\mathcal{S}$, the category of simplicial sets:
here

$$\underline{\mathcal{S}}(K,L)_n := \mathcal{S}(\Delta[n] \times K, L);$$

Composition : for $f \in \underline{\mathcal{S}}(K,L)_n$, $g \in \underline{\mathcal{S}}(L,M)_n$, so $f : \Delta[n] \times K \to L$, $g : \Delta[n] \times L \to M$,

$$g \circ f := (\Delta[n] \times K \stackrel{diag \times K}{\longrightarrow} \Delta[n] \times \Delta[n] \times K \stackrel{\Delta[n]\times f}{\longrightarrow} \Delta[n] \times L \stackrel{g}{\to} M);$$

Identity : $id_K : \Delta[0] \times K \stackrel{\cong}{\to} K$,

(ii) Top, 'the' category of spaces (of course, there are numerous variants but you can almost pick whichever one you like as long as the constructions work there):

$$\underline{Top}(X,Y)_n := Top(\Delta^n \times X, Y)$$

Composition and identities are defined analogously to in (i).

(iii) For each X, $Y \in Cat$, the category of small categories, then we similarly get $\mathcal{C}at(X,Y)$,

$$\mathcal{C}at(X,Y)_n = Cat([n] \times X, Y).$$

We leave the other structure up to the reader.

(iv) $\mathcal{C}rs$, the category of crossed complexes: see [28] and below, for background, and Tonks, [160], for a more detailed treatment of the simplicially enriched category structure;

$$\mathcal{C}rs(A,B) := Crs(\pi(n) \otimes C, D)$$

Composition has to be defined using an approximation to the identity, again see [160].

(v) $\mathcal{C}h^+_K$, the category of positive chain complexes of modules over a commutative ring K. (Details are left to the reader, or follow from the Dold-Kan theorem and example (vi) below.)

(vi) $Simp.Mod_K$, the category of simplicial K-modules. The structure uses tensor product with the free simplicial K-module on $\Delta[n]$ to define the 'hom' and the composition, so is very much like (i).

Notational remark: It is sometimes convenient to put the 'product with I' on the other side, giving homotopies and their higher order analogues in the form $h : K \times \Delta[n] \to L$, etc. As this is merely notational we will do this, when needed, without further comment. Of course the two formulations are completely equivalent.

In general any category of simplicial objects in a 'nice enough' category has a simplicial enrichment, although the general argument that gives the construction does not always make the structure as transparent as it might be.

There is an evident notion of $\mathcal{S}$-enriched functor, so we get a category of 'small' $\mathcal{S}$-categories, denoted $\mathcal{S}{-}Cat$. Of course, none of the above examples are 'small'.

We also have a notion of simplicially enriched groupoid, bearing the same relationship to $\mathcal{S}$-categories as ordinary groupoids do to categories.

We have denoted the category of simplicial sets, as before, by $\mathcal{S}$ and that of simplicially enriched groupoids by $\mathcal{S}-Grpd$. This latter category thus includes that of simplical groups, but it must be remembered that a simplicial object in the category of groupoids will, in general, have a non-trivial simplicial set as its 'object of objects', whilst in $\mathcal{S}-Grpd$, the corresponding simplicial object of objects will be constant. This corresponds to a groupoid in which each collection of 'arrows' between objects is a simplicial set, not just a set, and composition is a simplicial morphism, hence the term 'simplicially enriched'. We will often abbreviate the term 'simplicially enriched groupoid' to $\mathcal{S}$-groupoid, but the reader should again note that in some of the sources on this material the looser term 'simplicial groupoid' is used to describe these objects usually with a note to the effect that this is not a completely accurate term to use. Sometimes one gets a simplicial object in Cat or $Grpd$ and we need to see if it is 'really' a $\mathcal{S}$-category, resp., $\mathcal{S}$-groupoid. The following lemma helps. First some notation:

Let $Ob : Cat \to Sets$ be the functor that picks out the set of objects of a small category.

Lemma 3. *Let $\mathcal{B} : \mathbf{\Delta}^{op} \to Cat$ be a simplicial object in Cat such that $Ob(\mathcal{B})$ is a constant simplicial set with value B_0, say. For each pair $(x, y) \in B_0$, let*

$$\mathcal{B}(x, y)_n = \{\sigma \in \mathcal{B}_n |\ s(\sigma) = x, t(\sigma) = y\},$$

where, of course, s refers to the source or domain function in $\mathcal{B}_n$, since otherwise $s(\sigma)$ would have no meaning, similarly for t, being the target or codomain function.

(i) The collection $\{\mathcal{B}(x, y)_n |\ n \in \mathbb{N}\}$ has the structure of a simplicial set $\mathcal{B}(x, y)$ with face and degeneracies induced from those of $\mathcal{B}$.

(ii) The composition in each level of $\mathcal{B}$ induces

$$\mathcal{B}(x, y) \times \mathcal{B}(y, z) \to \mathcal{B}(x, z).$$

Similarly the identity map in $\mathcal{B}(x, x)$ is defined as id_x, the identity at x in the category $\mathcal{B}_0$.

(iii) The resulting structure is an $\mathcal{S}$-enriched category. ■

The proof is routine, but is worth writing out if you have not seen the result before.

2.6.2 The Dwyer-Kan $\mathcal{S}$-groupoid functor.

The loop groupoid functor of Dwyer and Kan, [49], is a functor

$$G : \mathcal{S} \to \mathcal{S}-Grpd,$$

which takes the simplicial set K to the simplicially enriched groupoid GK, where $(GK)_n$ is the free groupoid on the directed graph

$$K_{n+1} \rightrightarrows K_0 \ ,$$

where the two functions, s, source, and t, target, are $s = (d_1)^{n+1}$ and $t = d_0(d_2)^n$ with relations $s_0 x = id$ for $x \in K_n$. The face and degeneracy maps are given on generators by

$$s_i^{GK}(x) = s_{i+1}^K(x),$$

$$d_i^{GK}(x) = d_{i+1}^K(x),$$

for $x \in K_{n+1}$, $1 < i \leq n$ and $d_0^{GK}(x) = (d_1^K(x))(d_0^K(x))^{-1}$.

These definitions yield a simplicial groupoid as is easily checked and, as is clear, its simplicial set of objects is constant, so it also can be considered as a simplicially enriched groupoid, $G(K)$. (We note that we use both GK and $G(K)$ for this, inserting brackets when it helps reduce ambiguity.)

(NB. Beware there are serious 'typos' in the original paper, [49], relating to these formulae for the construction and in some of the related material.)

It is instructive to compute some examples and we will look at $G(\Delta[2])$ and $G(\Delta[3])$. These simplicially enriched groupoid are free groupoids in each simplicial dimension. Their structure can be clearly seen from the generating graphs. For instance, $G(\Delta[2])_0$ is the free groupoid on the graph

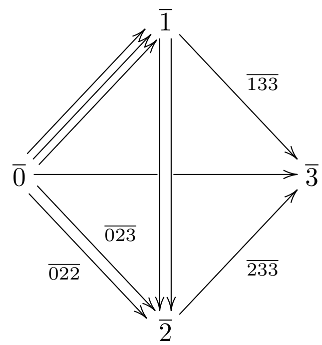
$$\begin{tikzcd}
& \bar{1} \arrow[dr, "\overline{12}"] & \\
\bar{0} \arrow[ur, "\overline{01}"] \arrow[rr, "\overline{02}"'] & & \bar{2}
\end{tikzcd}$$

whilst $G(\Delta[2])_1$ is the free groupoid on the graph

$$\begin{tikzcd}
& \bar{1} \arrow[dr, "\overline{122}"] & \\
\bar{0} \arrow[ur, shift left, "\overline{011}"] \arrow[ur, shift right, "\overline{012}"'] \arrow[rr, "\overline{022}"'] & & \bar{2}
\end{tikzcd}$$

Here it is worth noting that $\delta_0(\overline{012}) = (\overline{02}).(\overline{12})^{-1}$. Higher dimensions do not have any non-degenerate generators.

Again with $G(\Delta[3])$, in dimension 0, we have the free groupoid on the directed graph given by the 1-skeleton of $\Delta[3]$. In dimension 1, the generating directed graph is



Here only a few of the arrow labels have been given. Others are easy to provide (but moderately horrible to typeset in a sensible way!). Those from $\overline{0}$ to $\overline{1}$ are $\overline{012}$, $\overline{011}$ and $\overline{013}$; those from $\overline{1}$ to $\overline{2}$ are $\overline{122}$ and $\overline{123}$, and finally from $\overline{0}$ to $\overline{3}$, we have $\overline{033}$.

The next dimension is only a little more complicated. It has extra degenerate arrows such as $\overline{0112}$ and $\overline{0122}$ from $\overline{0}$ to $\overline{1}$, but also between these two vertices has $\overline{0123}$, coming from the non-degenerate 3-simplex of $\Delta[3]$. The full diagram is easy to draw (and again a bit tricky to typeset in a neat way), and is therefore left 'as an exercise'.

This loop groupoid functor has a right adjoint, $\overline{W}$, called the *classifying space* functor, but before we describe it in detail, we turn to the construction of a Moore complex of a simplically enriched groupoid, extending the one given earlier for simplical groups.

Given any $\mathcal{S}$-groupoid, G, its *Moore complex* NG is given by

$$NG_n = \bigcap_{i=1}^{n} Ker(d_i : G_n \to G_{n-1})$$

with differential $\partial : NG_n \to NG_{n-1}$ being the restriction of d_0. If $n \geq 1$, this is just a disjoint union of groups, one for each object in the object set, O, of G. If we write $G\{x\}$ for the simplicial group of elements that start and end at $x \in O$, then at object x, one has

$$NG\{x\}_n = (NG_n)\{x\}.$$

In dimension 0, one has $NG_0 = G_0$, so the $NG_n\{x\}$, for different objects x, are linked by the actions of the 0-simplices, acting by conjugation via repeated degeneracies.

Definition: If G is a $\mathcal{S}$-groupoid, its *fundamental groupoid* is $\pi_0 G = NG_0/\partial NG_1$.

This works because ∂NG_1 is a normal subgroupoid of G_0.

Definition: Again if G is a $\mathcal{S}$-groupoid, its n^{th} *homotopy groupoid* is $\pi_n(G) := Ker\,\partial_n / Im\,\partial_{n+1}$.

This is a groupoid with $\pi_n(G)(x, y)$ empty if $x \neq y$ in $Ob(G)$. Of course, $\pi_n(G)\{x\} = \pi_n(G\{x\})$, the n^{th} homotopy group of the vertex simplicial group, $G\{x\}$ of G at the object x.

For simplicity in the discussion that follows, we will often assume that the $\mathcal{S}$-groupoid is *reduced*, that is, its set, O, of objects is just a singleton set $\{*\}$, so, in fact, G is just a simplicial group.

Notational caution: The Dwyer-Kan construction behaves like a 'loop' construction. It lowers the index of each simplex, so a 2-simplex in K, for instance, gives a 1-simplex in the loop groupoid, GK. Thus $\pi_n(K) \cong \pi_{n-1}(GK)$, just as $\pi_n(X) \cong \pi_{n-1}(\Omega X)$. Sometimes this may cause a slight mismatch with the apparent dimensions of elements and thus some confusion.

2.6.3 The $\overline{W}$ construction.

We next need to make explicit the $\overline{W}$ construction.

Let H be an $\mathcal{S}$-groupoid, then $\overline{W}H$ is the simplicial set described by

- $(\overline{W}H)_0 = Ob(H_0)$, the set of objects of the groupoid of 0-simplices (and hence of the groupoid at each level);
- $(\overline{W}H)_1 = arr(H_0)$, i.e., the set of arrows of the groupoid H_0;
  and for $n \geq 2$,
- $(\overline{W}H)_n = \{(h_{n-1}, \ldots, h_0) \mid h_i \in arr(H_i) \text{ and } s(h_{i-1}) = t(h_i), 0 < i < n\}$.

Here s and t, as usual, are generic symbols for the domain and codomain mappings of all the groupoids involved. The face and degeneracy mappings between $(\overline{W}H)_1$ and $(\overline{W}H)_0$ are the source and target maps and the identity maps of H_0, respectively; whilst the face and degeneracy maps at higher levels are as follows:

- $d_0(h_{n-1}, \ldots, h_0) = (h_{n-2}\ldots, h_0)$;
- for $0 < i < n$,
  $d_i(h_{n-1}, \ldots, h_0) = (d_{i-1}h_{n-1}, d_{i-2}h_{n-2}, \ldots, d_0h_{n-i}.h_{n-i-1}, h_{n-i-2}, \ldots, h_0)$;
  and
- $d_n(h_{n-1}, \ldots, h_0) = (d_{n-1}h_{n-1}, d_{n-2}h_{n-2}, \ldots, d_1h_1)$,
  whilst
- $s_0(h_{n-1}, \ldots, h_0) = id_{s(h_{n-1})}, h_{n-1}, \ldots, h_0)$;
  and,
- for $0 < i \leq n$,
  $s_i(h_{n-1}, \ldots, h_0) = (s_{i-1}h_{n-1}, \ldots, s_0h_{n-i}, id_{t(h_{n-i})}, h_{n-i-1}, \ldots, h_0)$.

2.6.4 Simplicial Automorphisms and Regular Representations.

One original motivation for the study that led to the writing of this book was the Grothendieck programme, *Pursuing Stacks*, [76]. That involves non-Abelian cohomology, stacks and related structures. We will see some simple simplicial echoes of some of that in the following pages. It will not, however, be developed in detail in this book, but does suggest further directions for research. Here we will limit ourselves to some standard material adapted from Curtis, [42], Gabriel and Zisman, [66] and May, [117]. One of its potential applications is via the profinite completion functors that we will meet shortly.

As we saw, the usual enrichment of the category of simplicial sets is given by :
for each $n \geq 0$, the set of n-simplices is

$$\underline{\mathcal{S}}(K, L)_n = \mathcal{S}(K \times \Delta[n], L),$$

together with obvious face and degeneracy maps. Recall that composition is given by: for $f \in \underline{\mathcal{S}}(K, L)_n$, $g \in \underline{\mathcal{S}}(L, M)_n$, so $f : \Delta[n] \times K \to L$, $g : \Delta[n] \times L \to M$,

$$g \circ f := (K \times \Delta[n] \overset{K \times diag}{\longrightarrow} K \times \Delta[n] \times \Delta[n] \overset{f \times \Delta[n]}{\longrightarrow} L \times \Delta[n] \overset{g}{\to} M);$$

Identity : $id_K : K \times \Delta[0] \overset{\cong}{\to} K$,

For fixed K, $\underline{\mathcal{S}}(K, K)$ is a simplicial monoid and $\mathsf{aut}(K)$ will be the corresponding simplicial group of invertible elements.

If $f : K \times \Delta[n] \longrightarrow L$ is an n-simplex in $\underline{\mathcal{S}}(K, K)$, then we can form a diagram

$$\begin{tikzcd} K \times \Delta[n] \arrow[rr, "(f,p)"] \arrow[dr] & & L \times \Delta[n] \arrow[dl] \\ & \Delta[n] & \end{tikzcd}$$

in which the two slanting arrow are the obvious projections, (so $(f, p)(k, \sigma) = (f(k, \sigma), \sigma)$). Taking $K = L$, $f \in \mathsf{aut}(K)$ if and only if (f, p) is an isomorphism of simplicial sets.

Given a simplicial set K, and an n-simplex k in K, there is a representing map, also denoted,

$$k : \Delta[n] \longrightarrow K,$$

that sends the top dimensional generating simplex of $\Delta[n]$ to k. The enrichment above is part of an adjunction

$$\underline{\mathcal{S}}(K \times L, M) \cong \underline{\mathcal{S}}(L, \underline{\mathcal{S}}(K, M))$$

in which, given $\theta : K \times L \longrightarrow M$ and $\ell \in L_n$, the corresponding simplicial map

$$\overline{\theta} : L \longrightarrow \underline{\mathcal{S}}(K, M)$$

sends ℓ to the composite

$$K \times \Delta[n] \xrightarrow{K \times \ell} K \times L \xrightarrow{\theta} M.$$

In a simplicial group G, the multiplication is a simplicial map $\#_0 : G \times G \longrightarrow G$, and so, by the adjunction, we get a simplicial map

$$G \longrightarrow \underline{\mathcal{S}}(G, G)$$

and this is a simplicial monoid morphism. This gives the *right regular representation of* G,

$$\rho = \rho_G : G \longrightarrow \mathsf{aut}(G).$$

This representation needs careful interpretation. In dimension n, an element $g \in G_n$ acts by multiplication on the right on G, but even in dimension 0, this action is not as simple as one might think. (NB. Here $\mathsf{aut}(G)$ is the simplicial group of 'simplicial automorphisms of the *underlying simplicial set* of G, as, of course, multiplication by an element does not give a mapping that respects the group structure.) Here some simple examples are called for:

Suppose $g \in G_1$, then $\rho(g) \in \mathsf{aut}(G)_1 \subset \mathcal{S}(G, G)_1 = \mathcal{S}(G \times \Delta[1], G)$. In other words, $\rho(g)$ is a homotopy between $\rho(d_1 g)$ and $\rho(d_0 g)$. Of course, it is an invertible element of $\mathcal{S}(G, G)_1$ and this will have implications for its properties as a homotopy, and to use a geometric term, we might loosely refer to it as an *isotopy*.

In general, 0-simplices give simplicial maps corresponding to multiplication by that element, so that for $g \in G_0$, and $x \in G_n$,

$$\rho(g)(x) = x \#_0 s_0^{(n)}(g).$$

In dimension 1, we have that elements give isotopies, and in higher dimensions, we have 'isotopies of isotopies', and so on.

2.6.5 $\overline{W}$, W and twisted cartesian products.

Suppose we have simplicial sets Y, a potential 'fibre' and B, a potential 'base' which will be assumed to be pointed by a vertex, $*$. Inspired by the sort of construction that works for the construction of group extensions, we are going to try to construct a fibration sequence

$$Y \longrightarrow E \longrightarrow B.$$

Clearly the product $E = B \times Y$ will give such a sequence, but can we twist this cartesian product to get a more general construction? We will try setting $E_n = B_n \times Y_n$ and will change as little as possible in the data specifying faces and degeneracies. In fact we will take all the degeneracy maps to be exactly those of the cartesian product, and all but d_0 of the face maps likewise. This leaves just the zeroth face map.

In, say, a covering space considered as a fibration with discrete fibre, the fundamental group(oid) of the base acts by automorphisms / permutations on the fibre, and the fundamental group(oid) is generated by the edges, hence by elements of dimension one greater than that of the fibre, so we try a formula for d_0 of form

$$d_0(b,y) = (d_0 b, t(b)(d_0 y)),$$

where $t(b)$ is an automorphism of Y, determined by b in some way, hence giving a function $t : B_n \longrightarrow \mathsf{aut}(Y)_{n-1}$. Note here Y is an arbitrary simplicial set, not the underlying simplicial set of a simplicial group as was previously the case when we considered aut, but this makes little difference to the discussion.

Of course, with these tentative definitions, we must still have that the simplicial identities hold, but it is easy to check that these will hold exactly if t satisfies the following equations:

$$\begin{aligned}
d_i t(b) &= t(d_{i-1} b) \quad \text{for} \quad i > 0,\\
d_0 t(b) &= t(d_1 b) \#_0 t(d_0 b)^{-1},\\
s_i t(b) &= t(s_{i+1} b) \quad \text{for} \quad i \geq 0,\\
t(s_0 b) &= *.
\end{aligned}$$

A function t satisfying these equations will be called a *twisting function* and the simplicial set E, thus constructed, will be called a *regular twisted cartesian product*. We write $E = B \times_t Y$.

Of course a twisting function is not a simplicial map, but the formulae it satisfies look closely linked to those of the Dwyer-Kan loop group(oid) construction, page 50. In fact:

Proposition 16. *A twisting function $t : B \longrightarrow \mathsf{aut}(Y)$ determines a unique homomorphism of simplicial groupoids $t : GB \longrightarrow \mathsf{aut}(Y)$, and conversely.* ∎

Of course, since G is left adjoint to $\overline{W}$, we could equally well note

Corollary 3. *A twisting function t determines a unique simplicial morphism $t : B \longrightarrow \overline{W}(\mathsf{aut}(Y))$, and conversely.* ∎

Example : Simplicial covering spaces. Given any simplicial group, G, we can form its fundamental group, which is $\pi_0(G) = G_0/Im\{\partial : NG_1 \to G_0\}$. Given any group, π, we can form a constant simplicial group, $K(\pi, 0)$, having π in all dimensions and with all face and degeneracy maps being the identity homomorphism from π to itself. These two constructions are functorial and provide a pair of adjoint functors between the category of simplicial groups and that of groups. There is an obvious simplicial group homomorphism

$$G \to K(\pi_0, 0),$$

for $\pi_0 = \pi_0(G)$, which 'kills off' all higher terms in the Moore complex, NG. (We note that any $NK(\pi, 0)_n$ is trivial if $n \geq 1$ and is π in dimension 0.)

If K is a reduced simplicial set, (so K_0 is a singleton set), then GK is a simplicial group and $\pi_0 GK \cong \pi_1(K)$, see May, [117], for this in detail. In this case, the natural epimorphism,

$$GK \to K(\pi_0, 0)$$

is adjoint to

$$t : K \to \overline{W} K(\pi_0, 0),$$

which gives a natural twisting map $t : K \to K(\pi_0, 0)$, and hence a fibration (twisted cartesian product)

$$K \times_t K(\pi_0, 0) \to K,$$

with fibre the constant simplicial *set* $K(\pi_0, 0)$, i.e., the underlying simplicial set of the corresponding constant simplicial group. As a constant simplicial set is really 'just' a set, this gives us a fibration with 'discrete' fibre, and hence with unique lifting properties, i.e., a covering 'space'.

If $E \to K$ is any simplicial covering, (i.e., fibration with 'discrete' fibre), then $E \cong K \times_t F$ for some fibre F and twisting function t. We have $t : GK \to \mathsf{aut}(F)$, but F is constant, so this factors through $\pi_0 GK \to \mathsf{aut}(F)$, i.e., an action of $\pi_1 = \pi_1 K$, or equivalently $\pi_0(GK)$, on the set F_0. It is not difficult to extend this to identifying $\tilde{K} = K \times_t K(\pi_1, 0)$ as the universal covering and this correspondence with actions as being part of the Galois-Poincaré theory for simplicial coverings, for which the canonical source is probably Gabriel and Zisman, [66], or for links with groupoid coverings, Brown, [23], see also, Borceux and Janeldize, [18], p. 324 - 326. Of course, what we might want to do is to replace π_1 by its profinite completion and also to explore replacing this '1-type' by a general n-type.

2.6.6 n-types of spaces, simplicial sets and $\mathcal{S}$-group(oid)s

We earlier briefly mentioned 'n-equivalences' and 'n-types'. As homotopy types are enormously complex in structure, we may try to study them by 'filtering' that information in various ways, thus attempting to see how the information at the n^{th}-level depends on that at lower levels. The informational filtration by n-type is very algebraic and, as we will see later, very natural. It has two very satisfying interacting aspects. It gives complete models for a subclass of homotopy types, namely those whose homotopy groups vanish for all high enough n, but, at the same time, gives a set of approximating notions of equivalence that, on all 'spaces', give useful information on weak equivalences. We start with the topological notion:

Definition: Given a continuous function $f : (X, x_0) \to (Y, y_0)$ between connected pointed spaces, f is said to be a *n-equivalence* if the induced homomorphisms $\pi_k(f) : \pi_k(X, x_0) \to \pi_k(Y, y_0)$, for $1 \leq k \leq n$, are all isomorphisms. More generally, on relaxing the requirements on the spaces, a continuous function $f : X \to Y$ is a *n-equivalence* if it induces a bijection on π_0, that is, $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection, and for each $x_0 \in X$ and $1 \leq k \leq n$, $\pi_k(f) : \pi_k(X, x_0) \to \pi_k(Y, f(x_0))$ is an isomorphism.

There are alternative descriptions: f is an n-equivalence if and only if, for any polyhedron P of dimension $\leq n$, f induces a bijection

$$[P, f] : [P, X] \to [P, Y]$$

on homotopy classes. (We restrict to spaces having the homotopy type of a CW-complex here to avoid problems of weak equivalences that are not homotopy equivalences.)

We can form a category $Ho_n(Top)$ by formally inverting the n-equivalences within our favorite category of spaces, Top. As any weak equivalence is a n-equivalence (for all n) and any n-equivalence is an $(n-1)$-equivalence, there are functors

$$Ho(Top) \to Ho_n(Top)$$

and

$$Ho_n(Top) \to Ho_k(Top)$$

if $k \leq n$.

Definition: Two spaces are said to have the same *n-type* if they are isomorphic within $Ho_n(Top)$.

It should be clear that the definition does not imply that the two such spaces have to have an n-equivalence directly joining them as the isomorphism in $Ho_n(Top)$ may not be realisable in that way. It may be the result of a 'zig-zag' of actual maps with every other of them an n-equivalence. If the spaces are CW-complexes, and hence are specified with filterations by skeleta, which are built inductively by adding cells to previous levels, then we can do a lot better than arbitrary ziz-zags. Given n, the n-type of a CW-complex, X, will depend only on its $(n+1)$-skeleton, since the inclusion of $X^{(n+1)}$ into X will be an n-equivalence as follows from the definition of the homotopy groups.

It seems that, in his original thoughts on algebraic homotopy theory, Whitehead hoped to find algebraic models for n-types, that is to find algebraic descriptions of isomorphism classes of spaces within $Ho_n(Top)$. Classifying 1-types is 'easy ' as they have models that are just groups, so classification reduces to classifying groups up to isomorphism. This is still not an easy task, but there are a wide range of tools available for it. MacLane and Whitehead, [114], gave a complete algebraic model for 2-types. (As previously noted their 3-types are modern terminology's 2-types.) The model they proposed was the crossed module and we will be exploring the modern theory of crossed modules and their applications later on. We will also be looking at various related models for n-types.

It should be pointed out that although n-equivalence is defined in terms of the π_k, $0 \leq k \leq n$, the interactions between the various π_ks mean that not every sequence $\{\phi_k : \pi_k(X) \to \pi_k(Y)\}_{0\leq k\leq n}$ can be realised as the induced morphisms coming from some $f : X \to Y$ even if the ϕ_k are all isomorphisms.

For simplicial sets and simplicially enriched group(oid)s, the definitions of n-equivalence are analoguous:

Definition: For $f : G \to H$ a morphism of $\mathcal{S}$-groupoids, f is an *n-equivalence* if $\pi_0 f : \pi_0 G \to \pi_0 H$ is an equivalence of the fundamental groupoids of G and H and for each object $x \in Ob(G)$ and each k, $1 \leq k \leq n$,

$$\pi_k f : \pi_k(G\{x\}) \to \pi_k(H\{f(x)\})$$

is an isomorphism.

We write $Ho_n(\mathcal{S}-Grpd)$ for the corresponding category of n-types, i.e., $\mathcal{S}-Grpd(\Sigma_n^{-1})$, where Σ_n is the class of all n-equivalences of $\mathcal{S}$-groupoids. An *n-type of $\mathcal{S}$-groupoids* is an isomorphism class within $Ho_n(\mathcal{S}-Grpd)$.

We will also be needing the subcategory of $Ho_n(\mathcal{S}-Grpd)$ determined by the simplicial groups. This will be denoted $Ho_n(Simp.Grps)$. We will look in detail at various aspects of n-types of simplicial groups, and will adapt notation in obvious ways to handle the profinite and pro-$\mathcal{C}$ analogues when needed.

Repeated cautionary note: If K is a simplicial set, then as $\pi_k(K) \cong \pi_{k-1}(GK)$, the n-type of K corresponds to the $(n-1)$-type of GK.

2.7 Profinite homotopy

2.7.1 Homotopy procategories

In their work on étale homotopy, [6], Artin and Mazur proposed a working definition of prohomotopy. Starting with the categories of CW-complexes and Simplicial Sets, and sometimes, Kan complexes, they first formed the homotopy category and then the corresponding pro-category, $Pro-Ho(\mathcal{S})$ or whatever.

As pro-objects are diagrams, the objects on these categories could be considered as homotopy commutative diagrams of the corresponding objects, for instance, an object of $Pro-Ho(\mathcal{S})$ is repesentable as a homotopy commutative diagram of simplicial sets. The lack of information encoded in such a diagram as to the homotopies involved (they are required to exist, but are not specified in any way) and the consequent failure of $Pro-Ho(\mathcal{S})$ to correspond to a homotopy category of $Pro-\mathcal{S}$ meant that the development of homotopical algebra, or for us more importantly, some reasonable algebraic homotopy in this setting was impossible. The use of prosimplicial sets in a geometric topological context in Shape Theory led to the development of various homotopy structures on $Pro-\mathcal{S}$ and some of the related categories such as $Pro\text{–}Simp.Grps$ and $Pro\text{–}Ch_k^+$, the category of prochain complexes over a commutative ring. (A list of references for these will be given shortly.) These linked in to the good geometric homotopic intuitions of Shape Theory, and related Proper Homotopy Theory, but those intuitions also suggested tools for encoding the homotopies in the diagrams. (Some of these tools were developed as long ago as the 1940s, so had 'laid fallow' for years.) In this second generation of homotopy structures on procategories, the key ideas were (i) to base things firmly on a good class of 'weak equivalences' and (ii) to use explicit encoding of the homotopies, whether geometrically or categorically, and to require coherence conditions between them. The 'weak equivalences' usually included those morphisms in $Pro\text{–}\mathcal{S}$ that were 'levelwise weak equivalences'. By a result, known as the Reindexing Lemma, to be found in Artin and Mazur's lecture notes, [6], any morphism of pro-objects is isomorphic, in the category of such morphisms, to one that is 'levelwise' so is a pro-object in the category of morphisms of the original category. Such levelwise weak equivalences were pro-morphisms in which all the maps involved were weak equivalences in the sense of a homotopy theory on the original category. In 1973, Vogt, [162], showed that, in a category of diagrams of spaces, inverting 'levelwise' homotopy equivalences corresponded to homotopy coherent diagrams, that is, homotopy commutative diagrams in which the homotopies were specified and were compatible with each other up to higher homotopies, *ad infinitum*. This combined the two ideas mentioned before.

After a period of relative inactivity, the homotopy theory of pro-spaces has gone through a revival due to results by Morel, [123, 124] and Dehon, [43] and with Gaudens, [44], showing applications of profinite spaces to the study of the T-functor of Lannes. One consequence of this has been Isaksen's re-examination, [90, 91], of well behaved Quillen model category structures on the category of prosimplicial sets. The theory now looks as if it is heading for a fairly definitive form.

Other references for homotopy structures in this context are:

- J. Grossman, (1975), [74]: This concentrates on towers of spaces, but contains some very useful results;
- D. Edwards and H.Hastings, (1976): This was one of a series of very important advances at that time. It provided a rich theory with a lot of geometric-topological motivation and links with proper homotopy and strong shape theory;
- T. Porter, mid-1970s, [134–137] and for links with proper homotopy theory, [140]. This theory was based on K. Brown's cutdown version, [20], of Quillen's homotopical algebra, which is closely linked to Baues' algebraic homotopy theories mentioned earlier. It was motivated by attempts to bring the homotopy coherent machinery of Vogt, [162], to bear on prosimplicial homotopy theory. The link with the Edwards-Hasting's theory is explored in [138];

- E. Friedlander, (in particular, 1972-82), [61–63]: The emphasis in these papers and book is towards algebraic geometric applications. It is based on geometric tools for improving the Artin-Mazur étale homotopy theory.

Finally Isaksen's work, already cited, provides an excellent treatment of the technical problems of defining good homotopy theories for pro-simplicial sets. His further work with Fausk, [58], examines the overall question of finding model category structures in procategories, including that of G-spaces fro G a profinite group.

2.7.2 Profinite homotopy types

What should be a profinite homotopy type? For that matter, what should be a finite homotopy type? There are several 'obvious' answers. In the first, a 'space' or simplicial set, K, would be of finite homotopy type if $\pi_0 K$ is a finite set, and for each basepoint $*$ in K and each $k \geq 1$, $\pi_k(K)$ is a finite group. A second, slightly stronger, version would add that, for each basepoint $*$, only finitely many of the $\pi_k(K, *)$ should be non-trivial, and finally seemingly extra specially strong, the simplicial groupoid $G(K)$ should be such that it has finitely many components and for each component K_i, its homotopy type can be faithfully represented by a simplicial group whose Moore complex is of bounded finite length, and whose group of n-simplices is finite for each $n \geq 0$.

In, for instance, [123], the second version is used, but the coskeleton functors, cf. Artin and Mazur, [6], or Duskin, [48], can be used within $Pro{-}\mathcal{S}$, to replace a simplicial set K by its tower of coskeleta, $K^\natural$, within $Pro{-}\mathcal{S}$, and if K has finite homotopy type in the weaker first sense, then $K^\natural$ is a tower of simplicial sets, each of which has finite homotopy type in the second sense, hence for the purposes of pro-finite completions the first two are equivalent although not equal.

What about the third, where the homotopy type can be represented by a 'finite simplicial groupoid'? By restricting to components, we may assume without loss of generality that K is connected, and then may further assume it is reduced as we want to discuss its homotopy type. If there is a finite simplicial group, G, i.e., its Moore complex is of finite length and each G_n is finite, so that G and $G(K)$ are isomorphic in $Ho(Simp.Grps)$ then clearly as the homotopy groups of K are the homology groups of $NG(K)$ shifted one dimension, and these latter are isomorphic to the homology groups of NG, which are finite, since each G_n is finite, then the third condition clearly implies the second. In fact they are equivalent, as the following result of Ellis shows:

Theorem 4. *(Ellis, [56]) Suppose that $\pi_k(K)$ is trivial for all $k \geq c+1$, and that each of the homotopy groups $\pi_k(K)$ is finite for $k \leq c$, then the homotopy type of K is faithfully represented by a simplicial group whose Moore complex is of length at most $c-1$ and whose group of n-simplices is finite for each $n \geq 0$.* ■

It should be noted that an alternative notion of finite homotopy type has been used elsewhere. There it has been taken to mean that the homotopy type has a representative, which is a finite CW-complex. Of course, that in no way is strong enough for our purposes as the circle is a finite CW-complex yet has infinite fundamental group. We make the following formal definition:

Definition: A connected simplicial set K is said to be of *strongly finite homotopy type* if it has finitely many non-trivial homotopy groups all of which are finite. For a non-connected simplicial set K, we say it is of strongly finite homotopy type, if each component is.

Ellis notes, [56], the following variant of his result:

Proposition 17. *Suppose that each of the c non-trivial homotopy groups of K is a finite p-group, where p is a prime, then the homotopy type of K is faithfully represented by a simplicial group whose group of n-simplices is a finite p-group for each $n \geq 0$.* ■

We introduce a terminology to handle the corresponding simplicial groups and $\mathcal{S}$-groupoids:

Definition: Given a simplicial group G, we say G is *strongly finite* if each G_n is finite, and the Moore complex, (NG, ∂), is of finite length. We say it is *weakly finite* if each G_n is a finite group.

If G is a $\mathcal{S}$-groupoid, then we will say G is *strongly finite* if G has finitely many components and each of its vertex simplicial groups is strongly finite and similarly for 'weakly'.

We will follow our usual practice and extend the above to $\mathcal{C}$ settings by replacing 'finite' to '$\mathcal{C}$' wherever this makes sense.

Definition: A $\mathcal{S}$-groupoid G is *strongly homotopy finite* if it is homotopy equivalent to a strongly finite $\mathcal{S}$-groupoid; similarly for 'weakly'.

We can now say what we will mean by a pro-$\mathcal{C}$ homotopy type.

Definition: A $\mathcal{S}$-groupoid G is said to be a *pro-$\mathcal{C}$ $\mathcal{S}$-groupoid* if for each n the groupoid G_n is a pro-$\mathcal{C}$ groupoid, i.e. it is a profinite groupoid such that the set of objects is a profinite space and each vertex group is a pro-$\mathcal{C}$ group. Of course, a special case of this is the notion of a *pro-$\mathcal{C}$ simplicial group*, this being just that case when G has a single object.

The category of pro-$\mathcal{C}$ $\mathcal{S}$-groupoids is defined in the obvious way using continuous homomorphisms/functors. The definition of 'weak equivalence' in this setting is fairly straightforward.

Definition: A continuous simplicial map $f : G \to H$ of pro-$\mathcal{C}$ $\mathcal{S}$-groupoids is a *weak equivalence* if it induces a bijection on π_0 and for each object $x \in Ob(G)$ and each $k \geq 0$, the induced homomorphism $\pi_k(f) : \pi_k(G\{x\}) \to \pi_k(H\{f(x)\})$ is an isomorphism. The corresponding homotopy category is obtained by formally inverting the weak equivalences and will be denoted $Ho(Pro{-}\mathcal{C}.\mathcal{S}{-}Grpd)$.

A *pro-$\mathcal{C}$ homotopy type* is an isomorphism class within $Ho(Pro{-}\mathcal{C}.\mathcal{S}{-}Grpd)$.

It is only a bit more tricky to speak of a profinite simplicial set, K, i.e. a simplicial object in $Prof.$, as being pro-$\mathcal{C}$. If G is a profinite $\mathcal{S}$-groupoid, then $\overline{W}(G)$ is a profinite simplicial set in this sense, so K should be of pro-$\mathcal{C}$ homotopy type if it is continuously homotopically equivalent to a profinite simplicial set of the form $\overline{W}(G)$ for G a pro-$\mathcal{C}$ $\mathcal{S}$-groupoid. (This is discussed in some detail in Morel's paper, [124] for the case of pro-p 'simplicial groupoids'.) We note that if K is a pro-$\mathcal{C}$ simplicial in this sense then its set of components will be a profinite space and all its homotopy groups are pro-$\mathcal{C}$.

As we will be interested in the *algebraic* homotopy of these profinite homotopy types we will tend to concentrate on the profinite and pro-$\mathcal{C}$ simplicial groups and, to a lesser extent, groupoids. The profinite simplicial sets will rarely play any key role in the following development.

2.7.3 Profinite completion of homotopy types

Profinite completions of homotopy types were introduced initially by Artin and Mazur, [6], and their theory was extended and applied by Sullivan, [158]. A variant was introduced by Bousfield and Kan, [19], but was applicable mainly to nilpotent spaces (see later for the idea of a nilpotent action) and for 'pro-p completions', which is one of the most important cases. More recently, Morel, [123, 124], has reworked both of these sources to get a pro-p completion that works in general and which is more 'rigid' than the Artin-Mazur one. Morel's version has been examined by Isaksen, [92]. We will adapt Morel's theory to the general pro-$\mathcal{C}$ setting. That approach is ideally suited for this context as it is firmly based on the algebraic models of simplicial groups and $\mathcal{S}$-groupoids.

What should be a profinite completion? We have already seen various cases of them in an algebraic context, and will take the model from there. We have a category of 'finite' objects, within a category of all objects and look for a pro-adjoint to the inclusion of the category of finite ones into the bigger category. In the case of our algebraic models for homotopy types, we can take the finite objects to be the strongly (homotopy) finite $\mathcal{S}$-groupoids or their weak cousins. It does not, in fact, matter as the 'weak' ones are inverse limits of 'strong' ones as follows from Ellis's finiteness results together with truncation.

We extend the profinite completion functor on groups or groupoids dimensionwise to the simplical case, and so, if G is a simplicial group, for instance, then $\hat{G}$ will denote the profinite simplicial group obtained by $(\hat{G})_n = \widehat{(G_n)}$. The following is a simple consequence of the definition of simplicial morphisms as natural transformations.

Proposition 18. *For any finite simplicial group H, there is a natural bijection*

$$Simp.Grps(G, H) \cong Simp.Prof.Grps(\hat{G}, H).$$

■

Of course, the pro-$\mathcal{C}$ version, and the groupoid versions are similar.

Because of this proposition, there is an obvious way to profinitely complete a simplicial set, K. You just send it to $\overline{W}(\widehat{G(K)})$. The resulting map is adjoint to the natural unit of the adjunction, $G(K) \to \widehat{G(K)}$. Again the pro-$\mathcal{C}$ version, and the groupoid versions are similar. Thus the viewpoint we will be exploring is that one natural interpretation of the title of this book is the study of the homotopy properties of simplicial profinite groups and the application of their invariants. This will not be the only aspect we will examine as there are numeroous interesting and useful other points of view that should not be neglected, especially those coming form the crossed module, crossed complex thread.

3

Pro-$\mathcal{C}$ Crossed Modules.

3.1 Crossed modules and Pro-$\mathcal{C}$ crossed modules.

We first recall for convenience the definitions of crossed module (cf. Brown-Huebschmann [29]) and of a morphism of crossed modules.

3.1.1 Crossed Modules

Definition: A *crossed module* (C, G, δ) consists of groups C and G with a left action of G on C, written $(g, c) \to {}^{g}c$ for $g \in G$, $c \in C$, and a group homomorphism $\delta : C \to G$ satisfying the following conditions:
CM1) for all $c \in C$ and $g \in G$,

$$\delta({}^{g}c) = g\delta(c)g^{-1},$$

CM2) for all $c_1, c_2 \in C$,

$${}^{\delta(c_2)}c_1 = c_2c_1c_2^{-1}.$$

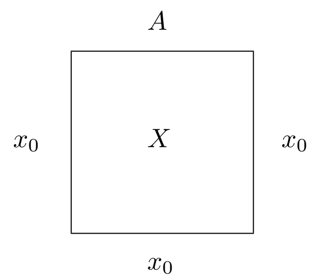
(CM2 is called the *Peiffer identity.*)

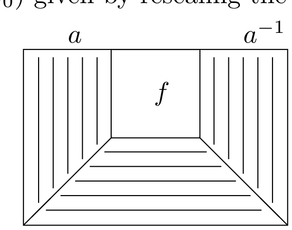
If (C, G, δ) and (C', G', δ') are crossed modules, a *morphism*, $(\mu, \eta) : (C, G, \delta) \to (C', G', \delta')$, *of crossed modules* consists of group homomorphisms $\mu : C \to C'$ and $\eta : G \to G'$ such that

(i) $\delta'\mu = \eta\delta$ and (ii) $\mu({}^{g}c) = {}^{\eta(g)}\mu(c)$ for all $c \in C$, $g \in G$.

Crossed modules and their morphisms form a category, of course. It will usually be denoted *CMod*.

Examples of crossed modules: When we have defined their profinite/ pro-$\mathcal{C}$ analogue, we will give some algebraic examples which exist in both the classical discrete setting and in the profinite cases, but for the moment we limit ourselves to the standard topological examples:

1. Let X be a pointed space, with $x_0 \in X$ as its base point, and A a subspace with $x_0 \in A$. Recall that the second relative homotopy group, $\pi_2(X, A, x_0)$, consists of relative homotopy classes of continuous maps
$$f : (I^2, \partial I^2, J) \to (X, A, x_0)$$
where ∂I^2 is the boundary of I^2, the square, $[0,1] \times [0,1]$, and $J = \{0,1\} \times [0,1] \cup [0,1] \times \{0\}$. Schematically f maps the square as:


so the top of the boundary goes to A, the rest to x_0 and the whole thing to X. The relative homotopies considered then have to preserve such structure, so intermediate mappings also send J to x_0, etc. Restriction of such an f to the top of the boundary clearly gives a homomorphism
$$\partial : \pi_2(X, A, x_0) \to \pi_1(A, x_0)$$
to the fundamental group of A, based at x_0. There is also an action of $\pi_1(A, x_0)$ on $\pi_2(X, A, x_0)$ given by rescaling the 'square' given by


where f is partially 'enveloped' in a region on which the mapping is behaving like a.
Of course, this gives a crossed module
$$\pi_2(X, A, x_0) \to \pi_1(A, x_0).$$
A direct proof is quite easy to give. One can be found in Hilton's book, [83] or in Brown-Higgins-Sivera, [28]. Alternatively one can use the argument in the next example.
2. Suppose $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration sequence of pointed spaces. Thus p is a fibration, $F = p^{-1}(b_0)$, where b_0 is the basepoint of B. The fibre F is pointed at f_0, say, and f_0 is taken as the basepoint of E as well.

There is an induced map on fundamental groups

$$\pi_1(F) \xrightarrow{\pi_1(i)} \pi_1(E)$$

and if a is a loop in E based at f_0, and b a loop in F based at f_0, then the composite path corresponding to aba^{-1} is homotopic to one wholly within F. To see this, note that $p(aba^{-1})$ is null homotopic. Pick a homotopy in B between it and the constant map, then lift that homotopy back up to E to one starting at aba^{-1}. This homotopy is the required one and its other end gives an element ${}^a b \in \pi_1(F)$ (abusing notation by confusing paths and their homotopy classes). With this action $(\pi_1(F), \pi(E), \pi_1(i))$ is a crossed module. This will not be proved here, but is not that difficult. Links with previous examples are strong.

If we are in the context of the above example, consider the inclusion map, f of a subspace A into a space X (both pointed at $x_0 \in A \subset X$). Form the corresponding fibration

$$i^f : M^f \to X$$

by forming the pulback

$$\begin{tikzcd} M^f \arrow[r, "\pi^f"] \arrow[d, "j^f"'] & X^I \arrow[d, "e_0"] \\ A \arrow[r, "f"'] & X \end{tikzcd}$$

so M^f consists of pairs (a, λ), where $a \in A$ and λ is a path from $f(a)$ to some point $\lambda(1)$. Set $i^f = e_1\pi^f$, so $i^f(a, \lambda) = \lambda(1)$. It is standard that i^f is a fibration and its fibre is the subspace $F_h(f) = \{(a, \lambda) \mid \lambda(1) = x_0\}$, often called the homotopy fibre of f. The base point of $F_h(f)$ is taken to be the constant path at x_0, (x_0, c_{x_0}).

If we note that

$$\pi_1(F_h(f)) \cong \pi_2(X, A, x_0)$$

$$\pi_1(M^f) \cong \pi_1(A, x_0)$$

(even down to the descriptions of the actions, etc.), the link with the previous example becomes clear, and thus furnishes another proof of the statement there.

3. The link between fibrations and crossed modules can also be seen in the category of simplicial groups. A morphism $f : G \to H$ of simplicial groups is a fibration if and only if each f_n is an epimorphism. This means that a fibration is determined by the fibre over the identity which is, of course, the kernel of f. The links between simplicial groups and simplicial sets mean that the analogue of π_1 is π_0. Thus the fibration f corresponds to

$$Ker\, f \overset{\lhd}{\to} G$$

and each level of this is a crossed module by our earlier observations. Taking π_0, it is easy to check that

$$\pi_0(Ker\, f) \to \pi_0(G)$$

is a crossed module. In fact any crossed module is isomorphic to one of this form.

The profinite analogue of this is now easy to give.

3.1.2 Pro$-\mathcal{C}$ crossed modules

Definition: A *pro$-\mathcal{C}$ crossed module* (C, G, δ) is a crossed module in which C and G are pro$-\mathcal{C}$ groups, G acts continuously on C and δ is a continuous group homomorphism.

If (C, G, δ) and (C', G', δ') are pro$-\mathcal{C}$ crossed modules and

$$(\mu, \eta) : (C, G, \delta) \to (C', G', \delta')$$

is a morphism between them in which both μ and η are *continuous* then we say (μ, η) is a *morphism of pro$-\mathcal{C}$ crossed modules.*

Pro$-\mathcal{C}$ crossed modules and the (continuous) morphisms between them form a category which we will denote $Pro{-}\mathcal{C}.CMod$.

There is, for a fixed profinite group G, a subcategory $Pro{-}\mathcal{C}.CMod/G$ of $Pro{-}\mathcal{C}.CMod$ which has as objects those pro$-\mathcal{C}$ crossed modules with G as the "base", i.e., all (C, G, δ) for this fixed G, and having as morphisms from (C, G, δ) to (C', G, δ') just those (μ, η) in $Pro{-}\mathcal{C}.CMod$ in which $\eta : G \to G$ is the identity homomorphism on G.

Remark: There is a functor

$$U_{CMod} : Pro{-}\mathcal{C}.CMod \to CMod,$$

which forgets the topological structure. The question of the existence of a left adjoint to U_{CMod}, i.e., of a pro$-\mathcal{C}$ completion functor for crossed modules, will be the subject of the next chapter.

We next turn to examples. In each of these examples removal of the topological conditions gives a corresponding example of a crossed module.

Examples: (i) Let H be a closed normal subgroup of a pro$-\mathcal{C}$ group G with $i : H \to G$ the inclusion, then we will say (H, G, i) is a *closed normal subgroup pair*. In this case, of course, G acts continuously on the left of H by conjugation and the inclusion homomorphism i makes (H, G, i) into a pro$-\mathcal{C}$ crossed module.

(ii) Suppose G is a pro$-\mathcal{C}$ group and M is a pseudocompact left $\widehat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-module; let $0 : M \to G$ be the trivial map sending everything in M to the identity element of G, then $(M, G, 0)$ is a pro$-\mathcal{C}$ crossed module.

As these two examples suggest, pro$-\mathcal{C}$ crossed modules lie between the two extremes of closed normal subgroups and pseudocompact modules. Their structure bears a certain resemblance to both - they are "external" closed normal subgroups but also are "twisted" pseudocompact modules.

Our third example gives yet another naturally occurring class of pro$-\mathcal{C}$ crossed modules for $\mathcal{C} = FGrps$, the category of finite groups.

(iii) Let G be a finitely generated profinite group, then $\mathrm{Aut}(G)$, the group of continuous automorphisms of G, is also profinite in the topology of uniform convergence, (cf. [3] and [153]). It should be remarked that for arbitrary profinite G, $\mathrm{Aut}(G)$, although totally disconnected in this topology, can fail to be compact.

Conjugation gives a continuous homomorphism

$$\partial : G \to \mathrm{Aut}(G).$$

Of course, $\mathrm{Aut}(G)$ acts continuously on G and ∂ is a profinite crossed module.

(iv) We suppose given a continuous morphism

$$\theta : M \to N$$

of pseudocompact left $\widehat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-modules and form the semi-direct product $N \rtimes G$. This is a pro$-\mathcal{C}$ group which we make act continuously on M via the projection from $N \rtimes G$ to G.

We define a continuous morphism

$$\partial : M \to N \rtimes G$$

by $\partial(m) = (\theta(m), 1)$, where 1 denotes the identity element of G, then $(M, N \rtimes G, \partial)$ is a pro$-\mathcal{C}$ crossed module.

(v) As a last example, let

$$1 \to K \xrightarrow{a} E \xrightarrow{b} B \to 1$$

be an extension of profinite groups with K a central subgroup of E. For each $g \in G$, use the existence of continuous sections, Corollary 1, to give an element $s(g) \in b^{-1}(g) \subseteq E$. Define an action of G on E by: if $x \in E$, $g \in G$, then

$${}^{g}x = s(g)xs(g)^{-1}.$$

This is well defined and continuous, since if $s(g)$, $s'(g)$ are two choices, $s(g) = ks'(g)$ for some $k \in K$, and K is central. (This also shows that this *is* an action.) The structure (E, G, b) is a profinite crossed module.

A particular important case, but we are here not in the profinite context as yet, is: for R a ring, let $E(R)$ be, as before, the group of elementary matrices of R, $E(R) \subseteq Gl(R)$ and $St(R)$, the corresponding Steinberg group with $b : St(R) \to E(R)$, the natural morphism, (see later or [119], for the definition). Then this gives a central extension

$$1 \to K_2(R) \to St(R) \to E(R) \to 1$$

and thus a crossed module. In fact

$$b : St(R) \to Gl(R)$$

is a crossed module. The group $Gl(R)/Im(b)$ is $K_1(R)$, the first algebraic K-group of the ring.

The following proposition also gives another range of examples of pro$-\mathcal{C}$ crossed modules; in the discrete case, it is an observation of R. Brown.

Proposition 19. *Let $\partial : A \to G$ and $\delta : B \to G$ be two pro$-\mathcal{C}$ crossed modules and let $(\phi, Id) : (A, G, \partial) \to (B, G, \delta)$ be a morphism in Pro$-\mathcal{C}.CMod/G$. Then defining a continuous B-action on A by ${}^b a = {}^{\delta(b)} a$, we have (A, B, ϕ) is a pro$-\mathcal{C}$ crossed module.* ■

The proof is an easy exercise in using the two crossed module axioms.

Remark: As mentioned earlier, one standard example of a crossed module occurs in topology with a pointed pair of spaces (X, A, x_0), then

$$\pi_2(X, A, x_0) \to \pi_1(A, x_0)$$

is a crossed module. When the pointed pair comes from an algebraic or combinatorial situation, then pro$-\mathcal{C}$ analogues of this can sometimes be constructed as we will see later.

Another standard example comes from a pointed fibration

$$F \to E \to B.$$

The $\pi_1(F, f_0) \to \pi_1(E, e_0)$ is a crossed module. The pro$-\mathcal{C}$ analogue can be approached via simplicial pro$-\mathcal{C}$ groups. In that context, the fibration is just an epimorphism, ϕ, and the fibre, F is replaced by the kernel of ϕ; see later 4.3.

Two useful constructions are those of the kernel and cokernel of a crossed module. We will see below that, for a pro-$\mathcal{C}$ crossed module, $\mathsf{X} = (\partial : C \to G)$, the image of ∂ is a closed normal subgroup of G so we can form the quotient group, $G/Im\,\partial = Coker\partial$, which is sometimes denoted $\pi_0(\mathsf{X})$. The kernel $Ker\,\partial = \pi_1(\mathsf{X})$ is a module over $\pi_0(\mathsf{X})$. A morphism of crossed modules $\mathsf{f} : \mathsf{X} \to \mathsf{X}'$ induces morphisms $\pi_i(\mathsf{f}) : \pi_i(\mathsf{X}) \to \pi_i(\mathsf{X}')$ for $i = 0, 1$ and is called a *weak equivalence* if these are isomorphisms.

3.2 Elementary Properties.

The first few of these will be, more or less, the converse of the examples we gave in the last section.

We suppose given a pro$-\mathcal{C}$ crossed module $\partial : C \to G$.

3.2.1 Images are normal.

Lemma 4. *Let $N = Im\ \partial$, then N is a closed normal subgroup of G.*

Proof: This is more or less immediate from axiom CM1. ■

3.2.2 Kernels are central

Lemma 5. *Let $A = Ker\ \partial$, then A lies in the centre of C, hence is an Abelian pro$-\mathcal{C}$ group.*

Proof: Since A is a kernel of a continuous homomorphism of pro$-\mathcal{C}$ groups, it is itself pro$-\mathcal{C}$. The proof that A is central is just that if $a \in A$ and $c \in C$, we have $cac^{-1}a^{-1} = c({}^{\partial a}c^{-1}) = 1_C$, since $\partial a = 1_C$.

The G-action then makes A into a pseudocompact $\widehat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-module. ■

In fact more is true:

3.2.3 Kernels are modules

Lemma 6. *The closed normal subgroup $N = Im\,\partial$ of G acts trivially on A, hence A is naturally a pseudocompact left $\widehat{\mathbb{Z}}_{\mathcal{C}}[\![G/N]\!]$-module.*

Proof: We note that N acts trivially on A. Suppose $n = \partial c$ and $a \in A$,

$$^{n}a = {}^{\partial c}a = cac^{-1} = a,$$

since a is central, so A is thus a topological $\widehat{\mathbb{Z}}_{\mathcal{C}}[\![G/N]\!]$-module. This together with our observations earlier imply that A is a pseudocompact $\widehat{\mathbb{Z}}_{\mathcal{C}}[\![G/N]\!]$-module. ■

These results explain, in part, the sense in which 'pro-$\mathcal{C}$ crossed module lies between the two extremes of closed normal subgroups and pseudocompact modules'. Given any pro-$\mathcal{C}$ crossed module, $\partial : C \to G$, there is a diagram

$$\begin{tikzcd}
Ker\,\partial \arrow[r] \arrow[dr] & C \arrow[r] \arrow[d, "\partial"] & Im\ \partial \arrow[dl, "i"] \\
 & G &
\end{tikzcd}$$

of crossed modules over G with the row exact. In the next few paragraphs, we will explore this sequence more closely, and will continue to use the notation $N = Im\ \partial$ and $A = Ker\ \partial$ as standard.

3.2.4 Categorical gloss on 'extreme cases'

We have a category $Pc_{\mathcal{C}}.G-Mod$ of pseudocompact G-modules and a functorial construction $(-, G, 0)$ from it to $Pro-\mathcal{C}.CMod/G$, namely: M goes to $(M, G, 0)$.

Proposition 20. *The category $Pc_{\mathcal{C}}.G-Mod$ is equivalent to a coreflective subcategory of $Pro-\mathcal{C}.CMod/G$.*

Proof: In other words $(-, G, 0)$ has a right adjoint and is equivalent to an inclusion of a subcategory. This becomes clear if you take (C, G, ∂) and $(M, G, 0)$ and look at some $\phi : (M, G, 0) \to (C, G, \partial)$. Of course, ϕ maps M into $Ker\,\partial$ and so determines $\overline{\phi} : M \to Ker\,\partial$ uniquely as a morphism of modules. The adjoint is thus : (C, G, ∂) goes to $(Ker\,\partial, G, 0)$. ■

This, of course, raises the question of the other 'extreme'. Let $Pro-\mathcal{C}.NSGrps(G)$ denote the category of closed normal subgroups of G with inclusions as morphisms.

Proposition 21. *The functor given by N goes to (N, G, i_N) has a left adjoint, namely (C, G, ∂) goes to $Im\ \partial$. This identifies $Pro-\mathcal{C}.NSGrps(G)$ with a reflective subcategory of $Pro-\mathcal{C}.CMod/G$.*

Proof: That Im gives a functor is easy to see. (If

$$\begin{tikzcd} C \arrow[rr, "\phi"] \arrow[dr, "\partial"'] & & D \arrow[dl, "\partial'"] \\ & G & \end{tikzcd}$$

is a morphism, then $Im\ \partial \subseteq Im\ \partial'$, etc.) Then use that if (N, G, i_N) is a normal inclusion crossed module, and

$$\begin{tikzcd} C \arrow[rr, "\phi"] \arrow[dr, "\partial"'] & & N \arrow[dl, "i_N"] \\ & G & \end{tikzcd}$$

commutes, then $Im\ \partial \subseteq N$. The statement of the proposition is just a neat re-packaging of this. ■

We will use this later in examining the relationships between different categories of crossed modules induced by a morphism of pro-$\mathcal{C}$ groups.

Of course, Pro–$\mathcal{C}.NSGrps(G)$ has a final object, namely the group G itself and this corresponds to (G, G, Id) being the final object of Pro–$\mathcal{C}.CMod/G$.

3.2.5 Abelianisation.

Lemma 7. *The Abelianisation of C has a natural pseudocompact $\widehat{\mathbb{Z}}_{\mathcal{C}}[\![G/N]\!]$-module structure on it.*

Proof: First we should point out that by "Abelianisation" we mean $C^{Ab} = C/[C,C]$, where $[C,C]$ is the *closed* normal subgroup of C generated by all commutators, thus C^{Ab} is Abelian pro$-\mathcal{C}$ and again it suffices to prove that N acts trivially on C^{Ab}. However, if $n \in N$, and $\partial c = n$, then for any $c' \in C$, we have that ${}^{n}c' = {}^{\partial c}c' = cc'c^{-1}$, hence ${}^{n}c'(c')^{-1} \in [C,C]$ or equivalently

$$^{n}(c'[C,C]) = c'[C,C],$$

so N does indeed act trivially on C^{Ab}. ■

Of course N^{Ab} also has the structure of a pseudocompact $\widehat{\mathbb{Z}}_{\mathcal{C}}[\![G/N]\!]$-module and thus a pro$-\mathcal{C}$ crossed module gives one three pseudocompact modules. These three are linked as shown by the following proposition.

Proposition 22. *Let (C,G,∂) be a pro$-\mathcal{C}$ crossed module. Then the induced morphisms*

$$A \to C^{Ab} \to N^{Ab} \to 0$$

form an exact sequence of pseudocompact $\widehat{\mathbb{Z}}_{\mathcal{C}}[\![G/N]\!]$-modules.

Proof: It is clear that the sequence

$$1 \to A \to C \to N \to 1$$

is exact and that the induced homomorphism from C^{Ab} to N^{Ab} is a continuous epimorphism. Since the composite homomorphism from A to N is trivial, A is mapped into $Ker(C^{Ab} \to N^{Ab})$ by the composite $A \to C \to C^{Ab}$. It is easily checked that this is onto and hence the sequence is exact as claimed. The pseudocompact module structures have already been outlined in earlier results. ■

3.2.6 The intersection $A \cap [C,C]$.

The kernel of the homomorphism from A to C^{Ab} is, of course, $A \cap [C,C]$ and this need not be trivial. Brown and Huebschmann ([29], p.160) give the following finite (and hence profinite) example: in examples of type (iii) (see last section) we have the profinite crossed module, $(G, Aut(G), \partial)$, for finitely generated, G. The kernel of ∂ is, of course, the centre ZG of G and $ZG \cap [G,G]$ can be non-trivial, for instance, if G is dicyclic or dihedral.

More information on this intersection was given in the discrete case in the paper, [57], by Ellis and Porter for those crossed modules, (C,G,∂), which are "free" in a sense soon to be made precise. There it is shown that for (C,G,∂) a free crossed module, $Ker\,\partial \cap [C,C]$ is $H_2(N)$, the second homology group of

$N = Im\, \partial$. This is in some ways a crossed module version of the Hopf formula for H_2 and we shall look at this more closely later. This result has also been proved by Huebschmann, [85].

Proposition 23. *If in the above exact sequence of pro$-\mathcal{C}$ groups*

$$1 \to A \to C \to N \to 1,$$

the epimorphism from C to N is continuously split (the splitting need not respect G action) then $A \cap [C,C]$ is trivial.

Proof: Given a continuous splitting $s : N \to C$, the group C can be written as $A \rtimes s(N)$. The commutators in C, therefore, all lie in $s(N)$ since A is Abelian, but then, of course, $A \cap [C,C]$ cannot contain any non-trivial elements. ∎

Remark: The discrete case of the above proposition is to be found in Brown-Huebschmann [29]. It should perhaps be pointed out that $C \to N$ will always have a continuous section, (cf. Corollary 1, page 12, Schatz [150], or Serre, [152]), but that this, of course, is not usually a group homomorphism.

3.2.7 Free pro-$\mathcal{C}$ groups: some difficulties.

The above proposition applies in particular when N is a free pro$-\mathcal{C}$ group. To ensure this, it is not always sufficient to require that G be a free pro$-\mathcal{C}$ group itself as the pro$-\mathcal{C}$ analogue of the Nielsen-Schreier theorem is in general false. For certain important classes $\mathcal{C}$, however, it holds, but sometimes in a limited form only. We refer the reader to Gildenhuys and Lim [70] and Lubotzky and van den Dries [109] for detailed discussions of the problem. We also note that Tate has proved a pro$-p$ version of the Nielsen-Schreier theorem: closed subgroups of free pro$-p$ groups are free pro$-p$ groups.

In applications the above situation arises mostly from presentations where G is a free pro$-\mathcal{C}$ group on a profinite space X and $N = N(R)$, the closed normal closure of a space of relations R and we recall the terminology introduced in 1.4, namely that if N is free pro$-\mathcal{C}$, we will say that $(X : R)$ is a *free pro$-\mathcal{C}$ presentation* of $F_{\mathcal{C}}(X)/N(R)$. If $\mathcal{C}$ is closed under extensions of groups, then Lubotzky and van den Dries ([109] p.29) prove that open subgroups of finitely generated free pro$-\mathcal{C}$ groups are free pro$-\mathcal{C}$, thus if $(X : R)$ is a finite presentation of a $\mathcal{C}$-group considered as a pro$-\mathcal{C}$ presentation, we have that it will be a free pro$-\mathcal{C}$ presentation since $F_{\mathcal{C}}(X)/N(R)$ is discrete, hence $N(R)$ is open.

3.3 Induced and restricted pro-$\mathcal{C}$ crossed modules.

We have already introduced the category *Pro$-\mathcal{C}$.CMod/G* for fixed pro$-\mathcal{C}$ group G. In the case of modules over a group G, as we have recalled in

Chapter 1, there are important functors from *G-Mod* to *H-Mod* and back, corresponding to a group homomorphism $\phi : G \to H$ and giving an adjoint pair; one of these is restriction along ϕ and gives a G-module structure to an H-module; the other is the induced H-module construction. Similar functors exist for pseudocompact modules over pro$-\mathcal{C}$ groups, G and H, corresponding to a *continuous* ϕ. We have seen (3.2.4) that $Pc_{\mathcal{C}}G\text{-}Mod$ is equivalent to a full subcategory of $Pro{-}\mathcal{C}.CMod/G$. In this section we examine the question of extending the restriction/induction adjoint pair to one defined between $Pro{-}\mathcal{C}.CMod/G$ and $Pro{-}\mathcal{C}.CMod/H$ corresponding to a continuous $\phi : G \to H$. (The case for abstract groups may be found in Brown-Higgins, [24]. An interpretation in terms of fibred categories is in [28].)

3.3.1 Restriction along a homomorphism ϕ.

Given a pro$-\mathcal{C}$ crossed module (C, H, ∂) over H and a continuous homomorphism $\phi : G \to H$ of pro-$\mathcal{C}$ groups, we can form the pullback:

$$\begin{tikzcd} D \arrow[r, "\psi"] \arrow[d, "\partial'"'] & C \arrow[d, "\partial"] \\ G \arrow[r, "\phi"'] & H \end{tikzcd}$$

in $Pro{-}\mathcal{C}$. Clearly the universal property of pullbacks gives a good universal property for this morphism of pro$-\mathcal{C}$ crossed modules, namely that any morphism $(\phi', \phi) : (C', G, \delta) \to (C, H, \partial)$ factors uniquely through (ψ, ϕ) and a morphism in $Pro{-}\mathcal{C}.CMod/G$ from (C', G, δ) to (D, G, ∂'). Of course this statement depends on verification that (D, G, ∂') is a pro$-\mathcal{C}$ crossed module and that the resulting maps are morphisms of pro$-\mathcal{C}$ crossed modules, but this is routine, and can be safely left as an exercise.

This construction also behaves nicely on morphisms of pro$-\mathcal{C}$ crossed modules over H and yields a functor

$$\phi_* : Pro - \mathcal{C}.CMod/H \to Pro - \mathcal{C}.CMod/G$$

which will be called *restriction along* ϕ.

3.3.2 The extreme cases: normal subgroups and modules.

Before turning to the adjoint situation of *extension along* ϕ, it is perhaps instructive to examine the above construction for the two extreme cases of closed normal subgroups and pseudocompact modules.

First let us note that if (C, H, ∂) is a pro$-\mathcal{C}$ crossed module with ∂ a monomorphism, then (C, H, ∂) is isomorphic to $(\partial(C), H,$ inclusion), i.e., to a closed normal subgroup example. We should point out that as ϕ_* is only defined up to isomorphism (as is any construction given by a universal property), one cannot hope that $\phi^*(N, H, i)$ will be exactly a "closed normal subgroup pair", but can ask when it is isomorphic to such a pair. Now writing $\phi_*(N, H, i) = (\phi_*(N), G, \phi_*(i))$ for convenience, we find $\phi_*(N) = \{(g, n) : \phi(g) = n\}$, so $\phi_*(N)$ is isomorphic to $\phi^{-1}(N) \subset G$, and ϕ_* is in this case a well known construction.

At the other extreme, if M is a pseudocompact $\widehat{\mathbb{Z}}_{\mathcal{C}}[\![H]\!]$-module, $\phi_*(M, H, 0)$ need not have a trivial structure map. In fact if $\phi_*(M, H, 0) = (D, G, \partial)$, D consists of pairs (g, m) with $\phi(g) = 0$ and so is isomorphic to $Ker\,\phi \times M$ as a group, with G acting continuously on $Ker\,\phi$ by conjugation and on M via ϕ. Thus we have

$$\phi^*(M, H, 0) \cong (Ker\,\phi \times \phi^\sharp(M), G, ipr_1),$$

where here we have written $\phi^\sharp(M)$ for the restriction of M along ϕ in the sense of module theory. Thus the two constructions do coincide when ϕ is a monomorphism.

The interpretation of the above situations may be helped by the following observations. Let $Pc_{\mathcal{C}}.G - Mod$, as before, be the category of pseudocompact $\widehat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-modules and consider the "inclusion" of it as a full subcategory of $Pro{-}\mathcal{C}.CMod/G$ given by M goes to $(M, G, 0)$. This functor has a right adjoint (cf. 3.2.4) which sends (C, G, ∂) to $Ker\,\partial$. Similarly, $Pro{-}\mathcal{C}.NSGps(G)$, denote the category of closed normal subgroups of the pro$-\mathcal{C}$ group G, with monomorphisms as the morphisms, then the functor from this category to $Pro{-}\mathcal{C}.CMod/G$ has a left adjoint, (again cf. 3.2.4) by which (C, G, ∂) goes to $(Im\,\partial, G)$. As we shall show that ϕ^* is a right adjoint to a functor ϕ_*, standard categorical arguments would have led us to expect that ϕ_* would "preserve" the construction in this second case, but that it might not do so in the earlier one.

3.3.3 Extension along ϕ.

We next consider the problem of induction, i.e., we suppose that we have a pro$-\mathcal{C}$ crossed module over G, say (C, G, ∂) and our continuous morphism $\phi : G \to H$ of pro$-\mathcal{C}$ groups and shall try to construct a "universal arrow" from (C, G, ∂) to some pro$-\mathcal{C}$ crossed module over H with ϕ in its base level.

Proposition 24. *Let (C, G, ∂) be a pro$-\mathcal{C}$ crossed module over G and let $\phi : G \to H$ be a continuous homomorphism of pro$-\mathcal{C}$ groups. Consider the pro$-\mathcal{C}$ group $\phi_*(C)$ topologically generated by the profinite space $C \times H$ with relations*

(i) $(c_1, h).(c_2, h) = (c_1c_2, h)$

(ii) $({}^gc, h) = (c, h\phi(g))$

(iii) $(c_1, h_1)(c_1, h_1)(c_1, h_1)^{-1} = (c_2, h_1(\phi\partial c_1)h_1^{-1}h_2)$

for all $h, h_1, h_2 \in H$, $c, c_1, c_2 \in C$ and $g \in G$.

Define a continuous homomorphism $\delta : \phi_*(C) \to H$ by extending $\delta(c,h) = h(\phi\partial c)h^{-1}$ to the whole of $\phi_*(C)$ and define a continuous $\widehat{\mathbb{Z}}_{\mathcal{C}}[\![H]\!]$-action on the left of $\phi_*(C)$ by ${}^h(c,h_1) = (c,hh_1)$ for $h,h_1 \in H$, $c \in C$, and a continuous homomorphism $\Psi : C \to \phi_*(C)$ by $\Psi(c) = (c,1)$, then

(a) $(\phi_*(C),H,\delta)$ is a pro–$\mathcal{C}$ crossed module over H,

(b) $(\Psi,\phi) : (C,G,\partial) \to (\phi_*(C),H,\partial)$ is a (continuous) morphism of pro–$\mathcal{C}$ crossed modules which has the following universal property:

Given any pro–$\mathcal{C}$ crossed module, (D,H,∂'), over H, and continuous morphism $(\theta,\phi) : (C,G,\partial) \to (D,H,\partial')$, (θ,ϕ) factorises in a unique way via (Ψ,ϕ) and a morphism $(\overline{\theta},Id_H)$, of pro–$\mathcal{C}$ crossed modules over H, i.e.,

$$\begin{tikzcd} C \arrow[r, "\theta"] \arrow[d, "\partial"'] & D \arrow[d, "\partial'"] \\ G \arrow[r, "\phi"'] & H \end{tikzcd} \quad = \quad \begin{tikzcd} C \arrow[r, "\Psi"] \arrow[d, "\partial"'] & \phi_*(C) \arrow[rr, "\overline{\theta}"] \arrow[dr, "\delta"] & & D \arrow[dl, "\partial'"'] \\ G \arrow[rr, "\phi"'] & & H & \end{tikzcd}$$

for a unique (θ,Id_H) in Pro–$\mathcal{C}$.CMod/H.

Proof: The statements about continuity are fairly trivial, leaving us to check the algebraic conditions. It is simple to verify *(a)*, in fact we have

$$\begin{aligned} \delta({}^h(c,h_1)) &= \delta(c,hh_1) \\ &= hh_1(\phi\partial c)(hh_1)^{-1} \\ &= h(\delta(c,h_1))h^{-1} \end{aligned}$$

and

$$\begin{aligned} {}^{\delta(c,h)}(c,h_1) &= {}^{h(\phi\partial c)h^{-1}}(c_1,h_1) \\ &= (c_1,h(\phi\partial c)h^{-1}h_1) \\ &= (c,h)(c_1,h_1)(c,h)^{-1} && \text{by (iii).} \end{aligned}$$

It is equally easy to check *(b)*, so this will be omitted.

Finally, given $(\theta,\phi) : (C,G,\partial) \to (D,H,\partial')$, we define $\bar{\theta} : \phi_*(C) \to D$ by

$$\bar{\theta}(c,h) = {}^h\theta(c).$$

It is clear that, as θ has to be H-equivariant, this must be the formula for $\bar{\theta}$ if $\bar{\theta}\Psi$ is to equal θ. That $\bar{\theta}$ is well defined, continuous and respects the defining relations for $\phi_*(C)$ can easily be checked. ∎

It is clear from the construction given above that ϕ_* gives a functor "extension along ϕ"

$$Pro-\mathcal{C}.CMod/G \to Pro-\mathcal{C}.CMod/H.$$

It is also fairly routine to check that ϕ_* is left adjoint to ϕ^*. This can be neatly formulated by introducing the notation $Pro\text{–}\mathcal{C}.CMod/\phi((C,G,\partial),(D,H,\partial'))$

for the set of morphisms, $(\eta, \phi) : (C, G, \partial) \to (D, H, \partial')$, in which the "base morphism" is the fixed ϕ. This gives a functor defined on $(Pro-\mathcal{C}.CMod/G)^{op} \times (Pro-\mathcal{C}.CMod/H)$ with values in $Sets$. For fixed (C, G, ∂) this functor is representable by $\phi_*(C, G, \partial)$ and for fixed (D, H, ∂'), it is representable by $\phi^*(D, H, \partial')$, these being just restatements in terms of representability of the universal properties of ϕ^* and ϕ_*. Of course this implies there is a natural isomorphism

$$Pro-\mathcal{C}.CMod/G((C, G, \partial), \phi^*(D, H, \partial'))$$
$$\cong Pro-\mathcal{C}.CMod/H(\phi_*(C, G, \partial), (D, H, \partial'))$$

as stated.

3.3.4 The extreme cases revisited.

Although as suggested earlier, this adjointness allows one to compare this construction of ϕ_* with the well known ones on closed normal subgroup pairs and on pseudocompact modules, it is in fact quite instructive to look at these using a "bare hands" approach.

Suppose $(M, G, 0)$ is a pseudocompact module (considered as a pro$-\mathcal{C}$ crossed module). The construction of $\phi_*(M, G, 0)$ here interprets as follows: $\phi_*(M)$ is topologically generated by $M \times H$ with relations of the form

(i)) $(m_1, h)(m_2, h)) = (m_1 m_2, h)$
(ii) $({}^g m, h) = (m, h\phi(g))$
and
(iii) $(m_1, h_1)(m_2, h_2)(m_1, h_1)^{-1} = (m_2, h_2)$.

Of course (iii) implies that $\phi_*(M)$ is Abelian, (i) and (ii) show it to be isomorphic to the usual induced module $\phi_\sharp(M)$, i.e., $\widehat{\mathbb{Z}}_{\mathcal{C}}[\![H]\!] \otimes_{\widehat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]} M$, (of course, one has to convert to additive notation to obtain the usual format) and $\delta : \phi_*(M) \to H$ is the trivial morphism, i.e., one has an isomorphism, $\phi_*(M, G, 0) \cong (\phi_\sharp(M), H, 0)$. When considering the corresponding question for a closed normal subgroup N of G, one is really only asking if $Ker(\delta : \phi_*(N) \to H)$ is trivial, but $Ker\,\delta$ will contain all (n, h) with $i(n) \in Ker\,\phi$ and so, in particular, in the case when ϕ is the canonical surjection from G to G/N, the ϕ_* construction will give a G/N-module and the trivial morphism from that to G/N. One, of course, immediately suspects that $\phi_*(N) \cong N^{Ab}$ and a direct verification of this using the universal property is easily made. Thus ϕ_* certainly does not preserve the subcategory of closed normal subgroups without additional conditions on ϕ itself.

3.3.5 Extension along an epimorphism.

The discussion of the case $\phi : G \to G/N$ above generalises to give a useful description of ϕ_* in all such cases.

Proposition 25. *Let (C, G, ∂) be a pro$-\mathcal{C}$ crossed module and $\phi : G \to H$ be an epimorphism with kernel K. Let $D = C/[C, K]$, where $[C, K]$ is the closed subgroup of C generated by all symbols $c({}^k c)^{-1}$ for $c \in C$, $k \in K$. Define $\delta(c[C, K]) = \phi(\partial c)$; then $\phi_*(C, G, \partial) \cong (D, H, \delta)$.*

Proof: We first check that $[C, K]$ is a G-invariant subgroup of C. We note that for $g \in G$, $k \in K$, and $c \in C$, we have

$$\begin{aligned} {}^g(c({}^k c)^{-1}) &= ({}^g c)\ {}^g({}^k c)^{-1} \\ &= ({}^g c)\ {}^{gkg^{-1}}({}^g c)^{-1} \quad \in [C, K], \end{aligned}$$

since $K \lhd G$. Hence G acts continuously on $C/[C, K]$, but K acts trivially, thus giving a continuous action of $H \cong G/K$ on $C/[C, K]$. It is clear that δ is a continuous H-equivariant homomorphism from $D = C/[C, K]$ to H. It is easily verified that (D, H, δ) is a pro$-\mathcal{C}$ crossed module.

Now finally suppose

$$\begin{tikzcd} C \arrow[r, "\mu"] \arrow[d, "\partial"'] & C' \arrow[d, "\partial'"] \\ G \arrow[r, "\phi"] & H \end{tikzcd}$$

is a morphism of pro-$\mathcal{C}$ crossed modules over ϕ as base morphism. Since $\mu({}^g c) = {}^{\phi(g)}\mu(c)$, we have that $\mu({}^k c) = \mu(c)$ for all $k \in K$, so $\mu([C, K])$ is trivial, hence μ factors via D as required. The diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\partial"'] & C/[C, K] \arrow[rr, "\overline{\mu}"] \arrow[dr, "\delta"] & & D \arrow[dl, "\partial'"'] \\ G \arrow[rr, "\phi"'] & & H & \end{tikzcd}$$

commutes since $\delta(c[C, K]) = \phi\partial c = \partial'\mu(c) = \partial'\overline{\mu}(c[C, K])$ for all $c \in C$. ∎

Corollary 4. *If $\phi : G \to H$ is an epimorphism with kernel K and $N \lhd G$ is a closed normal subgroup of G, $\phi_*(N, G, i)$ has top group isomorphic to $N/[N, K]$ (in the usual sense of the symbol) and so has kernel isomorphic to $(N \cap K)/[N, K]$.* ∎

3.3.6 Induction from final crossed modules.

We now turn to one of the most useful applications of induced pro-$\mathcal{C}$ crossed modules.

Consider the final object in the category *Pro-$\mathcal{C}$.CMod/G*. This is (G, G, Id). Although it seems somewhat trivial, it turns out to be extremely useful when its image by any ϕ_* is considered.

Suppose we have $\phi : G \to H$ as before and consider $\phi_*(G, G, Id)$. We can give a presentation for the "top group" of this; it has $G \times H$ as generators, and relations:

(i) $(g_1, h)(g_2, h) = (g_1g_2, h)$
(ii) $(g_1, h_1)(g_2, h_2)(g_1, h_1)^{-1} = (g_2, h_1(\phi(g_1))h_1^{-1}h_2)$,

the other relation listed (ii) in Proposition 25 being a consequence of these two in this special case.

We noted earlier, (in 3.3.3), that

$$\phi_* : Pro\!-\!\mathcal{C}.CMod/G \to Pro\!-\!\mathcal{C}.CMod/H$$

was left adjoint to ϕ^*, hence one has an isomorphism, natural in (D, H, ∂'),

$$Pro\!-\!\mathcal{C}.CMod/H(\phi_*(G, G, Id), (D, H, \partial'))$$
$$\cong Pro\!-\!\mathcal{C}.CMod/G((G, G, Id), \phi^*(D, H, \partial'))$$

and that both are naturally isomorphic to

$$Pro\!-\!\mathcal{C}.CMod/\phi((G, G, Id), (D, H, \partial')).$$

Now this last set, of course, consists of morphisms $\mu : G \to D$ such that $\partial'\mu = \phi$ (and $\mu(g_1g_2g_1^{-1}) = {}^{\phi(g_1)}\mu(g_2)$, but the second of these is a consequence of the first, together with the Peiffer identity), and so the above set consists precisely of the continuous homomorphisms $\mu : G \to D$ that make the diagram

$$\begin{tikzcd} G \arrow[rr, "\mu"] \arrow[dr, "\phi"'] & & D \arrow[dl, "\partial'"] \\ & H & \end{tikzcd}$$

commute. Introducing the category, Pro–$\mathcal{C}/H$, of pro–$\mathcal{C}$ groups over H, there is a forgetful functor

$$U_1 : Pro\!-\!\mathcal{C}.CMod/H \to Pro\!-\!\mathcal{C}/H$$

given by $U_1(C, H, \partial) = (\partial : C \to H)$. With this notation we have identified

$$Pro\!-\!\mathcal{C}.CMod/\phi((G, G, Id), (D, H, \partial'))$$

with Pro–$\mathcal{C}/H(\phi, U_1(D, H, \partial'))$ and have a natural isomorphism,

$$Pro\!-\!\mathcal{C}/H(\phi, U_1(D, H, \partial')) \cong Pro\!-\!\mathcal{C}.CMod/H(\phi_*(G, G, Id), (D, H, \partial')).$$

Corollary 5. *The functor U_1 has a left adjoint, F_1, given by sending $\phi : G \to H$ to $\phi_*(G, G, Id)$.* ■

This follows easily from our discussion.

3.3.7 Various forgetful functors and their right adjoints.

Clearly there are functors that forget more than does U_1. For instance one can, in addition, forget the group structure on the "top group", C, leaving one with just a continuous function with codomain the underlying profinite space, $U(H)$, of H. This gives one a forgetful functor

$$U_2 : Pro-\mathcal{C}.CMod/H \to Pro-\mathcal{C}.Spaces/U(H).$$

Corollary 6. *The functor U_2 has a left adjoint, F_2, given by sending $X \xrightarrow{f} U(H)$ to*

$$(\tilde{f})_*(F_\mathcal{C}(X), F_\mathcal{C}(X), Id)$$

where $F_\mathcal{C}$ denotes the free pro–$\mathcal{C}$ group functor and $\tilde{f}$ is the homomorphism uniquely related to f by the adjunction isomorphism:

$$Pro-\mathcal{C}.Spaces(X, U(H)) \cong Pro-\mathcal{C}(F_C(X), H).$$

Proof: We noted that U_2 was the composite of U_1 with the functor sending $(\phi : G \to H)$ to $(U(\phi) : U(G) \to U(H))$ in *Pro–$\mathcal{C}$.Spaces/*$U(H)$. This latter has a left adjoint, namely the functor which sends $f : X \to U(H)$ to $\tilde{f} : F_\mathcal{C}(X) \to H$. (This comes from the naturality in H of the isomorphism given in the statement of the corollary). Now U_2 has a left adjoint F_2 given as the composite of the left adjoint of "ϕ goes to $\tilde{U}(\phi)$" with F_1, hence F_2 sends f to $(\tilde{f})_*(F_\mathcal{C}(X), F_\mathcal{C}(X), Id)$ as stated. ∎

Definition: (a) Let $G \xrightarrow{\phi} H$ be a continuous morphism of pro–$\mathcal{C}$ groups, then $\phi_*(G, G,\mathrm{Id})$ is called the *free pro–$\mathcal{C}$ crossed module* on ϕ.

(b) Let H be a pro–$\mathcal{C}$ group, X a pro–$\mathcal{C}$ space and $f : X \to U(H)$ a continuous function from X to the underlying space of H. Let $\tilde{f} : F_\mathcal{C}(X) \to H$ be the homomorphism from the free pro–$\mathcal{C}$ group on X to H induced by f, then $(\tilde{f})_*(F_\mathcal{C}(X), F_\mathcal{C}(X),\mathrm{Id})$ is called the *free pro–$\mathcal{C}$ crossed module on f.*

In the next section we will study free pro–$\mathcal{C}$ crossed modules in more detail.

Remark: As was stated earlier, the discrete analogues of most of the results of this section are due to Brown and Higgins, [24] and a treatment in terms of fibred categories can be found in [28].

3.4 Free pro-$\mathcal{C}$ crossed modules.

Although these were introduced and shown to exist in the last section, their importance for later theory is such that they warrant a section on their own. In fact it will pay to prove their existence in a slightly different way as this provides a useful set of notation and terminology for later use.

Proposition 26. *A pro–$\mathcal{C}$ crossed module over G, (C, G, ∂), is isomorphic to the free pro–$\mathcal{C}$ crossed G-module on a continuous function $f : X \to G$ from a profinite space X to a pro–$\mathcal{C}$ group G if and only if the following universal property is satisfied: the given function f can be written ∂v for some continuous function $v : X \to C$ such that given any pro–$\mathcal{C}$ crossed module, (A, G, δ), over G and continuous function $w : X \to A$ such that $\delta w = f$, there is a unique morphism $\phi : C \to A$ of pro–$\mathcal{C}$ crossed modules over G such that $\phi v = w$.*

Proof: This is really nothing more nor less than the obvious reformulation of the universal property summarised by the isomorphisms given in the previous section.

In fact we consider the case

$$(C, G, \partial) = (\tilde{f})_*(F_{\mathcal{C}}(X), F_{\mathcal{C}}(X), Id).$$

The diagram

$$\begin{tikzcd} F_{\mathcal{C}}(X) \arrow[r, "\Psi"] \arrow[d, "Id"'] & \tilde{f}_*(F_{\mathcal{C}}(X)) = C \arrow[d, "\partial"] \\ F_{\mathcal{C}}(X) \arrow[r, "\tilde{f}"'] & G \end{tikzcd},$$

defining the induced pro–$\mathcal{C}$ crossed module shows that $\partial\Psi = \tilde{f}$, but on letting $\varepsilon : X \to F_{\mathcal{C}}(X)$ (strictly speaking, $\varepsilon : X \to UF_{\mathcal{C}}(X)$) be the insertion of generators, one obtains $\partial\Psi\varepsilon = \tilde{f}\varepsilon = f$, so we get a function $v : X \to C$ lifting f as required. Now given $w : X \to A$, we get $\tilde{w} : F_{\mathcal{C}}(X) \to A$ and $\delta w = f$ implies $\delta\tilde{w} = \tilde{f}$, so we have a morphism in *Pro–$\mathcal{C}$.CMod*

$$\begin{tikzcd} F_{\mathcal{C}}(X) \arrow[r, "\tilde{w}"] \arrow[d, "Id"'] & A \arrow[d, "\delta"] \\ F_{\mathcal{C}}(X) \arrow[r, "\tilde{f}"'] & G \end{tikzcd}$$

and hence a factorisation via $(\tilde{f})_*(F_{\mathcal{C}}(X), F_{\mathcal{C}}(X)$,Id) as required.

Thus if (C, G, ∂) is isomorphic to our previously constructed free pro–$\mathcal{C}$ crossed module, it has the required universal property. The converse follows by the usual "uniqueness up to isomorphism" of objects defined by universal properties. ■

3.4.1 A simplified construction.

We next give a slightly simplified description of the construction of the free pro–$\mathcal{C}$ crossed module on $f : X \to G$. The details of this construction were somewhat obscured by the added generality of the induction process and as they are important later, it is worth giving a simpler version here.

We suppose given a continuous function, $f : X \to G$. Let $E = F_{\mathcal{C}}(G \times X)$ be the free pro–$\mathcal{C}$ group on $G \times X$ and make G act on E by

$$^{g}(h, x) = (gh, x).$$

This is a continuous action. The function f induces a morphism $\theta : E \to G$ defined on generators by

$$\theta(g, x) = gf(x)g^{-1}.$$

The *Peiffer subgroup*, P, is the closed subgroup of E generated by the elements

$$uvu^{-1}({}^{\theta u}v)^{-1}$$

where $u, v \in E$.

Lemma 8. *The Peiffer subgroup P of E is a closed normal G-invariant subgroup of E.* ■

The proof is routine and will be omitted.

We note that $\theta(P) = \{1\}$, thus putting $C = E/P$, we obtain an induced continuous G-equivariant homomorphism

$$\theta_\sharp : C \to G.$$

Of course since we have killed off P, the Peiffer identity holds for $(C, G, \theta_\sharp)$. It is now fairly easy to check that $(C, G, \theta_\sharp)$ has the required universal property.

We note, for use in later chapters, the following two short exact sequences of pro–$\mathcal{C}$ groups

$$1 \to P \to E \to C \to 1,$$

and

$$1 \to I \to E \to N \to 1,$$

where $N = \theta_\sharp(C) = \theta(E)$.

3.4.2 A special case: when $f(X)$ topologically generates G

We next turn to a trivial, but very useful, consequence of our generalisation to pro–$\mathcal{C}$ groups of the description of $\theta_*(C, G, \partial)$ for θ an epimorphism. This provides a particularly neat description of the free pro–$\mathcal{C}$ crossed module on a continuous $f : X \to G$ where $f(X)$ topologically generates G.

Proposition 27. *Let $f : X \to G$ be a continuous function from a profinite space X to a pro–$\mathcal{C}$ group G such that $f(X)$ topologically generates G. Let $F_{\mathcal{C}}(X)$ denote the free pro–$\mathcal{C}$ group on X and denote by $R(X)$ the kernel of the induced map from $F_{\mathcal{C}}(X)$ to G. Then f induces a continuous homomorphism, $\bar{f} : F_{\mathcal{C}}(X)/[F_{\mathcal{C}}(X), R(X)] \to G$, and this is the free pro–$\mathcal{C}$ crossed module on f.*

Proof: As $\tilde{f} : F_{\mathcal{C}}(X) \to G$ is onto, we have from Proposition 25, that

$$(\tilde{f})_*(F_{\mathcal{C}}(X), F_{\mathcal{C}}(X), Id) \cong (F_{\mathcal{C}}(X)/[F_{\mathcal{C}}(X), R(X)], G, \bar{f}).$$

■

3.4.3 A generalisation.

As a final result describing free pro–$\mathcal{C}$ crossed modules, we note the following which allows one to use consequences of Proposition 27 in more than the limited number of cases that the surjectivity restriction might suggest.

Proposition 28. *If $\partial : C \to G$ is a free pro–$\mathcal{C}$ crossed module on some function $f : X \to G$ and $\partial C = N$ then the codomain restriction of ∂, $\partial' : C \to N$ is a free pro$-\mathcal{C}$ crossed module on a function $f' : T \times X \to N$ where T is the image of a continuous transversal of N in G and $1 \in T$.*

Proof: The transversal T is the image of G/N under a continuous map τ from G/N to G. Such a transversal exists by Corollary 2. The condition that $\tau(1) = 1$ can be obtained by translation.

Now define $f' : T \times X \to N$ by

$$f'(t, x) = tf(x)t^{-1};$$

f' is clearly continuous and its image topologically generates N.

Suppose $\delta : A \to N$ is an arbitrary pro$-\mathcal{C}$ crossed module over N and let $w : T \times X \to A$ be a function such that $\delta w = f'$. Define $w' : E \to A$ by

$$w'(g, x) = {}^{n(g)}w(\tau(\bar{g}), x),$$

where $\bar{g} = gN =$ the image of g in G/N, and $n(g) = g\tau(\bar{g})^{-1}$, and where E, as before, is the free pro$-\mathcal{C}$ group on $G \times X$. This description of $n(g)$ and $\tau(\bar{g})$ implies that $w'(g, x)$ is continuous on E.

The Peiffer subgroup P is normally generated by the Peiffer elements $uvu^{-1}({}^{\theta u}v)^{-1}$ with $u, v \in G \times X$. If $u = (h, y)$, $v = (g, x)$ and $g = n(g)\tau(\bar{g})$ as before, then since $\theta u \in N$, we have

$$w'({}^{\theta u}v) = {}^{\theta u.n(g)}w(\tau(\bar{g}), x) = {}^{\theta u}w'(v,)$$

whilst $\theta(u) = \delta w'(u)$ implies

$$w'({}^{\theta u}v) = (w'u)(w'v)(w'u)^{-1},$$

i.e., $w'(P)$ is trivial and w' induces a continuous homomorphism $\phi : C \to A$ satisfying $\delta\phi = \partial'$. A routine calculation shows that ϕ is the required unique morphism of pro$-\mathcal{C}$ crossed modules. ■

(We note that this proof, although essentially the same as that for the discrete case, does require careful handling of the transversal to ensure the continuity of w'. The process of generalisation from discrete groups to pro$-\mathcal{C}$ groups is often like this, many results generalise fairly directly but care is always needed over continuity).

Remark: The question arises as to whether or not the restriction that X be profinite is necessary. Putting it more precisely, suppose X is a general space and $f : X \to G$ is a continuous function to the underlying profinite space of some pro$-\mathcal{C}$ group G. Suppose further that (C, G, ∂) is a pro$-\mathcal{C}$ crossed module with exactly the universal property with respect to $f : X \to G$ that was given in 4.1 (but remember that here X need not be profinite). Is then (C, G, ∂) the free pro$-\mathcal{C}$ crossed module on some function $f' : X' \to G$ with X' profinite?

The answer is that it is and the obvious candidate works, namely X' can be taken to be the "profinite completion", $\widehat{X}$, of X. This has the universal property that maps from X to profinite spaces, Y, factor uniquely through a canonical map $\eta_X : X \to \widehat{X}$. In particular f factors as $X \stackrel{\eta_X}{\to} \widehat{X} \stackrel{f'}{\to} Y$ and (C, G, ∂) is the free pro$-\mathcal{C}$ crossed module on $f' : \widehat{X} \to G$.

The uniqueness of this factorisation now can be used fairly easily to check the claim made earlier in this remark.

3.4.4 Projective profinite crossed modules.

We will also need to use later on the notion of a projective profinite G-crossed module. A profinite G-crossed module (C, ∂) is said to be *projective* if it is a projective object in $Prof.CMod/G$. This amounts therefore to the following condition:

Given any epimorphism $\alpha : (A, \partial') \to (B, \partial'')$ in $Prof.CMod/G$ and a morphism $\gamma : (C, \partial) \to (B, \partial'')$, there is a lift of γ to a morphism $\gamma' : (C, \partial) \to (A, \eta_X \partial')$:

$$\begin{tikzcd}
A \arrow[rr, "\gamma'"] \arrow[dr, "\alpha", "onto"'] \arrow[ddr, "\partial'"'] & & C \arrow[dl, "\gamma"'] \arrow[ddl, "\partial"] \\
& B \arrow[d, "\partial''"] & \\
& G &
\end{tikzcd}$$

The easiest projective profinite crossed modules to construct are, of course, the free ones.

4

Identities amongst relations, simplicial groups and other connections

So far we have given constructions and results, but very little in the way of potential applications of these ideas. In later chapters we will use these concepts in both combinatorial and cohomological contexts, but, in both, the use will often depend on constructing pro$-\mathcal{C}$ crossed modules from pro$-\mathcal{C}$ presentations of a pro$-\mathcal{C}$ group and also from pro-$\mathcal{C}$ simplicial groups such as simplicial resolutions of a pro-$\mathcal{C}$ group. Let us first briefly explain the background for this from the abstract case.

4.1 Identities among relations and crossed modules

4.1.1 The complex of a presentation

Many of the constructions of combinatorial and cohomological group theory come directly or indirectly from low dimensional topology (cf. Brown, [21], or Stillwell, [157]). Often the basic construction used is that of the 2-dimensional CW-complex, $K(\mathcal{P})$, constructed from a presentation, $\mathcal{P} = (X : R)$, of a (discrete or abstract) group, G. This complex, $K(\mathcal{P})$, has a single vertex, a one-cell e^1_x for each $x \in X$ and 2-cell e^2_r for each relation $r \in R$. The attaching map for e^2_r represents $r \in F(X) = \pi_1(K(\mathcal{P})_1)$, the free group on X considered as the fundamental group of the 1-skeleton of $K(\mathcal{P})$.

The uses of this complex are many. For instance the chain complex on its universal cover, $\tilde{K}(\mathcal{P})$, provides a link between combinatorial information and cohomology as it provides a free $\mathbb{Z}G$-resolution of the trivial module $\mathbb{Z}$. The homotopy 2-type of $K(\mathcal{P})$, by the results of Whitehead, [164], can be completely captured by a crossed module associated with it. In general if K is a reduced pointed 2-complex and L its 1-skeleton, the crossed module

$$\partial : \pi_2(K, L) \to \pi_1(L),$$

completely determines the homotopy 2-type of K. In this group theoretic context

$$\pi_2(K(\mathcal{P}), K(\mathcal{P})^{(1)}) \to \pi_1(K(\mathcal{P})^{(1)})$$

is (up to isomorphism) simply the free crossed module on the inclusion function from R into $F(X)$.

This last fact shows how we might, in the pro-$\mathcal{C}$ context, still construct a crossed module encoding this type of information, even though the device of constructing a 2-complex, $K(\mathcal{P})$, is not directly available to us.

4.1.2 Group presentations, identities and 2-syzyzgies: the discrete case

To make sense of the profinite and pro-$\mathcal{C}$ crossed module constructions and their potential use, it will be necessary to have some examples of the constructions in the discrete case. In this section we will start by introducing identities for presentations of discrete groups in some more detail, but will also look at results on higher order 'syzygies', i.e. higher order analogues of the identities.

Presentations and Identities (cf. Brown-Huebschmann, [29])

We consider a presentation $\mathcal{P} = (X : R)$ of a group G. We thus have a short exact sequence,

$$1 \to N \to F \to G \to 1,$$

where $F = F(X)$, the free group on the set X, R is a subset of F and $N = N(R)$ is the normal closure in F of the set R. The group F acts on N by conjugation: ${}^u c = ucu^{-1},\ c \in N, u \in F$ and the elements of N are words in the conjugates of the elements of R:

$$c = {}^{u_1}(r_1^{\varepsilon_1})^{u_2}(r_2^{\varepsilon_2}) \dots {}^{u_n}(r_n^{\varepsilon_n})$$

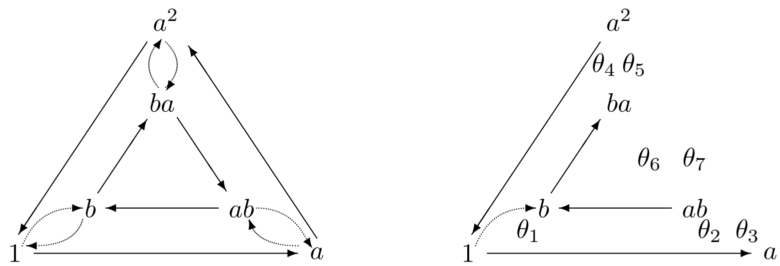
where each ε_i is $+1$ or 1. One also says such elements are *consequences* of R. Heuristically an identity among the relations of $\mathcal{P}$ is such an element c which equals 1. The problem of what this means is analogous to that of working with a relation, since, for example, in the presentation $(a : a^3)$ of C_3, the cyclic group of order 3, if a is thought of as being an element of C_3, then $a^3 = 1$, so why is this different from the situation with the 'presentation', $(a : a = 1)$? To get around that difficulty the free group on the generators $F(X)$ was introduced and, of course, in $F(\{a\})$, a^3 is not 1. A similar device, namely free crossed modules on the presentation will be introduced in a moment to handle the identities. Before that consider some examples which indicate that identities exist even in some quite common-or-garden cases.

Example 1: Suppose $r \in R$, but it is a power of some element $s \in F$, i.e. $r = s^m$. Of course, $rs = sr$ and

$${}^s r r^{-1} = 1$$

so ${}^s r.r^{-1}$ is an identity. In fact, there will be a unique $z \in F$ with $r = z^q$, q maximal with this property. This z is called the *root of* r and if $q > 1$, r is called a *proper power*.

Example 2: Consider one of the standard presentations of S_3, $(a, b : a^3, b^2, (ab)^2)$. Write $r = a^3$, $s = b^2$, $t = (ab)^2$. Here the presentation leads to F, free of rank 2, but $N(R) \subset F$, so it must be free as well, by the Nielsen-Schreier theorem. Its rank will be 7, given by the Schreier index formula or, geometrically, it will be the fundamental group of the Cayley graph of the presentation. This group is free on generators corresponding to edges outside a maximal tree as in the following diagram:



The Cayley graph of S_3 and a maximal tree in it.

The set of normal generators of $N(R)$ has 3 elements; $N(R)$ is free on 7 elements (corresponding to the edges not in the tree), but is specified as consisting of products of conjugates of r, s and t, and there are infinitely many of these. Clearly there must be some slight redundancy, i.e., there must be some identities among the relations!

A path around the outer triangle corresponds to the relation r; each other region corresponds to a conjugate of one of r, s or t. Consider a loop around a region. Pick a path to a start vertex of the loop, starting at 1. For instance the path that leaves 1 and goes along a, b and then goes around aaa before returning by $b^{-1}a^{-1}$ gives $abrb^{-1}a^{-1}$. Now the path around the outside can be written as a product of paths around the inner parts of the graph e.g. $(abab)b^{-1}a^{-1}b^{-1}(bb)(b^{-1}a^{-1}b^{-1}a^{-1})\ldots$ and so on. Thus r can be written in a non-trivial way as a product of conjugates of r, s and t. (An explicit identity constructed like this is given in [29].)

Example 3: In a presentation of the free Abelian group on 3 generators, one would expect the commutators, $[x, y]$, $[x, z]$ and $[y, z]$. The well-known identity, usually called the Jacobi identity, expands out to give an identity among these relations (again see [29], p.154 or Loday, [107].)

The idea that an identity is an equation in conjugates of relations leads one to consider formal conjugates of symbols that label relations. Abstracting this a bit, suppose G is a group and $f : Y \to G$, a function 'labelling' the elements of some subset of G. To form a conjugate, you need a thing being conjugated and an element 'doing' the conjugating, so form pairs (p, y), $p \in G$, $y \in Y$, to be thought of as ${}^py$, the *formal conjugate* of y by p. Consequences are words in conjugates of relations, *formal consequences* are elements of $F(G\times Y)$. There is a function extending f from $G\times Y$ to G given by

$$\bar{f}(p,y)=pf(y)p^{-1},$$

converting a formal conjugate to an actual one and this extends further to a group homomorphism

$$\phi : F(G\times Y)\to G$$

defined to be $\bar{f}$ on the generators. The group G acts on the left on $G\times Y$ by multiplication: $p.(p',y)=(pp',y)$. This extends to a group action of G on $F(G\times Y)$. For this action, ϕ is G-equivariant if G is given its usual G-group structure by conjugations / inner automorphisms.

This is, however, exactly the discrete case of the situation for the construction of free crossed modules, dealt with above, (see section 3.3.7).

We can now formally define the module of identities of a presentation $\mathcal{P}=(X:R)$. We form the free crossed module on $R\to F(X)$, which we will denote by $\partial : C(\mathcal{P})\to F(X)$. The *module of identities* of $\mathcal{P}$ is $Ker\,\partial$. By construction, the group presented by $\mathcal{P}$ is $G\cong F(X)/Im\,\partial$, where $Im\,\partial$ is just the normal closure of the set, R, of relations and we know that $Ker\,\partial$ is a G-module, (see section 3.2.3). We will usually denote the module of identities by $\pi_{\mathcal{P}}$.

The main problem is how to calculate $\pi_{\mathcal{P}}$ or equivalently $\pi_2(K(\mathcal{P}))$. One approach is via an associated chain complex. This can be viewed as the chains on the universal cover of $K(\mathcal{P})$, but can also be defined purely algebraically, however we will postpone this until we have developed a bit more theory.

Homotopical syzygies: There are both homotopical and homological syzygies. To start with we will concentrate on the homotopical versions, but will look at the homological ones later on.

We have built a complex, $K(\mathcal{P})$, from a presentation $\mathcal{P}$ of a group G. Any element in $\pi_2(K(\mathcal{P}))$ can, of course, be represented by a map from S^2 to $K(\mathcal{P})$ and by cellular approximation can be replaced, up to homotopy, by a cellular decomposition of S^2 and a cellular map $\phi : S^2\to K(\mathcal{P})$. We will adopt the terminology of Kapranov and Saito, [98], and Loday, [107], in referring to a pair consisting of a cellular subdivision of S^2 together with a cellular map, as above, as a *homotopical 2-syzygy*. Of course, such an object corresponds to an identity among the relations of $\mathcal{P}$, but is a specific representative of such an identity. A family $\{\phi_\lambda\}_{\lambda\in\Lambda}$ of such homotopical 2-syzygies is then called *complete* when the homotopy classes $\{[\phi_\lambda]\}_{\lambda\in\Lambda}$ generate $\pi_2(K(\mathcal{P}))$.

In this case, we can use the ϕ_λ to form the next stage of the construction of an Eilenberg-MacLane space, $K(G,1)$, by killing this π_2. More exactly, rename $K(\mathcal{P})$ as $X(2)$ and form

$$X(3):=X(2)\cup\bigcup_{\lambda\in\Lambda}e^3_\lambda,$$

by, for each $\lambda \in \Lambda$, attaching a 3-cell, e^3_λ to $X(2)$ using ϕ_λ. Of course, we then have

$$\pi_1(X(3)) \cong G, \quad \pi_2(X(3)) = 0.$$

Again $\pi_3(X(3))$ may be non-trivial, so we consider homotopical 3-syzygies. Such an object, s, will consist of an oriented polytope decomposition of S^3 together with a continuous map, f_s from S^3 to $X(3)$, which sends the i-skeleton of that decomposition to $X(i)$, $i = 0, 1, 2$. At this stage we have $X(0) = K(\mathcal{P})_0$, a point, $X(1) = K(\mathcal{P})_1$, and $X(2) = K(\mathcal{P})_2$. One wants enough such 3-syzygies, s, identified algebraically and combinatorially, so that the corresponding homotopy classes, $\{[f_s]\}$ generate $\pi_3(X(3))$.

It is clear, by induction, we get a notion of homotopical n-syzygy. We assume $X(n)$ has been built inductively by attaching cells of dimension $\leq n$ along homotopical k-syzygies for $k < n$, so that

$$\pi_1(X(n)) \cong G, \quad \pi_k(X(n)) = 0, \quad k = 2, \ldots, n-1,$$

then a *homotopical n-syzygy*, s, is an oriented polytope decomposition of S^n and a continuous cellular map $f_s : S^n \to X(n)$. After a choice of a set $\mathcal{R}_n$ of n-syzygies, so that $\{[s_s] \mid s \in \mathcal{R}_n\}$ generates $\pi_n(X(n))$ as a G-module, we can form $X(n+1)$ by attaching $n+1$-dimensional cells e^{n+1}_s along these f_s for $s \in \mathcal{R}_n$.

If we can do this in a sensible way, for all n, we say the resulting system of syzygies is *complete* and the limit space $X(\infty) = \bigcup X(n)$ is then a cellular model for BG, the classifying space of the group G.

This construction is, of course, just a homotopical version of the construction of a free resolution of the trivial G-module, $\mathbb{Z}$. Later we will consider how to form simplicial pro-$\mathcal{C}$ resolutions 'step-by-step' as another combinatorial way to replace $K(\mathcal{P})$ and more generally $K(G, 1)$.

Remark: Some additional aspects of this can be found in Loday's paper [107], in particular the link with the 'pictures' of Igusa, [88, 89].

Example and construction: Given any group G, we can find a presentation with $\{\langle g\rangle \mid g \neq 1, g \in G\}$ as set of generators and a relation $r_{g,g'} := \langle g\rangle\langle g'\rangle\langle g'g\rangle^{-1}$ for each pair (g, g') of elements of G. (We write $\langle 1\rangle = 1$ for convenience.)

The relation $r_{g,g'}$ gives a triangle

$$\begin{tikzcd} & \cdot \arrow[dr, "g'"] & \\ \cdot \arrow[ur, "g"] \arrow[rr, "g'g"'] & & \cdot \end{tikzcd}$$

and, for each triple (g, g', g'') , we get a homotopical 2-syzygy in the form of a tetrahedron.

Higher homotopical syzygies occur for any tuple, $(g_1, \ldots, g_n)$, of non-identity elements of G, by labelling a n-simplex. The limiting cellular space, $X(\infty)$, constructed from this context is just the usual model of the classifying space BG as geometric realisation of the nerve of G. The corresponding free resolution, $(C_*(G), d)$, is the classical *normalised bar resolution*. Using the bar resolution above dimension 2 together with the crossed module of the presentation at the base, one gets the standard free crossed resolution of the group, G. We will return to this later.

Syzygies for the Steinberg group (cf. Kapranov and Saito, [98]) Let R be an associative ring with 1. Recall that the *Steinberg group* $St_n(R)$ has generators $x_{ij}(a)$, labelling the elementary matrices $\varepsilon_{ij}(a)$, having

$$\varepsilon_{ij}(a)_{k,l} = \begin{cases} 1 & \text{if } k = l \\ a & \text{if } (k,l) = (i,j), a \in R \\ 0 & \text{otherwise,} \end{cases}$$

and relations

St1 $\quad x_{i,j}(a)x_{i,j}(b) = x_{i,j}(a+b)$;

St2 $\quad [x_{i,j}(a), x_{k,\ell}(b)] = \begin{cases} 1 & \text{if } i \neq \ell, j \neq k, \\ x_{i,\ell}(ab) & i \neq \ell, j = ik. \end{cases}$

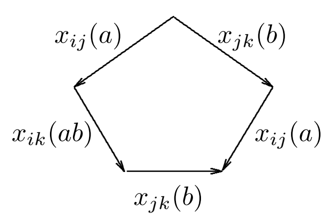
The identities / homotopical 2-syzygies are built from three types of polygon: a) a triangle, $T_{ij}(a,b)$ for each i, j, $i \neq j$, coming from St1;

b) a square,

$$\begin{tikzcd} \bullet \arrow[r, "x_{ij}(a)"] \arrow[d, "x_{kl}(b)"'] & \bullet \arrow[d, "x_{kl}(b)"] \\ \bullet \arrow[r, "x_{ij}(a)"'] & \bullet \end{tikzcd}$$

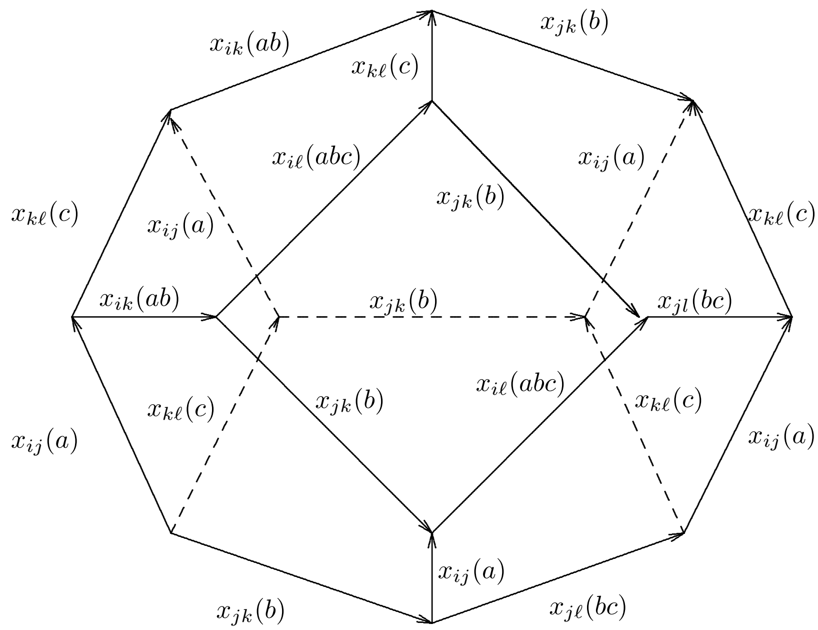
corresponding to the first case of St2 and

c) a pentagon, for the second:



Then for any pairs (i,j), (k,l), (m,p) with $x_{ij}(a)$, $x_{kl}(b)$, $x_{mp}(c)$, commuting by virtue of St2's first clause, we will have a homotopical syzygy in the form of a labelled cube.

There is also a homotopy 2-syzygy given by the associahedron labelled by generators as shown:



Remark: Kapranov and Saito, [98], have conjectured that the space $X(\infty)$ obtained by gluing labelled higher Stasheff polytopes together, is homotopically equivalent to the homotopy fibre of

$$f : BSt(R) \to BSt(A)^+,$$

where $(-)^+$ denotes Quillen's plus construction. The associahedron is a Stasheff polytope and, by encoding the data that goes to build the identities / syzygies schematically in a 'hieroglyph', Kapranov and Saito make a link between such hieroglyphs and polytopes.

Syzygies for the Braid groups: The understanding of the profinite Galois group, $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, of the algebraic closure of the field of rationals is linked to the Grothendieck-Teichmüller group, $\widehat{\mathcal{GT}}$ as defined by Drinfel'd, [47]. This corresponds to Grothendieck's 'Teichmüller tower', (cf. [77]), which is the system of moduli spaces, $\mathcal{M}_{g,n}$, of Riemann surfaces of genus g and with n marked points, on which $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts. Drinfel'd, [47], showed that $\widehat{\mathcal{GT}}$ can be viewed as a subgroup ot the automorphism group of $\widehat{Br}_n$, the profinite completion of Br_n, the braid group on n-strands, or, more exactly, on a tower of such braid groups. A full discussion of these links, and much more, can be found in the article by Lochak and Schneps, [103], that appears in [151], see also [104] for a cohomological interpretation of $\widehat{\mathcal{GT}}$ linked to braids.

The braid groups provide a good set of important examples of groups presentations having good geometric content and the above mentioned link with Grothendieck-Teichmüller theory gives additional motivation for discussing both them and their profinite and pro-$\mathcal{C}$ completions. Their usual presentation have been analysed for their higher syzygies (given in Loday, [107]), and we will use this to help illustrate the notion of higher syzygies. There are numerous sources which explore these braid groups geometrically and algebraically so we will refrain from doing so further here.

The Artin braid group, Br_{n+1}, defined using $n+1$ strands is given by

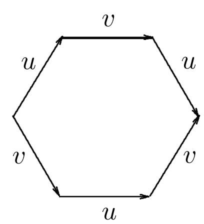
- generators: y_i, $i = 1, \ldots, n$;
- relations: $r_{ij} \equiv y_i y_j y_i^{-1} y_j^{-1}$ for $i + 1 < j$;
  $r_{ii+1} \equiv y_i y_{i+1} y_i y_{i+1}^{-1} y_i^{-1} y_{i+1}^{-1}$ for $1 \leq i < n$.

We will look at such groups only for small values of n.

By default, Br_2 has one generator and no relations, so is infinite cyclic.

The group Br_3: (We will simplify notation writing $u = y_1$, $v = y_2$.)

This then has presentation $\mathcal{P} = (u, v : r \equiv uvuv^{-1}u^{-1}v^{-1})$. It is also the 'trefoil group', i.e. the fundamental group of the complement of a trefoil knot. If we construct $X(2) = K(\mathcal{P})$, this is already a $K(Br_3, 1)$ space, having a trivial π_2. There are no higher syzygies. The model for the classifying space is given by the 2-cell with identifications:

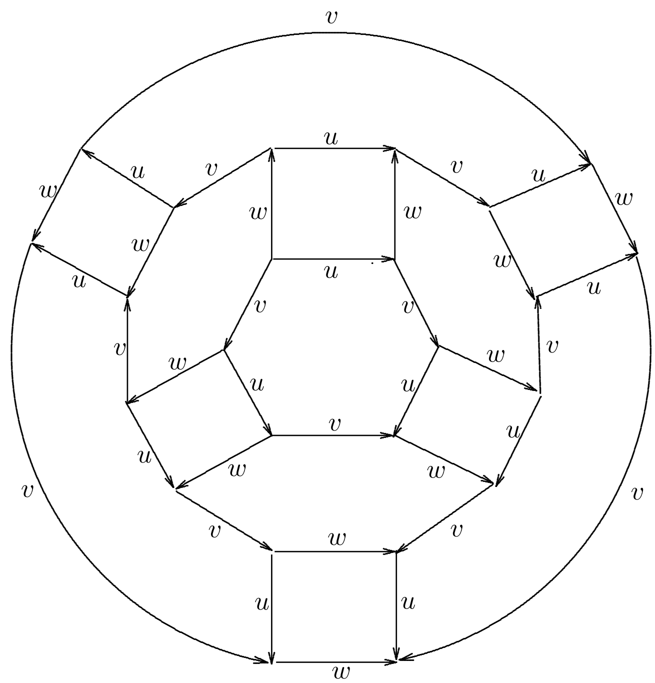


The group Br_4: simplifying notation as before, we have generators u, v, w and relations

$$\begin{aligned} r_u &\equiv vwvw^{-1}v^{-1}w^{-1}, \\ r_v &\equiv uwu^{-1}w^{-1}, \\ r_w &\equiv uvuv^{-1}u^{-1}v^{-1}. \end{aligned}$$

The 1-syzygies are made up of hexagons for r_u and r_w and a square for r_v. There is a fairly obvious way of fitting together squares and hexagons, namely as a permutohedron, and there is a labelling of such that gives a homotopical 2-syzygy as shown below.

A representing identity can be found using Igusa's method of pictures, [88, 89], which is explained in Loday's paper, [107] in which one can also find an explicit identity based on this 2-syzygy. Using this 2-syzygy, s, one can form $X(3) = X(2) \cup_s e_s^3$ and it is known that $X(3)$ is a $K(Br_4, 1)$ by using calculations of Deligne and Salvetti. An explicit construction of $X(3)$ can be obtained by quotienting the permutohedron by the indicated labelling.



As this example suggests, the presentation of Br_{n+1}, in general, gives a labelled permutohedron of higher dimensions and the $K(Br_{n+1}, 1)$ is the quotient of this by the labelling.

Remark: (i) The proofs of Deligne and Salvetti, mentioned by Loday, are based on the fact that, in the space $\mathbb{C}^{n+1}$, the complement of the union of the hyperplanes $\{x_i = x_j\}$, has fundamental group the braid group, Br_{n+1}.

(ii) Loday, [107], has introduced a parametrised version of the braid groups, $Br_n(R)$, where R is a ring, and gives higher syzygies for the resulting presentation. In [108], he and Stein prove that this $Br_n(R)$ is a semidirect product of $St_n(R)$ and Br_n, for a natural action of Br_n on the Steinberg group.

If one has a complete set of syzygies, $\{\mathcal{R}_n\}$, then one can build a space $X(\infty)$ with $X(n)$, as constructed above, as its n-skeleton or alternatively, a resolution of $\mathbb{Z}$ as G-module, a partial algebraic model of $X(\infty)$. The theory of crossed complexes and free crossed resolutions that will be intoduced in Chapter 6, provides an alternative algebraic model for $X(\infty)$ that is closely related to the resolution, yet provides a complete model of the homotopy type of $X(\infty)$, not just the chains on its universal cover. As it is algebraic, we can mimic that theory in the profinite case. Another perspective is given by R. A. Brown in [33], in terms of what is termed there 'generalised group presentations'.

4.2 Profinite analogues of identities

The above provides some motivation for our considerations here. Clearly we might approach identities for profinite presentations in an entirely analogous fashion, but we must be a bit careful as the Nielsen-Schreier theorem is not always available and we cannot use spatial models such as $K(\mathcal{P})$, $X(3)$, etc. of these extended presentations.

Given a profinite space of generators, X, for a pro$-\mathcal{C}$ group, G, one constructs $F_{\mathcal{C}}(X)$ and a continuous epimorphism from $F_{\mathcal{C}}(X)$ to G. It is commonplace that one regards elements of $F_{\mathcal{C}}(X)$ as being limits of "formal composites" of generators of G and that the kernel, $N(R)$, of the epimorphism is a good measure of the relations between the generators. One then views the choice of the space, R, of relations as being an effort to gain more knowledge of this kernel, $N(R)$, which, of course, consists of products of conjugates of elements in R and limits of such. Often, but not always, $N(R)$ is a free pro$-\mathcal{C}$ group (see the earlier discussion), but it will rarely be free on R itself nor on R together with its conjugates. To handle this the obvious thing to do is to take limits of "formal composites of formal conjugates of elements of R" and to examine the resulting object. Let us describe this idea more exactly.

We suppose $\mathcal{P} = (X : R)$ is a pro$-\mathcal{C}$ presentation of G, that is, G is a pro$-\mathcal{C}$ group isomorphic to $F_{\mathcal{C}}(X)/N(R)$, where $N(R)$ is the closed normal closure of the subspace $R \subset F_{\mathcal{C}}(X)$. The inclusion $R \hookrightarrow F_{\mathcal{C}}(X)$ gives us a corresponding free pro-$\mathcal{C}$ crossed module, $(C_{\mathcal{C}}(\mathcal{P}), F_{\mathcal{C}}(X), \partial)$. (It is in fact better to consider a presentation $\mathcal{P}$ as being a triple $(X : R, \rho)$ where $\rho : R \to F_{\mathcal{C}}(X)$ is a continuous map, not necessarily injective. This allows comparison of the properties of a given pair $(X : R)$ within different contexts, e.g. abstract, profinite and pro$-p$ presentations for a given set X with $R \subset F(X)$.)

The kernel, $\kappa_{\mathcal{C}}(\mathcal{P})$, of ∂ is thus reasonably interpreted as being the module of identities amongst the relations of $(X : R)$. Our previous results show that $\kappa_{\mathcal{C}}(\mathcal{P})$ is a pseudo-compact $\widehat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-module. Our results on exact sequences can be applied to give further information on $\kappa_{\mathcal{C}}(\mathcal{P})$.

4.2.1 The module of identities.

Since $\kappa_{\mathcal{C}}(\mathcal{P}) = Ker(C_{\mathcal{C}}(\mathcal{P}) \to F_{\mathcal{C}}(X))$, we have from Proposition 22, an exact sequence

$$\kappa_{\mathcal{C}}(\mathcal{P}) \to C_{\mathcal{C}}(\mathcal{P})^{Ab} \xrightarrow{\partial_*^{Ab}} N(R)^{Ab} \to 0,$$

of pseudocompact $\widehat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-modules. To learn more about $\kappa_{\mathcal{C}}(\mathcal{P})$, we need to examine the morphism from $\kappa_{\mathcal{C}}(\mathcal{P})$ to $C_{\mathcal{C}}(\mathcal{P})^{Ab}$ as well as the latter module.

If $\mathcal{C} = p$-groups (so pro$-\mathcal{C}$ = pro$-p$), if G is finite discrete, or in general if $(X; R)$ is what we have earlier, page 15 called a *free pro-$\mathcal{C}$ presentation*, then $N(R)$ is free pro$-\mathcal{C}$ and so the morphism $\kappa_{\mathcal{C}}(\mathcal{P}) \to C_{\mathcal{C}}(\mathcal{P})^{Ab}$ is monic by Proposition 23. We will only discuss this case, as the other type of situation would seem very hard to attack. We would however draw the attention of the reader to the following problem:

Calculate the second homology group of all non-free closed normal subgroups of free pro−$\mathcal{C}$ groups.

In particular, we note that the kernel of the epimorphism from a free profinite group F onto its maximal prosolvable quotient is not a free profinite group if F has rank greater than 1. What is H_2 of this kernel? The results of Ellis and Porter, [57], mentioned earlier, have identified H_2 with, in the notation of proposition 22, $A \cap [C, C]$, in the abstract case. The profinite analogue will be discussed later, in chapter 6.

4.2.2 The Abelianisation of the 'top' group in the free case.

Our information on $C_{\mathcal{C}}(\mathcal{P})^{Ab}$ is more complete. The following proposition is the pro−$\mathcal{C}$ analogue of Proposition 7 on p.162 of [29] for the abstract case. It has a useful corollary.

Proposition 29. *If (C, G, ∂) is a free profinite crossed module on a continuous function, $f : X \to G$, with X profinite, then C^{Ab} is a free pseudocompact $\widehat{\mathbb{Z}}_{\mathcal{C}}[\![G/Im\,\partial]\!]$-module on the image in C^{Ab} of the composite function*

$$X \xrightarrow{v} C \xrightarrow{p} C^{Ab},$$

where v is a lifting of f.

Proof: The fact that C^{Ab} is a pseudocompact $\widehat{\mathbb{Z}}_{\mathcal{C}}[\![G/Im\,\partial]\!]$-module has already been proved. Freeness is proved exactly as in Brown and Huebschmann, [29], for the discrete, purely algebraic case. One assumes given a pseudocompact $\widehat{\mathbb{Z}}_{\mathcal{C}}[\![G/Im\,\partial]\!]$-module, M say, and a continuous map from X to it. Considering the module as the corresponding crossed module with the trivial map to G, gives a unique map from C to M and then it is easy to check this factors through C^{Ab}. Uniqueness at each stage guarantees uniqueness overall. ■

Corollary 7. *If (C, G, ∂) is a free pro−$\mathcal{C}$ crossed G-module on $f : X \to G$ and v is a lifting of f to C, then v is injective.* ■

The proof in Brown-Huebschmann, [29], generalises without problem.

4.2.3 Identifying problems and problems of identification.

Collecting up facts, we find that if $\mathcal{P}$ is a free pro−$\mathcal{C}$ presentation, $\kappa_{\mathcal{C}}(\mathcal{P})$ is isomorphic to the kernel of a continuous map from $C_{\mathcal{C}}(\mathcal{P})^{Ab} \cong \widehat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]^{(R)}$ to the relation module $N(R)^{Ab}$. This latter can be shown to be a submodule of $\widehat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]^{(X)}$ and in this identification, the morphism concerned, ∂_*^{Ab}, is given by the pseudocompact analogue of the Jacobian matrix of the presentation, which will be examined later in section 6.2.3. As the identification of $N(R)^{Ab}$ and the properties of the transformation ∂_*^{Ab} require other techniques than those developed up to this point, they will be left until a later chapter, where the connections between "pro$-\mathcal{C}$ crossed complexes" and chain complexes of pseudocompact $\widehat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-modules are discussed.

4.2.4 Comparisons.

The question naturally arises of the connection between the pseudocompact identity module and the discrete one in the case of a finite $\mathcal{C}$-group G and finite presentation $(X : R)$. This presentation can be considered as a discrete one, $\mathcal{P}_{\text{disc}}$, or a pro$-\mathcal{C}$ one, $\mathcal{P}$. One thus obtains exact sequences corresponding to the two views taken:

$$\begin{tikzcd}
(i) \quad 0 \arrow[r] & \kappa_{\text{disc}}(\mathcal{P}_{\text{disc}}) \arrow[r] \arrow[d, "\alpha"'] & C_{\text{disc}}(\mathcal{P}_{\text{disc}})^{Ab} \arrow[r] \arrow[d, "\beta"'] & N_{\text{disc}}(R_{\text{disc}})^{Ab} \arrow[r] \arrow[d, "\gamma"'] & 0 \\
(ii) \quad 0 \arrow[r] & \kappa_{\mathcal{C}}(\mathcal{P}) \arrow[r] & C_{\mathcal{C}}(\mathcal{P})^{Ab} \arrow[r] & N(R)^{Ab} \arrow[r] & 0
\end{tikzcd}$$

and a map from (i) to (ii) as shown. Results on pro$-\mathcal{C}$ completions of free crossed modules contained in the next chapter imply that β and γ are canonical pro$-\mathcal{C}$ completion morphisms, but we will see later that α need not be; in fact $\kappa_{\text{disc}}(\mathcal{P})$ may be trivial, whilst $\kappa_{\mathcal{C}}(\mathcal{P})$ is non-trivial. However, the question of the relationship between $\kappa_{\text{disc}}(\mathcal{P})$ and $\kappa_{\mathcal{C}}(\mathcal{P})$, in general, remains unclear.

4.3 Profinite simplicial groups and crossed modules.

4.3.1 Replacing topological by simplicial constructions.

We saw in the last section how a presentation $\mathcal{P} = (X : R)$ of a group G, could be used to construct a complex, $K(\mathcal{P})$, giving combinatorial information on $\mathcal{P}$. The information contained in $K(\mathcal{P})$ can, as we have seen, be encoded in a free crossed module and we constructed a pro$-\mathcal{C}$ analogue of this when $\mathcal{P}$ was a pro$-\mathcal{C}$ presentation of a pro$-\mathcal{C}$ group, G.

The complex, $K(\mathcal{P})$, can be replaced by a simplicial group. This can be done "step-by-step" in the fashion of André, [4], and we will look at possible profinite analogues of this in a later chapter. For the moment we want to describe how one can pass from a given pro$-\mathcal{C}$ simplicial group, G, to a pro$-\mathcal{C}$ crossed module, $M(G, 1)$ and to examine what information $M(G, 1)$ retains about G. Later we will take this construction in several different directions, generalising it to give a pro$-\mathcal{C}$ crossed complex and pro$-\mathcal{C}$ cat^n-group, examining its compatability with pro$-\mathcal{C}$ completion, etc. For the moment we limit ourselves to the construction itself.

4.3.2 A crossed module from a simplicial group.

Given a pro$-\mathcal{C}$ simplicial group G, we form up $M(G,1)$ as the morphism

$$\partial : \frac{Ker\ d_1^1}{d_0^2(Ker\ d_1^2 \cap Ker\ d_2^2)} \to \frac{G_1}{d_0^2(Ker\ d_1^2)}.$$

We claim this is a pro$-\mathcal{C}$ crossed module. We first need to check that it makes sense, i.e., that $d_0(Ker\ d_1^2 \cap Ker\ d_2^2)$ and $d_0^2(Ker\ d_1^2)$ are closed normal subgroups of the respective denominators. Closedness is clear. To check normality we merely note that if $x = d_0^2 y$ for $y \in Ker\ d_1^2$ or $d_0(Ker\ d_1^2 \cap Ker\ d_2^2)$ then for any $g \in G_1$, $gxg^{-1} = d_0^2(s_0(g)ys_0(g)^{-1})$ and so is again in the relevant kernel. The continuous action of G_1 on $Ker\ d_1^1$ induces one of the quotient $G_1/d_0^2(Ker\ d_1^2)$ on $Ker d_1^1/d_0^2(Ker\ d_1^2 \cap Ker\ d_2^2)$. The morphism ∂ is induced by the inclusion of $Ker\ d_1^1$ into G_1, and hence is continuous.

Proposition 30. *Given a pro$-\mathcal{C}$ simplicial group $G.$, the above structure makes $M(G.,1)$ into a pro$-\mathcal{C}$ crossed module.*

Proof: The only things to check are the crossed module axioms. Writing $[g]$ for the coset corresponding to $g \in G_1$ or in $Ker\, d_1^1$ etc., we get for $x \in Ker\, d_1^1$,

(i) ${}^{[g]}[x] = [{}^{g}x] = [gxg^{-1}] = [g][x][g]^{-1}$, i.e., ∂ is equivariant, and for $x, y \in Ker\, d_1^1$,
(ii) ${}^{\partial[x]}[y] = [xyx^{-1}] = [x][y][x]^{-1}$.

The proof thus resides in checking that the (deliberate) ambiguity of the notation causes no problems. This is quite routine and can be safely left to the reader. ■

4.3.3 The kernel and cokernel.

Now that we have the crossed module $M(G,1)$, it is natural to work out the cokernel and kernel of ∂. The cokernel of ∂ can most easily be calculated by noting that $d_0 : G_1 \to G_0$ induces an isomorphism between $G_1/d_0(Ker\, d_1)$ and G_0. This means that we can replace $M(G,1)$ by an isomorphic pro$-\mathcal{C}$ crossed module

$$\partial : \frac{Ker\, d_1^1}{d_0^2(Ker\ d_1^2 \cap Ker\, d_2^2)} \to G_0,$$

where now ∂ is induced by d_0^1 and G_0 acts on the left-hand term via the continuous degeneracy homomorphism, $s_0 : G_0 \to G_1$, followed by conjugation. It follows that Coker ∂ is isomorphic to $G_0/d_0^1(Ker\, d_1^1)$ and analysis of the Moore complex of G in low dimensions shows that this quotient is the pro$-\mathcal{C}$ group, $\pi_0(G)$, of connected components of G. In particular if F is a free pro$-\mathcal{C}$ simplicial resolution of a pro$-\mathcal{C}$ group G, as used by Gildenhuys and Mackay [71], then, for example, $M(F,1)$ has G as its cokernel.

4.3.4 Ker ∂.

To calculate $Ker\,\partial$, it is again useful to use the second version of $M(G,1)$. The kernel of ∂ is then easily seen to be

$$\frac{Ker\ d_0^1 \cap Ker\ d_1^1}{d_0^2(Ker\ d_1^2 \cap Ker\ d_2^2)}$$

and the Moore complex supplies us with an interpretation of this as being $\pi_1(G)$.

Earlier we used the fact that if (K,L) is a connected CW-pair with L the 1-skeleton of K, then the boundary $\partial : \pi_2(K,L) \to \pi_1(L)$ of the long exact homotopy sequence of (K,L) is a crossed module with kernel $\pi_2(K)$ and cokernel $\pi_1(K)$. Here we have

$$\pi_1(G) \ = \ Ker M(G,1),$$

$$\pi_0(G) \ = \ Coker M(G,1),$$

since as we recalled earlier, simplicial groups model “loop groups” of homotopy types - but with a slippage of dimension.

4.3.5 “ . . . and back again”.

Given a pro$-\mathcal{C}$ crossed module (C,P,∂), can we find a simplicial group, G, whose associated $M(G,1)$ is continuously isomorphic to (C,P,∂)? Supposing we can - and we can - what is the connection between a given G and that reconstructed from $M(G,1)$?

Suppose $\mathsf{C} = (C,P,\partial)$ is a pro$-\mathcal{C}$ crossed module, we construct a pro$-\mathcal{C}$ simplicial group (in an apparently *ad hoc* fashion) $E(\mathsf{C})$ by

$$E(\mathsf{C})_0 = P, \qquad E(\mathsf{C})_1 = C \rtimes P,$$

$$s_0(p) = (1,p),\ d_0^1(c,p) = \partial c.p,\ d_1^1(c,p) = p.$$

Assuming $E(\mathsf{C})_n$ is defined and that it acts on C via the unique composed face map to $E(\mathsf{C})_0 = P$ followed by the given action of P on C, we set

$$\begin{aligned} E(\mathsf{C})_{n+1} &= C \rtimes E(\mathsf{C})_n; \\ d_0^{n+1}(c_{n+1},\dots,c_1,p) &= (c_{n+1},...,c_2,\partial c_1.p); \\ d_i^{n+1}(c_{n+1},\dots,c_{i+1},c_i,\dots,c_1,p) &= (c_{n+1},\dots,c_{i+1}c_i,\dots c_1,p) \\ &\qquad\qquad \text{for } 0 < i < n+1; \\ d_{n+1}^{n+1}(c_{n+1},\dots,c_1,p) &= (c_n,\dots,c_1,p); \\ s_i^n(c_n,\dots,c_1,p) &= (c_n,\dots,1,\dots,c_1,p), \end{aligned}$$

where the 1 is placed in the i^{th} position.

Clearly $Ker\, d_1^1 = \{(c,p) : p = 1\} \cong C$, whilst $Ker\, d_1^2 \cap Ker\, d_2^2 = \{(c_2, c_1, p) : (c_1, p) = (1,1) \text{ and } (c_2c_1, p) = (1,1)\} \cong \{1\}$, hence the "top term" of $M(E(\mathsf{C}), 1)$ is isomorphic to C itself, whilst $E(\mathsf{C})_0$ is P itself. The boundary map ∂ in this interpretation is the original ∂, since it maps $(c, 1)$ to $d_0(c)$, i.e., we have

Lemma 9. *There is a natural isomorphism*

$$\mathsf{C} \cong M(E(\mathsf{C}), 1).$$

■

4.3.6 Back yet again!

Suppose now that we pass from a pro$-\mathcal{C}$ simplicial group, G, to $M(G, 1)$ and then apply E. To compare G with $EM(G, 1)$, we first calculate the Moore complex of $E(\mathsf{C})$ for arbitrary C. This, by the calculations in 4.3.5, gives

$$1 \to 1 \to C \xrightarrow{\partial} P.$$

Comparing the Moore complex of G with that of $EM(G, 1)$ this gives us

$$N(G) = (\ldots \to N(G)_2 \to Ker\, d_1 \to G_1),$$

$$N(E(M(G,1)) = \ldots \to 1 \to \frac{Ker \quad d_1}{d_0 N(G)_2} \to G_0)$$

There is thus a type of truncation process going on that kills off $N(G)_2$ and all higher groups and their images. This will be thoroughly investigated in a later chapter, but for the moment we merely point out that there is a continuous simplicial morphism from G to $EM(G, 1)$ given at levels 0 and 1 by

$G_0 \to G_0$ by the identity,

$G_1 \to EM(G, 1)_1$ by g goes to $(gs_0d_1g^{-1}, d_1g)$

and at all higher levels is generated by these two.

It is clear that $EM(G, 1)$ captures information on the 1-type of G. In fact, $M(G, 1)$ is an algebraic model for the 1-type, since the constructions are functorial. We will investigate this thoroughly later on, but will see in the intervening pages many consequences of this idea.

4.3.7 Simplicial normal subgroups.

The above ideas interact neatly with a pretty little result which is the group theoretic version of Loday's observation that if $p : E \to B$ is a fibration with connected fibre, then the induced map from $\pi_1(F)$ to $\pi_1(E)$ has a crossed module structure. For simplicial groups, fibrations and epimorphisms are "the same" as we have already remarked, hence a normal simplicial subgroup corresponds to a fibre of a fibration, connectedness being built in automatically.

Of course normal subgroups are special crossed modules, and when dealing with simplicial groups the décalage in dimension will replace π_1 by π_0. Given these observations, one direction of the following result is not surprising.

Proposition 31. *Given a pro$-\mathcal{C}$ simplicial group G and a closed simplicial normal subgroup $N \lhd G$, the induced map*

$$i_* : \pi_0(N) \to \pi_0(G)$$

together with the induced action of $\pi_0(G)$ on $\pi_0(N)$ by conjugation of representing elements, is a pro$-\mathcal{C}$ crossed module. Conversely, given any pro$-\mathcal{C}$ crossed module (C, P, ∂), there is a simplicial pro$-\mathcal{C}$ group, G, and a closed normal simplicial subgroup, $N \lhd G$, such that (C, P, ∂) is isomorphic to $(\pi_0(N), \pi_0(G), i_*)$.

Proof: The continuous action of G on N gives a continuous map

$$G \times N \to N.$$

Applying π_0 and using that $\pi_0(G \times N) \cong \pi_0(G) \times \pi_0(N)$, we get a continuous action of $\pi_0(G)$ on $\pi_0(N)$. The two crossed module axioms can similarly be represented by commutative diagrams involving products, so again applying π_0 gives that the crossed module axioms are satisfied by $(\pi_0(N), \pi_0(G), i_*)$.

Conversely, given the pro$-\mathcal{C}$ crossed module (C, P, ∂), we build, using an idea of Loday, two new pro$-\mathcal{C}$ crossed modules $(1, C, inc)$ and $(C, C \rtimes P, inc)$ and a morphism between them

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & C \arrow[d, "inc"] \\ C \arrow[r, "\varepsilon"] & C \rtimes P \end{tikzcd}$$

where $\varepsilon(c) = (c, \partial c^{-1})$. This is clearly continuous and we leave it as an exercise to check ε is a homomorphism.

This morphism is a monomorphism making $\varepsilon(C)$ a normal subgroup of $C \rtimes P$. If we now apply the functor E to this and we set $E(C, C \rtimes P, inc) = G$ and $E(1, C, inc) = N$, then $N \lhd G$ and $(\pi_0(N), \pi_0(G), i_*)$ is isomorphic to (C, G, ∂). ■

Remark: In fact, in the above proof, G is a $K(P, 0)$, i.e., its only non-trivial homotopy group is P in dimension 0. Similarly N is a $K(C, 0)$.

4.3.8 The Brown-Loday lemma

We have seen how crossed modules give simplicial groups and vice versa. The relationship is very neatly illustrated by a result of Brown and Loday, [31]. This can be seen as a precursor of a whole lot of results that we will meet later and many more that we will not have space for. The proof uses low dimensional versions of several ideas that will play a role later on, in particular, semidirect decompositions and the simple observation that, in a general simplicial group, elements such as $s_1x.s_0y$ or $[s_1x, s_0y]$, are not, in general, themselves degenerate, although they are products of degenerate elements. We will state the result in a slightly more general forn than it was originally stated.

Let G be a simplicial group and $n \geq 1$. We will denote by D_n, the subgroup of G_n generated by the degenerate elements. (The original form of the result assumed $G_2 = D_2$, we will work rather with $NG_2 \cap D_2$.)

Proposition 32. *(The Brown-Loday lemma) Let N_2 be the (closed) normal subgroup of G_2 generated by elements of the form*

$$F_{(1),(0)}(x,y) = [s_1x, s_0y][s_0y, s_0x]$$

for $x, y \in NG_1 = Ker\, d_1$. Then $NG_2 \cap D_2 = N_2$ and consequently

$$\partial(NG_2 \cap D_2) = [Ker\, d_0, Ker\, d_1].$$

Before we prove this, we will explore several points.

• Suppose $z \in Ker\, d_0$, then $x = z.s_0d_1z^{-1}$ is in $Ker\, d_1$. Conversely given an $x \in Ker\, d_1$, $x.s_0d_0x^{-1} \in Ker\, d_0$. (This sets up a bijection between $Ker\, d_0$ and $Ker\, d_1$, but, of course, it is not usually an isomorphism.) Any $z \in Ker\, d_0$ thus has this form.

• The group G_2 has a semidirect product decomposition

$$G_2 \cong (NG_2 \rtimes s_0NG_1) \rtimes (s_1NG_1 \rtimes s_1s_0NG_0).$$

This is the case, $n = 2$ of a more general result (Conduché's lemma) that we will prove later on. This low dimensional case is easy to see. There is an obvious split epimorphism:

$$G_1 \underset{d_1}{\overset{s_0}{\leftrightarrows}} G_0\ ,$$

so $G_1 \cong NG_1 \rtimes s_0G_0$.

Similarly

$$G_2 \underset{d_2}{\overset{s_1}{\leftrightarrows}} G_1$$

yields $G_2 \cong NG_2 \rtimes s_1G_1$, but we also have

$$Ker\, d_2 \underset{d_1}{\overset{s_0}{\leftrightarrows}} Ker\, d_1\ ,$$

since if $d_2x = 1$, $d_1d_0x = d_0d_2x = 1$, so $d_0x \in Ker\, d_1$. Putting this together gives the decomposition we gave above.

• The previous two observations are linked by the mappings, defined for $x \in G_2$, by:

$$p_1(x) = x.s_1 d_2 x^{-1};$$
$$p_0(x) = x.s_0 d_1 x^{-1}.$$

These are mappings into the kernel of the projections of the decomposition, so given an arbitrary $x \in G_2$, $p_0 p_1(x) \in Ker\, d_1 \cap Ker\, d_2 = NG_2$, as can easily be checked. (This is the component of x in the Moore complex.) Now consider $x, y \in NG_1$ and the commutator, $[s_1 x, s_0 y]$. This latter is in G_2, so we look at its component in NG_2. This will be $p_1 p_0[s_1 x, s_0 y] = [s_1 x, s_0 y][s_0 y, s_0 x]$, i.e., the element $F_{(1),(0)}(x, y)$ of the statement of the proposition.

It is easily checked that $F_{(1),(0)}(x, y)$ is in $(NG_2 \cap D_2)$, and that

$$\partial F_{(1),(0)}(x, y) = s_0 d_0 x[y, s_0 d_0 x^{-1}.x] s_0 d_0 x^{-1} \in [Ker\, d_1, Ker\, d_0],$$

so $\partial N_2 \subseteq [Ker\, d_1, Ker\, d_0]$. Moreover, from our first observation, it is easy to see that any commutator $[y, z]$ with $y \in Ker\, d_1$ and $z \in Ker\, d_0$, can be written as $\partial F_{(1),(0)}(x', y')$ for suitable $x', y' \in NG_1$.

Proof of the proposition: Any element of G_2 has a decomposition (relative to the above semidirect product decomposition) in the form

$$g_2 = g.s_0(x).s_1(y).s_1 s_0(u),$$

with $g \in NG_2$, $x, y \in NG_1$ and $u \in NG_0 = G_0$, and moreover, these elements are determined by g_2 by applying various projections to it, e.g. $u = d_1 d_2(g_2)$, $y = p_0(d_2 g_2)$, etc.

Clearly we have $N_2 \subseteq NG_2 \cap D_2$. The semidirect decomposition shows that if $g_2 \in D_2$, it can be written as a product of a certain number of degenerate elements of form $s_0(x)$, $s_1(y)$ or $s_1 s_0(u)$ with $x, y \in NG_1$, $u \in NG_0$. We will use induction on the length of the expression thus representing an element in $NG_2 \cap D_2$. Suppose $g_2 \in NG_2 \cap D_2$ is of one of these three forms, then it must be trivial. For instance, $d_1 g_2 = d_2 g_2 = 1$ as it is in NG_2, so, if $g_2 = s_0(x)$, say, with $x \in NG_1$, then $x = d_1 s_0(x) = 1$. The other case is equally easy.

Now assume that if we have that $g \in NG_2 \cap D_2$ can be written as a product of fewer than n such elements, then

$$gN_2 = s_0(y_1) s_1(y_1') s_1 s_0(u_0) N_2,$$

i.e., g is congruent, mod N_2, to an element having trivial component in NG_2. Suppose now that an elemnt g_2 can be written as a product of n degenerate elements, say, $g_2 = s_\beta(x) g$ with $\beta = (0), (1)$, or $(1, 0)$ with g covered by our inductive hypothesis.

If $\beta = (0)$, so $g_2 = s_0(x) g$ then, of course,

$$\begin{aligned} g_2 N_2 &= s_0(x) s_0(y_1) s_1(y_1') s_1 s_0(y_0) N_2 \\ &= s_0(x y_1) s_1') s_1 s_0(y_0) N_2, \end{aligned}$$

and so has the right form.

If $g_2 = s_1(x)g$, then note $F_{(1)(0)}(x, y_1) = [s_1x, s_0y_1][s_0y_1, s_0x] \in N_2$, so $s_1(x)s_0(y_1)N_2 = s_0(xyx^{-1})s_1(x)N_2$. We can thus use this to pass $s_1(x)$ through the s_0 term and absorb it, mod N_2 in the rest.

Finally if $g_2 = s_1s_0(x)g$, then as $s_1s_0 = s_0s_0$, we can pass the first term through the others conjugating as we go, to get

$$g_2N_2 = s_0({}^{s_0(x)}y_1)s_1({}^{s_0(x)}y_1')s_1s_0(xy_0)N_2.$$

Thus, by induction, any element in D_2/N_2 has trivial component in $(NG_2 \cap D_2)/N_2$. We can conclude that $N_2 = NG_2 \cap D_2$. We finally note:

$$\partial(NG_2 \cap D_2) = \partial N_2 = [Ker\, d_0, Ker\, d_1].$$

■

In the next chapter, we will meet cat^1-groups, where the key condition is that $[Ker\, s, Ker\, t] = 1$. In the simplicial context $s = d_1$, $t = d_0$ and the $F_{(1)(0)}(x, y)$ elements correspond to liftings of 'Peiffer commutators', i.e., elements that map to the difference between the two expressions on the two sides of the Peiffer identity. To see this just note that $\partial F_{(1)(0)}(x, y) = {}^{\partial x}y.(xyx^{-1})^{-1}$. We will show that cat^1-groups are equivalent to crossed modules and then the above proposition gives that, for instance, if $NG_2 \cap D_2$ is trivial, $\partial : NG_1 \to NG_0$ is a crossed module.

This sort of condition will come in later when we discuss the relationship between crossed complexes and simplicial groups. Crossed complexes extend crossed modules by a chain complex and model more of the underlying homotopy type. We will also see higher dimensional analogues of the $F_{(1)(0)}(x, y)$ elements.

It is sometimes worth viewing $F_{(1),(0)}$ as a pairing operation

$$F_{(1),(0)} : NG_1 \times NG_1 \to NG_2,$$

which in the profinite context is clearly continuous. This is the first of many such pairings (*Peiffer pairings*) that have been considered in much more detail in the papers of Mutlu and Porter, [125–129] and are handled here in Chapter **??**.

5

Pro-$\mathcal{C}$ completions of crossed modules.

5.1 Cat^1-groups, and their pro-$\mathcal{C}$ analogues

The equivalence between crossed modules and internal categories in the category of groups has been known for some time (see the comments on this in Brown-Spencer, [32]). A neat reformulation of the latter type of object was given by Loday in [106] (see also Brown-Loday [31]). There one also finds the introduction of the convenient term "cat^1-group".

The first results on completions of crossed modules were given in Korkes' thesis, [100]. The treatment given here is based on [101].

5.1.1 Cat^1-groups.

A *cat^1-group* is a triple, (G, s, t), consisting of a group G and endomorphisms s, the *source map*, and t, the *target map* of G, satisfying the following axioms:

(i) $st = t$ and $ts = s$,

(ii) $[Ker\, s, Ker\, t] = 1$.

Here, of course, $[Ker\, s, Ker\, t]$ indicates the subgroup of G generated by the commutators $[g, h] = ghg^{-1}h^{-1}$ with $g \in Ker\, s$, $h \in Ker\, t$.

There is an obvious notion of a morphism between cat^1-groups: if (G, s, t), and (G', s', t') are cat^1-groups, a morphism

$$\phi : (G, s, t) \to (G', s', t')$$

is a group homomorphism, $\phi : G \to G'$, such that

$$s'\phi = \phi s$$

and

$$t'\phi = \phi t.$$

This gives a category, which we will denote $Cat^1(Grps)$, of cat^1-groups and morphisms between them.

5.1.2 Cat^1-groups and crossed modules

In [106], Loday shows that there is an equivalence between the categories *CMod* and $Cat^1(Grps)$. This equivalence is constructed as follows:

Given $\partial : C \to B$, crossed module, we form the semi-direct product, $G = C \rtimes B$, using the action of B on C. The structural maps s, t are given by

$$s(c, b) = (1, b) \text{ and } t(c, b) = (1, \partial(c)b)$$

for $c \in C$, $b \in B$. This satisfies the axioms for a cat^1-group as is easy, and quite instructive, to show. On the other hand, given a cat^1-group (G, s, t), we set $C = Ker\, s$, $B = Im\, s$, and $\partial = t|_C$, the restriction of t to C. The action of B on C is by conjugation within G. Again the axioms are easily checked.

These cat^1-groups have another interpretation that was mentioned earlier. Given any category with finite limits such as the category of groups or of profinite spaces, we have seen, (in section 1.5), the well known notion of an *internal category* within the given category. For instance an internal category in the category of all topological spaces has a space, Ob, of 'objects' and a space, Ar, of 'arrows'. The domain and codomain assignments give continuous mappings from Ar to Ob, whilst the assignment to each object of its identity arrow gives a continuous map in the other direction. There are the usual requirements that the domain of an identity arrow is the object itself and so on. Finally composition is defined on the space of composable arrows and is continuous. It is this notion that forms the basis for the notion of profinite groupoid that we saw briefly earlier. Here we note that the structure of a cat^1-group can be interpreted as being exactly that of an internal category in the category, *Grps*, of groups and homomorphisms.

Given a cat^1 group (G, s, t), the corresponding internal category has G as its group of arrows and $N = s(G)$ as its group of objects, s stands for 'source', i.e., 'domain' and t for 'target' ('codomain'). The rules $st = t$, etc., are then just 'the source of the identity on the target of an arrow is the target of the arrow', etc. Of course, here we are hiding the inclusion of N into G, which is the identity assignment. Finally the kernel-kernel commutator condition is exactly the condition that says that composition in the category is a group homomorphism. Explorations of these ideas can be found in several sources in the literature and so here it will be left as an exercise to search them out. (It is amusing to check, for instance, that any internal category in *Grps* is an internal groupoid in an obvious sense.) We will not be using this interpretaton very much and so have not included a detailed exposition here.

5.1.3 The pro$-\mathcal{C}$ versions

We next introduce the pro$-\mathcal{C}$ analogue of the above.

A *cat^1-pro$-\mathcal{C}$-group* is a cat^1-group, (G, s, t), in which G is a pro$-\mathcal{C}$ group and s and t are continuous endomorphisms of G. A morphism of cat^1-pro$-\mathcal{C}$-groups is a morphism, $\phi : (G, s, t) \to (G', s', t')$, of the underlying cat^1-groups such that ϕ is a continuous morphism of pro$-\mathcal{C}$ groups. This gives a category of cat^1-pro$-\mathcal{C}$-groups that we will denote $Cat^1(Pro\text{–}\mathcal{C})$. There is a forgetful functor from $Cat^1(Pro\text{–}\mathcal{C})$ to $Cat^1(Grps)$ which will be denoted by $\mathcal{U}_{\mathcal{C}}$.

Lemma 10. *There is an equivalence of categories*

$$Pro-\mathcal{C}.CMod \xrightarrow{\simeq} Cat^1(Pro-\mathcal{C})$$

compatible, via the forgetful functors, with the equivalence between $CMod$ and $Cat^1(Grps)$, i.e., the diagram

$$\begin{tikzcd}
Pro-\mathcal{C}.CMod \arrow[r, "\simeq"] \arrow[d, "\mathcal{U}_{CMod}"'] & Cat^1(Pro-\mathcal{C}) \arrow[d, "\mathcal{U}_{\mathcal{C}}"] \\
CMod \arrow[r, "\cong"] & Cat^1(Grps)
\end{tikzcd}$$

commutes.

Proof: In fact, if (C, B, ∂) is a pro$-\mathcal{C}$ crossed module, then $G = C \rtimes B$ is a pro$-\mathcal{C}$ group and the endomorphisms s and t, given earlier, are continuous, so the resulting (G, s, t) is a cat^1-pro$-\mathcal{C}$-group. Similarly if (G, s, t) is a cat^1-pro$-\mathcal{C}$-group then $(Ker\, s, Im\, s, t|_{Ker\, s})$ is a pro$-\mathcal{C}$ crossed module. ■

This lemma will enable us to prove the existence of a left adjoint for

$$\mathcal{U}_{\text{CMod}} : Pro-\mathcal{C}.CMod \to CMod$$

by constructing one for

$$\mathcal{U}_{\mathcal{C}} : Cat^1(Pro-\mathcal{C}) \to Cat^1(Grps).$$

This latter construction will need projective limits within $Cat^1(Pro\text{–}\mathcal{C})$ and so we will briefly look at their construction as it sheds some light on the pro$-\mathcal{C}$ completion functor that will result from their use.

Given a projective system $F : I \to Cat^1(Pro-\mathcal{C})$, one notes that F is a projective system of groups together with two endomorphisms of projective systems, $s, t : F \to F$ satisfying $st = t$ and $ts = s$, plus a commutator condition. We form Lim F by taking the limit of this underlying system of pro$-\mathcal{C}$ groups together with the induced endomorphisms, $Lim\, s$ and $Lim\, t$. Writing the result as $(\bar{F}, \bar{s}, \bar{t})$, we have merely to check the commutator condition $[Ker\, \bar{s}, Ker\, \bar{t}] = 1$. However $\bar{F}$ can be realised as a subgroup of the product $\prod_{i \in I} F(i)$, and $\bar{t}((x_i)) = (t(i)x_i)$, similarly for $\bar{s}$, so as the commutator subgroup $[Ker\, s(i), Ker\, t(i)]$ is trivial for each i in I, it is so for the limit as it can be calculated "pointwise".

5.2 Pro$-\mathcal{C}$ completions of cat^1-groups and crossed modules

5.2.1 Pro$-\mathcal{C}$ completions of cat^1-groups

Proposition 33. *A pro$-\mathcal{C}$ completion functor from $Cat^1(Grps)$ to $Cat^1(Pro-\mathcal{C})$ exists, (i.e., the forgetful functor $\mathcal{U}_{\mathcal{C}}$ has a left adjoint).*

Proof: An exact sequence

$$1 \to (K, s', t') \xrightarrow{u} (G, s, t) \xrightarrow{v} (H, s'', t'') \to 1$$

of cat^1-groups is an exact sequence

$$1 \to K \to G \to H \to 1$$

of the underlying groups and continuous maps compatible with the source and target maps. In this situation, we say that the cat^1 -group, (H, s'', t''), is the *quotient* of (G, s, t) by the normal sub-cat^1-group, (K, s', t'). The latter is of *finite index* in (G, s, t) if H is finite.

Given any (G, s, t), the set of its normal sub-cat^1-groups, (N, s', t'), of finite index with $G/N \in \mathcal{C}$ is directed by inclusion, so we can form an inverse system of finite quotients of (G, s, t) and can take its limit within the category of cat^1-pro$-\mathcal{C}$ groups. (As usual one considers each finite $\mathcal{C}$-cat^1-group as a pro$-\mathcal{C}$ one having the discrete topology).

Thus we can define a functor : $Cat^1(Grps) \to Cat^1(Pro - \mathcal{C})$ by

$$\widetilde{(G, s, t)} = Lim\{ \textit{ finite quotients of } (G, s, t)\}.$$

General considerations of category theory then imply that this functor is left adjoint to the forgetful functor from *Cat^1(Pro$-\mathcal{C}$)* to *Cat^1(Grps).* ∎

5.2.2 Pro$-\mathcal{C}$ completions for crossed modules.

Corollary 8. *A pro$-\mathcal{C}$ completion functor from CMod to Pro$-\mathcal{C}$.CMod exists, (i.e., the forgetful functor $\mathcal{U}_{CMod}$ has a left adjoint).*

Proof: In the diagram

$$\begin{tikzcd} Pro-\mathcal{C}.CMod \arrow[r, "\simeq"] \arrow[d, "\mathcal{U}_{CMod}"'] & Cat^1(Pro-\mathcal{C}) \arrow[d, "\mathcal{U}_{\mathcal{C}}"] \\ CMod \arrow[r, "\simeq"] & Cat^1(Grps) \end{tikzcd}$$

we have found a left adjoint to the (vertical) functor on the right. This induces, via the equivalence of categories, a left adjoint for the left hand (vertical) functor. ∎

One can attempt to use the functors defining the two equivalences to give an "explicit" description of this pro$-\mathcal{C}$ completion functor, but initially in what follows we shall merely use its existence and the universal property that it satisfies to compare it with the pro$-\mathcal{C}$ completion of the individual groups involved. This will, in fact, provide a description of the completion in most useful cases.

Notation: We will denote by $(\widetilde{C,G,\partial})$ or, less accurately, $(\tilde{C},\tilde{G},\tilde{\partial})$, the pro$-\mathcal{C}$ completion of the crossed module (C,G,∂).

5.3 Pro$-\mathcal{C}$ completions of groups and crossed modules.

It is natural to want to compare this pro$-\mathcal{C}$ completion, $(\tilde{C},\tilde{G},\tilde{\partial})$, with the pro$-\mathcal{C}$ completions, $\hat{C},\hat{G}$ and $\hat{\partial}$, of the individual pieces of data involved. One may even wonder why $(\hat{C},\hat{G},\hat{\partial})$ is not itself always the same as $(\tilde{C},\tilde{G},\tilde{\partial})$. To start the study of this problem we first look at $\tilde{G}$.

5.3.1 The two completions agree on the base.

Proposition 34. *For any crossed module* (C,G,∂)*,* $\tilde{G}\cong\hat{G}$*.*

Proof: This follows from an adjoint functor argument:
There is a forgetful functor

$$R : CMod \to Grps$$

given by $R(C,G,\partial)=G$ and also an analogous one

$$R_{p\mathcal{C}} : Pro-\mathcal{C}.CMod \to Pro-\mathcal{C}.$$

These have left adjoints L and $L_{p\mathcal{C}}$ defined by $L(G)=(G,G,id_G)$ and similarly for $L_{p\mathcal{C}}$.

We have a diagram of left and right adjoints

$$\begin{tikzcd} Pro-\mathcal{C}.CMod \arrow[r, shift left, "R_{p\mathcal{C}}"] \arrow[d, shift left, "\mathcal{U}_{CMod}"] & Pro-\mathcal{C} \arrow[l, shift left, "L_{p\mathcal{C}}"] \arrow[d, shift left, "\mathcal{U}_{\mathcal{C}}"] \\ CMod \arrow[r, shift left, "R"] \arrow[u, shift left, "(\tilde{\ })"] & Grps \arrow[l, shift left, "L"] \arrow[u, shift left, "(\hat{\ })"] \end{tikzcd}$$

The right adjoint diagram commutes, so the left adjoint diagram commutes up to isomorphism, i.e.,

$$(\widetilde{G,G,id_G})\simeq(\hat{G},\hat{G},id_{\hat{G}}),$$

but better we have a sequence of isomorphisms: for a pro$-\mathcal{C}$ group H,

$$\begin{aligned}
Pro{-}\mathcal{C}(R_{p\mathcal{C}}(\widetilde{C,G,\partial}),H) &\cong Pro{-}\mathcal{C}.CMod((\widetilde{C,G,\partial}),L_{p\mathcal{C}}(H))\\
&\cong CMod((C,G,\partial),U_{CMod}L_{p\mathcal{C}}(H))\\
&\cong CMod((C,G,\partial),LU_{Grps}(H)) \qquad \text{by observation}\\
&\cong Grps(R(C,G,\partial),U_{Grps}(H))\\
&\cong Grps(G,U_{Grps}(H))\\
&\cong Pro{-}\mathcal{C}(\hat{G},H),
\end{aligned}$$

hence $\hat{G} \cong \tilde{G}$, independently of what C is. ■

5.3.2 The cofinality condition

In order to study conditions which imply that $\tilde{C}$ and $\hat{C}$ are isomorphic, it is convenient to introduce a condition that we will call the *"cofinality condition"*.

Let (C,G,∂) be a crossed module and write $\Omega_G(C)$ for the directed subset of $\Omega(C)$, the set of finite index normal subgroups of C, consisting of those $W \in \Omega(C)$, $C/W \in \mathcal{C}$, which are G-invariant. We will say that (C,G,∂) satisfies the *cofinality condition* if $\Omega_G(C)$ is cofinal in $\Omega(C)$.

Proposition 35. *If $G \in \mathcal{C}$, then any crossed G-module, (C,G,∂), satisfies the cofinality condition.*

Proof: Given any $W \in \Omega(C)$, let

$$W' = \bigcap_{g\in G} {}^{g}W,$$

be the intersection of all translates of W under the G-action. Then W' is G-invariant and as G is in $\mathcal{C}$, W' is of finite index and $C/W' \in \mathcal{C}$. As W' is contained in W, this completes the proof. ■

5.3.3 Completions and the cofinality condition.

Theorem 5. *If (C,G,∂) satisfies the cofinality condition, then $\tilde{C} \cong \hat{C}$.*

Proof: Recall that one has an isomorphism

$$\hat{C} \cong Lim_{W\in\Omega(C)}C/W.$$

As $\Omega_G(C)$ is cofinal in $\Omega(C)$, we have that this is isomorphic to $Lim_{W\in\Omega_G(C)}C/W$, so when considering an element of $\hat{C}$, we can represent it as a compatible family $(c_W W)_{W\in\Omega_G}$ of elements with $c_W \in C$. Of course there is a natural map

$$\begin{tikzcd} C \arrow[r, "\phi_2"] \arrow[d, "\partial"'] & \tilde{C} \arrow[d, "\tilde{\partial}"] \\ G \arrow[r, "\phi_1"] & \tilde{G} \end{tikzcd}$$

coming from the adjointness. As $\hat{G}$ is pro-$\mathcal{C}$, we have a factorisation via

$$\begin{tikzcd} \hat{C} \arrow[rr] \arrow[dr, "\hat{\partial}"'] & & \tilde{C} \arrow[dl, "\tilde{\partial}"] \\ & \hat{G} & \end{tikzcd}$$

and the various universality properties imply that it suffices to prove that $(\hat{C}, \hat{G}, \hat{\partial})$ is a pro$-\mathcal{C}$ crossed module in order to prove that $\tilde{C} \cong \hat{C}$. Thus we need to show that the G-action on C extends to a $\hat{G}$-action on $\hat{C}$ such that $\hat{\partial}$ is $\hat{G}$-equivariant and the Peiffer relation holds.

We need to define therefore a map

$$\hat{G} \times \hat{C} \to \hat{C}.$$

This we can attempt to do either topologically or using the identification of the category, *Pro*$-\mathcal{C}$, with the category, pro($\mathcal{C}$), of projective systems in the category $\mathcal{C}$. For this we need, for each $W \in \Omega_G(C)$, to pick a $(V, W) \in \Omega(G) \times \Omega_G(C)$ such that there is a map

$$\psi_W : G/V \times C/W' \to C/W,$$

and that these maps are compatible with the bonding maps of the systems $\{G/V\}$ and $\{C/W\}$.

We pick $W' = W$ and $V = St_G(C/W)$. To see the reason for the latter choice, we note that since W is G-equivariant, there is a G-action on C/W, a finite group in $\mathcal{C}$. This gives a homomorphism

$$G \to \mathrm{Aut}_C(C/W)$$

giving $V = St_G(C/W)$ as its kernel and we note that $V \lhd_{fin} G$, since $\mathrm{Aut}_C(C/W)$ is finite.

We define ψ_W by the obvious rule

$$\psi_W(gV, cW) = {}^g cW.$$

Now assume $W' \subset W$, $W' \in \Omega_G(C)$, then we get a G-equivariant epimorphism

$$p_W^{W'} : C/W' \to C/W,$$

and since if $v \in St_G(C/W)$, ${}^v c.c^{-1} \in W'$, we have $V' = St_G(C/W') \subset V$ and an epimorphism in $q_v^{v'} : G/V' \to G/V$. Thus we have a commutative diagram

$$\begin{array}{ccc} G/V \times C/W & \longrightarrow & C/W \\ {\scriptstyle q_v^{v'} \times p_W^{W'}} \uparrow & & \uparrow {\scriptstyle p_W^{W'}} \\ G/V' \times C/W' & \longrightarrow & C/W' \end{array}$$

i.e., $\{\psi_W : W \in \Omega_G(C)\}$ is a map of projective systems. That it is an action is then clear.

To check the axioms we need an explicit description of $\hat{\partial} : \hat{C} \to \hat{G}$. Given $U \lhd G$, so that $G/U \in \mathcal{C}$, there is a composed homomorphism $C \to G \to G/U$. Take N to be its kernel, then since ∂ is G-equivariant and G/U is in $\mathcal{C}$, it follows that N is in $\Omega_G(C)$ and that $U \subset St_G(C/N)$. These observations readily imply that $\hat{\partial}$, defined by

$$\hat{\partial}_U(cN_U) = \partial c_U U,$$

is not only well defined, but is $\hat{G}$-equivariant.

The proof that the Peiffer relation holds now follows from the Peiffer identity in (C, G, ∂) and the descriptions of $\hat{\partial}$ and the $\hat{G}$-action. ■

Corollary 9. *If G is in $\mathcal{C}$ and (C, G, ∂) a crossed module, then $(\hat{C}, \hat{G}, \hat{\partial})$ is a crossed module, which is the pro$-\mathcal{C}$ completion of (C, G, ∂).* ■

5.3.4 Completions of crossed modules with nilpotent actions.

Preservation of certain crossed module structures by termwise pro$-\mathcal{C}$ completion is reminiscent of the preservation of nilpotent fibrations by completions, as exemplified by the nilpotent fibration lemma of Bousfield-Kan, [19]. Recalling that if $p : E \to B$ is a fibration with connected fibre F, then the induced map from $\pi_1(F)$ to $\pi_1(E)$ makes $(\pi_1(E), \pi_1(F), p_*)$ into a crossed module (cf., Loday, [105]), it is not surprising that there is a link between nilpotent actions and preservation of crossed module structures.

The usual definition of a nilpotent action of G on C is as follows (cf., Bousfield-Kan, [19]):

An action of a group G on a group C is said to be *nilpotent* if there is a finite sequence

$$C = C_1 \supset \ldots \supset C_j \supset \ldots \supset C_n = \{e\}$$

of subgroups of C such that for each j

(i) C_j is closed under the action of G,

(ii) C_{j+1} is normal in C_j and C_j/C_{j+1} is Abelian,

and (iii) the induced G-action on C_j/C_{j+1} is trivial.

We will say that the *G-nilpotent length of C*, in this case, *is less than or equal to n*, ($\ell_G(C) \leq n$).

Proposition 36. *If (C, G, ∂) is a crossed module so that the action of G on C is nilpotent, then $(\hat{C}, \hat{G}, \hat{\partial})$ is a crossed module which is the pro$-\mathcal{C}$ completion of (C, G, ∂).*

Proof: We check the cofinality condition using induction on the G-nilpotent length of C.

We first note that if $W \lhd C$ is such that C/W is in $\mathcal{C}$, it is sufficient to prove that $\bigcap {}^gW = V$, say, is such that C/V is in $\mathcal{C}$. If $\ell_G(C) = 1$, the group C is trivial. If $\ell_G(C) = 2$, then the group C is Abelian with trivial G-action. In neither case is there any difficulty. Next suppose we have that the result holds provided $\ell_G(C) < n$, more precisely we assume that if $\ell_G(C) < n$, then if W is normal in C and $C/W \in \mathcal{C}$, then $V = \bigcap^g W$ is also such that C/V is in $\mathcal{C}$.

Now look at C with $\ell_G(C) = n$, so that there is a sequence,

$$C = C_1 \supset C_2 \supset \ldots \supset C_n = \{e\},$$

as in the definition above. Taking the normal subgroup C_2, we get a short exact sequence

$$1 \to C_2 \to C_1 \xrightarrow{p} C_1/C_2 \to 1,$$

in which $\ell_G(C_2) < n$ and C_1/C_2 is Abelian with trivial G-action.

Next suppose $W \lhd C$ is such that $W/C \in \mathcal{C}$. For any $g \in G$, $p({}^gW) = p(W)$, since the G-action on C_1/C_2 is trivial. Moreover ${}^gW \cap C_2 = {}^g(W \cap C_2)$, since C_2 is closed under the G-action. Thus setting $V = \bigcap^g W$, we get $p(V) = p(W)$ and $V \cap C_2 = \bigcap^g (W \cap C_2)$. As $C_1/C_2 \cap W \in \mathcal{C}$, we apply the induction hypothesis to conclude $C_2/(C_2 \cap V) \in \mathcal{C}$. Similarly the quotient of C_1/C_2 by $p(V)$ is in $\mathcal{C}$ as it is the same as that by $p(W)$.

The group C/V is thus part of an exact sequence, the other groups of which are in $\mathcal{C}$, hence it also is in $\mathcal{C}$ as required. ∎

5.4 Pro-$\mathcal{C}$ completions of free crossed modules.

If F is a free group on a finite set X, then the pro$-\mathcal{C}$ completion of F is a free pro$-\mathcal{C}$ group on X. (If X is not finite, one has to handle X as a topological space and the statement of the result gets slightly more technical as we saw in XChapter 1.) Given this, it is natural to enquire if a similar thing holds for free crossed modules. The following result gives the answer.

Proposition 37. *If $\partial : C \to G$ is a free crossed module on a function $f : S \to G$, then (C, G, ∂) is the free pro$-\mathcal{C}$ crossed module on the profinite completion $\hat{f} : \hat{S} \to \hat{G}$ of f.*

We should remark that $\hat{f}$ is obtained from the composite continuous map,

$$S \to G \to \hat{G},$$

where S is given the discrete topology, via the factorisation

$$S \to \hat{S} \to \hat{G}.$$

As we noted earlier, the profinite completion of a space can, for instance, be obtained by taking the Boolean algebra of clopen, i.e., closed-open, subsets of the space and then forming the maximal ideal space of that Boolean algebra.

Proof of Proposition 37 We start by introducing some useful notation. We have already introduced $CMod/G$ and $Pro{-}\mathcal{C}.CMod/\hat{G}$ for the categories of crossed modules over G and pro$-\mathcal{C}$ crossed modules over $\hat{G}$ respectively. We also introduce categories: $Sets/G$ and $Spaces/\hat{G}$ to denote the category of functions with codomain G (resp. continuous functions with codomain $\hat{G}$ and domain a profinite space). There are forgetful functors from $CMod/G$ (resp. $Pro{-}\mathcal{C}.CMod/\hat{G}$) to $Sets/G$ (resp. $Spaces/\hat{G}$) and the existence of free crossed modules in the two instances correspond to the existence of left adjoints for these functors: thus

$$CMod/G((C(S),G,\partial_f),(D,G,\partial')) \cong Sets/G(S,G,f),U(D,G,\partial')),$$

where $(C(S),G,\partial_f)$ is the free crossed module on (S,G,f), and

$$\begin{aligned}&Pro-\mathcal{C}.CMod/\hat{G}(C_{\mathcal{C}}(X),\hat{G},\bar{\partial}_f),(E,\hat{G},\partial'))\\ &\qquad\qquad\qquad\qquad\cong Spaces/\hat{G}((X,\hat{G},f),U_{\mathcal{C}}(E,\hat{G},\partial')),\end{aligned}$$

where $(C_{\mathcal{C}}(X),\hat{G},\bar{\partial}_f)$ denotes the free pro$-C$ crossed module on $(X,\hat{G},f)$. Then we have

$$\begin{aligned}&Pro{-}\mathcal{C}.CMod/\hat{G}((C(S),G,\partial_f),(\tilde{D},\hat{G},\partial))\\ &\quad\cong CMod/G((C(S),G,\partial_f),U_\phi(\tilde{D},\hat{G},\partial))\\ &\quad\cong Sets/G((S,G,f),UU_\phi(\tilde{D},\hat{G},\partial))\\ &\quad\cong Spaces/\hat{G}((\hat{S},\hat{G},\hat{f}),U_{\mathcal{C}}(\tilde{D},\hat{G},\partial))\\ &\quad\cong Pro-\mathcal{C}.CMod/\hat{G}((C_{\mathcal{C}}(\hat{S}),\hat{G},\bar{\partial}_{\hat{f}}),(\tilde{D},\hat{G},\partial)).\end{aligned}$$

(Here we have used the base restriction functor U_ϕ along the homomorphism $\phi : G \to \hat{G}$. This functor is given by pullback along ϕ as in section 3.3). Thus

$$(C_{\mathcal{C}}(\hat{S}),\hat{G},\overline{\partial}_{\hat{f}}) \cong (C(S),G,\partial_f),$$

as required. ■

5.5 Remarks on the non exactness of pro-$\mathcal{C}$ completions

If G is a group and $N \lhd G$, it does not follow that $\hat{N}_{\mathcal{C}} \lhd \hat{G}_{\mathcal{C}}$, i.e., pro$-\mathcal{C}$ completion is not exact. This causes difficulties in algebraic geometry. Friedlander, [61], gave an example in which the pro$-L$ completion of a covering morphism does not yield an exact sequence under π_1 as expected. The base of the covering is a surface and the characteristic is 0, so, in other respects, the situation is extremely well behaved.

Anderson, [3], points out that things can go wrong even for finite groups. Let $S\ell(2,5)$ be, as usual, the group of 2×2 matrices of determinant 1 over the field $\mathbb{Z}_5$. The centre of $S\ell(2,5)$ is of order 2. Completing at the prime 2 kills $S\ell(2,5)$ but leaves the centre alone. Thus from

$$Z(S\ell(2,5)) \lhd S\ell(2,5),$$

one obtains

$$Z(S\ell(2,5)) \to \{1\}.$$

Considering this second example from the viewpoint of this monograph, we note that as $S\ell(2,5)$ is finite, the 2-completion of any normal pair $N \lhd S\ell(2,5)$ should be a pro-2 crossed module (by Proposition 35) and, of course, since $Z(S\ell(2,5))$ is cyclic of order 2, this is indeed so.

Friedlander's example is somewhat deeper. One expects fibrations to yield crossed modules under π_1, at least in topological cases, cf. [106]. Thus the fact that on completing away from the prime 2, a fibration sequence associated with a covering should yield a crossed module in π_1 and not a normal inclusion, should not be cause for surprise. Friedlander's results from [62] might perhaps be considered from this viewpoint.

A similar phenomenon occurs in the theory of group presentations. For abstract groups, one knows that for any one relator group, G, in which the relator is not a proper power, one has that the cohomological dimension of G, cdG, is 2. However for pro$-p$ groups, Gildenhuys, [68], has given an example of a pro$-p$ presentation with two generators and one relation which is not a proper power and yet is such that the group thus presented has infinite cohomological dimension. This, in crossed module terms, can be explained as follows: The free crossed module

$$C(\mathcal{P}) \xrightarrow{\partial} F(x,y)$$

of Gildenhuy's presentation has ∂ an inclusion. If we pro$-p$ complete this crossed module, we get the corresponding construction in pro$-p$ and the map in this free pro$-p$ crossed module is no longer an inclusion, it has kernel a cyclic $\hat{\mathbb{Z}}_p(G)$-module with a periodic resolution. A full and detailed treatment of this example must wait until we have developed more on identities in this context. This will require a detailed knowledge of the structure of continuous derivations to which we turn in the next chapter.

5.6 Completions of simplicial groups and crossed modules.

We know that the crossed module, $M(G,1)$, represents the 1-type of the simplicial group G. The interaction of pro$-p$ completion and n-types was studied by Bousfield and Kan, [19] p.113, in the case where the homotopy groups of G are finitely generated. They summarise the key result in this area as follows: "the homotopy type of $R_\infty X$ in dimensions $\leq k$" depends only on "the homotopy type of X in dimensions $\leq k$". The precise relation between this and the area we have been studying is given slightly earlier (p.109), where it is stated that if T is the R-completion functor for groups, then first applying T dimensionwise to GX, the loop group of X, and then using the $\overline{W}$-construction gives a space weakly homotopically equivalent to $R_\infty X$. We also recall that if G is finitely generated, the R-completion of G for $R = \mathbb{Z}_p$ is simply the pro$-p$ completion.

Given this, it is natural to ask the following question:
given a (discrete) simplicial group G, what is the relationship between $M(\hat{G}, 1)$ and $\widetilde{M(G,1)}$?
(Here we use $\hat{G}$ to denote the levelwise pro$-\mathcal{C}$ completion of G.)

The key to this is the fact that $M(-,1)$ and E define an equivalence between $CMod$ and a reflexive subcategory, $T_{1]}$, of *Simp.Grps*. This subcategory is defined by the condition that G is in it if, and only if, the Moore complex, NG, of G has trivial terms in dimensions 2 and above, $N(G)_i = \{1\}$ if $i \geq 2$. The reflector, $t_{1]} : Simp.Grps. \to T_{1]}$, is defined by the condition that $Nt_{1]}(G)$ is the same as the truncation of NG given by :

$$\begin{cases} NG_0 & \text{in dimension 0.} \\ NG_1/d_0NG_2 & \text{in dimension 1.} \\ 1 & \text{in higher dimensions.} \end{cases}$$

One can check directly that $t_{1]}G$ is isomorphic to $EM(G,1)$.

Suppose now that G is a (discrete) simplicial group and $\mathcal{M}$ a pro$-\mathcal{C}$ crossed module, then

$$Simp.Grps(G, U(E\mathcal{M})) \cong Simp.Pro-\mathcal{C}(G, E\mathcal{M}),$$

since $E\mathcal{M}$ is a simplicial pro$-\mathcal{C}$ group. This set is, in its turn, naturally isomorphic to $T^{\mathcal{C}}_{1]}(t^{\mathcal{C}}_{1]}(\hat{G}), E\mathcal{M})$, where $t^{\mathcal{C}}_{1]} : Simp.Pro-\mathcal{C} \to T^{\mathcal{C}}_{1]}$ is the pro$-\mathcal{C}$ analogue of the reflector that was mentioned earlier. Since $M(t^{\mathcal{C}}_{1]}(\hat{G}), 1) \cong M(\hat{G}, 1)$, this gives a natural isomorphism

$$Simp.Grps(G, U(E\mathcal{M})) \cong Pro-\mathcal{C}.CMod(M(\hat{G}, 1), \mathcal{M}).$$

The forgetful functor $U : Pro-\mathcal{C} \to Grps$, or more exactly its simplicial and crossed module extensions, satisfies

$$UE \cong EU,$$

so one also has

$$\begin{aligned} Simp.Grps(G, UE(\mathcal{M})) &\cong Simp.Grps(G, EU(\mathcal{M})) \\ &\cong CMod(M(G,1), U(\mathcal{M})) \\ &\cong Pro-\mathcal{C}.CMod(\widetilde{M(G,1)}, \mathcal{M}). \end{aligned}$$

We thus have:

Theorem 6. *There is a natural isomorphism*

$$M(\hat{G}, 1) \cong \widetilde{M(G, 1)}.$$

■

This clarifies and extends Bousfield and Kan's result in the case $n = 2$, since for a reduced homotopy type X, the pro$-\mathcal{C}$ completion, $\bar{W}\hat{G}X$, of X, has a 2-type represented by $M(\hat{G}X, 1)$ which is isomorphic to the pro$-\mathcal{C}$ completion of the crossed module $M(GX, 1)$, which represents the 2-type of X, by the results of MacLane and Whitehead [114], (cf. Loday [106] or Porter [139], or here Chapter 9).

6

Pro-$\mathcal{C}$ crossed complexes and chain complexes

6.1 Continuous Derivations and derived pseudocompact modules.

6.1.1 Definitions

Let $\phi : G \to H$ be a continuous homomorphism of pro$-\mathcal{C}$ groups. A *continuous ϕ-derivation*

$$\partial : G \to M$$

from G to a left pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![H]\!]$ -module M is a continuous mapping from G to M, which satisfies the equation

$$\partial(g_1 g_2) = \partial(g_1) + \phi(g_1)\partial(g_2)$$

for all g_1, $g_2 \in G$.

A *derived pseudocompact module* for ϕ consists of a left pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![H]\!]$-module, D_ϕ, and a continuous ϕ-derivation, $\partial_\phi : G \to D_\phi$ with the following universal property:

Given any left pseudocompact $\hat{\mathbb{Z}}[\![H]\!]$-module, M, and a continuous ϕ-derivation $\partial : G \to M$, there is a unique continuous morphism

$$\beta : D_\phi \to M$$

of pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![H]\!]$-modules such that $\beta\partial_\phi = \partial$.

The set of all continuous ϕ-derivations from G to M has a natural Abelian group structure. We denote this set by $Der_\phi(G, M)$. This gives a functor from $Pc_{\mathcal{C}}.H$-*Mod* to *Ab*, the category of Abelian groups. If (D_ϕ, ∂_ϕ) exists, then it sets up a natural isomorphism

$$Der_\phi(G, M) \cong Pc_{\mathcal{C}}.H-Mod(D_\phi, M),$$

i.e., (D_ϕ, ∂_ϕ) represents the ϕ-derivation functor.

6.1.2 Existence

The treatment of derived modules in the abstract group context that is found in Crowell's paper, [40], provides a basis for some of what follows. In particular it indicates how to prove the existence of (D_ϕ, ∂_ϕ) for any ϕ.

Form a pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![H]\!]$-module, D, by taking the free pseudocompact left $\hat{\mathbb{Z}}_{\mathcal{C}}[\![H]\!]$-module, $\hat{\mathbb{Z}}_{\mathcal{C}}[\![H]\!]^{(X)}$, on a space of generators, $X = \{\partial g : g \in G\}$, homeomorphic to the underlying profinite space of G. Within $\hat{\mathbb{Z}}_{\mathcal{C}}[\![H]\!]^{(X)}$ form the closed submodule, Y, generated by the elements

$$\partial(g_1 g_2) - \partial(g_1) - \phi(g_1)\partial(g_2).$$

Let $D = \hat{\mathbb{Z}}_{\mathcal{C}}[\![H]\!]^{(X)}/Y$ and define $d : G \to D$ to be the composite:

$$G \xrightarrow{\eta} \hat{\mathbb{Z}}_{\mathcal{C}}[\![H]\!]^{(X)} \xrightarrow{quotient} D,$$

where η is "inclusion of the generators", $\eta(g) = \partial g$. Thus d is continuous and, by construction, will be a continuous ϕ-derivation. The universal property is easily checked and hence (D_ϕ, ∂_ϕ) exists.

We will later on construct (D_ϕ, ∂_ϕ) in a different way which provides a more amenable description of D_ϕ, namely as a tensor product. As a first step towards this description, we shall give a simple description of D_G, that is, the pseudocompact derived module of the identity morphism of G. More precisely we shall identify (D_G, ∂_G) as being $(\hat{I}_{\mathcal{C}}(G), \partial)$, where $\hat{I}_{\mathcal{C}}(G)$ is the augmentation ideal of $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$ and $\partial : G \to \hat{I}_{\mathcal{C}}(G)$ is the usual map, $\partial(g) = g - 1$. Algebraically this is fairly simple to handle, however the problem of proving continuity of the induced homomorphisms will mean that we have to be careful, and approach the problem by a circuitous route.

6.1.3 Derivation modules and augmentation ideals: the case of finite groups

We first handle the classical case of G a finite $\mathcal{C}$-group. In this case, the identification of the universal property of

$$\hat{I}_{\mathcal{C}}(G) = Ker(\hat{\mathbb{Z}}_{\mathcal{C}}(G) \to \hat{\mathbb{Z}}_{\mathcal{C}})$$

is well known. The universal derivation is

$$d_G : G \to \hat{I}_{\mathcal{C}}(G)$$

given by $d_G(g) = g - 1$.

We introduce the notation $f_\delta : \hat{I}_{\mathcal{C}}(G) \to M$ for the continuous $\hat{\mathbb{Z}}_{\mathcal{C}}[G]$-module morphism corresponding to a (continuous) derivation

$$\delta : G \to M.$$

The factorisation $f_\delta d_G = \delta$ implies that f_δ must be defined by $f_\delta(g-1) = \delta(g)$. That this works follows from the fact that $\hat{I}_{\mathcal{C}}(G)$, as a $\hat{\mathbb{Z}}_{\mathcal{C}}$-module, is free on the set $\{g-1 : g \in G\}$ and that the relations in $\hat{I}_{\mathcal{C}}(G)$ are generated by those of the form

$$g_1(g_2-1) = (g_1g_2-1) - (g_1-1).$$

We will need two results on the augmentation ideal construction that are not commonly found in the literature.

The proofs are easy and so will be omitted.

Lemma 11. *Given finite groups G and H in $\mathcal{C}$ and a commutative diagram*

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\psi"'] & M \arrow[d, "\phi"] \\ H \arrow[r, "\delta'"'] & N \end{tikzcd} \qquad (*)$$

where δ, δ' are derivations, M is a left pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[G]$-module, N is a left pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[H]$-module and ϕ is a module map over ψ, i.e., $\phi(g.m) = \psi(g)\phi(m)$ for $g \in G$, $m \in M$. Then the corresponding diagram

$$\begin{tikzcd} \hat{I}_{\mathcal{C}}(G) \arrow[r, "f_\delta"] \arrow[d, "\psi"'] & M \arrow[d, "\phi"] \\ \hat{I}_{\mathcal{C}}(H) \arrow[r, "f_{\delta'}"'] & N \end{tikzcd} \qquad (**)$$

is commutative. ■

Lemma 12. *Given a commutative diagram*

$$\begin{tikzcd} & G \arrow[r, "\delta"] & M \arrow[dd, "\phi"] \\ K \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \\ & H \arrow[r, "\delta'"'] & N \end{tikzcd}$$

in which δ, δ' are derivations, M is a left pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[G]$-module, N is a left pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[H]$-module and ϕ is a module homomorphism for the restricted K-module structures on M and N, then the diagram

$$\begin{tikzcd}
& \hat{I}_{\mathcal{C}}(G) \arrow[r, "f_\delta"] & M \arrow[dd, "\phi"] \\
\hat{I}_{\mathcal{C}}(K) \arrow[ur, "\alpha_*"] \arrow[dr, "\beta_*"'] & & \\
& \hat{I}_{\mathcal{C}}(H) \arrow[r, "f_{\delta'}"'] & N
\end{tikzcd}$$

is a commutative diagram of left pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[G]$-modules. ■

6.1.4 Derivation modules and augmentation ideals: the general case.

Suppose G is a pro$-\mathcal{C}$ group and define $\hat{I}_{\mathcal{C}}(G)$ to be the kernel of the augmentation map from $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$ to $\hat{\mathbb{Z}}_{\mathcal{C}}$. There is an obvious G-derivation $d_G : G \to \hat{I}_{\mathcal{C}}(G)$ given by the usual rule, $d_G(g) = g - 1$. We have first to prove that this is continuous.

Lemma 13. *The map d_G is continuous as it is isomorphic to the inverse limit of the maps $d_{G/U} : G/U \to \hat{I}_{\mathcal{C}}(G/U)$ for U in $\Omega(G)$, the ordered set of $\mathcal{C}$-cofinite normal subgroups of G.*

Proof: In fact, for each such U, there is an exact sequence

$$O \to \hat{I}_{\mathcal{C}}(G/U) \to \hat{\mathbb{Z}}_{\mathcal{C}}[G/U] \stackrel{\varepsilon_U}{\to} \hat{\mathbb{Z}}_{\mathcal{C}} \to O$$

varying functorially with U. Taking the projective limit gives

$$O \to \varprojlim \hat{I}_{\mathcal{C}}(G/U) \to \hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!] \stackrel{\varepsilon}{\to} \hat{\mathbb{Z}}_{\mathcal{C}} \to O$$

(since lim is exact on $Pc_{\mathcal{C}}$.*G-Mod*), so

$$\hat{I}_{\mathcal{C}}(G) \cong \varprojlim \hat{I}_{\mathcal{C}}(G/U).$$

The maps $d_{G/U}$ defined by $d_{G/U}(gU) = gU - 1$ clearly give d_G "in the limit" and as these maps are themselves continuous, so is d_G. ■

Theorem 7. *If G is a pro-$\mathcal{C}$ group, $(D_G, d_G) \cong (\hat{I}_{\mathcal{C}}(G), d_G)$.*

Proof: Suppose given a continuous G-derivation $\delta : G \to M$, we have to construct a map $f : \hat{I}_{\mathcal{C}}(G) \to M$ of pseudocompact G-modules so that $fd_G = \delta$. (The classical rule $f(g-1) = \delta(g)$ must therefore be satisfied, but we do not as yet know that the $g-1$ topologically generate $\hat{I}_{\mathcal{C}}(G)$, so we cannot use this as a definition. We also cannot be certain, at this stage, of the relations between the $(g-1)$s. To avoid these difficulties we use the equivalence between the topological and the pro categorical approach and couch our proof in the latter terminology.)

We have $\delta : G \to M$ is continuous so for any open submodule $V \subset M$, there is an open normal subgroup, $U \in \Omega(G)$, and a map

$$\delta_{U,V} : G/U \to M/V$$

representing δ. Thus if $V' \subset V$, and we pick U' with $G/U' \in \mathcal{C}$ such that $\delta_{U',V'}$ also represents δ, there is some $U'' \subset U \cap U'$ and a commutative diagram

$$\begin{tikzcd} & G/U \arrow[r, "\delta_{U,V}"] & M/V \\ G/U'' \arrow[ur] \arrow[dr] & & \\ & G/U' \arrow[r, "\delta_{U',V'}"] & M/V' \arrow[uu] \end{tikzcd}$$

So as to be able to apply Lemma 12, it is necessary to have that $\delta_{U,V}$ and $\delta_{U',V'}$ are derivations. We claim that by, if necessary, passing to smaller open subgroups U, U', this can be assumed to be the case (i.e., the $\delta_{U,V}$ are cofinally derivations). We proceed as follows:

Given the open submodule $V \subset M$, let

$$St_G(M/V) = \{g \in G : gm + V = m + V \quad \text{for all} \quad m \in M\},$$

then as M/V is discrete of finite length, $St_G(M/V)$ is an open subgroup of G.

If $\delta_{U,V} : G/U \to M/V$ represents δ, then so does the composite

$$\delta_{U\cap W,V} : G/(U \cap W) \to G/U \to M/V$$

for any open normal subgroup W of G, and so we may assume that $\delta_{U,V}$ is defined with $U \subset St_G(M/V)$, but this means M/V is a G/U-module. It is now easy to check that since δ is a derivation and the diagram

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[d] & M \arrow[d] \\ G/U \arrow[r, "\delta_{U,V}"'] & M/V \end{tikzcd}$$

commutes, this map $\delta_{U,V}$ is a G/U-derivation. (Continuity is somewhat superfluous here, since M/V and G/U are both discrete.)

This implies that $\delta_{U,V}$ factors

$$\begin{tikzcd} G/U \arrow[r, "d_{G/U}"] \arrow[dr, "\delta_{U,V}"'] & I_{\mathcal{C}}(G/U) \arrow[d, "f_{U,V}"] \\ & M/V' \end{tikzcd}$$

with $f_{U,V}$ a continuous map of pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[G/U]$-modules.

By Lemma 12, the maps

$$f_{U,V} : I_{\mathcal{C}}(\hat{G}/U) \to M/V$$

form a map of pro-modules, hence we can take the limit of the above "pro-diagram" to get a diagram

$$\begin{tikzcd} G \arrow[r, "d_G"] \arrow[dr, "\delta"'] & \hat{I}_{\mathcal{C}}(G) \arrow[d, "f"] \\ & M \end{tikzcd}$$

The uniqueness of f follows from the uniqueness of the various $f_{U,V}$. This completes the verification that (D_G, d_G) and $(\hat{I}_{\mathcal{C}}(G), d_G)$ are isomorphic. ■

Corollary 10. *The subspace $Im\, d_G = \{g - 1 : g \in G\} \subset \hat{I}_{\mathcal{C}}(G)$ topologically generates $\hat{I}_{\mathcal{C}}(G)$ as a pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-module. Moreover the relations between these generators are generated by those of the form*

$$(g_1 g_2 - 1) - (g_1 - 1) - g_1(g_2 - 1).$$

■

It is useful to have also the following reformulation of the above Theorem stated explicitly.

Corollary 11. *There is a natural isomorphism*

$$Der_G(G, M) \cong Pc_{\mathcal{C}}.G\!-\!Mod(\hat{I}_{\mathcal{C}}(G), M).$$

■

6.1.5 Topological generation of $\hat{I}_{\mathcal{C}}(G)$.

The first of these two corollaries raises the question as to whether, if $X \subset G$ topologically generates G, does the set $G_X = \{x - 1 : x \in X\}$ topologically generate $\hat{I}_{\mathcal{C}}(G)$ as a pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-module.

Proposition 38. *If X topologically generates G, then G_X topologically generates $\hat{I}_{\mathcal{C}}(G)$.*

Proof: We first note the following elementary result:

Lemma 14. *If G is in Pro$-\mathcal{C}$ and M is a pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-module, then the semidirect product $M \rtimes G$ is in Pro$-\mathcal{C}$ and there is an isomorphism*

$$Der_G(G, M) \to \mathrm{Hom}/G(G, M \rtimes G)$$

where $\mathrm{Hom}/G(G, M \rtimes G)$ *is the set of continuous homomorphisms from G to $M \rtimes G$ over G, i.e., $\theta : G \to M \rtimes G$ such that for each $g \in G$, $\theta(g) = (\theta'(g), g)$ for some $\theta'(g) \in M$.*

Proof: Given such a θ and the corresponding θ', check that θ is a homomorphism if and only if θ' is a G-derivation. ■

Returning to the proof of the proposition, suppose M is a pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-module. It suffices to prove that the natural map from the group, $Pc_{\mathcal{C}}.G\text{-}Mod(\hat{I}_{\mathcal{C}}(G), M)$, of continuous $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-module morphisms into the group, $Cts(G_X, M)$, of continuous maps from G_X to M is one-to-one.

Suppose X topologically generates G, so there is a free pro$-\mathcal{C}$ group $F_{\mathcal{C}}(X)$ on X such that

$$\begin{tikzcd} F_{\mathcal{C}}(X) \arrow[rr, "\eta"] & & G \\ & X \arrow[ul] \arrow[ur, "i"'] & \end{tikzcd}$$

is a commutative diagram, with η an epimorphism. This implies that, for any pro$-\mathcal{C}$ group, H,

$$Cts(X, H) \cong Pro{-}\mathcal{C}(F_{\mathcal{C}}(X), H) \stackrel{(1-1)}{\longleftarrow} Pro{-}\mathcal{C}(G, H).$$

Now as before $\mathrm{Hom}/G(F_{\mathcal{C}}(X), M \rtimes G)$ denotes the set of continuous morphisms, θ, from $F_{\mathcal{C}}(X)$ to $M \rtimes G$ over G, i.e., so that

$$\begin{tikzcd} F_{\mathcal{C}}(X) \arrow[rr, "\theta"] \arrow[dr, "\eta"'] & & M \rtimes G \arrow[dl, "pr_1"] \\ & G & \end{tikzcd}$$

commutes. The epimorphism η induces a one-to-one function

$$\mathrm{Hom}/G(G, M \rtimes G) \to \mathrm{Hom}/G(F_{\mathcal{C}}(X), M \rtimes G).$$

Restricting this construction, we get a one-to-one map,

$$Cts(G_X, M) \to Cts/G(X, M \rtimes G),$$

(with the obvious extension of notation). Thus we obtain the following commutative diagram:

$$\begin{tikzcd} Pc_{\mathcal{C}}.G-Mod(\hat{I}_{\mathcal{C}}(G), M) \arrow[r, "(1-1)"] \arrow[d, "rest."'] & \mathrm{Hom}/G(F_{\mathcal{C}}(X), M \rtimes G) \arrow[d, "\cong"] \\ Cts(G_X, M) \arrow[r, "(1-1)"] & Cts/G(X, M \rtimes G) \end{tikzcd}$$

where *rest.* stands for restriction.

The commutativity of this diagram implies that this restriction map is one-to-one as required.

To complete the proof we indicate why this is sufficient. There is a natural isomorphism,

$$Cts(G_X, M) \cong Pc_{\mathcal{C}}.G - Mod(\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]^{(G_X)}, M),$$

where $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]^{(G_X)}$ denotes the free pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-module on the profinite space, G_X. For the pseudocompact module, $\hat{I}_{\mathcal{C}}(G)$, we obtain a restriction map

$$Pc_{\mathcal{C}}.G-Mod(\hat{I}_{\mathcal{C}}(G), \hat{I}_{\mathcal{C}}(G)) \stackrel{rest.}{\to} Pc_{\mathcal{C}}.G-Mod(\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]^{(G_X)}, \hat{I}_{\mathcal{C}}(G)),$$

and thus there is a map

$$\rho : \hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]^{(G_X)} \to \hat{I}_{\mathcal{C}}(G)$$

such that $\rho^* = rest.$ Since *rest.* is a monomorphism, ρ must thus be an epimorphism as required. ■

6.1.6 When G is free pro-$\mathcal{C}$.

One of the main uses of the above is in the case when $G \cong F_{\mathcal{C}}(X)$, i.e., is free pro$-\mathcal{C}$ on the space X.

Corollary 12. *If $G \cong F_{\mathcal{C}}(X)$ is the free pro$-\mathcal{C}$ group on the space X, then the set $\{x-1 : x \in X\}$ freely topologically generates $\hat{I}_{\mathcal{C}}(G)$ as a pseudocompact $\hat{\mathbb{Z}}[\![G]\!]$-module.*

Proof: Examination of the proof in this case shows

$$Cts(X, M) \cong Pc_{\mathcal{C}}.G - Mod(\hat{I}_{\mathcal{C}}(G), M).$$

■

6.1.7 (D_ϕ, d_ϕ), the general case.

We can now return to the identification of (D_ϕ, d_ϕ) in the general case.

Proposition 39. *If $\phi : G \to H$ is a continuous homomorphism of pro$-\mathcal{C}$ groups, then $D_\phi \cong \hat{\mathbb{Z}}_{\mathcal{C}}[\![H]\!]\hat{\otimes}_G\hat{I}_{\mathcal{C}}(G)$, the completed tensor product of $\hat{\mathbb{Z}}_{\mathcal{C}}[\![H]\!]$ and $\hat{I}_{\mathcal{C}}(G)$ over G.*

Proof: If M is a pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![H]\!]$-module, we will write $\phi^\sharp(M)$ for the restricted pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-module.

With this notation we have a chain of natural isomorphisms,

$$\begin{aligned} Der_\phi(G, M) &\cong Der_G(G, \phi^\sharp(M)) \\ &\cong Pc_{\mathcal{C}}.G\!-\!Mod(\hat{I}_{\mathcal{C}}(G), \phi^\sharp(M)) \\ &\cong Pc_{\mathcal{C}}.H\!-\!Mod(\hat{\mathbb{Z}}_{\mathcal{C}}[\![H]\!]\hat{\otimes}_G \hat{I}_{\mathcal{C}}(G), M), \end{aligned}$$

so by universality,

$$D_\phi \cong \hat{\mathbb{Z}}_{\mathcal{C}}[\![H]\!]\hat{\otimes}_G \hat{I}_{\mathcal{C}}(G),$$

as required. ∎

6.1.8 D_ϕ for ϕ, the counit $F_{\mathcal{C}}(X) \to G$.

The above will be particularly useful when ϕ is the "co-unit" map, $F_{\mathcal{C}}(X) \to G$, for X a space that topologically generates G. In fact we have the following:

Corollary 13. *Let $\phi : F_{\mathcal{C}}(X) \to G$ be a continuous epimorphism of pro$-\mathcal{C}$ groups, then there is a continuous isomorphism*

$$D_\phi \cong \hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]^{(X)}$$

of pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-modules. In this isomorphism, the generator e_x, of D_ϕ corresponding to $x \in X$, satisfies

$$d_\phi(x) = e_x$$

for all $x \in X$. ∎

6.2 Associated module sequences

6.2.1 Homological background

Given an exact sequence

$$1 \to K \to L \to Q \to 1$$

of abstract groups, then it is a standard result from homological algebra that there is an associated exact sequence of modules,

$$0 \to K^{Ab} \to \mathbb{Z}[Q] \otimes_L I(L) \to I(Q) \to 0.$$

There are several different proofs of this. Homological proofs give this as a simple consequence of the Tor^L-sequence corresponding to the exact sequence

$$0 \to I(L) \to \mathbb{Z}[L] \to \mathbb{Z} \to 0$$

together with a calculation of $\mathrm{Tor}_1^L(\mathbb{Z}[Q], \mathbb{Z})$. Such a proof has been generalised to the case when

$$1 \to K \to L \to Q \to 1$$

is an exact sequence of profinite or pro-p groups by Wambsganß-Türk, [163]. This, of course, yields an associated sequence of pseudocompact $\hat{\mathbb{Z}}[\![Q]\!]$-modules or of $\hat{\mathbb{Z}}_p[\![Q]\!]$-modules,

$$1 \to K^{Ab} \to \hat{\mathbb{Z}}_{\mathcal{C}}[\![Q]\!]\hat{\otimes}_L \hat{I}_{\mathcal{C}}(L) \to \hat{I}_{\mathcal{C}}(Q) \to 1$$

(for $\mathcal{C} = \mathit{FGrps}$ or p-*Groups*). His proof would seem to go through for more general $\mathcal{C}$. The second type of proof is more directly algebraic and has the advantage that it accentuates various universal properties of the sequence. The most thorough treatment of this would seem to be by Crowell, [40], for the discrete case. His proof generalises with a little care to handle pro$-\mathcal{C}$ groups; this we outline below.

6.2.2 The exact sequence in the pro-$\mathcal{C}$ case.

Proposition 40. *Let*

$$1 \to K \xrightarrow{\phi} L \xrightarrow{\psi} Q \to 1$$

be an exact sequence of pro$-\mathcal{C}$ groups and continuous homomorphisms. Then there is an exact sequence

$$0 \to K^{Ab} \xrightarrow{\tilde{\phi}} \hat{\mathbb{Z}}_{\mathcal{C}}[\![Q]\!]\hat{\otimes}_L \hat{I}_{\mathcal{C}}(L) \xrightarrow{\tilde{\psi}} \hat{I}_{\mathcal{C}}(Q) \to 0$$

of pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![Q]\!]-$modules.

Proof: By the universal property of D_ψ, there is a unique morphism

$$\tilde{\psi} : D_\psi \to \hat{I}_{\mathcal{C}}(Q)$$

such that $\tilde{\psi}\partial_\psi = \hat{I}_{\mathcal{C}}(\psi)\partial_L$.

Let $[K, K]$ denote the closed commutator subgroup of K and let $\delta : K \to K^{Ab} = K/[K, K]$ be the canonical pro$-\mathcal{C}$ abelianising morphism. We note that $\partial_\psi\phi : K \to D_\psi$ is a homomorphism (since

$$\begin{aligned}\partial_\psi\phi(k_1k_2) &= \partial_\psi\phi(k_1) + \psi\phi(k_1)\partial_\psi\phi(k_2)\\ &= \partial_\psi\phi(k_1) + \partial_\psi\phi(k_2),)\end{aligned}$$

so let $\tilde{\phi} : K^{Ab} \to D_\psi$ be the unique continuous morphism satisfying $\tilde{\phi}\delta = \partial_\psi\phi$. (Recall that K^{Ab} has a natural pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![Q]\!]$-module structure.)

That the composite $\tilde{\psi}\tilde{\phi} = 0$ follows easily from $\psi\phi = 0$. Since D_ψ is generated by symbols $d\ell$ and $\tilde{\psi}(d\ell) = \psi(\ell) - 1$, it follows that $\tilde{\psi}$ is onto. We next turn to "$Ker\ \tilde{\psi} \subseteq Im\ \tilde{\phi}$".

If we can prove $\alpha : D_\psi \to \hat{I}_{\mathcal{C}}(Q)$ is the cokernel of $\tilde{\phi}$ then we will have checked this inclusion and incidentally will have reproved that $\tilde{\psi}$ is onto.

Now let $D_\psi \to C$ be any continuous morphism such that $\alpha\tilde{\phi} = 0$. Consider the diagram

$$\begin{tikzcd}
K \arrow[r, "\phi"] \arrow[d, "\delta"'] & L \arrow[r, "\psi"] \arrow[d, "\partial_\psi"] & Q \arrow[d, "\partial_Q"] \\
K^{Ab} \arrow[r, "\tilde{\phi}"] & D_\psi \arrow[r, "\tilde{\psi}"] \arrow[dr, "\alpha"'] & \hat{I}_{\mathcal{C}}(Q) \\
& & C
\end{tikzcd}$$

The composite $\alpha\partial_\psi$ vanishes on the image of ϕ since $\alpha\partial_\psi\phi = \alpha\tilde{\phi}\delta$ and $\alpha\tilde{\phi}$ is assumed zero. Define $d : Q \to C$ by $d(q) = \alpha\partial_\psi(\ell)$ for $\ell \in L$ such that $\psi(\ell) = q$. As $\alpha\partial_\psi$ vanishes on $Im\ \phi$, this is well defined and

$$\begin{aligned}
d(q_1q_2) &= \alpha\partial_\psi(\ell_1\ell_2) \\
&= \alpha\partial_\psi(\ell_1) + \alpha(\psi(\ell_1)\partial_\psi(\ell_2)) \\
&= d(q_1) + q_1 d(q_2)
\end{aligned}$$

so d factors as $\bar{\alpha}\partial_Q$ in a unique way with $\bar{\alpha} : \hat{I}_{\mathcal{C}}(Q) \to C$. It remains to prove that $\alpha = \tilde{\psi}$, but

$$\begin{aligned}
\tilde{\psi}\partial_\psi &= I_C(\psi)\partial_L \\
&= \partial_Q\psi
\end{aligned}$$

by the naturality of ∂. Now finally note that $\bar{\alpha}\partial_Q = d$ and $d\psi = \alpha\partial_\psi$ to conclude that $\tilde{\psi}\partial_\psi$ and $\alpha\partial_\psi$ are equal. Equality of α and $\bar{\alpha}\tilde{\psi}$ then follows by the uniqueness clause of the universal property of (D_ψ, ∂_ψ).

Next we need to check that $K^{Ab} \to D_\psi$ is a monomorphism. To do this we exploit the fact that there is a continuous transversal, $s : Q \to L$, (Corollary 2, p. 13) satisfying $s(1) = 1$. This means that, as in Crowell, [40] p. 224, we can for each $\ell \in L$, $q \in Q$, find an element $q \times \ell$ uniquely determined by the equation

$$\phi(q \times \ell)) = s(q)\ell s(q\psi(\ell))^{-1},$$

which, of course, defines a continuous function from $Q \times L$ to K. Crowell's lemma 4.5 then shows

$$q \times \ell_1\ell_2 = (q \times \ell_1)(q\psi(\ell_1) \times \ell_2) \quad \text{for} \quad \ell_1, \ell_2 \in L.$$

Now let $M = \hat{\mathbb{Z}}_{\mathcal{C}}[\![Q]\!]^{(X)}$, with $X = \{\partial\ell : \ell \in L\}$, so that there is an exact sequence

$$M \to D_\psi \to 0.$$

As mentioned by Gildenhuys and Mackay, [71] p.462, the underlying topological group of $\hat{\mathbb{Z}}_{\mathcal{C}}[\![Q]\!]$ is the free Abelian pro$-\mathcal{C}$ group on the underlying topological space of Q. Similarly M, above, has, as underlying topological group, the free Abelian pro$-\mathcal{C}$ group on $Q \times X$.

Define a continuous map $\tau : M \to K^{Ab}$ of Abelian pro$-\mathcal{C}$ groups by

$$\tau(a, \partial\ell) = \delta(q \times \ell).$$

We check that if $p(m) = 0$, then $\tau(m) = 0$. Since $Ker\,p$ is topologically generated as a $\hat{\mathbb{Z}}_{\mathcal{C}}[\![Q]\!]$-module by elements of the form

$$\partial(\ell_1\ell_2) - \partial\ell_1 - \psi(\ell_1)\partial\ell_2,$$

it follows that as an Abelian pro$-\mathcal{C}$ group, $Ker\,p$ is topologically generated by the elements

$$(q, \partial(\ell_1\ell_2)) - (q, \partial\ell_1) - (q\psi(\ell_1), \partial\ell_2).$$

We claim that τ is zero on these elements; in fact

$$\begin{aligned}
\tau(q, \partial(\ell_1\ell_2)) &= \delta(q \times (\ell_1\ell_2)) \\
&= \delta(q \times \ell_1) + \delta(q\psi(\ell_1) \times \ell_2) \\
&= \tau(q, \ell_1) + \tau(q\psi(\ell_1), \ell_2).
\end{aligned}$$

Thus τ induces a continuous map $\eta : D_\psi \to K^{Ab}$ of Abelian pro$-\mathcal{C}$ groups.

Finally we check $\eta\tilde{\phi} =$ identity, so that $\tilde{\phi}$ is a monomorphism: let $b \in K^{Ab}$, $k \in K$ be such that $\delta(k) = b$, then

$$\begin{aligned}
\eta\tilde{\phi}(b) &= \eta\tilde{\phi}\delta(k) \\
&= \eta\partial_\psi(k) \\
&= \delta(1 \times \phi(k)),
\end{aligned}$$

but $1 \times \phi(k)$ is uniquely determined by

$$\phi(1 \times \phi(k)) = s(1)\phi(k)s(1\psi\phi(k))^{-1} = \phi(k),$$

since $s(1) = 1$, hence $1 \times \phi(k) = k$ and $\eta\tilde{\phi}(b) = \delta(k) = b$ as required. ■

A discussion of the way in which the original abstract version of this result interacts with the theory of covering spaces can be found in Crowell's paper already cited. We will very shortly see the connection of this module sequence with the Jacobian matrix of a group presentation and the Fox free differential calculus. It is this latter connection which requires that we have more or less explicit formulae for the maps $\tilde{\phi}$ and $\tilde{\psi}$ and hence requires that Crowell's detailed proof be generalised, not the easier homological proof.

6.2.3 Reidemeister-Fox derivatives and Jacobian matrices

At various points we will need to refer to Reidemeister-Fox derivatives as developed by Fox in a series of articles, see [60], and also summarised in Crowell and Fox, [41]. We will call these derivatives *Fox derivatives.* One of our aims here is to develop a theory for profinite algebraic homotopy in which actual calculations can be attempted. For instance, suppose we have a finitely presented (discrete) group, G and want to analyse its pro-p completion. We have a presentation of G and, perhaps, some geometric models for a $K(G,1)$. We also can call on various analogues of classical techniques for analysing group presentations such as those of Fox derivatives and the related Jacobian matrices. In this section we will examine such techniques in the discrete case. This will be useful when discussing homological syzygies, as well as being needed to understand the way in which our results in the profinite case can be considered extensions of these techniques.

(For this section, we will be restricting attention to the theory in the discrete case only.)

Suppose G is a group and M a G-module and let $\delta : G \to M$ be a derivation, (so $\delta(g_1g_2) = \delta(g_1) + g_1\delta(g_2)$ for all $g_1, g_2 \in G$), then, for calculations, the following lemma is very valuable, although very simple to prove.

Lemma 15. *If $\delta : G \to M$ is a derivation, then*
(i) $\delta(1_G) = 0$;
(ii) $\delta(g^{-1}) = -g^{-1}\delta(g)$ for all $g \in G$;
(iii) for any $g \in G$ and $n \geq 1$,

$$\delta(g^n) = (\sum_{k=0}^{n-1} g^k)\delta(g).$$

Proof: As was said, these are easy to prove.
$\delta(g) = \delta(1g) + 1\delta((g)$, so $\delta(1) = 0$, and hence (i); then

$$\delta(1) = \delta(g^{-1}g) = \delta(g^{-1}) + g^{-1}\delta(g)$$

to get (ii), and finally induction to get (iii). ∎

The Fox derivatives are derivations taking values in the group ring as a left module over itself. They are defined for $G = F(X)$, the free group on a set X. (We usually write F for $F(X)$ in what follows.)

Definition: For each $x \in X$, let

$$\frac{\partial}{\partial x} : F \to \mathbb{Z}F$$

be defined by
(i) for $y \in X$,

$$\frac{\partial y}{\partial x} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } y \neq x; \end{cases}$$

(ii) for any words, $w_1, w_2 \in F$,

$$\frac{\partial}{\partial x}(w_1 w_2) = \frac{\partial}{\partial x} w_1 + w_1 \frac{\partial}{\partial x} w_2.$$

This derivation will be called the *Fox derivative with respect to the generator* x.

Of course, a routine proof shows that the derivation property in (ii) defines $\frac{\partial w}{\partial x}$ for any $w \in F$.

Example: Let $X = \{u, v\}$, with $r \equiv uvuv^{-1}u^{-1}v^{-1} \in F = F(u, v)$, then

$$\begin{aligned} \frac{\partial r}{\partial u} &= 1 + uv - uvuv^{-1}u^{-1}, \\ \frac{\partial r}{\partial v} &= u - uvuv^{-1} - uvuv^{-1}u^{-1}v^{-1}. \end{aligned}$$

This relation is the typical braid group relation, here in Br_3, and we will come back to these simple calculations later.

It is often useful to extend a derivation $\delta : G \to M$ to a linear map from $\mathbb{Z}G$ to M by the simple rule that $\delta(g + h) = \delta(g) + \delta(h)$.

By the classical discrete versions of our results above, we have

$$Der(F, \mathbb{Z}F) \cong F{-}Mod(IF, \mathbb{Z}F),$$

and that

$$IF \cong \mathbb{Z}F^{(X)},$$

with the isomorphism matching each generating $x - 1$ with e_x, the basis element labelled by $x \in X$. (The universal derivation then sends x to e_x.)

For each given x, we thus obtain a morphism of F-modules:

$$d_x : \mathbb{Z}F^{(X)} \to \mathbb{Z}F$$

with

$$\begin{aligned} d_x(e_y) &= 1 \quad \text{if } y = x \\ d_x(e_y) &= 0 \quad \text{if } y \neq x, \end{aligned}$$

i.e., the 'projection onto the x^{th}-factor' or 'evaluation at $x \in X$' depending on the viewpoint taken of the elements of the free module, $\mathbb{Z}F^{(X)}$.

Suppose now that we have a group presentation, $\mathcal{P} = (X : R)$, of a group, G. Then we have a short exact sequence of groups

$$1 \to N \xrightarrow{\phi} F \xrightarrow{\gamma} G \to 1,$$

where $N = N(R)$, $F = F(X)$, i.e., N is the normal closure of R in the free group F. We also have a free crossed module,

$$C \xrightarrow{\partial} F,$$

constructed from the presentation and hence, two short exact sequences of G-modules with $\kappa(\mathcal{P}) = Ker\,\partial$, the module of identities of $\mathcal{P}$,

$$0 \to \kappa(\mathcal{P}) \to C^{Ab} \to N^{Ab} \to 0,$$

by the discrete analogue of Proposition 23 on page 74, and also

$$0 \to N^{Ab} \xrightarrow{\tilde{\phi}} IF \otimes_F \mathbb{Z}G \to IG \to 0.$$

We note that the first of these is exact because N is a free group, further

$$C^{Ab} \cong \mathbb{Z}G^{(R)},$$

and the map from this to N^{Ab} in the first sequence sends the generator e_r to $r[N, N]$.

We next revisit the derivation of the associated exact sequence (Proposition 40, page 130) in some detail to see what $\tilde{\phi}$ does to $r[N, N]$. We have $\tilde{\phi}(r[N, N]) = \partial_\gamma \phi(r) = \partial_\gamma(r)$, considering r now as an element of F, and by Corollary 13, on identifying D_γ with $\mathbb{Z}G^{(X)}$ using the isomorphism between IF and $\mathbb{Z}F^{(X)}$, we can identify $\partial_\gamma(x) = e_x$. We are thus left to determine $\partial_\gamma(r)$ in terms of the $\partial_\gamma(x)$, i.e., the e_x. The following lemma does the job for us.

Lemma 16. *Let $\delta : F \to M$ be a derivation and $w \in F$, then*

$$\delta w = \sum_{x \in X} \frac{\partial w}{\partial x} \delta x.$$

Proof: By induction on the length of w. ■

In particular we thus can calculate

$$\partial_\gamma(r) = \sum \frac{\partial r}{\partial x} e_x.$$

Tensoring with $\mathbb{Z}G$, we get

$$\tilde{\phi}(r[N, N]) = \sum \frac{\partial r}{\partial x} e_x \otimes 1.$$

There is one final step to get this into a usable form:

From the quotient map $\gamma : F \to G$, we, of course, get an induced ring homomorphism, $\gamma : \mathbb{Z}F \to \mathbb{Z}G$, and hence we have elements $\gamma(\frac{\partial r}{\partial x}) \in \mathbb{Z}G$. Of course,

$$\frac{\partial r}{\partial x} e_x \otimes 1 = e_x \otimes \gamma(\frac{\partial r}{\partial x}),$$

so we have, on tidying up notation just a little:

Proposition 41. *The composite map*

$$\mathbb{Z}G^{(R)} \to N^{Ab} \to \mathbb{Z}G^{(X)}$$

sends e_r *to* $\sum \gamma(\frac{\partial r}{\partial x})e_x$ *and so has a matrix representation given by* $J_{\mathcal{P}} = \big(\gamma(\frac{\partial r_i}{\partial x_j})\big)$. ■

Definition: The *Jacobian matrix* of a group presentation, $\mathcal{P} = (X : R)$ of a group G is

$$J_{\mathcal{P}} = \Big(\gamma(\frac{\partial r_i}{\partial x_j})\Big),$$

in the above notation.

The application of γ to the matrix of Fox derivatives simplifies expressions considerable in the matrix. The usual case of this is if a relator has the form rs^{-1}, then we get

$$\frac{\partial rs^{-1}}{\partial x} = \frac{\partial r}{\partial x} - rs^{-1}\frac{\partial s}{\partial x}$$

and if r or s is quite long this looks moderately horrible! However applying γ to the answer, the term rs^{-1} in the second of the two terms becomes 1. We can actually think of this as replacing rs^{-1} by $r - s$ when working out the Jacobian matrix.

Example: Br_3 revisited. We have $r \equiv uvuv^{-1}u^{-1}v^{-1}$, which has the form $(uvu)(vuv)^{-1}$. This then gives

$$\gamma(\frac{\partial r}{\partial u}) = 1 + uv - v \quad \text{and} \quad \gamma(\frac{\partial r}{\partial v}) = u - 1 - vu,$$

abusing notation to ignore the difference between u, v in $F(u, v)$ and the generating u, v in Br_3.

Homological 2-syzygies: In general we obtain a truncated chain complex:

$$\mathbb{Z}G^{(R)} \xrightarrow{d_2} \mathbb{Z}G^{(X)} \xrightarrow{d_1} \mathbb{Z}G \xrightarrow{d_0} \mathbb{Z} \to 0,$$

with d_2 given by the Jacobian matrix of the presentation, and d_1 sending generator e^1_x to $1 - x$, so $Im\, d_1$ is the augmentation ideal of $\mathbb{Z}G$.

Definition: A *homological 2-syzygy* is an element in $Ker\, d_2$..

A homological 2-syzygy is thus an element to be killed when building the third level of a resolution of G. We will shortly examine the generalisation of the above theory to the profinite case. We have all the ingredients from our previous work.

What are the links between homotopical and homological syzygies? In the discrete case, Brown and Huebschmann, [29], show they are isomorphic, as

$Ker\, d_2$ is isomorphic to the module of identities, but this result does not completely generalise to the profinite/pro-$\mathcal{C}$ setting.

Homological Syzygies for the braid group presentations:

Br_3: We have all the calculation for Br_3. The key part of the complex is the Jacobian matrix as that determines d_2:

$$d_2 = \begin{pmatrix} 1 + uv - v & u - 1 - vu \end{pmatrix}.$$

This has trivial kernel, but that comes most easily from the identification with homotopical syzygies.

Br_4: The presentation given earlier, page 94, yields a truncated chain complex with d_2

$$\mathbb{Z}G^{(r_u,r_v,r_w)} \xrightarrow{d_2} \mathbb{Z}G^{(u,v,w)}$$

with

$$d_2 = \begin{pmatrix} 0 & 1 + vw - w & v - 1 - wv \\ 1 - w & 0 & u - 1 \\ 1 + uv - v & u - 1 - vu & 0 \end{pmatrix}$$

and Loday, [107], has calculated that for the permutohedral 2-syzygy, s, one gets another term of the resolution, $\mathbb{Z}G^{(s)}$, and a $d_3 : \mathbb{Z}G^{(s)} \to \mathbb{Z}G^{(r_u,r_v,r_w)}$ given by

$$d_3 = \begin{pmatrix} 1 + vu - u - wuv & v - vwu - 1 - uv - vuwv & 1 + vw - w - uvw \end{pmatrix}.$$

6.3 Profinite crossed complexes.

Accurate encoding of homotopy types is tricky. Chain complexes, even of G-modules, can only record certain Abelian information. Simplicial groups, at the opposite extreme, can encode all connected homotopy types, but at the expense of such a large repetition of the essential information that makes calculation, at best, tedious and, at worst, virtually impossible. Complete information on truncated homotopy types can be stored in the cat^n-groups of Loday, [106]. We will look at these in chapter **??**. An intermediate model due to Blakers and Whitehead, [164], is that of a crossed complex. The algebraic and homotopy theoretic aspects of the theory of crossed complexes have been developed by Brown and Higgins, (cf. [26, 27], etc., in the bibliography and the forthcoming monograph by Brown, Higgins and Sivera, [28]) and by Baues, [9–11], see also chapter **??** of this monograph.

As we mentioned in the introduction, the use of universal covers of a $K(G, 1)$ in the study of the group G cannot immediately be adapted to the case of a pro$-\mathcal{C}$ group G, because the topological structure of G contains essential information on G. However the action of G on $K(G, 1)$ is very useful, so that a pro$-\mathcal{C}$ analogue would be a valuable tool to have. Although we have no $K(G, 1)$ available, we can build a pro$-\mathcal{C}$ homotopy type which will be an adequate replacement for it, at least in low dimensions, namely we can build a pro$-\mathcal{C}$ crossed complex.

6.3.1 Profinite crossed complexes: the Definition.

A *pro$-\mathcal{C}$ crossed complex*, which will be denoted C, consists of a sequence of pro$-\mathcal{C}$ groups and continuous morphisms

$$\mathsf{C} : \ldots \to C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \ldots \to C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0$$

satisfying the following:
CC1) $\delta_1 : C_1 \to C_0$ is a pro$-\mathcal{C}$ crossed module;
CC2) each C_n, $(n > 1)$, is a pseudocompact left $\hat{\mathbb{Z}}_{\mathcal{C}}[\![C_0/\delta_1 C_1]\!]$-module and each δ_n, $(n > 1)$ is a continuous morphism of pseudocompact left $\hat{\mathbb{Z}}_{\mathcal{C}}[\![C_0/\delta_1 C_1]\!]$-modules, (for $n = 2$, this means that δ_2 commutes with the action of C_0 and that $\delta_2(C_2) \subset C_1$ must be a pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![C_0/\delta_1 C_1]\!]$-module);
CC3) $\delta\delta = 0$.

The notion of a (continuous) morphism of pro$-\mathcal{C}$ crossed complexes is clear. It is a graded collection of morphisms preserving the various structure. We thus get a category, $Pro{-}\mathcal{C}.Crs$.

As we have that a crossed complex is a particular type of chain complex (of non-Abelian groups near the bottom), it is natural to define its homology groups in the obvious way.

Definition: If C is a (pro-$\mathcal{C}$) crossed complex, its n^{th} *homology group* is

$$H_n(\mathsf{C}) = \frac{Ker\,\delta_n}{Im\,\delta_{n+1}}.$$

These homology groups are, of course, functors from $Pro{-}\mathcal{C}.Crs$ to the category of Abelian (pro-$\mathcal{C}$) groups.

Definition: A morphism $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$ is called a *weak equivalence* if it induces isomorphisms on all homology groups.

There are good reasons for considering the homology groups of a crossed complex as being its homotopy groups as well as if the crossed complex comes from a simplicial group then the homotopy groups of the simplicial group are the same as the homology groups of the given crossed complex. We will examine this in detail later.

6.3.2 Example: pro-$\mathcal{C}$ crossed resolutions

Definition: A *pro$-\mathcal{C}$ crossed resolution* of a pro$-\mathcal{C}$ group G is a pro-$\mathcal{C}$ crossed complex, C, such that for each $n > 1$, $Im\,\delta_n = Ker\,\delta_{n-1}$ and there is a continuous isomorphism, $C_0/\delta_1 C_1 \cong G$.

A pro$-\mathcal{C}$ crossed resolution can be constructed from a pro$-\mathcal{C}$ presentation $\mathcal{P} = (X : R)$ as follows:

Let $C_{\mathcal{C}}(P) \to F_{\mathcal{C}}(X)$ be the free pro$-\mathcal{C}$ crossed module associated with $\mathcal{P}$. We set $C_1 = C_{\mathcal{C}}(\mathcal{P})$, $C_0 = F_{\mathcal{C}}(X)$, $\delta_1 = \partial$. Let $\kappa(\mathcal{P}) = Ker(\partial : C_{\mathcal{C}}(\mathcal{P}) \to$

$F_{\mathcal{C}}(X)$). This is the identity module of the presentation and is a pseudocompact left $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-module. As the category $Pc_{\mathcal{C}}.G\text{-}Mod$ has enough projectives, we can form a free pseudocompact resolution $\mathbb{P}$ of $\kappa(\mathcal{P})$. To obtain a pro$-\mathcal{C}$ crossed resolution of G, we join $\mathbb{P}$ to the crossed module by setting $C_n = P_{n-2}$ for $n > 2$, $\delta_n = d_{n-2}$ for $n > 2$ and the composite from P_0 to $C_{\mathcal{C}}(P)$ for $n = 2$.

We will return to this crossed resolution later.

6.3.3 The standard crossed pro-$\mathcal{C}$ resolution

We next look at a particular case of the above, namely the standard pro$-\mathcal{C}$ crossed resolution of G. In this, which we will denote by $\mathsf{C}G$, we have

(i) $C_0G =$ the free pro$-\mathcal{C}$ group on the underlying space of G. The element corresponding to $u \in G$ will be denoted by $[u]$.

(ii) C_1G is the free pro$-\mathcal{C}$ crossed module over C_0G on generators, written $[u, v]$, considered as elements of the space $G \times G$, in which the map δ_1 is defined on generators by

$$\delta[u, v] = [uv]^{-1}[u][v].$$

(iii) For $n > 2$, C_nG is the free pseudocompact left $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-module on the space G^{n+1}, but in which one has equated to zero any generator $[u_1, \ldots, u_{n+1}]$ in which some u_i is the identity element of G.

If $n > 3$, $\delta : C_nG \to C_{n-1}G$ is given by the usual formula

$$\begin{aligned}\delta[u_1, \ldots, u_{n+1}] = {}& {}^{[u_1]}[u_2, \ldots, u_{n+1}] \\ & + \sum_{i=1}^{n} (-1)^i [u_1, \ldots, u_i u_{i+1}, \ldots, u_{n+1}] + (-1)^{n+1}[u_1, \ldots, u_n].\end{aligned}$$

For $n = 2$, $\delta : C_2G \to C_1G$ is given by

$$\delta[u, v, w] = {}^{[u]}[v, w].[u, v]^{-1}.[uv, w]^{-1}[u, vw].$$

We will see later that this is the crossed analogue of the inhomogeneous bar resolution. A groupoid version can be found, for the abstract setting, in Brown-Higgins, [25], and the abstract group version in Huebschmann, [86]. In the first of these two references, it is pointed out that $\mathsf{C}G$, as constructed (but for abstract groups), is isomorphic to the crossed complex, $\underline{\pi}(BG)$, of the classifying space of G considered with its skeletal filtration. For any filtered space $\underline{X} = (X_n)_{n \in \mathbb{N}}$, the fundamental crossed complex $\underline{\pi}(\underline{X})$ is defined to have

$$\underline{\pi}(\underline{X})_n = (\pi_n(X_n, X_{n-1}, a))_{a \in X_0}$$

with $\underline{\pi}(\underline{X})_1$, the fundamental groupoid $\Pi_1 X_1 X_0$, and $\underline{\pi}(\underline{X})_2$, the family, $(\pi_2(X_2, X_1, a))_{a \in X_0}$, cf. 4.1.1. Thus this pro$-\mathcal{C}$ analogue will be of some potential importance in our overall plan of building algebraic pro$-\mathcal{C}$ analogues of the topological spaces that are so useful in combinatorial and cohomological group theory.

6.3.4 G-augmented pro$-\mathcal{C}$ crossed complexes.

Pro$-\mathcal{C}$ crossed resolutions of G are examples of G-augmented pro$-\mathcal{C}$ crossed complexes. A *G-augmented pro$-\mathcal{C}$ crossed complex* consists of a pair (C, ϕ) where C is a pro$-\mathcal{C}$ crossed complex and where $\phi : C_0 \to G$ is a continuous group homomorphism satisfying

(i) $\phi\delta_1$ is the trivial homomorphism;

(ii) $Ker\,\phi$ acts trivially on C_i for $i \geq 2$ and also on C_1^{Ab}.

A *morphism*

$$(\alpha, Id_G) : (\mathsf{C}, \phi) \to (\mathsf{C}', \phi')$$

of G-augmented pro$-\mathcal{C}$ crossed complexes consists of a morphism

$$\alpha : \mathsf{C} \to \mathsf{C}'$$

of pro$-\mathcal{C}$ crossed complexes such that $\phi'\alpha_0 = \phi$.

G-augmented pro$-\mathcal{C}$ crossed complexes form a category which we will denote by $Pro{-}\mathcal{C}.Crs_G$.

In the next section we will show how to construct G-augmented pro$-C$ crossed complexes from chain complexes of pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-modules.

6.4 From crossed complexes to chain complexes.

Notwithstanding the title of this section, we start by fulfilling our promise to construct a G-augmented pro$-\mathcal{C}$ crossed complex from a positive chain complex of pseudocompact left $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$ -modules. We will denote the category consisting of these latter objects by $Ch(Pc_{\mathcal{C}}.G - Mod)$.

6.4.1 From chain complexes to crossed complexes, ...

Proposition 42. *There is a functor*

$$\Delta_G : Ch(Pc_{\mathcal{C}}.G - Mod) \to Pro{-}\mathcal{C}.Crs_G$$

given by: if $\mathbb{M}$ is a chain complex of pseudocompact left $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-modules,

$$\mathbb{M} = \ldots \to M_n \to \ldots \to M_2 \to M_1 \to M_0$$

then $\Delta_G(\mathbb{M})$ is the pro$-\mathcal{C}$ crossed complex with

$$\Delta_G(\mathbb{M})_n = M_n \text{ if } n \geq 1$$
$$\Delta_G(\mathbb{M})_0 = M_0 \rtimes G$$

with differentials $\delta_n : \Delta_G(M)_n \to \Delta_G(M)_{n-1}$ given by $\delta_n = d_n$ if $n \geq 1$, and

$$\delta_1(m) = (1, d_1(m)) \in M \rtimes G \text{ for } m \in M_1.$$

The G-augmentation $M_0 \rtimes G \to G$ is given by the projection. ∎

The details of the verification are simple and so will be left out. The functor Δ_G is principally of interest because of the next result.

6.4.2 ... and back again.

Proposition 43. *The functor Δ_G has a left adjoint.*

Proof: We construct the left adjoint explicitly as follows:

Let $f. : (\mathsf{C}, \phi) \to \Delta_G(M.)$ be a morphism in $Pro{-}\mathcal{C}.Crs_G$, then we have the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_2 \arrow[r, "\delta_2"] \arrow[d, "f_2"] & C_1 \arrow[r, "\delta_1"] \arrow[d, "f_1"] & C_0 \arrow[r, "\phi"] \arrow[d, "f_0"] & G \arrow[d, "Id_G"] \\
\cdots \arrow[r] & M_2 \arrow[r, "\delta_2"] & M_1 \arrow[r, "\delta_1"] & M_0 \rtimes G \arrow[r, "pr_G"] & G
\end{tikzcd}$$

Since the right hand square commutes, f_0 is given by some formula

$$f_0(c) = (\phi(c), \partial(c)),$$

where $\partial : C_0 \to M_0$ is a ϕ-derivation. Thus $\partial = \tilde{f}_0 \partial_\phi$ for a unique continuous G-module morphism, $\tilde{f}_0 : D_\phi \to M_0$, and f_0 factors as

$$C_0 \xrightarrow{\bar{\phi}} D_\phi \rtimes G \xrightarrow{\tilde{f}_0 \rtimes G} M_0 \rtimes G,$$

where $\bar{\phi}(c) = (\phi(c), \partial_\phi(c))$.

The map $\partial_\phi \delta_1 : C_1 \to D_\phi$ is a continuous homomorphism since

$$\begin{aligned}
\partial_\phi \delta_1(c_1 c_2) &= \partial_\phi \partial_1(c_1) + \phi\partial_1(c_1)\partial_\phi\partial_1(c_2) \\
&= \partial_\phi\partial_1(c_1) + \partial_\phi\partial_1(c_2),
\end{aligned}$$

$\phi\partial_1$ being trivial (because (C, ϕ) is G-augmented).

Thus we obtain a map $d : C_1^{Ab} \to D_\phi$ given by $d(c[C,C]) = \partial_\phi\partial_1(c)$ for $c \in C_1$. The pro$-\mathcal{C}$ Abelian group C_1^{Ab} has a natural pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-module structure making d a continuous G-module morphism.

Similarly there is a unique continuous G-module morphism,

$$\tilde{f}_1 : C_1^{Ab} \to M_1,$$

satisfying

$$\tilde{f}_1(c[C,C]) = f_1(c).$$

Since for $c \in C_1$,

$$(1, d_1\tilde{f}_1(c)) = f_0(\delta_1 c) = (1, \tilde{f}_0\partial_\phi(\delta_1 c_1)),$$

we have that the diagram

$$\begin{tikzcd}
C_1^{Ab} \arrow[r, "\tilde{f}_1"] \arrow[d, "d"] & M_1 \arrow[d, "d_1"] \\
D_\phi \arrow[r, "\tilde{f}_0"] & M_0
\end{tikzcd}$$

commutes.

We also note that since $\delta_2 : C_2 \to C_1$ maps into $Ker\ \delta_1$, the composite

$$C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\text{can}} C_1^{Ab} \xrightarrow{d} D_\phi,$$

being given by $d(\delta_2(c)[C,C] = \partial_\phi \delta_1 \delta_2(c)$, is trivial and that $\tilde{f}_1 \delta_2(c[C,C]) = f_1 \delta_2(c) = d_2 f_2(c)$, thus we can define $\xi = \xi_G(\mathsf{C}, \phi)$ by

$$\begin{aligned} \xi_n &= C_n \text{ if } n \geq 2 \\ \xi_1 &= C_1^{Ab}, \\ \xi_0 &= D_\phi, \end{aligned}$$

the differentials being as constructed. We note that as $Ker\ \phi$ acts trivially on all C_n for $n \geq 2$, all the C_n have pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-module structures.

That ξ_G gives a functor

$$Pro{-}\mathcal{C}.Crs \to Ch(Pc_{\mathcal{C}}.G - Mod)$$

is now easy to check using the uniqueness clauses in the universal properties of D_ϕ and Abelianisation. Again uniqueness guarantees that the process "f goes to $\tilde{f}$" gives a natural isomorphism

$$Ch(Pc_{\mathcal{C}}.G - Mod)(\xi_G(\mathsf{C}, \phi), \mathbb{M}) \cong Pro{-}\mathcal{C}.Crs_G((\mathsf{C}, \phi), \Delta_G(\mathbb{M}))$$

as required. ■

It is relatively easy using the theory of restricted and induced pro$-\mathcal{C}$ crossed modules to extend the above natural isomorphism to handle morphisms of pro$-\mathcal{C}$ crossed complexes over different groups. This is left as an exercise for the diligent reader.

6.4.3 Pro$-\mathcal{C}$ crossed resolutions and chain resolutions

One of our motivations for introducing pro$-\mathcal{C}$ crossed complexes was that they enable us to model more of the sort of information encoded in a $K(G,1)$ than does the usual standard algebraic model, e.g. a chain complex such as the bar resolution. It is therefore of importance to see how this information changes under the functor ξ.

We start with a pro$-\mathcal{C}$ crossed resolution determined in low dimensions by a profinite presentation $\mathcal{P} = (X : R)$ of a pro$-\mathcal{C}$ group, G. Thus in this case $C_0 = F_{\mathcal{C}}(X)$ with $\phi : F_{\mathcal{C}}(X) \to G$, the 'usual' epimorphism, and $C_1 \to C_0$ is $C_{\mathcal{C}}(\mathcal{P}) \to F_{\mathcal{C}}(X)$, the free pro$-\mathcal{C}$ crossed module on $R \to F_{\mathcal{C}}(X)$. Using the results in subsection 4.2.2, we obtain $C_1^{Ab} \cong \hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]^{(R)}$, the free pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$-module on R. This maps down onto $N(R)^{Ab}$, the pro$-\mathcal{C}$ Abelianisation of the closed normal closure of R in $F_{\mathcal{C}}(X)$ via a map

$$\partial_* : \hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]^{(R)} \to N(R)^{Ab},$$

given by $\partial_*(e_r) = r[N(R), N(R)]$, where e_r is the generator of $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$ corresponding to $r \in R$.

There is also a short exact sequence

$$1 \to N(R) \xrightarrow{i} F_{\mathcal{C}}(X) \xrightarrow{\phi} G \to 1$$

and hence by Proposition 40, a short exact sequence

$$0 \to N(R)^{Ab} \xrightarrow{\tilde{i}} \hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!] \hat{\otimes}_F \hat{I}_{\mathcal{C}}(F) \xrightarrow{\tilde{\phi}} \hat{I}_{\mathcal{C}}(G) \to 0$$

(where we have written $F = F_{\mathcal{C}}(X)$).

By the Corollary to Proposition 39, we have

$$\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!] \hat{\otimes}_F \hat{I}_{\mathcal{C}}(F) \cong \hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]^{(X)}.$$

The required map $C_1^{Ab} \to D_\phi$ is the composite

$$\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]^{(R)} \xrightarrow{\partial_*} N(R)^{Ab} \xrightarrow{\tilde{i}} \hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]^{(X)}.$$

We have given an explicit description of ∂_* above, so to complete the description of d, it remains to describe $\tilde{i}$, but $\tilde{i}$ satisfies $\tilde{i}\delta = \partial_\phi i$, where $\delta : N(R) \to N(R)^{Ab}$, so $\tilde{i}(r[N(R), N(R)]) = d_\phi(r)$. Thus if r is an algebraic relator, i.e., if it is in the image of the dense subgroup generated by the elements of R, then $\partial(e_r)$ can be written as a *finite* sum of the form $\sum_x a_x e_x$ and the elements $a_x \in \hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$ are the analogues of the images of the Fox derivatives in the abstract case. If r is non-algebraic and has been specified as a limit of algebraic elements, the description of $d_\phi(r)$ is correspondingly more complicated.

This operator can best be viewed as the pro$-\mathcal{C}$ analogue of the Alexander matrix of a presentation of an abstract group. Clearly further study of this operator will depend on studying transformations between free pseudocompact modules over completed group rings, a subject we as yet know little about.

The above difficulty does not occur if X is finite (and hence discrete). Gildenhus and Mackay [71] in the proof of their proposition 2.2 (p. 466) constructed Fox derivatives in the context of pro$-p$ groups. They pointed out that their proof of the existence of these could be extended to the general case if it could be proved that if $X = \{x_1, \ldots, x_n\}$ and $F = F_{\mathcal{C}}(X)$, then $\hat{I}_{\mathcal{C}}(F)$ is freely generated by the set $(x_i - 1)$. This, of course, is corollary 12, hence continuous Fox derivatives exist in general and their results 2.2 and 3.3 are thus valid in more generality. We will return to this in more detail later on.

The rest of the pro$-\mathcal{C}$ crossed resolution does not change and so, on replacing $\hat{I}_{\mathcal{C}}(G)$ by $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!] \to \hat{\mathbb{Z}}_{\mathcal{C}}$, we obtain a free pseudocompact $\hat{\mathbb{Z}}[\![G]\!]$-resolution of the trivial module $\hat{\mathbb{Z}}_{\mathcal{C}}$,

$$\ldots \to \hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]^{(R)} \xrightarrow{d} \hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]^{(X)} \to \hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!] \to \hat{\mathbb{Z}}_{\mathcal{C}}$$

built up from the presentation. This is the pro$-\mathcal{C}$ analogue of the complex of chains on the universal cover, $\widetilde{K(G,1)}$, where $K(G,1)$ is constructed starting from a presentation $\mathcal{P}$.

6.4.4 Standard pro$-\mathcal{C}$ crossed resolutions and bar resolutions

We next turn to the special case of the standard pro$-\mathcal{C}$ crossed resolution of G discussed briefly in 6.3.3. Of course this is a special case of the previous one, but it pays to examine it in detail.

Clearly in $\xi = \xi(\mathsf{C}G, \phi)$, we have:
ξ_0 = the free pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$ -module on the underlying space of G, individual generators being written $[u]$, for $u \in G$;
ξ_1 = the free pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$ -module on $G \times G$, generators being written $[u, v]$;
$\xi_n = C_n G$, the free pseudocompact $\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]$ -module on G^{n+1}, etc.

The map $d_2 : \xi_2 \to \xi_1$ induced from δ_2 is given by

$$d_2[u, v, w] = u[v, w] - [u, v] - [uv, w] + [u, vw],$$

and the map $d_1 : \xi_1 \to \xi_0$ by

$$\begin{aligned} d_1([u, v]) &= d_\phi([uv]^{-1}[u][v]) \\ &= v^{-1}u^{-1}(-[uv] + [u] + u[v]), \end{aligned}$$

a unit times the usual bar resolution formula. Thus, as claimed earlier, the standard pro$-\mathcal{C}$ crossed resolution is the crossed analogue of the bar resolution in the pro$-\mathcal{C}$ context.

6.4.5 The pseudocompact module of identities.

Brown and Huebschmann, [29], p. 168, prove that for an abstract group G with presentation $\mathcal{P}$, the module of identities for $\mathcal{P}$ is naturally isomorphic to the second homology group, $H_2(\tilde{K}(\mathcal{P}))$, of the universal cover of $K(\mathcal{P})$, the 2-complex of the presentation.

Given a profinite presentation $\mathcal{P} = (X : R)$ of a pro$-\mathcal{C}$ group G, the analogue of $K(\mathcal{P})$, we have argued above, is the pro$-\mathcal{C}$ crossed module $C_{\mathcal{C}}(\mathcal{P}) \xrightarrow{d} F_{\mathcal{C}}(X)$ and the chains on the universal cover of $K(\mathcal{P})$ will be given by ξ_G of this, i.e., by the chain complex

$$\hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]^{(R)} \xrightarrow{d} \hat{\mathbb{Z}}_{\mathcal{C}}[\![G]\!]^{(X)},$$

however H_2 of this complex does not seem necessarily to be the same as the module of identities,

$$\kappa_{\mathcal{C}}(\mathcal{P}) = Ker(\partial : C_{\mathcal{C}}(\mathcal{P}) \to F_{\mathcal{C}}(X)).$$

In general there will be a short exact sequence

$$0 \to \kappa_{\mathcal{C}}(\mathcal{P}) \cap [C_{\mathcal{C}}(\mathcal{P}), C_{\mathcal{C}}(\mathcal{P})] \to \kappa_{\mathcal{C}}(\mathcal{P}) \to H_2(\xi(C_{\mathcal{C}}(\mathcal{P})) \to 0$$

and it seems likely that the term $\kappa(\mathcal{P}) \cap [C_{\mathcal{C}}(\mathcal{P}), C_{\mathcal{C}}(\mathcal{P})]$ is the second pro$-\mathcal{C}$ homology group of $N(R)$. As mentioned before $N(R)$ may not be a free pro$-\mathcal{C}$ group, and we know little or nothing about its pro$-\mathcal{C}$ homology.

This short exact sequence does however yield the analogue of the Brown-Huebschmann result for those presentations with $N(R)$, a free pre-$\mathcal{C}$ group as then Proposition 23, page 74 applies. We thus get

Proposition 44. *If $\mathcal{P} = (X : R)$ is a free pro-$\mathcal{C}$ presentation of G, then there is an isomorphism*

$$\kappa_{\mathcal{C}} \xrightarrow{\cong} H_2(\xi(C_{\mathcal{C}}(\mathcal{P})) = Ker(d : \hat{\mathbb{Z}}_p[\![G]\!]^R \to \hat{\mathbb{Z}}_p[\![G]\!]^X).$$

■

In particular, this is true if $\mathcal{C} = $ *p-groups* or if G is finite discrete.

6.4.6 Example:

By way of illustration, we use the above Proposition to calculate the identity module $\kappa_{\mathcal{C}}(\mathcal{P})$ when $\mathcal{P}$ is the presentation $(x_1, x_2 : x_1^p[x_2, x_1]^p)$ of a pro$-p$ group, G. We will calculate $d(e_r)$, e_r being the generator of $\hat{\mathbb{Z}}_p[\![G]\!]^1$ corresponding to $r = x_1^p[x_2, x^1]^p$, using Fox derivatives.

Gildenhuys, [68], has proved that G has infinite cohomological dimension, so that Lyndon's result on one relator groups has no pro$-p$ analogue. In the same paper, [68], Gildenhuys uses essentially the methods outlined below to analyse various other related one-relator pro$-p$ groups.

We know

$$d(e_r) = d_\phi(r) = \phi\left(\frac{\partial r}{\partial x_1}\right) e_1 + \phi\left(\frac{\partial r}{\partial x_2}\right) e_2,$$

thus αe_r will be in $\kappa(\mathcal{P})$ if and only if α annihilates both $\phi\left(\frac{\partial r}{\partial x_1}\right)$ and $\phi\left(\frac{\partial r}{\partial x_2}\right)$. Now

$$\phi\left(\frac{\partial r}{\partial x_1}\right) = \sum_0^{p-1} \phi(x_1)^k + (\phi(x_1)^p \phi(x_2^{-1}) \phi(x_1)^{-p}(1 - \phi(x_2)) \sum_0^{p-1} \phi(x_1)^k$$

and

$$\phi\left(\frac{\partial r}{\partial x_2}\right) = \phi(x_1)^p \phi(x_2^{-1})(\phi(x_1)^{p-1}).$$

We next repeat part of Gildenhuy's argument.

Clearly if we write $g_i = \phi(x_i)$, $i = 1, 2$, then $g_1^p = [g_1^p, g_2]$. If we denote the k^{th} term of the lower central series of G by G_k, one sees that g_1^p is in G_{k+1}

if it is in G_k, hence $g_1^p \in G_k$, but this intersection is $\{1\}$ since pro$-p$ groups are residually nilpotent, hence $g_1^p = 1$ and g_1 is of order p or 1. However, $x_1 \notin N(r)$, since $N(r) \subset F^p[F,F]$, where $F = F_p(X)$, so g_1 has order p.

Thus we have

$$\phi\left(\frac{\partial r}{\partial x_1}\right) = (2 - g_2^{-1})(1 + g_1 + \ldots + g_1^{p-1}),$$

$$\phi\left(\frac{\partial r}{\partial x_2}\right) = 0.$$

Since $(1-g_1)$ annihilates $\phi\left(\frac{\partial r}{\partial x_1}\right)$, we have that $\kappa_{\mathcal{C}}(\mathcal{P})$ contains the submodule generated by $(1 - g_1)e_r$.

To prove that $\kappa_{\mathcal{C}}(\mathcal{P})$ is in fact this submodule, we need to check that $2-g_2^{-1}$ is not a zero divisor. First we noted earlier, in section 1.10, that Lazard [102] proves that if $F_p(n)$ is the free pro$-p$ group on n-symbols, $\{x_1, \ldots, x_n\}$, then there is an isomorphism between $\hat{\mathbb{Z}}_p[\![F_p(n)]\!]$ and $A(n)$, the algebra of non-commutative formal series on n indeterminates, $t_1, \ldots, t_n$, with coefficients in $\hat{\mathbb{Z}}_p$, this isomorphism being given by x_i goes to $1 + t_i$ for each i.

There is an epimorphism

$$\rho : \hat{\mathbb{Z}}_p[\![F(2)]\!] \to \hat{\mathbb{Z}}_p[\![G]\!].$$

Suppose $a(2 - g^{-1}) = 0$ in $\hat{\mathbb{Z}}_p[\![G]\!]$, then we have there is a corresponding $\bar{a}(2 - x^{-1}) \in Ker\,\rho$. However in the isomorphism with $\hat{\mathbb{Z}}_p[\![t_1, t_2]\!] = A(2)$, $2 - x_2^{-1}$ goes to $(1 + 2t_2)(1 + t_2)^{-1}$, but $(1 + 2t_2)^{-1}$ exists, so $2 - x_2^{-1}$ is invertible, hence $\bar{a} \in Ker\,\rho$ and $a = 0$.

Thus $\kappa_{\mathcal{C}}(\mathcal{P}) \cong \{s(1 - g_1)e_r : s \in \hat{\mathbb{Z}}_p[\![G]\!]\}$.

Of course this accords well with Gildenhuys proof in [68] that G has infinite cohomological dimension, since this module has a periodic resolution exactly as does the corresponding resolution for C_p,

$$\ldots \to \hat{\mathbb{Z}}_p[\![G]\!] \xrightarrow{1-g_1} \hat{\mathbb{Z}}_p[\![G]\!] \xrightarrow{N} \hat{\mathbb{Z}}_p[\![G]\!] \xrightarrow{1-g_1} \ldots \quad ,$$

where N is the map multiplying by $1 + g_1 + \ldots + g_1^{p-1}$.

7

Pseudocompact coefficients

7.1 Introduction.

Often the coefficients for the cohomology of a profinite group, G, have been taken to be in a discrete G-module. While this yields a rich and useful theory, containing within it many analogues of results from the cohomology theory of abstract groups, it only tells half the story. One can glimpse other facets of the cohomology from time to time, for instance when Serre, [152] p.10, notes that if A is finite then $H^2(G, A)$ classifies extensions of G by A. Later Brumer, in [34], very nearly takes the plunge as he considers a homology theory which is pseudocompact in the values it takes and in his thesis, Wambsganß-Türk, [163], develops universal central extensions for a perfect profinite group, G, provided the Brumer homology $H_2(G; \hat{\mathbb{Z}})$ is finite and hence discrete.

In this chapter our aim is to indicate how easy and useful it is to be "free of one's inhibitions" and to define $H^n(G, A)$ when A is a pseudocompact $\hat{\mathbb{Z}}[\![G]\!]$-module, where $\hat{\mathbb{Z}}$ is the profinite completion of the integers and, as before, $\hat{\mathbb{Z}}[\![G]\!]$ is the pseudocompact completed group algebra of G. The generalisation of a large amount of theory from the abstract to the profinite case then turns out to be a relatively painless process, in fact in most cases a proof obtained by substituting "profinite group" for "group", "continuous morphism" for "morphism" etc. in a proof of the "classical" result works. We should however mention the proviso that the old proof be in a suitable form, for instance, if the construction of a homomorphism is given in terms of its values on generators, one must check that in the profinite case the generators provide a space which topologically generates the required group, that the definition of the map on the topological generators is continuous (which rarely causes problems) and that the analogues of the classical relations generate all the necessary relations in the profinite case. The conditions can often be quite tricky to check and in such cases it is sometimes simpler to use a different form of proof, more based on universal properties perhaps, which avoids these difficulties. To illustrate why we wish to develop this theory, we mention two applications, both of which will be handled in detail later on in this chapter.

(i) *Abelian extensions:* In the classical abstract theory, an Abelian extension

$$e : 1 \xrightarrow{\kappa} A \xrightarrow{\pi} G \to Q \to 1$$

of a group Q, by an Abelian group, A, determines a Q-module structure on A and a class in the cohomology group $H^2(Q, A)$. If Q is a profinite group and we wish the extension to be a profinite extension, i.e. for G and A to be profinite, with κ and π continuous, then we expect and find A to be a topological Q-module. By a result of Gildenhuys and Mackay [71], if A is a profinite Abelian group and a topological Q-module, then it is a pseudocompact $\hat{\mathbb{Z}}[\![G]\!]$-module. Using the well known result that continuous epimorphisms of profinite groups have continuous sections (cf. Schatz, [150], or Serre [152]), one easily checks that each such extension determines and is determined by a class in $H^2(Q, A)$, the continuous cohomology of Q with coefficients in A. Likewise one can discuss the theory of profinite central and stem extensions and their applications.

(ii) *Profinite homotopy types.* Classically the homotopy 2-type of a CW-complex, X, is determined by $\pi_1(X)$, $\pi_2(X)$ considered as a $\pi_1(X)$-module, and a class $k \in H^3(\pi_1(X), \pi_2(X))$. This class can be represented by a crossed module, (C, G, ∂), or a crossed 2-fold extension

$$0 \to \pi_2(X) \to C \xrightarrow{\partial} G \to \pi_1(X) \to 1.$$

(If X_1 denotes the 1-skeleton of X, i.e. the union of the 1- and 0-dimensional cells in the cellular decomposition of X, then this crossed extension can be written

$$\pi_2(X) \to \pi_2(X, X_1) \to \pi_1(X_1) \to \pi_1(X)$$

i.e. by part of the exact sequence of the pair (X, X_1).) These links between classes of crossed extensions, classes in the third cohomology group and 2-types have their profinite analogue, but this needs profinite coefficients. Later we will also look at the profinite analogues of Huebschmann's results, [86], which identify $H^n(Q, A)$ in terms of equivalence classes of crossed "long" extensions, again it is clear that this is only feasible if A is pseudocompact.

Finally we should mention the policy we have adopted when presenting and proving results. Many results in the theory of covering and representation groups, treated for instance in Beyl and Tappe [12], clearly generalise with little or no problem. Rather than giving a list of these, we have limited ourselves to showing how the minor modification necessary for a profinite version of classical universal central extension theory can be made. We hope that the message that such generalisations are quite easy will enable workers with profinite groups to *use* these ideas without need for the inclusion here of explicit statements and proofs. This is just one example of several; we have in each case chosen a particular aspect of the area and have given explanation and statements of some of the key theorems. Many of these theorems can safely be left to the reader to prove in detail and in that case, accompanying the statement, there will be found a precise reference to a proof of the analogous abstract group result that will generalise “painlessly”. In the rare cases where no such proof has been found, we have given the details either of a complete proof, if necessary, or of a bridging section to adapt some existing “abstract group” proof.

Remark: With $\mathcal{C}$ as in the previous chapters, it seems possible that a pro$-p$ version of much of the material here can be produced. There is a fair amount known on cohomology modulo a variety, notably in the work of Lue, [110, 111] and Stammbach, [155, 156], however, as a development of cohomology modulo the variety pro$-\mathcal{C}$ has not appeared in the literature, that idea needs further study before it can be evaluated as to its usefulness, etc. We thus have limited ourselves in this, and several of the following sections, to the profinite case only.

Notational Implication: The above means that $\hat{\mathbb{Z}}$ replaces $\hat{\mathbb{Z}}_{\mathcal{C}}$, $Prof.CMod$ replaces $Pro{-}\mathcal{C}.CMod$, etc.

7.2 Continuous cohomology of profinite groups.

7.2.1 The definition

The definition of $H^i(G, A)$ given by Serre, [152], clearly can be applied when A is not a *discrete* topological G-module. We repeat the definition (suitably adapted) for convenience.

Let G be a profinite group and A, a pseudocompact $\hat{\mathbb{Z}}[\![G]\!]$-module. We denote by $C^n(G, A)$, the Abelian group of continuous maps from G^n to A. The coboundary

$$d : C^n(G, A) \to C^{n+1}(G, A)$$

is given by the usual formula:

$$\begin{aligned} df(g_1, \ldots, g_{n+1}) = g_1 f(g_2, \ldots, g_{n+1}) \\ + \sum_{i=1}^{i=n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \\ + (-1)^{n+1} f(g_1, \ldots, g_n). \end{aligned}$$

Thus one obtains a complex $C^{\cdot}(G, A)$, whose cohomology groups $H^n(G, A)$ will be called the *cohomology groups of G with coefficients in A.*

7.2.2 Interpretation in low dimensions

In low dimensions, the interpretation of these groups is simple and follows the classical abstract case:

$$H^0(G, A) = A^G,$$

that is, the fixed submodule of A on which G acts trivially. Next

$$H^1(G, A) = Der(G, A)/Pder(G, A),$$

that is, it is the quotient of the module of continuous derivations from G to A by those of the form

$$d_a : G \to A$$

$$d_a(g) = (g - 1)a$$

for some fixed $a \in A$.

The group, $H^2(G, A)$, can be described as the group of classes of continuous factor sets from G to A and also as the group $Opext(G, A)$ of Abelian extensions of G by A, modulo congruence of extensions. The proof of this last fact can be manufactured from any of the standard proofs in the abstract case using the existence of continuous sections, (Corollary 1, page 12).

At the risk of over emphasizing the advantages of pseudocompact modules, let us again point out that in an extension by profinite groups of G by some topological module, A, since A appears as a kernel of a continuous homomorphism from E, say, to G it must be itself profinite, hence pseudocompact, and it will be discrete only if finite. Thus arbitrary discrete modules are not appropriate for handling extensions.

7.2.3 The bar resolution

The profinite analogue of the bar resolution $\mathsf{B}G$ of a profinite group G was mentioned in the last chapter in conjunction with the standard profinite crossed resolution $\mathsf{C}G$. Here we note that $\mathsf{B}G$ yields another description of the complex, $C^{\cdot}(G, M)$, above, namely

$$C(G, M) = \mathrm{Hom}(\mathsf{B}G, M)$$

for M in $Pc.G{-}Mod$.

This complex $\mathsf{B}G$ is made up of free pseudocompact $\hat{\mathbb{Z}}[\![G]\!]$-modules and yields a pseudocompact projective resolution of $\hat{\mathbb{Z}}$ with the trivial G-action, as in the abstract case (cf. MacLane, [112]). This thus leads to an isomorphism,

$$\mathrm{Ext}^n(\hat{\mathbb{Z}}, M) \cong H^n(G, M),$$

where the Ext^n is taken within the category $Pc.G{-}Mod$. This readily implies the existence of long exact sequences corresponding to short exact sequences of coefficients and by using the defining short exact sequence

$$0 \to \hat{I}(G) \to \hat{\mathbb{Z}}[\![G]\!] \to \hat{\mathbb{Z}} \to 0$$

of the completed augmentation ideal in the first variable of a Hom-Ext long exact sequence, one gets a second natural isomorphism

$$H^n(G, M) \cong Ext^{n-1}(\hat{I}(G), M)$$

in the standard way.

7.2.4 Induced and Coinduced Modules

As we saw in section 1.11, the usual results on induced and coinduced modules relative to an inclusion $H \hookrightarrow G$ between profinite groups go through with the obvious changes, although some care is needed when checking $\mathrm{Coin}_H^G M$ is pseudocompact. In particular, there is a profinite analogue of Shapiro's lemma: if $H \subseteq G$ and M is an H-module, then there is an isomorphism, natural in M,

$$H^*(H, M) \cong H^*(G, Coin_H^G M).$$

The proof, say in Brown, [21], p.73, adapts easily.

7.2.5 Limits of finite coefficients.

One natural question is the exact nature of the link between this cohomology with the standard profinite one. We know that if A is finite and discrete, then they are identical (since even the definitions coincide!). If we write a general pseudocompact module, A, as the limit of its finite length discrete quotient modules, $A = Lim_\alpha A_\alpha$, what is the relationship between $H^n(G, A)$ and the various $H^n(G, A_\alpha)$? There is a natural map

$$H^n(G, A) \to Lim_\alpha H^n(G, A_\alpha);$$

is it an isomorphism?

To answer these questions, we first look at what sensible topology if any one can put on the $C^n(G, A)$. (The work of Calvin Moore, [120–122], suggests that consideration of topologies on chain and cochain complexes may be of great importance in this area.) Suppose that A is finite and discrete, then $C^n(G, A)$, with the topology of pointwise convergence, is a compact Abelian group and the coboundary maps are continuous. As a consequence, $H^n(G, A)$ is also a compact Abelian group if we give it the quotient topology of the subspace topology of $Z^n(G, A)$. We only need this in order to note that if $A = Lim_\alpha A_\alpha$, then the derived functors $Lim^{(i)}$ of Lim will be zero when evaluated on $H^n(G, A_\alpha)$, since these latter groups are compact and the linking morphisms between them are continuous.

Proposition 45. *If $A = LimA_\alpha$ in the category $Pc.G - Mod$, with the A_α discrete, then the natural morphism*

$$H^n(G, A) \to Lim_\alpha H^n(G, A_\alpha)$$

is an isomorphism for all n.

Proof: Firstly we recall a result that is to be found in Jensen, [94], p.35. Let $A = \{A_\alpha\}$ be any projective system of R-modules and let M be an R-module. Then there are two spectral sequences

$$E_2^{(1)p,q} = Lim^{(p)}\mathrm{Ext}_R^q(M, A_\alpha)$$

and

$$E_2^{(2)p,q} = \mathrm{Ext}_R^p(M, Lim^{(q)}A_\alpha)$$

with the same limits.

We must adapt this before we can use it as, although $Pc.G-Mod$ is an Abelian category with very nice properties, it does not seem to be of the form $R-Mod$. Jensen's result is based on the construction of Roos of a (co)-complex $\Pi^\bullet A$, whose cohomology groups are the derived functors of Lim. Taking a projective resolution P of M, one has a bicomplex $\mathrm{Hom}(\mathsf{P}, \Pi^\bullet \underline{A})$. The two spectral sequences of Jensen's result are those associated with this bicomplex. Each step of this proof goes across to $Pc.G-Mod$ (as this category has exact products, which are preserved by the forgetful functor to G-*Mod*). We thus obtain for $M = \hat{\mathbb{Z}}$, $\underline{A} = \{A_\alpha\}$

$$E_2^{(1)p,q} = \begin{cases} Lim\ H^q(G, A_\alpha) & \text{if } p = 0 \\ 0 & \text{if } p \neq 0, \end{cases}$$

by the remarks preceding the statement of the theorem, and

$$E^{(2)p,q} = \begin{cases} H^p(G, LimA_\alpha) & \text{if } q = 0 \\ 0 & \text{otherwise.} \end{cases} .$$

Thus both spectral sequences collapse giving the required isomorphism. (An explicit calculation shows that this isomorphism is the obvious morphism.)

Remark: It is instructive to recall here the well known forms that $H^n(G, A)$ takes when A is discrete: (i) $H^q(G, A) \cong ColimH^q(G/U, A^U)$, where the colimit is taken over open normal subgroups of G and, as usual, $A^U = \{a : au = a \text{ for all } u \in U\}$ and (ii) $H^q(G, A) \cong Colim_{\mathsf{B}} H^q(G, B)$, where the colimit is taken over all $B \subset A$ of finite type.

7.3 Relative cohomology groups.

7.3.1 What are they?

Any treatment of relative cohomology groups, even in the abstract group situation, encounters a problem, namely that there is no single accepted definition of what they are! The setting is that G is a profinite group with H

as a closed (often normal) subgroup; the idea is that the cohomology group, $H^n(G, H; M)$, for a pseudocompact G-module, M, is made up of classes of n-cochains on G, which vanish on H, and have values in M, the difference in opinion is as to the extent to which these cochains vanish on H.

In this section we look at two of the possibilities in detail, especially in low dimensions, concentrating on the connections between the two theories. (For a survey of the possible theories, we suggest that the interested reader do as we did, and consults the collected Maths. Reviews on Infinite Groups, where most of the early papers on this subject are listed.)

The main contenders for the definitions of the relative cohomology, denoted $H^n(G, H; M)$, have been developed initially by the following authors.:

(i) M.Auslander [7]; L.Ribes, [146]; Takasu, [159]; Massey (although I have found no exact reference for this latter work other than a mention in another paper), and Gildenhuys and Ribes, [72].

(ii) Adamson, [1]; Hochschild, [84]; Ostberg, [132].

Both of these theories lead to relative groups $H^n(G, H; M)$. An alternative when H is normal is to consider the relative group $H^n(Q, G; M)$ for M a Q-module, with $Q = G/H$ as do, for instance, Loday [105], and, in more general contexts, Rinehart, [149], Van Osdol [161] and others. (A short discussion of the link between these two views together with further references can be found in Huebschmann's paper, [87].)

7.3.2 The Auslander-Ribes theory

We start with the Auslander-Ribes theory in its profinite version studied in [147]:

We consider G, a profinite group, H, a closed subgroup of G and M a discrete G-module. Restriction along the inclusion followed by extension yields the discrete module, $M^* = Coin_H^G(M)$, (cf. Serre, [152], p.13); this G-module M^* is defined as the set of continuous homomorphisms $m^* : G \to M$ such that $m^*(hx) = h.m^*(x)$ for all $h \in H$. The group G acts on M^* by the rule:

$$(gm^*)(x) = m^*(xg).$$

There is an injective homomorphism

$$i : M \to M^*$$

given by: $i(m)(x) = x.m$. This yields a homomorphism,

$$i_* : H^q(G, M) \to H^q(G, M^*) \cong H^q(H, M),$$

by Shapiro's lemma, which coincides with the restriction map.

In this general situation, one can consider the Abelian group, denoted $X(G, H; M)$ by Ribes [147], of continuous G-derivations from G to M that vanish on H, that is, $X(G, H; M)$ is $\{f : G \to M \mid f(xy) = xf(y) +$

$f(x), f$ is continuous and $f|_H = 0\}$. Since the category of *discrete* G-modules has enough injectives, one can consider the right derived functors of $X(G, H; -)$. It is these that Ribes calls the n^{th} *cohomology group* $H^n(G, H; M)$ *of the pair,* (G, H)*, with coefficients in* M. Now let $\Gamma(M) = Coker(i : M \to M^*)$, then one has

$$Hom_G(\hat{\mathbb{Z}}, \Gamma(M)) \cong X(G, H; M),$$

by calculation, (see [147]), so $H^n(G, H; M) \cong H^{n-1}(G, \Gamma(M))$.

In our context there is a slight problem: if M is a pseudocompact G-module, although the majority of the above definitions make sense, one cannot use injectives to define the right derived functors of $X(G, H; -)$ as $Pc.G{-}Mod$ does not have enough injectives. Of course, one can still *define*

$$H^n(G, H; M) = Ext^{n-1}(\hat{\mathbb{Z}}, \Gamma(M)),$$

since the group on the right makes sense in terms of *projective* resolutions of $\hat{\mathbb{Z}}$ or, using neither projectives nor injectives, in a Yoneda style description using $(n-1)$-fold extensions. We shall be needing both these types of treatment later.

The above suggests another means of defining the groups $H^n(G, H; M)$ as follows:

The inclusion $H \xrightarrow{j} G$ induces a map of bar resolutions

$$\mathsf{B}H \xrightarrow{B(j)} \mathsf{B}G,$$

over the change of rings, $\hat{\mathbb{Z}}[\![H]\!] \to \hat{\mathbb{Z}}[\![G]\!]$, and hence yields a monomorphism of complexes of pseudocompact G-modules

$$(\mathsf{B}H \otimes_H \hat{\mathbb{Z}}[\![G]\!]) \to \mathsf{B}G$$

which on generators is merely inclusion of H^n into G^n at level n. This, in turn, yields an epimorphism

$$C(G, M) = Hom(\mathsf{B}G, M) \to Hom((\mathsf{B}H \otimes_H \hat{\mathbb{Z}}[\![G]\!]), M),$$

which is split at each level and is natural in M. We write $K(M)$ for the kernel of this epimorphism, then we have

$$H^n(G, H; M) \cong H^{n-1}(K(M)).$$

This tells us that the elements of $H^n(G, H; M)$ can be represented by continuous $(n-1)$-cocycles $f : G^n \to M$ that vanish on the subspace, H^n, of G^n.

Another description of the homotopy type of $K(M)$ can be manufactured from this:

Writing $\mathsf{B}_G H$ for $\mathsf{B}H \otimes_H \hat{\mathbb{Z}}[\![G]\!]$, for short, the monomorphism

$$\mathsf{B}_G H \xrightarrow{B(j)} \mathsf{B}G$$

is locally split, i.e. is split at each level. Let $C(j)$ denote its cokernel. Then $K(M) \cong Hom_G(C(j), M)$. Alternatively (ignoring that $B(j)$ is a monomorphism), we might form its mapping cone/homotopy cokernel, $C_h(j)$, then $K(M) \simeq Hom(C_h(j), M)$. Both of these yield useful alternative descriptions of the elements of $H^n(G, H; M)$. These cohomology groups are also functorial in (G, H) in an obvious sense which we will not make precise here.

In low dimensions these relative cohomology groups have the following interpretation: a 1-cocycle is, as we have already noted, a continuous G-derivation $f : G \to M$ such that $f(h) = 0$ for all $h \in H$. As the principal G-derivations and principal H-derivations coincide, there are no coboundaries at this level, so $H^1(G, H; M) = X(G, H; M)$.

Depending on the model for the homotopy type of $K(M)$ that we use, we get a different argument leading to an interpretation of elements of $H^2(G, H; M)$. Let us consider $C_h(j)$, the homotopy cokernel of $B(j)$. In dimension n, this is $\mathsf{B}G_n \oplus \mathsf{B}_G H_{n-1}$ with differential given by

$$(x, y) \mapsto (d_G x + B(j)y, -d_H y).$$

Thus a homomorphism $f : C_h(j)_n \to M$ can be decomposed as $f = f_1 + f_2$, where $f_1 : \mathsf{B}G_n \to M$ and $f_2 : \mathsf{B}_G H_{n-1} \to M$. Such a cochain is a cocycle if $fd = 0$, i.e., if

$$f_1 d_G x + (f_1 B(j)y - f_2 d_H y) = 0$$

for all $x \in \mathsf{B}G_{n+1}$, $y \in \mathsf{B}H_n$. Restricting to the pair $(x, 0) \in C_h(j)$ gives that $f_1 : \mathsf{B}G_n \to M$ is an n-cocycle which, when restricted to $\mathsf{B}H$, gives a boundary. We will use this description in general later, but for the moment we restrict attention to the cases of $n = 1$ and 2.

Although we already have a description of the relative cocycles for $n = 1$, it is useful as a first step to see how this general description works out in that case. We have $f_1 : \mathsf{B}G_1 \to M$ is a cocycle (hence f_1 can be thought of as a derivation, $f_1 : G \to M$) such that $f_1 B(j)_1$ is a boundary, i.e. there is a morphism, $f_2 : \hat{\mathbb{Z}}[\![H]\!] \to M$, such that for any $y \in H$, $f_1(y) = f_2 d_H(y) = f_2(y - 1)$. Since $f_2(y-1)$ is $yf_2(1) - f_2(1)$, writing $m = f_2(1)$, we can replace f_1 by $\bar{f}_1$, given by $\bar{f}_1[x] = f_1[x] - mx + m$, so that $cls\, \bar{f}_1 = cls\, f_1$; $\bar{f}_1$ is again a derivation, $\bar{f}_1 : G \to M$, but now $\bar{f}_1(y) = 0$ for any $y \in H$ and we have a description as before: a class in $H^1(G, H; M)$ can be represented by a G-derivation $f : G \to M$ that vanishes on H.

When one looks at $n = 2$, one has a 2-cocycle $f : \mathsf{B}G_2 \to M$ such that $f \mid \mathsf{B}H_2$ is a coboundary. Thus using f one can build, in the usual way, an Abelian extension of G by M,

$$\mathcal{E}_f : 1 \to M \to E \xrightarrow{p} G \to 1,$$

say, such that the action of G on M is the given one; M is, of course, a closed normal subgroup of the profinite group E. To examine the implications of the condition: "$f|_{H^2}$ is a coboundary", we need to recall the way E is constructed from $f : G^2 \to M$.

We take the underlying space of E to be $M \times G$ and put on it the multiplication

$$(m_1, x_1) \cdot (m_2, x_2) = (m_1 + x_1 \cdot m_2 + f(x_1, x_2), x_1 x_2).$$

Now assume that there is some $h : H \to M$ such that for all $x_1, x_2 \in H$

$$\begin{aligned} f(x_1, x_2) &= (\partial h)(x_1.x_2) \\ &= x_1 h(x_2) - h(x_1, x_2) + h(x_1). \end{aligned}$$

Then the section $s(x) = (h(x), x)$ of p over the subgroup H is, in fact, a continuous splitting of the induced sequence $j^*(\mathcal{E}_f)$, i.e., the cocycle $(f, h) : C_h(j)_2 \to M$ yields an Abelian extension that is split over the subgroup H, *together with* a splitting of that induced sequence. It is easily checked that if (f, h) and (f', h') are two such cocycles, then they are cohomologous if and only if the two sequences $\mathcal{E}_f$ and $\mathcal{E}_{f'}$ are congruent in such a way that the two splittings differ by a "principal splitting", i.e. a splitting corresponding to a principal derivation.

We must wait until later to discuss interpretations of elements of $H^n(G, H; M)$ for $n \geq 3$ for this Auslander-Ribes theory.

7.3.3 The Adamson-Hochschild theory.

We next turn to the Adamson-Hochschild theory. Again our initial data consists of a profinite group G, a closed subgroup H and a G-module M. We have mentioned already that the usual results on restriction and induction of modules relative to the inclusion $H \hookrightarrow G$ go over to pseudocompact modules with no difficulty. We thus obtain an adjoint pair

$$Pc.G\!-\!Mod \underset{}{\overset{res_H^G}{\rightleftarrows}} Pc.H\!-\!Mod$$

where the unmarked functor is $-\hat{\otimes}_H \hat{\mathbb{Z}}[\![G]\!]$. The general theory of relative homological algebra thus applies (cf. Hochschild, [84], or MacLane, [112]).

For a pseudocompact G-module M, we define the relative cohomology groups for (G, H) with coefficients in M to be $Ext^n_{(R,S)}(\hat{\mathbb{Z}}, M)$ where, for ease of printing, we have written R for $\hat{\mathbb{Z}}[\![G]\!]$ and S for $\hat{\mathbb{Z}}[\![H]\!]$, and the relative Ext is constructed using a (R, S)-projective resolution of $\hat{\mathbb{Z}}$. We will denote these relative cohomology groups by $H^n((G, H); M)$ to distinguish them from those considered earlier.

In Hochschild's paper, [84], the abstract/discrete analogue of this construction is given and it is shown that these relative groups coincide with those introduced in 1954 by I.T. Adamson, [1]. Adamson exploited the fact that the set of cosets of H in G is a G-set and mixed this structure into a resolution in order to build the relative groups. Explicitly (in the notation of Hochschild's paper, [84]), for $n \geq 0$, let X_n be the free Abelian group generated by the $(n+1)$-tuples $(A_0, \ldots, A_n)$ of cosets $A_i = g_i H$ with $g_i \in G$. There is a G-module structure on X_n given by, for $g \in G$, $g \cdot (A_0, \ldots, A_n) = (gA_0, \ldots, gA_n)$. For $n = -1$, take $X_n = \mathbb{Z}$ to give the augmentation and for $n < -1$, $X_n = 0$. The differentials are given by the usual formulae, $d_0 : X_0 \to X_{-1}$ being the coefficient sum, then (X, d) is easily checked to be an acyclic $\mathbb{Z}G$-complex. Defining $h_{-1} : X_{-1} \to X_0$ by $h_{-1}(z) = z(H)$ and for $n \geq 0$, $h_n : X_n \to X_{n+1}$ by $h_n(A_0, \ldots, A_n) = (H, A_0, \ldots, A_n)$, yields a homotopy that is $\mathbb{Z}H$-linear, but not $\mathbb{Z}G$-linear. The cohomology groups $H^n((G, H); M)$ are then defined as those of the complex $Hom_{\mathbb{Z}G}(X, M)$ for M a G-module.

Of course, this description adapts easily to the case of a profinite group pair, (G, H), and a pseudocompact $\hat{\mathbb{Z}}[\![G]\!]$-module, M. All the maps (differentials and homotopies) so defined are continuous. As in the abstract group case, if $H \lhd G$, we have any (continuous) cocycle $f : X_n \to M$, has image in the fixed module M^H of M for the induced H action. (To see why, use the second complex introduced by Hochschild, [84] p.263.) This indicates why in case $H \lhd G$, one may identify $H^n((G, H); M)$ with $H^n(Q, M^H)$ where $Q = G/H$; a proof is given in Adamson, [1], or may easily be constructed from the above facts.

The resolution X of $\hat{\mathbb{Z}}$ is, of course, the relative version of the "unnormalized non-homogeneous bar resolution". It is easy to give a normalized version by requiring that each X_n be replaced by a quotient $\bar{X}_n$ obtained by dividing out by those $(n+1)$-tuples with an adjacent pair equal, $A_i = A_{i+1}$, for some i. In this latter form it is easily seen that each continuous cochain $f : \bar{X}_n \to M$ determines another, $\bar{f}$, say, with domain G^{n+1} obtained by composition with the quotient

$$\bar{f}(g_0, \ldots, g_n) = f(g_0 H, \ldots, g_n H)$$

and that not only is $\bar{f}$ normalized, but it satisfies the stronger conditions:

(i) $\bar{f}(g_0, \ldots, g_n) = \bar{f}(g_0 h_0, \ldots, g_n h_n)$ for any $h_0, \ldots, h_n \in H$,

(ii) $\bar{f}(g_0, \ldots, g_n) = 0$ if some $g_i g_{i-1}^{-1} \in H$.

In order to compare with Auslander-Ribes cocycles, it is useful to pass one stage further and to look at the conditions on the corresponding homogeneous cocycles.

The first condition is awkward to write down, but the second normalization condition is

$$\bar{f}([x_1| \ldots |x_n]) = 0 \text{ if any } x_i \in H.$$

Thus any Adamson-Hochschild cocycle is automatically an Auslander-Ribes one. To compare them, it will pay to have a description of the Adamson-Hochschild groups in low dimensions.

In dimension zero an Adamson-Hochschild cocycle is an element of M^H.

In dimension 1, the only difference is that an Adamson-Hochschild cochain $f : G \to M$ must take values in M^H if H is normal.

In dimension 2, an Adamson-Hochschild cocycle determines a relatively split pair of extensions

$$\begin{tikzcd} 1 \arrow[r] & M \arrow[r] & E \arrow[r, "p"] & G \arrow[r] & 1 \\ 1 \arrow[r] & M \arrow[r] \arrow[u, equal] & E|_H \arrow[r, shift right] \arrow[u] & H \arrow[r] \arrow[u] \arrow[l, shift right, "s"'] & 1 \end{tikzcd}$$

but, in addition, one has that the splitting s can be extended to a (continuous) transversal $\gamma : G \to E$ such that $\gamma(gh) = \gamma(g)s(h)$ for all $g \in G$, $h \in H$. (The discrete case is analysed by Hochschild, [84] p.263; this simpler description is given by Ostberg, [132].)

Ostberg, [132], deals with the interpretation of the relative H^3 in terms of relative abstract kernels, but we will briefly return to this later.

7.4 Extensions and long exact sequences.

In this section, we wish to investigate some of the long exact sequences linking the cohomology groups of the groups in a profinite extension:

$$1 \to N \to G \xrightarrow{\pi} Q \to 1.$$

7.4.1 Long exact sequences in cohomology

As we have seen, given an extension

$$1 \to N \to G \xrightarrow{\pi} Q \to 1$$

of profinite groups, one obtains an exact sequence

$$0 \to N^{Ab} \to \hat{\mathbb{Z}}[\![Q]\!]\hat{\otimes}_G \hat{I}(G) \xrightarrow{\tilde{\pi}} \hat{I}(Q) \to 1$$

of pseudocompact $\hat{\mathbb{Z}}[\![Q]\!]$-modules. This short exact sequence gives us a quick proof of the following:

Proposition 46. *Given an extension*

$$1 \to N \to G \xrightarrow{\pi} Q \to 1$$

of profinite groups, and a pseudocompact $\hat{\mathbb{Z}}[\![Q]\!]$-module, A, there is a long exact sequence

$$\begin{aligned} 0 &\to Der(Q, A) \to Der_\pi(G, A) \to Hom_Q(N^{Ab}, A) \\ &\to H^2(Q, A) \to H^2(G, \pi^*(A)) \to Ext^1_Q(N^{Ab}, A) \\ &\to H^3(Q, A) \to \dots, \end{aligned}$$

where the Ext^i_Q denote the derived functors of the hom-functor $Hom_Q(-,-)$ within Pc.Q-Mod .

Remark: Some people prefer the similar sequence that starts

$$0 \to H^1(Q, A) \to H^1(G, \pi^*(A)) \to Hom_Q(N^{Ab}, A) \to \dots \qquad .$$

This can easily be derived from the above by noting that

$$H^1(Q, A) \cong Der(Q, A)/Pder(Q, A),$$

whilst

$$H^1(G, \pi^*(A)) \cong Der(G, \pi^*(A))/Pder(G, \pi^*(A)),$$

but $Der(G, \pi^*(A))$ is identifiable as $Der_\pi(G, A)$, whilst the induced map from $Der(Q, A)$ to $Der_\pi(G, A)$ in that sequence maps $Pder(Q, A)$ isomorphically onto $Pder(G, \pi^*(A))$ modulo the above identification.

Proof of Proposition: The sequence is obtained from the $Hom - Ext$ long exact sequence of

$$1 \to N \to G \to Q \to 1$$

with coefficients in A, after identification of the various terms as follows:
a) $\hat{I}(Q) \cong D_Q$, so $Hom(\hat{I}(Q), A) \cong Der(Q, A)$;
b) $\hat{\mathbb{Z}}[\![Q]\!]\hat{\otimes}_G \hat{I}(G) \cong D_\pi$, so $Hom(\hat{\mathbb{Z}}[\![Q]\!]\hat{\otimes}_G \hat{I}(G), A) \cong Der_\pi(G, A)$.
c) The isomorphism (cf. section 2.5)

$$H^n(G, M) \cong Ext^{n-1}(\hat{I}(G), M)$$

for M in $Pc.G$-Mod can now be used to identify the $(3k+4)^{th}$ terms in the sequence and, modulo an evident natural isomorphism

$$Ext^i_Q(\hat{\mathbb{Z}}[\![Q]\!]\hat{\otimes}_G \hat{I}(G), -) \cong Ext^i_G(\hat{I}(G), \pi^*(-)),$$

also to handle the remaining terms. ∎

7.4.2 The Lyndon-Hochschild-Serre spectral sequence.

Given an extension

$$1 \to N \to G \to Q \to 1$$

and a G-module, M, the Lyndon-Hochschild-Serre spectral sequence gives in low dimensions a five term exact sequence,

$$0 \to H^1(Q, M^N) \to H^1(G, M) \to H^1(N, M)^Q \to H^2(Q, M^N) \to H^2(G, M).$$

Serre comments that if the extension is a sequence of profinite groups, and M is a discrete G-module, the usual derivation of the L-H-S spectral sequence generalises so the five term sequence is also valid in that case. Moore, [122], discusses such spectral sequences for general topological G-modules, but terms such as $H^p(Q, H^q(N, M))$ involve a more general class of coefficients than we have considered here. Thus no treatment of a profinite version of the L-H-S spectral sequence would seem to be possible wholly within the context of profinite groups and pseudocompact modules, however Boggi, [14], does give a form of the L-H-S spectral sequence with a slightly different choice of coefficients for cohomology.

Whatever way around these difficulties is used, the five term sequence makes perfect sense in our context and the question arises how one may derive its exactness without using Moore's version of the L-H-S argument. There seem to be various ways of doing this. In the next section, we look at the Auslander-Ribes relative cohomology groups again and their relation to this sequence will be examined, via their link with the Adamson-Hochschild groups given earlier.

7.4.3 Long exact sequences and relative cohomology

Let us assume a more general situation than that given by our extension. Suppose H is a closed subgroup of the profinite group G. For ease we will write, as before, $\mathsf{B}_G H = \mathsf{B}H\hat{\otimes}_H\hat{\mathbb{Z}}[\![G]\!]$.

We get a short exact sequence

$$0 \to \mathsf{B}_G H \to \mathsf{B}G \to C(j) \to 0,$$

of complexes of $\hat{\mathbb{Z}}[\![G]\!]$-modules. Applying $Hom_G(-, M)$ and taking the associated long exact sequence we get, after some fairly self evident identifications.

Proposition 47. *(cf. Gildenhuys and Ribes, [72]) Let H be a closed subgroup of a profinite group G and let M be a pseudocompact $\hat{\mathbb{Z}}[\![G]\!]$-module. There is a long exact sequence,*

$$\begin{aligned} 0 \to M^G \to M^H \xrightarrow{\delta} H^1(G,H;M) \xrightarrow{j} H^1(G;M) \xrightarrow{i} H^1(H;A) \\ \to H^2(G,H;M) \xrightarrow{j} H^2(G,M) \xrightarrow{i} \dots \end{aligned}$$

where the is are restriction maps induced by the inclusion $H \to G$ and the js are restriction maps induced by the inclusion $(G,1) \to (G,H)$. ■

It is worth noting that $M^G = H^0(G;M)$ and $M^H = H^0(H;M)$.

The link with the Lyndon-Hochschild-Serre 5-term exact sequence goes as follows:

Suppose $N \lhd G$ is a closed normal subgroup and $Q = G/N$ then we note that for a pseudocompact G-module, M,

$$\begin{aligned} H^1(Q, M^N) &= H^1((G,N);M) \\ &\cong X(G,N;M)/Im\,\delta \\ &= H^1(G,N;M)/Im\,\delta. \end{aligned}$$

Thus dividing out by $Im\,\delta$ yields the start of the L-H-S sequence

$$0 \to H^1(Q, M^N) \to H^1(G, M).$$

The next point to note is that the image of $H^1(G, M) \xrightarrow{i} H^1(N, M)$ lies in the fixed subgroup, $H^1(N, M)^Q$, for the induced Q-action. The standard proof, for instance in MacLane, [112] p.349, Lemma 9.1, extends to the profinite case with no difficulty.

The connection between

$$H^2(Q, M^N) \cong H^2((G, N); M)$$

and the corresponding Auslander-Ribes group, $H^2(G, N; M)$, is subtler than in dimension 1. We have so far two interconnected sequences (the top one has not at this stage been proven to be exact)

$$\begin{tikzcd}
0 \arrow[r] & H^1(Q, M^N) \arrow[r] \arrow[d, dash, "\cong"'] & H^1(G, M) \arrow[r] \arrow[d, dashed, dash, "\cong"'] & H^1(N, M)^Q \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & H^1(G, N; M)/Im\delta \arrow[r] & H^1(G, M) \arrow[r] & H^1(N, M) \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r, "\delta"] & H^2(Q, M^N) \arrow[r] \arrow[d, dotted, "?"] & H^2(G, M) \arrow[d] \\
\cdots \arrow[r, "\delta"] & H^2(G, N; M) \arrow[r] & H^2(G, M)
\end{tikzcd}$$

and it is the properties of the dotted arrow that are in doubt. However we have detailed knowledge of how the maps δ are defined and also we know that any Adamson-Hochschild cocycle yields an Auslander-Ribes one.

First we choose a continuous section

$$t : Q \to G$$

for the quotient map $\lambda : G \to Q$, so that $t(1) = 1$. Then given any continuous derivation $f : N \to M$, we can set $\bar{f}(g) = t(q)f(n)$ for $g \in G$, where $q = \pi(g) \in Q$ and $g = t(q)n$. The map $\bar{f} : G \to M$ is continuous and extends to a continuous homomorphism $\bar{f} : \mathsf{B}G_1 \to M$. As $\partial f = 0$, we have that $\partial \bar{f}$ vanishes on $B_G N_2$ and hence gives an Auslander-Ribes cocycle $f' : C(j)_2 \to M$, which yields the class of the desired relative 2-cocycle, i.e. $\delta(cls\ f) = cls\ f'$.

If f is a Q-invariant 1-cocycle, then the value of f' can be given quite simply; it is $f'(g_1, g_2) = t(q_1q_2)f(t(q_1q_2)^{-1}t(q_1)t(q_2))$, which is, of course, an Adamson-Hochschild 2-cocycle. In other words the restriction of δ to $H^1(N, M)^Q$ yields a map into

$$H^2((G, N), M) \cong H^2(Q, M^N),$$

as required. We thus need next to turn to exactness at these fourth terms. Suppose $f : C(j)_2 \to M$ is an Auslander-Ribes cocycle which becomes a coboundary on composition with the map $\mathsf{B}G_2 \to C(j)_2$, then $f(g_1, g_2) = g_1 h(g_2) - h(g_1 g_2) + h(g_1)$ for some $h : \mathsf{B}G_1 \to M$. However $f(n_1, n_2) = 0$ if $n_1, n_2 \in N$, so h, when restricted to $B_G N_1$, yields a derivation as required.

If f is an Adamson-Hochschild cocycle, one can say more, as routine calculations show that h is nearly Q-invariant, in fact $g^{-1}h({}^g n) - h(n)$ is a principal derivation for fixed $g \in G$. This shows that both sequences are exact as required, since it is easily checked that the composites are zero.

The usefulness of the exact sequence for Auslander-Ribes relative cohomology is that exactness is more or less self evident and that it is a "long" exact sequence not just a sequence of 5 terms. The Lyndon-Hochschild-Serre sequence *is* part of a longer sequence (cf. Huebschmann, [87]), but some of these terms are more difficult to handle, however the above comparison of the first five terms of the two sequences shows the extent to which the L-H-S sequence gives more detailed and useful information than does the other sequence; the interpretation of $H^2(G, N; M)$ is less immediate than that of $H^2(Q, M^N)$.

7.5 The Profinite Schur Multiplier

7.5.1 The direct approach

The full treatment of both the profinite Schur multiplier and the profinite Universal Coefficient Theorem really needs a discussion of profinite homology, so these topics will crop up again in the next chapter. However one of our principal aims in introducing the profinite Schur multiplier and in proving the Universal Coefficient Theorem in a profinite version is to provide an example where the fact that we can take pseudocompact/profinite coefficients in a cohomology group is necessary to avoid awkward restrictions on the validity of the results. As a full treatment of profinite homology is not strictly necessary for this, we have chosen to pick a shorter, more group theoretical, path to our goal by using a treatment based on that in Beyl and Tappe's notes, [12]. This has the advantage that many of the proofs in that source generalise very easily to the profinite case, although it does lead to some slight duplication with material in the next chapter.

The profinite version of the Schur multiplier has been introduced and used by Fröhlich, [64]. Its definition uses the profinite analogue of the Schur-Hopf formula. (Versions in Pro$-\mathcal{C}$, for $\mathcal{C}$ a Serre class of groups, have also been considered in the above mentioned source.)

Let G be a profinite group and let

$$e : 1 \to R \to F \to G \to 1$$

be a profinite presentation of G. We set

$$M(G) = \frac{R \cap [F,F]}{[F,R]},$$

where, as usual, $[F,R]$ is the *closed* normal subgroup generated by the elements $xyx^{-1}y^{-1}$, $x \in F$, $y \in R$.

This group is an Abelian profinite group and is easily shown to be independent (up to isomorphism) of the choice of e, cf. Beyl and Tappe, [12]. Thus when proving, say, functoriality of $M(G)$, it suffices to produce a functorial presentation, namely:

$$e(G) : 1 \to R_G \to F_G \stackrel{\pi_G}{\to} G \to 1,$$

where F_G is the free profinite group on the underlying profinite space of G and R_G is the kernel of the natural map $\pi_G : F_G \to G$. On the other hand explicit profinite presentations in the form $(X : R)$ allow one to link $M(G)$ with combinatorial group theoretic information on G such as the deficiency.

The functoriality of M means that given any extension

$$e : 1 \to N \stackrel{\kappa}{\to} G \stackrel{\pi}{\to} Q \to 1$$

of profinite groups, there is a continuous morphism

$$M(G) \stackrel{M(\pi)}{\to} M(Q)$$

of profinite Abelian groups. This fits into a 5-term exact sequence as in the abstract case, namely:

Proposition 48. *(cf. [12], p.31) Given the profinite extension*

$$e : 1 \to N \stackrel{\kappa}{\to} G \stackrel{\pi}{\to} Q \to 1,$$

there is a natural exact sequence

$$M(G) \xrightarrow{M(\pi)} M(Q) \xrightarrow{\theta_*(e)} \frac{N}{[N,G]} \xrightarrow{\kappa'} G^{Ab} \xrightarrow{\pi^{Ab}} Q^{Ab} \to 0.$$

If e' is another profinite extension congruent to e, then $\theta_*(e)$ and $\theta_*(e')$ are compatible via a commutative 5-term ladder diagram. ■

If $G = F$ is free, then $\theta_*(e)$ is “just” the inclusion

$$M(G) \cong \frac{N \cap [F,F]}{[N,F]} \to \frac{N}{[N,G]}.$$

The proof for the abstract case, given by Beyle and Tappe, [12] p.31, generalises to the profinite case without difficulty.

It is also worth noting (cf. [12], p.33) that if e is a (profinite) *central* extension, so $[N,G] = 1$, the sequence becomes

$$M(G) \stackrel{M(\pi)}{\rightarrow} M(Q) \stackrel{\theta_*(e)}{\rightarrow} N \stackrel{\kappa^{Ab}}{\rightarrow} G \stackrel{\pi^{Ab}}{\rightarrow} Q^{Ab}$$

These 5-term exact sequences have, of course, a homological interpretation. We will later on give this, using the adaptation of the classical arguments due to Wambsganß-Türk, [163], but will also give a group theoretic extension of this to an 8-term sequence using the Brown-Loday exterior product suitably extended to apply to profinite groups.

7.6 The Universal Coefficient Theorem

A profinite version of this for finite discrete coefficients was given in his thesis, [163] by Wambsganß-Türk. The key to his proof is the observation that the p-adic integers $\hat{\mathbb{Z}}_p$ form a local ring of global dimension 1, so as $\hat{\mathbb{Z}}$ can be decomposed as a product of various $\hat{\mathbb{Z}}_p$, we have $gl.\dim \hat{\mathbb{Z}} = 1$ and the usual proofs of the Universal Coefficient Theorem go over with little or no change. His proof generalises easily to pseudocompact coefficients, but as it is couched in terms of homology, we will state a form of the theorem in the style of Beyl and Tappe.

Theorem 8. *Let G and A be profinite groups with A Abelian, then there is a natural short exact sequence*

$$0 \to Ext^1(G^{Ab}, A) \stackrel{\psi}{\to} H^2(G, A) \stackrel{\theta_*}{\to} Hom(M(G), A) \to 0,$$

which is split. Identifying $H^2(G, A)$ with $Cext(G, A)$, the group of central extensions of G by A, the maps ψ and θ_* have the following descriptions.

(i) Let $ab : G \to G^{Ab}$ be the natural continuous morphism and let $[e] \in Ext^1(G^{Ab}, A)$ with

$$e : 0 \to A \to E \to G^{Ab} \to 0$$

then $\psi[e] = ab^*[e]$, the extension induced along ab.

(ii) If $e : 1 \to A \to E \to G \to 1$ is a central extension, then $\theta_*(e) : M(G) \to A$ corresponds to e, as in the 5-term exact sequence. ■

The proof in Beyl and Tappe, [12] p.34, goes through almost word for word - note that as $\hat{\mathbb{Z}}$ has global dimension 1, closed subgroups of free profinite Abelian groups are free profinite-Abelian; this is needed to replace the fact that a subgroup of a free Abelian group is free Abelian in their proof.

7.7 Universal Profinite Central Extensions.

7.7.1 Perfection!

Suppose G is a profinite group, we will say it is *perfect* if $[G, G] = G$, i.e. it is topological generated by commutators. If G is perfect profinite, then its Abelianization, G^{Ab}, is zero. This simple observation easily leads to the following facts about G (as in the abstract group case).

a) For any profinite Abelian group A, there is a natural isomorphism

$$H^2(G,A) \xrightarrow{\theta_*} Hom(M(G),A)$$

Thus we get

b) $H^2(G,-)$ is representable, i.e. there is an element $[e] \in H^2(G,M(G))$ so that $\theta_*(e) = Id_{M(G)}$ and hence if $[e'] \in H^2(G,A)$, there is a unique $\varphi : M(G) \to A$ (in fact $\varphi = \theta_*(e')$) satisfying $\theta_*[e] = [e']$.

c) Identifying $H^2(G,A)$ with $Cext(G,A)$, $[e]$ gives a central profinite extension

$$e : 1 \to M(G) \to E \to G \to 1$$

with the property that all other central extensions

$$e' : 1 \to A \to E' \to G \to 1$$

have the form $\varphi_*(e)$ up to isomorphism, for some unique continuous $\varphi : M(G) \to A$ determined by e'.

We say that an epimorphism $\pi : E \to G$ is a *perfect cover* if E is perfect and $Ker\,\pi \subseteq Z(E)$, the centre of E. In the extension e above, π is a perfect cover of G. In fact, looking at the 5-term exact sequence determined by e, we get

$$M(E) \xrightarrow{M(\pi)} M(G) \xrightarrow{\theta_*(e)} M(G) \to E \xrightarrow{\pi^{Ab}} G^{Ab} \to 0$$

and, as $\theta_*(e)$ is the identity, π^{Ab} is an isomorphism, but G^{Ab} is trivial.

e) There is a one-one correspondence between closed subgroups of $M(G)$ and isomorphism classes of perfect covers of G. If $U \leq M(G)$ is a closed subgroup, it gives an extension

$$e/U : 1 \to M(G)/U \to E/U \xrightarrow{\pi_U} G \to 1.$$

By the previous observations, this corresponds to the epimorphism $\theta_*(e/U) : M(G) \to M(G)/U$ given by the natural quotient map. Again using the 5-term exact sequence, but this time for e/U, one finds π_U is an isomorphism so E/U is again perfect.

This sets up a Galois correspondence between the closed subgroups of $M(G)$ and the perfect profinite covers of G, (cf. Beyl and Tappe, [12] p.114-115, and Kervaire, [99] for the abstract group case).

f) One can realise e quite easily; in fact e is isomorphic to the extension

$$e_0 : \frac{R \cap [F,F]}{[R,F]} \to \frac{[F,F]}{[R,F]} \to G,$$

where $R \to F \to G$ is a free profinite presentation of G. This is an extension since $\rho : [F,F] \to G$ is onto. The argument on p.115 of Beyl and Tappe goes over without change.

Remarks: a) The above depended in an elementary, but crucial, way on the possibility of taking coefficients in a profinite Abelian group.

b) The discussion of universal covering groups and Kervaire's results given here only scratches the surface. The treatment in depth of these notions given, for example, in Beyl and Tappe, [12] pp. 115 - 120, would seem to go over to the profinite case with little or no difficulty, however we have not checked that this is the case.

c) The above is an example of a Galois theory in the sense of Borceux and Janelidze, [18], and hence again suggests links to the general area of Grothendieck's *Pursuit of Stacks*, [76]. This, of course, raises the possibility of interpreting later results on cohomology and homology in this context. There is much more to be done to clarify these links, even in the abstract group case.

7.8 Cohomology and profinite crossed extensions

7.8.1 Cochains

Consider a pseudocompact G-module, M, and a non-negative integer n. We can form the chain complex, $K(M,n)$, having M in dimension n and zeroes elsewhere. We can also form a profinite crossed complex, $\mathsf{K}(M,n)$, that plays the role of the n^{th} Eilenberg-MacLane space of M in this setting. We call it the n^{th} *Eilenberg-MacLane crossed complex* of M:

If $n = 0$, $\mathsf{K}(M,n)_0 = G \ltimes M$, $\mathsf{K}(M,n)_i = 0$, $i > 0$.

If $n \geq 1$, $\mathsf{K}(M,n)_0 = G$, $\mathsf{K}(M,n)_n = M$, $\mathsf{K}(M,n)_i = 0$, $i \neq 0$ or n.

One way to view cochains is as chain complex morphisms. Thus on looking at $Ch(Pc.G{-}Mod)(\mathsf{B}G, K(M,n))$, one finds exactly $Z^{n+1}(G,M)$, the $(n+1)$-cocycles of the cochain complex $C(G,M)$. Using the adjointness between Δ_G and ξ_G given in Chapter 6, and the fact, which the attentive reader will, of course, have noticed, that $\mathsf{K}(M,n) = \Delta_G(K(M,n))$, we can also view $Z^{n+1}(G,M)$ as $Prof.Crs_G(\mathsf{C}G, \mathsf{K}(M,n))$.

In the category of chain complexes, one has that a homotopy from $\mathsf{B}G$ to $K(M,n)$ between 0 and f, say, is merely a coboundary, so that $H^{n+1}(G,M) \cong [\mathsf{B}G, K(M,n)]$, adopting the usual homotopical notation for the group of homotopy classes of maps from the bar resolution $\mathsf{B}G$ to $K(M,n)$. This description has its analogue in the crossed complex case as we shall see.

7.8.2 Homotopies

Let C, C' be two profinite crossed complexes with Q and Q' respectively as the cokernels of their bottom morphism. Suppose λ, $\mu : \mathsf{C} \to \mathsf{C}'$ are two morphisms inducing the same map $\varphi : Q \to Q'$.

A homotopy from λ to μ is a family, $h = \{h_k : k \geq 0\}$, of continuous maps $h_k : C_k \to C'_{k+1}$ for $k \geq 1$ satisfying the following conditions:

H1) $h_0 : C_0 \to C'_1$ is a continuous derivation along μ_0 (i.e. for $x, y \in C_0$,

$$h_0(xy) = h_0(x)({}^{\mu_0}h_0(y)),)$$

such that

$$\delta_1 h_0(x) = \lambda_0(x)\mu_0(x)^{-1}, \quad x \in C_0.$$

H2) $h_1 : C_1 \to C'_2$ is a continuous C_0-homomorphism with C_0 acting on C'_2 via λ_0 (or via μ_0, it makes no difference) such that

$$\delta_2 h_1(x) = \mu_1(x)^{-1}(h_0\delta_1(x)^{-1}\lambda_1(x)) \text{ for } x \in C_1.$$

H3) for $k \geq 2$, h_k is a continuous Q-homomorphism (with Q acting on the C'_k via the induced map $\varphi : Q \to Q'$) such that

$$\delta_{k+1}h_k + h_{k-1}\delta_k = \lambda_k - \mu_k.$$

We note that the condition that λ and μ induce the same map, $\varphi : Q \to Q'$, is, in fact, superfluous as this is implied by $H1$.

The properties of homotopies and the relation of homotopy are as one would expect. The continuous analogues of Huebschmann's results, [86] pp. 307-308, go through without difficulty and one finds $H^{n+1}(G, M) \cong [CG, \mathsf{K}(M, n)]$. Given that in higher dimensions, this is the same set exactly as $[\mathsf{B}G, K(M, n)]$ means that there is not much to check and so the proof has been omitted.

7.8.3 Huebschmann's description of cohomology classes.

The transition from this position to obtaining the profinite analogue of Huebschmann's descriptions of cohomology classes, [86], is now more or less formal. We will, therefore, only sketch the main points.

If G is a profinite group, M is a pseudocompact G-module and $n \geq 1$, a *profinite crossed n-fold extension* is an exact augmented profinite crossed complex,

$$0 \to M \to C_{n-1} \to \ldots \to C_1 \to C_0 \to G \to 1.$$

The notion of similarity of such extensions is analogous to that of n-fold extensions in the Abelian Yoneda theory, (cf. MacLane, [112]), as is the definition of a Baer sum. We leave the details to the reader. This yields an Abelian group, $Opext^n(G, M)$, of similarity classes of profinite crossed n-fold extensions of G by M.

Given a cohomology class in $H^{n+1}(G, M)$ realisable as a homotopy class of maps, $f : \mathsf{C}G \to \mathsf{K}(M, n)$, one uses f to form an induced crossed complex, much as in the Abelian Yoneda theory:

$$\begin{tikzcd}
 & J_n(G) \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\text{pushout}"] & C_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_0 \arrow[r] \arrow[d] & G \arrow[d, equal] \\
0 \arrow[r] & M \arrow[r] & M_{n-1} \arrow[r] & \cdots \arrow[r] & M_0 \arrow[r] & G
\end{tikzcd}$$

where $J_n(G)$ is $Ker(C_{n-1}G \to C_{n-2}G)$. (Thus J_nG is also $Im(C_nG \to C_{n-1}G)$ and as the map f satisfies $f\delta = 0$, it is zero on the subgroup $\delta(C_{n+1}G)$ (i.e. is constant on the cosets) and hence passes to $Im(C_nG \to C_{n-1}G)$ in a well defined way.) Arguments using lifting of maps and homotopies show that the assignment of this element of $Opext^n(G, M)$ to $cls(f) \in H^{n+1}(G, M)$ establishes an isomorphism between them. The continuity of the maps and homotopies is assured by the fact that they are constructed using the freeness clauses satisfied by the various parts of the crossed resolution $\mathsf{C}G$.

7.8.4 Interpretation in low dimensions.

The importance of having such a description of classes in $H^n(G, M)$ probably resides in low dimensions. To describe classes in $H^3(G, M)$, one has profinite crossed 2-fold extensions

$$0 \to M \to C_1 \xrightarrow{\partial} C_0 \to G \to 1,$$

where ∂ is a profinite crossed module and so M does need to be a pseudocompact G-module for this to work. In the abstract case, one has for any group G, a crossed 2-fold extension

$$0 \to Z(G) \to G \xrightarrow{\partial_G} Aut(G) \to Out(G) \to 1$$

where ∂_G sends $g \in G$ to the corresponding inner automorphism of G. An *abstract kernel* (in the sense of Eilenberg-MacLane, [52]) is a homomorphism $\psi : Q \to Out(G)$ and hence provides, by pulling back, a 2-fold extension of Q by the centre $Z(G)$ of G. In the profinite case, since $Aut(G)$ need not be profinite, abstract kernels may cause problems, however crossed 2-fold extensions work regardless.

Conrad, [38], has examined the possibility of generalising the notions of stem extensions to crossed 2-fold extensions for higher order analogues of the Schur multiplier. It seems likely that these notions and constructs go through to the profinite case, but we have not checked if this is so.

7.9 Relative crossed extensions

Our aim in this section is to give descriptions of the Auslander-Ribes relative groups for a pair, (G, H), of profinite groups with coefficients in a pseudocompact G-module, M. The descriptions we seek should be the higher order analogues of the descriptions given above of H^1 and H^2 in both cases, using n-fold crossed extensions in the profinite case, however we have not managed to find such an interpretation for the Adamson-Hochschild theory above H^3.

First let us point out that Ostberg, [132], has already begun this process (in the abstract group case) with a description of the Adamson-Hochschild H^3 using relative abstract kernels.

We will start by looking at the Auslander-Ribes theory:

7.9.1 Crossed interpretations of the Auslander-Ribes theory

We saw in section 7.3.2, that an Auslander-Ribes relative cohomology class could be realised by a continuous cochain

$$f : C_h(j)_n \to M,$$

where $C_h(j)_n = \mathsf{B}G_n \oplus \mathsf{B}_G H_{n-1}$. The differential of $C_h(j)$ being given by $d(x,y) = (d_G x + B(j)y, -d_H y)$, such a cochain is a cocycle if and only if, on writing $f = f_1 + f_2$, where $f_1 : \mathsf{B}G_n \to M$, $f_2 : \mathsf{B}_G H_{n-1} \to M$, f_1 is an n-cocycle and $f_{1|H}$ (i.e. $f_1 B(j)$) is a coboundary: $f_1 B(j) y = f_2 d_H y$.

Although we have already considered the case $n = 2$, it will pay to look at it again as the interpretation given in section 7.3.2 needs to be adapted slightly before it can be generalised to higher dimensions.

A cocycle $f : C_1 G \to M$ determines an extension as usual, however this can be viewed in a slightly different way. The continuous map f induces a map from $J_1 G$ to M, but

$$J_1 G \to C_0 G \to G$$

is the extension

$$N \to F_G \to G,$$

which determines an Abelian extension,

$$0 \to N^{Ab} \to F/[N,N] \to G \to 1.$$

The map $f : J_1 G \to M$ factors via a map $f^{Ab} : N^{Ab} \to M$, and hence induces an extension in the usual way

$$\begin{tikzcd}
0 \arrow[r] & N^{Ab} \arrow[r] \arrow[d] & F/[N,N] \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "="] & 1 \\
0 \arrow[r] & M \arrow[r] & E_f \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

with the left hand square a pushout.

If the G-action on M is trivial, then the map $f^{Ab} : N^{Ab} \to M$ factors further via $N/[G,N]$, and one gets a central extension.

Now turning to the relative case, as we have not defined a homotopy cokernel in the context of profinite crossed complexes, we must be careful. We have a relative cocycle, (f_1, f_2), where $f_1 : C_1 G \to N$ is a G-cocycle and where $f_2 : C_0 H \to M$ satisfies $f_2 \partial = f_1 C_1(j)$, i.e., we have a diagram

$$\begin{tikzcd}
C_1H \arrow[r, "C_1(j)"] \arrow[d, "\partial"'] & C_1G \arrow[r, "f_1"] \arrow[d] & M \\
C_0H \arrow[r, "C_0(j)"'] \arrow[d] \arrow[urr, "f_2"] & C_0G \arrow[d] & \\
H \arrow[r, "j"'] & G &
\end{tikzcd}$$

Passing to the Abelianisations yields

$$\begin{tikzcd}
N_H^{Ab} \arrow[r, "N(j)^{Ab}"] \arrow[d] & N_G^{Ab} \arrow[r, "f_1^{Ab}"] \arrow[d] & M \\
F_H \arrow[r] \arrow[d] \arrow[urr, "f_2"] & F_G \arrow[d] & \\
H \arrow[r, "j"'] & G &
\end{tikzcd}$$

and as before the induced extensions $\mathcal{E}_f$ over G and $\mathcal{E}'$ over H will be given by pushouts along f_1^{Ab} and $f_1^{Ab}N^{Ab}(j)$ respectively. However the induced extension $\mathcal{E}'$ over H is easily checked to be congruent to the restricted one, $j^*(\mathcal{E}_f)$, which, by our arguments in earlier sections, is split. How can we see it is split directly? This is, in fact, easy, since we have a diagram

$$\begin{tikzcd}
N_H^{Ab} \arrow[r] \arrow[d] & M \arrow[d, "i"] \\
F_H \arrow[r] \arrow[d] \arrow[ur, dotted, "f_2"] & E' \arrow[d] \\
H \arrow[r, "j"'] & H
\end{tikzcd}
\qquad \text{(Top square a pushout)}$$

and f_2 induces a continuous map, $E' \to M$, splitting i.

Now this treatment generalises easily to higher dimensions. In general, if $(f_1, f_2) : C_h(j)_n \to M$ is a relative n-cocycle, we have a profinite crossed n-fold extension $\mathcal{E}$ determined by f_1 and the diagram

$$\begin{tikzcd}
0 \arrow[r] & J_n(G) \arrow[r] \arrow[d, "f'"'] & C_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_0 \arrow[r] \arrow[d] & G \arrow[d, equal] \\
0 \arrow[r] & M \arrow[r] & M_{n-1} \arrow[r] & \cdots \arrow[r] & C_0G \arrow[r] & G
\end{tikzcd}$$

with the first square a pushout. Corresponding to $f_1C(j)$, there is a split profinite crossed n-fold extension of H with the splitting determined by f_2, i.e., what could be called a *relatively split crossed n-fold extension*. Choosing a different but cohomologous pair, $(f_1^{'}, f_2^{'})$ say, changes the n-fold extensions by a congruence, but the splittings over H will not necessarily be preserved, instead they will be "homotopic" i.e. differing by a map from the last-but-three term in the extension. With this description one should be able to give a simple description of the boundary terms in the Auslander-Ribes exact sequence. As we have no immediate application in mind, we will leave the final details to the reader.

7.9.2 Adamson-Hochschild theory

We now turn more briefly to the Adamson-Hochschild theory. This, in general, is much more difficult to interpret. In fact apart from Ostberg's description of classes in H^3 given in [132], it is difficult to find any definite results. Huebschmann, [87], has an interpretation in a general case, but as his description involves automorphism groups, it does not easily generalise to the profinite case. Loday, [105], gives a neat description if H is a normal subgroup of G and H acts trivially on M, but in this case M is a Q-module and, by the results of Adamson, [1], $H^3((G,H);M) \cong H^3(Q,M)$, so we can give a description of elements in this case already. (Loday's results are, however, of interest because of their link with relative universal central extensions, see also Conrad, [37].) No really adequate interpretation of relative cohomology classes (in the sense of Adamson and Hochschild) seems to have yet appeared in print - at least as far as we have seen. The importance of such an interpretation would be the possibility of working out more information on the next level within the L-H-S spectral sequence. All the above refers to the "abstract" group case, but it applies equally to the profinite case.

We finish with a brief look at Ostberg's relative crossed 2-fold extensions. (He uses the language of abstract kernels, but, in the profinite case, crossed 2-fold extensions are more appropriate.)

A profinite relative crossed 2-fold extension is a diagram,

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & C_1 \arrow[r, "\partial"] & C_0 \arrow[r, "p"] & G \arrow[r] & 1 , \\
0 \arrow[r] & M \arrow[r] \arrow[u, dash, "="] & C_1' \arrow[r, "\partial'"'] \arrow[u] & C_0' \arrow[r, "p'"'] \arrow[u] & H \arrow[r] \arrow[u, "j"'] \arrow[ul, dotted, "s"] & 1
\end{tikzcd}$$

with the two rows being profinite crossed 2-fold extensions of G or H by the pseudocompact G-module M, together with a continuous homomorphism $s : H \to C_0$ such that $ps = j$ and such that there is a continuous transversal $\gamma : G \to C_0$ satisfying $\gamma j = s$ and $\gamma(gh) = \gamma(g)s(h)$, $\gamma(hg) = s(h)\gamma(g)$ for all $g \in G$, $h \in H$.

Although higher order versions of this can be written down, it is difficult to check that such gadgets contain equivalent information to relative cocycles. There would seem to be a need for further invesitgation here.

7.10 Profinite 2-types and profinite cohomology

We now turn to a discussion of the second type of application that was mentioned as motivation in the introduction to this chapter. In classifying homotopy types and in obstruction theory, one frequently has invariants that are elements in cohomology groups of the form $H^m(X, \pi)$, where typically π is the n^{th} homotopy group of some space. When dealing with profinite homotopy types, π will be a profinite group, usually Abelian with a π_1 action, i.e. we are exactly in the situation described in this chapter, except that X is a profinite homotopy type not a profinite group. Of course, provided that X is connected, we can replace X by a profinite simplicial group, bringing us even nearer to the situation of this chapter. A complete "work-out" of these ideas is however still lacking, but in one case we can handle this in complete detail, namely for describing profinite 2-types. We shall work within the category of profinite simplicial groups.

7.10.1 Profinite 2-types

Recall, from chapter 2, that a morphism

$$f : G \to H$$

of simplicial groups is called a *2-equivalence* if it induces isomorphisms

$$\pi_0(f) : \pi_0(G) \to \pi_0(H,)$$

and

$$\pi_1(f) : \pi_1(G) \to \pi_1(H).$$

Clearly there is no obstruction to extending this definition to profinite or pro-$\mathcal{C}$ simplicial groups, and we will consider this 'done'. We can form a quotient category, $Ho_2(Prof.Simp.Grps)$, of $Prof.Simp.Grps$ by formally inverting the 2-equivalences. Then we say two profinite simplicial groups, G and H, have the same *profinite 2-type* if they are isomorphic in $Ho_2(Prof.Simp.Grps)$.

This is, of course, just a special case of the general notion of n-type in which "n-equivalences" are inverted, thus forming the quotient category $Ho_n(Prof.Simp.Grps)$. An n-equivalence is a morphism, f, inducing isomorphisms, $\pi_i(f)$, for $i = 0, 1, \ldots n-1$. (Do not forget the cautionary note on page 59.)

7.10.2 Profinite 1-types

Before examining profinite 2-types in detail, it will pay to think about 1-types. A morphism f as above is a 1-equivalence if it induces an isomorphism on π_0, i.e. $\pi_0(f)$ is an isomorphism. Given any profinite group G, there is a profinite simplicial group, $K(G,0)$ consisting of G in each dimension with face and degeneracy maps all identities. Given a profinite simplicial group H, having $G \cong \pi_0(H)$, the natural quotient map

$$H_0 \to \pi_0(H) \cong G,$$

extends to a natural 1-equivalence between H and $K(\pi_0(H),0)$.

It is fairly routine to check that

$$\pi_0 : Prof.Simp.Grps \to Prof.Grps$$

has $K(-,0)$ as an adjoint and that, as the unit is a natural 1-equivalence, and the counit an isomorphism, this adjoint pair induces an equivalence between the category $Ho_1(Prof.Simp.Grps)$ of 1-types and the category, $Prof.Grps$, of profinite groups. In other words,

profinite groups are algebraic models for profinite 1-types.

7.10.3 Algebraic models for profinite n-types?

So much for profinite 1-types. Can one provide algebraic models for profinite 2-types or, in general, profinite n-types? We touched on this in Chapter 2.1. The criteria that any such "models" might satisfy are debatable. Perhaps ideally, or even unrealistically, there should be an isomorphism class of algebraic "gadget" for each 2-type. An alternative weaker solution is to say that a notion of equivalence between the models is possible, only equivalence classes, not isomorphism classes, correspond to 2-types, but the notion of equivalence is algebraically defined. It is this weaker possibility that corresponds to our aim here.

7.10.4 Algebraic models for profinite 2-types.

In section 4.3, we discussed how to pass from profinite simplicial groups to profinite crossed modules, and back again. Recall that if G is a profinite simplicial group, then we can form a profinite crossed module

$$\partial : \frac{NG_1}{d_0(NG_2)} \to G_0,$$

where the action of G_0 is via the degeneracy, $s_0 : G_0 \to G_1$, and ∂ is induced by d_0. (As before we will denote this profinite crossed module by $M(G,1)$.) The kernel of ∂ is

$$\frac{Ker\, d_0 \cap Ker\, d_1}{d_0(NG_2)} \cong \pi_1(G),$$

whilst its cokernel is

$$\frac{G_0}{d_0(NG_1)} \cong \pi_0(G),$$

and so we have a profinite crossed 2-fold extension

$$0 \to \pi_1(G) \to \frac{NG_1}{d_0(NG_2)} \to G_0 \to \pi_0(G) \to 1$$

and hence a cohomology class $k(G) \in H^3(\pi_0(G), \pi_1(G))$.

Suppose now that $f : G \to H$ is a morphism of profinite simplicial groups, then one obtains a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(G) \arrow[r] \arrow[d, "\pi_1(f)"'] & \frac{NG_1}{d_0(NG_2)} \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d, "f_0"] & \pi_0(G) \arrow[r] \arrow[d, "\pi_0(f)"] & 1 \\
0 \arrow[r] & \pi_1(H) \arrow[r] & \frac{NH_1}{d_0(NH_2)} \arrow[r] & H_0 \arrow[r] & \pi_0(H) \arrow[r] & 1
\end{tikzcd}$$

If, therefore, f is a 2-equivalence, $\pi_0(f)$ and $\pi_1(f)$ will be isomorphisms and the diagram shows that, modulo these isomorphisms, $k(G)$ and $k(H)$ are the same cohomology class, i.e. the 2-type of G determines π_0, π_1 and this cohomology class, k in $H^3(\pi_0, \pi_1)$.

Conversely, suppose we are given a profinite group π, a pseudocompact $\hat{\mathbb{Z}}[\![\pi]\!]$-module, M, and a cohomology class $k \in H^3(\pi, M)$, then we can realise k by a profinite 2-fold extension

$$0 \to M \to C \xrightarrow{\partial} G \to \pi \to 1$$

(by the results of section 7.8.3). The profinite crossed module, $\mathcal{X} = (C, G, \partial)$, determines a profinite simplicial group $E(\mathcal{X})$ as in section 4.3 and

$$M(E(\mathcal{X}), 1) \cong \mathcal{X}.$$

Suppose we had chosen an equivalent profinite 2-fold extension

$$0 \to M \to C' \xrightarrow{d'} G' \to \pi \to 1$$

The equivalence guarantees that there is a zig-zag of maps of 2-fold extensions joining it to that considered earlier. We need only look at the case of a direct basic equivalence:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & C \arrow[r, "\partial"] \arrow[d] & G \arrow[r] \arrow[d] & \pi \arrow[r] \arrow[d, "="] & 1 \\
0 \arrow[r] & M \arrow[r] & C' \arrow[r, "\partial'"] & G' \arrow[r] & \pi \arrow[r] & 1
\end{tikzcd}$$

giving a map of profinite crossed modules, $\varphi : \mathcal{X} \to \mathcal{X}'$, where $\mathcal{X}' = (C', G', \partial')$. This induces a morphism of simplicial groups,

$$E(\varphi) : E(\mathcal{X}) \to E(\mathcal{X}'),$$

that is, of course, a 2-equivalence. If there is a longer zig-zag between $\mathcal{X}$ and $\mathcal{X}'$ then the intermediate profinite crossed modules give intermediate profinite simplicial groups and a zig-zag of 2-equivalences so that $E(\mathcal{X})$ and $E(\mathcal{X}')$ are isomorphic in $Ho_2(Prof.Simp.Grps)$, i.e. they have the same 2-type. This argument can, of course, be reversed.

If G and H have the same 2-type, they are isomorphic within the category $Ho_2(Prof.Simp.Grps)$, so they are linked in $Prof.Simp.Grps$ by a zig-zag of 2-equivalences, hence the corresponding cohomology classes in $H^3(\pi_0(G), \pi_1(G))$ are the same up to identification of $H^3(\pi_0(G), \pi_1(G))$ and $H^3(\pi_0(H), \pi_1(H))$. This proves the profinite analogue of the result of MacLane and Whitehead, [114], that we mentioned earlier, 36, giving an algebraic model for 2-types of connected CW-complexes.

Theorem 9. *(MacLane and Whitehead, [114]) Profinite 2-types are classified by a profinite group π_0, a pseudocompact π_0-module, π_1 and a class in $H^3(\pi_0, \pi_1)$.* ■

We have handled this in such a way as to derive an equivalence of categories:

Proposition 49. *There is an equivalence of categories,*

$$Ho_2(Prof.Simp.Grps) \cong Ho(Prof.CMod),$$

where $Ho(Prof.CMod)$ is formed from $Prof.CMod$ by formally inverting those maps of crossed modules that induce isomorphisms on both the kernels and the cokernels. ■

The corresponding abstract group case can be found in various sources, for instance, Baues, [9].

We could equally well attempt to look at the cohomology classes of crossed n-fold extensions, but these will not correspond to all (profinite) n-types as there is extra structure in the general n-type. We will look at this shortly.

7.11 Summary and Conclusions.

In this chapter we have attempted to give a "survey" of some of those parts of the cohomology theory of groups that generalise to the profinite case if one replaces the usual discrete coefficients of the cohomology by pseudocompact ones. We hope we have convinced the potential user that this theory is quite accessible, it being, for the most part, a moderately simple process to transform the "abstract" theory into the profinite/pseudocompact context. Versions of much of this material can also be developed for pro$-\mathcal{C}$ groups.

The main algebriac application that we have given is to the theory of universal central extensions of perfect profinite groups, but, as we have indicated, much of the cohomological treatment of stem extensions and of representation groups (given in Beyl and Tappe, [12], for instance) can be adapted on much the same lines. We have only aimed to indicate what is possible not to give an exhaustive survey parallel to existing treatments in the abstract group case.

Our treatment of the relative cohomology groups will hopefully draw attention to the need for a unified approach. Many authors have worked in this area, but a detailed comparison of the Auslander-Ribes and Adamson-Hochschild approaches does not, as yet, exist except in low dimensions as here. The importance of this for understanding the possible extensions of the 5-term exact sequence and also for non-Abelian cohomology, seems clear. This is true even for the abstract case; in the profinite case, certain of the constructions hit the additional difficulty that automorphism groups of profinite groups need not be profinite, and perhaps this indicates a need to work with prodiscrete localic groups or similar objects that may allow automorphism objects to be constructed.

When it comes to the theory of profinite crossed extensions, we have again not been exhaustive in our treatment. No definitive version in the abstract case yet exists. For the would-be user of this, for instance, in the case of 2-fold extensions, we would draw their attention to Loday's paper, [105], and also to Conrad's two articles [37] and [38], which treat central and stem 2-fold extensions and higher versions of the Schur multiplier respectively. These would seem to have considerable potential for application. Of course, for the development of the general theory, Huebschmann's work should be consulted, see the bibliography.

The profinite generalisation of the results of MacLane and Whitehead, [114], further supports the view that pseudocompact coefficients are needed to get a complete picture of the potential of profinite cohomology. Analogues for n-types in general will be given in Chapter **??**.

Finally we would like to raise the tantalising problem of the link between this cohomology theory and profinite presentations of (pro)finite groups. The powerful results of Golod and Safarevic (cf. [152]) showed that the cohomology theory of finite p-groups yields detailed information on the relative numbers of generators and relations needed. Now one has the additional information, is it possible to analyse this information yet further perhaps to give even better knowledge of the deficiency of a p-group?

8

Homology of Profinite Groups

Many of the applications of the new crossed module techniques have been in the homology rather than the cohomology of groups. In particular the development by Brown and Loday, [31], of a non-Abelian tensor product for groups and its subsequent purely algebraic treatment by Ellis, [53, 54], has yielded eight term exact homology sequences with explicit descriptions of the extra terms. This work grew out of the crossed module and crossed square versions of van Kampen's theorem, [31].

As we suggested in the previous chapter, the natural way to handle central extensions is via a profinite valued homology theory. Such a theory was defined by Brumer, [34], but although Wambsganß-Türk, [163], and Korkes have pushed his results further along the classical lines of homological algebra, no treatment of this theory is at present in the published literature. In this chapter we not only include a brief resumé of this foundational material, but will push the homology theory further along the lines already developed by Brown, Loday, Ellis and others in the case of abstract groups, introducing tensor and exterior products of profinite groups, and applying them to give further information on central extensions, an eight term homology exact sequence, and so on.

We will assume that the reader has some basic background knowledge of homological algebra and the homology of groups.

8.1 Tensor and Torsion Products of Pseudo-Compact $\hat{\mathbb{Z}}[\![G]\!]$-modules and Homology Groups of Profinite Groups.

8.1.1 Tensor products of modules

We start by examining in more detail the construction of the tensor product of pseudocompact modules as given by Brumer, [34]. We have already used it earlier.

Definition: Let M be a right pseudocompact $\hat{\mathbb{Z}}[\![G]\!]$-module and N a left pseudocompact $\hat{\mathbb{Z}}[\![G]\!]$-module, then their *(completed) tensor product* is a pseudocompact $\hat{\mathbb{Z}}$-module, $M\hat{\otimes}_G N$ defined as an inverse limit of the tensor products of their finite quotient modules M/U and N/V, where $U \in \Omega(M)$, $V \in \Omega(N)$, that is, U and V are taken from the directed sets of all open submodules of M and N, respectively, that are of finite colength:

$$M\hat{\otimes}_G N = Lim\{(M/U) \otimes_G (N/V) : U \in \Omega(M), V \in \Omega(N)\}.$$

Since each M/U and N/V is a $\hat{\mathbb{Z}}[\![G]\!]$-module of finite length, their tensor product over G is also of finite length and hence $M\hat{\otimes}_G N$ is a pseudocompact $\hat{\mathbb{Z}}$-module. Of course, if either M is a left pseudocompact $\hat{\mathbb{Z}}[\![H]\!]$-module, or N is a right pseudocompact $\hat{\mathbb{Z}}[\![H]\!]$-module, then $M\hat{\otimes}_G N$ will inherit that structure.

8.1.2 Representing elements in completed tensors.

If $m \in M$ then, since $M \cong Lim\{M/U : U \in \Omega(M)\}$, we can represent m by a system of elements $m = (m_U)_{U\in\Omega(M)}$, where $m_U = m + U \in M/U$. If $m = (m_U)$ and $n = (n_V)$ then we will, of course, denote by $m \otimes n$ the element $\{m_U \otimes n_V : U \in \Omega(M), V \in \Omega(N)\}$ in $M\widehat{\otimes}_G N$.

There is a continuous function

$$f : M \times N \to M\widehat{\otimes}_G N$$

defined by $f(m,n) = m \otimes n$. This mapping is bilinear and is the universal bilinear map with domain $M \times N$.

8.1.3 Torsion products

Brumer, [34], notes that $Pc.G - Mod$ has enough projectives, see also, [148]. Using this, it is routine to define torsion products.

Definition: Let M be a right and N a left pseudocompact $\hat{\mathbb{Z}}[\![G]\!]$-module. The n^{th} *torsion product*, $Tor_n^G(M,N)$ and N is defined to be the n^{th} left derived functor of $M\hat{\otimes}-$ evaluated at N.

One thus takes a projective resolution P of N within the category $Pc.G-Mod$ and one puts

$$Tor_n^G(M,N) = H_n(M\widehat{\otimes}_G \mathsf{P}).$$

The usual properties of Tor_n^G follow as in the non-topological case.

8.1.4 Homology

Again turning to Brumer, we find the following:

Definition: Let G be a profinite group and let M be a right pseudocompact $\hat{\mathbb{Z}}[\![G]\!]$-module. The n^{th} *homology group of* G *with coefficients in* M, is given by

$$H_n(G,M) = Tor_n^G(M,\hat{\mathbb{Z}}).$$

If M is also a left pseudocompact $\hat{\mathbb{Z}}[\![G]\!]$-module, then each $H_n(G,M)$ will be one as well. (To see this note that $M\widehat{\otimes}_G\mathsf{P}$ will be within $Pc.H{-}Mod$, hence so will be its homology).

If $G = Lim\, G_i$ and $M = Lim\, M_i$ where the M_i are pseudocompact G_i-modules, then Brumer proves ([34], p.455) that

$$Tor_n^G(M,\hat{\mathbb{Z}}) \cong Lim\, Tor_n^{G_i}(M_i,\hat{\mathbb{Z}})$$

and thus, as a particular case, one obtains

$$H_n(G,M) = Lim\,\{H_n(G/U, M/M\hat{I}(U)) : U \in \Omega(G)\}$$

A "rival" definition was given in [133] by Poitou. The aim of his theory was towards a duality theory rather than towards homological dimension, as was that of Brumer. Poitou's definition is:

$$H_n(G,M) = Lim\,\{H_n(G/U, M^U) : U \in \Omega(G)\},$$

where, as usual, $M^U = \{m \in M : um = m \text{ for all } u \in U\}$.

8.2 A profinite Hopf formula

In section 7.5, in the last chapter, we introduced the profinite version of the Schur multiplier:

$$M(G) = \frac{R \cap [F,F]}{[F,R]}.$$

Of course, we should expect that this is isomorphic to $H_2(G,\hat{\mathbb{Z}})$ as in the classical theory for abstract groups. We would also expect the proof to be more or less the same. We start this section with a resumé of this proof.

8.2.1 The profinite Stallings 5-term exact homology sequence.

We have already seen the following (Proposition 40, p.130):

If

$$1 \to K \xrightarrow{\phi} G \xrightarrow{\psi} Q \to 1$$

is an exact sequence of profinite groups and continuous homomorphisms, then there is an exact sequence

$$0 \to K^{Ab} \xrightarrow{\tilde{\phi}} \hat{\mathbb{Z}}[\![Q]\!]\widehat{\otimes}_G \hat{I}(G) \xrightarrow{\tilde{\psi}} \hat{I}(Q) \to 0$$

of pseudocompact $\hat{\mathbb{Z}}[\![Q]\!]$-modules.

The Stallings 5-term exact homology sequence (see Stallings [154]) has a profinite analogue:

Proposition 50. *Given an exact sequence of profinite groups and continuous homomorphisms, as above, then for M a pseudocompact $\hat{\mathbb{Z}}[\![Q]\!]$-module, the following sequence is exact:*

$$H_2(G,M) \to H_2(Q,M) \to M\widehat{\otimes}_Q K^{Ab} \to M\widehat{\otimes}_G \hat{I}(G) \to M\widehat{\otimes}_Q \hat{I}(Q) \to 0.$$

■

The usual proof for the abstract group case, which involves the long exact $Tor_*^Q(M,-)$-sequence, goes over with no difficulty, so a detailed proof will not be given here. This is, as we will see later, the same result as Proposition 48, when $M = \hat{\mathbb{Z}}$. One of the key lemmas in the proof of the above is that there is a natural isomorphism

$$Tor_{i-1}^Q(M, \hat{I}(Q)) \cong H_i(Q,M), \qquad i \geq 2.$$

This is proved using the $Tor_*^Q(M,-)$-sequence applied to the short exact augmentation sequence,

$$0 \to \hat{I}(Q) \to \hat{\mathbb{Z}}[\![Q]\!] \xrightarrow{\varepsilon} \hat{\mathbb{Z}} \to 0.$$

In the case $i = 1$, a weaker result holds, namely that

$$0 \to H_1(Q,M) \to M\widehat{\otimes}_Q \hat{I}(Q) \to M\widehat{\otimes}_Q \hat{\mathbb{Z}}[\![Q]\!] \to M\widehat{\otimes}_Q \hat{\mathbb{Z}}$$

is exact, so that $H_1(Q,M) = Ker(M\widehat{\otimes}_Q \hat{I}(Q) \to M)$, since $M\widehat{\otimes}_Q \hat{\mathbb{Z}}[\![Q]\!] \cong M$. This map from $M\hat{\otimes}_Q \hat{I}(Q)$ to M is, of course, given on generators by $m \otimes a = ma$, so if the action of Q on M is trivial, this map is zero and $H_1(Q,M) \cong M\widehat{\otimes}_Q \hat{I}(Q)$. Thus, in this situation (and in particular when $M = \hat{\mathbb{Z}}$), we get a 5-term exact sequence

$$H_2(G,M) \to H_2(Q,M) \to M\hat{\otimes} K^{Ab} \to H_1(G,M) \to H_1(Q,M) \to 0.$$

The particular case of the above in which $M = \hat{\mathbb{Z}}$ is of most interest. We write $H_i(G)$ for $H_i(G, \hat{\mathbb{Z}})$. This then gives, as a special case of the above, an exact sequence

$$H_2(G) \to H_2(Q) \to \hat{\mathbb{Z}}\hat{\otimes} K^{Ab} \to H_1(G) \to H_1(Q) \to 0.$$

This sequence is isomorphic, as we shall show, to that given by Proposition 48, page 163:

$$M(G) \to M(Q) \to \frac{K}{[K,G]} \to G^{Ab} \to Q^{Ab} \to 0$$

where $M(G)$ is the profinite Schur multiplier.

The first point to note, that $H_1(G) \cong G^{Ab}$, is an easy consequence of the isomorphism $H_1(G) \cong \hat{\mathbb{Z}}\widehat{\otimes}_G \hat{I}(G)$ as in the discrete case.

We next note the following:

Proposition 51. *If F is a free profinite group, then for any pseudocompact $\hat{\mathbb{Z}}F$-module, M, and any $n \geq 2, H_n(F, M) = 0$.*

Proof: By Corollary 12, $\hat{I}(F)$ is a free pseudocompact $\hat{\mathbb{Z}}[\![F]\!]$-module, hence

$$0 \to \hat{I}(F) \to \hat{\mathbb{Z}}[\![F]\!] \to \hat{\mathbb{Z}} \to 0$$

is a free resolution of $\hat{\mathbb{Z}}$. Of course, as in the abstract case, this implies that $\mathrm{Tor}_n^F(M, \hat{\mathbb{Z}}) = 0$ for $n \geq 2$; an alternative way is to use the isomorphism

$$H_i(F, M) \cong Tor_{i-1}(M, \hat{I}(F)) \text{ for } i \geq 2$$

and the freeness of $\hat{I}(F)$. ■

8.2.2 Comparison of the 5-term sequences

To continue our comparison of the two five term exact sequences, we need to investigate $\hat{\mathbb{Z}}\hat{\otimes}K^{Ab}$ and the quotient, $\dfrac{K}{[K,G]}$.

Let M be a pseudocompact $\hat{\mathbb{Z}}[\![Q]\!]$-module with trivial Q-action, and

$$p : K/[K, K] \to K/[K, G],$$

the obvious natural morphism. It is sufficient to prove that

$$p^* : Hom_{\hat{\mathbb{Z}}}(\frac{K}{[K, G]}, M) \to Hom_{\hat{\mathbb{Z}}[\![Q]\!]}(K^{Ab}, M)$$

is an isomorphism, since the right hand group is isomorphic to $Hom_{\hat{\mathbb{Z}}}(K^{ab}\hat{\otimes}\hat{\mathbb{Z}}, M)$, in the usual way. Here $Hom_{\hat{\mathbb{Z}}}$ denotes the continuous $\hat{\mathbb{Z}}$-module morphisms. (We recall, from section 3.2.5, that K^{Ab} is a pseudocompact $\hat{\mathbb{Z}}[\![Q]\!]$-module, since $K \to G$ is a profinite crossed module).

Given $\theta : K^{Ab} \to M$, over $\hat{\mathbb{Z}}[\![Q]\!]$, then the kernel of θ, $Ker\,\theta$, contains $[K, G]/[K, K]$, since if $[k, g]$ is a commutator with $k \in K$, $g \in G$,

$$\begin{aligned}\theta([k, g][K, K] &= \theta(k[K, K]) + \psi(g)\theta(k^{-1}[K, K]) \\ &= \theta(k[K, K]) - \psi(g)\theta(k[K, K]) \\ &= \ 0,\end{aligned}$$

since the action of $\psi(g)$ on M is trivial. Hence there exists a morphism $\bar{\theta}$: $K/[K, G] \to M$ induced by θ, i.e., p^* is onto. To complete the proof, we have to show it is one-one, but as p is an epimorphism in the category of pseudocompact $\mathbb{Z}[\![Q]\!]$-modules, this is immediate.

We record this as a lemma for future reference.

Lemma 17. *The natural isomorphism from K^{Ab} to $K/[K, G]$ induces a natural isomorphism*

$$K^{Ab}\hat{\otimes}\hat{\mathbb{Z}} \to K/[K, G].$$

■

8.2.3 Profinite Hopf formula

This lemma leads, as in the classical abstract case, to an isomorphism between $H_1(G)$ and $M(G)$, the profinite Schur multiplier. The following was noted by both Türk, [163], and Korkes, [100].

Proposition 52. *Let the profinite group G have a presentation $(X : R)$, then writing F for $F(X)$ the free profinite group on X, and $N(R)$ for the closed normal closure of R, there is a natural isomorphism*

$$H_2(G,\hat{\mathbb{Z}}) \cong \frac{N(R)\cap[F,F]}{[F,N(R)]} = M(G).$$

Proof: Writing N for $N(R)$, we get a short exact sequence

$$1 \to N \to F \to G \to 1$$

and hence a five-term exact sequence for M a pseudocompact $\hat{\mathbb{Z}}[\![G]\!]$-module,

$$H_2(F,M) \to H_2(G,M) \to M\hat{\otimes}_G N^{Ab} \to M\hat{\otimes}_F \hat{I}(F)) \to M\hat{\otimes}_G \hat{I}(G) \to 0.$$

We have already noted (Proposition 51) that $H_2(F,M) = 0$, so

$$H_2(G,M) \cong Ker(M\hat{\otimes}_G N^{Ab} \to M\hat{\otimes}_F \hat{I}(F)).$$

In the case where M is the trivial $\hat{\mathbb{Z}}[\![G]\!]$-module, $\hat{\mathbb{Z}}$, we have calculated that this is equivalent to

$$H_2(G,\hat{\mathbb{Z}}) \cong Ker\left(\frac{N}{[F,N]} \to \frac{F}{[F,F]}\right)),$$

i.e.,

$$H_2(G,\hat{\mathbb{Z}}) \cong \frac{N\cap[F,F]}{[F,N]},$$

as required. ■

8.2.4 8-term sequences?

This profinite Hopf formula shows, as promised, that $H_2(G) \cong M(G)$ thus linking up the homologically derived 5-term exact sequence with that derived earlier by non-homological means. The homological sequence has a well-known extension to an eight term exact sequence if the initial short exact sequence

$$1 \to N \to G \to Q \to 1$$

is a weak stem central extension. The sixth term, first noted by Ganea, [67], is $G^{Ab} \otimes N$. The next two terms are H_3 of Q and G respectively, giving

$$H_3(G) \to H_3(Q) \to G^{Ab} \otimes N \to H_2(G) \to H_2(Q) \\ \to N/[G,N] \to H_1(G) \to H_1(Q) \to 0$$

Gut [80] also has this eight term sequence as well as a ten term sequence. The blockage to non-central versions of this is the difficulty of deciding what a non-Abelian tensor product of groups should be. We will turn to this in the profinite case shortly.

The central version of the eight term exact sequence for profinite group homology has been derived from the abstract case by Türk, [163]. We will give a more general result with no centrality conditions later in this chapter, based on the work of Korkes, [100]. This partially fills the gap between the 8-term exact sequence of the homological case and the "elementary" Schur multiplier approach which only yields a 5-term sequence.

The main applications we will give of this material are to a homological treatment of universal central extensions via a profinite version of the Brown-Loday tensor product, which will be given shortly, and a profinite 'Stallings theorem' on the lower central series quotients.

8.2.5 A profinite Stallings theorem.

As one would expect, the definition of the lower central series of a profinite group is obtained from that in the abstract case by taking the closure of the commutator subgroups. For completeness we give it in full:

Let G be a profinite group, and define subgroups G_i, $i = 0, 1, \ldots$ by the rules: $G_0 = G$, $G_{n+1} = [G, G_n]$ = closed subgroup generated by the relevant commutators $[x, y]$, $x \in G$, $y \in G_n$. The series $G_0 \supseteq G_1 \supseteq \ldots \supseteq G_n \supseteq \ldots$ is the *lower central series* of G. Note each G_n is a closed normal subgroup of G.

Proposition 53. *Let $g : G \to H$ be a continuous morphism of profinite groups such that g induces an isomorphism $g_* : G^{Ab} \to H^{Ab}$ and an epimorphism $g_* : H_2(G) \to H_2(H)$, then g induces isomorphisms*

$$g_n : G/G_n \to H/H_n, \qquad n \geq 0.$$

Proof: For $n \geq 2$, we have exact sequences

$$1 \to G_{n-1} \to G \to G/G_{n-1} \to 1$$

and

$$1 \to H_{n-1} \to H \to H/H_{n-1} \to 1.$$

These give us a commutative exact 'ladder' in homology

$$\begin{tikzcd}
H_2(G) \arrow[r] \arrow[d, "\beta_1"] & H_2(G/G_{n-1}) \arrow[r] \arrow[d, "\beta_2"] & G_{n-1}/[G, G_{n-1}] \arrow[r] \arrow[d, "\beta_3"] & G^{Ab} \arrow[r] \arrow[d, "\beta_4"] & (G/G_{n-1})^{Ab} \arrow[r] \arrow[d, "\beta_5"] & 0 \\
H_2(H) \arrow[r] & H_2(H/H_{n-1}) \arrow[r] & H_{n-1}/[H, H_{n-1}] \arrow[r] & H^{Ab} \arrow[r] & (H/H_{n-1})^{Ab} \arrow[r] & 0,
\end{tikzcd}$$

where the β_i are induced by g. We are given that β_4 is an isomorphism, whilst β_1 is an epimorphism. We note that the morphism

$$g_n : G/G_n \to H/H_n$$

is trivially an isomorphism for $n = 0$ and that the hypothesis of the statement of the result includes the case $n = 1$. We assume as induction hypothesis that g_{n-1} is an isomorphism. This implies that β_2 and β_5, in the 'ladder', are isomorphisms and, by the Five Lemma, so is β_3. We thus know that in the following diagram

$$\begin{tikzcd}
G_{n-1}/G_n \arrow[r] \arrow[d, "\beta_3"'] & G/G_n \arrow[r] \arrow[d, "g_n"'] & G/G_{n-1} \arrow[d, "g_{n-1}"'] \\
H_{n-1}/H_n \arrow[r] & H/H_n \arrow[r] & H/H_{n-1}
\end{tikzcd}$$

with exact rows, g_{n-1} is an isomorphism (by hypothesis), and β_3 is by the above, hence g_n is. ■

8.3 Non-Abelian tensor products of profinite groups

As mentioned earlier, the problem of extending the five-term exact sequence to the left is easily resolved if the extension involved is central. Ganea, [67], found a tensor product term that does the job. In general this suggests that one needs a non-Abelian tensor product. Such is provided in the abstract case via the theory of crossed squares and by Brown-Loday, [31], in their work on van Kampen theorems for cat^n-groups. The non-Abelian tensor product is not, however, limited in its usefulness to plugging a hole in an exact sequence! It provides a new neat description of H_2 of a group that completes that given by C. Miller, [118], and involves "universal commutator relations". It also gives a "Hopf-like" description of H_3. For perfect groups it provides insight into the structure of the universal central extensions. All of these aspects have their analogues in the profinite case. Most will be treated later in this chapter.

8.3.1 The non-Abelian tensor product

Let G and H be profinite groups and further suppose that G acts continuously on the left on H and H similarly acts continuously on the left of G, the two actions being compatible in the sense that for $g, g' \in G$, $h, h' \in H$,

$$^{(^g h)}g' = g(^h(g^{-1}g'g)) \qquad\qquad ^{(^h g)}h' = h(^g(h^{-1}h'h)).$$

Thus we are also considering G and H acting on themselves by conjugation. The profinite tensor product of G and H, written $G\hat{\otimes}H$ is the profinite group generated by the profinite space $G \times H$, a typical generator being written $(g \otimes h)$, on which G and H act diagonally and in which the relations

$$(1) \quad gg' \otimes h = {}^{g}(g' \otimes h)(g \otimes h)$$

$$(2) \quad g \otimes hh' = (g \otimes h)^{h}(g \otimes h')$$

are satisfied for all $g, g' \in G$, $h, h' \in H$.

8.3.2 Tensor relations, commutators and crossed pairings.

We note the similarity between these relations and the commutator relations within a group G,

$$[gg', h] = {}^{g}[g', h][g, h]$$

and

$$[g, hh'] = [g, h]^{h}[g, h'].$$

Of course ${}^{g}[g', h] = [{}^{g}g', {}^{g}h]$, so the action is of the same diagonal type as well. This observation will be useful later on.

If M, N are closed normal subgroups of G then there is a commutator map

$$[\ ,\ ] : M \times N \to G.$$

The remark above shows that this has certain good properties. In the abstract group case, Brown and Loday have extracted these properties to obtain the definition of a crossed pairing (see Brown-Loday, [31]).

Definition: If G, H and A are profinite groups with continuous left actions of G and H on themselves, each other and on A, a *continuous crossed pairing* is a continuous function

$$f : G \times H \to A$$

satisfying

(i) $f(gg', h) = {}^{g}f(g'h)f(g, h)$,

(ii) $f(g, hh') = f(g, h)^{h}f(g, h')$,

(iii) $f({}^{m}g, {}^{m}h) = {}^{m}f(g, h)$,

for any g, $g' \in G$, h, $h' \in H$, any elements m of G or H (or equivalently any $m \in G * H$, the free product of G and H).

Proposition 54. *The function*

$$\phi : G \times H \to G \hat{\otimes} H,$$

given by $\phi(g, h) = g \otimes h$, is a continuous crossed pairing.

Moreover, given any continuous crossed pairing

$$\phi' : G \times H \to A,$$

there is a unique continuous homomorphism $\theta : G \hat{\otimes} H \to A$ satisfying $\phi = \theta\phi$, (ϕ is a universal continuous crossed pairing).

Proof: As the inclusion of $G \times H$ into the free profinite group, $F(G \times H)$, is continuous and the quotient from that second group to $G \hat{\otimes} H$ is as well, the map ϕ is clearly continuous. The conditions for ϕ to be a crossed pairing are easily checked; for example

$$(i) \qquad \begin{aligned} \phi(gg', h) &= gg' \otimes h \\ &= {}^{g}(g' \otimes h).g \otimes h \\ &= {}^{g}\phi(g', h)\phi(g, h) \end{aligned}$$

by definition. The other two are just as easy.

Given $\phi' : G \times H \to A$, it is clear that if θ exists, it must satisfy $\theta(g \otimes h) = \phi'(g, h)$, so uniqueness is again easy. The existence of θ follows from the fact that the extension of ϕ' to $F(G \times H)$ must vanish on the relations of the given presentation of $G \hat{\otimes} H$. ■

8.4 Universal profinite central extensions

8.4.1 Universal profinite central extensions revisited

We have already briefly considered the existence of universal profinite central extensions in the previous chapter, p. 164. There we used cohomology and the Schur multiplier. This gives, for a perfect profinite group, G, with profinite presentation sequence

$$R \to F \to G,$$

the usual form

$$\frac{R \cap [F, F]}{[R, F]} \to \frac{[F, F]}{[R, F]} \to G$$

for the universal profinite central extension. (This theory when $H_2(G) = M(G)$ is finite can also be found in Wambsganß-Türk, [163]). For convenience we will briefly reformulate the basic definitions here.

An extension

$$1 \to N \xrightarrow{\alpha} G \xrightarrow{\beta} Q \to 1 \qquad \ldots \qquad (*)$$

of profinite groups is said to be *central* if the natural action of Q on N makes it a trivial pseudocompact Q-module. Of course, this happens exactly when $N \subset Z(G)$, the centre of G, since for any choice $t : Q \to G$ of transversal for β, the Q-action on N is given by

$$\alpha({}^{q}n) = t(q)\alpha(n)t(q)^{-1}.$$

A central profinite extension (*) of Q by N is said to be *universal* if given any central profinite extension

$$1 \to M \to H \to Q \to 1$$

of Q, there exists a unique continuous homomorphism from G to H (over Q) i.e. there exists a unique $\varphi : G \to H$ such that $\beta'\varphi = \beta$.

A central profinite extension (*) is *split* if there is a continuous homomorphism $s : Q \to G$ such that $\beta s =$ identity on Q, i.e., the epimorphism of the extension is split.

The results that follow are the profinite analogues of the results in the abstract case given by Milnor, [119]. The proofs are easy adaptations of those given there, but are included for convenience.

Proposition 55. *Let*

$$1 \to N \to G \to Q \to 1$$

and

$$1 \to M \to H \to Q \to 1$$

be two central profinite extensions and suppose that G is perfect profinite. Then there is at most one continuous homomorphism from G to H over G.

Proof: Suppose α_1 and α_2 are continuous homomorphisms from G to H over Q, then for g_1, $g_2 \in G$, we have

$$\alpha_1(g_1) = \alpha_2(g_1)m_1$$

and

$$\alpha_1(g_2) = \alpha_2(g_2)m_2,$$

where m_1, $m_2 \in M$ and hence are central in H.

Thus

$$\alpha_1([g_1, g_2]) = \alpha_2([g_1, g_2]) \qquad \text{for all } g_1, g_2 \in G,$$

but as $G = [G, G]$, this means $\alpha_1 = \alpha_2$. ■

Corollary 14. *If G is not perfect profinite in the above, there exists a profinite central extension*

$$1 \to M \to H \to Q \to 1$$

with at least two homomorphisms α_1, $\alpha_2 : G \to M$ over Q.

Proof: As G is not profinite perfect, there is a nonzero continuous f from G to some Abelian profinite group M. Take $H = M \times Q$ with $\beta'(q, m)$, for $q \in Q$, $m \in M$. We set

$$\alpha_1(g) = (\beta(g), 1), \qquad \alpha_2(g) = (\beta(g), f(g)).$$

These are clearly continuous and over Q. ■

Proposition 56. *If*

$$1 \to N \to G \to Q \to 1$$

is a central profinite extension of a perfect profinite group Q, then the closed commutator subgroup $G' = [G, G]$ of G is perfect and maps onto Q. Thus

$$1 \to N \cap G' \to G' \to Q \to 1$$

is a perfect profinite extension of Q.

Proof: As $Q = [Q,Q]$, it is clear that $\beta(G') = Q$. Thus every element $g \in G$ can be written as a product $g'a$ for $g' \in G$, $a \in N$. Therefore any (topological) generator $[g_1, g_2]$ can be rewritten $[g_1', g_2']$ for some g_1',$g_2' \in G'$. Thus $G' = [G', G']$ as required. ■

8.4.2 The link with tensor products.

We saw in an earlier section that given a perfect profinite group Q, and a profinite presentation sequence

$$1 \to R \to F \to Q \to 1$$

then the sequence

$$1 \to \frac{R \cap [F,F]}{[F,R]} \to \frac{[F,F]}{[F,R]} \to Q \to 1 \qquad \ldots \qquad (\dagger)$$

is a universal central extension. (The proof from Beyl and Tappe, [12], p.115, extends without difficulty.) We can see this directly now that we have a few more results at our disposal.

The extension

$$1 \to \frac{R}{[F,R]} \to \frac{F}{[F,R]} \xrightarrow{\theta} Q \to 1$$

will be central whatever Q we have, but, if Q is perfect profinite, then θ induces an epimorphism from $[F,F]/[R,F]$ to Q giving us the sequence ($\dagger$) above. It remains to prove universality.

Suppose

$$1 \to M \to H \xrightarrow{p} Q \to 1$$

is a central extension of Q, then there is a lifting to a morphism $f : F \to H$ over Q, i.e. $pf = \theta$, because F is free profinite. Furthermore, as the given extension is central $f([F,R]) = 1$ and f induces a morphism from $F/[F,R]$ to H over Q. Restricting this morphism to $[F,F]/[R,F]$ gives a morphism of sequences. Why is this morphism unique? By an earlier result, it is sufficient to prove that $[F,F]/[R,F]$ is perfect profinite, but this follows from the previous proposition.

Of course, the left-hand term of this universal profinite central extension is $M(Q)$, i.e. is isomorphic to $H_2(Q)$, the second homology group of Q. We thus have a construction of profinite central extensions when Q is profinite. It is easy to see that if

$$1 \to N \to G \to Q \to 1$$

is a universal profinite central extension, then G must be perfect profinite, but this implies Q must be perfect profinite, so it is only the perfect profinite groups that have universal profinite central extensions. It is worth noting that Q may be perfect profinite, without being perfect as an abstract group.

The disadvantage of the above "classical" style construction using a profinite presentation is that it suggests that the universal profinite central extension (u.p.c.e.) can only be constructed if one has already got a profinite presentation. The non-Abelian profinite tensor product gives a way of constructing this u.p.c.e. starting from G.

Proposition 57. *Let G be a perfect profinite group. Then the continuous map*

$$[\ ,\ ] : G\hat{\otimes}G \to G$$

is a universal extension of G.

Proof: We first look at continuity. The map, $[\ ,\ ]$, is defined on generators by $[\ ,\ ](g_1 \otimes g_2) = [g_1, g_2]$. This certainly gives a continuous map defined on $G \times G$ and thus on $F(G \times G)$. We have to check what happens to relations, but if g, h,$k \in G$,

$$[gh, k] = {}^g[h,l][g,k]$$

and

$$[g, hk] = [g,h]^h[g,k],$$

i.e., $[\ ,\ ] : G \times G \to G$ is a continuous crossed pairing, which thus induces $[\ ,\ ]$ defined on $G\hat{\otimes}G$.

Now suppose

$$1 \to N \to H \xrightarrow{f} G \to 1 \qquad (*)$$

is a central extension of G. We know (Corollary 2, p. 13) that f has a continuous transversal, $s : G \to H$. Using s, we define a mapping

$$\varphi : G \times G \to H$$

by $\varphi(g_1, g_2) = [s(g_1), s(g_2)]$. As (*) is central, φ is a crossed pairing. [To see this we check: First we note that for g_1, $g_2 \in G$, $s(g_1g_2)s(g_2)^{-1}s(g_1)^{-1} \in N \subseteq Z(H)$, so that $s(g_1g_2) = s(g_1)s(g_2)n$ for some $n \in Z(H)$ and hence

$$\begin{aligned}
\varphi(gg'g'') &= [s(gg'), s(g'')] \\
&= [s(g)s(g'), s(g'')] \\
&= {}^{s(g)}[s(g'), s(g'')][s(g), s(g'')] \\
&= {}^{s(g)}\varphi(g', g'')\varphi(g, g'').
\end{aligned}$$

It remains only to recall that as (*) is central, G acts continuously on H via ${}^gh = s(g)hs(g)^{-1}$ and the action is independent of the choice of s.]

We thus have a continuous homomorphism

$$\varphi : G\hat{\otimes}G \to H$$

such that $f\varphi = [\ ,\ ]$. In fact this φ is independent of the choice of s, as is easily seen. This however is not quite enough to check universality. For that we must check φ is unique with the property that $f\varphi = [\ ,\ ]$ and for this it is sufficient to check if $G\hat{\otimes}G$ is perfect.

Remembering that G is perfect profinite, we see that $G\hat{\otimes}G$ is generated by the elements $g \otimes g'$ and g, $g' \in [G,G]$ (because this is exactly G). Thus by the defining relations for $G\hat{\otimes}G$, it is generated by the space of elements of the form $g \otimes g'$ with g, g' simple commutators. However, if g_1, g_2, g_1', $g_2' \in G$, we have

$$[g_1 \otimes g_2, g_1' \otimes g_2] = [g_1, g_2] \otimes [g_1', g_2']$$

so $G\hat{\otimes}G$ is generated by its own commutators, i.e., it is perfect profinite. ■

Remark: In the abstract case, this proposition can be found in Dennis [45], as well as in Brown and Loday, [30].

8.4.3 Exterior product

The commutators $[g, g]$, $g \in G$ are, of course, all trivial. This observation leads one to study an exterior product $G\widehat{\wedge}G$ derived from $G\hat{\otimes}G$ as follows:

Definition: The *exterior square* or *exterior product* of G, denoted $G\widehat{\wedge}G$, is the profinite group topologically generated by symbols $g \wedge h$ for $(g, h) \in G \times G$ with the following relations,
(1) $gg' \wedge h = {}^{g}(g' \wedge h)(g \wedge h)$,
(2) $g \wedge hh' = (g \wedge h)^h(g \wedge h')$,
(3) $g \wedge h = 1$ if $g = h$.

Thus $G\widehat{\wedge}G$ is a quotient of $G\hat{\otimes}G$ by the symmetric elements $g \otimes g$. The commutator map

$$[\ ,\ ] : G\hat{\otimes}G \to G,$$

thus factors via a second continuous commutator map

$$[\ ,\ ]' : G\widehat{\wedge}G \to G.$$

Proposition 58. *If G is perfect profinite then*

$$[\ ,\ ]' : G\widehat{\wedge}G \to G$$

is the universal central extension of G.

Proof: Much as in the proof of Proposition 57, we construct for a central extension,

$$1 \to N \to H \xrightarrow{f} G \to 1 \quad \ldots \qquad (*),$$

a symmetric crossed pairing

$$\varphi : G \times G \to H$$

using a continuous transversal $s : G \to H$. Again as G is perfect profinite, so is $G\widehat{\wedge}G$, and the proof structure is more or less identical to that of Proposition 57. ■

Corollary 15. *If G is a perfect profinite group, then there is a natural isomorphism*

$$H_2G \cong Ker([\ ,\ ]' : G\widehat{\wedge}G \to G)$$

Proof: The homology group H_2G is isomorphic to the kernel of the universal central extension of G. ■

This produces an invariant description of H_2G. We will see an alternative approach to $G\widehat{\wedge}G$ in the next section in which again commutators will be very important.

8.4.4 Tensors and exterior products

Comparing the two constructions above, one immediately asks for the relation between $G\hat{\otimes}G$ and $G\widehat{\wedge}G$ in general. Clearly there is a quotient map from $G\hat{\otimes}G$ to $G\widehat{\wedge}G$. If G is perfect profinite, then we have that this quotient map is an isomorphism. We will examine the general question when we have looked at other questions and have introduced a generalisation of the exterior product.

Another problem is, of course, the relationship of this tensor product of (possibly non-Abelian) profinite groups, with the usual tensor product of Abelian profinite groups. The following proposition provides the necessary technical results to answer this question. (The abstract group analogue is in Brown-Loday, [31]). We use in this the profinite coproduct $G\hat{*}H$ of profinite groups G and H. We will study this in more detail in section 8.5.3.

Proposition 59. *Let G, H be groups equipped with compatible actions on each other.*
a) The profinite coproduct $G\hat{*}H$ acts continuously on $G\hat{\otimes}H$ so that

$${}^p(g\otimes h) = {}^pg\otimes {}^ph \qquad g\in G, h\in H, p\in G\hat{*}H.$$

b) There are continuous homomorphisms

$$\lambda : G\hat{\otimes}H \to G, \qquad \lambda' : G\hat{\otimes}H \to H$$

such that

$$\lambda(m\otimes n) = m^n m^{-1}, \quad \lambda'(m\otimes n) = {}^m n n^{-1}$$

c) The triples $(G\hat{\otimes}H, G, \lambda)$, $(G\hat{\otimes}H, H, \lambda')$ with the given actions are profinite crossed modules.
d) If $\ell\in G\hat{\otimes}H$, $g'\in G$, $h'\in H$, then

$$(\lambda\ell)\otimes h' = \ell({}^{h'}\ell)^{-1}, \qquad g'\otimes(\lambda'\ell) = ({}^{g'}\ell)\ell^{-1}.$$

e) The actions of G on $Ker\,\lambda'$ and H on $Ker\,\lambda$ are trivial.
f) If ℓ, $\ell'\in G\hat{\otimes}H$, then

$$[\ell,\ell'] = \lambda\ell\otimes\lambda'\ell'$$

and, in particular,

$$[g\otimes h, g'\otimes h'] = (g({}^hg)^{-1})\otimes(({}^{g'}h')h'^{-1}).$$

Proof: Quite a lot of this follows immediately from the defining relations for $G\hat{\otimes}H$ and the universal property. We note for future use that if $g,\ g' \in G$, $h,\ h' \in H$, then we can form hgh^{-1} and ghg^{-1} within $G\hat{*}H$ and $G\hat{*}H$ acts continuously on both G and H, furthermore:

$$\begin{aligned}{}^{({}^h g)}(g') &= {}^h g g' ({}^h g)^{-1}\\ &= {}^h(g\,{}^{h^{-1}}g' g^{-1})\\ &= {}^{hgh^{-1}}g',\end{aligned}$$

similarly ${}^{({}^g h)}(h') = {}^{ghg^{-1}}h'$.

The first result that may cause some difficulty is the Peiffer identity in c). Given the construction of $G\hat{\otimes}H$, it suffices to prove this on the generators. First we look at $gg' \otimes hh'$ and expand it in two different ways:

$$\begin{aligned}gg' \otimes hh' &= {}^g(g' \otimes hh')(g \otimes hh')\\ &= {}^g((g' \otimes h)^h(g' \otimes h'))(g \otimes h)^h(g' \otimes h')\\ &= {}^g(g' \otimes h)^{gh}(g' \otimes h')(g \otimes h)^h(g' \otimes h')\end{aligned}$$

and also

$$\begin{aligned}gg' \otimes hh' &= (gg' \otimes h)^h(gg' \otimes h')\\ &= {}^g(g' \otimes h)(g \otimes h)^{hg}(g' \otimes h')^h(g' \otimes h').\end{aligned}$$

On comparing these we get

$${}^{gh}(g' \otimes h')(g \otimes h) = (g \otimes h)^{hg}(g' \otimes h').$$

Now suppose $\ell = g \otimes h$, $\ell' = g' \otimes h'$, and we examine ${}^{\lambda\ell}\ell'$.

$$\begin{aligned}{}^{g^h g^{-1}}(g' \otimes h') &= {}^{g^h g^{-1}}g' \otimes {}^{g^h g^{-1}}h'\\ &= {}^{ghg^{-1}h^{-1}}g' \otimes {}^{ghg^{-1}h^{-1}}h' \qquad \text{from our earlier observation}\\ &= gh({}^{g^{-1}h^{-1}}g' \otimes {}^{g^{-1}h^{-1}}h')\\ &= (g \otimes h)^{hg}({}^{g^{-1}h^{-1}}g' \otimes {}^{g^{-1}h^{-1}}h')(g \otimes h)^{-1}\\ &= (g \otimes h)(g' \otimes h')(g \otimes h)^{-1}\end{aligned}$$

as required. The general case follows.

For d) again it is sufficient to look at the case $\ell = g \otimes h$. Then

$$\begin{aligned}\lambda\ell \otimes h' &= g^h g^{-1} \otimes h'\\ &= {}^g({}^h g^{-1} \otimes h')(g \otimes h')\\ &= {}^{gh}(g^{-1} \otimes h^{-1}h'h)(g \otimes h')\\ &= {}^g(g^{-1} \otimes h)^{-1g}(g^{-1} \otimes h'h)(g \otimes h')\end{aligned}$$

$$\begin{aligned}
&= (g \otimes h)^g (g^{-1} \otimes h')^{gh'} (g^{-1} \otimes h)(g \otimes h') \\
&= (g \otimes h)(g \otimes h')^{-1 gh'} (g^{-1} \otimes n)(g \otimes h') \\
&= (g \otimes h)^{h' g} (g^{-1} \otimes n) \\
&= (g \otimes h)^{h'} (g \otimes h)^{-1} = l({}^{h'} l)^{-1}
\end{aligned}$$

as required. The proof of the second formula is similar.

The proof of e) is now straightforward. For instance if $\lambda' \ell = 1$ then

$$^{m'}\ell \ell^{-1} = m' \otimes \lambda' \ell = m' \otimes 1 = 1,$$

so $^{m'}\ell = l$.

Finally if ℓ, $\ell' \in \mathbb{G} \otimes H$, then

$$\begin{aligned}
\lambda \ell \otimes \lambda' \ell' &= \ell^{\lambda' \ell'} \ell^{-1} \\
&= \ell \ell' \ell^{-1} \ell'^{-1} = [\ell, \ell']
\end{aligned}$$

by the Peiffer identity. ■

8.4.5 The case of the trivial action

Proposition 60. *If G acts trivially on H and H acts trivially on G then*

$$G \hat{\otimes} H = G^{Ab} \hat{\otimes} H^{Ab},$$

the ordinary pseudocompact tensor product of the corresponding Abelianisations.

Proof: If the actions are trivial, then both λ and λ' are trivial homomorphisms (by b) of Proposition 59). By f), $G \hat{\otimes} H$ is Abelian, and by e), the actions of G and H on $G \hat{\otimes} H$ are trivial. As this action is diagonal, any generator $gg'g^{-1} \otimes h = {}^g(g' \otimes h)$ must be congruent modulo the relations to the generator $g' \otimes h$ and similarly in the H-variable. Hence the natural morphisms from G to G^{Ab} and H to H^{ab} induce an isomorphism

$$G \hat{\otimes} H \to G^{Ab} \hat{\otimes} H^{Ab},$$

where, for the moment, the righthand term is the "non-Abelian" profinite tensor product of the groups G^{Ab} and H^{Ab}, i.e., we still have to check that this latter is the pseudocompact module tensor product, as they are differently defined. However, writing the defining relations for $G \hat{\otimes} H$ in an additive form reduces them to bilinearity conditions, so in fact there is nothing to worry about. ■

8.4.6 The commutator map

Another interesting consequence of the above results is the following.

Suppose that we take $G = H$ with conjugation as the action. Then $\lambda = \lambda' = [\ ,\ ] : G\hat{\otimes}G \to G$, the map that we considered earlier. Part d) above can therefore be rephrased in this special case.

Proposition 61. *The commutator map* $[\ ,\ ] : G\hat{\otimes}G \to G$ *together with the diagonal action of* G *by conjugation gives a profinite crossed module.*

Proof: This is an immediate corollary of the earlier result. ∎

There are several useful further consequences of this. Firstly we have that the kernel of this commutator map is central in $G\hat{\otimes}G$. We write $J_2(G) = Ker\,([\ ,\ ] : G\hat{\otimes}G \to G)$, and, for convenience, write $k = [\ ]$, then there is an exact sequence

$$1 \to J_2(G) \to G\hat{\otimes}G \to G \to G^{Ab} \to 1.$$

This is a crossed extension of G^{Ab} by $J_2(G)$. If G is perfect, we have already noted that $J_2(G)$ is $H_2(G)$ and this extension reduces to the universal central extension. In general, this group $J_2(G)$ will be useful in comparing $G\hat{\otimes}G$ and $G\widehat{\wedge}G$. We will shortly be able to provide a similar crossed extension of G^{Ab} by $H_2(G)$, but before that we will have to look at profinite analogues of Miller's results, [118], which we will do in section 8.5.

Another consequence of these propositions is that they provide part of a result yielding a second universal property for $G\hat{\otimes}H$. To handle this we need to introduce crossed squares and their analogue in the profinite case.

8.4.7 Crossed squares: an introduction

We saw earlier that (profinite) crossed modules were like (closed) normal subgroups except that the inclusion map is replaced by a homomorphism that need not be a monomorphism. We even saw that all (profinite) crossed modules are, up to isomorphism, obtainable by applying π_0 to a (profinite) simplicial "inclusion crossed module".

Given a pair of (closed) normal subgroups M, N of a (profinite) group G, we can form a square

$$\begin{tikzcd} M \cap N \arrow[r] \arrow[d] & N \arrow[d] \\ M \arrow[r] & G \end{tikzcd}$$

in which each morphism is a profinite inclusion crossed module and there is a commutator map

$$h : M \times N \to M \cap N$$

$$h(m, n) = [m, n].$$

This forms a *crossed square* of (profinite) groups. We will be dealing with crossed squares and their higher dimensional analogues later. Here we will give an interim definition of crossed squares. The notion is due to Guin-Walery and Loday [79]. This slightly shortened form of the definition is adapted from Brown-Loday [31].

A *profinite crossed square* (more correctly *crossed square of profinite groups*) is a commutative square of profinite groups and continuous homomorphisms

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"] & P \end{tikzcd}$$

together with continuous actions of the group P on L, M and N (and hence continuous actions of M on L and N via μ and of N on L and M via ν) and a continuous function $h : M \times N \to L$. This structure is to satisfy the following axioms:
(i) the maps λ , λ' preserve the actions of P, furthermore with the given continuous actions, the maps μ, ν and $\kappa = \mu\lambda = \mu'\lambda'$ are profinite crossed modules;
(ii) $\lambda h(m,n) = m^n m^{-1}$, $\lambda' h(m,n) = {}^m n n^{-1}$;
(iii) $h(\lambda\ell, n) = \ell^n \ell^{-1}$, $h(m, \lambda'\ell) = {}^m\ell\ell^{-1}$;
(iv) $h(mm', n) = {}^m h(m', n)h(m, n)$, $h(m, nn') = h(m, n)^n h(m, n')$;
(v) $h({}^p m, {}^p n) = {}^p h(m, n)$;
for all $\ell \in L$, $m, m' \in M$, $n, n' \in N$ and $p \in P$.

There is an evident notion of morphism of crossed squares and we obtain a category $Prof.Crs^2$, the *category of profinite crossed squares*. It clearly also exists in pro-$\mathcal{C}$ versions.

8.4.8 Examples

(a) Given any profinite simplicial group G and two closed simplicial normal subgroups M and N, the square

$$\begin{tikzcd} M \cap N \arrow[r] \arrow[d] & N \arrow[d] \\ M \arrow[r] & G \end{tikzcd}$$

with inclusions and with $h = [\ ,\ ] : M \times N \to G$ is a simplicial "inclusion crossed square" of profinite simplicial groups. Applying π_0 to the diagram gives a profinite crossed square and, as we will show later on, all profinite crossed squares arise in this way (up to isomorphism).

(b) The conditions on the h-map are exactly those of a continuous crossed pairing, so it should come as no surprise that given profinite crossed modules (M, P, μ), (N, P, ν), the square

$$\begin{array}{ccc} M\widehat{\otimes}N & \xrightarrow{\lambda} & M \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ N & \xrightarrow[\nu]{} & P \end{array}$$

together with the universal continuous crossed pairing

$$M \times N \to M\hat{\otimes}N$$

and the action of M on N and N on M via P, gives one a profinite crossed square.

c) Any profinite simplicial group G yields a profinite crossed square, $M(G,2)$ defined by

$$\begin{array}{ccc} \dfrac{NG_2}{d_0(Ker\, d_1 \cap Ker\, d_2)} & \longrightarrow & Ker\, d_1 \\ \downarrow & & \downarrow \\ Kerd_2 & \longrightarrow & G_1 \end{array}$$

for suitable maps (see later in section **??**, page **??**). This is, in fact, part of the construction that shows that all connected profinite 3-types are modelled by crossed squares.

8.4.9 Universal properties

The definition of a continuous morphism of profinite crossed squares is hopefully clear. It is a map of diagrams continuous at each corner, commuting with actions and the h-maps. Given this, the following proposition is not surprising and it gives a second interpretation of the universal property of $M\hat{\otimes}N$.

Proposition 62. *Let (M,P,μ), (N,P,ν) be profinite crossed modules and let*

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ N & \xrightarrow[\nu]{} & P \end{array} \quad \left(\text{simplified notation} \begin{pmatrix} L & M \\ N & P \end{pmatrix}\right)$$

be an arbitrary profinite crossed square extending the two given profinite crossed modules. Then there is a unique map of profinite crossed squares

$$\theta : \begin{pmatrix} M\widehat{\otimes}N & M \\ N & P \end{pmatrix} \to \begin{pmatrix} L & M \\ N & P \end{pmatrix}$$

which is the identity on M, N and P.

Proof: It is clear that $h : M \times N \to L$ is a continuous crossed pairing, hence induces a morphism from $M\hat{\otimes}N$ to L. The fact that this is a map of crossed squares is immediate from the properties of the h-map. Uniqueness follows from the uniqueness clause of the universal property of $\hat{\otimes}$ together with the fact that $\theta(h(m,n)) = \theta(m) \otimes \theta(n)$. ■

8.4.10 The universal property as a pushout

Brown and Loday [31], in the abstract case, found a beautifully elegant reformulation of this (their Proposition 2.15, p.318). Here is the profinite analogue. Their proof goes over without change, but we have included it for completeness.

Proposition 63. *The following diagram of inclusions of profinite crossed squares is a pushout in the category of profinite crossed squares*

$$\begin{array}{ccc}
\begin{pmatrix} 1 & 1 \\ 1 & P \end{pmatrix} & \longrightarrow & \begin{pmatrix} 1 & M \\ 1 & P \end{pmatrix} \\
\downarrow & & \downarrow \\
\begin{pmatrix} 1 & 1 \\ N & P \end{pmatrix} & \longrightarrow & \begin{pmatrix} M\widehat{\otimes}N & M \\ N & P \end{pmatrix}.
\end{array}$$

Proof: We will show the equivalence of this with the universal property of the previous proposition. First we note that there are two forgetful functors from $Prof.Crs^2$ to $Prof.CMod$ namely

$$U_1 \begin{pmatrix} L & M \\ N & P \end{pmatrix} = (M, P, \mu)$$

and

$$U_2 \begin{pmatrix} L & M \\ N & P \end{pmatrix} = (N, P, \nu).$$

These both have right adjoints, given by

$$G_1(M, P, \mu) = \begin{pmatrix} M & M \\ P & P \end{pmatrix},$$

resp.

$$G_2(N, P, \nu) = \begin{pmatrix} N & P \\ N & P \end{pmatrix}.$$

In the first of these the h-map is given by $h(m, p) = m({}^p m)^{-1}$, in the second by $h(p, n) = {}^p n n^{-1}$. Thus U_1 and U_2 preserve colimits and hence pushouts. (These functors also have left adjoints, so also preserve limits).

Using this we see that if the diagram

$$\begin{tikzcd} \begin{pmatrix} 1 & 1 \\ 1 & P \end{pmatrix} \arrow[r] \arrow[d] & \begin{pmatrix} 1 & M \\ 1 & P \end{pmatrix} \arrow[d] \\ \begin{pmatrix} 1 & 1 \\ N & P \end{pmatrix} \arrow[r] & \begin{pmatrix} L & M' \\ N' & P' \end{pmatrix}. \end{tikzcd}$$

is a pushout in $Prof.Crs^2$, then $M \cong M'$, $N \cong N'$, $P \cong P'$, but then there is a unique continuous morphism

$$\theta : \begin{pmatrix} M\widehat{\otimes}N & M \\ N & P \end{pmatrix} \to \begin{pmatrix} L & M \\ N & P \end{pmatrix}.$$

As this gives a commutative square of the same type as above, the universal property of pushouts gives that there is a φ such that $\theta\varphi$ is the identity. The tensor universal property then shows that $\varphi\theta$ is also the identity, so θ is an isomorphism as required. The proof that this pushout property implies the universal property of the completed tensor product is straightforward. ∎

We saw earlier that profinite crossed modules formed a category equivalent to that of profinite cat^1-groups, that is, category objects in $Prof.Grps$. The category $Prof.Crs^2$ is similarly equivalent to $Cat^2(Prof.Grps)$, i.e., double category objects in $Prof.Grps$ or profinite cat^2-groups (see later) . In this interpretation, the left and right adjoints to the forgetful functors U_1 and U_2 correspond to discrete category and "chaotic" or "indiscrete" category structures respectively.

We will return to profinite crossed squares and their higher order analogues later on.

8.5 Profinite Associated Groups

8.5.1 C. Miller and exterior products

The exterior product in the abstract case grew out of the work of Brown and Loday on generalised versions of the van Kampen theorem, [30, 31]. The interpretations we have given of the tensor product in the previous section are in terms of crossed squares, yet the exterior product was in essence known in the 1950s in the work of C. Miller, [118], in 1953. She considered a group that is, roughly speaking, the group of all relations satisfied by commutators of a group G. Her treatment proved several useful results which can be interpreted as giving results about exterior products. We will adapt her arguments to the profinite situation. Unlike much of the previous sections, the proofs in this section are not immediate analogues of the proofs in the abstract case. They need more care.

8.5.2 Profinite Associated Groups

Let G be as usual a profinite group and let $\langle G, G\rangle$ denote the free profinite group on symbols $\langle g, g'\rangle$ with $g, g' \in G$, thus $\langle G, G\rangle = F(G \times G)$, but we want the notation $\langle g, g'\rangle$ to mirror the commutator notation.

There is a continuous homomorphism

$$\phi : \langle G, G\rangle \to [G, G]$$

given by $\phi\langle g, g'\rangle = [g, g'] = gg'g^{-1}g'^{-1}$, thus ϕ is onto.

We set

$$Z(G) = \{z \in \langle G, G\rangle : \phi(z) = 1\},$$

the kernel of ϕ. Let $B(G)$ be the closed normal subgroup of $\langle G, G\rangle$ topologically generated by the following words of $\langle G, G\rangle$:

$$\begin{aligned}
w_1(g, g) &= \langle g, g\rangle && \text{for each } g \in G;\\
w_2(g, g') &= \langle g, g'\rangle\langle g', g\rangle && \text{for each pair } g, g' \in G;\\
w_3(g, g', x) &= \langle gg', x\rangle\langle g, x\rangle^{-1}\langle {}^g g', {}^g x\rangle^{-1} && \text{for each triple } g, g', x \in G;\\
w_{3'}(g, x, g') &= \langle g, xg'\rangle\langle {}^x g, {}^x g'\rangle^{-1}\langle g, x\rangle^{-1} && \text{for each triple } g, g', x \in G;\\
w_4(x, g, g') &= {}^x\langle g, g'\rangle\langle g, g'\rangle^{-1}\langle x, [g, g']\rangle^{-1} && \text{for each triple } x, g, g' \in G,
\end{aligned}$$

where $\quad {}^x\langle g, g'\rangle = \langle {}^x g, {}^x g'\rangle = \langle xgx^{-1}, xg'x^{-1}\rangle.$

Of course these correspond to the universal commutator relations:

$$\begin{array}{lrl}
1) & \langle g, g'\rangle \equiv 1 & \text{if } g = g',\\
2) & \langle g, g'\rangle^{-1} \equiv \langle g', g\rangle, & \\
3) & \langle gg', x\rangle \equiv \langle {}^g g', {}^g x\rangle\langle g, x\rangle, & \\
3') & \langle g, xg'\rangle \equiv \langle g, x\rangle\langle {}^x g, {}^x g'\rangle, & \\
4) & {}^x\langle g, g'\rangle \equiv \langle x, [g, g']\rangle\langle g, g'\rangle, &
\end{array}$$

for all g, g' and x in G and $\equiv$ is congruence in $\langle G, G\rangle$ mod $B(G)$.

As $B(G)$ is a closed normal subgroup of $Z(G)$, so we can form the *profinite associated group* $H(G)$ of G as the quotient

$$H(G) = Z(G)/B(G).$$

If $f : A \to A'$ is a continuous homomorphism of profinite groups, then f induces a continuous homomorphism,

$$\bar{f} : \langle A, A\rangle \to \langle A', A'\rangle$$

and $\bar{f}$ sends $Z(A)$ to $Z(A')$ and $B(A)$ into $B(A')$, thus f induces a continuous

$$f_* : H(A) \to H(A'),$$

giving a functor H defined on $Prof.Grps$.

8.5.3 Associated groups of profinite coproducts

The aim of the next few paragraphs will be to investigate the behaviour of H on profinite coproducts. First we will give the definition of these. (Ribes, [147], calls this a *profinite amalgamated product* and Binz-Neukirch-Wenzel, [13], use the term *free profinite product.*)

Definition: Let C_α, $\alpha \in I$, be profinite groups. The *profinite coproduct* of the $C_\alpha s$ consists of a system, $\{C, \{\phi_\alpha\}_{\alpha\in I}\}$, where C is a profinite group C and continuous *monomorphisms*

$$\phi_\alpha : C_\alpha \to C$$

with the usual universal property: given any profinite group A and continuous homomorphisms

$$\theta_\alpha : C_\alpha \to A \qquad \alpha \in I$$

then there is a unique continuous homomorphism $\theta : C \to A$ such that $\theta\phi_\alpha = \theta_\alpha$ for all $\alpha \in I$.

Note the restriction that the ϕ_α be monomorphisms. The subgroup $\bigcup_\alpha \phi_\alpha(C_\alpha)$ algebraically generates a dense subgroup of C and as each C_α is compact Hausdorff, each ϕ_α is topologically an embedding, (cf. Ribes [147]). We will write $C = \hat{*}_\alpha C_\alpha$ and if I contains just two elements, $C = C_1 \hat{*} C_2$ as before.

We are restricting attention to profinite groups here. There are clearly $pro-\mathcal{C}$ coproducts that can be defined, and Ribes shows in [147] that $pro-\mathcal{C}$ coproducts always exist provided $\mathcal{C}$ is the class of finite groups or finite p-groups; however strange things happen if one tries to form profinite coproducts with amalgamated subgroup (i.e., pushouts). Ribes, [146], gives examples, for $\mathcal{C}$ equal to 'finite groups' and for $\mathcal{C}$ being 'p-groups', where, although the pushout exists, the "inclusions" of the various cofactors are not monomorphisms. A detailed discussion of the problems met with in these cases can be found in [148], Chapter 9.

Proposition 64. *If $G = M \hat{*} N$ is the profinite coproduct of profinite groups M and N, then $H(G) \cong H(M) \times H(N)$.*

Proof: Let $i : M \to G$ and $j : N \to G$ be the natural continuous injections and let $\alpha : G \to M$, $\beta : G \to N$ be the natural continuous projections obtained functorially by the homomorphism $N \to 1$ inducing

$$\alpha : G = M \hat{*} N \to M \hat{*} 1 \cong M,$$

and similarly for β. This gives a commutative diagram

$$
\begin{tikzcd}
H(M) \arrow[dr, "i'"] \arrow[dd, "="'] & & H(N) \arrow[dl, "j'"'] \arrow[dd, "="] \\
& H(G) & \\
H(M) \arrow[ur, "\alpha'"] & & H(N) \arrow[ul, "\beta'"']
\end{tikzcd}
$$

This shows that i' and j' are embeddings and that $i'H(M)$ and $j'H(N)$ have intersection $\{1\}$. Moreover, if we can show that $H(G)$ is the profinite group product $i'H(M)j'H(N)$ then clearly we will have $H(G) = i'H(M) \times j'H(N)$ as required. Thus we are left to check that $i'H(M)$ and $j'H(N)$ together topologically generate $H(G)$.

We first let $G_1 = M * N$, the *algebraic* coproduct of the underlying groups M and N. The map from G_1 into G has dense image. Now if X is any profinite space and X_1 a dense subspace of X, then $F(X_1)$, the free profinite group on X_1, is isomorphic to $F(\hat{X}_1)$, the free profinite group on the profinite completion $\hat{X}_1$ of X_1 and there is an epimorphism

$$F(X_1) \to F(X)$$

induced by the continuous surjection from $\hat{X}_1$ to X. We will apply this with X_1 equal to $G_1 \times G_1$.

Abusing notation slightly, we will write $\langle G_1, G_1 \rangle$ for $F(G_1 \times G_1)$ and hence will get an epimorphism

$$\tau : \langle G_1, G_1 \rangle \to \langle G, G \rangle,$$

since the space $G_1 \times G_1$ topologically generates $\langle G, G \rangle$.

Now let $Z(G_1)$ be the kernel of the commutator map

$$\phi : \langle G_1, G_1 \rangle \to [G, G],$$

then the induced map from $Z(G_1)$ to $Z(G)$ is onto. Let $B(G_1)$ be generated by the same words as $B(G)$, but with g, g', x, etc. in G_1, then we can write $H(G_1) = Z(G_1)/B(G_1)$ and have a continuous epimorphism

$$H(G_1) \to H(G).$$

Writing $q : M \to G_1$, $p : N \to G_1$ for the natural injections, it will suffice to prove that $q'(H(M))$ and $p'(H(N))$ algebraically generate a dense subgroup of $H(G_1)$, since there is a commutative diagram

$$
\begin{tikzcd}
& H(G_1) & \\
H(M) \arrow[ur, "q'"] \arrow[dr, "i'"'] & & H(N) \arrow[ul, "p'"'] \arrow[dl, "j'"] \\
& H(G) &
\end{tikzcd}
$$

so the image under the epimorphism of the dense subgroup will be the subgroup algebraically generated by $i'(HM))$ and $j'(H(N))$, hence this latter subgroup will be dense.

Now if $\langle g, g'\rangle$ is a generator of $\langle G_1, G_1\rangle$, say with

$$g = x_1y_1x_2y_2\ldots x_ny_n \quad \text{and} \quad g' = x_1'y_1'x_2'y_2'\ldots x_m'y_m$$

with the $x_i, x_j' \in M$ and the $y_i, y_i' \in N$, then a repeated application of the universal commutator relations 3) and $3'$) above reduces $\langle g, g'\rangle$ modulo $B(G_1)$ to a product of symbols of the form ${}^z\langle x, x'\rangle, {}^z\langle y, y'\rangle, {}^z\langle x, y\rangle$ and ${}^z\langle y, x\rangle$ where $x, x' \in M$, $y, y' \in N$ and $z \in G_1$. Each element of this form can be written as a product of terms of the same type, but without the exponent z appearing, by repeated use of the simple rules:

$$
\begin{array}{ll}
5) & {}^{x_0}\langle x, x'\rangle = \langle {}^{x_0}x, {}^{x_0}x'\rangle, \\
4') & {}^{y_0}\langle x, x'\rangle = \langle {}^{y_0}x, {}^{y_0}x'\rangle \qquad \equiv \langle y_0, [x, x']\rangle\langle x, x'\rangle \mod B(G_1), \\
6) & {}^{x_0}\langle x, y\rangle \equiv \langle x_0x, y\rangle\langle y, x_0\rangle \mod B(G_1), \\
7) & {}^{y_0}\langle x, y\rangle \equiv \langle y_0, x\rangle\langle x, y_0y\rangle \mod B(G_1),
\end{array}
$$

and four similar rules, obtained by swapping the rôles of x and y, x_0 and y_0, etc. Each is a simple consequence of the basic commutator relations.

We thus note that each generator $\langle g, g'\rangle$ and hence any element k in a dense subgroup of $\langle G_1G_1\rangle$ is congruent $(\mathrm{mod} B(G_1))$ to a product of terms $\langle x, x'\rangle$, $\langle y, y'\rangle$, $\langle x, y\rangle$ and $\langle y, x\rangle$.

In fact for arbitrary such k,

$$k \equiv \eta \equiv \eta'\tau \mod B(G_1)$$

with τ a product of terms $\langle y, y'\rangle$ and η' a product of terms $\langle x, x'\rangle$, $\langle x, y\rangle$ and $\langle y, x\rangle$. We can do this because of the two derived moves

$$
\begin{array}{ll}
8) & \langle y, y'\rangle\langle x_0, y_0\rangle \equiv \langle [y, y'], x_0\rangle\langle x_0, [y, y']y_0\rangle\langle y, y'\rangle, \\
9) & \langle y, y'\rangle\langle y_0, x_0\rangle \equiv \langle [y, y']y_0, x_0\rangle\langle x_0, [y, y']\rangle\langle y, y'\rangle, \\
10) & \langle y, y'\rangle\langle x_2, x'\rangle = \langle y^{y'}y^{-1}, {}^x x^{-1}x'\rangle\langle x, x'\rangle\langle y, y'\rangle,
\end{array}
$$

which can be checked to be consequences of the basic relations (cf., Miller, [118], p.589). These moves have "duals" obtained by reversing the rôles of x and y; using these dual moves enables us to push all occurrences of $\langle x, x\rangle$ terms to the left.

This gives

$$k \equiv \gamma\eta''\tau$$

where η'' involves only terms $\langle x, y\rangle$ or $\langle y, x\rangle$ and γ a product of terms $\langle x, x'\rangle$. Since $\langle y, x\rangle = \langle x, y\rangle^{-1}$, we have

$$k \equiv \gamma\rho\tau \qquad \bmod B(G_1)$$

with $\gamma \in \langle qM, qM\rangle$, $\tau \in \langle pN, pN\rangle$ and $\rho \in D$, the subgroup generated by all symbols $\langle x, y\rangle$ with $x \neq 1 \in M$, $y \neq 1 \in N$.

Now if $\psi(k) = 1$ then writing $\bar{k} = \psi(k)$ etc., we get $\bar{\gamma}\bar{\rho}\bar{\tau} = \bar{k} = 1 \in [G, G] \subset G$. Projecting onto M yields $\bar{\gamma} = 1$, similarly $\bar{\tau} = 1$ and hence $\bar{\rho} = 1$. We have

$$\rho = \langle x_1, y_1\rangle^{\varepsilon_1}\langle x_2, y_2\rangle^{\varepsilon_2} \dots \langle x_m, y_m\rangle^{\varepsilon_m},$$

where $\varepsilon_2 = \pm 1$, $x_i \neq 1$, $y_i \neq 1$ for all i. We see that $\bar{\rho}$ can be written as a finite sequence of elements in G_1, but $G_1 = M * N$, the algebraic coproduct (free product) of M and N, so $\bar{\rho} = 1$ only if $m = 0$ and $\rho = 1$, as in the "abstract" case. Finally $\rho = 1$ implies

$$k \equiv \gamma\tau \text{ with } \bar{\gamma} = 1 \text{ and } \bar{\tau} = 1.$$

Thus $q'H(M)$ and $p'H(N)$ together algebraically generate a dense subgroup of $H(G_1)$, $H(G_1) = q'H(M)p'H(N)$ in $Prof.Grps$ and as the map $H(G_1) \to H(G)$ is onto, we get that the images of $H(M)$ and $H(N)$ generate $H(G)$ as required. ■

8.5.4 The generating relations of $B(G)$ as universal commutator relations.

The generating relations of $B(G)$ are designed to behave like "universal commutators". We list three more consequences that illustrate this.

a) $\langle g, x^{-1}\rangle \equiv {}^{g}\langle g^{-1}, x\rangle$,
b) $\langle g, x\rangle^{-1} = {}^{x}\langle g, x^{-1}\rangle$,
c) $\langle g, x\rangle\langle g', x'\rangle\langle g, x\rangle^{-1} \equiv {}^{[g,x]}\langle g', x'\rangle$,

These can easily be deduced from the earlier list of consequences of the relations (or some indications of proofs can be found in Miller, [118] p.589).

8.5.5 Exterior squares revisited

The correspondence

$$\langle g, g'\rangle \longleftrightarrow g \wedge g'$$

extends to give an isomorphism between $\langle G, G\rangle / B(G)$ and $G \widehat{\wedge} G$. This implies of course that $H(G) \cong Ker(G \widehat{\wedge} G \to G)$. There is a nice homotopical interpretation of this kernel (cf., Ellis, [54]). We briefly mentioned that crossed modules and crossed squares correspond to 2-types and 3-types of simplicial groups and hence to 3-types and 4-types of reduced simplicial sets. (Recall that an n-type is a truncated homotopy type, so there are no non-trivial homotopy groups above a certain level). We shall be looking in detail at this later, but here we need only note that given a (profinite) crossed square

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P, \end{tikzcd}$$

the third homotopy group of the corresponding 3-type is $Ker\lambda \cap Ker\lambda'$. For instance if (M, P, μ), (N, P, ν) are given, we can form profinite universal crossed squares

$$\begin{tikzcd} M\widehat{\otimes}N \arrow[r] \arrow[d] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P, \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M\widehat{\wedge}N \arrow[r] \arrow[d] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P, \end{tikzcd}$$

and obtain the profinite analogues of the groups Ellis denotes $\pi_3(M \otimes N)$ and $\pi_3(M \wedge N)$. In particular $\pi_3(G \wedge G) \cong H(G)$ corresponds to the π_3 of the crossed square

$$\begin{tikzcd} G\widehat{\wedge}G \arrow[r] \arrow[d] & G \arrow[d, "="] \\ G \arrow[r, "="'] & G, \end{tikzcd}$$

8.5.6 *H* of a free f.g. profinite group is trivial

Our next aim is to prove that H of a free profinite group is trivial. In the abstract case, Miller proves this (in [118]) by first noting that $H(F(1)) = 0$, where $F(1)$ is free on one generator. This step is easy also in the profinite case; one argues as follows:

Suppose $F(1) = \langle x : \quad \rangle$. We will examine $\langle x^r, x^s \rangle$ for $r, s \in \mathbb{Z}$.

(i) If $r = s = 1$, then $\langle x^r, x^s \rangle = \langle x, x \rangle \equiv 1$ (by 1)

(ii) If $r = 1$, $s = -1$ then $\langle x^r, x^s \rangle = \langle x, x^{-1} \rangle \equiv {}^{x^{-1}}\langle x, x \rangle \equiv 1$ (by b) above.

Similarly, if $r = 1$, and $s = 1$;

(iii) If $r > 0$ and $s = -t < 0$ then $\langle x^r, x^s \rangle = {}^{x^t}\langle x^r, x^t \rangle$, so we can reduce to examining $r, t > 0$ in this case.

Similarly, if $r < 0$ and $s > 0$

(iv) If $s = 0$, then since for any g',

$$\begin{aligned} \langle x^r, g' \rangle &= \langle x^r, 1g' \rangle \\ &\equiv \langle x^r, 1 \rangle \langle x^r, g' \rangle, \end{aligned}$$

we must have $\langle x^r, 1 \rangle \equiv 1$, and similarly $\langle 1, x^s \rangle \equiv 1$.

Finally, if $r, s > 1$, $\langle x^r, x^s\rangle \equiv \langle x^r, x\rangle\langle x^r, x^{s-1}\rangle$, and $\langle x^r, x\rangle \equiv \langle x^{r-1}, x\rangle\langle x, x\rangle$, so by induction $\langle x^r, x^s\rangle \equiv 1$ in this case as well, thus within $\langle F(1), F(1)\rangle$, a dense subset of the space of generators is within $B(F(1))$, so $F(1) \wedge F(1) = 1$ and $H(F(1)) = 1$ as a result.

If $F(n)$ is generated by a set of n elements, then $F(n) = F(n-1)\hat{*}F(1)$, so by a simple inductive argument, $H(F(n)) \cong 1$. As a consequence if F is a finitely generated free profinite group, then $H(F) = 1$.

Our next aim will be to extend this to arbitrary free profinite groups. We cannot here use the sort of argument used by Miller, [118], in the abstract group case, since profinite spaces are inverse not direct limits of finite sets, so we next turn to the interaction of H with inverse limits.

8.5.7 *H* and inverse limits.

The interaction of homology with inverse limits is usually much more subtle than that with direct limits. In the latter case the two constructions commute, but in the former one gets spectral sequences measuring the lack of commutation of the two constructions, see, for instance, Jensen, [94]. On looking at such spectral sequences, it is of note that many terms are of the form $Lim^{(i)}H_j$, i.e. they are obtained from the homology of an inverse system of modules or chain complexes, by applying the derived functors of limit. Although the H construction is clearly a homology (as it is constructed as $Z(G)/B(G)$), the theory of derived functors of the limit on non-Abelian groups is much less easy to use than that on modules. However for the moment if we pretended to use a spectral sequence argument, we would find that the H_js would be profinite or pseudocompact, hence that $Lim^{(i)}H_j$ would vanish if i was positive. Thus our supposed spectral sequence would collapse and the two operations would commute: $Lim\, H(G(i) = H(Lim\, G(i))$. Of course this is no argument, but it does suggest that a different, perhaps more direct, attack would be worth making.

Let, therefore, $G = Lim\, G(i)$ be a profinite group expressed as a limit of finite quotient groups. For every $G(i), i \in I$, we form $\langle G(i), G(i)\rangle$, a free profinite group on the set

$$\{\langle g_i, g_i'\rangle : g_i, g_i' \in G(i)\}$$

There are natural epimorphisms

$$\phi(i) : \langle G(i), (G(i)\rangle \to [G(i), G(i)]$$

as before and we write $Z(G(i)) = Ker\, \phi(i)$, hence obtaining an exact sequence of the groups

$$1 \to Z(G(i)) \to \langle G(i), G(i)\rangle \to [G(i), G(i)] \to 1$$

This is natural in $G(i)$ so forms a "pro exact sequence", that is an inverse system of exact sequences, of which we will take the limit,

$$1 \to Lim\, Z(G(i)) \to Lim\, \langle G(i), G(i) \rangle \to Lim\, [G(i), G(i)] \to 1$$

The resulting sequence is exact since the groups involved were all profinite.

Next we recall the result of Gildenhuys and Lim, [70], that $F(Lim\, X(i)) \cong Lim\, F(X_i)$, so $Lim\, \langle G(i), G(i) \rangle \cong \langle G, G \rangle$. As each $[G(i), G(i)]$ is within the corresponding $G(i)$ and limits of groups are calculated using a subgroup of the product, $\Pi G(i)$, with pointwise multiplication and inversion, it is easily checked that $[G, G] \cong Lim\, [G(i), G(i)]$, the isomorphism being compatible with that of $\langle G, G \rangle$ with $Lim\, \langle G(i), G(i) \rangle$. Thus $Z(G)$ is naturally isomorphic to $Lim\, Z(G(i))$.

A similar argument shows that $B(G) \cong Lim\, B(G(i))$ and so we obtain:

Proposition 65. *If $G \cong Lim\, G(i)$, then*

$$G \widehat{\wedge} G \cong Lim\, (G(i) \widehat{\wedge} G(i))$$

and

$$H(G) \cong Lim\, H(G(i)).$$

■

Corollary 16. *If F is a free profinite group, then the natural commutator map*

$$F \widehat{\wedge} F \to [F, F]$$

is an isomorphism.

Proof: Our calculations showed $H(F) \cong 1$, and $H(F)$ is the kernel of the commutator map. ■

8.5.8 Profinite tensor square of finite groups are finite.

Ellis, [54], shows that the tensor product of finite groups is finite. This suggests that if G is finite, $G \hat{\otimes} G$ and hence $G \hat{\wedge} G$ should be finite as well. To adapt his argument to prove this profinite analogue would lead us too far afield for the present, however we can note the following.

Proposition 66. *If G is profinite, and $G = Lim\, G(i)$, then*

$$G \hat{\otimes} G \cong Lim\, (G(i) \hat{\otimes} G(i)).$$

Proof: One can easily adapt the above proof for $\widehat{\wedge}$. (By omitting w_1 from the relations in $B(G)$, this gives a group isomorphic to $G \hat{\otimes} G$; now rerun the above discussion of H and inverse limits.) ■

8.6 Exterior products and Homology

8.6.1 Exterior squares and free profinite groups

In Corollary 57, we saw that if G is a perfect profinite group, then the kernel of the commutator map from $G\widehat{\wedge}G$ to G is isomorphic to H_2G. The restriction on G is, in fact, not necessary as we will see. To work towards this result, we need to see how the profinite exterior product interacts with presentations.

Proposition 67. *If F is a free profinite group and $i : R \to F$, the inclusion map of a closed normal free profinite subgroup, R, then the canonical continuous homomorphisms*

$$i\widehat{\wedge}1 : R\widehat{\wedge}R \to F\widehat{\wedge}R \quad \textit{and} \quad i\widehat{\wedge}i : R\widehat{\wedge}R \to F\widehat{\wedge}F$$

are normal inclusions.

Proof: (see Ellis, [54], for the abstract group case): Using the commutative squares

$$\begin{tikzcd} R\widehat{\wedge}R \arrow[r, "\simeq"] \arrow[d, "i\widehat{\wedge}1"'] & {[R,R]} \arrow[d] \\ F\widehat{\wedge}R \arrow[r] & {[F,R]} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R\widehat{\wedge}R \arrow[r, "\simeq"] \arrow[d, "i\widehat{\wedge}i"'] & {[R,R]} \arrow[d] \\ F\widehat{\wedge}F \arrow[r] & {[F,F]} \end{tikzcd} ,$$

one has that the two homomorphisms are monomorphisms. Normality follows from the fact that if $f \in F$, r, r', $r'' \in R$, then

$$(f \wedge r)(r' \wedge r'')(f \wedge r)^{-1} = ({}^{[f,r]}r' \wedge {}^{[f,r]}r'').$$

■

Remark: As the proof uses that

$$R\widehat{\wedge}R \to [R,R]$$

is an isomorphism (or at least a monomorphism), it is difficult to see how to avoid the restriction that R be free profinite. This returns us to the problem of calculating $H_2(N)$ if N is a closed normal subgroup of a free profinite group, F.

8.6.2 Exterior pairings

When we introduced the profinite tensor product, we also looked at continuous crossed pairings, but when we considered the exterior product no corresponding pairing was defined since at that stage we did not need it. It is now convenient to fill this gap.

Let M and N be closed normal subgroups of a profinite group G. Given another profinite group H, a *continuous exterior pairing* from $M \times N$ to H is a continuous map

$$h : M \times N \to H$$

such that, for all $m, m' \in M$, $n, n' \in N$,
1) $h(mm', n) = h({}^m m', {}^m n)h(m, n)$,
2) $h(m, nn') = h(m, n)h({}^n m, {}^n n')$,
3) $h(m, n) = 1, \quad$ whenever $m = n$.

The universal continuous exterior pairing is, of course, given by

$$M \times N \to M \widehat{\wedge} N,$$

$$(m, n) \mapsto m \wedge n.$$

It will later be important that the commutator map from $M \times N$ to $M \cap N$ is a continuous exterior pairing.

We note that to verify a property for continuous exterior pairings, it suffices to check it for the universal one. For instance, arguments for Proposition 59, page 191, used only the rules of the tensor product, hence these arguments apply equally well to the exterior product. We state the corresponding results below:

Proposition 68. *Let M and N be closed subgroups of a profinite group G.*
a) The group G acts continuously on $M \widehat{\wedge} N$ so that

$${}^g(m \wedge n) = {}^g m \wedge {}^g n \qquad g \in G, m \in M, n \in N.$$

b) There are continuous homomorphisms

$$\lambda : M \widehat{\wedge} N \to M, \qquad \lambda' : M \widehat{\wedge} N \to N$$

such that $\lambda(m \wedge n) = [m, n] = \lambda'(m \wedge n)$.
c) The triples $(M \widehat{\wedge} N, M, \lambda)$ and $(M \widehat{\wedge} N, N, \lambda')$ are profinite crossed modules with the given actions.
d) If $l \in M \widehat{\wedge} N, m \in M$ and $n \in N$, then

$$\lambda l \wedge n = l({}^n l)^{-1}, \qquad m \wedge \lambda' l = ({}^m l)l^{-1}$$

e) The actions of M on $Ker\lambda$ and of N on $Ker\lambda'$ are trivial.
f) If $l, l' \in M \widehat{\wedge} N$, then

$$[l, l'] = \lambda l \wedge \lambda' l'$$

and in particular $[m \wedge n, m' \wedge n'] = [m, n] \wedge [m', n']$.

Proof: As remarked above, one can easily adapt the proof of proposition 59 to give a proof here. ∎

Corollary 17. *Let M be a closed normal subgroup of a profinite group, G and let $h : G \times M \to H$ be a continuous exterior pairing, then for any $m, m' \in M, g, g' \in G$, the following equations hold:*

$$(A) \qquad h(m, [g, m']) = h(^m g, ^m m')h(g, m')$$

and

$$(B) \qquad [h(g, m), h(g', m')] = h([g, m], [g', m']).$$

Proof: For (A) we first note that as M is contained in G, the left hand term makes sense. In the universal example $h(g, m) = g \wedge m$ and $h(m, [g, m']) = m \wedge \lambda'(g, m')$, so this equation is then a consequence of d) in the proposition above. In general there is a continuous homomorphism $\varphi : G \wedge M \to H$ such that $h(g, m) = \varphi(g \wedge m)$, and the equation holds in H because φ preserves the equation in the universal example.

Similarly for (B), using f) above in the universal example gives the required form. ∎

8.6.3 Profinite presentations and exterior products.

Given a free profinite presentation

$$R \to F \xrightarrow{\alpha} G$$

of a profinite group, G, we can construct an exact sequence,

$$1 \to R \wedge R \to F \wedge R \to C \to 1$$

for some profinite group C. The next result is the first step to identifying C.

Proposition 69. *(cf. Ellis, [54]) Given a free profinite presentation, as above, of a profinite group, G, there is a continuous exterior pairing*

$$h : F \times R \to \hat{I}(G)\widehat{\otimes}_G R^{Ab}$$

with the following properties:
(i) $h(r, r') = 1$ for all pairs $(r, r') \in R \times R$, i.e., h vanishes on $R \times R$.
(ii) given any continuous exterior pairing $h' : F \times R \to H$ that vanishes on $R \times R$, there is a unique factorisation via

$$\psi : \hat{I}(G)\widehat{\otimes}_G R^{Ab} \to H$$

(i.e. $h' = \psi h$, and h is a "universal continuous pairing relative to R").

Proof: We define $h : F \times R \to \hat{I}(G)\widehat{\otimes}_G R^{Ab}$ by

$$h(f, r) = (\alpha f - 1) \otimes \underline{r}$$

for all $(f,r) \in F \times R$ and $\underline{r} \in R^{Ab}$ represented by r. The continuous map, h, is an exterior pairing, since, if $fu \in F$ and $r \in R$, we have:

$$\begin{aligned} h(fu,r) &= (\alpha(fu)-1) \otimes \underline{r} \\ &= (\alpha(fu)-\alpha(f)) \otimes \underline{r} + (\alpha(f)-1) \otimes r \\ &= (\alpha(fuf^{-1})-1) \otimes \underline{f}\underline{r}\underline{f}^{-1} + (\alpha(f)-1) \otimes \underline{r} \\ &= h({}^f u, {}^f r)h(f,r). \end{aligned}$$

Similarly for $f \in F$, $r, r' \in R$,

$$h(f, rr') = h(f,r)h({}^r f, {}^r r')$$

and for $r \in R$, as $\alpha(r) = 1$, $h(r,r) = 0$. Of course, this also shows that h vanishes on $R \times R$, so (i) is satisfied.

As h is a continuous exterior pairing, it induces a unique continuous homomorphism, φ, from $F \widehat{\wedge} R$ to $\hat{I}(G)\widehat{\otimes}_G R^{Ab}$, so that on denoting the natural universal continuous exterior pairing by

$$k : F \times R \to F \widehat{\wedge} R,$$

we have $h = \varphi k$. We also have an isomorphism

$$l : R \widehat{\wedge} R \to [R,R]$$

$$l(r \wedge r') = [r, r'],$$

since R is assumed to be free. We assemble these maps into the commutative diagram

$$\begin{tikzcd} & R \times R \arrow[r, "inc"] \arrow[d, "k"] & F \times R \arrow[r, "h"] \arrow[d, "k"] & \hat{I}(G)\widehat{\otimes} R^{Ab} \\ R\widehat{\wedge}R \arrow[r, "\cong"] & {[R,R]} \arrow[r, tail, "inc"] & F\widehat{\wedge}R \arrow[ur, "\phi"'] & \end{tikzcd}$$

[Statement (ii) of the proposition is thus equivalent to the identification of $\hat{I}(G)\hat{\otimes}R^{Ab}$ as the cokernel, C, in the exact sequence we constructed at the start of this section. Its universal property in terms of continuous exterior products is, however, easier to check, although, of course, equivalent to that of the cokernel.]

Suppose $h' : F \times R \to H$ is a continuous exterior pairing that vanishes on $R \times R$. We pick a continuous section $s : G \to F$ for α and another $t : R^{ab} \to R$ for the quotient homomorphism, $\beta : R \to R^{Ab}$. Define a map

$$\Psi : \hat{I}(G) \times R^{Ab} \to H$$

on generators by $\Psi(g-1, \underline{r}) = h'(s(g), t(\underline{r}))$. This is clearly continuous and as the set of elements of form $g-1$ topologically generates $\hat{I}(G)$, this defines

Ψ on the whole of $\hat{I}(G) \times R^{Ab}$. If we pick other sections s', t' then for $g \in G$, $(\underline{r}) \in R^{Ab}$, there are elements r', r'' with $r'' \in [R, R]$ such that

$$s'(g) = s(g)r', \qquad t'(\underline{r}) = t(\underline{r})r'',$$

thus we have a possibly different map, Ψ', given by

$$\Psi'(g - 1, \underline{r}) = h'(s'(g), t'(\underline{r})),$$

but

$$\begin{aligned} h'(s'(g), t'(\underline{r})) &= h'(s(g)r', t(\underline{r})r'') \\ &= h'({}^{s(g)}r', {}^{s(g)}t(\underline{r})r'')h'(s(g), t(\underline{r})r'') \\ &= h'(s(g), t(\underline{r})r'') \qquad\qquad \text{since } h'(R \times R) = 1 \\ &= h'(s(g), t(\underline{r}))h'({}^{t(r)}s(g), {}^{t(r)}r''). \end{aligned}$$

The term ${}^{t(r)}r'' \in [R, R]$, so this second term will vanish if $h'(f, -)$ vanishes on commutators, but

$$h'(f, [r, r']) = h'({}^f r, {}^f r')h'(r, r')^{-1},$$

which is trivial, since h' vanishes on $R \times R$. Thus $\Psi = \Psi'$ and Ψ is independent of the choice of s and t.

To define Ψ on the tensor, we really need to check it is bilinear. This initially looks awkward since H is an arbitrary profinite group, but the above equality shows that

$$[h'(m, n), h'(m', n')] = h'([m, n], [m', n']),$$

and if $m \in F$ and $n \in R$, both commutators are in R and h' vanishes on $R \times R$ so all commutators are trivial and the image of h' is Abelian, hence we may as well assume H is Abelian. It will also be expedient to rewrite the operation in H as addition. The exterior pairing rules now give

$$\begin{aligned} h'(mm', n) &= h'({}^m m', {}^m n) + h'(m, n), \\ h'(m, nn') &= h'(m, n) + h'({}^n m, {}^n n'), \\ h'(m, n) &= 0 \qquad\qquad \text{if } f = r. \end{aligned}$$

We can now look at $\Psi((g - 1)g', \underline{r})$; this is

$$\begin{aligned} \psi(gg' - g, \underline{r}) &= \psi(gg' - 1, \underline{r}) - \psi(g' - 1, \underline{r}) \\ &= h'(s(gg'), t(\underline{r}) - h'(s(g'), t(\underline{r})) \\ &= h'({}^{s(g)}s(g'), {}^{s(g)}t(\underline{r})) + h'(s(g), t(\underline{r})) - h'(s(g'), t(\underline{r})) \\ &= h'(s(g), [s(g'), t(\underline{r})]) + h'(s(g), t(\underline{r})) \end{aligned}$$

by (A) of our previous corollary. If Ψ is to be G-bilinear, we must hope that this is the same as $h'(s(g), {}^{s(g')}t(\underline{r}))$, but this latter is

$h'(s(g), [s(g'), t(\underline{r})]t(\underline{r})) = h'(s(g), [s(g'), t(\underline{r})]) + h'({}^{[s(g'),t(\underline{r})]}s(g), {}^{[s(g'),t(\underline{r})]}t(\underline{r})),$

which expression we will call (b), and

$$\alpha({}^{[s(g'),t(\underline{r})]}s(g)) = \alpha([s(g'), t(\underline{r})]s(g)[s(g'), t(\underline{r})]^{-1}),$$

so as $t(\underline{r}) \in R = Ker\,\alpha$, this is $s(g)$, whilst $\beta([[s(g), t(\underline{r})], t(\underline{r})]) = 1$, since the commutator is in R^{Ab}. Thus the second term of (b) is $h(s(g), t(\underline{r}))$ by the same argument as we used to prove independence of Ψ from the choice of s and t.

To sum up, we have shown Ψ to be G-bilinear and hence to define a uniquely determined $\psi : \hat{I}(G)\widehat{\otimes}_G R^{Ab} \to H$. It is clear that $h' = \psi h$ and that ψ is uniquely determined by this property. ■

Corollary 18. *Given a free profinite presentation*

$$1 \to R \to F \to G \to 1$$

of a profinite group, G, then there is a short exact sequence of profinite groups

$$1 \to R\widehat{\wedge}R \to F\widehat{\wedge}R \xrightarrow{\phi} \hat{I}(G)\widehat{\otimes}_G R^{Ab} \to 1.$$

■

This follows immediately from the previous result.

8.6.4 Preparation for Miller's theorem

We need one more general result about the profinite exterior product before we can complete the link-up with homology.

Suppose that K, M, N are closed normal subgroups of a profinite group G with $K \subset M \cap N$. Then there are induced homomorphisms

$$i_1 : K\widehat{\wedge}N \to M\widehat{\wedge}N \qquad\qquad i_2 : M\widehat{\wedge}K \to M\widehat{\wedge}N$$

given by $i_1(k \wedge n) = k \wedge n$, $i_2(m \wedge k) = m \wedge k$, with the obvious abuse of notation!

Lemma 18. *a) $(K\widehat{\wedge}N, M\widehat{\wedge}N, i_1)$ and $(M\widehat{\wedge}K, M\widehat{\wedge}N, i_2)$ are profinite crossed modules.*
b) The images $K_1 = Im\, i_1$, $K_2 = Im\, i_2$ are closed normal subgroups of $M\widehat{\wedge}N$.
c) There is a natural isomorphism:

$$(M/_K)\widehat{\wedge}(N/_K) \cong \frac{M\widehat{\wedge}N}{K_1K_2}.$$

Proof: a) The profinite group $M\widehat{\wedge}N$ acts continuously on $K\widehat{\wedge}N$ by conjugation:

$$^{m\wedge n}(k\wedge n') = {}^{[m,n]}k \wedge {}^{[m,n']}n',$$

but this makes the same sense if $k\wedge n$ is considered as an element of $K\widehat{\wedge}N$ or of $M\widehat{\wedge}N$. This then essentially says that i_1 preserves the action. The other crossed module axiom is also more or less trivial: if $m = k' \in K$ in the above, this same equation provides the necessary verification.

b) follows immediately from a).

c) Consider the mapping

$$h : M \times N \to M/_K \widehat{\wedge} N/_K$$

given by $h(m,n) = p(m)\wedge p(n)$, where $p : G \to G/_K$ is the natural epimorphism. We note that the usual generating set of $M/_K\widehat{\wedge}\, N/_K$ is contained in the image of h.

$$\begin{aligned} h(mm',n) &= p(m)p(m')\wedge p(n)\\ &= ({}^{p(m)}p(m') \wedge {}^{p(m)}p(n))(p(m)\wedge p(n))\\ &= h({}^{m}m', {}^{m}n)h(m,n), \end{aligned}$$

similarly for the other exterior pairing relations, so h induces a homomorphism,

$$\varphi : M\widehat{\wedge}N \to (M/_K)\widehat{\wedge}(N/_K),$$

which by our earlier comment is onto. Clearly $Ker\,\varphi$ contains both K_1 and K_2, hence also K_1K_2, their group product, i.e., φ induces an epimorphism

$$\bar{\varphi} : \frac{M\widehat{\wedge}N}{K_1K_2} \to (M/_K)\widehat{\wedge}(N/_K).$$

There is a map,

$$k : M/_K \times N/_K \to \frac{M\widehat{\wedge}N}{K_1K_2},$$

defined as follows: pick a continuous section $s : G/_K \to G$ and set $k(\bar{m},\bar{n}) = (s(\bar{m})\wedge s(\bar{n}))K_1K_2$. We check this does not depend on the choice of s. If s' is another continuous section, there are elements k', $k'' \in k$ such that $s'(\bar{m}) = s(\bar{m})k'$, $s'(\bar{n}) = s(\bar{n})k''$, so

$$s'(\bar{m})\wedge s'(\bar{n}) = s'(\bar{m})k' \wedge s'(\bar{n})k''$$

and a simple argument (which by now should be routine) shows this is congruent modulo K_1K_2 to $s(\bar{m})\wedge s(\bar{n})$, i.e., k is independent of the choice of s. Similarly

$$k(\bar{m}\bar{m}',\bar{n}) = (s(\bar{m}\bar{m}')\wedge s(\bar{n}))K_1K_2$$

and although $s(\bar{m}\bar{m}')$ need not be the same as $s(\bar{m})s(\bar{m}')$, they differ by an element of K and so again a, by now, routine argument gives that k is an exterior pairing.

If $\ell : M/_K \times N/_K \to H$ is any continuous exterior pairing, we can define a homomorphism

$$\psi : \frac{M\widehat{\wedge}N}{K_1K_2} \to H$$

by $\psi(m\wedge n)K_1K_2) = \ell(p(m), p(n))$. This is well defined and satisfies $\psi k = \ell$ (a routine verification left to the reader). The uniqueness of ψ with this property is easily checked, so in fact $\bar{\varphi}$ is an isomorphism as required. ∎

Proposition 70. *Suppose we have a presentation sequence*

$$1 \to R \to F \to G \to 1$$

of a profinite group G, then

$$G\widehat{\wedge}G \cong [F, F]/[F, R].$$

Proof: This is really a corollary of the preceding lemma. (Note we do not need to impose the restriction that R be free profinite.) We note that as $G \cong F/R$, $G\widehat{\wedge}G \cong (F/R)\widehat{\wedge}(F/R)$, so if, in that result, we take $M = N = F$, $K = R$, then

$$Im(R\widehat{\wedge}F \to F\widehat{\wedge}F) = Im(F\widehat{\wedge}R \to F\widehat{\wedge}F) = [F, R],$$

so we get $G\widehat{\wedge}G \cong [F, F]/[F, R]$, as required. ∎

8.6.5 Miller's Theorem

Suppose we have, as above, a presentation sequence

$$1 \to R \to F \to G \to 1$$

of a profinite group G, the previous result gives

$$Ker([\quad,\quad]' : G\widehat{\wedge}G \to G)$$

is isomorphic to $Ker([F, F]/[F, R] \to G)$, i.e., we have:

Theorem 10. *(cf. Miller, [118]) There is a natural isomorphism*

$$H_2(G) \cong Ker(G\widehat{\wedge}G \to G).$$

Proof: The result follows from the profinite Hopf formula. ∎

8.6.6 $H_3(G)$?

If
$$1 \to R \to F \to G \to 1$$
is a presentation sequence for a profinite group G, then the exact sequence (see chapter 7):
$$0 \to R^{Ab} \to \hat{\mathbb{Z}}[\![G]\!]\widehat{\otimes}_F \hat{I}(F) \to \hat{I}(G) \to 0$$
has as its middle term, a free pseudocompact G-module on the space of generators of F. This then gives us a partial resolution of $\hat{\mathbb{Z}}$ as a pseudocompact G-module, namely

$$\begin{tikzcd}
 & \hat{\mathbb{Z}}[\![G]\!]\widehat{\otimes}_F \hat{I}(F) \arrow[rr] \arrow[dr] & & \hat{\mathbb{Z}}[\![G]\!] \arrow[r] & \hat{\mathbb{Z}} \\
R^{Ab} \arrow[ur] & & \hat{I}(G) \arrow[ur] & &
\end{tikzcd}$$

In an earlier chapter, we have already used the bottom part of this to note that
$$\mathrm{Tor}^G_{i-1}(M, \hat{I}(G)) \cong H_i(G, M) \quad \text{if } i \geq 2$$
and
$$0 \to H_1(G, M) \to M\widehat{\otimes}_G \hat{I}(G) \xrightarrow{\delta} M\widehat{\otimes}_G \hat{\mathbb{Z}}[\![G]\!] \to M\widehat{\otimes}_G \hat{\mathbb{Z}}$$
is exact. Here we need rather to work with resolutions of the coefficient module M, which in our application will be $\hat{\mathbb{Z}}$, giving formulae for $H_i(G, \hat{\mathbb{Z}})$, i.e., $H_i(G)$.

The exact sequence
$$0 \to \hat{I}(G) \to \hat{\mathbb{Z}}[\![G]\!] \to \hat{\mathbb{Z}} \to 0$$
together with the fact that $H_i(G, \hat{\mathbb{Z}}[\![G]\!]) = 0$, $\hat{\mathbb{Z}}[\![G]\!]$ being, of course, free, gives us that
$$H_i(G) \cong H_{i-1}(G, \hat{I}(G)) \text{ for } i \geq 2.$$
Using a similar argument with the exact sequence
$$0 \to R^{Ab} \to \hat{\mathbb{Z}}[\![G]\!]\widehat{\otimes}_F \hat{I}(F) \to \hat{I}(G) \to 0$$
gives
$$H_j(G, \hat{I}(G)) \cong H_{j-1}(G, R^{Ab}) \text{ for } j \geq 2,$$
which together give us
$$H_i(G) \cong H_{i-2}(G, R^{Ab}) \text{ for } i \geq 3.$$
In particular we have

$$H_3(G) \cong H_1(G, R^{Ab}).$$

We also know from the earlier exact sequence that

$$H_1(G, R^{Ab}) = Ker(\delta : R^{Ab}\hat{\otimes}\hat{I}(G) \to R^{Ab}).$$

This homomorphism δ is induced by the inclusion of $\hat{I}(G)$ into $\hat{\mathbb{Z}}[\![G]\!]$ followed by the identification of $\hat{\mathbb{Z}}[\![G]\!]\widehat{\otimes}_G R^{Ab}$ with R^{Ab} itself. Thus $\delta(g-1)\otimes\bar{r}$ is, in the notation used earlier, $\beta(s(g)t(\bar{r})s(g)^{-1})$ and $Im\,\delta$ is $[F,R]/[R,R]$.

Notation: At this point it is helpful to introduce the notation $\pi_3(F\widehat{\wedge}R)$ for $Ker(F\widehat{\wedge}R \to [F,R])$. The reason for the π_3 is that this corresponds to the π_3 of the homotopy type represented by the exterior product crossed square (see Ellis, [54], and later results here). Following Ellis, [54] in the abstract group case, gives us a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & \pi_3(F\widehat{\wedge}R) \arrow[d] & H_3(G) \arrow[d] & \\
1 \arrow[r] & R\widehat{\wedge}R \arrow[r] \arrow[d] & F\widehat{\wedge}R \arrow[r] \arrow[d] & \hat{I}(G)\widehat{\otimes}R^{Ab} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & {[R,R]} \arrow[r] & {[F,R]} \arrow[r] \arrow[d] & {[F,R]/[R,R]} \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}$$

but this implies that $\pi_3(F\widehat{\wedge}R)$ and $H_3(G)$ are isomorphic, provided that R is free profinite, so as to guarantee that the left hand arrow is an isomorphism. We are thus ready to prove the profinite version of Brown and Loday's result giving an analogue for $H_3(G)$ of the Hopf-formula for $H_2(G)$.

Proposition 71. *Given any profinite group G, let*

$$1 \to R \to F \to G \to 1$$

be the profinite presentation of G given by $F = F(G)$, the free profinite group on the underlying space of G, then there is an isomorphism of pseudocompact modules,

$$H_3(G) \xrightarrow{\cong} Ker(F\widehat{\wedge}R \to R).$$

Proof: We note that we have the result from the above discussion, provided that the presentation is free, i.e., that R is free and thus in particular if G is finite.

We also know that if $G = Lim\, G_i$,

$$H_3(G) \cong Lim\, H_3(G_i)$$

by the result of Brumer, [34] p.455, mentioned earlier in this chapter. It therefore remains to prove the analogous result for $\pi_3(F\widehat{\wedge}R)$.

Assuming the G_i are finite, by the result of Gildenhuys and Lim, [70], used earlier, $F \cong Lim\, F(G_i)$. Thus our situation is that, at each level, we have a presentation

$$1 \to R_i \to F_i \to G_i \to 1,$$

so on taking the limit we get

$$R \cong Lim\, R_i,$$

and each R_i is free on finitely many generators. Adapting an earlier argument we have:

$$F\widehat{\wedge}R \cong Lim\,(F_i\widehat{\wedge}R_i),$$

where for convenience we have written F_i for $F(G_i)$, and $H_3(G) \cong Ker\,(F\widehat{\wedge}R \to R)$, as required. ■

Remark: The problem of showing that the above is independent of the presentation, i.e., given *any* profinite presentation of G, the analogous statement holds, seems to depend on techniques that seem tricky. These are related to problems involving analogues of Tietsze's theorem in the profinite case:
given two profinite presentations of a profinite group, are they "homotopic" in any reasonable sense and is such a homotopy realisable by some "profinite Tietsze transformations"?
This may be resolvable using crossed module / simplicial techniques, but it is not at all certain how. The result that follows, together with earlier remarks on H_2 of non-free normal subgroups of free profinite groups may contribute to the understanding of this area. It is the analogue for profinite groups of a result of Ellis and Porter, [57], in the abstract case.

8.6.7 The crossed module analogue of the Hopf formula.

In chapter 3, we noted the existence of two exact sequences associated to a free profinite crossed module, (C, G, ∂), on a continuous function $f : X \to G$. We recall the notation:

$E = F(G \times X)$, the free profinite group on $G \times X$;

$P =$ the Peiffer subgroup of E, that is, the closed normal subgroup generated by the Pieffer elements

$$uvu^{-1}({}^{\theta u}v)^{-1},$$

where $\theta(g, x) = gf(x)G^{-1}$;

$C = E/P$;

$N = Im(\partial : C \to G)$;
$I = \mathrm{Ker}(E \to N)$;
The two exact sequences are then

$$1 \to P \to E \to C \to 1, \qquad (a)$$

and

$$1 \to I \to E \to N \to 1. \qquad (b)$$

Proposition 72. *Let $\partial : C \to G$ be a projective profinite crossed module with $\partial C = N$, say. Then*

$$H_2(N) \cong Ker\,\partial \cap [C, C].$$

Proof: By the Hopf formula, and the profinite presentation (b),

$$H_2(N) \cong \frac{I \cap [E, E]}{[E, I]}.$$

Now let $\partial' : C' \to G$ be the free profinite crossed module on $\partial : C \to G$ (considered as a continuous function). This gives an epimorphism

$$\begin{tikzcd} C' \arrow[rr, "\psi"] \arrow[dr, "\partial'"'] & & C \arrow[dl, "\partial"] \\ & G & \end{tikzcd}$$

and as (C, G, ∂) is assumed projective, ψ splits, but then as ψ is a crossed module map, it is itself a profinite crossed module and $Ker\,\psi$ is central. This implies that ψ induces an isomorphism

$$[C', C'] \xrightarrow{\cong} [C, C].$$

We are therefore reduced to proving the proposition for free profinite crossed modules.

We have from Proposition 28, that $\partial : C \to N$ is free profinite and epimorphic, hence using (b) again and Proposition 27, we get

$$C \cong \frac{E}{[E, I]},$$

so

$$[C, C] \cong \frac{[E, E]}{[E, I]},$$

with ∂ being induced by the projection of E onto N.

Finally we note that

$$\mathrm{Ker}\partial \cap [C, C] \cong \frac{I \cap [E, E]}{[E, I]},$$

which completes the proof. ■

Remark: For the proof we only used the fact that (b) was a profinite presentation, not any details of how it was constructed.

8.6.8 Ratcliffe's lemma.

In [145], Ratcliffe proved a lemma (his lemma 2.1) by manipulation of commutators. As was noted by Ellis and Porter in [57] for the abstract case, this result on commutators can be elegantly proved using arguments similar to the above.

Proposition 73. *(Ratcliffe [145] 2.1 for the abstract version). Given a continuous function $f : X \to G$ with G a profinite group, and (C, G, ∂), the free profinite crossed module on f, then using the notation as above:*

$$[E, I] \cong P \cap [E, E]$$

Proof: Since $C \cong E/P$, $[C, C] = [E, E]/(P \cap [E, E])$, but we noted earlier that $[C, C] = [E, E]/[E, I]$, and the result follows. ■

The interest of these two results is twofold. We have noted that if $\mathcal{P} = (X : R)$ is a profinite presentation of a profinite group G, the identity module $\kappa(\mathcal{P})$ is the kernel of $\partial : C(\mathcal{P}) \to F(X)$ (cf., section 4.2.1). In the abstract case, $H_2(Im\,\partial) = 0$, since the Nielsen-Schreier theorem implies $Im\,\partial$ is free; in the profinite case

$$H_2(Im\,\partial) = \kappa(\mathcal{P}) \cap [C(\mathcal{P}), C(\mathcal{P})],$$

and will be present in all $\kappa(\mathcal{P})$ regardless of what $\mathcal{P}$ is. In the abstract case, one can interpret this in terms of "Y-sequences" and "Peiffer sequences", cf., Brown-Huebschmann, [29]. It would be interesting to see if any profinite analogue of this was possible.

8.7 An eight term exact sequence in profinite homology

We are now able to fulfil our promise to extend the Stallings exact sequence some terms to the left. In the abstract case, this particular formulation was first developed by Brown and Loday, [31], using topological methods. We will follow the treatment in Ellis, [54], again adapting in the obvious way to the profinite case.

8.7.1 G and two closed normal subgroups.

Given a profinite group G and two closed normal subgroups M, N, we have induced maps

$$\begin{pmatrix} M\widehat{\wedge}N \\ \downarrow \\ M \cap N \end{pmatrix} \xrightarrow{\alpha} \begin{pmatrix} G\widehat{\wedge}G \\ \downarrow \\ G \end{pmatrix} \to \begin{pmatrix} G/M\widehat{\wedge}G/M \\ \downarrow \\ G/M \end{pmatrix} \times \begin{pmatrix} G/N\widehat{\wedge}G/N \\ \downarrow \\ G/N \end{pmatrix}$$

If H is any profinite group with H_0, H_1 closed normal subgroups, let $\langle H_0, H_1\rangle_H$ be the subgroup of $H\widehat{\wedge}H$ generated by the elements $h_0 \wedge h_1$ with $h_0 \in H_0$, $h_1 \in H_1$, thus the image of the top part of α, α_1, is $\langle M, N\rangle_G$, there are isomorphisms,

$$G/_M\widehat{\wedge}G/_M \cong (G\widehat{\wedge}G)/_{\langle M, G\rangle_G}$$

and

$$G/_N\widehat{\wedge}G/_N \cong (G\widehat{\wedge}G)/_{\langle N, G\rangle_G}$$

and the images of $M\widehat{\wedge}N \to M \cap N$ and $\langle M, N\rangle_G \to G$ are both $[M, N]$. The final ingredient that we will need is that given any M, N, G, as here,

$$1 \to M \cap N \xrightarrow{\alpha} G \xrightarrow{\beta} G/_M \times G/_N \to 1$$

is exact provided $G = MN$, the group product of the normal subgroups. (The only non-obvious thing here is that β is onto, but the fact that any $g \in G$ can be written as a limit of elements of the form mn ensures that it is.

Proposition 74. *Given a profinite group G and two closed normal subgroups so that $G = MN$, then there is an exact sequence*

$$\pi_3(M\widehat{\wedge}N) \to \pi_3(G\widehat{\wedge}G) \to \pi_3(G/_M\widehat{\wedge}\, G/_M) \times \pi_3(G/_N\widehat{\wedge}\, G/_N)$$
$$\to \{M \cap N\}/_{[M, N]} \to G^{Ab} \to (G/_M)^{Ab} \times (G/_N)^{Ab} \to 1$$

Proof: We first note the group theoretic "snake lemma":

Lemma 19. *Given a commutative diagram*

$$\begin{tikzcd}
1 \arrow[r] & A_1 \arrow[r] \arrow[d, "\alpha"] & B_1 \arrow[r] \arrow[d, "\beta"] & C_1 \arrow[r] \arrow[d, "\gamma"] & 1 \\
1 \arrow[r] & A_0 \arrow[r] & B_0 \arrow[r] & C_0 \arrow[r] & 1
\end{tikzcd}$$

of (profinite) groups and (continuous) homomorphisms, where the rows are exact and the images of α, β, γ are normal in the respective codomains, then there is a six-term exact sequence

$$1 \to Ker\,\alpha \to Ker\,\beta \to Ker\,\gamma \xrightarrow{\delta} Coker\,\alpha \to Coker\,\beta \to Coker\,\gamma \to 1,$$

where the maps are either induced from those of the original diagram or are defined in the usual way (for δ). The homomorphism δ is continuous.

Proof of lemma: This is standard except for continuity of δ which follows if one picks a continuous section of the epimorphism from B_1 to C_1. ■

Return to the main Proof: Consider the diagram above and replace

$$M\widehat{\wedge}N \to G\widehat{\wedge}G$$

by $\langle M,N\rangle_G \to G\widehat{\wedge}G$. We note there is an epimorphism from $M\widehat{\wedge}N$ to $\langle M,N\rangle_G$ compatible with the commutator maps so that $\pi_3(M\widehat{\wedge}N)$ maps down onto

$$Ker(\langle M,N\rangle_G \to G).$$

Now the result follows from the snake lemma. ■

8.7.2 A "Mayer-Vietoris type" sequence and a possible identification of $\pi_3(M\widehat{\wedge}N)$?

We note that given the earlier results on $H_2(G)$ and $\pi_3(G\widehat{\wedge}G)$, the above sequence is

$$\pi_3(M\widehat{\wedge}N) \to H_2(G) \to H_2(G/M)\oplus H_2(G/N) \to (M\cap N)/[M,N]$$

$$\to H_1(G) \to H_1(G/M)\oplus H_1(G/N) \to 0$$

If $G = G_1\hat{*}G_2$ the amalgamated free product of G_1 and G_2 over A, then (provided the difficulties noted by Ribes, [147], do not occur in this example), taking M to be the image of G_1 and N to be that of G_2, $M\cap N = A$ and $[M,N] \cong [A,A]$, so the term $(M\cap N)/[M,N]$ is $H_1(A)$. The identification of $\pi_3(M\widehat{\wedge}N)$ as $H_2(A)$ is then probable if still conjectural.

8.7.3 A special case.

If $G = N$, the condition that $G = MN$ is clearly satisfied and we get the five-term exact sequence yet again, but this time with an extra term $\pi_3(G\widehat{\wedge}M)$ on the left hand end. Before we discuss this term by itself, we will show how it is involved in another exact sequence.

We have $M \lhd G$ and so have an exact sequence

$$1 \to M \to G \xrightarrow{\beta} Q \to 1,$$

where $Q = G/_M$. Pick a profinite presentation sequence,

$$1 \to R \to F \xrightarrow{p} G \to 1,$$

for G and let $S = Ker(F \xrightarrow{\beta p} Q)$, then

$$1 \to R \to S \to M \to 1$$

and

$$1 \to S \to F \to Q \to 1$$

are exact, (but beware, given the problem with subgroups of free profinite groups, we cannot assume S is free). Next note that

$$\begin{tikzcd}
 & F\widehat{\wedge}R \arrow[r] \arrow[d] & F\widehat{\wedge}S \arrow[r] \arrow[d] & G\widehat{\wedge}M \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & R \arrow[r] & S \arrow[r] & M \arrow[r] & 1
\end{tikzcd}$$

is exact, since $G\hat{\wedge}M \cong (F/_{R})\widehat{\wedge}(F/_{S})$, which in turn can be identified with $(F\widehat{\wedge}S)/\langle F, R\rangle_F$. Now using the snake lemma on this, we get

Proposition 75. *In the above situation there is an exact sequence, (A),*

$$\pi_3(F\widehat{\wedge}R) \to \pi_3(F\widehat{\wedge}S) \to \pi_3(G\widehat{\wedge}M)$$

$$\xrightarrow{\partial} \frac{R}{[F,R]} \to \frac{S}{[F,S]} \to \frac{M}{[G,M]} \to 0$$

of Abelian profinite groups. ■

The special case of the "Mayer-Vietoris" sequence for $G = N$ gave us an exact sequence, (B),

$$\pi_3(G\widehat{\wedge}M) \xrightarrow{\theta} H_2(G) \to H_2(Q) \to \frac{M}{[G,M]} \to H_1(G) \to H_1(Q) \to 0$$

This suggests that somehow the two sequences are related. Of course $H_2(G) \cong (R \cap [F,F])/[F,R] \subset R/[F,R]$, similarly for $H_2(Q)$. The map from $\pi_3(G\widehat{\wedge}M)$ to $R/[F,R]$ is the continuous boundary link coming from the snake lemma, and so we can give it an explicit description:
If $x \in \pi_3(G\widehat{\wedge}M)$, there is some $y \in F\widehat{\wedge}S$ mapping to x and the image of y in S is in R, of course, $\partial(x) = y[F,R]$. Again, of course, the image of the map from $F\hat{\wedge}S$ to S is $[F,S]$, so

$$\partial(x) \in \frac{R \cap [F,F]}{[F,R]}.$$

In sequence (B), the map θ is induced by the inclusion of $G\widehat{\wedge}M$ into $G\widehat{\wedge}G$. If $x \in \pi_3(G\widehat{\wedge}M)$, then for $y \in F\widehat{\wedge}S$ covering it, $\theta(x) = y[F,R]$ according to the calculations made earlier in the previous section.

Putting this together we get

$$\begin{tikzcd}
\pi_3(G\hat{\wedge}M) \arrow[r, "\theta"] \arrow[d, "\cong"'] & H_2(G) \arrow[r] \arrow[d, "inc"] & H_2(Q) \arrow[r] \arrow[d, "inc"] & M/[G,M] \arrow[r] \arrow[d, "="] & \cdots \\
\pi_3(G\hat{\wedge}M) \arrow[r, "\partial"] & R/[F,R] \arrow[r] & S/[F,S] \arrow[r] & M/[G,M] &
\end{tikzcd}$$

commutes with both rows exact, but this means that we can weld the two sequences together to get an eight-term exact sequence.

Proposition 76. *(due to Brown-Loday in the abstract group case) Given an exact sequence*

$$1 \to M \to G \to Q \to 1,$$

there is an exact sequence

$$H_3(G) \to H_3(Q) \to \pi_3(G\widehat{\wedge}M) \to H_2(G) \to H_2(Q)$$
$$\to M/_{[G, M]} \to H_1(G) \to H_1(Q) \to 0$$

■

8.7.4 Ganea's and other related results

Various such exact sequences have been proposed in the abstract case. Ganea, [67], looked at the case when M is central in G. In this case G operates trivially on M, so $G\widehat{\otimes}M$ is isomorphic to $G^{Ab}\widehat{\otimes}M^{Ab}$ and hence to $G^{Ab}\widehat{\otimes}M$. The Ganea term is $G^{Ab} \otimes M$, which, of course, maps down onto $G\widehat{\wedge}M$ and $\pi_3(G\widehat{\wedge}M) \cong G\widehat{\wedge}M$ in this central case. Eckmann and Hilton, [50], (again in the abstract case) showed that there was a sequence as above with $\pi_3(G\widehat{\wedge}M)$ replaced by a "Coker σ" term where σ is a homomorphism defined using a spectral sequence. With Stammbach, [51], they investigated "weak stem extensions" where the map $\mu : M \otimes M \to G^{Ab} \otimes M$ is zero. These include the stem extensions where $N \subset [G, G]$. In this case the mysterious coker σ term is $G^{Ab}\widehat{\otimes}M$. For general central extensions, they considered the diagram

$$\begin{tikzcd}
M\widehat{\otimes}M \arrow[r, "\bar{\chi}"] \arrow[d, "\mu"'] & H_2M \arrow[d, "\mu"] \\
G^{Ab}\widehat{\otimes}M \arrow[r, "\chi"] & H_2G
\end{tikzcd}$$

where χ, $\bar{\chi}$ are commutator maps (cf. Ganea [67]) and is the map $\chi(g \otimes m) = g \wedge m$ if we identify H_2G with the kernel of $G \wedge G \to G$, similarly for $\bar{\chi}$. Now take

$$U = \mu(Ker\bar{\chi})$$

and they prove that the missing term corresponding to $\pi_3(G\widehat{\wedge}M)$ is $(G^{Ab}\widehat{\otimes}M)/U$. It is clear that this is isomorphic to $\pi_3(G\widehat{\wedge}M)$ in this case, as $G\widehat{\wedge}M$ is obtained from $G\widehat{\otimes}M$ by requiring that $m \wedge m = 1$ for all $m \in M$.

8.7.5 Ellis's work

The proofs in this section have largely been based on the purely algebraic proofs of Ellis. The amount of adjustment needed to handle the profinite case has been minimal, except where the lack of the subgroup theorem for free profinite groups has complicated matters. We have, we hope, laid the basis for a fuller understanding of homology and its relationship with commutators and tensor and exterior products. We have not handled all the available results in this direction, however, and would mention Ellis's work on multirelative homology, [55]. Other workers have results using simplicial methods. So far we have concentrated on the homological algebra techniques. We will now be starting to bridge the gap between these techniques and simplicial techniques, linking homological with homotopical algebra and algebraic homotopy.

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