The character map in equivariant twistorial Cohomotopy implies the Green-Schwarz mechanism with heterotic M5-branes

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Abstract

The celebrated Green-Schwarz mechanism in heterotic string theory has been suggested to secretly underly a higher gauge theoretic phenomenon. Here we prove that its lift to (i) the Hořava-Witten Green-Schwarz mechanism for (ii) heterotic line bundles in heterotic M-theory with (iii) M5-branes parallel to MO9-planes on (iv) A_1-singularities is accurately encoded in the higher gauge theory for higher gauge group of the equivariant homotopy type of the $\mathbb{Z}_2$-equivariant A_∞-loop group of twistor space.

In this formulation, the flux densities of the heterotic gauge field, the B-field on the M5-brane, and of the C-field in the M-theory bulk are all unified into the character image of a single cocycle in equivariant twistorial Cohomotopy theory; and that cocycle enforces the flux quantization condition on fields: the integrality of the gauge flux, the half-shifted integrality of the C-field flux and the integrality of the dual C-field flux (i.e., of the Page charge in the bulk and of the Hopf-WZ term on the M5-brane). This result is in line with the Hypothesis H that M-brane charges are quantized in tangentially twisted Cohomotopy theory.

The mathematical content of our proof is, first, the construction of the equivariant twisted non-abelian character map via an equivariant twisted non-abelian de Rham theorem, which we prove; and, second, the computation of the equivariant relative minimal model of the $\mathbb{Z}_2$-equivariant $\mathrm{Sp}(1)$-parametrized twistor fibration. We lay out the relevant background in equivariant rational homotopy theory and explain how this brings about the subtle flux quantization relations in heterotic M-theory.

Contents

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1 Introduction

Flux 2-forms and gauge groups. The flux density of electromagnetism (the Faraday tensor) is famously a differential 2-form F_2^{EM} on spacetime; and Maxwell's equations say that, away from magnetic monopoles, this 2-form is closed. More generally, the flux density of the nuclear force fields is a differential 2-form F_2^{YM} with values in a Lie algebra, and the Yang-Mills equations imply that its de Rham differential satisfies a *Bianchi identity*. Moreover, (quantum) consistency requires *flux quantization* conditions [SS24b][SS24a]. Together, these constrain F_2 to be the curvature 2-form of a connection on a principal bundle with structure group (gauge group) the Lie group $G = \mathrm{U}(1)$ in the case of electromagnetism and $G = \mathrm{SU}(n)$ in the case of nuclear forces (see [BMSS83][Al85][Fra97][MaS00][Na03][Ma16][RS17]).

Gauge group	**Bianchi identity**
$\mathrm{U}(1)$	$d\,F_2^{\mathrm{EM}} \;=\; 0$
$\mathrm{SU}(n)$	$d\,F_2^{\mathrm{YM}} \;=\; \big[A \wedge F_2^{\mathrm{YM}}\big]$

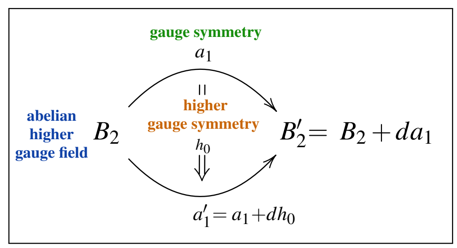
(1)

Higher flux forms and higher gauge groups. While string theory [GSW12][IU12], in its low energy spectrum, contains such gauge fields A, as well as the similar gravitational field (e, ω), it also contains *higher form fields*, the most prominent of which is the Kalb-Ramond *B-field* [KR74] with flux density a differential 3-form H_3. In type II string theory this 3-form is closed, but in heterotic string theory [GHMR85][AGLP12] the de Rham differential of H_3 is the difference of characteristic 4-forms of the gravitational and ordinary gauge fields. This differential relation (2) is the hallmark of the celebrated ("first superstring revolution" [Schw07]) *Green-Schwarz mechanism* [GS84] (reviews in [GSWe85, §2][Wi00, §2.2][GSW12]) for anomaly cancellation.

Higher gauge group	**Higher Bianchi identity**
$\mathbf{B}\mathrm{U}(1)$	$d\,H_3^{\mathrm{II}} \;=\; 0$
String^{c_2}	$d\,H_3^{\mathrm{het}} \;=\; c_2(A) - \tfrac{1}{2}p_1(\omega)$

(2)

It is well-understood [Ga86][FW99][CJM02] that flux quantization implies the B-field in type II string theory to be a *higher gauge field* [BaSc07][BH11][Sc13] for gauge 2-group [BL04][BCSS07][Sc13, §1.2.5.2, §5.1.4] the circle 2-group $\mathbf{B}\mathrm{U}(1)$, hence a higher gauge connection on a $\mathbf{B}\mathrm{U}(1)$-principal 2-bundle [FSSt10, §3.2.3][FSS12b, §2.5][NSS12a][FSS13a, §3.1][FSS20d, §4.3]. (Equivalently: a *Deligne cocycle* [De71, §2.2][MM74, §3.1.7][AM77, §III.1][Bei85][Bry93, §I.5][Ga97], a *Cheeger-Simons character* [CS85] or a *bundle gerbe connection* [Mu96][SSW07][SWa07].)



Non-abelian higher gauge theory. Therefore, comparison of (2) with (1) suggests [SSS09a][SSS12][FSS14a][FSS20c] that the B-field in heterotic string theory is unified with the gauge and gravitational fields into a single higher gauge field for a *non-abelian* higher gauge group, for which the Green-Schwarz mechanism (2) becomes the corresponding higher Bianchi identity.

A choice of gauge 2-group which makes this work is $\mathrm{String}^{c_2}(n)$ [SSS12][FSS14a][FSS20a]. This is a higher analogue of $\mathrm{Spin}^c(n)$ (e.g. [LM89, §D]) and a twisted cousin of $\mathrm{String}(n)$ [BCSS07] (review in [FSS14a, §App.]). The theory of higher gauge symmetry, on which this analysis is based, is obtained by applying general principles of categorification and homotopification to a suitable formulation of ordinary gauge theory. The most encompassing is non-abelian differential cohomology [Sc13][SS20b][FSS20d].

Formulation of gauge theory	Degree one		Higher degree	
	abelian	non-abelian	abelian	non-abelian
Cartan-Ehresmann connections	[BMSS83][Ma16] [MaS00][RS17]			[SSS09a][FSSt10] [SSS12][FSS14a]
holonomy parallel transport	[Bar91][CP94] [SW09][Dum10]		[CS85]	[BaSc07][SW11] [SW13]
Differential cohomology	[Bry93, §II]		[Ga97] [FSS12b, §2.6]	[FSS15b] [FSS20d]

Table 1 – Higher gauge theory.

This success suggests that further rigorous analysis of higher non-abelian differential cohomology should shed light on elusive aspects of string- and M-theory, much as core structure of Yang-Mills theory had first been found by mathematicians from rigorous analysis of degree-1 non-abelian cohomology (to Yang's famous surprise[1]).

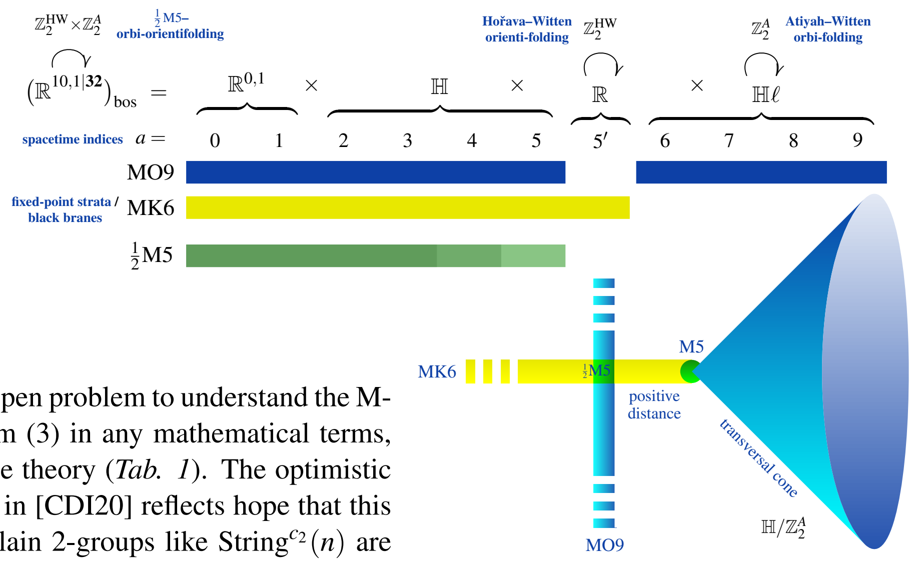
[1] "I found it amazing that gauge theory are exactly connections on fiber bundles, which the mathematicians developed without reference to the physical world. I added: 'this is both thrilling and puzzling, since you mathematicians dreamed up these concepts out of nowhere.' [Chern] immediately protested: 'No, no. These concepts were not dreamed up. They were natural and real.' " [Ya83, p. 567][Zh93, p. 9].

Non-perturbative higher flux forms and equivariant higher gauge groups. Any viable theory of physical reality must be non-perturbative [BaSh10][Br14]. Despite the pivotal role of non-perturbative phenomena in the foundations of physics, such as in color confinement [Gr11] (existence of ordinary matter), nucleosynthesis (becoming of ordinary matter), Higgs metastability (existence of ordinary spacetime), QCD cosmology (becoming of ordinary spacetime), its theoretical understanding remains open: called the "Holy Grail" by nuclear physicists [Hol99, p. 1][Gu08, §13.1.9] and a *Millennium Problem* by the Clay Mathematics Institute [CMI][JW00].

However, the celebrated ("second superstring revolution" [Schw96]) *M-theory* conjecture [Wi95][Du96][Du98][Du99] indicates a potential solution to this problem (e.g. [AHI12][RZ16, §4]). Specifically, the *Hořava-Witten mechanism* [HW96][DM97][BDS00][Mos08] in heterotic M-theory [HW95][DOPW99][DOPW00][Ov02] proposes a non-perturbative completion of the ordinary Green-Schwarz mechanism (2), given by coupling the (higher) heterotic gauge fields to the flux 4-form G_4 of the M-theory C-field [CJS78][Wi97a], as shown on the right here:

Equivariant higher gauge group §2	**Bulk/boundary higher Bianchi identity** in bulk of heterotic M-theory	on M5-brane parallel to MO9	
$\Omega_{\mathcal{G}/\!\!/\times B\mathrm{Sp}(1)}\big(\Upsilon(\mathbb{C}P^3 /\!\!/ \mathbb{Z}_2^A)/\!\!/\mathrm{Sp}(1)\big)$ **loop A_∞-group of twistor space, $\mathbb{Z}_2^A$-equivariant & Sp(1)-parametrized** (70)	$dC_3 = G_4 - \tfrac{1}{4}p_1(\omega) + c_2(A)$ [HW96][DFM03, (3.9)], see (161)	$dH_3^{\mathrm{M5}} = c_2(A) - \tfrac{1}{2}p_1(\omega)$ [OST14, (1.2)][OSTY14, (2.18)], see (163)	(3)

In the worldvolume theory of M5-branes at a distance parallel to an MO9-plane intersecting an ADE-singularity (see [GKST01, (6.13)] [DHTV15, §6.1.1] [SS19a, Fig. V] [FSS19d] [FSS20b], also discussed as *E-string theories* [HLV14, Fig. 1] [KKLPV14, Fig. 5] [GHKKLV18, Fig. 8]), this reproduces [OST14, (1.2)][OSTY14, (2.18)] the plain Green-Schwarz mechanism, now in more realistic 5+1 spacetime dimensions (review in [In14, (4.1)] [Shi18, §7.2.8][CDI20, p. 18]).

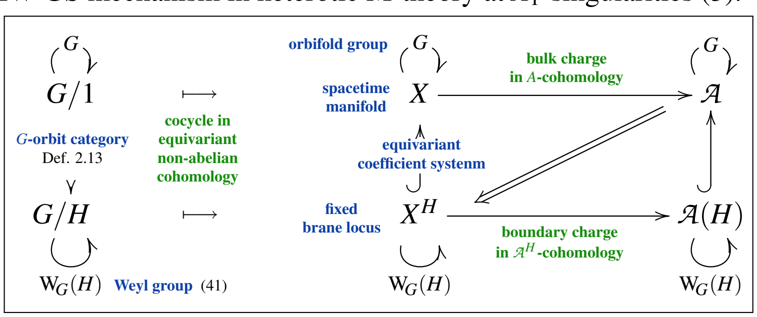


The open problem. It has been an open problem to understand the M-theoretic Green-Schwarz mechanism (3) in any mathematical terms, specifically in terms of higher gauge theory (*Tab. 1*). The optimistic terminology of *2-group symmetries* in [CDI20] reflects hope that this open problem has a solution, but plain 2-groups like $\mathrm{String}^{c_2}(n)$ are insufficient for accommodating **(a)** the C-field flux G_4 (requiring at least a 3-group [FSS14a]) and **(b)** the bulk/brane-relation in (3) (requiring equivariance [HSS18][SS19a][BSS19]).

Hypothesis H. We have recently formulated a precise hypothesis, called *Hypothesis H* [FSS19b][FSS19c][SS20a] (following [Sa13, §2.5], exposition in [SS23d]) on flux quantization in M-theory and have proven in [FSS20c] that this Hypothesis correctly reproduces the HW-GS mechanism in the heterotic M-theory bulk on the left of (3).

Solution: Flux quantization in equivariant non-abelian cohomology. Here we consider *Hypothesis H* in $\mathbb{Z}_2^A$-*equivariant* non-abelian cohomology and demonstrate that charge quantization in the resulting *equivariant twistorial Cohomotopy theory* implies the bulk/brane HW-GS mechanism in heterotic M-theory at A_1-singularities (3).

This is our main Theorem 1.1 below. Key here is the observation [HSS18][SS19a] that Elmendorf's theorem ([El83], see [SS21, Prop. 4.5.1] and Prop. 2.26 below) in equivariant homotopy theory ([tD79, §8][May96][Blu17][SS20b], see §2.2 below) exhibits flux quantization in equivariant cohomology theories as assigning bulk-boundary charges to branes at orbi-singularities.



We now outline how this works: **Flux quantization and cohomology.** The phenomenon of *charge-* or *flux quantization* (going back to [Di31][Al85], developed in [Fr00][Sa10b][Sz12], we follow [FSS20d] with exposition in [SS24b]) is a deep correspondence between **(i)** cohomology theories in mathematics hupf and **(ii)** constraints on flux densities in physics. It is worthwhile to recall the key examples (*Table 2*), as we are about to unify all these:

(1.a) The archetypical example is Dirac's charge quantization [Di31] of the electromagnetic field in the presence of **magnetic monopoles**, This says that the cohomology class of the ordinary electromagnetic flux density (the Faraday tensor regarded as a differential 2-form on spacetime) must be the image under the *de Rham homomorphism* of a class in *ordinary integral cohomology* (see [Al85, §2][Fra97, §16.4e] for surveys).

(1.b) Similarly, the existence of **gauge instantons** in nuclear physics (e.g., [Na03, §10.5.5]), and thus (by [AM89][Su10][Su15]) also of **Skyrmions** [RZ16], means that the class of the characteristic 2nd Chern forms built from the non-abelian nuclear force flux density must be the image under the *Chern-Weil homomorphism* of a class in *degree-1 non-abelian cohomology*, represented by a principal gauge bundle (review in [FSS20d]). This is at the heart of the striking confluence of Yang-Mills theory with principal bundle theory (reviewed in [EGH80][BMSS83][MaS00][Na03][Ma16]).

(1.c) The analogous phenomenon also appears in the description of gravity: Here the existence of **gravitational instantons** means ([EF67][BB76]) that the class of the characteristic 1st Pontrjagin form, built from the gravitational field strength tensor, must be in the image under the Chern-Weil homomorphism of the tangent bundle of spacetime. This is a consequence of identifying the gravitational field strength with the Riemann tensor, hence is a consequence of the equivalence principle (see [EGH80][Na03] for surveys).

(2.a) To unify these three situations (electromagnetic, nuclear and gravitational force), the K-theory proposal in string theory [MM97][Wi98][FH00][GS19] asserts that, due to **D-brane** charge, the joint class of the NS [NS71] and RR [Ra71] field flux forms must be in the image under the *Chern character* map of a class in the generalized cohomology theory called *twisted topological K-theory* (see [Fr02][GS19][FSS20d, §5.1] and references therein).

Quantized charge	**Expression**	**Charged object** [2]	**Quantizing cohomology theory**	see
Magnetic flux	$[F_2(A)]$	Gauge monopole	Ordinary cohomology in degree 2	[Di31] [Fra97, §16.4e]
2nd Chern form	$[c_2(A)]$	Gauge instanton	Non-abelian cohomology $H^1(-;G_{\text{gauge}})$	[Ch50] [Ch51, §III] [FSS20d, §4.2]
1st Pontrjagin form	$[\frac{1}{2}p_1(\omega)]$	Gravitational instanton	Non-abelian cohomology $H^1(-;\text{Spin})$	
NS-flux	$[H_3]$	NS5-brane	Ordinary cohomology in degree 3	[Ga86] [FW99][CJM02]
RR-flux	$[F_{2\bullet}]_{H_3}$	D-branes	Twisted topological K-theory	[Wi98][Fr00] [GS19]
Shifted C-field flux	$[G_4 + \frac{1}{4}p_1(\omega)]$	M5-brane	Twisted 4-Cohomotopy	[FSS19b, §3.4]
Hopf-WZ/ Page charge	$\begin{bmatrix} H_3 \wedge (G_4 + \frac{1}{4}p_1(\omega)) \\ +2G_7 \end{bmatrix}$	M2-brane	Twisted 7-Cohomotopy	[FSS19c]
M-heterotic C-field flux	$[G_4 - \frac{1}{4}p_1(\omega)] = [F_2 \wedge F_2]$	Heterotic M5-brane	Twistorial Cohomotopy	[FSS20c]
Heterotic B-field flux	$dH_3 = \frac{1}{2}p_1(\omega) - c_2(A)$	Heterotic NS5-brane	$\mathbb{Z}_2^A$-equivariant twistorial Cohomotopy	§4

Table 2 – Charge quantization in gauge-, string-, and M-theory. Quantization conditions correspond to cohomology theories.

(2.b) To provide the non-perturbative completion of this unified theory, the M-theory conjecture [Wi95][Du96][Du98][Du99] asserts that these NS/RR-fluxes are a perturbative approximation to *C-field fluxes* [CJS78][Wi97a], quantized due to **M-brane** charge. However, actually formulating M-theory has remained an open problem (e.g., [Du96, 6][HLW98, p. 2][Du98, p. 6][NH98, p. 2][Du99, p. 330][Mo14, 12] [CP18, p. 2][Wi19, @21:15] [Du19, @17:14]). Proposals for cohomological charge quantization of the C-field have led to interesting advancements [DMW00][DFM03][HS05][Sa05a][Sa05b][Sa06][Sa10b][FSS14a], but the situation had remained inconclusive.

Hypothesis H. However, a homotopy-theoretic reanalysis [FSS13][FSS16a][HSS18][BMSS19] (review in [FSS19a, (57)]) of the κ-symmetric functionals that actually define the super p-branes on super-Minkowski target spacetimes (the "brane scan", see [HSS18, §2]) has revealed that the character map for M-brane charge must land in (the rational image of) the non-abelian cohomology theory called *Cohomotopy theory* [Bo36][Sp49] [Pe56][Ta09][KMT12], just as proposed in [Sa13, §2.5]. Incorporating into this flat-space analysis the twisting of Cohomotopy theory by non-flat spacetime tangent bundles via the (unstable) J-homomorphism ([Wh42], review in [We14]) leads to:

> **Hypothesis H**: The M-theory C-field is flux-quantized in *tangentially twisted Cohomotopy theory*.

This hypothesis has been shown [FSS19b][FSS19c][SS19a][SS19b] to correctly imply various subtle effects of charge quantization in the M-theory bulk, including the bulk of Hořava-Witten's heterotic M-theory (3) [FSS20c].

The character map in non-abelian cohomology. Technically, charge quantization in a cohomology theory means to require the classes of the flux forms to lift through the *character map* [FSS20d] (Table 2); hence through the classical Chern character in the case of K-theory ([Hil55, §12][AH61, §1.10][Hil71, §V]), more generally through the Chern-Dold character for generalized cohomology theories ([Do65, Cor. 4][Bu70], review in [Hil55, p. 50][FSS20d, §4.1]) in the traditional sense of [Wh62][Ad75], and generally through the twisted non-abelian character map ([FSS20d], exposition in [SS24b, §3.2], details below in §3.4).

local coefficient bundle $A /\!\!/ G \to BG$		$H^{\tau}(X;A)$ (twist in $H(X,BG)$; spacetime manifold; classifying space) twisted non-abelian cohomology	$\xrightarrow{\mathrm{ch}_A}$ twisted non-abelian character map	$H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}(X;\mathfrak{l}A)$ (twist in $H_{\mathrm{dR}}(X;\mathfrak{l}BG)$; spacetime manifold; Whitehead L_∞-algebra) twisted non-abelian de Rham cohomology
	[FSS20d] Def. 5.4	$[c]_\tau$ cocycle in twisted A-cohomology	$\longmapsto$	$\mathrm{ch}_A([c])$ flux densities satisfying Bianchi identities
$B^n\mathbb{Z} \to *$	[FSS20d] Ex. 4.9	$H^n(X;\mathbb{Z})$ ordinary cohomology	$\xrightarrow{\mathrm{dR}}$ de Rham homomorphism	$\big\{ F_n \in \Omega^n_{\mathrm{dR}}(X) \,\big\vert\, dF_n = 0 \big\}_{/\sim}$
$B\mathrm{U}(n) \to *$	[FSS20d] Thm. 4.26	$H^1\big(X;\mathrm{U}(n)\big)$ non-abelian cohomology	$\xrightarrow{\mathrm{cw}}$ Chern-Weil homomorphism	$\left\{ \begin{array}{l} \vdots, \\ c_2(A), \\ c_1(A) \end{array} \in \Omega^{2\bullet}_{\mathrm{dR}}(X) \,\middle\vert\, \begin{array}{l} \vdots \\ d\,c_2(A) = 0 \\ d\,c_1(A) = 0 \end{array} \right\}_{/\sim}$
$(\mathbb{Z}\times B\mathrm{U}) /\!\!/ B\mathrm{U}(1) \to B^2\mathrm{U}(1)$	[FSS20d] Prop. 5.5	$\mathrm{KU}^{\tau}(X)$ twisted complex K-theory	$\xrightarrow{\mathrm{ch}^{\tau}}$ twisted Chern character	$\left\{ \begin{array}{l} F_{2\bullet}, \\ H_3 \end{array} \in \Omega^{\bullet}_{\mathrm{dR}}(X) \,\middle\vert\, \begin{array}{l} dF_{2\bullet+2} = H_3 \wedge F_{2\bullet} \\ dH_3 = 0 \end{array} \right\}_{/\sim}$
$S^4 /\!\!/ \widehat{B\mathrm{Sp}(2)} \to \widehat{B\mathrm{Sp}(2)}$	[FSS20d] Ex. 5.23a	$\pi^{\tau}(X)$ J-twisted 4-Cohomotopy	$\xrightarrow{\mathrm{ch}^{\tau}_{\pi}}$ twisted FSS-character	$\left\{ \begin{array}{l} 2G_7, \\ G_4 \end{array} \in \Omega^{\bullet}_{\mathrm{dR}}(X) \,\middle\vert\, \begin{array}{l} d\,2G_7 = -G_4 \wedge G_4 + \big(\tfrac{1}{4}p_1(\omega)\big)^2 \\ d\;\; G_4 = 0 \end{array} \right\}_{/\sim}$
$\mathbb{C}P^3 /\!\!/ \widehat{B\mathrm{Sp}(2)} \to \widehat{B\mathrm{Sp}(2)}$	[FSS20d] Ex. 5.23b	$\mathscr{T}^{\tau}(X)$ twistorial Cohomotopy	$\xrightarrow{\mathrm{ch}^{\tau}_{\mathscr{T}}}$ twisted FSS-character	$\left\{ \begin{array}{l} H_3 \\ F_2 \\ 2G_7, \\ G_4 \end{array} \in \Omega^{\bullet}_{\mathrm{dR}}(X) \,\middle\vert\, \begin{array}{l} d\;\; H_3 = G_4 - \tfrac{1}{4}p_1(\omega) - F_2 \wedge F_2 \\ d\;\; F_2 = 0 \\ d\,2G_7 = -G_4 \wedge G_4 + \big(\tfrac{1}{4}p_1(\omega)\big)^2 \\ d\;\; G_4 = 0 \end{array} \right\}_{/\sim}$

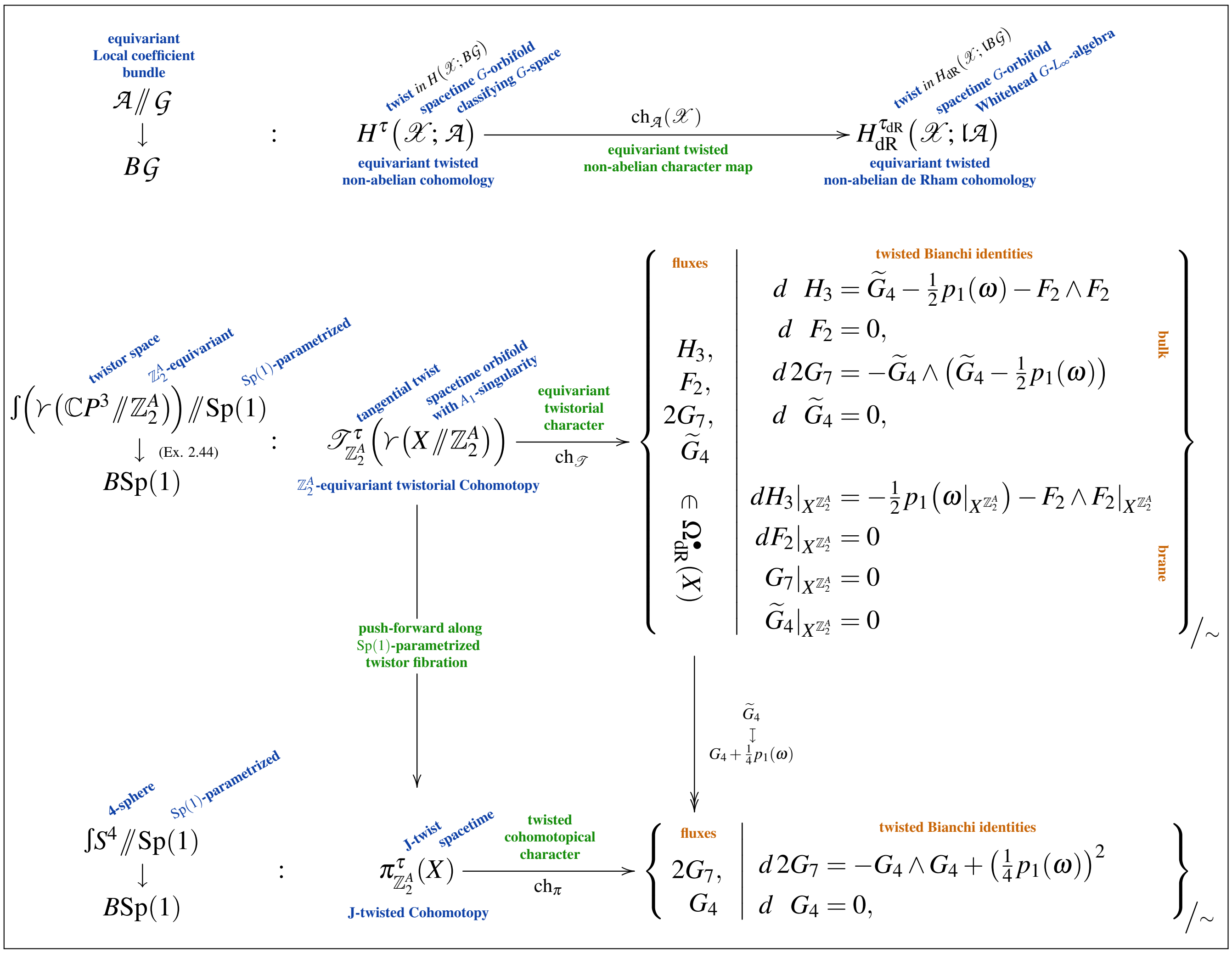
Table 2 – Character maps in twisted non-abelian cohomology. Flux quantization means to lift flux forms through ch_A to A-cocycles.

Lifting through the character map quantizes fluxes. While the condition to lifting through the de Rham homomorphism (second line in *Tab. 2*) is just the integrality of the periods of the flux form, hence of the total charge, the obstructions to lifting, say, through the twisted Chern character (fourth line) are richer: these are organized by the Atiyah-Hirzebruch spectral sequence in twisted K-theory [AH61] or rather in differential twisted K-theory [GS17][GS19a][GS19], and their analysis provided the original consistency checks that D-brane charge should be quantized in twisted K-theory (e.g. [MMS01][ES06]).

Analogous generalized tools (Postnikov systems, e.g. [Wh78, §XI][GJ99, §VI], and rational minimal models, e.g. [Ha83]) exist for the analysis of obstructions to lifts through non-abelian character maps. Particularly in Cohomotopy theory [GS20], they reveal that charge quantization in twistorial Cohomotopy theory (last line in *Tab. 2*) imposes, among a list of other constraints expected in M-theory (see [FSS19b, Table 1]), the charge quantization (3) expected in the bulk of heterotic M-theory [FSS19b, §3.4][FSS20c][FSS20d, §5.3].

Here we generalize this analysis to *equivariant* twistorial Cohomotopy and prove the following result (in §3):

Theorem 1.1. **(i)** *The character map (Def. 3.78) in $\mathbb{Z}_2^A$-equivariant twistorial Cohomotopy (Def. 2.48), on $\mathbb{Z}_2^A$-orbifolds (Def. 2.36) with* $\mathrm{Sp}(1)$*-structure τ and -connection ω (Example 3.70), is of the following form (3.79):*



(ii) *Moreover, a necessary condition for the fluxes to lift through this character map is their shifted integrality:*

$$\big[\widetilde{G}_4\big] := \big[G_4 + \tfrac{1}{4}p_1(\omega)\big] \in H^4(X;\mathbb{Z}) \to H^4(X;\mathbb{R}), \qquad \big[F_2\big] \in H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{R}). \tag{4}$$

Thus, charge quantization in $\mathbb{Z}_2^A$-equivariant twistorial Cohomotopy enforces the twisted Bianchi identities (3) of the Green-Schwarz mechanism in heterotic M-theory with M5-branes parallel to MO9-planes on A_1-singularities, for heterotic line bundles [AGLP12][FSS20c, p. 5] with gauge group $\mathrm{U}(1) \simeq \mathrm{S}\big(\mathrm{U}(1)^2\big) \subset E_8$.

The computation at the heart of our proof. At the heart of the proof of Theorem 1.1 is the computation (Prop. 3.56 below) of the equivariant relative minimal model ([Tri82, §5][Scu02, §11][Scu08, §4], recalled as Def. 3.40 below) of the $\mathbb{Z}_2$-equivariant $\mathrm{Sp}(1)$-parametrized twistor fibration in equivariant rational homotopy theory.

The equivariant twistor fibration. The *twistor fibration* $t_{\mathbb{H}}$ ([At79, §III.1][Br82], see [FSS20c, §2]) is the map from $\mathbb{C}P^3$ ("twistor space") to $\mathbb{H}P^1 \simeq S^4$ which sends complex lines to the right quaternionic lines that they span:

$$
\begin{array}{ccccccc}
S^2 \simeq \mathbb{H}^\times/\mathbb{C}^\times & \xrightarrow{\mathrm{fib}(t_{\mathbb{H}})} & & & & & \\
& & \mathbb{C}P^3 & \simeq & (\mathbb{C}^4 \setminus \{0\})/\mathbb{C}^\times & \ni & \{ v \cdot z \,|\, z \in \mathbb{C}^\times \} \\
& {\scriptstyle \text{twistor fibration}} & \big\downarrow {\scriptstyle t_{\mathbb{H}}} & & \big\downarrow & & \\
& & \mathbb{H}P^1 & \simeq & (\mathbb{H}^2 \setminus \{0\})/\mathbb{H}^\times & \ni & \{ v \cdot q \,|\, q \in \mathbb{H}^\times \}
\end{array}
\qquad (5)
$$

The fiber of the twistor fibration is hence $\mathbb{H}^\times/\mathbb{C}^\times \simeq \mathbb{C}P^1 \simeq S^2$.

(i) There is the evident action of $\mathrm{Sp}(2)$, on both $\mathbb{C}P^3$ and $\mathbb{H}P^1$, by left multiplication of homogeneous representatives with unitary quaternion 2×2 matrices (52):

$$
\begin{array}{ccc}
\mathrm{Sp}(2) \times \mathbb{C}P^3 & \longrightarrow & \mathbb{C}P^3\,, \\
(A\;,\;[v]) & \longmapsto & [A \cdot v]
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Sp}(2) \times \mathbb{H}P^1 & \longrightarrow & \mathbb{H}P^1\,, \\
(A\;,\;[v]) & \longmapsto & [A \cdot v]
\end{array}
\qquad (6)
$$

and the twistor fibration (being given by quotienting on the right) is manifestly equivariant under this left action.

(ii) Consider the following subgroups:

$$
\mathbb{Z}_2^A := \left\{ 1 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \sigma := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \right\} \subset \mathrm{Sp}(2)\,, \qquad \sigma \,:\, [z_1 : z_2 : z_3 : z_4] \;\mapsto\; [z_3 : z_4 : z_1 : z_2]\,, \qquad (7)
$$

$$
\mathrm{Sp}(1) := \left\{ q\cdot := \left(\begin{smallmatrix} q & 0 \\ 0 & q \end{smallmatrix}\right) \,\middle|\, q \in S(\mathbb{H}) \right\} \subset \mathrm{Sp}(2)\,. \qquad (8)
$$

Since these manifestly commute with each other, the homotopy quotient $\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(1)$ of twistor space (5) by $\mathrm{Sp}(1)$ still admits the structure of a G-space (as in [tD79, §8][May96][Blu17]) for $G = \mathbb{Z}_2^A$, fibered over $B\mathrm{Sp}(1)$ (see Ex. 2.44 below for details).

The equivariant minimal relative dgc-algebra model of twistor space. Our Prop. 3.56 gives its equivariant minimal model:

$$
\begin{array}{ccc}
\underset{\substack{\text{twistor space} \\ \text{homotopy-quotiented by } \mathrm{Sp}(1) \\ \text{with residual } \mathbb{Z}_2^A\text{-action}}}{\overset{\mathbb{Z}_2^A \circlearrowright}{\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(1)}}
& : &
\begin{array}{ccc}
\overset{\mathbb{Z}_2 \circlearrowright}{\mathbb{Z}_2/1} & \xmapsto{\text{bulk}} & \mathbb{R}\big[\tfrac{1}{4}p_1\big]\left[\begin{array}{c} h_3, \\ f_2 \\ \omega_7, \\ \widetilde{\omega}_4 \end{array}\right] \Big/ \left(\begin{array}{l} d\,h_3 = \widetilde{\omega}_4 - \tfrac{1}{2}p_1 - f_2 \wedge f_2 \\ d\,f_2 = 0 \\ d\,\omega_7 = -\widetilde{\omega}_4 \wedge \big(\widetilde{\omega}_4 - \tfrac{1}{2}p_1\big) \\ d\,\widetilde{\omega}_4 = 0 \end{array}\right) \\
\Big\downarrow {\scriptstyle \mathbb{Z}_2\text{-orbit category}} & & \Big\downarrow {\scriptstyle \text{minimal } \mathbb{Z}_2^A\text{-equivariant model relative to } B\mathrm{Sp}(1)} \\
\mathbb{Z}_2/\mathbb{Z}_2 & \xmapsto{\text{singularity}} & \mathbb{R}\big[\tfrac{1}{4}p_1\big]\left[\begin{array}{c} h_3, \\ f_2 \end{array}\right] \Big/ \left(\begin{array}{l} d\,h_3 = \quad - \tfrac{1}{2}p_1 - f_2 \wedge f_2 \\ d\,f_2 = 0 \end{array}\right)
\end{array}
\end{array}
\qquad (9)
$$

normalized (as in [FSS19b][FSS19c][FSS20c]) such that:

(a) all closed generators shown are rational images of *integral* and *integrally in-divisble* cohomology classes;

(b) $\omega := \widetilde{\omega} - \frac{1}{4}p_1$ is fiberwise the volume form on $\mathbb{H}P^1 \simeq S^4$, and f_2 is fiberwise the volume form on $\mathbb{C}P^1 \simeq S^2$.

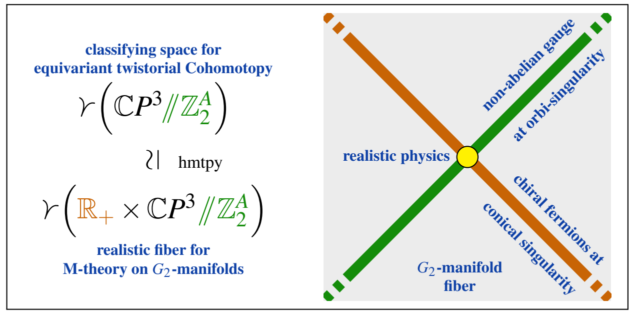
As a non-trivial example of a (relative) minimal model in rational equivariant homotopy theory, this may be of interest in its own right. Such examples computed in the literature are rare (we have not come across any). Here we are concerned with a most curious aspect of this novel example: Under substituting the algebra generators in (9) with differential forms on a $\mathbb{Z}_2^A$-orbifold (essentially the non-abelian character map, Def. 3.78), the relations in (9) are just those expected for flux densities in M-theory at an orbi-conifold singularity (3) – the details are in §4:

$$
\underset{\substack{\text{dgc-algebra generators of} \\ \text{equivariant relative minimal model}}}{\big(\tfrac{1}{4}p_1,\, \widetilde{\omega}_4,\, \omega_7,\, f_2,\, h_3\big)}
\quad\longleftrightarrow\quad
\Big(\underset{\substack{\text{Pontrjagin form} \\ \text{(gravitational flux density)}}}{\tfrac{1}{4}p_1(\omega)}\,,\; \overset{\substack{\text{shifted} \\ \text{C-field flux density}}}{G_4 + \tfrac{1}{4}p_1(\omega)}\,,\; \underset{\substack{\text{dual} \\ \text{C-field flux density}}}{2G_7}\,,\; \overset{\text{gauge flux}}{F_2}\,,\; \underset{\text{B-field flux}}{H_3} \Big).
$$

M-Theory on G_2-manifolds? It may be noteworthy that the classifying space (9) for $\mathbb{Z}_2^A$-equivariant twistorial Cohomotopy, which, by Theorem 1.1, implements charge quantization in heterotic M-theory (3), is homotopy equivalent to the $\mathbb{Z}_2^A$-orbifold of the metric cone (topologically: a cylinder) over complex projective 3-space:

(a) This metric cone $\mathbb{R}_+ \times \mathbb{C}P^3$ is one of three known [BS89][GPP90] simply-connected conical G_2-manifolds, the other two being the metric cone on $S^3 \times S^3$ and on $\mathrm{SU}(2)/(\mathrm{U}(1)\times\mathrm{U}(1))$, respectively.

(b) Its G_2-metric is invariant [ABS20] under the left $\mathrm{Sp}(2)$-action (6), so that its orbifold quotient $\prec\big(\mathbb{R}_+ \times \mathbb{C}P^3 /\!\!/ \mathbb{Z}_2^A\big)$ is a G_2-orbi-conifold with an A_1-type orbi-singularity intersecting a conical singularity.



Exactly such intersections of ADE-orbifold singularities with conical singularities in G_2-manifolds are thought to be the type of fiber spaces over which KK-compactification of (non-heterotic) M-theory produces chiral fermions charged under non-abelian gauge groups in the resulting 4-dimensional effective field theory ([AW01][Wi01] [AW01][Ach02], review in [AG04]). This might suggest that here we are seeing an aspect of duality between (flux quantization in) heterotic M-theory and M-theory on G_2-manifolds. We hope to address this elsewhere.

Outline.

In §2 we introduce equivariant non-abelian cohomology theory (in equivariant generalization of [FSS20d, §2]) and the example of equivariant twistorial Cohomotopy theory $\mathcal{T}^{\tau}_{\mathbb{Z}_2^A}(-)$ (Def. 2.48).

In §3 we introduce equivariant non-abelian de Rham cohomology theory and the equivariant non-abelian character map (in equivariant generalization of [FSS20d, §3-5]) and compute the $\mathbb{Z}_2^A$-equivariant relative minimal model of $\mathrm{Sp}(1)$-parametrized twistor space (Prop. 3.56).

In §4 we discuss the character map in equivariant twistorial Cohomotopy theory and conclude the proof of the main Theorem 1.1, in equivariant generalization of [FSS20d, §5.3].

Notation. For various types of symmetry groups and their quotients, we use the following notation:

T	Compact Borel equivariance group	Def. 2.11	$\int (X /\!\!/ T)$	Borel equivariant homotopy type	Ex. 2.8
G	Finite proper equivariance group		$\prec(X /\!\!/ G)$	Orbifold	Ex. 2.20
			$\int \prec(X /\!\!/ G)$	Proper equivariant homotopy type	Def. 2.23
$T \times G$	Borel & proper equivariance group		$\int\big(\prec(X /\!\!/ G)\big) /\!\!/ T$	Proper G-equivariant & Borel T-equivariant homotopy type	Ex. 2.43
$\mathscr{G}$	Simplicial group/∞-group	Not.2.2	$\mathscr{A} /\!\!/ \mathscr{G}$	Homotopy quotient	Prop. 2.7
$\mathcal{G}$	G-equivariant ∞-group	Rem. 2.42	$\mathcal{A} /\!\!/ \mathcal{G}$	G-equivariant homotopy quotient	(66)

Our notation for equivariant homotopy theory follows [SS20b]. The symbol "$\prec$" refers to proper equivariant objects ("orbi-singular objects"), parametrized over the orbit category (Def. 2.13) of the equivariance group (35):

Symbol		Meaning	Details
$G\mathrm{Actions}(\mathrm{TopologicalSpaces})$	G-actions on topological spaces	Category of topological spaces equipped with continuous action of the equivariance group G	Def. 2.11
$G\mathrm{Orbits}$	G-orbits	Category of canonical orbits G/H of the equivariance group, with equivariant maps between them	Def. 2.13
$\mathcal{G}\mathrm{SimplicialSets}$	G-equivariant simplicial sets	Category of contra-variant functors from G-orbits to simplicial sets	Def. 2.19
$\mathcal{G}\mathrm{VectorSpaces}_{\mathbb{R}}^{\vee}$	G-equivariant dual vector spaces	Category of co-variant functors from G-orbits to vector spaces	Def. 3.5
$\mathcal{G}\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$	G-equivariant dgc-algebras	Category of co-variant functors from G-orbits to connective differential graded-commutative algebras	Def. 3.30
$\mathcal{G}\mathrm{HomotopyTypes}$	G-equivariant homotopy types	Homotopy category of projective model category of contra-variant functors from G-orbits to simplicial sets	Def. 2.22

2 Equivariant non-abelian cohomology

In §2.1 we recall basics of ∞-groups and their ∞-actions and establish some technical Lemmas.
In §2.2 we recall basics of proper equivariant homotopy theory and introduce our running Example 2.44.
in §2.3 we introduce equivariant non-abelian cohomology theory.
in §2.4 we introduce twisted equivariant non-abelian cohomology theory.

Throughout, we illustrate all concepts in the running example of the $\mathbb{Z}_2^A$-equivariant and Sp(1)-parametrized twistor fibration (Example 2.44), the induced equivariant twistorial Cohomotopy theory (Def. 2.48) and its character image in equivariant de Rham cohomology (Example 3.74). We highlight that here both flavors of equivariance are involved.

	Borel equivariance	**Proper equivariance**
Equivariance group §2	$T = \mathrm{Sp}(1)$	$G = \mathbb{Z}_2$
Equivariant dR-cohomology §3	Borel-Weil-Cartan model	Bredon-type theory
Physical effect §4	Flux quantization: shift of G_4 by $\frac{1}{4}p_1$ & GS-mechanism	Inclusion of M5-brane locus into spacetime

We make free use of basic concepts from category theory and homotopy theory (for joint introduction see [Rie14][Ri20]), in particular of model category theory ([Qu67], review in [Ho99][Hir02][Lu09a, A.2]). Relevant concepts and facts are recalled in [FSS20d, §A].

For $\mathscr{C}$ a category, and $X, A \in \mathscr{C}$ a pair of objects, we write

$$\mathscr{C}(X,A) \in \mathrm{Sets} \tag{10}$$

for its set of morphisms from X to A. This assignment is, of course, a contravariant functor in its first argument, to be denoted:

$$\mathscr{C}(-;A) : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathrm{Sets}\,. \tag{11}$$

Elementary as it is, this is of profound interest whenever $\mathscr{C}$ is the *homotopy category* of a homotopy topos [TV05] [Lu09a][Re10], in which case the contravariant hom-functors (11) are *non-abelian cohomology theories* [To02] [Sc13][SS20b][FSS20d]. These subsume generalized and ordinary cohomology theories ([FSS20d, §2]), as well as their equivariant enhancements, which we consider below.

2.1 Homotopy theory of ∞-group actions

Plain homotopy theory.

Notation 2.1 (Classical homotopy category). **(i)** We write

$$\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\,,\ \mathrm{SimplicialSets}_{\mathrm{Qu}} \in \mathrm{ModelCategories} \tag{12}$$

for the classical model category structures on topological spaces and on simplicial sets, respectively ([Qu67, §II.3], review in [Hir15][GJ99]).

(ii) The classical Quillen equivalence

$$\mathrm{TopologicalSpaces}_{\mathrm{Qu}} \underset{\mathrm{Sing}}{\overset{|-|}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq_{\mathrm{Qu}}}}}} \mathrm{SimplicialSets}_{\mathrm{Qu}} \tag{13}$$

induces an equivalence between the corresponding homotopy categories, which we denote:

$$\mathrm{SimplicialSets} \xrightarrow[\text{localization}]{\gamma} \mathrm{HomotopyTypes} \;:=\; \mathrm{Ho}\big(\mathrm{SimplicialSets}_{\mathrm{Qu}}\big)\,. \tag{14}$$

(iii) We denote the localization functor from topological spaces to this classical homotopy category by "ʃ": [3]

$$\begin{array}{ccc} \mathrm{TopologicalSpaces} & \xrightarrow[\text{localization at weak homotopy equivalences}]{\text{shape } ʃ} & \mathrm{HomotopyTypes}\,. \\ & {\scriptstyle \text{form singular simplicial set (13)}} \searrow \quad \mathrm{SimplicialSets} \quad \nearrow {\scriptstyle \gamma \text{ localization (14)}} & \end{array} \tag{15}$$

[3]The "esh"-symbol "ʃ" stands for *shape* [Sc13, 3.4.5][Sh15, 9.7][SS20b, §3.1.1], following [Bo75], which for the well-behaved topological spaces of interest here is another term for their *homotopy type* [Lu09a, 7.1.6][Wa17, 4.6].

Borel-equivariant homotopy theory. We recall basics of Borel-equivariant homotopy theory, but in the generality of equivariance for ∞-group actions (for the broader picture see [NSS12a][SS20b, §2.2]).

Notation 2.2 (Model category of simplicial groups)**. (i)** We write

$$\mathrm{SimplicialGroups} \;:=\; \mathrm{Groups}\big(\mathrm{SimplicialSets}\big) \tag{16}$$

for the category of simplicial groups.
(ii) This becomes ([Qu67, §II.3.7]) a model category

$$\mathrm{SimplicialGroups}_{\mathrm{proj}} \;\in\; \mathrm{ModelCategories}$$

by taking the weak equivalences and fibrations to be those of $\mathrm{SimplicialSets}_{\mathrm{Qu}}$ (Notation 2.1).
(iii) We denote the homotopy category of this model structure by

$$\mathrm{SimplicialGroups}_{\mathrm{proj}} \xrightarrow[\substack{\text{localization at} \\ \text{weak homotopy equivalences}}]{\gamma} \mathrm{Groups}_\infty \;:=\; \mathrm{Ho}\big(\mathrm{SimplicialGroups}_{\mathrm{proj}}\big)\;. \tag{17}$$

and denote the generic object here by

$$\mathscr{G} \;\in\; \mathrm{SimplicialGroups} \xrightarrow{\gamma} \mathrm{Groups}_\infty\,.$$

Example 2.3 (Shapes of topological groups are ∞-groups)**.** For $T \in \mathrm{TopologicalGroups}$, its singular simplicial set (13) is canonically a simplicial group (16)

$$\mathrm{Sing}(T) \;\in\; \mathrm{SimplicialGroups}\,, \tag{18}$$

and, since the weak equivalence of simplicial groups are those of the underlying simplicial sets, its image in the homotopy category is the shape $\int T$ (15), now equipped with induced ∞-group structure (Notation 2.2):

$$\begin{array}{ccc} \mathrm{TopologicalGroups} & \xrightarrow[\text{localization at weak homotopy equivalences}]{\infty\text{-group shape } \int} & \mathrm{Groups}_\infty\,. \\ {\scriptstyle\text{form singular simplicial group (13), (18)}} \searrow & & \nearrow {\scriptstyle \gamma\ \text{localization (17)}} \\ & \mathrm{SimplicialGroups} & \end{array} \tag{19}$$

Notation 2.4 (Model category of reduced simplicial sets)**. (i)** We write

$$\mathrm{ReducedSimplicialSets} \hookrightarrow \mathrm{SimplicialSets}$$

for the full subcategory on those $S \in \mathrm{SimplicialSets}$ that have a single 0-cell, $S_0 = *$.
(ii) This becomes ([GJ99, §V, Prop. 6.2]) a model category with weak equivalences and cofibrations those of $\mathrm{SimplicialSets}_{\mathrm{Qu}}$ (Notation 2.1):

$$\mathrm{ReducedSimplicialSets}_{\mathrm{GJ}} \;\in\; \mathrm{ModelCategories}\,.$$

(iii) Since reduced simplicial sets model those homotopy types (14) which are *pointed and connected* (e.g. [NSS12b, Prop. 3.16]), we denote the corresponding homotopy category by

$$\mathrm{ReducedSimplicialSets}_{\mathrm{GJ}} \xrightarrow{\gamma} \mathrm{HomotopyTypes}^{*/}_{\geq 1} \;:=\; \mathrm{Ho}\big(\mathrm{ReducedSimplicialSets}_{\mathrm{GJ}}\big)\,. \tag{20}$$

Proposition 2.5 (Classifying space/loop space construction [GJ99, §V, Prop. 6.3][St12][NSS12b, §3.5])**.** *There exists a Quillen equivalence between the model categories of reduced simplicial sets (Notation 2.4) and that of simplicial groups (Notation 2.2)*

$$\mathrm{SimplicialGroups}_{\mathrm{proj}} \underset{\overline{W}}{\overset{\simeq_{\mathrm{Qu}}}{\rightleftarrows}} \mathrm{ReducedSimplicialSets} \tag{21}$$

whose derived adjunction is given by forming homotopy types of based loop spaces and of classifying spaces:

$$\underset{\infty\text{-groups}}{\mathrm{Groups}_\infty} \underset{\substack{B(-) := \mathbb{R}\overline{W}(-) \\ \text{classifying space}}}{\overset{\substack{\text{based loop } \infty\text{-group} \\ \Omega(-)}}{\underset{\simeq}{\rightleftarrows}}} \underset{\substack{\text{pointed \& connected} \\ \text{homotopy types}}}{\mathrm{HomotopyTypes}^{*/}_{\geq 1}} \tag{22}$$

Notation 2.6 (Homotopy theory of simplicial group actions)**.** For $\mathscr{G} \in \text{SimplicialGroups}$ (Notation 2.2)
(i) we write

$$\mathscr{G}\text{Actions} := \text{SimplicialFunctors}\big(B\mathscr{G}, \text{SimplicialSets}\big)$$

for the category of simplicial functors from the simplicial groupoid with a single object and $\mathscr{G}$ as its hom-object to the simplicial category of simplicial sets.
(ii) This becomes a model category by taking the weak equivalences and fibrations to be those of underling simplicial sets (evaluating at the single vertex of $B\mathscr{G}$):

$$\mathscr{G}\text{Actions}_{\text{proj}} \in \text{ModelCategories}$$

and we denote its homotopy category by:

$$\mathscr{G}\text{Actions}_{\text{proj}} \xrightarrow{\ \gamma\ } \text{Ho}\big(\mathscr{G}\text{Actions}_{\text{proj}}\big) =: \mathscr{G}\text{Actions}_{\infty}\,.$$

The following, Prop. 2.7, is pivotal for discussion of twisted non-abelian cohomology, notably for the notion of equivariant local coefficient bundles below in Def. 2.45; for more background and context see [NSS12a, §4][SS20b, §2.2][FSS20d, Prop. 2.28].

Proposition 2.7 (∞-Group actions equivalent to fibrations over classifying space [DDK80, Prop. 2.3][Sh15])**.** *For $\mathscr{G} \in \text{SimplicialGroups}$ (Notation 2.2), the simplicial Borel construction (e.g. [NSS12b, Prop. 3.37]) is the right adjoint of a Quillen equivalence*

$$\mathscr{G}\text{Actions}_{\text{proj}} \underset{\underset{\text{simplicial Borel construction}}{\mathscr{G}\,\circlearrowright\, X \;\mapsto\; \frac{X\times W\mathscr{G}}{\mathscr{G}}}}{\overset{\simeq_{\text{Qu}}}{\rightleftarrows}} \text{SimplicialSets}^{/\overline{W}\mathscr{G}}_{\text{Qu}} \tag{23}$$

between the projective model structure on simplicial $\mathscr{G}$-actions (Notation 2.6) and the slice model structure ([Hir02, §7.6.4]) of the classical model strcuture on simplicial sets (12) *over* $\overline{W}\mathscr{G}$ (21)*. Its derived equivalence of homotopy categories*

$$\underset{\infty\text{-group }\mathscr{G}}{\infty\text{-actions of}}\quad \mathscr{G}\text{Actions}_{\infty} \underset{\underset{\text{homotopy quotient}}{\mathscr{G}\,\circlearrowright\, A \;\mapsto\; A/\!\!/\mathscr{G}}}{\overset{\overset{\text{homotopy fiber}}{\text{hofib}_*(p) \,\leftarrow\!\shortmid\, (E \xrightarrow{p} BG)}}{\underset{\simeq}{\rightleftarrows}}} \text{Ho}\Big(\text{SimplicialSets}^{/\overline{W}\mathscr{G}}_{\text{Qu}}\Big) \quad \underset{\text{over classifying space }B\mathscr{G}}{\text{homotopy types fibered}} \tag{24}$$

is given in one direction by forming homotopy fibers of fibrations over BG and in the other by forming homotopy quotients of ∞-actions ([NSS12b, Prop. 3.73]):

$$\underset{\mathscr{G}\,\circlearrowright\, A}{\mathscr{G}\ \infty\text{-action on }A} \quad\longleftrightarrow\quad \begin{array}{ccc} A & \xrightarrow{\ \text{hofib}_*(\rho_A)\ } & A/\!\!/\mathscr{G} \\ & \underset{\mathscr{G}\text{-classifying space}}{A\text{-fibration over}} & \big\downarrow{\scriptstyle \rho_A} \\ & & B\mathscr{G}\,. \end{array} \tag{25}$$

Example 2.8 (Homotopy type of Borel construction)**.**
For $T \in \text{TopologicalGroups}$ and $T \circlearrowright X \in T\text{Actions}\big(\text{TopologicalSpaces}\big)$ (Def. 2.11), passage to singular simplicial sets (13) yields a simplicial action (Notation 2.6). The corresponding fibration (Prop. 2.7) is given by the topological shape (15) of the Borel construction:

$$\begin{array}{ccc} \int X & \xrightarrow{\ \text{hofib}(\rho_X)\ } & \int\big(\frac{X\times ET}{T}\big) =: \int\big(X/\!\!/T\big). \\ & & \big\downarrow{\scriptstyle \rho_X} \\ & & \int BT \end{array}$$

Lemma 2.9 (Pasting law [Lu09a, Lem 4.4.2.1])**.** *For $\mathscr{C}$ a model category, and given a pasting composite of two commuting squares*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \big\downarrow & & \big\downarrow & \text{(hpb)} & \big\downarrow \\ D & \longrightarrow & E & \longrightarrow & F \end{array}$$

such that the right square is homotopy Cartesian, then the left square is homotopy Cartesian if and only if the total rectangle is.

Lemma 2.10 (Homotopy fibers of homotopy-quotiented morphisms)**.**
Let $\mathcal{G} \in \mathrm{Groups}_\infty$ *(Notation 2.2) and* $(A, \rho_A) \xrightarrow{(f,\rho_f)} (A', \rho_{A'}) \in \mathcal{G}\mathrm{Actions}^{*/}_{\infty}$ *a morphism of* ∞*-actions (Notation 2.6) preserving an* $\mathcal{G}$*-fixed point* $\mathrm{pt} : * \to A \xrightarrow{f} A'$ *(see also [SS20b, Def. 2.97]). Then:*
(a) *The homotopy fiber of the homotopy-quotiented morphism* $f /\!\!/ \mathcal{G}$ (24) *coincides with the homotopy fiber of* f

$$\mathrm{hofib}_*\big(f /\!\!/ \mathcal{G}\big) \;\simeq\; \mathrm{hofib}_*(f)\;. \tag{26}$$

(b) *The homotopy fiber of* f *is canonically equipped with an* ∞*-action by* $\mathcal{G}$*:*

$$\big(\mathrm{hofib}_*(f),\, \rho_h\big) \;\in\; \mathcal{G}\mathrm{Actions}_\infty\;.$$

(c) *The corresponding homotopy quotient is equivalent to the homotopy fiber of the homotopy-quotiented morphism parametrized over BG, namely the following homotopy pullback:*

$$\begin{array}{ccc} \mathrm{hofib}_*(f) /\!\!/ \mathcal{G} \;\simeq\; \mathrm{hofib}_{B\mathcal{G}}\big(f /\!\!/ \mathcal{G}\big) & \longrightarrow & A /\!\!/ \mathcal{G} \\ \big\downarrow & \text{(hpb)} & \big\downarrow {\scriptstyle f /\!\!/ \mathcal{G}} \\ B\mathcal{G} & \xrightarrow[\mathrm{pt}' /\!\!/ \mathcal{G}]{} & A' /\!\!/ \mathcal{G}\,. \end{array} \tag{27}$$

Proof. Consider the following pasting diagrams:

$$\begin{array}{ccccc} \mathrm{hofib}_*\big(f /\!\!/ \mathcal{G}\big) & \longrightarrow & \mathrm{hofib}_{B\mathcal{G}}\big(f /\!\!/ \mathcal{G}\big) & \longrightarrow & A /\!\!/ \mathcal{G} \\ \big\downarrow & \text{(hpb)} & \big\downarrow & \text{(hpb)} & \big\downarrow {\scriptstyle f /\!\!/ \mathcal{G}} \\ * & \longrightarrow & B\mathcal{G} & \xrightarrow[\mathrm{pt}' /\!\!/ \mathcal{G}]{} & A' /\!\!/ \mathcal{G} \end{array} \;\;\simeq\;\; \begin{array}{ccccc} \mathrm{hofib}_*(f) & \longrightarrow & A & \longrightarrow & A /\!\!/ \mathcal{G} \\ \big\downarrow & \text{(hpb)} & \big\downarrow {\scriptstyle f} & \text{(hpb)} & \big\downarrow {\scriptstyle f /\!\!/ \mathcal{G}} \quad {\scriptstyle \rho_A} \\ * & \xrightarrow[\mathrm{pt}']{} & A' & \longrightarrow & A' /\!\!/ \mathcal{G} \\ & & \big\downarrow & \text{(pb)} & \big\downarrow {\scriptstyle \rho_{A'}} \\ & & * & \longrightarrow & B\mathcal{G} \end{array} \tag{28}$$

With the right Cartesian square (27) given, the pasting law (Lem. 2.9) identifies the top left objects on both sides as shown; in particular, the left square on the right gives (29). But, since the composite bottom morphism is the same basepoint inclusion on both sides, this implies:

$$\mathrm{hofib}_*\big(f /\!\!/ \mathcal{G}\big) \;\simeq\; \mathrm{hofib}_*(f)\;. \tag{29}$$

Moreover, the left Cartesian square on the left of (28) exhibits, by Prop. 2.7, a $\mathcal{G}$-action on $\mathrm{hofib}_*\big(f /\!\!/ \mathcal{G}\big)$ with homotopy quotient given by

$$\mathrm{hofib}_*\big(f /\!\!/ \mathcal{G}\big) /\!\!/ \mathcal{G} \;\simeq\; \mathrm{hofib}_{B\mathcal{G}}\big(f /\!\!/ \mathcal{G}\big). \tag{30}$$

The combination of the equivalences (26) and (30) yields the claimed equivalence in (27). □

2.2 Proper equivariant homotopy theory

We now recall relevant basics of proper[4] equivariant homotopy theory [tD79, §8][May96][Blu17] and introduce the examples of interest here.

G-Actions.

[4]Here by "proper equivariance" we refer to the fine notion of equivariant homotopy/cohomology in the sense of Bredon, as opposed to the coarse notion in the sense of Borel. For in-depth conceptual discussion of this distinction see [SS20b]. Besides the colloquial meaning of "proper", the action of our finite equivariance groups is necessarily *proper* in the technical sense of general topology (see Lemma 2.34 below), whence this terminology nicely matches that recently advocated in [DHLPS19].

Definition 2.11 (Group actions on topological spaces)**. (i)** For a given compact topological group, which serves the symmetry group of *Borel equivariance* in the following, generically to be denoted

$$\underset{\text{Borel equivariance group}}{T} \in \text{CompactTopologicalGroups}\,, \tag{31}$$

we write

$$T\text{Actions}\big(\text{TopologicalSpaces}\big) \in \text{Categories} \tag{32}$$

for the category whose objects are topological spaces X equipped with a continuous T-action

$$T \circlearrowright X \;:\; \begin{array}{ccc} T \times X & \xrightarrow{\text{continuous}} & X \\ (t\,,x) & \longmapsto & t \cdot x \end{array} \quad \text{such that:} \; \underset{x \in X}{\forall}\, \mathrm{e}\cdot x = x \;\; \text{and} \;\; \underset{\substack{x\in X \\ t_1, t_2 \in G}}{\forall} \big(t_1 \cdot (t_2 \cdot x)\big) = (t_1 \cdot t_2)\cdot x \tag{33}$$

and whose morphisms are T-equivariant continuous functions, which we denote as follows:

$$\overset{T}{\circlearrowright} X_1 \xrightarrow{\;f\;} \overset{T}{\circlearrowright} X_2 \qquad \Leftrightarrow \qquad \underset{\substack{x\in X \\ t \in T}}{\forall}\; f(t\cdot x) = t \cdot f(x)\,. \tag{34}$$

(ii) Throughout, our *proper* equivariance group is a finite group, to be denoted:

$$\underset{\text{proper equivariance group}}{G} \in \text{FiniteGroups}\,. \tag{35}$$

This finite group can be viewed as a topologically discrete topological group and we have the corresponding category (32) of continuous actions:

$$G\text{Actions}\big(\text{TopologicalSpaces}\big) \in \text{Categories}\,. \tag{36}$$

(iii) The full subcategory of the latter category on those objects, where also the topological space being acted on is discrete, is that of G-actions on sets:

$$G\text{Actions}\big(\text{Sets}\big) \hookrightarrow G\text{Actions}\big(\text{TopologicalSpaces}\big)\,. \tag{37}$$

(iv) Regarding the direct product group of the Borel equivariance group (31) with the proper equivariance group (35) as a compact topological group

$$\underset{\text{Borel \& proper equivariance group}}{T \times G} \in \text{CompactTopologicalGroups}\,,$$

we have the category of topological actions of this product group. This contains the previous categories, (32) and (36), as full subcategories (via equipping a space with trivial action)

$$T\text{Actions}\big(\text{TopologicalSpaces}\big) \hookrightarrow \big(T\times G\big)\text{Actions}\big(\text{TopologicalSpaces}\big) \hookleftarrow G\text{Actions}\big(\text{TopologicalSpaces}\big)\,. \tag{38}$$

Example 2.12 (Representation spheres)**.** Let $V \in T\text{Representations}^{\text{fin}}_{\mathbb{R}}$ be a finite-dimensional linear representation of a compact topological group (31). Then the one-point compactification of V (the topological sphere of the same dimension, e.g. [Ke55, p. 150]) inherits a topological T-action (Def. 2.11) via stereographic projection, denoted

$$S^V \in T\text{Actions}\big(\text{TopologicalSpaces}\big)$$

and called the *representation sphere* of V (e.g. [Blu17, §1.1.5][SS19a, §3]).

Definition 2.13 (Orbit category)**.** The *category of G-orbits* or *orbit category* of the equivariance group G (35)

$$G\text{Orbits} \hookrightarrow G\text{Actions}\big(\text{Sets}\big) \in \text{Categories}$$

is (up to equivalence of categories) the full subcategory of discrete G-actions (37) on the coset spaces G/H (which are discrete spaces, since G is assumed to be finite) for all subgroup inclusions $H \overset{\iota}{\hookrightarrow} G$.

Example 2.14 (Explicit parameterization of morphisms of GOrbits)**.** The hom-sets (10) in the G-orbit category (Def. 2.13) from any G/H_1 to any G/H_2 are in bijection with sets of conjugations, inside G, of H_1 into subgroups of H_2, modulo conjugations in H_2:

$$G\mathrm{Orbits}\big(G/H_1,\, G/H_2\big) \;\simeq\; \frac{\big\{\phi : H_1 \hookrightarrow H_2,\; g \in G \,|\, \mathrm{Ad}_{g^{-1}} \circ \iota_1 = \iota_2 \circ \phi\big\}}{\big((\phi, g) \sim (\mathrm{Ad}_{h_2^{-1}} \circ \phi, g h_2) \,|\, h_2 \in H_2\big)}\,. \tag{39}$$

(Here "Ad" denotes the adjoint action of the group on itself, and $H_i \overset{\iota_i}{\hookrightarrow} G$ are the two subgroup inclusions.)

Example 2.15 (Orbit category of $\mathbb{Z}_2$)**.** The orbit category (Def. 2.13) of the cyclic group $\mathbb{Z}_2 := \{\mathrm{e}, \sigma \,|\, \sigma \circ \sigma = \mathrm{e}\}$ is

$$\mathbb{Z}_2\mathrm{Orbits} \;\simeq\; \left\{ \overset{\circlearrowleft \mathbb{Z}_2}{\mathbb{Z}_2/1} \xrightarrow{\;\exists !\;} \overset{\circlearrowleft 1}{\mathbb{Z}_2/\mathbb{Z}_2} \right\}.$$

Hence its hom-sets (10) are:

$$\begin{array}{ll} \mathbb{Z}_2\mathrm{Orbits}\big(\mathbb{Z}_2/1\,,\mathbb{Z}_2/1\;\big) \simeq \mathbb{Z}_2\,, & \mathbb{Z}_2\mathrm{Orbits}\big(\mathbb{Z}_2/\mathbb{Z}_2\,,\mathbb{Z}_2/\mathbb{Z}_2\big) \simeq 1\,, \\ \mathbb{Z}_2\mathrm{Orbits}\big(\mathbb{Z}_2/1\,,\mathbb{Z}_2/\mathbb{Z}_2\big) \simeq *\,, & \mathbb{Z}_2\mathrm{Orbits}\big(\mathbb{Z}_2/\mathbb{Z}_2\,,\mathbb{Z}_2/1\;\big) \simeq \varnothing\,. \end{array} \tag{40}$$

Example 2.16 (Automorphism groups in orbit category)**.** For G a finite group and $H \subset G$ a subgroup, the endomorphisms of $G/H \in G$Orbits (Def. 2.13) form the *Weyl group* $\mathrm{W}_G(H)$ (e.g. [May96, p. 13]) of H in G,

$$\mathrm{End}_{G\mathrm{Orbits}}(G/H) \;\simeq\; \mathrm{Aut}_{G\mathrm{Orbits}}(G/H) \;=\; \mathrm{W}_G(H) \;:=\; \mathrm{N}_G(H)/H\,, \tag{41}$$

namely the quotient group by H of the normalizer $\mathrm{N}_G(H)$ of H in G. For instance:

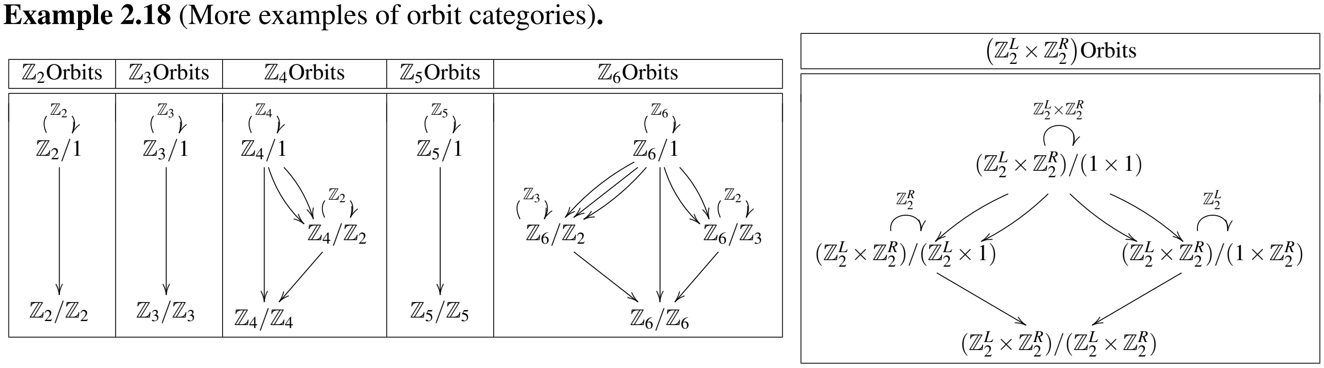
$$\mathrm{W}_G(1) \;=\; G, \quad \mathrm{W}_G(G) \;=\; 1; \qquad \text{generally:} \quad \underset{\text{normal}}{H \subset G} \;\Rightarrow\; \mathrm{W}_G(H) \;=\; G/H\,.$$

Generally:

Example 2.17 (Hom-sets in orbit category via Weyl groups)**.** For any two subgroups $K, H \subset G$, the hom-set (10) in the G-orbit category (Def. 2.13) between their corresponding coset spaces is, as a right $\mathrm{W}_G(H)$-set via Example 2.16, a disjoint union of copies of $\mathrm{W}_G(H)$, one for each way of conjugating K into a subgroup of H:

$$G\mathrm{Orbits}\big(G/K\,, G/H\big) \;\simeq \coprod_{\substack{g \in G/\mathrm{N}_G(K) \\ \text{s.t. } g^{-1}Kg \subset H}} g\mathrm{W}_G(H) \quad \in \mathrm{W}_G(H)\mathrm{Actions}\big(\mathrm{Sets}\big)\,. \tag{42}$$

Example 2.18 (More examples of orbit categories)**.**



Equivariant homotopy types.

Definition 2.19 (Equivariant simplicial sets)**.** We write

$$G\mathrm{SimplicialSets} \;:=\; \mathrm{Functors}\big(G\mathrm{Orbits}^{\mathrm{op}}\,, \mathrm{SimplicialSets}\big)$$

for the category of functors from the opposite of G-orbits (Def. 2.13) to simplicial sets.

Example 2.20 (Systems of fixed loci of topological G-actions)**.** Let $G \circlearrowright X \in G\mathrm{Actions}\big(\mathrm{TopologicalSpaces}\big)$ (Def. 2.11). For $H \subset G$ any subgroup, a G-equivariant function (34)

$$
\overset{G}{\circlearrowright} G/H \xrightarrow{\;f\;} \overset{G}{\circlearrowright} X
\qquad \Leftrightarrow \qquad
f([\mathrm{e}]) \in \underset{\text{H-fixed locus}}{X^H} := \left\{ x \in X \;\middle|\; \underset{h \in H}{\forall} \big(h \cdot x = x\big) \right\} \subset X
\tag{43}
$$

from the corresponding G-orbit (Def. 2.13) is determined by its image $f([\mathrm{e}]) \in X$ of the class of the neutral element, and that image has to be fixed by the action of $H \subset G$ of X. Therefore, the corresponding G-equivariant mapping spaces

$$
\mathrm{Maps}\big(G/H, X\big)^G \simeq X^H
$$

are the topological subspaces of H-fixed points inside X, the *H-fixed loci in $G \circlearrowright H$*. By functoriality of the mapping-space construction, these fixed point loci are exhibited as arranging into a contravariant functor on the G-orbit category (Def. 2.13):

$$
\curlyvee(X /\!\!/ G): \qquad G\mathrm{Orbits}^{\mathrm{op}} \xrightarrow{\;\mathrm{Maps}(-,X)^G\;} \mathrm{TopologicalSpaces}
\tag{44}
$$

$$
\begin{array}{ccccl}
G/H_1 & \longmapsto & X^{H_1} & & \text{H_1-fixed locus} \\
{\scriptstyle [(\mathrm{id},g)]}\big\downarrow & & \simeq \big\uparrow {\scriptstyle g\cdot(-)} & & \text{residual action on H_2-fixed locus} \\
G/H_1 & \longmapsto & X^{H_1} & & \\
{\scriptstyle [(\phi,e)]}\big\downarrow & & \big\uparrow {\scriptstyle \phi^*} & & \text{inclusion of H_2-fixed locus} \\
G/H_2 & \longmapsto & X^{H_2} & & \text{H_2-fixed locus}
\end{array}
$$

Here we used Example 2.14 to make explicit the nature of the continuous functions between fixed point spaces that this functor assigns to morphisms of GOrbits. In particular, we see from Example 2.16 that the residual action on the H-fixed locus X^H is by the Weyl group $\mathrm{W}_G(H)$ (41). Postcomposing (44) with the singular simplicial set functor (13) yields an equivariant simplicial set (Def. 2.19), to be denoted (the notation follows [SS20b, §3.2, 5.1]):

$$
G \circlearrowright X \;\longmapsto\; \mathrm{Sing}\big(\curlyvee\big(X /\!\!/ G\big)\big) := \mathrm{Sing}\Big(\mathrm{Maps}\big(-, X\big)^G\Big) \in \underset{\curlyvee}{G}\mathrm{SimplicialSets}\,.
\tag{45}
$$

Proposition 2.21 (Model category of equivariant simplicial sets [Hir02, Thm. 11.6.1][Gui06, Thm. 3.3][St16, §2.2])**.** *The category of equivariant simplicial sets (Def. 2.19) carries a model category structure whose*

(a) W – *weak equivalences are the weak equivalences of* $\mathrm{SimplicialSets}_{\mathrm{Qu}}$ *over each* $G/H \in G$ Orbits*;*

(b) Fib – *fibrations are the weak equivalences of* $\mathrm{SimplicialSets}_{\mathrm{Qu}}$ *over each* $G/H \in G$ Orbits*.*

We denote this model category by $\underset{\curlyvee}{G}\mathrm{SimplicialSets}_{\mathrm{proj}} \in \mathrm{ModelCategories}\,.$

Definition 2.22 (Equivariant homotopy types)**.** We denote the homotopy category of the projective model structure on equivariant simplicial sets (Prop. 2.21) by

$$
\underset{\curlyvee}{G}\mathrm{SimplicialSets}_{\mathrm{proj}} \xrightarrow[\text{localization}]{\;\gamma\;} \underset{\curlyvee}{G}\mathrm{HomotopyTypes} := \mathrm{Ho}\big(\underset{\curlyvee}{G}\mathrm{SimplicialSets}_{\mathrm{proj}}\big)\,.
\tag{46}
$$

The key source of equivariant homotopy types is the shapes of orbi-singularized homotopy quotients of topological spaces by continuous group actions (we follow [SS20b, §3.2] in terminology and notation):

Definition 2.23 (Equivariant shape)**.** The composite of forming systems of fixed loci (Example 2.20) with localization to equivariant homotopy types (Def. 2.22) is the *equivariant shape* operation, generalizing the plain shape (15):

$$
\begin{array}{ccc}
G\mathrm{Actions}\big(\mathrm{TopologicalSpaces}\big) & \xrightarrow[\text{localization at fixed locus-wise weak homotopy equivalences}]{\overset{\text{equivariant shape}}{G \circlearrowright X \;\longmapsto\; \int \curlyvee(X /\!\!/ G)}} & \underset{\curlyvee}{G}\mathrm{HomotopyTypes}\,. \\
& {\scriptstyle \text{form singular equivariant simplicial set (45)}} \searrow \qquad \nearrow {\scriptstyle \gamma \text{ localization (46)}} & \\
& \underset{\curlyvee}{G}\mathrm{SimplicialSets} &
\end{array}
\tag{47}
$$

Example 2.24 (Smooth equivariant homotopy types)**.** A topological space X equipped with trivial G-action has equivariant shape (Def. 2.23) given by the functor on the orbit category which is constant on its ordinary shape (15)

$$\begin{array}{ccc} \text{TopologicalSpaces} & \xrightarrow[\text{shape}]{\quad \int \quad} & \text{HomotopyTypes} \\ \big\downarrow {\scriptstyle \text{equip with trivial action}} & & \big\downarrow {\scriptstyle \text{Smth}} \; {\scriptstyle \text{form constant functor on orbit category}} \\ G\text{Actions}\big(\text{TopologicalSpaces}\big) & \xrightarrow[\text{equivariant shape}]{\quad \int(-/\!\!/ G) \quad} & G\text{HomotopyTypes}\,. \end{array} \tag{48}$$

For brevity, we will mostly leave this embedding notationally implicit and write

$$X \;:=\; \text{Smth}\textstyle\int X \;\in\; G\text{HomotopyTypes}\,. \tag{49}$$

Elmendorf's theorem. In fact, every equivariant homotopy type (Def. 2.22) is the equivariant shape (Def. 2.23) of some topological space with G-action (Def. 2.11). This is the content of Elmendorf's theorem ([El83], see Prop. 2.26 below). Due to this fact, topological G-actions in equivariant homotopy theory are often conflated with their G-equivariant shape, and jointly referred to as *G-spaces* (e.g., [tD79, §8][Blu17, §1]).

Proposition 2.25 (Model category of simplicial G-actions and fixed loci [Gui06, Thm. 3.12][St16, Prop. 2.6])**.** *The category* $G\text{Actions}\big(\text{SimplicialSets}\big)$ *of G-actions $G \circlearrowright S$ on simplicial sets (analogous to Def. 2.11) carries a model category structure whose weak equivalences and fibrations are those that become so in the classical model structure on simplicial sets* (12) *under the functor (analogous to Example 2.20)*

$$\begin{array}{ccc} G\text{Actions}\big(\text{SimplicialSets}\big) & \xrightarrow{\;\text{Maps}(-,-)^G\;} & G\text{SimplicialSets} \\ G \circlearrowright S & \longmapsto & \big(G/H \;\mapsto\; S^H\big) \end{array} \tag{50}$$

which sends a G-action $G \circlearrowright S$ to its system of H-fixed loci parametrized over $G/H \in G\text{Orbits}$.

We denote this model category by

$$G\text{Actions}\big(\text{SimplicialSets}\big)_{\text{fine}} \;\in\; \text{ModelCategories}\,.$$

Proposition 2.26 (Elmendorf's theorem via model categories [St16, Thm. 3.17][Gui06, Prop. 3.15])**.** *The functor assigning systems of simplicial fixed loci* (50) *is the right adjoint in a Quillen equivalence*

$$G\text{Actions}\big(\text{SimplicialSets}\big)_{\text{fine}} \; \underset{\text{Maps}(-,-)^G}{\overset{(-)(G/1)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq_{\text{Qu}}}}}} \; G\text{SimplicialSets}_{\text{proj}} \tag{51}$$

between the fine model structure on simplicial G-actions (Prop. 2.25) and the model category of equivariant simplicial sets (Prop. 2.21).

Examples of equivariant homotopy types.

Example 2.27 (G^{ADE}-equivariant 4-sphere)**.** Let

$$G := G^{\text{ADE}} \;\subset\; \text{Spin}(3) \;\simeq\; \text{Sp}(1)$$

be a finite subgroup of the Spin group in dimension 3; these are famously classified along an ADE-pattern (reviewed in [HSS18, Rem. A.9]). Via the exceptional isomorphism with the quaternionic unitary group, this induces a canonical smooth action (Def. 2.35) on the Euclidean 4-space underlying the space of quaternions (reviewed as [HSS18, Prop. A.8]) and hence also on the corresponding representation 4-sphere (Example 2.12):

$$\overset{G^{\text{ADE}}}{\underset{\mathbb{R}^4}{\curvearrowright}} \;,\quad \overset{G^{\text{ADE}}}{\underset{S^4}{\curvearrowright}} \;\in\; G^{\text{ADE}}\text{Actions}\big(\text{SmoothManifolds}\big)\,.$$

(a) The corresponding ADE-orbifolds (Def. 2.36)

$$\prec\!\left(S^4 /\!\!/ G^{\mathrm{ADE}}\right) \;\in\; G^{\mathrm{ADE}}\mathrm{Orbifolds}$$

appear in spacetime geometries of 1/2BPS black M5-branes [dMFO10, §8.3][HSS18, §2.2.6][SS19a, §4.2] (discussed in §4 below).

(b) The corresponding G^{ADE}-equivariant homotopy types (Def. 2.22) (their equivariant shape, Def. 2.23)

$$\underset{\text{of 4-sphere}}{\overset{G^{\mathrm{ADE}}\text{-equivariant shape}}{\int \prec\!\left(S^4 /\!\!/ G^{\mathrm{ADE}}\right)}} \;\in\; G^{\mathrm{ADE}}\mathrm{HomotopyTypes}$$

are the coefficients of ADE-equivariant Cohomotopy theory [HSS18, §5.2][SS19a, §3] (lifted to equivariant twistorial Cohomotopy theory below in Def. 2.48).

Example 2.28 ($\mathbb{Z}_2^A$-equivariant twistor space)**.** Consider the quaternion unitary group (e.g. [FSS20c, §A]) with its two commuting subgroups from (7) and (8):

$$\mathbb{Z}_2^A,\, \mathrm{Sp}(1) \;\subset\; \mathrm{Sp}(2) \;:=\; \left\{ g \in \mathrm{Mat}_{2\times 2}(\mathbb{H}) \,\middle|\, g \cdot g^\dagger \,=\, 1 \right\}. \tag{52}$$

Their canonical action on $\mathbb{H}^2 \simeq_{\mathbb{R}} \mathbb{R}^8$ by left matrix multiplication induces an action (6) on $\mathbb{C}P^3$ ("twistor space"). The fixed locus (43) of the subgroup $\mathbb{Z}_2^A$ (7) under this action is evidently given by those $[z_1 : z_2 : z_3 : z_4] \in \mathbb{C}P^3$ such that $z_1 + \mathrm{j}\cdot z_2 = z_3 + \mathrm{j}\cdot z_4 \in \mathbb{H}$. Since these are exactly the elements that are sent by the twistor fibration $t_{\mathbb{H}}$ (5) to the base point $[1:1] \in \mathbb{H}P^1$, the $\mathbb{Z}_2^A$-fixed locus in twistor space $\mathbb{C}P^3$ coincides with the S^2-fiber of the twistor fibration $t_{\mathbb{H}}$ (5):

$$\left(\mathbb{C}P^3\right)^{\mathbb{Z}_2^A} \;\simeq\; S^2 \xhookrightarrow{\;\mathrm{fib}(t_{\mathbb{H}})\;} \mathbb{C}P^3. \tag{53}$$

Hence the $\mathbb{Z}_2$-equivariant homotopy type (15) of twistor space with its $\mathbb{Z}_2^A$ action (6) is given by the following functor on the $\mathbb{Z}_2$-orbit category (2.15):

$$\underset{\text{of twistor space}}{\overset{\mathbb{Z}_2^A\text{-equivariant shape}}{\int \prec\!\left(\mathbb{C}P^3 /\!\!/ \mathbb{Z}_2^A\right)}} \;:\; \begin{array}{ccc} \overset{\curvearrowright \mathbb{Z}_2}{\mathbb{Z}_2/1} & \longmapsto & \overset{\curvearrowright \mathbb{Z}_2^A}{\int \mathbb{C}P^3} \\ \big\downarrow & & \big\uparrow{\scriptstyle \mathrm{fib}(t_{\mathbb{H}})}\ \ {\scriptstyle \text{fiber inclusion of twistor fibration}} \\ \mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \int S^2 \end{array} \tag{54}$$

Equivariant homotopy groups.

Definition 2.29 (Equivariant groups)**. (i)** We write

$$\prec\! G\mathrm{Groups} \;:=\; \mathrm{Functors}\!\left(G\mathrm{Orbits}^{\mathrm{op}},\, \mathrm{Groups}\right)$$

for the category of contravariant functors on the G-orbit category (Def. 2.13) with values in groups.

(ii) We write

$$\prec\! G\mathrm{AbelianGroups} \;:=\; \mathrm{Functors}\!\left(G\mathrm{Orbits},\, \mathrm{AbelianGroups}\right)$$

for the sub-category of contravariant functors with values in abelian groups.

Example 2.30 (Equivariant singular homology groups)**.** For $\mathcal{X} \in \prec\! G\mathrm{HomotopyTypes}$ (Def. 2.22), $A \in \mathrm{AbelianGroups}$, the ordinary A-homology groups in degree $n \in \mathbb{N}$ of the stages of $\mathcal{X}$ form an equivariant abelian group in the sense of Def. 2.29, to be denoted:

$$\underline{H}_n(\mathcal{X}; A) \;:\; G/H \longmapsto H_n\big(\mathcal{X}(G/H); A\big).$$

Definition 2.31 (Equivariant homotopy groups)**.**

(i) For $\mathcal{X} \in \prec\! G\mathrm{HomotopyTypes}$ (Def. 2.22), $\prec(*/\!\!/G) \xrightarrow{\;x\;} \mathcal{X}$ a base-point, and $n \in \mathbb{N}$, we say that the nth *equivariant homotopy group* of $\mathcal{X}$ at x is the equivariant group (Def. 2.29) which is stage-wise the ordinary nth homotopy group, to be denoted:

$$\underline{\pi}_n(\mathcal{X}, x) \;:=\; \Big(G/H \;\mapsto\; \pi_n\big(\mathcal{X}(G/H), x(G/H)\big)\Big). \tag{55}$$

(ii) Similarly, for $G \circlearrowright X \in G\mathrm{Actions}\big(\mathrm{TopologicalSpaces}\big)$ (Def. 2.11), $G \circlearrowright * \xrightarrow{x} G \circlearrowright X$ a fixed base point, and $n \in \mathbb{N}$, we say that the nth *equivariant homotopy group* of $G \circlearrowright X$ is that (55) of its equivariant shape (15):

$$\underline{\pi}_n\big(X,x\big) := \underline{\pi}_n\Big(\int \curlyvee\big(X /\!\!/ G\big), \int \curlyvee\big(x /\!\!/ G\big)\Big) = \Big(G/H \mapsto \pi_n\big(X^H, x\big)\Big). \tag{56}$$

Definition 2.32 (Equivariant connected homotopy types)**.** We write

$$\underset{\curlyvee}{G}\mathrm{HomotopyTypes}_{\geq 1} \hookrightarrow \underset{\curlyvee}{G}\mathrm{HomotopyTypes} \tag{57}$$

for the full subcategory on those equivariant homotopy types $\mathcal{X}$ (Def. 2.22) which

(a) are equivariantly connected, in that $\mathcal{X}(G/H) \in \mathrm{HomotopyTypes}$ is connected for all $H \subset G$;

(b) admit an equivariant base point $\curlyvee\big(* /\!\!/ G\big) \to \mathcal{X}$.

Definition 2.33 (Equivariant 1-connected homotopy types)**.** **(i)** We write

$$\underset{\curlyvee}{G}\mathrm{HomotopyTypes}_{\geq 2} \hookrightarrow \underset{\curlyvee}{G}\mathrm{HomotopyTypes}_{\geq 1} \hookrightarrow \underset{\curlyvee}{G}\mathrm{HomotopyTypes} \tag{58}$$

for the further full subcategory on those equivariant homotopy types $\mathcal{X}$ (Def. 2.22) which

(a) are equivariantly connected and admit an equivariant base point (Def. 2.32);

(b) have trivial first equivariant homotopy group (Def. 2.31) at that base point:

$$\underline{\pi}_1(\mathcal{X}, x) = \underline{1}.$$

(ii) By the Hurewicz theorem, this implies that the equivariant real cohomology groups (Example 2.30) of these objects are trivial in degrees ≤ 1

$$X \in \underset{\curlyvee}{G}\mathrm{HomotopyTypes}_{\geq 2} \quad \Rightarrow \quad \Big(\underline{H}^0(X) \simeq \underline{\mathbb{R}} \ \text{ and } \ \underline{H}^1(X) \simeq 0\Big).$$

(iii) We write

$$\underset{\curlyvee}{G}\mathrm{HomotopyTypes}^{\mathrm{fin}_{\mathbb{R}}}_{\geq 2} \hookrightarrow \underset{\curlyvee}{G}\mathrm{HomotopyTypes}_{\geq 2} \hookrightarrow \underset{\curlyvee}{G}\mathrm{HomotopyTypes}$$

for the further full subcategory of those equivariant 1-connected homotopy types (58) which are of *finite type* over $\mathbb{R}$, in that all their equivariant real homology groups (Example 2.30) are finite-dimensional:

$$\underset{\substack{H \subset G \\ n \in \mathbb{N}}}{\forall} \dim_{\mathbb{R}}\Big(H_n\big(\mathcal{X}(G/H); \mathbb{R}\big)\Big) < \infty.$$

***G*-Orbifolds.** Given a smooth manifold X equipped with a smooth group action $G \circlearrowright X$, there are several somewhat different mathematical notions of what exactly counts as the corresponding *quotient orbifold* (review in [MM03][Ka08, §6][IKZ10]).

- First, there is the singular quotient space X/G that dominates the early literature on orbifolds [Sa56][Sa57] [Th80][Hae84] as well as the contemporary physics literature [BL99, §1.3].
- Second, there is the smooth stacky homotopy quotient $X /\!\!/ G$ that has become the popular model for orbifolds among Lie theorists [MP97][Moe02][Ler08][Am12].
- Third, there is the fine incarnation of orbifolds *orbisingular homotopy quotients* $\curlyvee\big(X /\!\!/ G\big)$ in singular cohesive homotopy theory [SS20b], which unifies the above two perspectives and lifts them to make orbifolds carry proper equivariant differential cohomology theories.

Here we extract from [SS20b] the essence of this latter fine perspective that is necessary and convenient for the present purpose, as Def. 2.36 below.

Lemma 2.34 (Fixed loci of finite smooth actions are smooth manifolds)**.** *If $G \circlearrowright X \in G\mathrm{Actions}\big(\mathrm{TopologicalSpaces}\big)$ (Def. 2.11) is such that X admits the structure of a smooth manifold and such that the action* (33) *of G is smooth, then the fixed loci $X^H \hookrightarrow X$* (43) *are themselves smooth submanifolds.*

Proof. Since G is assumed to be finite (35), its smooth action is proper (e.g. [Lee12, Cor. 21.6]). But in smooth manifolds with proper smooth G-action, every closed submanifold inside a fixed locus has a G-equivariant tubular neighborhood [Bre72, §VI, Thm. 2.2][Ka07, Thm. 4.4]. This applies, in particular, to individual fixed points, where it says that each such has a neighborhood in the fixed locus diffeomorphic to an open ball. □

Definition 2.35 (Smooth group actions on smooth manifolds)**. (i)** We write

$$G\mathrm{Actions}\big(\mathrm{SmoothManifolds}\big) \longrightarrow G\mathrm{Actions}\big(\mathrm{TopologicalSpaces}\big)$$

for the category of smooth manifolds equipped with G-actions on the underlying topological spaces (Def. 2.11) which are smooth.

(ii) Similarly, if the compact Borel-equivariance group (31) is equipped with smooth structure making it a Lie group

$$T \;\in\; \mathrm{CompactLieGroups} \longrightarrow \mathrm{CompactTopologicalGroups}\,,$$

we write

$$\big(T\times G\big)\mathrm{Actions}\big(\mathrm{SmoothManifolds}\big) \longrightarrow \big(T\times G\big)\mathrm{Actions}\big(\mathrm{TopologicalSpaces}\big)$$

for the category of smooth manifolds equipped with $T\times G$-actions on the underlying topological spaces (Def. 2.11) which are smooth.

Definition 2.36 (G-Orbifolds [SS20b])**. (i)** We write

$$G\mathrm{Orbifolds} \;:=\; \mathrm{Functors}\big(G\mathrm{Orbits}^{\mathrm{op}},\, \mathrm{SmoothManifolds}\big) \tag{59}$$

for the category of contravariant functors from G-orbits (Def. 2.13) to smooth manifolds.

(ii) By Lemma 2.34, the system of fixed loci (44) of a smooth action $G \curvearrowright X$ (Def. 2.35) takes values in smooth manifolds

$$G \curvearrowright X \text{ smoothly} \quad\Rightarrow\quad \curlyvee\big(X /\!\!/ G\big) \;:\; G\mathrm{Orbits}^{\mathrm{op}} \dashrightarrow \mathrm{SmoothManifolds} \longrightarrow \mathrm{TopologicalSpaces}\,, \tag{60}$$

and hence witnesses an object $\curlyvee\big(X /\!\!/ G\big) \in G\mathrm{Orbifolds}$ (2.36) which is a smooth geometric refinement of the underlying equivariant homotopy type (Def. 2.23), in that we have the following commuting diagram of functors:

$$
\begin{array}{ccc}
G\mathrm{Actions}\big(\mathrm{SmoothManifolds}\big) & \xrightarrow{\;G \curvearrowright X \;\mapsto\; \curlyvee(X /\!\!/ G)\;} & G\mathrm{Orbifolds} \\
{\scriptsize\substack{\text{forget smooth structure} \\ (60)}}\big\downarrow & & \big\downarrow \int \;{\scriptsize\substack{\text{equivariant shape} \\ \text{(Def. 2.23)}}} \\
G\mathrm{Actions}\big(\mathrm{TopologicalSpaces}\big) & \underset{(45)}{\xrightarrow{\;G \curvearrowright X \;\mapsto\; \int \curlyvee(X /\!\!/ G)\;}} & \mathcal{G}\mathrm{HomotopyTypes}\,.
\end{array}
$$

2.3 Equivariant non-abelian cohomology theories

We introduce the general concept of equivariant non-abelian cohomology theories, in direct generalization of [FSS20d, §2.1], and consider some examples. This is in preparation for the twisted case in the next subsection.

In equivariant generalization of [FSS20d, §2.1], we set:

Definition 2.37 (Equivariant non-abelian cohomology)**.** Let $\mathcal{X}, \mathcal{A} \in \mathcal{G}\mathrm{HomotopyTypes}$ (Def. 2.22).

(i) The *proper G-equivariant non-abelian cohomology* of $\mathcal{X}$ with coefficients in $\mathcal{A}$ is the hom-set (10)

$$\overset{\substack{\text{equivariant} \\ \text{non-abelian cohomology}}}{H\big(\mathcal{X};\mathcal{A}\big)} \;:=\; \mathcal{G}\mathrm{HomotopyTypes}\big(\mathcal{X}\,,\,\mathcal{A}\big)\,.$$

(ii) For $X \in G\mathrm{Actions}\big(\mathrm{TopologicalSpaces}\big)$ (Def. 2.11), with induced equivariant homotopy type $\int\curlyvee\big(X /\!\!/ G\big)$ (15), we write

$$\overset{\substack{\text{equivariant} \\ \text{non-abelian cohomology}}}{H_G\big(X;\mathcal{A}\big)} \;:=\; H\Big(\int\curlyvee\big(X /\!\!/ G\big);\mathcal{A}\Big) \;:=\; \mathcal{G}\mathrm{HomotopyTypes}\Big(\int\curlyvee\big(X /\!\!/ G\big)\,,\,\mathcal{A}\Big)\,.$$

(iii) We call the corresponding contravariant functor

$$G\mathrm{Actions}\big(\mathrm{TopologicalSpaces}\big)^{\mathrm{op}} \xrightarrow{\;\int \prec(-/\!\!/G)\;} \underset{\prec}{G}\mathrm{HomotopyTypes}^{\mathrm{op}} \xrightarrow{\;H(-;\mathcal{A})\;} \mathrm{Sets} \qquad \text{with composite } H_G(-;\mathcal{A}) \tag{61}$$

the *equivariant non-abelian cohomology theory* with coefficients in $\mathcal{A}$.

Equivariant ordinary cohomology.

Example 2.38 (Equivariant representation ring)**.** For H a finite group and $\mathbb{F}$ a field, write

$$\mathrm{Rep}_{\mathbb{F}}(X) \;\in\; \mathrm{Rings} \longrightarrow \mathrm{AbelianGroups} \tag{62}$$

for the additive abelian group underlying the representation ring of H (i.e., the Grothendieck group of the semi-group of finite-dimensional $\mathbb{F}$-linear H-representations under tensor product of representations, review in [BSS19, §2.1]). Under the evident restriction of representations to subgroups and under conjugation action on representations, these groups arrange into a contravariant functor on the G-orbit category (Def. 2.13)

$$\begin{array}{ccccc} \underline{\mathrm{Rep}_{\mathbb{F}}} : G\mathrm{Orbits}^{\mathrm{op}} & \longrightarrow & \mathrm{AbelianGroups} & \in & \underset{\prec}{G}\mathrm{AbelianGroups} \\ G/H & \longmapsto & \mathrm{Rep}_{\mathbb{F}}(H) & & \end{array} \tag{63}$$

and hence consitute an equivariant abelian group (Def. 2.29).

Example 2.39 (Bredon cohomology [Bre67a, p. 3][Bre67b, Thm. 2.11 & (6.1)][GM95, p. 10])**.**
Given $\underline{A} \in \underset{\prec}{G}\mathrm{AbelianGroups}$ (Def. 2.29) and $n \in \mathbb{N}$:
(i) There is the *Eilenberg-MacLane G-space*

$$\mathcal{K}(\underline{A},n) \;\in\; \underset{\prec}{G}\mathrm{HomotopyTypes} \tag{64}$$

in equivariant connected homotopy types (Def. 2.22), characterized by the fact that it admits a fixed point with equivariant homotopy groups (Def. 2.31) given by

$$\underline{\pi}_k\big(\mathcal{K}(\underline{A},n)\big) \;\simeq\; \begin{cases} \underline{A} & \mid \;\; k=n, \\ 0 & \mid \;\; \text{otherwise.} \end{cases}$$

(ii) The *ordinary equivariant cohomology* or *Bredon cohomology* in degree n of $X \in G\mathrm{Actions}\big(\mathrm{TopologicalSpaces}\big)$ (Def. 2.11) with coefficients in $\underline{A}$ is its equivariant non-abelian cohomology (Def. 2.37) with coefficients in $\mathcal{K}(\underline{A},n)$ (64):

$$\underbrace{H^n_G\big(X;\underline{A}\big)}_{\substack{\text{Bredon cohomology} \\ \text{(equivariant ordinary cohomology)}}} \;\simeq\; H_G\big(X;\,\mathcal{K}(\underline{A},n)\big) \;=\; H\Big(\prec\big(X/\!\!/G\big)\,,\,\mathcal{K}(\underline{A},n)\Big).$$

Equivariant Cohomotopy.

Example 2.40 (Equivariant non-abelian Cohomotopy [tD79, §8.4][Pe94][Cr03] [SS19a])**.** For $G \curvearrowright V$ a linear G-representation on a finite-dimensional real vector space V, the *representation sphere* (e.g. [Blu17, Ex. 1.1.5])

$$S^V \;:=\; V^{\mathrm{cpt}} \;\in\; G\mathrm{Actions}\big(\mathrm{TopologicalSpaces}\big) \xrightarrow{\;\int \prec(-/\!\!/G)\;} \underset{\prec}{G}\mathrm{HomotopyTypes}$$

defines an equivariant homotopy type (15). This is the coefficient space for the equivariant non-abelian cohomology theory (Def. 2.37) called (unstable) *equivariant Cohomotopy* in RO-degree V:

$$\underbrace{\pi^V_G(X)}_{\substack{\text{equivariant} \\ \text{Cohomotopy}}} \;:=\; H_G\Big(X;\,\prec\big(S^V/\!\!/G\big)\Big) \;\simeq\; H\Big(\prec\big(X/\!\!/G\big);\,\prec\big(S^V/\!\!/G\big)\Big).$$

Equivariant non-abelian cohomology operations.

Definition 2.41 (Equivariant non-abelian cohomology operations)**.** For $\mathcal{A}, \mathcal{B} \in \underset{\prec}{G}\mathrm{HomotopyTypes}$ (Def. 2.22), a *cohomology operation* from equivariant non-abelian $\mathcal{A}$-cohomology to $\mathcal{B}$-cohomology (Def. 2.37) is a natural transformation

$$H(-;\mathcal{A}) \xrightarrow{\ \phi_*\ } H(-;\mathcal{B})$$

of the corresponding equivariant non-abelian cohomology theories (61). By the Yoneda lemma, such operations are induced by post-composition with morphisms between equivariant coefficient spaces:

$$\mathcal{A} \xrightarrow{\ \phi\ } \mathcal{B} \quad \in \underset{\prec}{G}\mathrm{HomotopyTypes}\,. \tag{65}$$

2.4 Equivariant twisted non-abelian cohomology theories

We introduce equivariant twisted non-abelian cohomology, in direct generalization of [FSS20d, §2.2], and introduce the main example of interest here (Def. 2.48 below).

Equivariant ∞-Actions.

Remark 2.42 (Equivariant ∞-actions)**. (i)** In equivariant generalization of Prop. 2.5 (and as a special case of [NSS12a, Thm. 2.19][NSS12b, Thm. 3.30, Cor. 3.34]), every equivariantly pointed and equivariantly connected equivariant homotopy type (Def. 2.32) is, equivalently, the equivariant classifying space $B\mathcal{G}$ of an *equivariant ∞-group*

$$\mathcal{G} \;\in\; \underset{\prec}{G}\mathrm{EquivariantGroups}_\infty \;:=\; \mathrm{Ho}\Big(\mathrm{Functors}\big(G\mathrm{Orbits}^{\mathrm{op}}, \mathrm{SimplicialGroups}\big)_{\mathrm{proj}}\Big)\,.$$

(ii) In equivariant generalization of Prop. 2.7 (and as a special case of [NSS12a, §4][SS20b, §2.2]), ∞-actions of such equivariant ∞-groups on equivariant homotopy types $\mathcal{A}$ are, equivalently, homotopy fibrations of equivariant homotopy types over $B\mathcal{G}$ with homotopy fiber $\mathcal{A}$, hence a system of non-equivariant homotopy fibration (25) parametrized by the $G/H \in G\mathrm{Orbits}$ (Def. 2.13), denoted as follows [5]

$$\begin{array}{ccccc}
 & \mathcal{A} \xrightarrow{\ \mathrm{hofib}(\rho_{\mathcal{A}})\ } & \mathcal{A}/\!\!/\mathcal{G} & & \\
 & \substack{\text{equivariant homotopy fibration}\\ \text{associated to } \infty\text{-action of } \mathcal{G} \text{ on } \mathcal{A}} & \big\downarrow{\scriptstyle \rho_{\mathcal{A}}} & & \in \underset{\prec}{G}\mathrm{HomotopyTypes} \\
 & & B\mathcal{G} & & \\
 & \mathcal{A}(G/H) \xrightarrow{\ \mathrm{hofib}(\rho_{\mathcal{A}}(G/H))\ } & \mathcal{A}(G/H)/\!\!/\mathcal{G}(G/H) & & \\
G/H \;\longmapsto & \substack{\text{homotopy fibration}\\ \text{associated to } \infty\text{-action of } \mathcal{G}(G/H) \text{ on } \mathcal{A}(G/H)} & \big\downarrow{\scriptstyle \rho_{\mathcal{A}}(G/H)} & & \in \mathrm{HomotopyTypes} \\
 & & B\mathcal{G}(G/H) & &
\end{array} \tag{66}$$

A key source of equivariant ∞-actions are equivariant parametrized homotopy types, in the following sense:

Example 2.43 (Equivariant parametrized homotopy types)**.** Consider $T \in \mathrm{CompactTopologicalGroups}$ (31), $G \in \mathrm{FiniteGroups}$ (35), and $X \in \big(T \times G\big)\mathrm{Actions}\big(\mathrm{TopologicalSpaces}\big)$ (38).
(i) Since the two group actions separately commute with each other, we may consider forming the combined
(a) proper equivariant shape (Def. 2.23) with respect to the G-action;
(b) ordinary shape (15) of the homotopy quotient (Borel construction, Example 2.8) with respect to the T-action:

$$\underset{\prec}{G}\mathrm{HomotopyTypes} \;\ni\; \Big(\big(\prec(X/\!\!/G)\big)/\!\!/T\Big)\;;\;\; G/H \;\longmapsto\; \int\big(X^H/\!\!/T\big)\,. \tag{67}$$

This is the G-equivariant homotopy type (Def. 2.22) given on $G/H \in G\mathrm{Orbits}$ (Def. 2.13) by the Borel homotopy quotient construction (Example 2.8) of the T-action on the $G \supset H$-fixed locus (Example 2.20).

[5] Here and in the following we indicate the ambient category of a given diagram. The notation "Diagram ∈ Category" means that each vertex of the diagram is an object in that category, and each arrow is a morphism in that category.

(ii) With the classifying space BT regarded as a smooth G-equivariant homotopy type (i.e., with trivial G-action, Example 2.24) the G-equivariant T-parametrized space (67) sits in an equivariant fibration (66) over BT with homotopy fiber the G-equivariant shape of X (Def. 2.23):

$$
\begin{array}{ccccc}
 & \int \curlyvee\big(X /\!\!/ G\big) & \xrightarrow{\ \mathrm{hofib}\left(\rho_{\curlyvee(X /\!\!/ G)}\right)\ } & \int\Big(\big(\curlyvee(X /\!\!/ G)\big) /\!\!/ T\Big) & \\
 & & & \big\downarrow {\scriptstyle \rho_{\curlyvee(X /\!\!/ G)}} & \in \ \underset{\curlyvee}{G}\mathrm{HomotopyTypes} \\
 & & & BT & \\
 & \int X^H & \xrightarrow{\ \mathrm{hofib}(\rho_{X^H})\ } & \int\big(X^H /\!\!/ T\big) & \\
G/H \ \longmapsto & & & \big\downarrow {\scriptstyle \rho_{X^H}} & \in \ \mathrm{HomotopyTypes} \\
 & & & BT &
\end{array}
$$

We may refer to these objects as *proper G-equivariant and Borel T-equivariant homotopy types* , but for brevity and due to their above fibration over BT, we will say *G-equivariant T-parametrized homotopy types*.

Example 2.44 ($\mathbb{Z}_2^A$-equivariant $\mathrm{Sp}(1)$-parametrized twistor fibration)**.** Recall the $\mathbb{Z}_2^A$-equivariant twistor fibration (5) from Example 2.28. Since the $\mathrm{Sp}(2)$-subgroups $\mathbb{Z}_2^A$ (7) and $\mathrm{Sp}(1)$ (8) commute with each other, the quotient by the action of $\mathrm{Sp}(1)$ of the Cartesian product of the twistor fibration (5) with (the identity map on) the total space $E\mathrm{Sp}(2)$ of the universal principal $\mathrm{Sp}(2)$-bundle still has a residual equivariance under $\mathbb{Z}_2^A$:

$$
\begin{array}{ccccc}
 & & \overset{\mathbb{Z}_2^A}{\curvearrowright} & \overset{\text{twistor fibration}}{} & \overset{\mathbb{Z}_2^A}{\curvearrowright} \\
\dfrac{S^2 \times E\mathrm{Sp}(2)}{\mathrm{Sp}(1)} & \xrightarrow{\ \frac{\mathrm{fib}(t_{\mathbb{H}}) \times \mathrm{id}}{\mathrm{Sp}(1)}\ } & \dfrac{\mathbb{C}P^3 \times E\mathrm{Sp}(2)}{\mathrm{Sp}(1)} & \xrightarrow{\ \frac{t_{\mathbb{H}} \times \mathrm{id}}{\mathrm{Sp}(1)}\ } & \dfrac{S^4 \times E\mathrm{Sp}(2)}{\mathrm{Sp}(1)} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\dfrac{E\mathrm{Sp}(2)}{\mathrm{Sp}(1)} & = & \dfrac{E\mathrm{Sp}(2)}{\mathrm{Sp}(1)} & = & \dfrac{E\mathrm{Sp}(2)}{\mathrm{Sp}(1)}
\end{array}
\quad \in \ \mathbb{Z}_2^A\mathrm{Actions}\big(\mathrm{TopologicalSpaces}\big)^{/\frac{E\mathrm{Sp}(2)}{\mathrm{Sp}(1)}} \tag{68}
$$

Hence, using Example 2.28 and identifying the Borel construction of homotopy quotients (e.g. [NSS12b, Prop. 3.73], here for subgroups $H \subset G$)

$$
\underset{\text{Borel construction}}{\frac{X \times EG}{H}} \ \simeq \ \underset{\text{homotopy quotient}}{X /\!\!/ H} \ \in \mathrm{HomotopyTypes}\,, \tag{69}
$$

the $\mathbb{Z}_2^A$-equivariant homotopy type (Def. 2.22) of the middle vertical morphism in (68) exhibits a $\mathbb{Z}_2^A$-equivariant $\mathrm{Sp}(1)$-parametrized homotopy type (in the sense of Example 2.43) of this form:

$$
\begin{array}{ccccc}
 & & & \overset{\mathbb{Z}_2^A}{\curvearrowright} \ \int \mathbb{C}P^3 /\!\!/ \mathrm{Sp}(1) & \\
\int\Big(\curlyvee\big(\mathbb{C}P^3 /\!\!/ \mathbb{Z}_2^A\big)\Big) /\!\!/ \mathrm{Sp}(1) \ : & \overset{\mathbb{Z}_2}{\curvearrowright}\ \mathbb{Z}_2/1 & \longmapsto & \big\uparrow {\scriptstyle \mathrm{fib}(t_{\mathbb{H}}) /\!\!/ \mathrm{Sp}(1)} \quad \searrow {\scriptstyle \rho_{\int \mathbb{C}P^3}} & \int B\mathrm{Sp}(1) \\
\underset{\text{twistor space}}{\underset{\&\ \mathrm{Sp}(1)\text{-parametrized}}{\mathbb{Z}_2^A\text{-equivariant}}} \ \searrow {\scriptstyle \rho_{\int \curlyvee(\mathbb{C}P^3 /\!\!/ \mathbb{Z}_2^A)}} & \big\downarrow & & & \| \\
\int\Big(\curlyvee\big(\ast /\!\!/ \mathbb{Z}_2^A\big)\Big) /\!\!/ \mathrm{Sp}(1) \ : & & & \int S^2 /\!\!/ \mathrm{Sp}(1) & \\
 & \mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \searrow {\scriptstyle \rho_{\int S^2}} & \\
 & & & & \int B\mathrm{Sp}(1)\,.
\end{array}
\tag{70}
$$

The analogous statement holds for the vertical morphism on the right of (68), so that the full square on the right of (68) exhibits a morphism in $\mathbb{Z}_2^A$-equivariant $\mathrm{Sp}(1)$-parametrized homotopy types (Example 2.43) of this form:

$$
\begin{array}{ccc}
\overset{\substack{\mathbb{Z}_2^A\text{-equivariant} \\ \mathrm{Sp}(1)\text{-parametrized} \\ \text{twistor space}}}{\int\Big(\curlyvee\big(\mathbb{C}P^3 /\!\!/ \mathbb{Z}_2^A\big)\Big) /\!\!/ \mathrm{Sp}(1)} & \xrightarrow[\ \int \curlyvee(t_{\mathbb{H}} /\!\!/ \mathbb{Z}_2^A) /\!\!/ \mathrm{Sp}(1)\ ]{\substack{\mathbb{Z}_2^A\text{-equivariant} \\ \mathrm{Sp}(1)\text{-parametrized} \\ \text{twistor fibration}}} & \overset{\substack{\mathbb{Z}_2^A\text{-equivariant} \\ \mathrm{Sp}(1)\text{-parametrized} \\ \text{4-sphere}}}{\int\Big(\curlyvee\big(S^4 /\!\!/ \mathbb{Z}_2^A\big)\Big) /\!\!/ \mathrm{Sp}(1)} \\
 & \searrow \qquad \swarrow & \\
 & B\mathrm{Sp}(1) &
\end{array}
\quad \in \mathrm{Ho}\Big(\underset{\curlyvee}{\mathbb{Z}_2}\mathrm{SimplicialSets}_{\mathrm{proj}}^{/\int B\mathrm{Sp}(1)}\Big)\,, \tag{71}
$$

where $B\mathrm{Sp}(1) := \mathrm{Smth}\int B\mathrm{Sp}(1)$ (Example 2.24).

Twisted equivariant non-abelian cohomology.

In twisted generalization of Def. 2.37 and in equivariant generalization of [FSS20d, §2.2], we set:

Definition 2.45 (Twisted equivariant non-abelian cohomology)**.** Let

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{hofib}(\rho_{\mathcal{A}})} & \mathcal{A}/\!\!/\mathcal{G} \\ & \substack{\text{equivariant}\\ \text{local coefficient}\\ \text{bundle}} & \big\downarrow{\scriptstyle \rho_{\mathcal{A}}} \\ & & B\mathcal{G} \end{array} \quad \in \; \mathcal{G}\mathrm{HomotopyTypes} \tag{72}$$

be an homotopy fibration as in Remark 2.42, to be regarded now as an *equivariant local coefficient bundle*, and let $\mathcal{X} \in \mathcal{G}\mathrm{HomotopyTypes}$ (Def. 2.22) equipped with an *equivariant twist*

$$[\tau] \;\in\; H\big(\mathcal{X}; B\mathcal{G}\big) \tag{73}$$

in equivariant non-abelian cohomology (Def. 2.37) with coefficients in $B\mathcal{G}$. We say that the *τ-twisted equivariant non-abelian cohomology* of $\mathcal{X}$ with coefficients in $\mathcal{A}$ is the hom-set from τ to ρ_A in the homotopy category of the slice model structure (see [FSS20d, Ex. A.10]) over $B\mathcal{G}$ of the projective model structure on equivariant simplicial sets (Prop. 2.21):

$$\underbrace{H^\tau\big(\mathcal{X}; \mathcal{A}\big)}^{\substack{\text{twisted equivariant}\\ \text{non-abelian cohomology}}} \;:=\; \mathrm{Ho}\Big(\mathcal{G}\mathrm{SimplicialSets}^{/B\mathcal{G}}_{\mathrm{proj}}\Big)\big(\tau\,,\rho_{\mathcal{A}}\big)\,.$$

Twisted equivariant ordinary cohomology.

Example 2.46 (Twisted Bredon cohomology)**.** Let $G \circlearrowright X \in G\mathrm{Actions}\big(\mathrm{TopologicalSpaces}\big)$ (Def. 2.11) with a base point $G \circlearrowright * \xrightarrow{x} G \circlearrowright X$, let $\underline{A} \in \mathcal{G}\mathrm{AbelianGroups}$ (Def. 2.29), and let

$$r \,:\; \underline{\pi}_1(X) \times \underline{A} \longrightarrow \underline{A}$$

be an action of the equivariant fundamental group (Def. 2.31) of X on $\underline{A}$. For $n \in \mathbb{N}$, there is an equivariant local coefficient bundle (72)

$$\begin{array}{ccc} \mathcal{K}(\underline{A},n) & \longrightarrow & \mathcal{K}(\underline{A},n)/\!\!/\underline{\pi}_1(X) \\ & \substack{\text{equivariant ordinary}\\ \text{local coefficients}} & \big\downarrow{\scriptstyle \rho} \\ & & B\underline{\pi}_1(X) \end{array}$$

with typical fiber the equivariant Eilenberg-MacLane space (64), such that the twisted equivariant non-abelian cohomology with local coefficients in ρ coincides (by [Go97a, Cor. 3.6][MuSe10, Thm. 5.10]) with traditional r-twisted Bredon cohomology in degree n ([MoSv93, Def. 2.1][MuMu96, Def. 3.8][MuPa02]):

$$\overbrace{H^{n+r}_G\big(X;\underline{A}\big)}^{\substack{\text{twisted}\\ \text{Bredon cohomology}}} \;\simeq\; H^\tau\big(X;\, \mathcal{K}(\underline{A},n)\big)\,.$$

Equivariant tangential structure. In equivariant generalization of [FSS20d, Example 2.33], we have:

Definition 2.47 (Equivariant tangential structure)**.** Let $G \circlearrowright X \in G\mathrm{Actions}\big(\mathrm{SmoothManifolds}\big)$ (Def. 2.35) of dimension $n := \dim(X)$, and let $\mathscr{G} \xrightarrow{\phi} B\mathrm{GL}(n)$ be a topological group homomorphism. An *equivariant tangential $(\mathscr{G},\phi)$-structure* (or just $\mathscr{G}$-structure, for short) on the orbifold $\prec\big(X/\!\!/G\big)$ (Def. 2.36) is a class in the equivariant twisted non-abelian cohomology (Def. 2.45) of the equivariant shape (Def. 2.23) of the orbifold with equivariant local coefficients (72) in

$$\begin{array}{ccc} \mathrm{GL}(n)/\!\!/\mathscr{G} & \longrightarrow & B\mathscr{G} \\ & & \big\downarrow{\scriptstyle B\phi} \\ & & B\mathrm{GL}(n) \end{array}$$

and with twist given by the classifying map τ_{Fr} of the frame bundle:

$$(\mathscr{G},\phi)\mathrm{Structures}\big(\curlyvee\big(X /\!\!/ G\big)\big) \;:=\; H^{\tau_{\mathrm{Fr}}}\Big(\curlyvee\big(X /\!\!/ G\big);\, \mathrm{GL}(n) /\!\!/ \mathscr{G}\Big).$$

Equivariant twistorial Cohomotopy. In equivariant generalization of [FSS20d, Ex. 2.44] we have:

Definition 2.48 (Equivariant twistorial Cohomotopy theory)**.** Let $X^8 \in \mathbb{Z}_2\mathrm{Actions}\big(\mathrm{TopologicalSpaces}\big)$ (Def. 2.11) be a smooth spin 8-manifold equipped with tangential structure (see [FSS19b, Ex. 2.33]) for the subgroup $\mathrm{Sp}(1) \subset \mathrm{Sp}(2) \subset \mathrm{Spin}(8)$ (where the first inclusion is (7) and the second is again given by left quaternion multiplication, e.g. [FSS19b, Ex. 2.12])

$$[\tau] \;\in\; H_{\mathbb{Z}_2^A}\big(X^8; B\mathrm{Sp}(1)\big)\,.$$

We say that:

- **(a)** its $\mathbb{Z}_2^A$*-equivariant twistorial Cohomotopy* $\mathscr{T}^{\tau}_{\mathbb{Z}_2^A}(-)$ is the τ-twisted equivariant non-abelian cohomology theory (Def. 2.45) with local coefficients in the $\mathbb{Z}_2^A$-equivariant $\mathrm{Sp}(1)$-parametrized twistor space;
- **(b)** its $\mathbb{Z}_2^A$*-equivariant J-twisted Cohomotopy* $\pi^{\tau}_{\mathbb{Z}_2^A}(-)$ is the τ-twisted equivariant non-abelian cohomology theory (Def. 2.45) with local coefficients in the $\mathbb{Z}_2^A$-equivariant $\mathrm{Sp}(1)$-parametrized 4-sphere;
- **(c)** the twisted equivariant cohomology operation $\mathscr{T}^{\tau}_{\mathbb{Z}_2^A}(-) \longrightarrow \pi^{\tau}_{\mathbb{Z}_2^A}(-)$ is that induced by the $\mathbb{Z}_2^A$-equivariant $\mathrm{Sp}(1)$-parametrized twistor fibration;

all as induced by the (morphism of) local coefficient bundles (71) in Example 2.44:

$$\underset{\substack{\text{equivariant}\\ \text{twistorial Cohomotopy}}}{\mathscr{T}^{\tau}_{\mathbb{Z}_2^A}(X)} \;:=\; H^{\tau}_{\mathbb{Z}_2^A}\Big(X; \textstyle\int \curlyvee\big(\mathbb{C}P^3 /\!\!/ \mathbb{Z}_2^A\big)\Big) \xrightarrow[\big(\int\curlyvee(t_{\mathbb{H}} /\!\!/ \mathbb{Z}_2^A) /\!\!/ \mathrm{Sp}(1)\big)_*]{\substack{\text{push-forward along}\\ \text{equivariant parametrized}\\ \text{twistor fibration}}} H^{\tau}_{\mathbb{Z}_2^A}\Big(X; \textstyle\int \curlyvee\big(S^4 /\!\!/ \mathbb{Z}_2^A\big)\Big) =: \underset{\substack{\text{equivariant}\\ \text{J-twisted Cohomotopy}}}{\pi^{\tau}_{\mathbb{Z}_2^A}(X)}\,. \tag{74}$$

3 Equivariant non-abelian de Rham cohomology

We had shown in [FSS20d, §3] how the fundamental theorem of dgc-algebraic rational homotopy theory ([BG76, §9.4, §11.2]), augmented by differential-geometric observations [GM13, §9], provides a non-abelian de Rham theorem for L_∞-algebra valued differential forms, which serve as the recipient of non-abelian character maps.

The equivariant generalization of this fundamental theorem had been obtained in [Scu08] (following [Tri82]) without having found much attention yet. Here we review, in streamlined form and highlighting examples and applications, the underlying theory of injective equivariant dgc-algebras/L_∞-algebras in §3.1 and how these serve to model equivariant rational homotopy theory in §3.2. Then we use this in §3.3 to prove the equivariant non-abelian de Rham theorem (Prop. 3.63) including its twisted version (Prop. 3.67); which, in turn, we use in §3.4 to construct the equivariant non-abelian character map (Def. 3.76) and its twisted version (Def. 3.78).

3.1 Equivariant dgc-algebras and equivariant L_∞-algebras

We discuss here the generalization of the homotopy theory of connective dgc-algebras and of connective L_∞-algebras (following [FSS20d, §3.1]) to G-equivariant homotopy theory, for any finite equivariance group G (35). While the homotopy theory of equivariant connective dgc-algebras has been developed in [Tri82][Scu02][Scu08], previously little to no examples or applications have been worked out. Here we develop equivariantized twistor space as a running example (culminating in Prop. 3.56 below).

While the general form of the homotopy theory of plain dgc-algebras generalizes to equivariant dgc-algebras, the crucial new aspect is that equivariantly not every connective cochain complex, and hence not every connective dgc-algebra, is fibrant. The fibrant equivariant cochain complexes must be degreewise injective, which is now a non-trivial condition (Prop. 3.12 below). The key effect on the theory is that equivariant minimal Sullivan models (Def. 3.40) – which still exist and still have the expected general properties – are no longer given just by iterative adjoining of (equivariant systems of) generators, but by adjoining of injective resolutions (Example 3.28) of systems of generators. This has interesting effects, as shown in Example 3.42, which is at the heart of the proof of Prop. 3.56 and thus of Theorem 1.1.

Plain homological algebra. For plain (i.e., non-equivariant) dgc-algebra, we follow the conventions of [FSS20d, §3.1]. In particular, we make use of the following notation:

Notation 3.1 (Generators/relations presentation of cochain complexes)**.**
We may denote any $V \in \mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0, \mathrm{fin}}$ by generators (a graded linear basis) and relations (the linear relations given by the differential). For instance:

$$\mathbb{R}\langle c_2\rangle \big/ (d\, c_2 = 0) \quad \simeq \quad \big(0 \longrightarrow 0 \longrightarrow \mathbf{1} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \big),$$

$$\mathbb{R}\left\langle \begin{array}{c} c_3', \\ c_3, \\ b_2 \end{array} \right\rangle \Big/ \left(\begin{array}{l} d\, c_3' = 0 \\ d\, c_3 = 0 \\ d\, b_2 = c_3 \end{array} \right) \quad \simeq \quad \big(0 \longrightarrow 0 \longrightarrow \mathbf{1} \hookrightarrow \mathbf{2} \longrightarrow 0 \longrightarrow \cdots \big).$$

Notation 3.2 (Generators/relations presentation of dgc-algebras)**.** We may denote the Chevalley-Eilenberg algebra $\mathrm{CE}(\mathfrak{g}) \in \mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0, \mathrm{fin}}$ of any $\mathfrak{g} \in L_\infty\mathrm{Algebras}_{\mathbb{R}, \mathrm{fin}}^{\geq 0}$ ([FSS20d, Def. 3.25]) by generators (a graded linear basis) and relations (the polynomial relations given by the differential). For instance (see [FSS20d, Ex. 3.67, 3.68]):

$$\mathbb{R}[c_2] \big/ (d\, c_2 = 0) \quad \simeq \quad \mathrm{CE}(\mathfrak{b}\mathbb{R}) \qquad \text{and} \qquad \mathbb{R}\left[\begin{array}{c} \omega_7, \\ \omega_4 \end{array} \right] \Big/ \left(\begin{array}{l} d\, \omega_7 = -\omega_4 \wedge \omega_4 \\ d\, \omega_4 = 0 \end{array} \right) \quad \simeq \quad \mathrm{CE}\big(\mathfrak{l}S^4\big)\,.$$

Similarly, for T a finite-dimensional compact and simply-connected Lie group with Lie algebra

$$\mathfrak{t} \simeq \Big\{ \langle t_a \rangle_{a=1}^{\dim(T)}, [-,-] \Big\} \in \mathrm{LieAlgebras}_{\mathbb{R}, \mathrm{fin}}\,,$$

the abstract Chern-Weil isomorphism (e.g. [FSS20d, §4.2]) reads:

$$\Big(\mathbb{R}\big[\{r_2^a\}_{a=1}^{\dim(T)}\big]/\big(d\,r_2^a = 0\big)\Big)^T \;\simeq\; \mathrm{CE}(\mathfrak{l}BT)\,, \tag{75}$$

where on the left $(-)^T$ denotes the T-invariant elements with respect to the coadjoint action on the dual vector space of the Lie algebra.

Equivariant vector spaces.

Example 3.3 (Linear representations as functors)**.** For G any finite group, write BG for the category with a single object and with G as its endomorphisms (hence its automorphisms). Then functors on BG with values in vector spaces are, equivalently, linear G-representations with G acting either from the left or from the right, depending on whether the functor is contravariant or covariant:

$$\begin{aligned} G\mathrm{Representations}^l_{\mathbb{R}} &\simeq \mathrm{Functors}\big(BG^{\mathrm{op}}\,,\,\mathrm{VectorSpaces}_{\mathbb{R}}\big)\,,\\ G\mathrm{Representations}^r_{\mathbb{R}} &\simeq \mathrm{Functors}\big(BG\,,\,\mathrm{VectorSpaces}_{\mathbb{R}}\big)\,. \end{aligned} \tag{76}$$

Example 3.4 (Irreducible $\mathbb{Z}_2$-representations)**.** We write

$$\mathbf{1},\,\mathbf{1}_{\mathrm{sgn}} \;\in\; \mathbb{Z}_2\mathrm{Representations}^r_{\mathbb{R}}$$

for the two irreducible right representations (Example 3.3) of $\mathbb{Z}_2$, namely the trivial representation and the sign representation, respectively.

Definition 3.5 (Equivariant vector spaces)**.** We write

$$\begin{aligned} G\mathrm{VectorSpaces}^{\mathrm{fin}}_{\mathbb{R}} &:= \;\mathrm{Functors}\big(G\mathrm{Orbits}^{\mathrm{op}}\,,\,\mathrm{VectorSpaces}^{\mathrm{fin}}_{\mathbb{R}}\big)\,,\\ G\mathrm{VectorSpaces}^{\vee,\mathrm{fin}}_{\mathbb{R}} &:= \;\mathrm{Functors}\big(G\mathrm{Orbits}\,,\,\mathrm{VectorSpaces}^{\mathrm{fin}}_{\mathbb{R}}\big) \end{aligned} \tag{77}$$

for the categories of contravariant or covariant functors, respectively, from the G-orbit category (Def. 2.13) to the category of finite-dimensional vector spaces over the real numbers.

Notice that forming linear dual vector spaces constitutes an equivalence of categories

$$\mathrm{VectorSpaces}^{\mathrm{fin}}_{\mathbb{R}} \xrightarrow[\simeq]{(-)^\vee} \big(\mathrm{VectorSpaces}^{\mathrm{fin}}_{\mathbb{R}}\big)^{\mathrm{op}}$$

and hence induces an equivalence:

$$\begin{aligned} \big(G\mathrm{VectorSpaces}_{\mathbb{R}}\big)^{\mathrm{op}} &= \Big(\mathrm{Functors}\big(G\mathrm{Orbits}^{\mathrm{op}}\,,\,\mathrm{VectorSpaces}^{\mathrm{fin}}_{\mathbb{R}}\big)\Big)^{\mathrm{op}}\\ &\simeq \mathrm{Functors}\Big(G\mathrm{Orbits}\,,\,\big(\mathrm{VectorSpaces}^{\mathrm{fin}}_{\mathbb{R}}\big)^{\mathrm{op}}\Big)\\ &\simeq \mathrm{Functors}\Big(G\mathrm{Orbits}\,,\,\mathrm{VectorSpaces}^{\mathrm{fin}}_{\mathbb{R}}\Big)\\ &= G\mathrm{VectorSpaces}^{\vee,\mathrm{fin}}_{\mathbb{R}}. \end{aligned}$$

This justifies extending the notation (77) to vector spaces which are not necessarily finite-dimensional

$$\begin{aligned} G\mathrm{VectorSpaces}_{\mathbb{R}} &:= \;\mathrm{Functors}\big(G\mathrm{Orbits}^{\mathrm{op}}\,,\,\mathrm{VectorSpaces}_{\mathbb{R}}\big)\\ G\mathrm{VectorSpaces}^{\vee}_{\mathbb{R}} &:= \;\mathrm{Functors}\big(G\mathrm{Orbits}\,,\,\mathrm{VectorSpaces}_{\mathbb{R}}\big) \end{aligned}$$

and to speak of the latter as the category of *equivariant dual vector spaces* (denoted Vec^*_G in [Tri82]).

Example 3.6 (Equivariant dual vector spaces of real cohomology groups)**.** For $\mathcal{X} \in G\mathrm{HomotopyTypes}$ (Def. 2.22) and $n \in \mathbb{N}$, the stage-wise real cohomology groups in degree n form an equivariant dual vector space (Def. 3.5)

$$\underline{H}^n\big(\mathcal{X};\mathbb{R}\big) \;:\; G/H \longmapsto H^n\big(\mathcal{X}(G/H);\mathbb{R}\big)\,.$$

If these are stage-wise finite-dimensional, then these are the linear dual equivariant vector spaces of the equivariant singular real homology groups $\underline{H}_n\big(\mathcal{X};\mathbb{R}\big)$ from Example 2.30.

Example 3.7 ($\mathbb{Z}_2$-equivariant dual vector spaces)**.** A (finite-dimensional) dual $\mathbb{Z}_2$-equivariant vector space (Def. 3.5) is a diagram of (finite-dimensional) vector spaces indexed by the $\mathbb{Z}_2$-orbit category (Example 2.15)

$$\left(\begin{array}{ccc} \overset{\mathbb{Z}_2}{\circlearrowright} & & \overset{\mathbb{Z}_2}{\circlearrowright} \\ \mathbb{Z}_2/1 & \mapsto & \mathbf{N} \\ \downarrow & & \downarrow \phi \\ \mathbb{Z}_2/\mathbb{Z}_2 & \mapsto & V \end{array}\right) \;\in\; \mathbb{Z}_2 \mathrm{VectorSpaces}^{\vee}_{\mathbb{R}}$$

hence constitutes:

– a right $\mathbb{Z}_2$-representation $\mathbf{N}$ (Example 3.3),
– a vector space V (finite-dimensional),
– a linear map ϕ from the underlying vector space of $\mathbf{N}$ to V.

Example 3.8 (Restriction of equivariant vector spaces to Weyl group linear representation)**.** For $H \subset G$ a subgroup, with Weyl group $\mathrm{W}_G(H) = \mathrm{Aut}_{G\mathrm{Orbits}}(G/H)$ (Example 2.16), the canonical inclusion of categories

$$B\mathrm{W}_G(H) \xhookrightarrow{\;i_H\;} G\mathrm{Orbits} \tag{78}$$

induces restriction functors of equivariant vector spaces (Def. 3.5) to linear representations (Example 3.3):

$$\begin{array}{rcl} \mathrm{W}_G(H)\mathrm{Representations}^{l}_{\mathbb{R}} & \xleftarrow{\;i_H^*\;} & G\mathrm{VectorSpaces}_{\mathbb{R}}\,, \\ \mathrm{W}_G(H)\mathrm{Representations}^{r}_{\mathbb{R}} & \xleftarrow{\;i_H^*\;} & G\mathrm{VectorSpaces}^{\vee}_{\mathbb{R}}\,. \end{array} \tag{79}$$

Example 3.9 (Regular equivariant vector space)**.** For any subgroup $K \subset G$ we have an equivariant dual vector space (Def. 3.5) given by the $\mathbb{R}$-linear spans of the hom-sets (10) out of G/K in the orbit category (Def. 2.13):

$$\mathbb{R}\big[G\mathrm{Orbits}(G/K\,,-)\big] \;\in\; G\mathrm{VectorSpaces}^{\vee}_{\mathbb{R}}\,.$$

For any further subgroup $H \subset G$, its restriction (Example 3.8) to a linear representation from the right (Example 3.3) of the Weyl group of H (Def. 2.16) is

$$i_H^*\big(\mathbb{R}\big[G\mathrm{Orbits}(G/K\,,-)\big]\big) \;=\; \mathbb{R}\big[G\mathrm{Orbits}(G/K\,,G/H)\big] \;\in\; \mathrm{W}_G(H)\mathrm{Representations}^{r}_{\mathbb{R}}\,,$$

where $\mathrm{W}_G(H)$ acts in linear extension of its canonical right action on the hom-set of the orbit category (Example 2.16).

Lemma 3.10 (Extension of linear representations to equivariant vector spaces)**.** *For any $H \subset G$, the restriction of equivariant vector spaces to linear representations (Example 3.8) has a right adjoint*

$$\mathrm{W}_G(H)\mathrm{Representations}^{r}_{\mathbb{R}} \underset{\mathrm{Inj}_H}{\overset{i_H}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} G\mathrm{VectorSpaces}^{\vee}_{\mathbb{R}}\,,$$

where

$$\mathrm{Inj}_H(V^*) \;\in\; G\mathrm{VectorSpaces}^{\vee}_{\mathbb{R}} \;=\; \mathrm{Functors}\big(G\mathrm{Orbits}\,,\mathrm{VectorSpaces}_{\mathbb{R}}\big)$$

is given by

$$\mathrm{Inj}_H(V^*) \;:\; G/K \longmapsto \mathrm{W}_G(H)\mathrm{Representations}^{r}_{\mathbb{R}}\Big(\mathbb{R}\big[G\mathrm{Orbits}(G/K\,,G/H)\big]\,,V^*\Big) \tag{80}$$

$$= \bigoplus_{\substack{g \,\in\, G/\mathrm{N}_G(K) \\ \text{s.t. } g^{-1}Kg \,\subset\, H}} V^*\,. \tag{81}$$

Here the regular $\mathrm{W}_G(H)$-representation in the first argument on the right of (80) *is from Example 3.9.*

Proof. Formula (80) is a special case of the general formula for right Kan extension [Ke82, (4.24)], here applied to the inclusion (78) regarded in $\mathrm{VectorSpaces}_{\mathbb{R}}$-enriched category theory. Its equivalence to (81) follows with Example 2.17. See also [Tri82, (4.1)][Scu08, Lemma 2.3]. ☐

Injective equivariant dual vector spaces. Recall the general definition of injective objects (e.g. [HS71, p. 30]), applied to equivariant dual vector spaces:

Definition 3.11 (Injective equivariant dual vector spaces)**.** An object $I \in \underset{\curvearrowright}{G}\mathrm{VectorSpaces}^{\vee}_{\mathbb{R}}$ (Def. 3.5) is called *injective* if morphisms into it extend along all injections, hence if every solid diagram of the form

$$\begin{array}{ccccc} W & \xleftarrow{} & \exists & \dashrightarrow & I \quad \text{injective object} \\ & \underset{\text{injection}}{\hookleftarrow} & V & \nearrow & \end{array} \tag{82}$$

admits a dashed morphism that makes it commute, as shown. We write

$$\underset{\curvearrowright}{G}\mathrm{VectorSpaces}^{\vee,\mathrm{inj}}_{\mathbb{R}} \hookrightarrow \underset{\curvearrowright}{G}\mathrm{VectorSpaces}^{\vee}_{\mathbb{R}}$$

for the full sub-category on the injective objects.

Proposition 3.12 (Injective envelope of equivariant dual vector spaces [Tri82, p. 2][Scu02, Prop. 7.34][Scu08, Lem. 2.4, Prop. 2.5])**.** *For $V \in \underset{\curvearrowright}{G}\mathrm{VectorSpaces}^{\vee}_{\mathbb{R}}$ (Def. 3.5), the direct sum of extensions $\mathrm{Inj}_{(-)}$ (Def. 3.10)*

$$\mathrm{Inj}(V) \;:=\; \bigoplus_{[H\subset G]} \mathrm{Inj}_H\big(V_H\big) \quad \in \underset{\curvearrowright}{G}\mathrm{VectorSpaces}^{\vee}_{\mathbb{R}}\,, \tag{83}$$

of those components at stage H which vanish on all deeper stages

$$V_H \;:=\; \left\{ \begin{array}{ll} \bigcap\limits_{[K\supsetneq H]} \ker\Big(V(G/H) \xrightarrow{V(G/(H\hookrightarrow K))} V(G/K) \Big) & \mid \;\; H \neq G \\ V(G/G) & \mid \;\; H = G \end{array} \right. \tag{84}$$

receives an injection

$$V \hookrightarrow \mathrm{Inj}(V) \tag{85}$$

that extends the canonical inclusion of the V_H, and which is an injective envelope (e.g. [HS71, §I.9]) of V in $\underset{\curvearrowright}{G}\mathrm{VectorSpaces}^{\vee}_{\mathbb{R}}$. In particular:

(i) *the summands $\mathrm{Inj}_H(V)$ (Example 3.10) are injective objects (Def. 3.11);*

(ii) *V is injective (Def. 3.11) precisely if* (85) *is an isomorphism.*

Example 3.13 (Ground field is injective as equivariant dual vector space)**.** The equivariant dual vector space (Def. 3.5) which is constant on the ground field

$$\underline{\mathbb{R}} \;:=\; \mathrm{const}_{G\mathrm{Orbits}}(\mathbb{R}) \;:\; G/H \longmapsto \mathbb{R}$$

is isomorphic to the right extension (Lemma 3.10) $\underline{\mathbb{R}} \simeq \mathrm{Inj}_G(\mathbf{1})$ of $\mathbb{R} \simeq \mathbf{1} \in 1\mathrm{Representations}_{\mathbb{R}}$, and hence is injective, by Prop. 3.12.

Example 3.14 (Injective $\mathbb{Z}_2$-equivariant dual vector spaces)**.** For $G = \mathbb{Z}_2$ (Example 2.15) the irreducible representations

$$\mathbf{1}, \mathbf{1}_{\mathrm{sgn}} \;\in \mathbb{Z}_2\mathrm{Representations}_{\mathbb{R}}\,, \qquad \mathbf{1} \;\in 1\mathrm{Representations}_{\mathbb{R}} \;\simeq\; \mathrm{VectorSpaces}_{\mathbb{R}}$$

of the respective Weyl groups (Example 2.16, Example 3.4) induce by right extension (Def. 3.10) the following three $\mathbb{Z}_2$-equivariant vector spaces (Example 3.7), which, by Prop. 3.12, are the direct summand building blocks of all injective $\mathbb{Z}_2$-equivariant dual vector spaces:

$$\mathrm{Inj}_1(\mathbf{1}) \;:\; \begin{array}{ccc} \overset{\mathbb{Z}_2}{\curvearrowright} & & \\ \mathbb{Z}_2/1 & \longmapsto & \mathbf{1} \\ \big\downarrow & & \big\downarrow{\scriptstyle 0} \\ \mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & 0\,, \end{array} \qquad\qquad \mathrm{Inj}_1(\mathbf{1}_{\mathrm{sgn}}) \;:\; \begin{array}{ccc} \overset{\mathbb{Z}_2}{\curvearrowright} & & \\ \mathbb{Z}_2/1 & \longmapsto & \mathbf{1}_{\mathrm{sgn}} \\ \big\downarrow & & \big\downarrow{\scriptstyle 0} \\ \mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & 0\,, \end{array} \tag{86}$$

and

$$
\mathrm{Inj}_{\mathbb{Z}_2}(\mathbf{1}) \quad : \quad
\begin{array}{ccc}
\overset{\mathbb{Z}_2}{\circlearrowright} & & \\
\mathbb{Z}_2/1 & \longmapsto & \mathbf{1} \\
\big\downarrow & & \big\downarrow {\scriptstyle \mathrm{id}} \\
\mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathbf{1}\,.
\end{array}
\tag{87}
$$

To see this, use (40) in (80) to get, for two cases,

$$
\mathrm{Inj}_{1}(\mathbf{1}) \quad : \quad
\begin{array}{ccl}
\overset{\mathbb{Z}_2}{\circlearrowright} & & \\
\mathbb{Z}_2/1 & \longmapsto & \mathbb{Z}_2\mathrm{Representations}_{\mathbb{R}}\Big(\underbrace{\mathbb{R}\big[\mathbb{Z}_2\mathrm{Orbits}(\mathbb{Z}_2/1\,,\,\mathbb{Z}_2/1)\big]}_{\simeq \mathbf{1}\oplus\mathbf{1}_{\mathrm{sgn}}}, \mathbf{1}\Big) \simeq \mathbf{1} \\
\big\downarrow & & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \big\downarrow {\scriptstyle 0} \\
\mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathbb{Z}_2\mathrm{Representations}_{\mathbb{R}}\Big(\underbrace{\mathbb{R}\big[\mathbb{Z}_2\mathrm{Orbits}(\mathbb{Z}_2/\mathbb{Z}_2\,,\,\mathbb{Z}_2/1)\big]}_{\simeq 0}, \mathbf{1}\Big) \simeq 0
\end{array}
$$

and

$$
\mathrm{Inj}_{\mathbb{Z}_2}(\mathbf{1}) \quad : \quad
\begin{array}{ccl}
\overset{\mathbb{Z}_2}{\circlearrowright} & & \\
\mathbb{Z}_2/1 & \longmapsto & 1\mathrm{Representations}_{\mathbb{R}}\Big(\underbrace{\mathbb{R}\big[\mathbb{Z}_2\mathrm{Orbits}(\mathbb{Z}_2/1\,,\,\mathbb{Z}_2/\mathbb{Z}_2)\big]}_{\simeq \mathbf{1}}, \mathbf{1}\Big) \simeq \mathbf{1} \\
\big\downarrow & & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \big\downarrow {\scriptstyle \mathrm{id}} \\
\mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & 1\mathrm{Representations}_{\mathbb{R}}\Big(\underbrace{\mathbb{R}\big[\mathbb{Z}_2\mathrm{Orbits}(\mathbb{Z}_2/\mathbb{Z}_2\,,\,\mathbb{Z}_2/\mathbb{Z}_2)\big]}_{\simeq \mathbf{1}}, \mathbf{1}\Big) \simeq \mathbf{1}\,.
\end{array}
$$

Lemma 3.15 (Tensor product preserves injectivity of finite-dim dual vector G-spaces [Go97b, Lem. 3.6, Rem 1.2] [Scu02, Prop. 7.36])**.** *Let $V, W \in \underset{\curlyvee}{G}\mathrm{VectorSpaces}_{\mathbb{R}}^{\vee,\mathrm{fin}}$ (Def. 3.5). If V and W are both injective (Def. 3.11), then so is their tensor product $V \otimes W : G/H \longmapsto V(G/H) \otimes W(G/H)$.*

Equivariant smooth differential forms. In preparation of discussing equivariant de Rham cohomology, consider:

Example 3.16 (Equivariant smooth differential forms)**.** Let $G \circlearrowright X \in G\mathrm{Actions}\big(\mathrm{SmoothManifolds}\big)$ (Def. 2.35) and $n \in \mathbb{N}$. Then there is the equivariant dual vector space (Def. 3.30)

$$
\Omega^n_{\mathrm{dR}}\big(\curlyvee\big(X /\!\!/ G\big)\big) \;\in\; \underset{\curlyvee}{G}\mathrm{VectorSpaces}^{\vee}_{\mathbb{R}}
$$

given by the system of vector spaces of smooth differential n-forms (e.g. [BT82]) of the fixed submanifolds (60), with pullback of differential forms along residual actions and along inclusions of fixed loci:

$$
\underset{\text{differential } n\text{-forms}}{\underset{\text{of equivariant smooth}}{\overset{\text{equivariant dual vector space}}{\Omega^n_{\mathrm{dR}}\big(\curlyvee\big(X /\!\!/ G\big)\big)}}} \quad : \quad
\begin{array}{ccc}
\overset{g_1 \in W_G(H_1)}{\circlearrowright} & & \overset{X^{g_1^*}}{\circlearrowright} \\
G/H_1 & \longmapsto & \Omega^n_{\mathrm{dR}}\big(X^{H_1}\big) \;\; \text{ordinary differential forms on fixed submanifold} \\
{\scriptstyle p}\big\downarrow & & \big\downarrow {\scriptstyle X^{p^*}} \;\; \text{pullback along inclusion of fixed loci} \\
G/H_2 & \longmapsto & \Omega^n_{\mathrm{dR}}\big(X^{H_2}\big) \\
\underset{g_2 \in W_G(H_2)}{\circlearrowright} & & \underset{X^{g_2^*}}{\circlearrowright}
\end{array}
$$

Remark 3.17 (Equivariant smooth differential forms are injective)**.** The following Lemmas 3.19, 3.20, 3.21 show that the equivariant dual vector spaces of smooth differential n-forms (Def. 3.16) are injective objects (Def. 3.11), at least if the equivariance group is of order 4 or cyclic of prime order:

$$
G \;\in\; \big\{\, \mathbb{Z}_p \,|\, p \text{ prime} \big\} \,\cup\, \big\{ \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \big\}\,.
$$

From the proofs of these lemmas, given below, it is fairly clear how to approach the proof of the general case. But since this is heavy on notation if done properly, and since we do not need further generality for our application here, we will not go into that.

Notation 3.18 (Extension of smooth differential forms away from fixed loci)**.**
For $G \curvearrowright X \in G\mathrm{Actions}(\mathrm{SmoothManifolds})$ (Def. 2.35) and $H \subset G$, choose a tubular neighborhood (e.g. [Ko96, §1.2]) $\mathscr{N}_X(X^H) \subset X$ of the fixed locus (which exists by Lemma 2.34). Then multiplication of smooth n-forms on X^H with a choice of bump function in the neighborhood coordinates induces a linear section, which we denote ext_H, of the operation of restricting differential forms to the fixed locus:

$$\Omega^n_{\mathrm{dR}}(X^H) \overset{\mathrm{ext}_H}{\dashrightarrow} \Omega^n_{\mathrm{dR}}(X) \xrightarrow{(-)|_{X^H}} \Omega^n_{\mathrm{dR}}(X^H)\,, \qquad (-)|_{X^H} \circ \mathrm{ext}_H = \mathrm{id}$$

Lemma 3.19 ($\mathbb{Z}_p$-Equivariant smooth differential forms are injective)**.** *Let the equivariance group $G = \mathbb{Z}_p$ be a cyclic group of prime order. Then, for $\mathbb{Z}_p \curvearrowright X \in \mathbb{Z}_p\mathrm{Actions}(\mathrm{SmoothManifolds})$ (Def. 2.35), the equivariant dual vector space of $\mathbb{Z}_p$-equivariant smooth differential n-forms (Def. 3.33) is injective (Def. 3.11):*

$$\Omega^n_{\mathrm{dR}}\big(\Upsilon(X /\!\!/ \mathbb{Z}_p)\big) \;\in\; \underset{\curlyvee}{G}\mathrm{VectorSpaces}_{\mathbb{R}}^{\vee,\mathrm{inj}}\,. \qquad (88)$$

Proof. By extension of differential forms away from the fixed locus (Notation 3.18), we obtain the following isomorphism of equivariant dual vector spaces to a direct sum of injective extensions (Lemma 3.10)

$$\begin{array}{ccccc}
 & \underset{\text{equivariant smooth differential } n\text{-forms}}{\Omega^n_{\mathrm{dR}}\big(\Upsilon(X/\!\!/\mathbb{Z}_p)\big)} & \xrightarrow{\simeq} & \underset{\text{differential } n\text{-forms on fixed locus}}{\mathrm{Inj}_{\mathbb{Z}_p}\Big(\Omega^n_{\mathrm{dR}}\big(X^{\mathbb{Z}_p}\big)\Big)} \oplus \underset{\text{differential } n\text{-forms whose restriction to the fixed locus vanishes}}{\mathrm{Inj}_1\Big(\big\{\omega \in \Omega^n_{\mathrm{dR}}(X) \,\big|\, \omega|_{X^{\mathbb{Z}_p}} = 0\big\}\Big)} \\
\overset{\mathbb{Z}_p \circlearrowright}{\mathbb{Z}_p/1} & \alpha & \longmapsto & \Big(\alpha|_{X^{\mathbb{Z}_p}} \;,\; \alpha - \mathrm{ext}_{\mathbb{Z}_p}\big(\alpha|_{X^{\mathbb{Z}_p}}\big)\Big) \\
\downarrow & \downarrow & & \downarrow \qquad\qquad \downarrow \\
\mathbb{Z}_p/\mathbb{Z}_p & \alpha|_{X^{\mathbb{Z}_p}} & \longmapsto & \Big(\alpha|_{X^{\mathbb{Z}_p}} \;,\; 0\Big)\,,
\end{array}$$

where we used, since p is assumed to be prime, that the only subgroups of G are 1 and $\mathbb{Z}_p$ itself (Example 2.18). By Prop. 3.12, this implies the claim (88). □

Lemma 3.20 ($\mathbb{Z}_4$-Equivariant smooth differential forms are injective)**.** *Let the equivariance group $G = \mathbb{Z}_4$ be the cyclic group of order 4. Then, for $\mathbb{Z}_4 \curvearrowright X \in \mathbb{Z}_4\mathrm{Actions}(\mathrm{SmoothManifolds})$ (Def. 2.35), the equivariant dual vector space of $\mathbb{Z}_4$-equivariant smooth differential n-forms (Def. 3.33) is injective (Def. 3.11):*

$$\Omega^n_{\mathrm{dR}}\big(\Upsilon(X /\!\!/ \mathbb{Z}_4)\big) \;\in\; \underset{\curlyvee}{G}\mathrm{VectorSpaces}_{\mathbb{R}}^{\vee,\mathrm{inj}}\,. \qquad (89)$$

Proof. Since the subgroups of $\mathbb{Z}_4$ are linearly ordered $1 \subset \mathbb{Z}_2 \subset \mathbb{Z}_4$ (Example 2.18), the proof of Lemma 3.19 generalizes immediately. Using extensions of differential n-forms (Notation 3.18), both from $X^{\mathbb{Z}_4}$ as well as from $X^{\mathbb{Z}_2}$, we obtain the following isomorphism of equivariant dual vector spaces to a direct sum of injective extensions (Lemma 3.10)

$$\begin{array}{ccccc}
 & \underset{\text{equivariant smooth differential } n\text{-forms}}{\Omega^n_{\mathrm{dR}}\big(\Upsilon(X/\!\!/\mathbb{Z}_4)\big)} & \xrightarrow{\simeq} & \underset{\text{differential } n\text{-forms on deep fixed locus}}{\mathrm{Inj}_{\mathbb{Z}_4}\Big(\Omega^n_{\mathrm{dR}}\big(X^{\mathbb{Z}_4}\big)\Big)} \oplus \underset{\text{differential } n\text{-forms on shallow fixed locus whose restriction to the deep fixed locus vanishes}}{\mathrm{Inj}_{\mathbb{Z}_2}\Big(\big\{\omega \in \Omega^n_{\mathrm{dR}}(X^{\mathbb{Z}_2}) \,\big|\, \omega|_{X^{\mathbb{Z}_4}} = 0\big\}\Big)} \oplus \underset{\text{differential } n\text{-forms whose restriction to the shallow fixed locus vanishes}}{\mathrm{Inj}_1\Big(\big\{\omega \in \Omega^n_{\mathrm{dR}}(X) \,\big|\, \omega|_{X^{\mathbb{Z}_2}} = 0\big\}\Big)} \\
\overset{\mathbb{Z}_4 \circlearrowright}{\mathbb{Z}_4/1} & \alpha & \longmapsto & \Big(\alpha|_{X^{\mathbb{Z}_4}} \;,\; \big(\alpha - \mathrm{ext}_{\mathbb{Z}_4}(\alpha|_{X^{\mathbb{Z}_4}})\big)|_{X^{\mathbb{Z}_2}} \;,\; \alpha - \mathrm{ext}_{\mathbb{Z}_2}\big(\alpha|_{X^{\mathbb{Z}_2}}\big)\Big) \\
\downarrow & \downarrow & & \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \\
\mathbb{Z}_4/\mathbb{Z}_2 & \alpha|_{X^{\mathbb{Z}_2}} & \longmapsto & \Big(\alpha|_{X^{\mathbb{Z}_4}} \;,\; \alpha|_{X^{\mathbb{Z}_2}} - \big(\mathrm{ext}_{\mathbb{Z}_4}(\alpha|_{X^{\mathbb{Z}_4}})\big)|_{X^{\mathbb{Z}_2}} \;,\; 0\Big) \\
\downarrow & \downarrow & & \downarrow \qquad\qquad \downarrow \\
\mathbb{Z}_4/\mathbb{Z}_4 & \alpha|_{X^{\mathbb{Z}_4}} & \longmapsto & \Big(\alpha|_{X^{\mathbb{Z}_4}} \;,\; 0 \;,\; 0\Big)
\end{array}$$

By Prop. 3.12, this implies the claim (89). $\square$

Lemma 3.21 ($\mathbb{Z}_2 \times \mathbb{Z}_2$-Equivariant smooth differential forms are injective)**.** *Let the equivariance group $G = \mathbb{Z}_2^L \times \mathbb{Z}_2^R$ be the Klein 4-group. Then, for $\mathbb{Z}_2^L \times \mathbb{Z}_2^R \circlearrowright X \in \mathbb{Z}_2^L \times \mathbb{Z}_2^R \mathrm{Actions}(\mathrm{SmoothManifolds})$ (Def. 2.35), the equivariant dual vector space of equivariant smooth differential n-forms (Def. 3.33) is injective (Def. 3.11):*

$$\Omega^n_{\mathrm{dR}}\big(\curlyvee\big(X /\!\!/ \mathbb{Z}_2^L \times \mathbb{Z}_2^R\big)\big) \;\in\; G\mathrm{VectorSpaces}_{\mathbb{R}}^{\vee,\mathrm{inj}}\,. \tag{90}$$

Proof. We obtain an isomorphism to a direct sum of injective extensions (Lemma 3.10), much as in the proofs of Lemmas 3.19 and 3.20,

$$\underset{\text{equivariant smooth differential } n\text{-forms}}{\Omega^n_{\mathrm{dR}}\big(\curlyvee(X /\!\!/ \mathbb{Z}_4)\big)} \xrightarrow{\simeq} \underset{\text{differential } n\text{-forms on deep fixed locus}}{\mathrm{Inj}_{\mathbb{Z}_4}\Big(\Omega^n_{\mathrm{dR}}\big(X^{\mathbb{Z}_4}\big)\Big)} \oplus \overset{\text{differential } n\text{-forms on shallow fixed loci whose restriction to the deep fixed locus vanishes}}{\begin{array}{c} \mathrm{Inj}_{\mathbb{Z}_2^L}\Big(\Big\{\omega \in \Omega^n_{\mathrm{dR}}\big(X^{\mathbb{Z}_2^L}\big) \,\Big\vert\, \omega|_{X^{\mathbb{Z}_4}} = 0\Big\}\Big) \\ \oplus \\ \mathrm{Inj}_{\mathbb{Z}_2^R}\Big(\Big\{\omega \in \Omega^n_{\mathrm{dR}}\big(X^{\mathbb{Z}_2^R}\big) \,\Big\vert\, \omega|_{X^{\mathbb{Z}_4}} = 0\Big\}\Big) \end{array}} \oplus \mathrm{Inj}_1\left(\overset{\text{differential } n\text{-forms whose restriction to the shallow fixed loci vanishes}}{\left\{\omega \in \Omega^n_{\mathrm{dR}}(X) \,\middle\vert\, \begin{array}{l} \omega|_{X^{\mathbb{Z}_2^L}} = 0 \\ \omega|_{X^{\mathbb{Z}_2^K}} = 0 \end{array}\right\}}\right)$$

$$\begin{array}{ccccccc}
\overset{\mathbb{Z}_4}{\circlearrowright} G/1 & & \alpha & \longmapsto & \Big(\alpha|_{X^{\mathbb{Z}_4}} \,, & \overbrace{\Big(\alpha - \mathrm{ext}_{\mathbb{Z}_2^L \times \mathbb{Z}_2^R}\big(\alpha|_{X^{\mathbb{Z}_2^L \times \mathbb{Z}_2^R}}\big)\Big)}^{=:\beta}\Big|_{X^{\mathbb{Z}_2^L} + |X^{\mathbb{Z}_2^R}} \,, & \begin{array}{l}\beta - \mathrm{ext}_{\mathbb{Z}_2^R}\big(\beta|_{X^{\mathbb{Z}_2^R}}\big) \\ \quad - \mathrm{ext}_{\mathbb{Z}_2^L}\big(\beta|_{X^{\mathbb{Z}_2^L}}\big)\end{array}\Big) \\
\downarrow \quad \searrow G/\mathbb{Z}_2^R & & \downarrow & & \downarrow & \downarrow & \downarrow \\
G/\mathbb{Z}_2^L \quad \swarrow & & \alpha|_{X^{\mathbb{Z}_2}} & \longmapsto & \Big(\alpha|_{X^{\mathbb{Z}_4}} \,, & \Big(\alpha - \mathrm{ext}_{\mathbb{Z}_2^L \times \mathbb{Z}_2^R}\big(\alpha|_{X^{\mathbb{Z}_2^L \times \mathbb{Z}_2^R}}\big)\Big)\Big|_{X^{\mathbb{Z}_2^L} + |X^{\mathbb{Z}_2^R}} \,, & 0 \Big) \\
\downarrow & & \downarrow & & \downarrow & \downarrow & \downarrow \\
G/\mathbb{Z}_2^L \times \mathbb{Z}_2^R & & \alpha|_{X^{\mathbb{Z}_4}} & \longmapsto & \Big(\alpha|_{X^{\mathbb{Z}_4}} \,, & 0 \,, & 0 \Big)
\end{array}$$

and hence conclude the result, again by Prop. 3.12. The only further subtlety to take care of here is that the two extensions $\mathrm{ext}_{\mathbb{Z}_2^L}$ and $\mathrm{ext}_{\mathbb{Z}_2^R}$ (Notation 3.18) need to be chosen compatibly, such as to ensure that each preserves the property of a form to vanish on the corresponding other fixed locus:

$$\Big(\mathrm{ext}_{\mathbb{Z}_2^L}\Big(\beta|_{X^{\mathbb{Z}_2^L}}\Big)\Big)\Big|_{\mathbb{Z}_2^R} = 0\,, \qquad \Big(\mathrm{ext}_{\mathbb{Z}_2^R}\Big(\beta|_{X^{\mathbb{Z}_2^R}}\Big)\Big)\Big|_{\mathbb{Z}_2^L} = 0\,.$$

This is achieved by choosing an *equivariant* tubular neighborhood (by [Bre72, §VI, Thm. 2.2][Ka07, Thm. 4.4]) around the intersection $X^{\mathbb{Z}_2^R} \cap X^{\mathbb{Z}_2^L}$ and using this to choose the extension away from $X^{\mathbb{Z}_2^L}$ to be orthogonal to that away from $X^{\mathbb{Z}_2^R}$. $\square$

Equivariant graded vector spaces.

Definition 3.22 (Equivariant graded vector spaces)**.** We write

$$G\mathrm{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0} := G\mathrm{VectorSpaces}_{\mathbb{R}}^{\mathbb{N}} \simeq \mathrm{Functors}\big(G\mathrm{Orbits}^{\mathrm{op}}, \mathrm{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0}\big)$$

for the category of $\mathbb{N}$-graded objects in equivariant vector spaces (Def. 3.5).

Definition 3.23 (Equivariant rational homotopy groups)**.** For $\mathcal{X} \in G\mathrm{HomotopyTypes}_{\geq 1}$ (Def. 2.32) and $n \in \mathbb{N}$, the rationalized nth equivariant homotopy group (Def. 2.31) hence the stage-wise rationalized simplicial homotopy group (Def. 2.31)

$$\underline{\pi}_n(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{R} \;:\; G/H \longmapsto \pi_n\big(\mathcal{X}(G/H)\big) \otimes_{\mathbb{Z}} \mathbb{R}\,,$$

form an equivariant graded vector space (Def. 3.22):

$$\underline{\pi}_{\bullet+1}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{R} \;\in\; G\mathrm{VectorSpaces}_{\mathbb{R}}\,.$$

Example 3.24 ($\mathbb{Z}_2^A$-Equivariant rational homotopy groups of twistor space)**.** The $\mathbb{Z}_2$-equivariant rational homotopy groups (Def. 3.23) of $\mathbb{Z}_2^A$-equivariant twistor space (Example 2.28) are, by (54), given by the rational homotopy groups of $\mathbb{C}P^3$ and, on the fixed locus, of S^2. Hence these look as follows (using, e.g., [FSS20c, Lemma 2.13] with [FSS20d, Prop. 3.65]):

$\underline{\pi}_{\bullet}^{\mathbb{Z}/2}(\mathbb{C}P^3) \otimes_{\mathbb{Z}} \mathbb{R} \simeq$ (91)

$\mathbb{Z}_2/H$	$(\mathbb{C}P^3)^H$	$\pi_2 \otimes \mathbb{R}$	$\pi_3 \otimes \mathbb{R}$	$\pi_4 \otimes \mathbb{R}$	$\pi_5 \otimes \mathbb{R}$	$\pi_6 \otimes \mathbb{R}$	$\pi_7 \otimes \mathbb{R}$	$\pi_8 \otimes \mathbb{R}$	$\pi_9 \otimes \mathbb{R}$	$\cdots$
$\mathbb{Z}_2/1$	$\mathbb{C}P^3$	**1**	0	0	0	0	**1**	0	0	$\cdots$
$\mathbb{Z}_2/\mathbb{Z}_2$	S^2	**1**	**1**	0	0	0	0	0	0	$\cdots$

Equivariant cochain complexes.

Definition 3.25 (Equivariant cochain complexes)**.** We write

$$\underset{\sim}{G}\mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0} :=\ \mathrm{Functors}\big(G\mathrm{Orbits}\,,\,\mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0}\big)$$

for the category of functors from the G-orbit category (Def. 2.13) to the category of connective cochain complexes (i.e., in non-negative degrees with differential of degree +1) over the real numbers.

Definition 3.26 (Delooping of equivariant cochain complexes)**.** For $V \in \underset{\sim}{G}\mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0}$ (Def. 3.25), we denote its delooping as

$$\mathfrak{b}V \;:\; G/H \longmapsto \Big(0 \longrightarrow V^0(G/H) \xrightarrow{d_V^0} V^1(G/H) \xrightarrow{d_V^1} V^2(G/H) \longrightarrow \cdots \Big).$$

As an instance of the general notion of mapping cones (e.g. [Scha11, Def. 3.2.2]), we get:

Example 3.27 (Cone on an equivariant cochchain complex)**.** For $V \in \underset{\sim}{G}\mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0}$ (Def. 3.25), we say that the *cone* on its delooping $\mathfrak{b}V$ (Def. 3.26) is the equivariant cochain complex $\mathfrak{e}V \in \underset{\sim}{G}\mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0}$ given by

$$\mathfrak{e}V := \mathrm{Cone}(\mathfrak{b}V) \,:\, G/H \longmapsto \left(\begin{array}{ccccccccc} V^0(G/H) & \xrightarrow{-d_V^0} & V^1(G/H) & \xrightarrow{-d_V^1} & V^2(G/H) & \xrightarrow{-d_V^2} & V^3(G/H) & \xrightarrow{-d_V^3} & \cdots \\ \oplus & \searrow^{\mathrm{id}} & \oplus & \searrow^{\mathrm{id}} & \oplus & \searrow^{\mathrm{id}} & \oplus & \searrow^{\mathrm{id}} & \\ 0 & \xrightarrow[0]{} & V^0(G/H) & \xrightarrow[d_V^0]{} & V^1(G/H) & \xrightarrow[d_V^1]{} & V^2(G/H) & \xrightarrow[d_V^2]{} & \cdots \end{array} \right).$$

This sits in the evident cofiber sequence:

$$\begin{array}{ccc} V & \xleftarrow{\mathrm{cofib}(i_{\mathfrak{b}V})} & \mathfrak{e}V \\ & & \uparrow{\scriptstyle i_{\mathfrak{b}V}} \\ & & \mathfrak{b}V \end{array} \qquad \in \;\; \underset{\sim}{G}\mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0}\,. \tag{92}$$

As an instance of the general notion of injective resolutions (e.g. [Scha11, §4.5]), we have:

Example 3.28 (Injective resolution of equivariant dual vector spaces)**.**

Let $V \in \underset{\sim}{G}\mathrm{VectorSpaces}_{\mathbb{R}}^{\vee}$ (Def. 3.5). Then, by Prop. 3.12, we obtain an *injective resolution* (e.g. [HS71, p. 129]) of V given by the equivariant cochain complex (Def. 3.25) which in degree 0 is the injective envelope (83) of V, and whose differentials are, recursively, the injective envelope inclusions (85) of the quotients by the image of the previous degree.

$$\begin{array}{ccc} \vdots & & \vdots \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathrm{Inj}\big(\mathrm{coker}(d^1)\big) \\ \uparrow & & \uparrow{\scriptstyle d^2} \\ 0 & \longrightarrow & \mathrm{Inj}\big(\mathrm{coker}(d^0)\big) \\ \uparrow & & \uparrow{\scriptstyle d^1} \\ 0 & \longrightarrow & \mathrm{Inj}\big(\mathrm{Inj}(V)/V\big) \\ \uparrow & & \uparrow{\scriptstyle d^0} \\ V & \hookrightarrow & \mathrm{Inj}(V) \end{array} \qquad =: \mathrm{Inj}^{\bullet}(V) \;\in\; \underset{\sim}{G}\mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0}$$

This is such that for any $A^\bullet \in G\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}$ which is degreewise injective (Def. 3.11) and any morphism of equivariant dual vector spaces

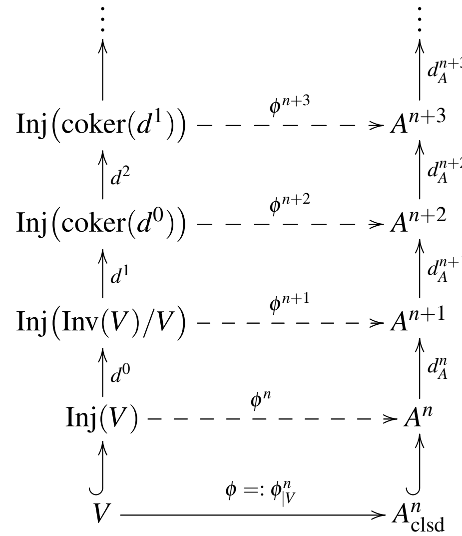
$$\big\{ V \xrightarrow{\ \phi\ } A^n_{\mathrm{clsd}} \big\} \quad \in \quad G\text{VectorSpaces}^{\vee}_{\mathbb{R}}$$

from V to the subspace of closed elements (cocycles) in A^n, there exists an extension to a morphism

$$\big\{ \mathfrak{b}^n \mathrm{Inj}^\bullet(V) \xrightarrow{\ \phi^\bullet\ } A^\bullet \big\} \quad \in \quad G\text{CochainComplexes}_{\mathbb{R}}^{\geq 0} \tag{93}$$

of equivariant cochain complexes (as shown on the right) given recursively by using injectivity of A^{n+i+1} to obtain dashed extensions (82):

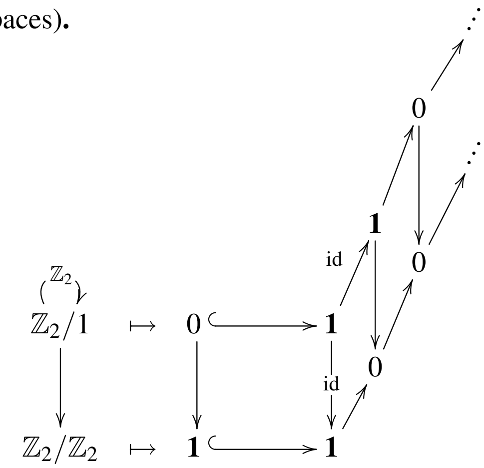
$$\begin{array}{ccc} \mathrm{Inj}^{i+1}(V) & \overset{\phi^{n+i+1}}{\dashrightarrow} & A^{n+i+1} \,. \\ \uparrow & \nearrow_{d_A \circ \phi^i} & \\ \mathrm{Inj}^i(V)/\mathrm{im}(d^{i-1}) & & \end{array}$$



Example 3.29 (Injective resolution of $\mathbb{Z}_2$-equivariant dual vector spaces)**.** Consider the $\mathbb{Z}_2$-equivariant dual vector space (Example 3.7) given by

$$\left(\begin{array}{ccc} \overset{\mathbb{Z}_2}{\circlearrowright} & & \\ \mathbb{Z}_2/1 & \longmapsto & 0 \\ \downarrow & & \downarrow 0 \\ \mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathbf{1} \end{array} \right) \quad \in \quad \mathbb{Z}_2\, G\text{VectorSpaces}^{\vee}_{\mathbb{R}} \,. \tag{94}$$

Recalling the three injective atoms of $\mathbb{Z}_2$-equivariant dual vector spaces from Example 3.14, we find that the injective resolution (Example 3.28) of (94) is the $\mathbb{Z}_2$-equivariant cochain complex shown on the right.



In terms of generators-and-relations (Notation 3.1), this says:

$$\mathrm{Inj}^\bullet \left(\begin{array}{ccc} \overset{\mathbb{Z}_2}{\circlearrowright} & & \\ \mathbb{Z}_2/1 & \longmapsto & 0 \\ \Big\downarrow & & \Big\downarrow \\ \mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathbb{R}\langle c_0 \rangle / (d\, c_0 = 0) \end{array} \right) \;=\; \left(\begin{array}{ccc} \overset{\mathbb{Z}_2}{\circlearrowright} & & \\ \mathbb{Z}_2/1 & \longmapsto & \mathbb{R}\left\langle \begin{array}{c} c_0', c_1 \\ c_0 \end{array} \right\rangle \Big/ \left(\begin{array}{l} d\, c_0' = c_1 \\ d\, c_1 = 0 \\ d\, c_0 = 0 \end{array} \right) \\ \Big\downarrow & & \Big\downarrow \\ \mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathbb{R}\langle c_0 \rangle \big/ \big(d\, c_0 = 0 \big) \end{array} \right) . \tag{95}$$

Equivariant dgc-algebras.

Definition 3.30 (Equivariant dgc-Algebras)**.** We write

$$G\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \;:=\; \text{Functors}\big(G\text{Orbits}\,,\, \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \big)$$

for the category of functors from the G-orbit category (Def. 2.13) to the category of connective dgc-algebras over the real numbers.

Definition 3.31 (Equivariant cochain cohomology groups)**.** For $A \in G\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$ (Def. 3.30) and $n \in \mathbb{N}$, we write

$$\underline{H}^n(A) \;\in\; G\text{VectorSpaces}^{\vee}_{\mathbb{R}}$$

for the equivariant dual vector space (Def. 3.5) of cochain cohomology groups

$$\underline{H}^n(A) \;:\; G/H \;\longmapsto\; H^n\big(A(G/H)\big) \,.$$

Example 3.32 (Equivariant base dgc-algebra)**.**
We write $\underline{\mathbb{R}} \in G\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$ for the equivariant dgc-algebra (Def. 3.30) which is constant on the ground field $\mathbb{R}$:

$$\underline{\mathbb{R}} \;:\; G/H \longmapsto \mathbb{R}.$$

For the case $G = \mathbb{Z}_2$ (Example 2.15), this is shown on the right.

$$\begin{array}{ccc} \overset{\mathbb{Z}_2}{\circlearrowright} & & \\ \mathbb{Z}_2/1 & \longmapsto & \mathbb{R} \\ \downarrow & & \downarrow \mathrm{id} \\ \mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathbb{R} \end{array}$$

Example 3.33 (Equivariant smooth de Rham complex)**.** For $G \circlearrowright X \in G\text{Actions}\big(\text{SmoothManifolds}\big)$ (Def. 2.35). there is the equivariant dgc-algebra (Def. 3.30)

$$\Omega^\bullet_{\mathrm{dR}}\big(\,\curlyvee(X /\!\!/ G)\big) \;\in\; G\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$$

of equivariant smooth differential forms (Example 3.16) equipped with the wedge product and de Rham differential formed stage-wise, as in the ordinary smooth de Rham complex (e.g. [BT82]) of the fixed loci.

Example 3.34 (Free equivariant dgc-algebra on equivariant cochain complex)**.** For $V^\bullet \in G\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}$ (Def. 3.25):
(i) We obtain the free equivariant dgc-algebra (Def. 3.30)

$$\mathrm{Sym}(V^\bullet) \;\in\; G\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\,,$$

given over each $G/H \in G\text{Orbits}$, by the free dgc-algebra on the cochain complex at that stage:

$$\mathrm{Sym}(V^\bullet) \;:\; G/H \mapsto \mathrm{Sym}\big(V^\bullet(G/H)\big),$$

with all structure maps induced by the functoriality of the non-equivariant Sym-construction.
(iii) This extends to a functor

$$G\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \underset{\text{CchnCmplx}}{\overset{\text{Sym}}{\underset{\bot}{\rightleftarrows}}} G\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}\,, \tag{96}$$

which is left adjoint to the evident assignment of underlying equivariant cochain complexes.

In terms of generators and relations (Notation 3.1, 3.2), passing to free dgc-algebras means to replace angular brackets by square brackets:

Example 3.35 (Free $\mathbb{Z}_2$-equivariant dgc-algebra on injective resolution)**.** In the case $G = \mathbb{Z}_2$ (Example 2.15), the free $\mathbb{Z}_2$-equivariant dgc-algebra (Example 3.34) on the n-fold delooping (Def. 3.26) of the injective resolution (95) from Example 3.29 is:

$$\mathrm{Sym} \circ \mathfrak{b}^n \circ \mathrm{Inj}^\bullet \left(\begin{array}{ccc} \overset{\mathbb{Z}_2}{\circlearrowright} & & \\ \mathbb{Z}_2/1 & \longmapsto & 0 \\ \downarrow & & \downarrow \\ \mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathbb{R}[c_0]/(d\,c_0 = 0) \end{array} \right) = \left(\begin{array}{ccc} \overset{\mathbb{Z}_2}{\circlearrowright} & & \\ \mathbb{Z}_2/1 & \longmapsto & \mathbb{R}\left[\begin{array}{l} c'_n, c_{n+1} \\ c_n \end{array}\right] \Big/ \left(\begin{array}{c} d\,c'_n = c_{n+1} \\ d\,c_{n+1} = 0 \\ d\,c_n = 0 \end{array} \right) \\ \downarrow & & \downarrow \\ \mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathbb{R}\langle c_n\rangle/(d\,c_n = 0) \end{array} \right). \tag{97}$$

In equivariant generalization of [FSS20d, Def. 3.25], we have:

Definition 3.36 (Equivariant L_∞-algebras)**.** We write

$$\begin{array}{ccc} G L_\infty\text{Algebras}_{\mathbb{R},\mathrm{fin}}^{\geq 0} & \xhookrightarrow{\;\;\mathrm{CE}\;\;} & \big(G\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\big)^{\mathrm{op}} \\ \underline{\mathfrak{g}} & \longmapsto & \mathrm{CE}\big(\underline{\mathfrak{g}}\big) \end{array} \tag{98}$$

for the opposite of the full subcategory of equivariant dgc-algebras (Def. 3.30) on those that are stage-wise Chevalley-Eilenberg algebras of L_∞-algebras (connective and finite-type over the real numbers, as in [FSS20d, Def. 3.25]).

In generalization of Example 3.33, we have:

Example 3.37 (Proper G-equivariant and Borel-Weil-Cartan T-equivariant smooth de Rham complex)**.** Let $(T \times G) \circlearrowleft X \in (T \times G)\mathrm{Actions}(\mathrm{SmoothManifolds})$ (Def 2.35), where $T \in \mathrm{CompactLieGroups}$ is finite-dimensional with Lie algebra denoted (as in Notation 3.1)

$$\mathfrak{t} \simeq \left\{ \langle t_a \rangle_{a=1}^{\dim(T)}, [-,-] \right\} \in \mathrm{LieAlgebras}_{\mathbb{R},\mathrm{fin}}. \tag{99}$$

Consider the equivariant dgc-algebra (Def. 3.30)

$$\Omega^\bullet_{\mathrm{dR}}\Big(\big(\curlyvee(X /\!\!/ G) \big) /\!\!/ T \Big) \in \underset{\curlyvee}{G}\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}$$

of T-invariants in the tensor product of proper G-equivariant smooth differential forms (Example 3.16) with the free symmetric graded algebra on

$$\mathfrak{b}^2\mathfrak{t}^\vee \simeq \langle r_2^a \rangle_{a=1}^{\dim(T)},$$

(the linear dual space of (99) in degree 2) and equipped with the sum of the de Rham differential

$$d_{\mathrm{dR}} : \omega \wedge r_2^{a_1} \wedge \cdots \wedge r_2^{a_p} \longmapsto \big(d_{\mathrm{dR}} \omega \big) \wedge r_2^{a_1} \wedge \cdots \wedge r_2^{a_p}$$

and the operator

$$r_2^a \wedge \iota_{t_a} : \omega \wedge r_2^{a_1} \wedge \cdots \wedge r_2^{a} \longmapsto \big(\iota_{t_a} \omega \big) \wedge r_2^a \wedge r_2^{a_1} \wedge \cdots \wedge r_2^a,$$

where

- $\omega \in \Omega^\bullet_{\mathrm{dR}}(-)$,
- ι_{t_a} denotes the contraction of differential forms with the vector field that is the derivative of the action $T \times X \to X$ along t_a,
- summation over the index $a \in \{1, \cdots, \dim(T)\}$ is understood, and
- the T-action on $\mathfrak{t}^\vee$ is the coadjoint action and on that differential forms is by pullback along the given action on X:

$$\underset{\substack{\text{proper } G\text{-equivariant \& Borel } T\text{-equivariant} \\ \text{smooth de Rham complex}}}{\Omega^\bullet_{\mathrm{dR}}\Big(\big(\curlyvee(X /\!\!/ G) \big) /\!\!/ T \Big)} : G/H \longmapsto \underset{\text{Cartan model for } T\text{-equivariant Borel cohomology of } H\text{-fixed locus } X^H}{\Big(\Omega^\bullet_{\mathrm{dR}}\big(X^H\big) \Big[\{ r_2^a \}_{a=1}^{\dim(T)} \Big], d_{\mathrm{dR}} + r_2^a \wedge \iota_{t_a} \Big)^T}. \tag{100}$$

This is, stage-wise over $G/H \in G\mathrm{Orbits}$ (Def. 2.13), the *Cartan model* dgc-algebra for Borel T-equivariant de Rham cohomology ([AB84][MQ86, §5][Ka93][GS99], review in [Me06][KT15][Pe17]), here formed for the fixed submanifolds (Lemma 2.34) of the all the subgroups of the G-action.

Homotopy theory of equivariant dgc-algebras.

Proposition 3.38 (Projective model structure on connective equivariant dgc-algebras [Scu02, Theorem 3.2])**.** *There is the structure of a model category on* $\underset{\curlyvee}{G}\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}$ *(Def. 3.30) whose*

W – *weak equivalences are the quasi-isomorphisms over each* $G/H \in G$ Orbits*;*

Fib – *fibrations are the degreewise surjections whose degreewise kernels are injective (Def. 3.11).*

We denote this model category by

$$\Big(\underset{\curlyvee}{G}\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}} \Big)_{\mathrm{proj}} \in \mathrm{ModelCategories}.$$

A key technical subtlety of the model structure on equivariant dgc-algebras (Prop. 3.38), compared to its non-equivariant version ([BG76, §4.3][GM96, §V.3.4][FSS20d, Prop. 3.36]), is that not all objects are fibrant anymore, since equivariantly the injectivity condition (Def. 3.11) is non-trivial (Prop. 3.12). However, we have the following class of examples of fibrant objects:

Proposition 3.39 (Equivariant smooth de Rham complex is projectively fibrant)**.**
For $G \circlearrowright X \in G\mathrm{Actions}\big(\mathrm{SmoothManifolds}\big)$ *(Def. 2.35), the equivariant smooth de Rham complex (Example 3.33) is a fibrant object in the projective model structure (Prop. 3.38)*

$$\Omega^\bullet_{\mathrm{dR}}\big(\curlyvee(X /\!\!/ G)\big) \xrightarrow{\ \in \mathrm{Fib}\ } 0 \quad \in \quad \big(\underset{\curlyvee}{G}\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}},$$

at least if G *is of order 4 or cyclic of prime order.*

Proof. By Prop. 3.38, the statement is equivalent to the claim that the equivariant dual vector spaces of equivariant smooth differential n-forms are injective. This is indeed the case, by Lemmas 3.19, 3.20, 3.21 (Remark 3.17). ☐

Next we turn to discussion of fibrant and cofibrant equivariant dgc-algebras.

Minimal equivariant dgc-algebras.

Definition 3.40 (Minimal equivariant dgc-algebras [Tri82, Construction 5.10][Scu02, §11][Scu08, §4])**.**
Let $A \in \underset{\curlyvee}{G}\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}$ (Def. 3.30) be such that, for all $k \in \mathbb{N}$, the underlying $\mathrm{ChnCmplx}(A)^k \in \underset{\curlyvee}{G}\mathrm{CochainComplexes}^{\geq 0}_{\mathbb{R}}$ is injective (Def. 3.11).
(i) For $n \in \mathbb{N}$, an *elementary extension* $A \hookrightarrow A[\mathfrak{b}^n V]_\phi$ of A in degree n is a pushout of the image under Sym (Example 3.34) of the cone inclusion (Example 3.27) of the $(n+1)$-fold delooping (Def. 3.26) of the injective resolution $\mathrm{Inj}^\bullet(V)$ (Example 3.28)

$$\begin{array}{ccc} A\big[\mathfrak{b}^n V_n\big]_{\phi_n} & \longleftarrow & \mathrm{Sym}\big(\mathfrak{e}\mathfrak{b}^n \mathrm{Inj}^\bullet(V_n)\big) \\ \big\uparrow & {\scriptstyle (\mathrm{po})} & \big\uparrow {\scriptstyle \mathrm{Sym}\left(i_{\mathfrak{b}^{n+1}\mathrm{Inj}^\bullet(V_n)}\right)} \\ A & \xleftarrow{\ \widetilde{\phi^\bullet_n}\ } & \mathrm{Sym}\big(\mathfrak{b}^{n+1}\mathrm{Inj}^\bullet(V_n)\big) \end{array} \quad \in \underset{\curlyvee}{G}\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}} \tag{101}$$

along the adjunct $\widetilde{\phi}^\bullet$ (96) of an injective extension (93)

$$A^\bullet \xleftarrow{\ \phi^\bullet_n\ } \mathfrak{b}^{n+1}\mathrm{Inj}^\bullet(V_n) \ \in \ \underset{\curlyvee}{G}\mathrm{CochainComplexes}^{\geq 0}_{\mathbb{R}} \tag{102}$$

of a given *attaching map* out of a given equivariant dual vector space V_n (Def. 3.5):

$$A^{n+1}_{\mathrm{clsd}} \xleftarrow{\ \phi_n\ } V_n \quad \in \ \underset{\curlyvee}{G}\mathrm{VectorSpaces}^{\vee}_{\mathbb{R}}. \tag{103}$$

(ii) An inclusion

$$B^\bullet \overset{\mathrm{min}}{\hookrightarrow} A^\bullet \quad \in \underset{\curlyvee}{G}\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}} \tag{104}$$

of degreewise injective (Def. 3.11) equivariant dgc-algebras (Def. 3.30) which are equivariantly 1-connected

$$B^0 \simeq \underline{\mathbb{R}}, \quad B^1 \simeq \underline{\mathbb{R}}$$

is called *relative minimal* if it is isomorphic under $B^\bullet$ to the result of a sequence of elementary extensions (101) in strictly increasing degrees (noticing with Lemma 3.15, that the result of an elementary extension (101) is again degreewise injective).
(iii) An equivariant dgc-algebra $A^\bullet$, such that the unique inclusion of the equivariant ground field $\underline{R}$ (which is clearly 1-connected and injective, by Example 3.13) is a relative minimal dgc-algebra (104)

$$\underline{\mathbb{R}} \overset{\mathrm{min}}{\hookrightarrow} A^\bullet \in \underset{\curlyvee}{G}\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}, \tag{105}$$

is called a *minimal equivariant dgc-algebra*.

Definition 3.41 (Minimal equivariant L_∞-algebra)**.** Any minimal equivariant dgc-algebra A (Def. 3.40) is the equivariant Chevalley-Eilenberg algebra (98)

$$A \simeq \mathrm{CE}\big(\underline{\mathfrak{g}}^A\big)$$

of an equivariant L_∞-algebra $\underline{\mathfrak{g}}^A \in \underset{\curlyvee}{G}L_\infty\mathrm{Algebras}^{\geq 0}_{\mathbb{R},\mathrm{fin}}$ (Def. 3.36), defined uniquely up to isomorphism. We say that the *underlying* graded equivariant vector space (Def. 3.22)

$$\underline{\mathfrak{g}}^A_\bullet \;\in\; \underset{\curvearrowright}{G}\mathrm{GradedVectorSpaces}^{\geq 0}_{\mathbb{R}}$$

of this equivariant L_∞-algebra is the linear dual of the spaces of generators $V_n^A \in \underset{\curvearrowright}{G}\mathrm{VectorSpaces}^{\vee}_{\mathbb{R}}$ (103) of the elementary extensions (101) that exhibit the minimality of A:

$$\underline{\mathfrak{g}}^A_n := \big(V_n^A\big)^{\vee} \quad \in \underset{\curvearrowright}{G}\mathrm{VectorSpaces}_{\mathbb{R}}\,.$$

Example 3.42 (A minimal $\mathbb{Z}_2$-equivariant dgc-algebra)**.** We spell out the construction of an equivariant minimal dgc-algebra (Def. 3.40), for $G = \mathbb{Z}_2$ (Example 2.15), which involves three basic cases of the elementary extensions (101):

(i) In the first stage, begin with the equivariant base algebra $\underline{\mathbb{R}}$ (Example 3.32) and consider the attaching map (103) in degree 2 given by

$$\phi_2 \;:\; \begin{array}{ccccc} \overset{\curvearrowright^{\mathbb{Z}_2}}{\mathbb{Z}_2/1} & \longmapsto & \mathbb{R} & \xleftarrow{\;0 \,\leftarrow\!\shortmid\, c_3\;} & \mathbb{R}\langle c_3\rangle \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathbb{R} & \xleftarrow{\;0 \,\leftarrow\!\shortmid\, c_3\;} & \mathbb{R}\langle c_3\rangle \end{array} \tag{106}$$

By Example 3.14, the equivariant dual vector space on the right is already injective (87), so that we may extend this attaching map immediately to an equivariant cochain map (102)

$$\phi_2^\bullet \;:\; \begin{array}{ccccc} \overset{\curvearrowright^{\mathbb{Z}_2}}{\mathbb{Z}_2/1} & \longmapsto & \mathbb{R} & \xleftarrow{\;0 \,\leftarrow\!\shortmid\, c_3\;} & \mathbb{R}\langle c_3\rangle\big/(d\,c_3 = 0) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathbb{R} & \xleftarrow{\;0 \,\leftarrow\!\shortmid\, c_3\;} & \mathbb{R}\langle c_3\rangle\big/(d\,c_3 = 0)\,, \end{array}$$

where on the right we are using the generators-and-relations Notation 3.1. By Example 3.35, its adjunct morphism of equivariant dgc-algebras is

$$\widetilde{\phi}_2^\bullet \;:\; \begin{array}{ccccc} \overset{\curvearrowright^{\mathbb{Z}_2}}{\mathbb{Z}_2/1} & \longmapsto & \mathbb{R} & \xleftarrow{\;0 \,\leftarrow\!\shortmid\, c_3\;} & \mathbb{R}[c_3]\big/(d\,c_3 = 0) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathbb{R} & \xleftarrow{\;0 \,\leftarrow\!\shortmid\, c_3\;} & \mathbb{R}[c_3]\big/(d\,c_3 = 0)\,. \end{array}$$

Since all these diagrams so far are constant on the orbit category, the resulting pushout (101) is computed over both objects $\mathbb{Z}_2/H \in \mathbb{Z}_2\mathrm{Orbits}$ as in non-equivariant dgc-theory, and thus yields this minimal equivariant dgc-algebra:

$$\begin{array}{ccc} \overset{\curvearrowright^{\mathbb{Z}_2}}{\mathbb{Z}_2/1} & \longmapsto & \mathbb{R}[f_2]\big/(d\,f_2 = 0) \\ \downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\ \mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathbb{R}[f_2]\big/(d\,f_2 = 0)\,. \end{array} \tag{107}$$

(ii) Consider next the following attaching map (103) in degree 3 to the equivariant dgc-algebra (107):

$$\phi_3 \;:\; \begin{array}{ccccc} \overset{\curvearrowright^{\mathbb{Z}_2}}{\mathbb{Z}_2/1} & \longmapsto & \mathbb{R}[f_2]\big/(d\,f_2 = 0) & \longleftarrow & 0 \\ \downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\ \mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathbb{R}[f_2]\big/(d\,f_2 = 0) & \xleftarrow{\;f_2\wedge f_2 \,\leftarrow\!\shortmid\, c_4\;} & \mathbb{R}\langle c_4\rangle\,. \end{array} \tag{108}$$

Here the equivariant dual vector space on the right is *not* injective: Its injective envelope is given in Example 3.29, and the free dgc-algebra on this is given in Example 3.35, which says that the required extension (102) of the attaching map ϕ is hence of this form:

$$
\widetilde{\phi}_3^\bullet \;:\;
\begin{array}{ccccc}
\overset{\curvearrowright^{\mathbb{Z}_2}}{\mathbb{Z}_2/1} & \longmapsto & \mathbb{R}[f_2]/(d f_2 = 0) & \xleftarrow[f_2 \wedge f_2 \,\leftarrow\!\shortmid\, c_4]{0 \,\leftarrow\!\shortmid\, c_5} & \mathbb{R}\begin{bmatrix} c_5 \\ c_4 \end{bmatrix} \Big/ \begin{pmatrix} d c_5 = 0 \\ d c_4 = c_5 \end{pmatrix} \\
\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow \\
\mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathbb{R}[f_2]/(d f_2 = 0) & \xleftarrow{f_2 \wedge f_2 \,\leftarrow\!\shortmid\, c_4} & \mathbb{R}[c_4]/(d c_4 = 0)\,.
\end{array}
$$

The pushout (101) along this map is the following, yielding the next stage of the minimal equivariant dgc-algebra on the rear left:

$$
\begin{array}{ccccc}
\mathbb{R}\begin{bmatrix} h_3, \\ \omega_4, \\ f_2 \end{bmatrix} \Big/ \begin{pmatrix} d h_3 = \omega_4 - f_2 \wedge f_2 \\ d \omega_4 = 0 \\ d f_2 = 0 \end{pmatrix} & \xleftarrow[\substack{\omega_4 \,\leftarrow\!\shortmid\, b_4 \\ h_3 \,\leftarrow\!\shortmid\, b_3}]{\substack{0 \,\leftarrow\!\shortmid\, c_5 \\ f_2 \wedge f_2 \,\leftarrow\!\shortmid\, c_4}} & & & \mathbb{R}\begin{bmatrix} c_5, & b_4, \\ c_4, & b_3 \end{bmatrix} \Big/ \begin{pmatrix} d c_5 = 0\,,\; d b_4 = c_5 \\ d c_4 = c_5\,,\; d b_3 = b_4 - c_4 \end{pmatrix} \\
 & \nwarrow & & \nwarrow & \\
 & & \mathbb{R}[f_2]/(d f_2 = 0) & \xleftarrow[f_2 \wedge f_2 \,\leftarrow\!\shortmid\, c_4]{0 \,\leftarrow\!\shortmid\, c_5} & \mathbb{R}\begin{bmatrix} c_5 \\ c_4 \end{bmatrix} \Big/ \begin{pmatrix} d c_5 = 0 \\ d c_4 = c_5 \end{pmatrix} \\
\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow \\
\mathbb{R}\begin{bmatrix} h_3, \\ f_2 \end{bmatrix} \Big/ \begin{pmatrix} d h_3 = - f_2 \wedge f_2 \\ d f_2 = 0 \end{pmatrix} & \xleftarrow[h_3 \,\leftarrow\!\shortmid\, b_3]{f_2 \wedge f_2 \,\leftarrow\!\shortmid\, c_4} & & & \mathbb{R}[c_4, b_3]/(d c_4 = c_5\,,\, d b_3 = - c_4) \\
 & \nwarrow & & \nwarrow & \\
 & & \mathbb{R}[f_2]/(d f_2 = 0) & \xleftarrow{f_2 \wedge f_2 \,\leftarrow\!\shortmid\, c_4} & \mathbb{R}[c_4]/(d c_4 = 0)\,.
\end{array}
$$

(iii) Finally, consider the following further attaching map (103) to the previous stage, in degree 7:

$$
\phi_7 \;:\;
\begin{array}{ccccc}
\overset{\curvearrowright^{\mathbb{Z}_2}}{\mathbb{Z}_2/1} & \longmapsto & \mathbb{R}\begin{bmatrix} h_3, \\ \omega_4, \\ f_2 \end{bmatrix} \Big/ \begin{pmatrix} d h_3 = \omega_4 - f_2 \wedge f_2 \\ d \omega_4 = 0 \\ d f_2 = 0 \end{pmatrix} & \xleftarrow{-\omega_4 \wedge \omega_4 \,\leftarrow\!\shortmid\, c_8} & \mathbb{R}\langle c_8 \rangle \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathbb{R}\begin{bmatrix} h_3, \\ f_2 \end{bmatrix} \Big/ \begin{pmatrix} d h_3 = \quad - f_2 \wedge f_2 \\ d f_2 = 0 \end{pmatrix} & \longleftarrow & 0\,.
\end{array}
\tag{109}
$$

Here the equivariant dual vector space on the right is again injective, by (86) in Example 3.14. Therefore, the corresponding elementary extension (101) is by pushout along the following morphism of dgc-algebras

$$
\widetilde{\phi}_7^\bullet \;:\;
\begin{array}{ccccc}
\overset{\curvearrowright^{\mathbb{Z}_2}}{\mathbb{Z}_2/1} & \longmapsto & \mathbb{R}\begin{bmatrix} h_3, \\ \omega_4, \\ f_2 \end{bmatrix} \Big/ \begin{pmatrix} d h_3 = \omega_4 - f_2 \wedge f_2 \\ d \omega_4 = 0 \\ d f_2 = 0 \end{pmatrix} & \xleftarrow{-\omega_4 \wedge \omega_4 \,\leftarrow\!\shortmid\, c_8} & \mathbb{R}[c_8]/(d c_8 = 0) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathbb{R}\begin{bmatrix} h_3, \\ f_2 \end{bmatrix} \Big/ \begin{pmatrix} d h_3 = \quad - f_2 \wedge f_2 \\ d f_2 = 0 \end{pmatrix} & \longleftarrow & 0\,.
\end{array}
$$

This pushout is the identity on $\mathbb{Z}_2/\mathbb{Z}_2$, and is an ordinary cell attachment of plain dgc-algebras on $\mathbb{Z}_2/1$, hence yields the following equivariant dgc-algebra, which is thereby seen to be minimal (Def. 3.40):

$$A \; := \; \begin{array}{ccc} \overset{\mathbb{Z}_2}{\curvearrowright} \mathbb{Z}_2/1 & \longmapsto & \mathbb{R}\left[\begin{array}{l}\omega_7, \\ h_3, \\ \omega_4, \\ f_2\end{array}\right] \Big/ \left(\begin{array}{l} d\,\omega_7 = -\omega_4 \wedge \omega_4 \\ d\,h_3 = \omega_4 - f_2 \wedge f_2 \\ d\,\omega_4 = 0 \\ d\,f_2 = 0 \end{array}\right) \\ \big\downarrow & & \big\downarrow \\ \mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathbb{R}\left[\begin{array}{l} h_3, \\ f_2 \end{array}\right] \Big/ \left(\begin{array}{l} d\,h_3 = \quad - f_2 \wedge f_2 \\ d\,f_2 = 0 \end{array}\right). \end{array} \tag{110}$$

In summary, the graded equivariant dual vector space of generators (Def. 3.41) of this minimal equivariant dgc-algebra is the following:

$$\underline{\mathfrak{g}}^A_\bullet \;=\; \begin{array}{|c||c|c|c|c|c|c|c|c|c|} \hline \mathbb{Z}_2/H & \underline{\mathfrak{g}}^A_2 & \underline{\mathfrak{g}}^A_3 & \underline{\mathfrak{g}}^A_4 & \underline{\mathfrak{g}}^A_5 & \underline{\mathfrak{g}}^A_6 & \underline{\mathfrak{g}}^A_7 & \underline{\mathfrak{g}}^A_8 & \underline{\mathfrak{g}}^A_9 & \cdots \\ \hline \hline \mathbb{Z}_2/1 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \cdots \\ \hline \mathbb{Z}_2/\mathbb{Z}_2 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline \end{array} \;\in\; \mathbb{Z}_2\mathrm{GradedVectorSpaces}^{\geq 0}_{\mathbb{R}}\,. \tag{111}$$

(106) (108) (109)

Lemma 3.43 (Minimal equivariant dgc-algebras are projectively cofibrant [Scu08, Thm. 4.2])**.** *All elementary extensions* (101) *are cofibrations*

$$A \xrightarrow{\;\in \mathrm{Cof}\;} A\big[\mathfrak{b}^n V_n\big]_{\phi_n} \qquad \in \Big(\underset{\curlyvee}{G}\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\Big)_{\mathrm{proj}}.$$

Hence all relative minimal equivariant dgc-algebra inclusions (104) *are cofibrations and, in particular, all minimal equivariant dgc-algebras* (105) *are cofibrant objects in the model category* $\Big(\underset{\curlyvee}{G}\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\Big)_{\mathrm{proj}}$ *(Prop. 3.38).*

Proposition 3.44 (Existence of equivariant minimal models [Scu02, Thm. 3.11, Cor. 3.9])**.**
Let $A \in \underset{\curlyvee}{G}\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}$ *(Def. 3.30) be cohomologically 1-connected, in that the equivariant cochain cohomology groups (Def. 3.31) are trivial in degrees* ≤ 1*:*

$$\underline{H}^0(A) \simeq \underline{\mathbb{R}} \qquad \textit{and} \qquad \underline{H}^1(A) \simeq 0\,. \tag{112}$$

(i) *There exists a minimal equivariant dgc-algebra (Def. 3.40) equipped with a quasi-isomorphism*

$$A_{\mathrm{min}} \xrightarrow[\;\in \mathrm{W}\;]{p^{\mathrm{min}}_A} A\,. \tag{113}$$

(ii) *This is unique up to isomorphism, in that for* $A'_{\mathrm{min}} \xrightarrow{\in \mathrm{W}} A$ *any other such, there is a commuting diagram of the form*

$$\begin{array}{ccc} A_{\mathrm{min}} & \xrightarrow{\quad\simeq\quad} & A'_{\mathrm{min}} \\ & {\scriptstyle \in \mathrm{W}}\searrow \quad A \quad \swarrow{\scriptstyle \in \mathrm{W}} & \end{array}$$

with the top morphism an isomorphism of equivariant dgc-algebras.

Remark 3.45 (Existence of equivariant relative minimal models(?))**.** By analogy with the theory of (relative) minimal models in non-equivariant dgc-algebraic rational homotopy theory (e.g., [BG76, §7][Ha83][FHT00, Thm. 14.12][FSS20d, Prop. 3.50]), it is to be expected that Prop. 3.44 holds in greater generality:

(a) The existence of equivariant minimal models should hold more generally for fixed locus-wise nilpotent G-spaces (not necessarily fixed-locus wise simply-connected).

(b) There should exist also equivariant *relative* minimal models, unique up to relative isomorphism, of any morphism between fixed locus-wise nilpotent spaces of $\mathbb{R}$-finite homotopy type.

While a proof of these more general statements should be a fairly straightforward generalization of the proofs of the existing results, it does not seem to be available in the literature. Nonetheless, for our main example of interest (Example 2.44) we explicitly find the equivariant relative minimal model (in Prop. 3.56 below).

3.2 Equivariant rational homotopy theory

We review the fundamentals of equivariant rational homotopy theory [Tri82][Tri96][Go97b][Scu02][Scu08] and prove our main technical result (Prop. 3.56 below). Throughout we make free use of plain (non-equivariant) dgc-algebraic rational homotopy theory [BG76] (review in [FHT00][He07][GM13][FSS17][FSS20d, §3.2]).

Notice that the minimal Sullivan model dgc-algebras in plain rational homotopy theory (see [Ha83][FH17]), whose equivariant generalization we consider in Def. 3.40 below, essentially coincide with what in the supergravity literature are known as "FDA"s, following [vN82][D'AF82][CDF91] (see, e.g., [ADR16]). For translation, see [FSS13][FSS16a][FSS16b][HSS18][BMSS19][FSS19a].

Equivariant rationalization. Equivariant rational homotopy theory is concerned with the following concept:

Definition 3.46 (Equivariant rationalization [May96, §II.3][Tri82, §2.6])**.**
Let $\mathcal{X} \in \underset{\prec}{G}\mathrm{HomotopyTypes}_{\geq 2}$ (Def. 2.33).
(i) $\mathcal{X}$ is called *rational* (here: over the real numbers, see [FSS20d, Rem. 3.51]) if all its equivariant homotopy groups (Def. 2.31) carry the structure of equivariant vector spaces (here: over the real numbers, Def. 3.5):

$$\mathcal{X} \text{ is rational over the reals} \qquad \Leftrightarrow \qquad \underline{\pi}_{\bullet+1}(\mathcal{X}) \;\in\; \underset{\prec}{G}\mathrm{VectorSpaces}_{\mathbb{R}} \longrightarrow \underset{\prec}{G}\mathrm{Groups}\,. \tag{114}$$

(ii) A *rationalization* of $\mathcal{X}$ (here: over the real numbers) is a morphism

$$\mathcal{X} \xrightarrow{\;\eta^{\mathbb{R}}_{\mathcal{X}}\;} L_{\mathbb{R}}\mathcal{X} \qquad \in \; \underset{\prec}{G}\mathrm{HomotopyTypes} \tag{115}$$

to a rational equivariant homotopy type (114) which induces isomorphisms on all equivariant rational cohomology groups (Example 3.6):

$$\underline{H}^\bullet\big(L_{\mathbb{R}}\mathcal{X};\mathbb{R}\big) \xrightarrow[\;\simeq\;]{\;(\eta^{\mathbb{R}}_{\mathcal{X}})^*\;} \underline{H}^\bullet\big(\mathcal{X};\mathbb{R}\big).$$

In other words: equivariant rationalization is plain rationalization (e.g. [FSS20d, Def. 3.55]) at each stage $G/H \in$ GOrbits.

Proposition 3.47 (Uniqueness of equivariant rationalization [May96, §II, Thm. 3.2])**.** *Equivariant rationalization (Def. 3.46) of equivariantly simply-connected equivariant homotopy types exists essentially uniquely.*

Equivariant PL de Rham theory.

Definition 3.48 (Equivariant PL de Rham complex)**.** Write

$$\begin{array}{ccc} \underset{\prec}{G}\mathrm{SimplicialSets} & \xrightarrow{\;\Omega^\bullet_{\mathrm{PLdR}}\;} & \big(\underset{\prec}{G}\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\mathrm{op}} \\ \mathcal{X} & \longmapsto & \Big(G/H \;\mapsto\; \Omega^\bullet_{\mathrm{PLdR}}\big(\mathcal{X}(G/H)\big)\Big) \end{array}$$

for the functor from equivariant simplicial sets (Def. 2.19) to the opposite of equivariant dgc-algebras (Def. 3.30). This applies the plain PL de Rham functor [Su77][BG76, p. 1.-7][FSS20d, Def. 3.56] (assigning dgc-algebras of piecewise polynomial differential forms) to diagrams of simplicial sets parametrized over the orbit category.

Proposition 3.49 (Equivariant PL de Rham theorem [Tri82, Thm. 4.9])**.** *For any* $\mathcal{X} \in \underset{\prec}{G}\mathrm{SimplicialSets}$ *(Def. 2.19) and* $\mathcal{A}_{\mathbb{R}} \in \underset{\prec}{G}\mathrm{VectorSpaces}_{\mathbb{R}}$ *(Def. 3.5), we have a natural isomorphism*

$$H^\bullet\big(\mathcal{X};\mathcal{A}_{\mathbb{R}}\big) \;\simeq\; H^\bullet\big(\Omega^\bullet_{\mathrm{PLdR}}(\mathcal{X};\mathcal{A}_{\mathbb{R}})\big)$$

between the Bredon cohomology of $\mathcal{X}$ *(Example 2.39) with coefficients in* $\mathcal{A}_{\mathbb{R}}$*, and the cochain cohomology of the equivariant PL de Rham complex of* $\mathcal{X}$ *(Def. 3.48) with coefficients in* $\mathcal{A}_{\mathbb{R}}$*.*

Proposition 3.50 (Quillen adjunction between equivariant simplicial sets and equivariant dgc-algebras [Scu08, Prop. 5.1])**.** *The equivariant PL de Rham complex construction (Def. 3.48) is the left adjoint in a Quillen adjunction*

$$\big(\mathcal{G}\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\big)^{\mathrm{op}}_{\mathrm{proj}} \underset{\mathrm{exp}}{\overset{\Omega^\bullet_{\mathrm{PLdR}}}{\underset{\bot_{\mathrm{Qu}}}{\rightleftarrows}}} G\mathrm{SimplicialSets}_{\mathrm{proj}}$$

between the projective model structure on equivariant simplicial sets (Prop. 2.21) and the opposite of the projective model structure on connective equivariant dgc-algebras (Prop. 3.38).

The fundamental theorem of dgc-algebraic equivariant rational homotopy theory.

Proposition 3.51 (Fundamental theorem of dgc-algebraic equivariant rational homotopy theory [Scu08, Thm. 5.6])**.** *On equivariant 1-connected $\mathbb{R}$-finite homotopy types (Def. 2.33):*
(i) *The derived PL de Rham adjunction (Prop. 3.50) restricts to an equivalence of homotopy categories*

$$\big(\mathcal{G}\mathrm{HomotopyTypes}^{\mathrm{fin}_{\mathbb{R}}}_{\geq 2}\big)^{\mathbb{R}} \underset{\mathbb{R}\mathrm{exp}}{\overset{\mathbb{L}\Omega^\bullet_{\mathrm{PLdR}}}{\underset{\simeq}{\rightleftarrows}}} \mathrm{Ho}\Big(\big(\mathcal{G}\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\big)^{\mathrm{op}}_{\mathrm{proj}}\Big)^{\geq 2}_{\mathrm{fin}}$$

between those simply-connected $\mathbb{R}$-finite equivariant homotopy types (Def. 2.33) which are rational (Def. 3.46) over the real numbers and formal duals of cohomologically connected 1-connected (112) *equivariant dgc-algebras.*
(ii) *The derived adjunction unit is equivariant rationalization (Def. 3.46):*

$$\mathcal{X} \in \mathcal{G}\mathrm{HomotopyTypes}^{\mathrm{fin}_{\mathbb{R}}}_{\geq 2} \quad \Rightarrow \quad \begin{array}{ccc} \mathcal{X} & \xrightarrow{\mathbb{D}\eta^{\mathrm{PLdR}}_{\mathcal{X}}} & \mathbb{R}\mathrm{exp}\circ\mathbb{L}\Omega^\bullet_{\mathrm{PLdR}}(\mathcal{X})\,. \\ \| & & \downarrow \simeq \\ \mathcal{X} & \xrightarrow{\eta^{\mathbb{R}}_{\mathcal{X}}} & L_{\mathbb{R}}\mathcal{X} \end{array} \tag{116}$$

Remark 3.52. That the equivariant derived PLdR-unit (116) models equivariant rationalization is not made explicit in [Scu08], but it follows immediately from the fact that:
(a) by definition, the equivariant PLdR adjunction is stage-wise over $G/H \in G\mathrm{Orbits}$ the plain PLdR adjunction;
(b) the derived unit of the plain PLdR-adjunction models plain rationalization by the non-equivariant fundamental theorem (e.g. [FSS20d, Prop. 3.60]); and
(c) that equivariant rationalization (Def. 3.46) is stage-wise plain rationalization.

Equivariant rational Whitehead L_∞-algebras

Definition 3.53 (Equivariant Whitehead L_∞-algebra)**.** For $\int \curlyvee (X /\!\!/ G) \in \mathcal{G}\mathrm{HomotopyTypes}^{\mathrm{fin}_{\mathbb{R}}}_{\geq 2}$ (Def. 2.33), we say that its *equivariant Whitehead L_∞-algebra*

$$\mathfrak{l}\curlyvee(X /\!\!/ G) \in \mathcal{G}L_\infty\mathrm{Algebras}^{\geq 0}_{\mathbb{R},\mathrm{fin}}$$

is the equivariant L_∞-algebra (Def. 3.36) whose equivariant Chevalley-Eilenberg algebra (98) is the minimal model (well-defined by Prop. 3.44) of the equivariant PL de Rham complex (Def. 3.48) of $\int \curlyvee (X /\!\!/ G)$:

$$\mathrm{CE}\big(\mathfrak{l}\curlyvee(X /\!\!/ G)\big) \;:=\; \Omega^\bullet_{\mathrm{PLdR}}(\mathcal{X})_{\mathrm{min}} \xrightarrow[\in \mathrm{W}]{p^{\mathrm{min}}} \Omega^\bullet_{\mathrm{PLdR}}(\mathcal{X}) \;\in\; \mathcal{G}\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\,. \tag{117}$$

Proposition 3.54 (Equivariant rational homotopy groups in the equivariant Whitehead L_∞-algeba [Tri82, Thm. 6.2 (2)])**.** *For $\int \curlyvee (X /\!\!/ G) \in \mathcal{G}\mathrm{HomotopyTypes}^{\mathrm{fin}_{\mathbb{R}}}_{\geq 2}$ (Def. 2.33), the equivariant rational homotopy groups of ΩX (Example 3.23) are equivalent to the underlying equivariant graded vector space (Def. 3.41) of the equivariant Whitehead L_∞-algebra (Def. 3.53) of $\curlyvee (X /\!\!/ G)$:*

$$\underbrace{\big(\mathfrak{l}\curlyvee(X /\!\!/ G)\big)_\bullet}_{\text{equivariant Whitehead } L_\infty\text{-algebra}} \;\simeq\; \underbrace{\underline{\pi}_\bullet(\Omega X)\otimes_{\mathbb{Z}}\mathbb{R}}_{\text{equivariant rational homotopy groups of equivariant loop space}}\,. \tag{118}$$

Examples of equivariant Whitehead L_∞-algebras.

Proposition 3.55 ($\mathbb{Z}_2$-Equivariant minimal model of twistor space)**.** *The equivariant minimal model (Def. 3.40) of the $\mathbb{Z}_2^A$-equivariant twistor space (Example 2.28) is the following $\mathbb{Z}_2$-equivariant dgc-algebra (Def. 3.30):*

$$
\mathrm{CE}\big(\mathfrak{l}\,\Upsilon\big(\mathbb{C}P^3 /\!\!/ \mathbb{Z}_2\big)\big) \;:\;
\begin{array}{ccc}
\mathbb{Z}_2/\mathbb{Z}_2 & \mapsto & \mathbb{R}\left[\begin{array}{c} h_3, \\ f_2 \end{array}\right] \Big/ \left(\begin{array}{ll} d\,h_3 = & - f_2 \wedge f_2 \\ d\,f_2 = 0 & \end{array} \right) \\
\big\uparrow & & \big\uparrow \\
\mathbb{Z}_2/1 & \mapsto & \mathbb{R}\left[\begin{array}{c} h_3, \\ f_2 \\ \omega_7, \\ \omega_4 \end{array}\right] \Big/ \left(\begin{array}{l} d\,h_3 = \omega_4 - f_2 \wedge f_2 \\ d\,f_2 = 0 \\ d\,\omega_7 = -\omega_4 \wedge \omega_4 \\ d\,\omega_4 = 0 \end{array} \right) \\
\circlearrowleft_{\mathbb{Z}_2} & &
\end{array}
\;\in\; \mathbb{Z}_2\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \qquad (119)
$$

Proof. **(i)** Checking that (119) is indeed a minimal equivariant dgc-algebra is the content of Example 3.42, where this minimal algebra is obtained in (110).

(ii) It remains to see that (119) has indeed the algebraic homotopy type of the rationalized equivariant twistor space, under the fundamental theorem (Prop. 3.51). By (54), this amounts to showing that the right vertical morphism of ordinary dgc-algebras in (119) is a dgc-algebraic model (under the non-equivariant fundamental theorem of rational homotopy theory, [BG76, §8] reviewed as [FSS20d, Prop. 3.59]) of the inclusion of the fiber of the twistor fibration (5). But, by [FSS20d, Lem. 3.71]), the dgc-algebra model for this fiber is the cofiber of the minimal relative model of the twistor fibration. The latter is given in [FSS20c, Lem. 2.13], and its cofiber manifestly coincides with (119).

(iii) As a consistency check, notice that the equivariant rational homotopy groups of twistor space (91) do match the generators (111) of this minimal model; as it must be, by Prop. 3.54. □

Proposition 3.56 ($\mathbb{Z}_2$-Equivariant relative minimal model of Sp(1)-parametrized twistor space)**.** *The equivariant relative minimal model (Def. 3.40) of the $\mathbb{Z}_2^A$-equivariant* Sp(1)*-parametrized twistor space (Example 2.44) is the following $\mathbb{Z}_2$-equivariant dgc-algebra (Def. 3.30) under* $\mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(1)\big) = \mathbb{R}\big[\tfrac{1}{4}p_1\big]\big/\big(d\,\tfrac{1}{4}p_1 = 0\big)$*:*

$$
\mathrm{CE}\Big(\Big(\mathfrak{l}_{B\mathrm{Sp}(1)}\big(\underbrace{\Upsilon}_{\text{twistor space}}(\mathbb{C}P^3 \underbrace{/\!\!/ \mathbb{Z}_2^A}_{\text{orbifolded wrt } \mathbb{Z}_2^A})\big) \underbrace{/\!\!/ \mathrm{Sp}(1)}_{\text{parametrized wrt Sp(1)}}\Big)\Big) \;:\;
\begin{array}{ccc}
\circlearrowright^{\mathbb{Z}_2} & & \\
\mathbb{Z}_2/1 & \longmapsto & \mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(1)\big)\left[\begin{array}{c} h_3, \\ f_2 \\ \omega_7, \\ \widetilde{\omega}_4 \end{array}\right] \Big/ \left(\begin{array}{l} d\,h_3 = \widetilde{\omega}_4 - \tfrac{1}{2}p_1 - f_2 \wedge f_2 \\ d\,f_2 = 0 \\ d\,\omega_7 = -\widetilde{\omega}_4 \wedge \big(\widetilde{\omega}_4 - \tfrac{1}{2}p_1\big) \\ d\,\widetilde{\omega}_4 = 0 \end{array} \right) \\
\big\downarrow & & \big\downarrow \\
\mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(1)\big)\left[\begin{array}{c} h_3, \\ f_2 \end{array}\right] \Big/ \left(\begin{array}{ll} d\,h_3 = & -\tfrac{1}{2}p_1 - f_2 \wedge f_2 \\ d\,f_2 = 0 & \end{array} \right),
\end{array}
\qquad (120)
$$

where

(a) *all closed generators are normalized such as to be rational images of integral and integrally in-divisible classes;*

(b) $\omega := \widetilde{\omega} - \tfrac{1}{4}p_1$ *is fiberwise the pullback along* $\mathbb{C}P^3 \xrightarrow{t_{\mathbb{H}}} S^4$ (5) *of the volume element on* S^4*;*

(c) f_2 *is fiberwise the volume element on* $S^2 \xrightarrow{\mathrm{fib}(t_{\mathbb{H}})} \mathbb{C}P^3$.

Proof. **(i)** To see that (120) is relative minimal, observe that it is obtained from the equivariant base dgc-algebra

$$
\begin{array}{ccc}
\circlearrowright^{\mathbb{Z}_2} & & \\
\mathbb{Z}_2/1 & \longmapsto & \mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(1)\big) = \mathbb{R}\big[\tfrac{1}{4}p_1\big]\big/\big(d\,\tfrac{1}{4}p_1 = 0\big) \\
\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\
\mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(1)\big)
\end{array}
$$

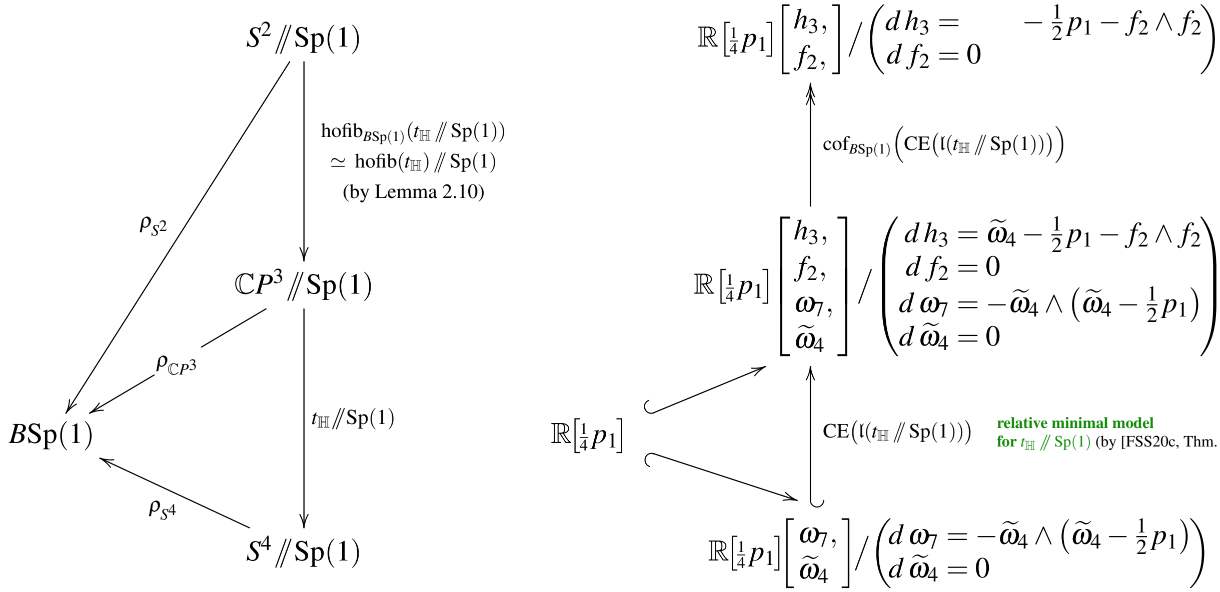
by the same three cell attachments as in the construction of the absolute minimal model of Example, 3.42 for the plain equivariant twistor space (Prop. 3.55), subject only to these replacements:

$$
\begin{array}{rcl}
f_2 \wedge f_2 & \longmapsto & f_2 \wedge f_2 + \tfrac{1}{2}p_1 \\
\omega_4 \wedge \omega_4 & \longmapsto & \widetilde{\omega}_4 \wedge \big(\widetilde{\omega}_4 - \tfrac{1}{2}p_1\big)
\end{array}
$$

in the attaching maps ϕ_3 (108) and ϕ_7 (109), respectively.

(ii) By the fundamental theorem (Prop. 3.51), it remains to see that (120) is weakly equivalent to the relative equivariant PL de Rham complex of equivariant parametrized twistor space:

(ii.1) First observe that the relative minimal model $\mathrm{CE}\big(\mathfrak{l}(t_{\mathbb{H}} /\!\!/ \mathrm{Sp}(1))\big)$ for the *non*-equivariant Sp(1)-parametrized twistor fibration $t_{\mathbb{H}}$, relative to the minimal model of $S^4 /\!\!/ \mathrm{Sp}(1)$ relative to $B\mathrm{Sp}(1)$, is as follows, with generators normalized as stated in the claim above:



(121)

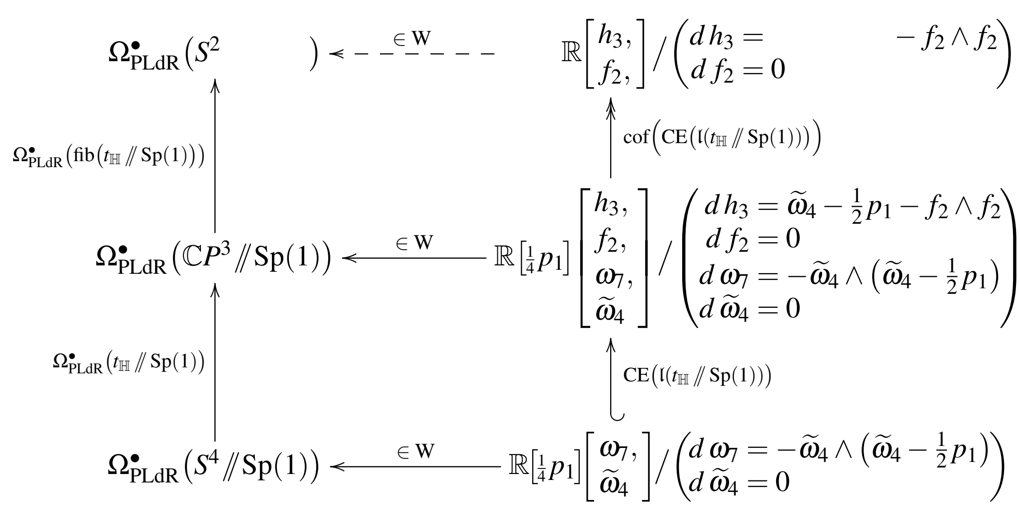
This is the statement of [FSS20c, Thm. 2.14], using the following notational simplifications in the present case:

(a) the Euler 8-class χ_8 appearing in [FSS20c, (39)] vanishes here under restriction along $B\mathrm{Sp}(1) \to B\mathrm{Sp}(2)$;

(b) we have applied to [FSS20c, (49)] the dgc-algebra isomorphism given by

$$h_3 \leftrightarrow h_3\,, \qquad f_2 \leftrightarrow f_2\,, \qquad \omega_7 \leftrightarrow \omega_7\,, \qquad \omega_4 \leftrightarrow \widetilde{\omega}_4 - \tfrac{1}{4}p_1\,. \tag{122}$$

(ii.2) This being a non-equivariant relative minimal model, it comes with horizontal weak equivalences of non-equivariant dgc-algebras as shown in the bottom square of the following commuting diagram (by, e.g., [FHT00, Thm. 14.12]), which induces (by the *fiber lemma* [BK72, §II] in the form [FHT00, Prop. 15.5][FHT15, Thm. 5.1]) a weak equivalence on plain cofibers (which is forms on S^2, by Lemma 2.10), as shown in the following top square:



(123)

(Here we are using that with $t_{\mathbb{H}}$ also $t_{\mathbb{H}} /\!\!/ \mathrm{Sp}(1) := \frac{t_{\mathbb{H}} \times W\mathrm{Sp}(1)}{\mathrm{Sp}(1)}$ is a fibration, by the right Quillen functor (23) in Prop. 2.7, and that all spaces involved are simply-connected, so that all the technical assumptions in [FHT15, (5.1)] are indeed met.)

(ii.3) Then observe that

$$H^\bullet\big(S^2 /\!\!/ \mathrm{Sp}(1);\mathbb{R}\big) \;\simeq\; \mathbb{R}\big[\omega_2,\, \tfrac{1}{4}p_1\big] \big/ \big((\omega_2)^2\big) \;\simeq\; H^\bullet\big(B\mathrm{Sp}(1);\mathbb{R}\big) \otimes_{\mathbb{R}} H^\bullet\big(S^2;\mathbb{R}\big). \tag{124}$$

This follows readily from the Gysin exact sequence (e.g. [Sw75, §15.30])

$$\cdots \to H^\bullet\big(B\mathrm{Sp}(1);\mathbb{R}\big) \xrightarrow{\rho_{S^2}^*} H^\bullet\big(S^2/\!\!/\mathrm{Sp}(1);\mathbb{R}\big) \xrightarrow{\int_{S^2}} H^{\bullet-2}\big(B\mathrm{Sp}(1);\mathbb{R}\big) \xrightarrow[=0]{c\cup(-)} H^{\bullet+1}\big(B\mathrm{Sp}(1);\mathbb{R}\big) \to \cdots \tag{125}$$

for the S^2-fiber sequence $S^2 \xrightarrow{\mathrm{hofib}(\rho_{S^2})} S^2 /\!\!/ \mathrm{Sp}(1) \xrightarrow{\rho_{S^2}} B\mathrm{Sp}(1)$ that corresponds to the $\mathrm{Sp}(1)$-action on S^2, by Prop. 2.7; and using that $H^\bullet\big(B\mathrm{Sp}(1);\mathbb{R}\big) \simeq \mathbb{R}\big[\tfrac{1}{4}p_1\big]$ (e.g. [FSS20d, Lemma 4.24]) is concentrated in degrees divisible by 4 (so that, in particular, the Euler class $c \in H^3\big(B\mathrm{Sp}(1);\mathbb{R}\big) \simeq 0$ in (125) vanishes).
But using (124) in (123) implies that also the induced map on *relative* fibers (27) over $B\mathrm{Sp}(1)$ is a weak equivalence:

$$\begin{array}{ccccc}
\mathbb{Z}_2/\mathbb{Z}_2 & & \Omega^\bullet_{\mathrm{PLdR}}\big(S^2 /\!\!/ \mathrm{Sp}(1)\big) & \overset{\in \mathrm{W}}{\longleftarrow\!\!-\!\!-\!\!-} & \mathbb{R}\big[\tfrac{1}{4}p_1\big]\left[\begin{array}{l} h_3, \\ f_2, \end{array}\right] \Big/ \left(\begin{array}{l} d\,h_3 = \quad\; -\tfrac{1}{2}p_1 - f_2\wedge f_2 \\ d\,f_2 = 0 \end{array}\right) \\
\big\uparrow & & \Big\uparrow {\scriptstyle \Omega^\bullet_{\mathrm{PLdR}}\left(\mathrm{fib}_{B\mathrm{Sp}(1)}(t_{\mathbb{H}} /\!\!/ \mathrm{Sp}(1))\right) \,\simeq\, \Omega^\bullet_{\mathrm{PLdR}}\left(\mathrm{fib}(t_{\mathbb{H}}) /\!\!/ \mathrm{Sp}(1)\right)} & & \Big\uparrow {\scriptstyle \mathrm{cof}_{B\mathrm{Sp}(1)}\left(\mathrm{CE}\left(\mathfrak{l}(t_{\mathbb{H}} /\!\!/ \mathrm{Sp}(1))\right)\right)} \\
\mathbb{Z}_2/1 & & \Omega^\bullet_{\mathrm{PLdR}}\big(\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(1)\big) & \overset{\in \mathrm{W}}{\longleftarrow} & \mathbb{R}\big[\tfrac{1}{4}p_1\big]\left[\begin{array}{l} h_3, \\ f_2, \\ \omega_7, \\ \widetilde{\omega}_4 \end{array}\right] \Big/ \left(\begin{array}{l} d\,h_3 = \widetilde{\omega}_4 - \tfrac{1}{2}p_1 - f_2\wedge f_2 \\ d\,f_2 = 0 \\ d\,\omega_7 = -\widetilde{\omega}_4 \wedge \big(\widetilde{\omega}_4 - \tfrac{1}{2}p_1\big) \\ d\,\widetilde{\omega}_4 = 0 \end{array}\right) \\
\circlearrowleft_{\mathbb{Z}_2} & & & &
\end{array} \tag{126}$$

(ii.4) By Lemma 2.10 applied to (70), we see that the left morphism in (126) is equivalently the inclusion of the fixed-locus in the $\mathbb{Z}_2^A$-equivariant $\mathrm{Sp}(1)$-parametrized twistor space (Example 2.44). Thus, by the stage-wise definition of the equivariant PL de Rham complex (Def. 3.48), it follows that the left morphism in (126) is the PL de Rham complex of $\mathbb{Z}_2^A$-equivariant $\mathrm{Sp}(1)$-parametrized twistor space (as indicated by alignment with the $\mathbb{Z}_2^A$-orbit category on the far left of (123)). Finally this means, by the fundamental theorem (Prop. 3.51), that the commuting square in (123) exhibits the claimed equivariant dgc-algebra (9) as indeed modeling the equivariant rational homotopy type of the $\mathbb{Z}_2^A$-equivariant $\mathrm{Sp}(1)$-parametrized twistor space. (The images on the left of the generators on the right of (123) are indeed all invariant under the $\mathbb{Z}_2^A \subset \mathrm{Sp}(2)$-action, by [BMSS19, Lemma 5.5]). □

3.3 Equivariant non-abelian de Rham theorem

We introduce properly equivariant non-abelian de Rham cohomology with coefficients in equivariant L_∞-algebras, in direct generalization of the non-equivariant discussion in [FSS20d, §3.3]. Our key example here is the non-abelian cohomology of equivariant twistorial differential forms (Example 3.74 below). The main result is the proper equivariant non-abelian de Rham theorem (Prop. 3.63) and its twisted version (Prop. 3.67). The specialization to traditional Borel-equivariant abelian de Rham cohomology is the content of Prop. 3.72 below.

Flat equivariant L_∞-algebra valued differential forms.
In equivariant generalization of [FSS20d, Def. 3.77], we set:

Definition 3.57 (Flat equivariant L_∞-algebra valued differential forms)**.** Let $\underline{\mathfrak{g}} \in \underset{\curlyvee}{G}L_\infty\mathrm{Algebras}^{\geq 0}_{\mathbb{R},\mathrm{fin}}$ (Def. 3.36) and $G \circlearrowright X \in G\mathrm{Actions}\big(\mathrm{SmoothManifolds}\big)$ (Def. 2.35). Then the set of *flat equivariant* $\underline{\mathfrak{g}}$*-valued differential forms* on X is the hom-set (10)

$$\Omega_{\mathrm{dR}}\big(\curlyvee(X/\!\!/G);\underline{\mathfrak{g}}\big)_{\mathrm{flat}} \;:=\; \underset{\curlyvee}{G}\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\Big(\mathrm{CE}\big(\underline{\mathfrak{g}}\big)\,,\, \Omega^\bullet_{\mathrm{dR}}\big(\curlyvee(X/\!\!/G)\big)\Big)$$

of equivariant dgc-algebras (Def. 3.30) from the equivariant Chevalley-Eilenberg algebra (98) of $\underline{\mathfrak{g}}$ to the equivariant smooth de Rham complex (Def. 3.33) of X.

In equivariant generalization of [FSS20d, Def. 3.92], we set:

Definition 3.58 (Flat twisted equivariant L_∞-algebra valued differential forms on G-orbifold)**.** Consider an *equivariant L_∞-algebraic local coefficient bundle* in the form of a fibration of equivariant L_∞-algebras (Def. 3.36) whose equivariant Chevalley-Eilenberg algebras (98), are relative minimal (Def. 3.40)

$$\begin{array}{ccc} \underline{\mathfrak{g}} & \xrightarrow[\text{equivariant } L_\infty\text{-algebraic local coefficient bundle}]{\mathrm{fib}(\underline{\mathfrak{p}})} & \widehat{\underline{\mathfrak{b}}} \\ & & \big\downarrow{\scriptstyle \underline{\mathfrak{p}}} \\ & & \underline{\mathfrak{b}} \end{array} \quad \in \; \prec GL_\infty\mathrm{Algebras}^{\geq 0}_{\mathbb{R},\mathrm{fin}} \, . \tag{127}$$

Then, for $G \circlearrowright X \in G\mathrm{Actions}\big(\mathrm{SmoothManifolds}\big)$ (Def. 2.35) equipped with an *equivariant non-abelian de Rham twist*

$$\tau_{\mathrm{dR}} \;\in\; \Omega_{\mathrm{dR}}\big(\prec\!\big(X /\!\!/ G\big);\, \underline{\mathfrak{b}}\big)_{\mathrm{flat}} \tag{128}$$

given by a flat equivariant $\underline{\mathfrak{b}}$-valued differential form (Def. 3.57) on X, the set of *flat τ_{dR}-twisted equivariant $\underline{\mathfrak{g}}$-valued* differential forms on X is the hom-set (10) in the co-slice category of $\prec G\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}$ (Def. 3.30) under $\mathrm{CE}(\underline{\mathfrak{g}})$ from $\mathrm{CE}(\underline{\mathfrak{p}})$ to τ_{dR}:

$$\begin{aligned} \Omega^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\big(\prec\!\big(X /\!\!/ G\big)\,,\, \underline{\mathfrak{g}}\big)_{\mathrm{flat}} &:= \big(\prec G\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\mathrm{CE}(\underline{\mathfrak{b}})/}\big(\mathrm{CE}\big(\underline{\mathfrak{p}}\big)\,,\, \tau_{\mathrm{dR}}\big) \\ &= \left\{ \begin{array}{ccc} \Omega^\bullet_{\mathrm{dR}}\big(\prec\!\big(X /\!\!/ G\big)\big) & \xleftarrow{\;\;\text{flat twisted equivariant } \underline{\mathfrak{g}}\text{-valued differential form}\;\;} & \mathrm{CE}\big(\widehat{\underline{\mathfrak{b}}}\big) \\ & {}_{\text{twist } \tau_{\mathrm{dR}}}\nwarrow \qquad \nearrow^{\mathrm{CE}(\underline{\mathfrak{p}}) \text{ local coefficients}} & \\ & \mathrm{CE}\big(\underline{\mathfrak{b}}\big) & \end{array} \right\} . \end{aligned} \tag{129}$$

Equivariant non-abelian de Rham cohomology.

Notation 3.59 (Cylinder orbifold)**.** For $G \circlearrowright X \in G\mathrm{Actions}\big(\mathrm{SmoothManifolds}\big)$ (Def. 2.35), let the product manifold $X \times \mathbb{R}$ be equipped with the G-action given by

$$\begin{array}{rcl} G \times (X \times \mathbb{R}) & \longrightarrow & X \times \mathbb{R} \\ (g, \quad (x,\ t)) & \longmapsto & (g \cdot x, t) \, . \end{array}$$

We say that the resulting G-orbifold (Def. 2.36) $\prec\!\big((X \times \mathbb{R}) /\!\!/ G\big) \in G\mathrm{Orbifolds}$ is the *cylinder orbifold* of $\prec\!\big(X /\!\!/ G\big)$, and we write

$$\prec\!\big(X /\!\!/ G\big) \simeq \prec\!\big((X \times \{0\}) /\!\!/ G\big) \xhookrightarrow{\;i_0\;} \prec\!\big((X \times \mathbb{R}) /\!\!/ G\big) \xhookleftarrow{\;i_1\;} \prec\!\big((X \times \{1\}) /\!\!/ G\big) \simeq \prec\!\big(X /\!\!/ G\big) \tag{130}$$

for the canonical inclusion maps and

$$\prec\!\big((X \times \mathbb{R}) /\!\!/ G\big) \xrightarrow{\;p_X\;} \prec\!\big(X /\!\!/ G\big) \tag{131}$$

for the canonical projection map.

In equivariant generalization of [FSS20d, Def. 3.83], we set:

Definition 3.60 (Coboundaries between flat equivariant L_∞-algebra valued differential forms)**.**
Let $\underline{\mathfrak{g}} \in \prec GL_\infty\mathrm{Algebras}^{\geq 0}_{\mathbb{R},\mathrm{fin}}$ (Def. 98) and $G \circlearrowright X \in G\mathrm{Actions}\big(\mathrm{SmoothManifolds}\big)$ (Def. 2.35).
(i) Then, given flat differential forms $A_0, A_1 \in \Omega_{\mathrm{dR}}\big(\prec\!(X /\!\!/ G);\, \underline{\mathfrak{g}}\big)_{\mathrm{flat}}$ (Def. 3.57), a *coboundary* between them

$$A_0 \overset{\widetilde{A}}{\Longrightarrow} A_1$$

is a flat equivariant $\underline{\mathfrak{g}}$-valued differential form (Def. 3.57) on the cylinder orbifold (Notation 3.59)

$$\widetilde{A} \;\in\; \Omega_{\mathrm{dR}}\Big(\overbrace{\prec\!\big((X \times \mathbb{R}) /\!\!/ G\big)}^{\text{cylinder orbifold}};\, \underline{\mathfrak{g}}\Big)_{\mathrm{flat}} \tag{132}$$

such that this restricts to the given pair of forms

$$i_0^*(\widetilde{A}) = A_0 \quad \text{and} \quad i_1^*(\widetilde{A}) = A_1 \tag{133}$$

along the canonical inclusions (130).

(ii) We denote the relation given by existence of such a coboundary by $A_1 \sim A_2$.

Lemma 3.61 (Equivalence of equivariant smooth and PL de Rham complex of smooth orbifold)**.** *Let $G \circlearrowright X \in G\mathrm{Actions}(\mathrm{SmoothManifolds})$ (Def. 2.35). Then the corresponding equivariant PL de Rham complex (Def. 3.48) is isomorphic to the equivariant smooth de Rham complex (Example 3.33) in the homotopy of equivariant dgc-algebras (Prop. 3.38):*

$$\Omega^\bullet_{\mathrm{dR}}\big(\prec(X /\!\!/ G)\big) \;\simeq\; \Omega^\bullet_{\mathrm{PLdR}}\big(\prec(X /\!\!/ G)\big) \;\in\; \mathrm{Ho}\Big(\big(G\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}\Big). \tag{134}$$

Proof. Observe that the analogous non-equivariant statement holds by [FSS20d, Lem. 3.90], using [GM13, Cor. 9.9], and that its proof proceeds by analyzing natural constructions applied to a choice of smooth triangulation of the given smooth manifold X.

Now, for a smooth manifold equipped with a smooth G-action $G \circlearrowright X$, we may choose a G-equivariant smooth triangulation, by the equivariant triangulation theorem [Il78][Il83]. Given this, the remainder of the non-equivariant proof applies stage-wise over the orbit category. Since the weak equivalences of equivariant dgc-algebras are the stage-wise weak equivalences of non-equivariant dgc-algebras (Prop. 3.38), the claim follows. □

In equivariant generalization of [FSS20d, Def. 3.84], we set:

Definition 3.62 (Equivariant non-abelian de Rham cohomology)**.** Let $G \circlearrowright X \in G\mathrm{Actions}(\mathrm{SmoothManifolds})$ (Def. 2.35) and $\underline{\mathfrak{g}} \in G L_\infty\mathrm{Algebras}^{\geq 0}_{\mathbb{R},\mathrm{fin}}$ (Def. 3.36). The *equivariant non-abelian de Rham cohomology* of $G \circlearrowright X$ with coefficients in $\underline{\mathfrak{g}}$ is the quotient of the set of flat equivariant differential forms (Def. 3.57) by the coboundary relation (Def. 3.60):

$$H_{\mathrm{dR}}\big(\prec(X /\!\!/ G); \mathfrak{g}\big) \;:=\; \Big(\Omega_{\mathrm{dR}}\big(\prec(X /\!\!/ G); \mathfrak{g}\big)_{\mathrm{flat}}\Big)_{/\sim}.$$

In equivariant generalization of [FSS20d, Thm. 3.87], we have:

Proposition 3.63 (Equivariant non-abelian de Rham theorem)**.** *Let $\mathcal{A} \in G\mathrm{HomotopyTypes}^{\mathrm{fin}_{\mathbb{R}}}_{\geq 2}$ (Def. 2.33) and $G \circlearrowright X \in G\mathrm{Actions}(\mathrm{SmoothManifolds})$ (Def. 2.35), such that its equivariant shape (Def. 2.23) is also equivariantly simply-connected and of $\mathbb{R}$-finite type: $\int \prec(X /\!\!/ G) \in G\mathrm{HomotopyTypes}^{\mathrm{fin}_{\mathbb{R}}}_{\geq 2}$. Then, at least if G has order 4 or is cyclic of prime order (Remark 3.17), there is an equivalence between:*

(a) *real equivariant non-abelian cohomology (Def. 2.37) with coefficients in the equivariant rationalization $L_{\mathbb{R}}\mathcal{A}$ (Def. 3.46) and*

(b) *equivariant non-abelian de Rham cohomology (Def. 3.62) of the G-orbifold $\prec(X /\!\!/ G)$ (Def. 2.36) with coefficients in the equivariant Whitehead L_∞-algebra $\mathfrak{l}\mathcal{A}$ (Def. 3.53):*

$$\underbrace{H\Big(\int \prec(X /\!\!/ G); L_{\mathbb{R}}\mathcal{A}\Big)}_{\text{equivariant non-abelian real cohomology}} \;\simeq\; \underbrace{H_{\mathrm{dR}}\Big(\prec(X /\!\!/ G); \mathfrak{l}\mathcal{A}\Big)}_{\text{equivariant non-abelian de Rham cohomology}}. \tag{135}$$

Proof. Consider the following sequence of bijections:

$$\begin{aligned}
H\big(\prec(X /\!\!/ G); L_{\mathbb{R}}\mathcal{A}\big) &:= G\mathrm{HomotopyTypes}\big(\prec(X /\!\!/ G), L_{\mathbb{R}}\mathcal{A}\big) \\
&\simeq \mathrm{Ho}\Big(\big(G\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}\Big)\Big(\Omega^\bullet_{\mathrm{PLdR}}(\mathcal{A}), \Omega^\bullet_{\mathrm{PLdR}}\big(\prec(X /\!\!/ G)\big)\Big) \\
&\simeq \mathrm{Ho}\Big(\big(G\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}\Big)\Big(\mathrm{CE}(\mathfrak{l}\mathcal{A}), \Omega^\bullet_{\mathrm{dR}}\big(\prec(X /\!\!/ G)\big)\Big) \\
&\simeq \big(G\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}\Big(\mathrm{CE}(\mathfrak{l}\mathcal{A}), \Omega^\bullet_{\mathrm{dR}}\big(\prec(X /\!\!/ G)\big)\Big)_{/\sim_{\text{right homotopy}}} \\
&\simeq \Big(\Omega_{\mathrm{dR}}\big(\prec(X /\!\!/ G); \mathfrak{l}\mathcal{A}\big)_{\mathrm{flat}}\Big)_{/\sim} \\
&=: H_{\mathrm{dR}}\big(\prec(X /\!\!/ G); \mathfrak{l}\mathcal{A}\big).
\end{aligned}$$

The first step is Def. 2.37, while the second step is the fundamental theorem (Prop. 3.51). In the third step we are:
(a) post-composing in the homotopy category with the isomorphism $\Omega^\bullet_{\mathrm{PLdR}}(-) \simeq \Omega^\bullet_{\mathrm{dR}}(-)$ (134);
(b) pre-composing with the isomorphism $\mathrm{CE}\big(\mathfrak{l}\mathcal{A}\big) \simeq \Omega^\bullet_{\mathrm{PLdR}}\big(\mathcal{A}\big)$ exhibiting the minimal model (117).

Now the domain object $\mathrm{CE}(\mathfrak{l}\mathcal{A})$ is cofibrant (by Lemma 3.43) and the codomain object $\Omega^\bullet_{\mathrm{dR}}\big(\prec(X /\!\!/ G)\big)$ is fibrant (by Prop. 3.39). Consequently, the hom-set in the homotopy category is equivalently given ([Qu67, §I.1 Cor. 7], see [FSS20d, Prop. A.16]) by right-homotopy classes of equivariant dgc-algebra homomorphisms between these objects, shown in the fourth step.

To exhibit these right homotopies, we may choose as path-space object ([Qu67, Def. I.4], see [FSS20d, A.11]) the equivariant de Rham complex on the cylinder orbifold (Notation 3.59): this qualifies as a path space object by stage-wise application of [FSS20d, Lem. 3.88] and using again the argument of Lemmas 3.19, 3.20, 3.21 for equivariant fibrancy. But with this choice of path space object, the right homotopy relation manifestly coincides (by stage-wise application of [FSS20d, Lem. 3.89]) with the coboundary relation on equivariant non-abelian forms (Def. 3.60). which is the fifth step above. With this, the last step is Def. 3.62.

In conclusion, the composite of this chain of bijections gives the claimed bijection (135). □

Twisted equivariant non-abelian de Rham cohomology.
In equivariant generalization of [FSS20d, Def. 3.97], we set:

Definition 3.64 (Coboundaries between flat twisted equivariant L_∞-algebra valued differential forms)**.** Given an equivariant L_∞-algebraic local coefficient bundle (127)

$$
\begin{array}{ccc}
\underline{\mathfrak{g}} & \xrightarrow{\ \mathrm{fib}(\underline{\mathfrak{p}})\ } & \widehat{\underline{\mathfrak{b}}} \\
 & \substack{\text{equivariant } L_\infty\text{-algebraic} \\ \text{local coefficient bundle}} & \big\downarrow{\scriptstyle \underline{\mathfrak{p}}} \\
 & & \underline{\mathfrak{b}}
\end{array}
\quad \in \ \underset{\prec}{G} L_\infty \mathrm{Algebras}^{\geq 0}_{\mathbb{R},\mathrm{fin}}\,, \tag{136}
$$

and given $G \circlearrowright X \in G\mathrm{Actions}\big(\mathrm{SmoothManifolds}\big)$ (Def. 2.35) equipped with an equivariant non-abelian de Rham twist (128)

$$
\tau_{\mathrm{dR}} \ \in \ \Omega_{\mathrm{dR}}\Big(\prec\big(X /\!\!/ G\big); \underline{\mathfrak{b}}\Big)\,,
$$

(i) we say that a *coboundary* between a pair

$$
A_0, A_1 \ \in \ \Omega^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\Big(\prec\big(X /\!\!/ G\big); \underline{\mathfrak{g}}\Big)
$$

of flat equivariant τ_{dR}-twisted $\underline{\mathfrak{g}}$-valued differential forms (Def. 3.57) is such a form on the cylinder orbifold (Notation 3.59)

$$
\widetilde{A} \ \in \ \Omega^{p_X^*(\tau_{\mathrm{dR}})}_{\mathrm{dR}}\Big(\underbrace{\prec\big((X \times \mathbb{R}) /\!\!/ G\big)}^{\text{cylinder orbifold}}; \underline{\mathfrak{g}}\Big)
$$

twisted by the pullback of the given twist to the cylinder orbifold (along the canonical projection (131)), such that this restricts to the given pair of forms

$$
i_0^*\big(\widetilde{A}\big) \ = \ A_0 \quad \text{and} \quad i_1^*\big(\widetilde{A}\big) \ = \ A_1 \tag{137}
$$

along the canonical inclusions (130).
(ii) We denote the relation that there exists such a coboundary by $A_0 \sim A_1$.

In equivariant generalization of [FSS20d, Def. 3.98], we set:

Definition 3.65 (Twisted equivariant non-abelian de Rham cohomology)**.** Let $G \circlearrowright X \in G\mathrm{Actions}\big(\mathrm{SmoothManifolds}\big)$ (Def. 2.35) and let $\underline{\mathfrak{g}} \to \widehat{\underline{\mathfrak{b}}} \to \underline{\mathfrak{b}}$ be an equivariant L_∞-algebraic local coefficient bundle (127), and let

$$
\big[\tau_{\mathrm{dR}}\big] \ \in \ H_{\mathrm{dR}}\big(\prec(X /\!\!/ G); \underline{\mathfrak{b}}\big)_{\mathrm{flat}} \tag{138}
$$

be the equivariant non-abelian de Rham cohomology class (Def. 3.62) of an equivariant twist (128). Then we say that the *equivariant τ_{dR}-twisted de Rham cohomology* of the G-orbifold $\prec\big(X /\!\!/ G\big)$ (Def. 2.36) with coeffcients in

$\underline{\mathfrak{g}}$ is the quotient of the set of *equivariant* τ_{dR}*-twisted* $\underline{\mathfrak{g}}$*-valued differential forms* (Def. 3.58) by the coboundary relation from Def. 3.64:

$$H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\Big(\prec(X /\!\!/ G);\, \underline{\mathfrak{g}}\Big) \;:=\; \Omega^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\Big(\prec(X /\!\!/ G);\, \underline{\mathfrak{g}}\Big)_{/\sim}.$$

Notation 3.66 (Equivariant local coefficient bundle with relative minimal model)**.** Given an equivariant local coefficient bundle (72)

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\;\mathrm{hofib}(\rho_{\mathcal{A}})\;} & \mathcal{A} /\!\!/ \mathcal{G} \\ & {\scriptstyle \text{equivariant local coefficient bundle}} & \big\downarrow {\scriptstyle \rho_{\mathcal{A}}} \\ & & B\mathcal{G} \end{array} \quad \in \; \mathcal{G}\mathrm{HomotopyTypes}^{\mathrm{fin}_{\mathbb{R}}}_{\geq 2} \tag{139}$$

all of whose objects are equivariantly 1-connected and of $\mathbb{R}$-finite type (Def. 2.33), assume (Remark 3.45) that ρ_A admits an equivariant relative minimal model (Def. 3.40). This is to be denoted as follows:

$$\begin{array}{ccccc}
 & & \mathrm{CE}\big(\mathfrak{l}\mathcal{A}\big) & \xleftarrow{\;\mathrm{cofib}(\mathrm{CE}(\mathfrak{l}\rho_{\mathcal{A}}))\;} & \mathrm{CE}\big(\mathfrak{l}_{B\mathcal{G}}(\mathcal{A}/\!\!/\mathcal{G})\big) \\
\Omega^\bullet_{\mathrm{PLdR}}\big(\mathcal{A}\big) & \xleftarrow{\;\Omega^\bullet_{\mathrm{PLdR}}(\mathrm{hofib}(\rho_{\mathcal{A}}))\;} & \Omega^\bullet_{\mathrm{PLdR}}\big(\mathcal{A}/\!\!/\mathcal{G}\big) & \xleftarrow{\;p^{\mathrm{min}_{B\mathcal{G}}}_{\mathcal{A}/\!\!/\mathcal{G}} \in \mathrm{W}\;} & \big\uparrow {\scriptstyle \mathrm{CE}(\mathfrak{l}\rho_{\mathcal{A}})} \;{\scriptstyle \text{equivariant relative minimal model}} \\
 & {\scriptstyle \text{equivariant dgc-algebra model of local coefficient bundle}} & \big\uparrow {\scriptstyle \Omega^\bullet_{\mathrm{PLdR}}(\rho_{\mathcal{A}})} & & \mathrm{CE}\big(\mathfrak{l}B\mathcal{G}\big) \;{\scriptstyle \text{equivariant minimal model}} \\
 & & \Omega^\bullet_{\mathrm{dR}}\big(B\mathcal{G}\big) & \xleftarrow{\;p^{\mathrm{min}}_{B\mathcal{G}} \in \mathrm{W}\;} &
\end{array} \tag{140}$$

Notice that the corresponding fibration of equivariant L_∞-algebras (Def. 3.36) serves as an equivariant L_∞-algebraic local coefficient bundle (127).

In equivariant generalization of [FSS20d, Thm. 3.104], we have:

Proposition 3.67 (Twisted equivariant non-abelian de Rham theorem)**.** *Consider the following*

- *Let* $\rho_{\mathcal{A}}$ *be an equivariant local coefficient bundle of equivariantly 1-connected G-spaces of finite* $\mathbb{R}$*-homotopy type, which admits an equivariant relative minimal model; all as in Notation 3.66.*
- *Moreover, let* $G \circlearrowright X \in G\mathrm{Actions}\big(\mathrm{SmoothManifolds}\big)$ *(Def. 2.35) be such that also its equivariant shape (Def. 2.23) is equivariantly 1-connected and of* $\mathbb{R}$*-finite type,* $\int \prec \big(X /\!\!/ G\big) \in \mathcal{G}\mathrm{HomotopyTypes}^{\mathrm{fin}_{\mathbb{R}}}_{\geq 2}$ *and let this be equipped with an equivariant twist* τ *(73) with coefficients in the equivariant rationalization (Def. 3.46) of* $B\mathcal{G}$*.*
- *Write* τ_{dR} *for a representative of the image under the equivariant non-abelian de Rham theorem (Prop. 3.63) of the class of this twist in equivariant* $\mathfrak{l}B\mathcal{A}$*-valued de Rham cohomology (Def. 3.62) that the equivariant local coefficient bundle* (139) *admits an equivariant relative minimal model (Def. 3.40)*

$$\begin{array}{ccc} H\Big(\int \prec \big(X /\!\!/ G\big);\, L_{\mathbb{R}} B\mathcal{G}\Big) & \underset{\text{equivariant non-abelian de Rham theorem}}{\simeq} & H_{\mathrm{dR}}\Big(\prec\big(X /\!\!/ G\big);\, \mathfrak{l}B\mathcal{G}\Big). \\ {\scriptstyle \text{rational twist}} & & {\scriptstyle \text{de Rham twist}} \\ [\tau] & \longmapsto & [\tau_{\mathrm{dR}}] \end{array} \tag{141}$$

Then there is an equivalence between:
(a) *the* τ*-twisted equivariant real non-abelian cohomology (Def. 2.45) with local coefficients in* $\rho_{\mathcal{A}}$*, and*
(b) *the* τ_{dR}*-twisted equivariant de Rham cohomology (Def. 3.65) with local coefficients in* $\mathfrak{l}_{B\mathcal{G}}\rho_{\mathcal{A}}$ (140)*:*

$$\underset{\text{twisted equivariant non-abelian real cohomology}}{H^{\tau}\Big(\int \prec \big(X /\!\!/ G\big);\, L_{\mathbb{R}}\mathcal{A}\Big)} \;\simeq\; \underset{\text{twisted equivariant non-abelian de Rham cohomology}}{H^{\tau_{\mathrm{dR}}}\Big(\prec\big(X /\!\!/ G\big);\, \mathfrak{l}\mathcal{A}\Big)}. \tag{142}$$

Proof. The proof proceeds in direct joint generalization of the proofs of Prop. 3.63 (equivariant case) and [FSS20d, Thm. 3.104] (twisted case).
First, by the fundamental theorem (Prop. 3.51), the twisted real cohomology is given by morphisms in the homotopy category of the co-slice model category of this form:

$$\Omega^\bullet_{\mathrm{PLdR}}\Big(\int \curlyvee (X /\!\!/ G)\Big) \xleftarrow{\quad} \Omega^\bullet_{\mathrm{PLdR}}(\mathcal{A} /\!\!/ \mathcal{G}) \quad \text{with} \quad \Omega^\bullet_{\mathrm{PLdR}}(\tau) : \Omega^\bullet_{\mathrm{PLdR}}(B\mathcal{G}) \to \Omega^\bullet_{\mathrm{PLdR}}\Big(\int \curlyvee (X /\!\!/ G)\Big),\ \Omega^\bullet_{\mathrm{PLdR}}(\rho_{\mathcal{A}}) : \Omega^\bullet_{\mathrm{PLdR}}(B\mathcal{G}) \to \Omega^\bullet_{\mathrm{PLdR}}(\mathcal{A} /\!\!/ \mathcal{G}) \quad \in \mathrm{Ho}\Big(\big(\curlyvee G\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}\Big). \tag{143}$$

Second, by

(a) post-composition with the isomorphism $\Omega^\bullet_{\mathrm{PLdR}}(-) \simeq \Omega^\bullet_{\mathrm{dR}}(-)$ (134),

(b) pre-composition with the equivalence from the equivariant relative minimal model (140),

this becomes equivalent to morphisms of this form:

$$\Omega^\bullet_{\mathrm{dR}}\Big(\int \curlyvee (X /\!\!/ G)\Big) \xleftarrow{\quad} \Omega^\bullet_{\mathrm{PLdR}}(\mathcal{A} /\!\!/ \mathcal{G}) \quad \text{with} \quad \tau_{\mathrm{dR}} : \mathrm{CE}(\mathfrak{l}B\mathcal{G}) \to \Omega^\bullet_{\mathrm{dR}}\Big(\int \curlyvee (X /\!\!/ G)\Big),\ \mathrm{CE}(\mathfrak{l}\rho_{\mathcal{A}}) : \mathrm{CE}(\mathfrak{l}B\mathcal{G}) \to \Omega^\bullet_{\mathrm{PLdR}}(\mathcal{A} /\!\!/ \mathcal{G}) \quad \in \mathrm{Ho}\Big(\big(\curlyvee G\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}\Big). \tag{144}$$

But, in this form,

(a) the codomain τ_{dR} is a fibrant object in the coslice model category, since $\Omega^\bullet_{\mathrm{dR}}(-)$ is fibrant in the un-sliced model structure (Prop. 3.39);

(b) the relative minimal model domain $\mathrm{CE}(\mathfrak{l}\rho_{\mathcal{A}})$ is cofibrant, by Lemma 3.43.

It follows ([Qu67, §I.1 Cor. 7], see [FSS20d, Prop. A.16]) that a morphism of the form (144) in the homotopy category is equivalently the right homotopy class of an actual homomorphism of equivariant dgc-algebras in the coslice, hence is equivalently the right homotopy class of a flat equivariant twisted $\mathfrak{l}\mathcal{A}$-valued differential form, by Def. 3.58.

Finally, in joint generalization of the proof of Prop. 3.63 (equivariant case) and [FSS20d, Lem. 3.105] (twisted case), we see that a path space object ([Qu67, Def. I.4], see [FSS20d, A.11]) exhibiting these right homotopies in the coslice is given by pullback to the equivariant smooth de Rham complex of the cylinder orbifold (132). But with that choice, right homotopies are manifestly the same as coboundaries of flat equivariant twisted $\mathfrak{l}\mathcal{A}$-valued differential forms (Def. 3.64), and hence the claim follows. □

Twisted non-abelian Borel-Weil-Cartan equivariant de Rham cohomology. Finally, we combine traditional Borel(-Weil-Cartan) T-equivariant de Rham cohomology ([AB84][MQ86, §5][Ka93][GS99], review in [Me06][KT15][Pe17]), with proper G-equivariance and generalize it to non-abelian L_∞-algebra coefficients.

By Prop. 2.7 and Remark 2.42, any Borel T-equivariantized G-orbifold carries a canonical twist in equivariant non-abelian cohomology $H^1(-,T) \simeq H(-,BT)$. The following is the de Rham image of that twist:

Definition 3.68 (Canonical de Rham twist on Borel T-equivariant G-orbifolds)**.**
Let $(T\times G) \circlearrowright X \in (T\times G)\mathrm{Actions}(\mathrm{SmoothManifolds})$ (Def 2.35) for $T \in \mathrm{CompactLieGroups}$ finite-dimensional and simply-connected, with Lie algebra $\mathfrak{t}$ (99), regarded as a smooth G-equivariant L_∞-algebra (Def. 3.36). We say that the *canonical de Rham twist* on the corresponding T-parametrized G-orbifold is the canonical inclusion of equivariant dgc-algebras (Def. 3.30) from the minimal model for the classifying space of T (regarded as a smooth G-equivariant homotopy type, Example 2.24) into the proper G-equivariant & Borel T-equivariant smooth de Rham complex (Example 3.37):

$$\begin{array}{ccc} \Omega^\bullet_{\mathrm{dR}}\Big(\big(\curlyvee(X /\!\!/ G)\big) /\!\!/ T\Big) & & \overbrace{\Big(\Omega^\bullet_{\mathrm{dR}}(X^H) \otimes \mathbb{R}\Big[\{r_2^a\}_{a=1}^{\dim(T)}\Big],\, d_{\mathrm{dR}} + r_2^a \wedge \iota_{t_a}\Big)^T}^{\text{Cartan model for } T\text{-equivariant Borel cohomology of } H\text{-fixed locus } X^H} \\ \uparrow{\scriptstyle \tau^{\mathrm{can}}_{\mathrm{dR}}} & : \quad G/H \quad \longmapsto & \uparrow \\ \mathrm{CE}(\mathfrak{l}BT) & & \Big(\mathbb{R}\Big[\{r_2^a\}_{a=1}^{\dim(T)}\Big]\Big)^T \end{array}$$

where on the bottom we used the abstract Chern-Weil isomorphism (75) in the form discussed in [FSS20d, §4.2].

Example 3.69 (Equivariant Cartan map)**.** In the situation of Def. 3.68, consider the case when the T-action is free, hence that $X := P$ is the total space of a G-equivariant T-principal bundle $P \to B := P/T$ (e.g. [KT15, p .2]). Then, for any choice of G-invariant N-principal connection $\nabla \in N\mathrm{Connections}(P)^G$, we have the following weak equivalence (in the sense of Prop. 3.38) of G-equivariant dgc-algebras (Def. 3.30) in the co-slice under the minimal model dgc-algebra of the classifying space (75):

$$
\begin{array}{ccc}
\Omega^\bullet_{\mathrm{dR}}\Big(\big(\prec(X/\!\!/G)\big)/\!\!/T\Big) & \xrightarrow{\ \in \mathrm{W}\ } & \Omega^\bullet_{\mathrm{dR}}\big(\prec(B/\!\!/G)\big) \\
& {\scriptstyle \tau^{\mathrm{can}}_{\mathrm{dR}}} \nwarrow \quad \nearrow {\scriptstyle \mathrm{cw}_T} & \\
& \mathrm{CE}\big(\mathfrak{l}BT\big) &
\end{array}
\; : \; G/H \longmapsto \;
\begin{array}{ccc}
\Big(\Omega^\bullet_{\mathrm{dR}}\big(X^H\big)\Big[\{r_2^a\}_{a=1}^{\dim(T)}\Big]\Big)^T & \xrightarrow[\ r_2^a \,\mapsto\, F_\nabla^a\ ]{\ \omega \,\mapsto\, \omega_{\mathrm{hor}}\ } & \Omega^\bullet_{\mathrm{dR}}\big(B^H\big) \\
& \nwarrow \qquad \nearrow {\scriptstyle c \,\mapsto\, c(F_\nabla)}\ {\scriptstyle \text{Chern-Weil hom.}} & \\
& \Big(\mathbb{R}\Big[\{r_2^a\}_{a=1}^{\dim(T)}\Big]\Big)^T &
\end{array}
$$

This is from the proper G-equivariant Borel T-equivariant smooth de Rham complex of X (Example 3.37) to the proper G-equivariant smooth de Rham complex over X/T (Example 3.33), which is stage-wise over G/H the *Cartan map* quasi-isomorphism [GS99, §5] (review in [Me06, (20), (30)]) from the Cartan model of X^H (100) to the ordinary smooth de Rham complex of $B^H = (X/N)^H$. This sends the Cartan model generators r_2^a to the curvature form component F_∇^a of the given connection, and hence restricts on universal real characteristic classes, represented by invariant polynomials c, to the Chern-Weil homomorphism assigning characteristic forms: $c \mapsto c(F_\nabla)$.

Example 3.70 (Tangential de Rham twists on G-orbifolds with T-structure)**.** In further specialization of Example 3.69, let $X \circlearrowleft B \in G\mathrm{Actions}\big(\mathrm{SmoothManifolds}\big)$ (Def. 2.35) be equipped with G-equivariant $T \subset \mathrm{GL}(\dim(X))$-structure (see [SS20b, p. 9] for pointers), namely with a G-equivariant reduction of its $\mathrm{GL}(\dim(X))$-frame bundle to a T-principal T-frame bundle $T\mathrm{Fr}(X)$:

$$
\begin{array}{ccc}
\overset{T\times G}{\circlearrowright} & & \overset{T\times G}{\circlearrowright} \\
{\scriptstyle T\text{-frame bundle}}\ \ T\mathrm{Fr}(X) & \xhookrightarrow[\ T\text{-structure}\ ]{\ G\text{-equivariant}\ } & \mathrm{Fr}(X)\ \ {\scriptstyle \text{frame bundle}} \\
& \searrow \quad \swarrow & \\
& X & \\
& \underset{G}{\circlearrowright} &
\end{array}
$$

Then Example 3.69 induces on the G-orbifold $\prec\big(X/\!\!/G\big)$ (Def. 2.36) an equivariant non-abelian de Rham twist (138) encoding all the real characteristic forms of the given G-equivariant T-structure on X (the *tangential twist*):

$$
\begin{array}{ccc}
\Omega^\bullet_{\mathrm{dR}}\Big(\big(\prec(T\mathrm{Fr}(X)/\!\!/G)\big)/\!\!/T\Big) & \xrightarrow[\ \in \mathrm{W}\ ]{\ \text{Cartan map equivalence}\ } & \Omega^\bullet_{\mathrm{dR}}\big(\prec(X/\!\!/G)\big). \\
{\scriptstyle \text{canonical de Rham twist on orbifold's } T\text{-frame bundle}}\ \ \nwarrow {\scriptstyle \tau^{\mathrm{can}}_{\mathrm{dR}}} & & {\scriptstyle \mathrm{cw}_T} \nearrow \ \ {\scriptstyle \text{tangential de Rham twist on } G\text{-orbifold}} \\
& \mathrm{CE}\big(\mathfrak{l}BT\big) &
\end{array}
$$

In further generalization of Def. 3.65, we set:

Definition 3.71 (Proper G-equivariant & Borel T-equivariant twisted non-abelian de Rham cohomology)**.** Let $(T\times G) \circlearrowleft X \in \big(T\times G\big)\mathrm{Actions}\big(\mathrm{SmoothManifolds}\big)$ (Def. 2.35) for T finite-dimensional, compact and simply-connected, and let

$$
\begin{array}{ccc}
\underline{\mathfrak{g}} & \xrightarrow{\ \mathrm{hofib}(\underline{\mathfrak{p}})\ } & \underline{\widehat{\mathfrak{b}}} \\
& & \big\downarrow {\scriptstyle \underline{\mathfrak{p}}} \\
& & \mathfrak{l}BT
\end{array}
\tag{145}
$$

be an equivariant L_∞-algebraic local coefficient bundle (127) over the Whitehead L_∞-algebra of BT (i.e., whose Chevalley-Eilenberg algebra is (75)).

(i) We say that the set of *flat, canonically twisted, proper G-equivariant & Borel T-equivariant,* $\underline{\mathfrak{g}}$*-valued differential forms* on X is the hom-set (10) in the co-slice of G-equivariant dgc-algebras (Def. 3.30) from $\mathrm{CE}\big(\underline{\mathfrak{p}}\big)$ (98) to the canonical de Rham twist (Def. 3.68) on the corresponding T-parametrized G-orbifold:

$$
\begin{aligned}
\Omega^{\tau^{\mathrm{can}}_{\mathrm{dR}}}_{\mathrm{dR}}\Big(\big(\prec(X/\!\!/G)\big)/\!\!/T;\underline{\mathfrak{g}}\Big) &:= \Big(\big(\underset{\prec}{G}\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}\Big)^{\mathrm{CE}(\mathfrak{l}BT)/}\Big(\mathrm{CE}\big(\underline{\mathfrak{p}}\big)\,,\tau^{\mathrm{can}}_{\mathrm{dR}}\Big) \\
&= \left\{
\begin{array}{ccc}
\Omega^\bullet_{\mathrm{dR}}\Big(\big(\prec(X/\!\!/G)\big)/\!\!/T\Big) & \xleftarrow{\ \text{flat canonically-twisted proper } G\text{-equivariant \& Borel } T\text{-equivariant } \underline{\mathfrak{g}}\text{-valued differential form}\ } & \mathrm{CE}\big(\underline{\widehat{\mathfrak{b}}}\big) \\
{\scriptstyle \tau^{\mathrm{can}}_{\mathrm{dR}}} \nwarrow & & \nearrow {\scriptstyle \mathrm{CE}(\underline{\mathfrak{p}})} \\
& \mathrm{CE}\big(\mathfrak{l}BT\big) &
\end{array}
\right\}.
\end{aligned}
\tag{146}
$$

(ii) A *coboundary* between two such elements is defined, as in Def. 3.60, by a concordance form on the cylinder orbifold:

$$\widetilde{A} \;\in\; \Omega_{\mathrm{dR}}^{p_X^*(\tau_{\mathrm{dR}}^{\mathrm{can}})}\Big(\big(\curlyvee\big((X\times\mathbb{R})\sslash G\big)\big)\sslash T;\,\underline{\mathfrak{g}}\Big). \tag{147}$$

The corresponding twisted equivariant non-abelian de Rham cohomology is defined, as in Def. 3.65, to be the set of coboundary-classes of the elements in the set (146):

$$H_{\mathrm{dR}}^{\tau_{\mathrm{dR}}^{\mathrm{can}}}\Big(\big(\curlyvee(X\sslash G)\big)\sslash T;\,\underline{\mathfrak{g}}\Big) \;:=\; \Omega_{\mathrm{dR}}^{\tau_{\mathrm{dR}}^{\mathrm{can}}}\Big(\big(\curlyvee(X\sslash G)\big)\sslash T;\,\underline{\mathfrak{g}}\Big)_{/\sim}.$$

In Borel-equivariant generalization of [FSS20d, Prop. 3.86], we have:

Proposition 3.72 (Reproducing traditional Borel-Weil-Cartan equivariant de Rham cohomology)**.** *For the case of trivial proper equivariance, $G=1$, consider $T \circlearrowright X \in G\mathrm{Actions}\big(\mathrm{SmoothManifolds}\big)$ (Def. 2.35) and let the equivariant L_∞-algebraic coefficient bundle* (145) *be the trivial bundle with fiber the line Lie n-algebra $\mathfrak{b}^{n+1}\mathbb{R}$ ([FSS20d, Ex. 3.27]). Then the canonically twisted proper G-equivariant & Borel T-equivariant non-abelian de Rham cohomology of X (Def. 3.71) reduces to the traditional Borel-Weil-Cartan equivariant de Rham cohomology (the cochain cohomology of the Cartan model complex* (100)*) in degree n:*

$$\underset{\text{de Rham cohomology}}{\overset{\text{Borel-Weil-Cartan equivariant}}{H^n_{\mathrm{dR},T}(X)}} \;\simeq\; H_{\mathrm{dR}}\big(X\sslash T;\,\mathfrak{b}^n\mathbb{R}\big).$$

Proof. From unravelling the definitions it is clear that, under the given assumptions, the defining set of cochains (146) reduces to the set of closed degree n elements in the Cartan model complex (100) on $X = X^1$. Hence, given any pair of such, it is sufficient to see that the coboundaries according to (147) exist precisely if a coboundary with respect to the Cartan model differential $d_{\mathrm{dR}} + r_2^a \wedge \iota_{t_a}$ exists.

In the case when the second summand $r_2^a \wedge \iota_{t_a}$ vanishes, this is shown by the proof in [FSS20d, Prop. 3.86], using the fiberwise Stokes theorem for fiber integration over $[0,1]\subset\mathbb{R}$. Inspection shows that this proof generalizes verbatim in the presence of the second summand in the Cartan differential, using that this second summand evidently anti-commutes with the fiber integration operation:

$$r^a \wedge \iota_{t_a}\int_{[0,1]}\widetilde{C} \;=\; -\int_{[0,1]} r^a\wedge\iota_{t_a}\widetilde{C}.$$

□

Remark 3.73 (Localization in gauge theory)**.** Prop. 3.72 means that the equivariant de Rham cohomology considered here subsumes the traditional Borel-equivariant de Rham cohomology that is used, for instance, in localization of gauge theories (see [Ne04][Pe12][PZ+17]), and generalizes it to finite proper equivariance groups and to non-abelian coefficients.

In equivariant generalization of [FSS20d, Ex. 3.96], we have:

Example 3.74 (Flat equivariant twistorial differential forms)**.** Consider the equivariant relative Whitehead L_∞-algebra (120) of $\mathbb{Z}_2^A$-equivariant & $\mathrm{Sp}(1)$-parametrized twistor space (70) (from Thm. 3.56) as an equivariant L_∞-algebraic local coefficient bundle (127)

$$\begin{array}{ccc} \mathfrak{l}\curlyvee\big(\mathbb{C}P^3\sslash\mathbb{Z}_2^A\big) & \longrightarrow & \mathfrak{l}_{B\mathrm{Sp}(1)}\big(\curlyvee\big(\mathbb{C}P^3\sslash\mathbb{Z}_2^A\big)\sslash\mathrm{Sp}(1)\big) \\ & & \Big\downarrow{\scriptstyle \rho_{\curlyvee(\mathbb{C}P^3\sslash\mathbb{Z}_2^A)}} \\ & & \mathfrak{l}B\mathrm{Sp}(1) \end{array} \tag{148}$$

Let $X \in \mathbb{Z}_2\mathrm{Actions}\big(\mathrm{SmoothManifolds}\big)$ (Def. 2.35) be a spin 8-manifold with fixed locus (43) denoted

$$\curlyvee\big(X\sslash\mathbb{Z}_2\big) \quad : \quad \begin{array}{ccc} \overset{\mathbb{Z}_2}{\circlearrowright}\;\mathbb{Z}_2/1 & \longmapsto & \overset{\mathbb{Z}_2}{\circlearrowright}\;X^{11} \\ \big\downarrow & & \big\uparrow \\ \mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & X^{\mathbb{Z}_2^A} \end{array} \tag{149}$$

and equipped with $\mathbb{Z}_2$-invariant $\mathrm{Sp}(1)$-structure τ, compatible $\mathbb{Z}_2^A$-invariant $\mathrm{Sp}(1)$-connection $\nabla \in \mathrm{Sp}(1)\mathrm{Connections}(X)$, and corresponding tangential de Rham twist (Example 3.70)

$$
\begin{array}{ccc}
\Omega^\bullet_{\mathrm{dR}}\big(\rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ \mathbb{Z}_2)\big) & \xleftarrow{\;\tau_{\mathrm{dR}}\;} & \mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(1)\big). \\
\tfrac{1}{4}p_1(\nabla) & \longleftarrow\!\shortmid & \tfrac{1}{4}p_1
\end{array}
$$

Then the set of flat τ_{dR}-twisted equivariant differential forms (Def. 129) with local coefficients in (148) is of the following form:

$$
\begin{aligned}
&\overset{\text{flat equivariant twistorial differential forms on } \mathbb{Z}_2\text{-orbifold } X}{\Omega^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\Big(\rotatebox[origin=c]{70}{$\prec$}\big(X /\!\!/ \mathbb{Z}_2\big);\, \mathfrak{l}\rotatebox[origin=c]{70}{$\prec$}\big(\mathbb{C}P^3 /\!\!/ \mathbb{Z}_2^A\big)\Big)_{\mathrm{flat}}} \\
&= \left\{
\begin{array}{c} H_3, \\ F_2, \\ 2G_7, \\ \widetilde{G}_4 \end{array}
\in \Omega^\bullet_{\mathrm{dR}}\big(X^{11}\big)
\;\middle|\;
\begin{array}{ll}
\overset{\text{twisted Bianchi identities in bulk } \mathbb{Z}_2^A\text{-orientifold}}{d\;H_3 = \widetilde{G}_4 - \tfrac{1}{2}p_1(\nabla) - F_2 \wedge F_2,} & \overset{\text{restriction to } \mathbb{Z}_2^A\text{-fixed locus}}{dH_3\vert_{X^{\mathbb{Z}_2^A}} = -\tfrac{1}{2}p_1\big(\nabla\vert_{X^{\mathbb{Z}_2^A}}\big) - F_2 \wedge F_2\vert_{X^{\mathbb{Z}_2^A}}} \\
d\;F_2 = 0, & \\
d\,2G_7 = -\widetilde{G}_4 \wedge \big(\widetilde{G}_4 - \tfrac{1}{2}p_1(\nabla)\big) & G_7\vert_{X^{\mathbb{Z}_2^A}} = 0, \\
d\,\widetilde{G}_4 = 0, & \widetilde{G}_4\vert_{X^{\mathbb{Z}_2^A}} = 0
\end{array}
\right\}.
\end{aligned}
\tag{150}
$$

This follows as an immediate consequence of Prop. 3.56, according to which an element $\mathscr{F}$ of this set of forms is a morphism of equivariant dgc-algebras of the following form (see around (156) for further discussion):

$$
\mathscr{F} :
\begin{array}{ccccc}
\overset{\curvearrowright^{\mathbb{Z}_2}}{\mathbb{Z}_2/1} & \longmapsto & \Omega^\bullet_{\mathrm{dR}}(X) & \xleftarrow[\substack{2G_7 \,\leftarrow\!\shortmid\, \omega_7 \\ \widetilde{G}_4 \,\leftarrow\!\shortmid\, \widetilde{\omega}_4}]{\substack{H_3 \,\leftarrow\!\shortmid\, h_3 \\ F_2 \,\leftarrow\!\shortmid\, f_2}} & \mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(1)\big)\left[\begin{array}{c} h_3, \\ f_2 \\ \omega_7, \\ \widetilde{\omega}_4 \end{array}\right] \Big/ \left(\begin{array}{l} d\,h_3 = \widetilde{\omega}_4 - \tfrac{1}{2}p_1 - f_2 \wedge f_2 \\ d\,f_2 = 0 \\ d\,\omega_7 = -\widetilde{\omega}_4 \wedge \big(\widetilde{\omega}_4 - \tfrac{1}{2}p_1\big) \\ d\,\widetilde{\omega}_4 = 0 \end{array}\right) \\
\big\downarrow & & \big\downarrow \;\; \substack{\alpha \\ \downarrow \\ \alpha\vert_{X^{\mathbb{Z}_2}}} & & \big\downarrow \\
\mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \Omega^\bullet_{\mathrm{dR}}\big(X^{\mathbb{Z}_2}\big) & \longleftarrow & \mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(1)\big)\left[\begin{array}{c} h_3, \\ f_2 \end{array}\right] \Big/ \left(\begin{array}{l} d\,h_3 = \quad - \tfrac{1}{2}p_1 - f_2 \wedge f_2 \\ d\,f_2 = 0 \end{array}\right).
\end{array}
\tag{151}
$$

3.4 Equivariant non-abelian character map

The Chern character in K-theory is just one special case of a plethora of character maps in a variety of flavors of generalized cohomology theories. In fact, as highlighted in [FSS20d][SS24b], from the point of view of homotopy-theoretic non-abelian cohomology theory – where all cohomology classes are represented by (relative, parametrized) homotopy classes of maps into a classifying space (fibered, parametrized ∞-stack) – character maps are naturally realized as the non-abelian cohomology operations induced by *rationalization* of the classifying space (followed by a de Rham-Dold-type equivalence that brings the resulting rational cohomology theory into canonical shape).

Seen through the lens of Elmendorf's theorem (Prop. 2.26), rationalization in proper equivariant homotopy theory (Def. 3.46) is stage-wise, on fixed loci, given by rationalization in non-equivariant homotopy theory. Consequently, the equivariant character maps are fixed loci-wise given by non-equivariant characters, hence are fixed loci-wise given by rationalization (followed by a de Rham equivalence).

For this reason we will be brief here and refer to [FSS20d] for background and further detail. We just make explicit now the concrete model of the equivariant non-abelian character map by means of the equivariant PL de Rham Quillen adjunction from Prop. 3.50. and then we discuss one example (in §4): the character map in equivariant twistorial Cohomotopy theory.

The character map in equivariant non-abelian cohomology.
In equivariant generalization of [FSS20d, Def. 4.1], we set:

Definition 3.75 (Rationalization in equivariant non-abelian cohomology)**.** Let $\mathcal{A} \in \underset{\curlyvee}{G}\mathrm{HomotopyTypes}_{\geq 2}^{\mathrm{fin}_{\mathbb{R}}}$ (Def. 2.33). Then we say that *rationalization in $\mathcal{A}$-cohomology* is the equivariant non-abelian cohomology operation (Def. 2.41) from $\mathcal{A}$-cohomology to real $L_{\mathbb{R}}\mathcal{A}$-cohomology which is induced (65) by the rationalization unit (115) on $\mathcal{A}$:

$$H\big(-;\mathcal{A}\big) \xrightarrow{\;(\eta^{\mathbb{R}}_{\mathcal{A}})_*\;} H\big(-;L_{\mathbb{R}}\mathcal{A}\big)\,.$$

In equivariant generalization of [FSS20d, Def. 4.2], we set:

Definition 3.76 (Equivariant non-abelian character map)**.** Let $G \curvearrowright X \in G\mathrm{Actions}\big(\mathrm{SmoothManifolds}\big)$ (Def. 2.35) and $\underline{\mathfrak{g}}$ (Def. 3.36). Then the *equivariant non-abelian character map* on equivariant non-abelian $\mathcal{A}$-cohomology (Def. 2.37) over the orbifold $\curlyvee\big(X \,/\!\!/\, G\big)$ (Def. 2.36) is the composite of the rationalization cohomology operation (Def. 3.75) with the equivariant non-abelian de Rham theorem (Prop. 3.63) over the orbifold $\curlyvee(X /\!\!/ G)$ (Def. 2.36)

$$\underset{\text{character map}}{\overset{\text{equivariant non-abelian}}{\mathrm{ch}_{\mathcal{A}}(X)}} \;:\; \underset{\substack{\text{equivariant non-abelian}\\ \mathcal{A}\text{-cohomology}}}{H\Big(\curlyvee\big(X/\!\!/G\big);\mathcal{A}\Big)} \xrightarrow[\;(\eta^{\mathbb{R}}_{\mathcal{A}})_*\;]{\text{rationalization}} H\Big(\curlyvee\big(X/\!\!/G\big);L_{\mathbb{R}}\mathcal{A}\Big) \xrightarrow[\;\simeq\;]{\substack{\text{equivariant non-abelian}\\ \text{de Rham theorem}}} \underset{\substack{\text{equivariant non-abelian de Rham cohomology}\\ \text{with coefficient in equivariant Whitehead } L_\infty\text{-algebra}}}{H_{\mathrm{dR}}\Big(\curlyvee\big(X/\!\!/G\big);\mathfrak{l}\mathcal{A}\Big)}. \qquad (152)$$

The character map in twisted equivariant non-abelian cohomology.
In equivariant generalization of [FSS20d, Def. 5.2], we set:

Definition 3.77 (Rationalization in twisted equivariant non-abelian cohomology)**.** Let $\rho_{\mathcal{A}}$ be an equivariant local coefficient bundle of equivariantly 1-connected G-spaces of finite $\mathbb{R}$-homotopy type, which admits an equivariant relative minimal model; all as in Notation 3.66. Then *rationalization* in twisted equivariant non-abelian cohomology with local coefficients in $\rho_{\mathcal{A}}$ (Def. 2.45) is the equivariant non-abelian cohomology operation

$$\big(\eta^{\mathbb{R}}_{\rho_{\mathcal{A}}}\big)_* \;:\; H^{\tau}\big(X;\mathcal{A}\big) \xrightarrow{\;(\mathbb{D}\eta^{\mathrm{PLdR}}_{\rho_{\mathcal{A}}}\circ(-))\circ\mathbb{L}(\eta^{\mathbb{R}}_{B\mathcal{G}})_!\;} H^{L_{\mathbb{R}}\tau}\big(X;L_{\mathbb{R}}\mathcal{A}\big)$$

which is induced (as shown in [FSS20d, (264)]) by the pasting composite with the naturality square on $\rho_{\mathcal{A}}$ of the rationalization unit (Def. 3.46). By the fundamental theorem (Prop. 3.51), this means explicitly: the left derived base change (e.g. [FSS20d, Ex. A.18]) along the PLdR-adjunction unit (Prop. 3.50) on $B\mathcal{G}$ followed by composition with the following commuting square, regarded as a morphism in the slice over its bottom right object:

$$\mathbb{D}\eta^{\mathbb{R}}_{\rho_{\mathcal{A}}} \;:=\; \left(\begin{array}{ccccc}
\mathcal{A}/\!\!/\mathcal{G} & \xrightarrow{\;\eta^{\mathrm{PLdR}}_{\mathcal{A}/\!\!/\mathcal{G}}\;} & \exp\circ\Omega_{\mathrm{PLdR}}\big(\mathcal{A}/\!\!/\mathcal{G}\big) & \xrightarrow{\;p^{\mathrm{min}_{B\mathcal{G}}}_{\mathcal{A}/\!\!/\mathcal{G}}\;} & \exp\circ\mathrm{CE}\big(\mathfrak{l}_{B\mathcal{G}}(\mathcal{A}/\!\!/\mathcal{G})\big)\\
\Big\downarrow{\scriptstyle \rho_{\mathcal{A}}} & & \Big\downarrow{\scriptstyle \exp\circ\Omega^{\bullet}_{\mathrm{PLdR}}(\rho_{\mathcal{A}})} & & \Big\downarrow{\scriptstyle \exp\circ\mathrm{CE}(\mathfrak{l}\rho_{\mathcal{A}})}\\
B\mathcal{G} & \xrightarrow{\;\eta^{\mathrm{PLdR}}_{B\mathcal{G}}\;} & \exp\circ\Omega^{\bullet}_{\mathrm{PLdR}}\big(B\mathcal{G}\big) & \xrightarrow{\;p^{\mathrm{min}}_{B\mathcal{G}}\;} & \exp\circ\mathrm{CE}\big(\mathfrak{l}(B\mathcal{G})\big)
\end{array}\right).$$

Here the left hand side is the naturality square of the equivariant PL de Rham adjunction (Prop. 3.50), while the right hand side is the image under exp of the relative minimal model (140). (Hence the composite represents the naturality square of the derived PL de Rham adjunction unit, see e.g. [FSS20d, Ex. A.21]).

In equivariant generalization of [FSS20d, Def. 5.4], we set:

Definition 3.78 (Twisted equivariant non-abelian character map)**.** Let $G \curvearrowright X \in G\mathrm{Actions}\big(\mathrm{SmoothManifolds}\big)$ (Def. 2.35), and let $\rho_{\mathcal{A}}$ be an equivariant local coefficient bundle of equivariantly 1-connected G-spaces of finite $\mathbb{R}$-homotopy type, which admits an equivariant relative minimal model; all as in Notation 3.66. Then the *twisted equivariant non-abelian character map* is the twisted equivariant cohomology operation

$$
\underset{\text{twisted equivariant non-abelian character}}{\mathrm{ch}^{\tau}_{\mathcal{A}}} \;:\; \underset{\text{twisted equivariant non-abelian }\mathcal{A}\text{-cohomology}}{H^{\tau}\big(\curlyvee(X /\!\!/ G);\, \mathcal{A}\big)} \xrightarrow[\;(\eta^{\mathbb{R}}_{\rho_{\mathcal{A}}})_*\;]{\text{rationalization}} H^{L_{\mathbb{R}}\tau}\big(\curlyvee(X /\!\!/ G);\, L_{\mathbb{R}}\mathcal{A}\big) \xrightarrow[\;\simeq\;]{\text{equivariant twisted non-abelian de Rham theorem}} \underset{\text{twisted equivariant non-abelian de Rham cohomology}}{H^{\tau_{\mathrm{dR}}}\big(\curlyvee(X /\!\!/ G);\, \mathfrak{l}\mathcal{A}\big)} \tag{153}
$$

from twisted equivariant non-abelian cohomology (Def. 2.45) with local coefficients in $\rho_{\mathcal{A}}$ to twisted equivariant non-abelian de Rham cohomology (Def. 3.65) with coefficients in $\mathfrak{l}\rho_{\mathcal{A}}$ (as in Notation 3.66).

Finally, we have:

Remark 3.79 (Proof of Theorem 1.1)**.** We collect together our results:
(i) That the Bianchi identities in the twistorial character map are as shown on p. 6 follows by Prop. 3.56, as discussed in Example 3.74.
(ii) That the quantization conditions in the twistorial character are as shown in (4) follows by observing that the twisted equivariant character map (Def. 3.78) is fixed-locus wise equivalent to the corresponding non-equivariant twisted character map [FSS20d, Def. 5.4] (for instance by the fundamental theorem, Prop. 3.51, using that the equivariant PL de Rham adjunction is stage-wise given by the non-equivariant PL de Rham adjunction, Prop. 3.50).
(iii) In particular, at global stage $\mathbb{Z}_2^A/1 \in \mathbb{Z}_2\mathrm{Orbits}$ on the bulk $X^1 = X$, the equivariant twistorial character restricts to the non-equivariant twistorial character map for which the claimed flux quantization condition have been proven in [FSS20b, Prop. 3.13][FSS20c, Thm. 4.8][FSS20c, Cor. 3.11], see also [FSS20d, §5.3].

4 M-brane charge-quantization in equivariant twistorial Cohomotopy

We conclude by matching the content of Theorem 1.1 to the expected flux quantization and Green-Schwarz mechanism for heterotic M5-branes. First, we recall the traditional physics story about branes at orbi-singularities, and point out (following [HSS18][SS19a]) how the equivariant non-abelian cohomology theory developed above has just the right properties to be a plausible candidate for making precise the famous but informal notion that physical bulk degrees of freedom get enhanced by degrees of freedom located right at the branes/on the singular loci (see [BMSS19, §1] for pointers).

The M5-brane at an ADE-singularity. The full mathematical nature of "M-branes" is a large and largely open subject (for pointers see [Sa10b][HSS18, §2]). One securely understood aspect is the *black M-brane* solutions of 11-dimensional supergravity [Gue92][Du99, §5], these being direct analogs of black hole solutions in 4-dimensional Einstein gravity. One finds [AFCS99, §3, 5.2][dMFO10, §8.3] that:
(a) Near the horizon of such black M5-brane(s) of charge $N \in \mathbb{N}$, spacetime is described by an extremely curved throat geometry of diameter

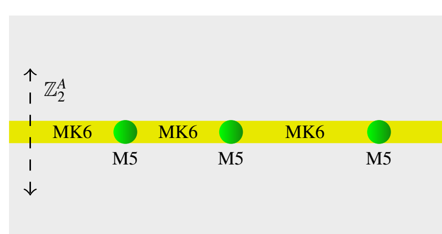
$$
\underset{\text{throat diameter of black M5}}{\ell^{\mathrm{M5}}_{\mathrm{th}}} \;=\; \underset{\text{number/charge of branes}}{N^{1/3}} \cdot \underset{\text{Planck length in 11d}}{\ell_P}\,. \tag{154}
$$

(b) At distances r large compared to this radius, spacetime looks like a flat orbifold, with an ADE-type singularity (Example 2.27) running through the brane locus (if the brane preserves any supersymmetry, hence that it is $1/2$BPS, see [HSS18, Def. 3.38]):

black M5 ($1/2$ BPS)

near horizon limit, $r \sim \ell^{\mathrm{M5}}_{\mathrm{th}}$: $\mathrm{AdS}_7 \times S^4 /\!\!/ G^{\mathrm{ADE}}$ — throat geometry intersecting orbifold ADE-singularity

far-horizon limit, $r \gg \ell_P$: $\mathbb{R}^{6,1} \times \mathbb{R}^4 /\!\!/ G^{\mathrm{ADE}}$ — flat orbifold with ADE-singularity



(155)

Since the totality of the ADE-singularity here (Example 2.27) is also [IMSY98, (47)][As00, (18)] the far-horizon geometry of the *KK-monopole* solution [So83][GP83] to 11-dimensional supergravity [To95, (1)] [Sen98] (the "MK6-brane", see [HSS18, §2.2.5]), the situation (155) may be interpreted as saying that the 1/2-BPS M5-brane "probes" the ADE-singularity, being a "domain wall" inside the MK6-brane [DHTV15, §3] (see [EGKRS00, §5.1] [BH97, §2.4] for the corresponding situation of NS5-branes inside D6-branes in type II string theory).

Much attention has been devoted to the limiting case of the near horizon limit (155) where a vast number $N \gg 1$ of M5-branes are coincident on each other, in which case perturbative quantum fields propagating inside the throat geometry are thought to capture much of the quantum physics of these objects (by AdS/CFT duality [AGMOO99] applied to M5-branes [NT99][NP02][CP18][ACR 20]). In fact, one needs N to be of order $N \gtrsim (\mathrm{nm}/\ell_P)^3 \sim 10^{75}$ (compare Avogadro's number $\sim 10^{24}$) in order for the throat size $\ell_{\mathrm{th}}^{\mathrm{M5}}$ (154) to be at least mesoscopic, hence for the classical near-horizon limit in (155) to have physical meaning in the first place.

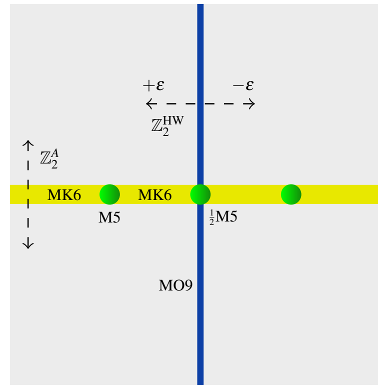
Black branes and proper equivariant cohomology. Conversely, this entails that for microscopic (single) M5-branes, the "far" horizon limit in (155) actually applies at every physically sensible distance, and whatever non-trivial quantum-gravitational physics is associated with the microscopic M5-brane must all be crammed inside the orbifold singularity. We conclude that a mathematical model of microscopic M-brane physics ought to:
(a) see physical spacetime stratified into smooth and orbi-singular loci; and
(b) model physical fields that may acquire "extra degrees of freedom" which are "hidden inside" the singular loci.

We highlight (following [HSS18][SS19a]) that exactly these demands are satisfied by flux quantization in proper equivariant non-abelian cohomology theories, in the sense of §2.3 and §2.4. Namely, the parametrization of objects in proper equivariant homotopy theory over the orbit category (§2.2 §3.1) records:
(a) for domain objects (spacetimes) the strata of orbi-singular loci (Example 2.20), and
(b) for co-domain objects (field coefficients) the degrees of freedom available on each stratum. For example, the "inner structure" of a $\mathbb{Z}_2$-equivariant (Example 2.15) L_∞-algebra valued differential form $\mathscr{F}$ (Def. 3.58, such as in Example 3.74 above, see (151)) looks as follows:

$$
\underset{\text{a flat } \mathbb{Z}_2\text{-equivariant } \underline{\mathfrak{g}}\text{-valued differential form}}{\Omega_{\mathrm{dR}}\Big(\curlyvee\big(X /\!\!/ \mathbb{Z}_2\big); \underline{\mathfrak{g}}\Big) \ni F} \quad : \quad
\begin{array}{ccccc}
\overset{\circlearrowright \mathbb{Z}_2}{\mathbb{Z}_2/1} & \longmapsto & \Omega^\bullet_{\mathrm{dR}}\big(X_{\mathrm{bulk}}\big) & \xleftarrow[F_{\mathrm{bulk}}]{\text{bulk flux densities}} & \mathrm{CE}\big(\mathfrak{g}_{\mathrm{bulk}}\big) \;\; \text{flux coefficients on bulk spacetime} \\
\big\downarrow & & \big\downarrow \;\text{restriction to orbi-singular locus} & & \big\downarrow \;\text{emergence of brane DOFs} \\
\mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \Omega^\bullet_{\mathrm{dR}}\big(X_{\mathrm{brane}}\big) & \xleftarrow[F_{\mathrm{brane}}]{\text{brane flux densities}} & \mathrm{CE}\big(\mathfrak{g}_{\mathrm{brane}}\big) \;\; \text{flux coefficients on brane locus}
\end{array}
\qquad (156)
$$

Remark 4.1 (Emerging brane DOFs and injective resolutions in minimal models)**.**
(i) Here the mathematical reflection of new degrees of freedom appearing on the brane is the appearance of new generators of the equivariant coefficient L_∞-algebra $\underline{\mathfrak{g}}$ (Def. 3.36), namely (Prop. 3.54) of (rational) homotopy groups of the coefficient space $\exp(\mathrm{CE}(\underline{\mathfrak{g}}))$, which appear only on the fixed locus, not in the bulk (such as the π_3 in Example 3.24).
(ii) Interestingly, it is precisely this type of generators that the mathematical formalism of dgc-algebraic rational homotopy theory regards as special. These are the generators that are not injective (Example 3.14) and which hence contribute to the equivariant flux DOFs via their injective resolution (by Def. 3.40), as illustrated by Example 3.42(ii). This is precisely the mathematical subtlety that distinguishes equivariant minimal models from non-equivariant minimal models.

The heterotic M5-brane at an ADE-singularity. In heterotic M-theory (Hořava-Witten theory [HW95][Wi96][HW96][DOPW99] [DOPW00][Ov02]) the brane configuration M5 || MK6 (155) encounters, in addition, the fixed locus of an orientation-reversing ("orienti-fold") $\mathbb{Z}_2^{\mathrm{HW}}$-action on spacetime (the MO9-*plane*, see [HSS18, §2.2.1] for pointers). The joint fixed locus of the resulting *orbi-orientifold* (157) [SS19a, §4.1] is identified (e.g. [DHTV15, §6.1][AF17]) with the lift to M-theory of the heterotic NS5-brane [Le10], or equivalently/dually, of the $\frac{1}{2}$NS5-brane [HZ98][HZ99, §3][GKST01, §6] [DHTV15, §6][AF17, p. 18]) of type I' string theory, hence also called the $\frac{1}{2}$M5-*brane* [HSS18, Ex. 2.2.7][FSS19d, 4][SS19a, 4.1][FSS20b, (1)]; while the full M5-branes appear in mirror pairs at positive distance from the MO9 (the "tensor branch" of their worldvolume theory, e.g. [DHTV15, Fig. 1]).



Mathematically, this means [SS19a, (67)] that we have the following exact sequence of orbi-orientifold groups [DFM11, p. 4], acting on the transversal $\mathbb{R}^{\mathbf{5}}$ (and hence on its representation sphere $S^{\mathbf{5}}$, by Example 2.12) as indicated in the bottom row here:

$$1 \longrightarrow \underset{\substack{\textbf{orbifold} \\ \mathbb{R}^{\mathbf{1}_{\mathrm{triv}}+\mathbf{4}_{\mathbb{H}}}}}{\mathbb{Z}_2^A} \xhookrightarrow{\text{index-2 subgroup}} \underset{\mathbb{R}^{\mathbf{1}_{\mathrm{sgn}}+\mathbf{4}_{\mathbb{H}}}}{\overbrace{\mathbb{Z}_2^{\mathrm{HW}}}^{\{\mathrm{e},\sigma\}} \times \mathbb{Z}_2^A} \underset{\simeq}{\xrightarrow{\substack{(\mathrm{e},q)\mapsto(\mathrm{e},+q) \\ (\sigma,q)\mapsto(R,-q)}}} \underset{\mathbb{R}^{\mathbf{5}}}{\overbrace{\mathbb{Z}_2^{\mathrm{HW+refl}}}^{\{\mathrm{e},R\}} \times \mathbb{Z}_2^A} \longrightarrow\!\!\!\!\rightarrow \underset{\substack{\textbf{orientifold} \\ \mathbb{R}^{\mathbf{5}_{\mathrm{sgn}}}}}{\mathbb{Z}_2^{\mathrm{HW+refl}}} \longrightarrow 1\,. \tag{157}$$

(with the middle two groups labelled **orbi-orientifold**)

The Hořava-Witten Green-Schwarz mechanism in 11d/10d. The mathematical nature of the MO9-plane in Hořava-Witten theory has remained somewhat mysterious. The original suggestion of [HW96, (3.9)] is that near one MO9-plane at $\varepsilon = 0$ the C-field flux is of the form

$$(\mathrm{id} - \iota_{\partial_\varepsilon})G_4 \;=\; \theta_\varepsilon \cdot \big(\tfrac{1}{4}p_1(\omega) - c_2(A)\big) \qquad \text{at and near the MO9?} \tag{158}$$

where ε denotes the coordinate function along the HW-circle and θ_ε is its Heaviside step function. This is motivated from the fact that, under double dimensional reduction to heterotic string theory via the relation (e.g. [MSa04, (4.4)])

$$\int_\varepsilon G_4 \;=\; H_3^{\mathrm{het}}\,, \tag{159}$$

the Ansatz (158) reduces to the Green-Schwarz relation known to hold in heterotic string theory [GS84] (reviews in [GSWe85, §2][Wi00, §2.2][GSW12]):

$$d\,H_3^{\mathrm{het}} \;=\; c_2(A) - \tfrac{1}{4}p_1(\omega)$$

via the following standard transformation (left implicit in [HW96, above (1.13)]):

$$dH_3^{\mathrm{het}} \;=\; d\int_\varepsilon G_4 \;=\; -\int_\varepsilon dG_4 \;=\; -\int_\varepsilon \delta_\varepsilon d\varepsilon \wedge \big(\tfrac{1}{4}p_1(\omega) - c_2(A)\big) \;=\; c_2(A) - \tfrac{1}{4}p_1(\omega)\,.$$

But for (159) to hold, we need G_4 to contain the summand $d\varepsilon \wedge H_3^{\mathrm{het}}$. Since this is not closed, in general, while G_4 is not supposed to have other non-closed components besides (158), G_4 must contain the full exact summand

$$dH_3 \;:=\; d\big((\varepsilon - 1)H_3^{\mathrm{het}}\big) \tag{160}$$

(which makes sense locally). But this, finally, modifies (158) to

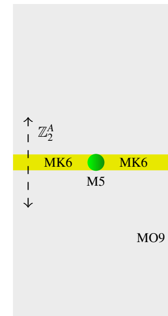
$$G_4 \;=\; \theta_\varepsilon\big(\tfrac{1}{4}p_1(\omega) - c_2(A)\big) + dH_3 \qquad \text{at and near the MO9.}$$

and hence, away from the MO9 locus, to

$$G_4 \;=\; \tfrac{1}{4}p_1(\omega) - c_2(A) + dH_3 \qquad \text{away from the MO9.} \tag{161}$$

While (161) differs by the exact term dH_3 from the original (158) proposed in [HW96, (3.9)], it actually coincides, away from the MO9 locus, with the proposal for a more fine-grained mathematical model for the C-field in [DFM03, (3.9)].

The spacetime ADE-orbifold away from the MO9. Hence we now:



(a) assume, along with [DFM03, (3.9)], that (161) is the correct nature of the C-field away from an MO9 locus, differing from the original proposal [HW96, (3.9)] by a local exact term (which is exactly the local gauge freedom that ought to be available), and

(b) focus on heterotic M5-branes away from the MO9-locus, hence on the *tensor branch* of their worldvolume field theory (e.g., [DHTV15, §6.1.1]).

Mathematically, this means that we pass to the *semi-complement orbifold* [SS19a, (80)], namely to the complement of the fixed locus of $\mathbb{Z}_2^{\mathrm{HW}}$ in (157). The resulting $\mathbb{Z}_2^{\mathrm{HW}} \times \mathbb{Z}_2^A$-equivariant shape (Def. 2.23) is again equivalent to that of the plain MK6 $\mathbb{Z}_2^A$-orbifold (155).

Therefore, this means that we may model the relevant spacetime as a $\mathbb{Z}_2^A$-orbifold (Def. 2.36), whose fixed locus is interpreted as the heterotic M5-brane locus

$$\mathbb{Z}_2\mathrm{Orbifolds} \ni \underset{\text{spacetime orbifold}}{\prec\big(X /\!\!/ \mathbb{Z}_2^A\big)} \;:\; \begin{array}{ccc} \overset{\circlearrowright \mathbb{Z}_2}{\mathbb{Z}_2/1} & \longmapsto & \overset{\circlearrowright \mathbb{Z}_2^A}{X_{\mathrm{bulk}}} \;\; \text{bulk spacetime} \\ \downarrow & & \uparrow \\ \mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & X_{\mathrm{M5}} \;\; \text{brane locus} \end{array} \tag{162}$$

The Green-Schwarz mechanism on heterotic M5-branes. Upon reduction to the 6d worldvolume of a heterotic M5-brane, the Bianchi identity (161) of the Hořava-Witten Green-Schwarz mechanism in the 11d bulk becomes

$$dH_3^{\mathrm{M5}} = c_2(A_{6d}) - \tfrac{1}{4}p_1(\omega_{6d}) \qquad \text{GS on M5 parallel to MO9.} \tag{163}$$

The derivation of (163) for the tensor branch of M5-branes parallel to an MO9-plane is due to [OST14, (1.2)] [OSTY14, (2.18)], recalled in [In14, (4.1)][CDI20, p. 18]. The same formula for M5-branes at A-type singularities is discussed in [Shi18, 7.2.8]. See also the original discussion of KK-compactification of the Green-Schwarz mechanism in heterotic string theory on a $K3$ surface to 6d [GSWe85][Sag92].

Conclusion. By comparison, theorem 1.1 provides a detailed mathematical reflection of this traditional picture:

- **(i)** The $\mathbb{Z}_2$-orbifold (162) entering Theorem 1.1 reflects the heterotic bulk M-theory spacetime with the tensor-branch $1/2$M5-brane at the A_1-type singular locus.
- **(ii)** The Bianchi identities (150) given by Theorem 1.1 reproduce the expected bulk flux relation (161) and the brane/boundary Green-Schwarz relation (163) as in (3).
- **(iii)** The integrality conditions (4) reflect the expected flux quantization conditions in heterotic M-theory [FSS20c].

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