Twisted Cohomotopy implies twisted String structure on M5-branes

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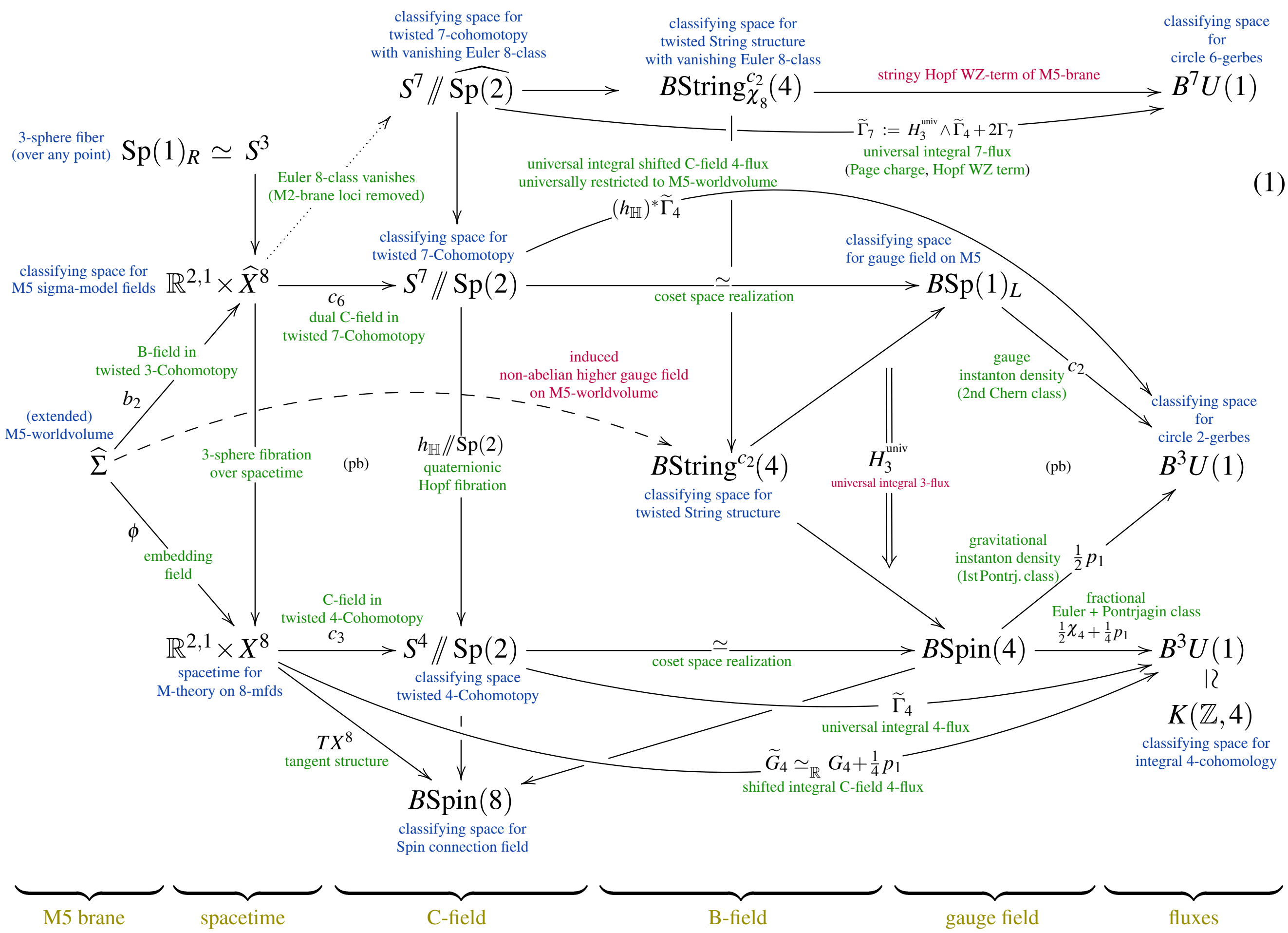
Abstract

We show that charge-quantization of the M-theory C-field in twisted Cohomotopy implies emergence of a higher $\mathrm{Sp}(1)$-gauge field on single heterotic M5-branes, which exhibits worldvolume String^{c_2}-structure.

Towards an understanding of the elusive mathematical formulation of M-theory [Du99][BBS06] the following assumption has been proposed [Sa13][FSS19b][FSS19c]:

Hypothesis H: *The M-Theory C-field is charge-quantized in twisted Cohomotopy theory, i.e. the C-field flux densities G_4, G_7 are in the image of the twisted cohomotopical character [FSS20, §5.3].*

Assuming this, fields of the M5-brane sigma-model [FSS19d] are found to be encoded by classifying maps shown in green in the following homotopy-commutative diagram of topological spaces [FSS19c] (based on [FSS19b]):



(1)

This diagram is best read from bottom-left to top right. Detailed explanation follows, recalling the green maps from [FSS19c], and showing how the purple maps come about, and what this means for the M5 sigma-model.

Open problem M5. It is a widely acknowledged open problem (e.g., [Mo12, p. 77][La12, p. 49][Hu13, p. 1][La19, 6.3]) to find a formulation of the non-abelian higher degree gauge field theory which is expected to arise on coincident M5-branes in M-theory ("non-abelian gerbe theory" [Wi02, p. 6, 15]). There are two aspects to this:

Q. 1 – Non-abelian gauge structure on M5-branes.	**Q. 2 – Higher gauge structure on M5-branes.**
How does a non-abelian gauge group arise on coincident M5-branes? (Given that the traditional argument for non-abelian gauge fields on D-branes via perturbative open string scattering does not apply.)	*How does this get promoted to a* higher gauge group *for a* non-abelian gerbe gauge theory*?* (Given that the gauge potential on the M5-brane is, locally, not a 1-form as in Yang-Mills theory, but a 2-form.)

Open problem M. These are, of course, just aspects of the broader problem of providing a formulation of M-theory itself. This, too, has remained wide open (e.g., [Du96, 6][HLW98, p. 2][Du98, p. 6][NH98, p. 2][Du99, p. 330][Mo14, 12][CP18, p. 2][Wi19][Du19]). Hence it is a rather plausible possibility that properly understanding M5-branes *in* M-theory is impossible without first understanding fundamental principles of M-theory itself.

Hypothesis H. A candidate for a fundamental mathematical principle of M-theory has been proposed [Sa13, 2.5] under the name *Hypothesis H* [FSS19b]. This hypothesis is motivated from hidden structures found in the "brane scan" or rather "brane bouquet" [FSS13b][HSS18, 2.1] which were revealed by a re-analysis of the κ-symmetric super p-brane sigma-models through the lens of super homotopy theory [FSS15][FSS16] (reviewed in [FSS19a, 7]). The hypothesis has since been checked to rigorously imply a fair number of phenomena expected to be characteristic of M-theory [FSS19b][FSS19c][SS19a][BSS19][SS19b][SS20] (with exposition in [Sc20]).

The single heterotic M5-brane... In particular, we proved in [FSS19d] that the κ-symmetric Perry-Schwarz action functional [PS97][Sc97][APPS97] for the single M5-brane in heterotic M-theory (Hořava-Witten theory [HW95][Wi96][HW96][LLO97][LOW99][DOPW99][DOPW00][Ov02][Ov18]) *emerges* from *Hypothesis H* (following an analogous but much simpler derivation of the action functional for the single M2-brane in [HSS18, 6.2]).

...is already non-abelian. But the *heterotic* M5-brane is special in that even a single such brane is expected to carry a non-abelian (higher) gauge field, namely for gauge group the quaternionic unitary group $\mathrm{Sp}(1) \simeq \mathrm{SU}(2)$. This has been argued by identifying heterotic NS5-branes with "small instantons" in the $\mathfrak{so}(32)$-heterotic gauge field [Wi95][AG96][AM97], and it has dually [Se99] been argued as an effect of gauge enhancement for coincident 5-branes on orientifolds [Mu97]: D5-branes in type I string theory [GP96], M5-branes in heterotic M-theory [GH96].

This second perspective says that a single heterotic M5-brane may be understood as consisting of two coincident M5-branes that are bound as mirror images of the orientifold group action, thus revealing the non-abelian $\mathrm{Sp}(1)$ gauge field on the brane as an instance of the general phenomenon expected on coincident M5-branes. Thus it makes sense to investigate **Q. 1** and **Q. 2** already in the case of single heterotic M5-branes.

Coincident M5s → and their **worldvolume fields:**	$N = \mathbf{1}$ M5	$N_{\mathrm{HET}} = \mathbf{1}$ $\mathrm{M5}_{\mathrm{HET}}$ $\shortparallel$ $\mathrm{N} = \mathbf{2}/\mathbb{Z}_2$ M5	$N > \mathbf{2}$ M5
Gauge field	none	$\mathrm{Sp}(1)$-bundle	unclear
Higher gauge field	$U(1)$-gerbe $= BU(1)$ 2-bundle	nonabelian gerbe: $\mathrm{String}^{c_2}(4)$ 2-bundle	unclear

argued in [Wi96][AG96][GP96][GH96]

derived in Theorem 1 from Hypothesis H

Results. We prove here that *Hypothesis H* implies, for the topological sector of single heterotic M5-branes:

(1) the emergence of a non-abelian $\mathrm{Sp}(1)$-gauge field structure on the brane worldvolume;

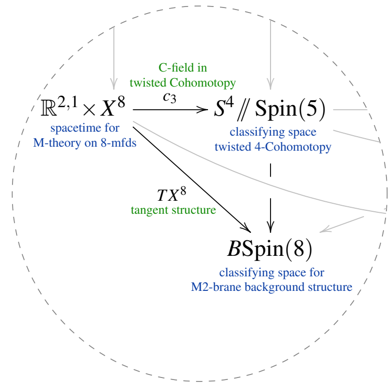
(2) the emergence of a lift to a higher non-abelian gauge field (non-abelian gerbe field), specifically to a String^{c_2}-structure on the M5-brane worldvolume (as in [Sa11, 2.1], generalizing the notion of [CHZ11]);

(3) the worldvolume 3-flux H_3 being the corresponding non-abelian Chern-Simons 2-gerbe [FSS13a][NSS12] (cf. [CJMSW05][Wa13]) reflecting a Green-Schwarz-type mechanism on the worldvolume [SSS12][FSS12a].

All these results follow from inspection of the single encapsulating homotopy-commutative diagram (1) in Theorem 1, using just basic homotopy theory (see pointers in [NSS12]), the results of [FSS19b][FSS19c] and, for interpreting the factorization as String^{c_2}-structure, the concepts developed in [FSS12a]. We explain all this on the following pages.

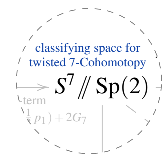
First we explain the existence of the outer part of diagram (1). All statements, proofs and references that we appeal to here are given in [FSS19b].

The classifying space of twisted 4-cohomotopy is the homotopy quotient $S^4 /\!\!/ \mathrm{O}(5)$ of the 4-sphere by its canonical action of the orthogonal group. Up to homotopy equivalence this may be taken to be the Borel construction space $\big(S^4 \times E\mathrm{O}(5)\big)/\mathrm{O}(5)$, where $E\mathrm{O}(5)$ is the total space of the universal $O(5)$-principal bundle. This is manifestly equipped with a map to ("fibered over") the classifying space $B\mathrm{O}(5) = (E\mathrm{O}(5))/\mathrm{O}(5)$. Pulling this back along $B\mathrm{Spin}(5) \to B\mathrm{O}(5)$ and then pushing forward along $B\mathrm{Spin}(5) \to B\mathrm{Spin}(8)$ yields the vertical map shown on the right.

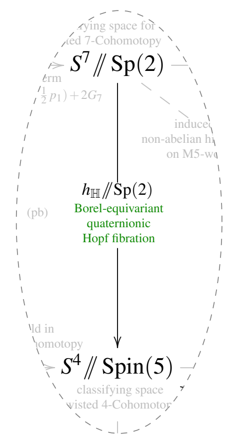


A cocycle in twisted Cohomotopy on a spin-manifold X^8 is a continuous function c_3 to this classifying space, such that composition with this vertical map classifies (the frame bundle of) the tangent bundle TX^8.

The classifying space of twisted 7-Cohomotopy is, similarly, the homotopy quotient $S^7 /\!\!/ \mathrm{O}(8) \simeq \big(S^7 \times E\mathrm{O}(8)\big)/\mathrm{O}(8)$ of S^7 by the canonical action of the orthogonal group $\mathrm{O}(8)$. We consider this action restricted to the **quaternionic unitary group** $\mathrm{Sp}(2)$ for the canonical action $\mathrm{Sp}(2) \longrightarrow \mathrm{O}(8)$ given by left multiplication of quaternion 2×2 matrices on the quaternion 2-space $\mathbb{H}^2 \simeq_{\mathbb{R}} \mathbb{R}^8$.



The quaternionic Hopf fibration $S^7 \xrightarrow{h_{\mathbb{H}}} S^4$ is equivariant with respect to this group action, via the exceptional isomorphism $\mathrm{Sp}(2) \simeq \mathrm{Spin}(5)$. Via Borel-equivariance this induces the map, shown on the right, between the classifying spaces for twisted Cohomotopy in degrees 7 and 4, all fibered over $B\mathrm{Spin}(8)$.

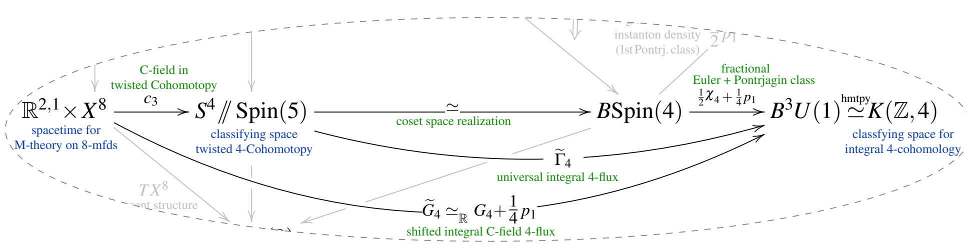


The full equivariance group of the quaternionic Hopf fibration is the central product group $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \simeq \mathrm{Spin}(5) \cdot \mathrm{Spin}(3)$. This is hence singled out as the structure group of 8-manifolds which admit twisted Cohomotopy compatibly in degrees 4 and 7, related by the fully Borel-equivariant quaternionic Hopf fibration:

$$
\begin{tikzcd}
 & S^7 /\!\!/ \mathrm{Sp}(2)\cdot\mathrm{Sp}(1) \arrow[rr, "h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)\cdot\mathrm{Sp}(1)", "\text{pushed forward along maximally Borel-equivariant quaternionic Hopf fibration}"'] \arrow[dr] & & S^4 /\!\!/ \mathrm{Spin}(5)\cdot\mathrm{Spin}(3) \arrow[dl] \\
X^8 \arrow[ur, "\text{cocycle in twisted 7-Cohomotopy}\ c_6"] \arrow[rr, "TX^8"'] & & B\mathrm{Spin}(8) &
\end{tikzcd}
$$

Here we restrict attention to the subgroup $\mathrm{Sp}(2) \simeq \mathrm{Sp}(2) \cdot \mathbb{Z}_2 \subset \mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ only for sake of exposition.

The homotopy equivalence of classifying spaces $S^4 /\!\!/ \mathrm{Spin}(5) \simeq B\mathrm{Spin}(4)$ is induced by the coset space realization of the 4-sphere $S^4 \simeq \mathrm{Spin}(5)/\mathrm{Spin}(4)$.

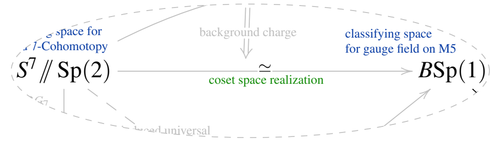


Under this equivalence, **the integral cohomology ring** $H^4(B\mathrm{Spin}(4);\mathbb{Z}) \simeq \mathbb{Z}[\frac{1}{2}p_1, \frac{1}{2}\chi_4 + \frac{1}{4}p_1]$ is identified as containing the *shifted Euler class* $\widetilde{\Gamma}_4 := \frac{1}{2}\chi_4 + \frac{1}{4}p_1$; the plain Euler class $\frac{1}{2}\chi_4$ is the fiberwise volume element on S^4. **Hypothesis H identifies** the C-field flux with the pullback of this class along the given cohomotopy cocycle c_3.

This way **the integral cohomology structure of $B\mathrm{Spin}(4)$ implies the shifted flux quantization of the C-field**:

$$\tfrac{1}{2}\chi_4 + \tfrac{1}{4}p_1 \in H^4(B\mathrm{Spin}(4);\mathbb{Z}) \quad \Leftrightarrow \quad G_4 + \tfrac{1}{4}p_1 \in H^4(X^8;\mathbb{Z})\,.$$

The other homotopy equivalence of classifying spaces $S^7 /\!\!/ \mathrm{Sp}(2) \simeq B\mathrm{Sp}(1)$ is similarly induced by the coset space realization of the 7-sphere as $S^7 \simeq \mathrm{Sp}(2)/\mathrm{Sp}(1)$. This generalizes to the full equivariance group as the homotopy quotient $S^7 /\!\!/ \big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \;\simeq\; B\big(\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\big)$.

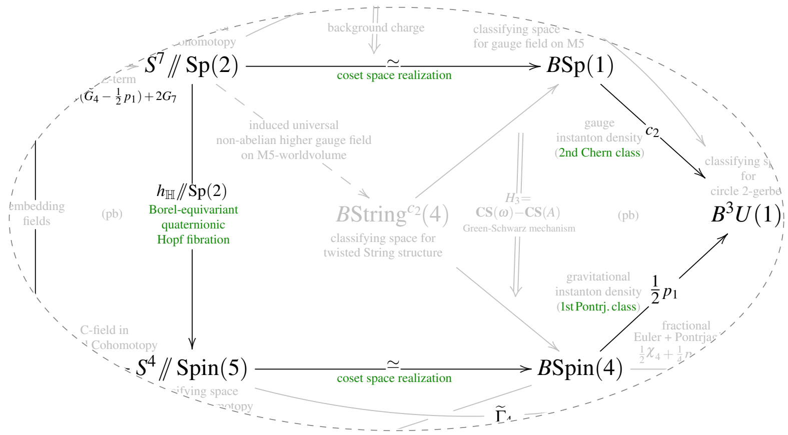


The homotopy-commutativity of the outer square is seen as follows:
With the exceptional isomorphisms

$$
\begin{aligned}
\mathrm{Spin}(4) &\simeq \mathrm{Spin}(3)\times\mathrm{Spin}(3)\\
&\simeq \;\; \mathrm{Sp}(1) \;\times\; \mathrm{Sp}(1)
\end{aligned}
\tag{2}
$$

and the induced cohomology identifications

$$
\begin{aligned}
H^\bullet\big(B\mathrm{Sp}(1);\mathbb{Z}\big) &\simeq \mathbb{Z}\big[c_2\big]\\
H^\bullet\big(B\mathrm{Spin}(4);\mathbb{Z}\big) &\simeq \mathbb{Z}\big[c_2^L, c_2^R\big]
\end{aligned}
\tag{3}
$$



the universal Pontrjagin class $\frac{1}{2}p_1$ on $B\mathrm{Spin}(4)$ is identified with the sum of the second Chern classes of the two $\mathrm{Sp}(1)$-factors (by [FSS19b, Lemma 3.9]), and the Borel-equivariant quaternionic Hopf fibration is identified with the inclusion of the left $\mathrm{Sp}(1)$-factors (by [FSS19b, Prop. 2.22]), as shown here:

$$
\begin{array}{ccccccccc}
& S^7 /\!\!/ \mathrm{Sp}(2) & \underset{\text{coset space realization}}{\overset{\simeq}{\longleftrightarrow}} & B\mathrm{Sp}(1) & \overset{=}{\longleftrightarrow} & B\mathrm{Sp}(1)_L & \times & \ast & \\
{\scriptstyle\text{Borel-equivariant quaternionic Hopf fibration}} & \Big\downarrow{\scriptstyle h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)} & & \Big\downarrow & & \Big\|{\scriptstyle\text{inclusion of left factor}} & & \Big\downarrow & \text{[FSS19b, 2.22]}\\
& S^4 /\!\!/ \mathrm{Spin}(5) & \underset{\simeq}{\overset{\text{coset space realization}}{\longleftrightarrow}} & B\mathrm{Spin}(4) & \underset{\simeq}{\overset{\text{exceptional isomorphism}}{\longleftrightarrow}} & B\mathrm{Sp}(1)_L & \times & B\mathrm{Sp}(1)_R &
\end{array}
\tag{4}
$$

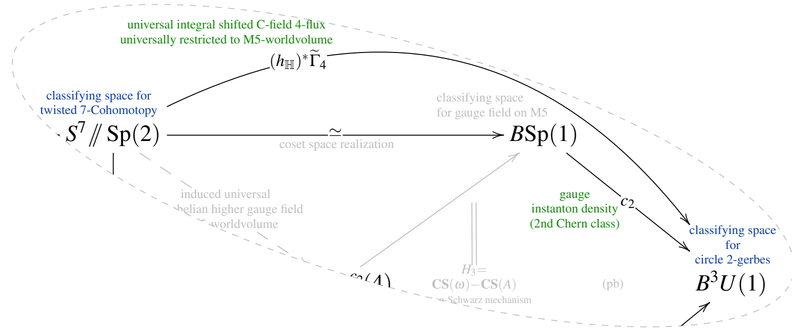
$$
\begin{array}{llcllll}
& H^4(B\mathrm{Spin}(4);\mathbb{Z}) & \overset{\simeq}{\longleftrightarrow} & H^4(B\mathrm{Sp}(1);\mathbb{Z}) \oplus H^4(B\mathrm{Sp}(1);\mathbb{Z}) & & & \\
{\scriptstyle\text{first Pontrjagin class/ C-field background charge}} & \frac{1}{2}p_1 & = & c_2^L & + & c_2^R \;{\scriptstyle\text{total Chern class}} & \\
{\scriptstyle\text{shifted fractional Euler class/ shifted integral C-field flux}} & \widetilde{\Gamma}_4 = \frac{1}{2}\chi_4 + \frac{1}{4}p_1 & = & c_2^L \;{\scriptstyle\text{left Chern class}} & & & \text{[FSS19b, 3.9]}\\
{\scriptstyle\text{C-field flux relative to background charge}} & \widetilde{\Gamma}_4 - \frac{1}{2}p_1 & = & & & -c_2^R \;{\scriptstyle\text{right Chern class}} &
\end{array}
$$

This makes it manifest that the pullback of the Pontrjagin class along the Borel-equivariant quaternionic Hopf fibration kills the right Chern class summand and retains the left Chern class summand, which is again the plain Chern class on the single $\mathrm{Sp}(1)$-factor that is the domain of the Borel-equivariant quaternionic Hopf fibration:

$$
\big(h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)\big)^\ast \left(\tfrac{1}{2}p_1\right) \;=\; c_2\,. \tag{5}
$$

Since $B^3U(1) \simeq K(\mathbb{Z},4)$, this equality of cohomology classes comes from a homotopy between their representative maps to classifying spaces, and this is the homotopy commutativity of the outer square above.

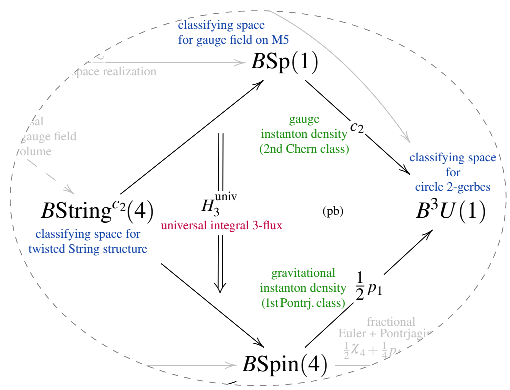
The pullback of the integral 4-flux class $\widetilde{\Gamma}_4$ along the Borel-equivariant quaternionic Hopf fibration is also seen, from (4), to equal the Chern class. The meaning of this for the C-field flux becomes transparent when equivalently considering the pullback of the difference $\widetilde{\Gamma}_4 - \frac{1}{2}p_1$: The last line of (4) shows that this difference *vanishes* after the pullback. Under the interpretation of vanishing of twisted 4-Cohomotopy as its factorization through $h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)$, this is the statement that $\frac{1}{2}p_1$ is the *background charge* of the C-field [FSS19b, 3.5].



We now explain the factorization through twisted String structure in the inner part of the diagram (1). The terminology and conceptualization of twisted String structures and their relation to the Green-Schwarz mechanism [GS84] follows [Sa11][SSS12][FSS12a][FSS12b].

The homotopy fiber product of the classifying maps for the Pontrjagin- and Chern-classes (hence the homotopy-pullback of one along the other, denoted "(pb)" in the diagram) is, by definition, the classifying space for $c_2^{\mathrm{Sp}(1)}$-twisted String(4)-structure [Sa11][SSS12, Def. 2.8]:

$$\overset{\substack{\text{classifying space for}\\ c_2\text{-twisted}\\ \text{String}(4)\text{-structure}}}{B\mathrm{String}^{c_2}(4)} \;:=\; \underbrace{\overset{\substack{\text{classifying space for}\\ \text{Spin}(4)\text{-structure}}}{B\mathrm{Spin}(4)} \underset{B^3U(1)}{\times} \overset{\substack{\text{classifying space for}\\ \text{Sp}(1)\text{-gauge fields}}}{B\mathrm{Sp}(1)}}_{\substack{\text{homotopy fiber product}\\ \text{over classifying space for circle 2-gerbes}}}\,. \tag{6}$$

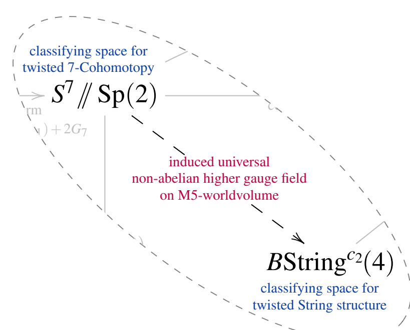


This being a *homotopy*-fiber product means that its defining square diagram is filled by a universal homotopy H_3^{univ}, as shown above, that is a coboundary between cocycle representatives of these two characteristic classes after their pullback to the homotopy-fiber product space $B\mathrm{String}^{c_2}(4)$. As such, H_3^{univ} is the homotopy-theoretic reflection of the *Green-Schwarz mechanism* [SSS12]: Upon enhancing all classifying spaces of topological structures in (6) to their corresponding *moduli stacks* of differential structures (according to [FSS10][SSS12][FSS12a][FSS12b], see (11) below) this homotopy is given, locally, by a 3-form flux H_3 whose de Rham differential is the heterotic Bianchi identity:

$$\underbrace{d\,H_3}_{\text{3-flux form}} \;=\; \underbrace{\mathrm{Tr}(R \wedge R)}_{\substack{= \frac{1}{2}p_1(R)\\ \text{Pontrjagin form}}} - \underbrace{\mathrm{Tr}(F \wedge F)}_{\substack{= c_2(F)\\ \text{Chern form}}}\,. \tag{7}$$

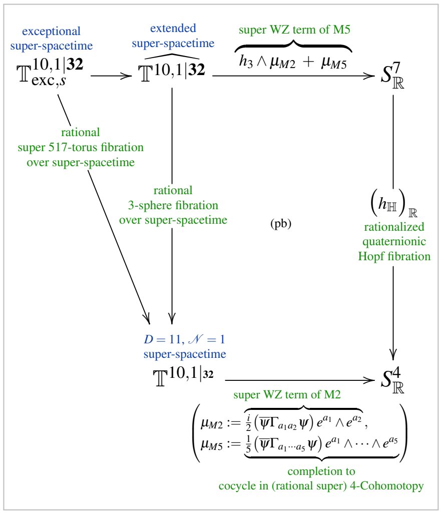
Here R and F denote, as usual, the local curvature forms of connections on the given Spin(4)- and Sp(1)-bundle, respectively, and we take the traces to be integrally normalized.

The dashed factorization in the inner part of (1) through the homotopy fiber product (6), shown on the right equivalently with domain $S^7 /\!\!/ \mathrm{Sp}(2)$, is now the immediate consequence of the commutativity of the outer square, as established above (5), due to the defining universal property of homotopy-fiber products.



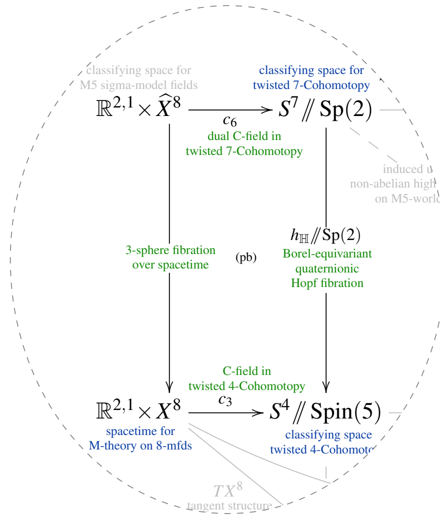
In conclusion, this shows that Sp(1) gauge- and $\mathrm{String}^{c_2}(4)$ higher gauge structure emerges whenever a cocycle in twisted 4-Cohomotopy factors through the classifying space $S^7 /\!\!/ \mathrm{Sp}(2)$ of twisted 7-Cohomotopy via the Borel-equivariant quaternionic Hopf fibration.

Next we explain, with [FSS15][FSS19c][FSS19d], how this space $S^7 /\!\!/ \mathrm{Sp}(2)$ is, under *Hypothesis H*, the classifying space for the higher gauge field in the M5-brane sigma-model with target space $\mathbb{R}^{2,1} \times X^8$.



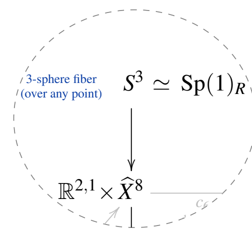
In rational super-spaces

The 3-sphere fibration over spacetime associated with the given cocycle c_3 in twisted Cohomotopy theory is the homotopy-pullback of the Borel-equivariant quaternionic Hopf fibration along c_3. This is the direct analog in fibered topological spaces of the construction in rational super-spaces (survey in [FSS19a]) which induces the super WZ term of the M5-brane sigma model [FSS15], exhibiting 3-spherical T-duality of the M5 [FSS18] [SS18], and from that the full κ-symmetric Lagrangian of the M5-sigma model [FSS19d].



In topological spaces

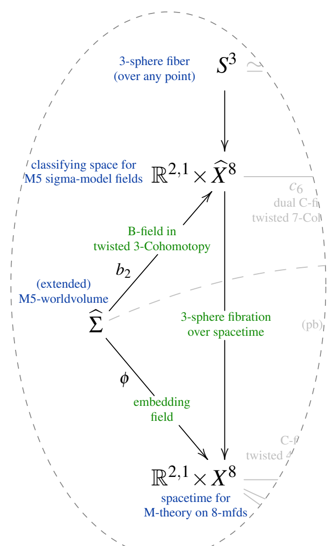
The 3-sphere fiber S^3 is identified with $\mathrm{Sp}(1)_R$ under the coset space realizations (4). This follows using basic facts of homotopy theory (the pasting law for homotopy pullbacks, the commutativity of homotopy fibers with homotopy pullbacks, and the looping/delooping equivalence, see [NSS12] for background): First, since homotopy-pullback preserves homotopy fibers, it is identified with the homotopy-fiber of the Borel-equivariant quaternionic Hopf fibration, which by (4) is obtained as follows:



$$\underbrace{\mathrm{fib}\big(h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)\big)}_{\substack{\text{homotopy fiber of} \\ \text{Borel-equivariant} \\ \text{quaternionic Hopf fibration}}} \;\simeq\; \begin{array}{c} \mathrm{fib}\big(\;\ast \longrightarrow B\mathrm{Sp}(1)_R\big) \\ \times \\ \mathrm{fib}\big(B\mathrm{Sp}(1)_L \underset{\mathrm{id}}{\longrightarrow} B\mathrm{Sp}(1)_L\big) \end{array} \;\simeq\; \begin{array}{c} \mathrm{Sp}(1)_R \\ \times \\ \ast \end{array} \;\simeq\; \mathrm{Sp}(1)_R \tag{8}$$

This implies that given any space $\widehat{\Sigma}$ equipped with a map ϕ to spacetime $\mathbb{R}^{2,1} \times X^8$, a lift b_2, up to specified homotopy (which is notationally suppressed in the diagram on the right), of ϕ to the extended spacetime $\mathbb{R}^{2,1} \times \widehat{X}^8$ is locally on $\widehat{\Sigma}$ a map to $S^3 \simeq \mathrm{Sp}(1)_R$.

Such a map b_2 is a cocycle in twisted 3-Cohomotopy, with the twist being given by the pullback of the C-field cocycle c_3 in 4-Cohomotopy along the embedding field map ϕ. The pasting composition of the homotopy-commutative diagram shown on the right with the homotopy-commutative square shown above (4) identifies the pullback of the shifted integral C-field flux to $\widehat{\Sigma}$ with that of the first fractional Pontrjagin class: Concretely, the image of this data in rational homotopy theory (see [FSS16, A] [BSS18, 2.1]) is:



$$\text{a pair} \qquad \big(\underbrace{\phi}_{\substack{\text{map to}\\ \text{spacetime}}},\; \underbrace{H_3}_{\substack{\text{3-form on}\\ \text{worldvolume}}}\big) \tag{9}$$

$$\text{such that} \qquad d\,H_3 \;=\; \phi^*\big(\widetilde{G}_4 - \tfrac{1}{2}p_1\big)\,. \tag{10}$$

This is the field content of the M5-brane sigma model with worldvolume $\widehat{\Sigma}$ and target space $\mathbb{R}^{2,1} \times X^8$. Thus $\mathbb{R}^{2,1} \times \widehat{X}^8$ is identified with the classifying space for M5-brane sigma-model fields, for given background C-field.

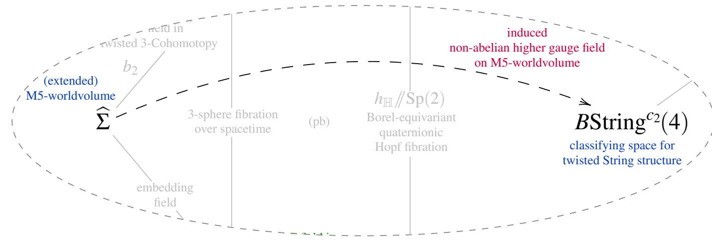
Now we are in position to state the main result of the present article:
If the worldvolume field ϕ is indeed an embedding of Spin manifolds, so that $\phi^* \frac{1}{2} p_1(TX^8) = \frac{1}{2} p_1(T\widehat{\Sigma})$, then **the integral lift of the trivialization** (10)**, is a *twisted String structure* on the M5-brane** [Sa11, 2.1] namely a trivialization of $\frac{1}{2} p_1$ relative to a background 4-class $\phi^* \widetilde{G}_4$ in integral cohomology.

Here with *Hypothesis H* we see further further substructure in this phenomenon, in that the pullback of the shifted integral C-field flux to the M5 worldvolume is identified with the second Chern class of an $\mathrm{Sp}(1)$-gauge field (4) $\phi^* \widetilde{G}_4 \simeq c_2$, whence we have specifically c_2-twisted String structure on the worldvolume. In direct analogy with Spin^c-structure, this is called String^{c_2}*-structure* [Sa11, 2.1], with classifying space $B\mathrm{String}^{c_2}$ (6).

In conclusion, the above discussion shows:

(i) the homotopy-commutativity of the total outer part of the diagram (1);

(ii) the existence of the dashed factorization map (6) induced by the universal property of the homotopy fiber product $B\mathrm{String}^{c_2}$, which identifies the field content (9) of the M5-brane sigma model with that of an embedding field together with a higher $\mathrm{Sp}(1)$-gauge field with gauge 2-group String^{c_2}.



Hence we have proven the following, for the mathematical formulation of the M-theory C-field subject to *Hypothesis H* according to [FSS19b], and the corresponding formulation of the B-field in the M5-brane sigma-model according to [FSS19c]:

Theorem 1. *Assuming C-field charge-quantization in twisted 4-Cohomotopy for M-theory on 8-manifolds, then the induced B-field charge quantization in twisted 3-Cohomotopy on the M5-brane worldvolume is equivalently charge-quantization in* $\mathrm{String}^{c_2}(4)$*-cohomology, according to the homotopy-commutative diagram* (1)*.*

Outlook – Differential String structure on M5-branes. We have focused here on discussion of the topological sector of all fields, classified by a non-abelian generalized but "topological" cohomology; while the full field content is in *differential non-abelian cohomology* [FSS20]. For example, the classifying space $B\mathrm{Spin}(8)$ in the above discussion is to be promoted to the smooth moduli stack $\mathbf{B}\mathrm{Spin}(8)_{\mathrm{conn}}$ of principal Spin connections [FSS10]. While we do not discuss such differential form data here, one impact of Theorem 1 is that it makes immediate how this discussion should proceed: namely directly by promoting (1) from a diagram in spaces to a diagram of the corresponding smooth ∞-stacks according to [FSS10][FSS15] (reviewed in [FSS13a]); hence, in particular, promoting the topological twisted String structure classified by the space $B\mathrm{String}^{c_2}(4)$ (6) to twisted *differential* string structure classified by a smooth 2-stack $\mathbf{B}\mathrm{String}^{c_2}_{\mathrm{conn}}$, according to [SSS12][FSS12a][FSS12b].

	Classifying spaces	$\xleftarrow[\text{(trad.: "geometric realization")}]{\text{pass to shape}}$	*Moduli stacks*	
for Spin bundles Spin structure	$B\mathrm{Spin}(n)$		$\mathbf{B}\mathrm{Spin}(n)_{\mathrm{conn}}$	for Spin connections
for $B^{n-2}U(1)$-bundles (ordinary) n-cohomology	$B^{n-1}U(1)$	$\overset{\text{promote}}{\longmapsto}$	$\mathbf{B}^{n-1}U(1)_{\mathrm{conn}}$	for abelian gerbe connections differential n-cohomology (ordinary)
for String^{c_2} 2-bundles twisted String structure	$B\mathrm{String}^{c_2}(n)$		$\mathbf{B}\mathrm{String}^{c_2}(n)_{\mathrm{conn}}$	for String 2-connections twisted differential String structure
for ΩS^n ∞-bundles n-Cohomotopy	$B\Omega S^n \simeq S^n$		$\mathbf{B}\Omega S^n_{\mathrm{conn}}$	for ΩS^n ∞-connections differential n-Cohomotopy

(11)

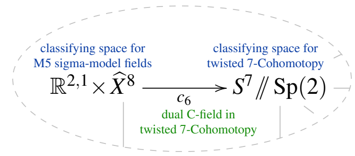
Following the method of [SSS08][FSS10] for constructing such differential refinements, the idea of differential String structures on M5-branes has recently been explored in [SäS17] in an attempt to find a higher gauge theoretic interpretation of the action functionals for non-abelian $D = 6$, $\mathcal{N} = (1,0)$ gauge theories proposed in [SSW11]. But here the conceptual origin and precise flavor of the string gauge field on the M5 seems to have remained open.

Theorem 1 solves this issue by pinpointing specifically $\mathrm{String}^{c_2}(4)$-structure (6) and explaining how this connects to the broader structure of the M5-brane in M-theory, showing that this is the charge quantization on the

M5 that relates to the web of anomaly cancellation conditions in M-theory[FSS19b][FSS19c][SS20]. First consequences have been drawn in [Ro20].

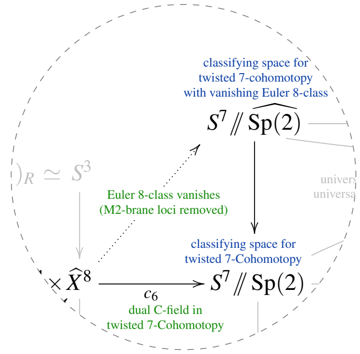
With the stringy field content on the M5-brane in hand, we close by discussing the *Hopf WZ-term* in the M5-brane action functional [In00] *recast as a functional on a* String^{c_2} *higher gauge field* (noticing that the Hopf WZ term induces the full Lagrangian density, by the method of [FSS19d]).

The cocycle c_6 in twisted 7-Cohomotopy, on the classifying space $\widehat{X}^8$ for M5-brane sigma-model fields, arises from the construction of the latter as a homotopy pullback of the Borel-equivariant quaternionic Hopf fibration. This c_6 is the *dual of the C-field*, with flux form G_7, in the situation that the C-field itself trivializes [FSS19b], as it does after pullback along ϕ to the M5, by (10).



The *flux of the dual C-field* on X^8 must measure the *Page charge* [DS91, (41)] of solitonic M2-branes in the spacetime $\mathbb{R}^{2,1}\times X^8$, each stretched along the $\mathbb{R}^{2,1}$ factor. But this requires care:

By the *Poincaré-Hopf theorem* for 7-Cohomotopy [FSS19b, 2.6, 3.7], it follows that if the loci of these M2-branes are assumed to be removed from spacetime (as the locus of the magnetic monopole is in traditional Dirac charge-quantization [Fr11, 16.4e]) then the Euler 8-class χ_8 of X^8 vanishes. This is the situation of M-theory on 8-manifolds [SVW96][FSS19b, 3.8 & Rmk. 3.1].



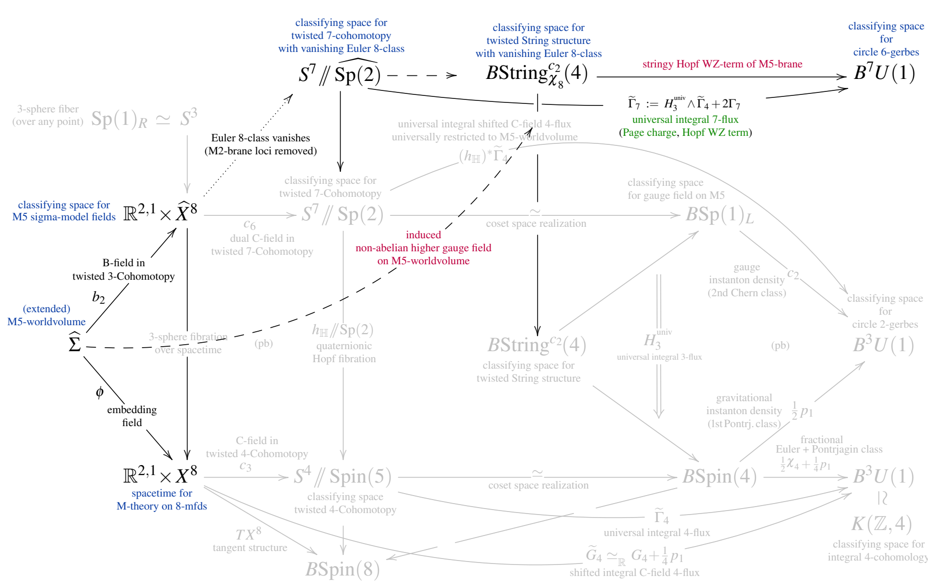
The vanishing of the Euler 8-class is homotopy-theoretically imposed [FSS19c, Def. 4.2] by homotopy pullback of the whole diagram (1) to the homotopy fiber $B\widehat{\mathrm{Sp}(2)}$ of the classifying map for χ_8:

Thus in the case of vanishing Euler 8-class of X^8, the cocycle c_6 in twisted 7-Cohomotopy on the classifying space for M5-brane sigma model fields lifts to the dotted map shown above.

$$
\begin{tikzcd}
\underset{\text{with vanishing Euler 8-class}}{\overset{\text{classifying space for twisted 7-Cohomotopy}}{\big(S^7/\!\!/\widehat{\mathrm{Sp}(2)}\big)}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \overset{\text{classifying space for twisted 7-Cohomotopy}}{S^7/\!\!/\mathrm{Sp}(2)} \arrow[d] & \\
B\widehat{\mathrm{Sp}(2)} \arrow[r, "\text{homotopy fiber of universal Euler 8-class}"'] & B\mathrm{Spin}(8) \arrow[r, "\chi_8", "\text{universal Euler 8-class}"'] & K(\mathbb{Z},8)
\end{tikzcd}
\tag{12}
$$

The universal Hopf WZ-term of the M5-brane is, assuming *Hypothesis H*, a class $\widetilde{\Gamma}_7$ in integral 7-cohomology on this space (12), as shown in green below. This is the result of [FSS19c, Theorem 4.8]:

Equivalently, this $\widetilde{\Gamma}_7$ is the universal *Page charge* density sourced by M2-branes, and as such is the "dual" of the integral shifted 4-flux $\widetilde{\Gamma}_4$. The integrality of $\widetilde{\Gamma}_7$ is crucial both for the Hopf WZ term to be well defined as a Wess-Zumino term for the M5-brane, as well as its interpretation as M2-brane charge being consistent.



Now, the dashed factorization in (1) directly implies, under pullback to $B\widehat{\mathrm{Sp}(2)}$ (12) that:

The Hopf WZ term descends to a class on $\mathrm{String}^{c_2}_{\chi_8}$ higher gauge fields as shown in purple in the above.

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Data availability. Data sharing is not applicable to this article as no new data was created in this study.

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