Rational Parametrised Stable Homotopy Theory

Dissertation

zur

Erlangung der naturwissenschaftlichen Doktorwürde
(Dr. sc. nat.)

vorgelegt der

Mathematisch-naturwissenschaftlichen Fakultät

der

Universität Zürich

von

Vincent S. Braunack-Mayer

von

Flums-Grossberg SG

Promotionskommission

Prof. Dr. Alberto Cattaneo (Vorsitz)
Prof. Dr. Giovanni Felder
Prof. Dr. Thomas Willwacher

Zürich 2018

Abstract

Rational Parametrised Stable Homotopy Theory

by Vincent S. BRAUNACK-MAYER

This thesis is a study of various aspects of parametrised stable homotopy theory, its rationalisation and its application to twisted differential cohomology. We construct simplicial model categories of parametrised spectra over a fixed base space and relate these to one another via base change adjunctions. We construct model categories for the global homotopy theory of parametrised spectra that have good formal properties, and apply these models to establish a framework for studying twisted differential cohomology via sheaves of parametrised spectra.

Building from Quillen's rational homotopy theory, we show that the torsion-free information of a spectrum parametrised by a 1-connected space is completely encoded by algebraic data. Working over 1-connected base spaces, the rational homotopy theory of parametrised spectra is naturally identified with the homotopy theories of unbounded dg Lie representations and of unbounded dg comodules. We also discuss a dual picture based on Sullivan's approach to rational homotopy theory, which gives a more direct relationship between parametrised spectra and dg modules over dg algebras. This dual picture provides a way of algebraically presenting twisted cohomology theories and characterises a full subcategory of finitely-presented objects in the rational homotopy category of parametrised spectra in terms of perfect dg modules.

Finally, we conclude by applying our rational homotopy theory results to twisted differential cohomology. We show that that the path method of higher Lie integration is one half of a smooth Sullivan–de Rham adjunction relating smooth ∞-stacks to algebra. Stabilising this adjunction gives a means by which dg modules over real dg algebras produce twisted differential cohomology, an important mechanism by which we can pass from infinitesimal/rational approximations to true homotopy theory. This thesis lays down firm mathematical foundations for future explorations of the role of twisted differential cohomology in quantum field theory and M-theory.

Zusammenfassung

Rational Parametrised Stable Homotopy Theory

von Vincent S. BRAUNACK-MAYER

Diese Arbeit beschäftigt sich mit mehreren Aspekten der parametrisierten stabilen Homotopietheorie, ihrer Rationalisierung und ihrer Anwendung auf die verdrehte Differential-Kohomologie. Zuerst konstruieren wir simpliziale Modellkategorien von Spektren parametrisiert über einen festen Basisraum. Diese Modellkategorien stehen über Basenwechsel-Adjunktionen in Beziehung zueinander. Zudem konstruieren wir Modellkategorien für die globale Homotopietheorie parametrisierter Spektren, die gute formale Eigenschaften besitzen, und wenden diese Modelle an, um einen Rahmen für das Studium der verdrehten Differential-Kohomologie durch Garben parametrisierter Spektren zu erstellen.

Ausgehend von Quillens rationaler Homotopietheorie zeigen wir, dass der torsionsfreie Teil eines parametrisierten Spektrums, das durch einen einfach zusammenhängenden Raum parametrisiert ist, vollständig von algebraischen Daten erfasst wird. Die rationale Homotopietheorie parametrisierter Spektren mit einfach zusammenhängenden Basisräumen wird natürlich mit den Homotopie-Theorien von unbegrenzten dg-Lie-Darstellungen und von unbegrenzten dg-Komodulen identifiziert. Daher studieren wir ein duales Bild basierend auf Sullivans Ansatz zur rationalen Homotopietheorie, der eine direktere Beziehung zwischen parametrisierten Spektren und dg-Modulen bietet. Dieses duale Bild ergibt eine Regel, die es erlaubt, verdrehte Kohomologietheorien algebraisch darzustellen und charakterisiert eine volle Unterkategorie von endlich dargestellten Objekten in der rationalen Homotopiekategorie parametrisierter Spektren durch perfekte dg-Module.

Schliesslich wenden wir unsere rationale Homotopietheorie auf die verdrehte Differential-Kohomologie an. Wir formulieren die Wegintegrationsmethode höherer Lie-Theorie als Teil einer Sullivan–de Rham Adjunktion, die glatte ∞-Stacks mit dg-Algebren in Verbindung bringt. Die Stabilisierung dieser Adjunktion liefert eine Methode, mit dem ein dg-Modul eine verdrehte differentielle Kohomologietheorie erzeugt, wodurch wir von infinitesimalen/rationalen Approximationen zur echten Homotopietheorie übergehen können. Diese Arbeit legt mathematische Grundlagen für zukünftige Untersuchungen der Rolle der verdrehten Differential-Kohomologie in der Quantenfeldtheorie und der M-Theorie.

“The question you raise, ‘How can such a formulation lead to computations?’ doesn’t bother me in the least! Throughout my whole life as a mathematician, the possibility of making explicit, elegant computations has always come out by itself, as a byproduct of a thorough conceptual understanding of what was going on. Thus I never bothered about whether what would come out would be suitable for this or that, but just tried to understand—and it always turned out that understanding was all that mattered.”

- A. GROTHENDIECK

To L.

For all the obvious reasons.
And also some subtle ones.

Acknowledgements

Perhaps I should have listened when I was told that completing a doctoral degree is not an easy thing. I am incredibly grateful for the support of all the fine people who accompanied me on this journey. Without their help in keeping a level head and finding meaning in the madness, I would surely have shared a Sisyphean fate.

Firstly, I wish to sincerely thank my supervisor Alberto Cattaneo for this patience and collegiality. He gave me the means and the freedom to pursue my own path, and I thank him for always amiably listening to my hare-brained ideas.

Secondly, I wish to thank my informal co-supervisor Urs Schreiber. I took a leap into the unknown when I contacted him midway through, and I humbly thank him for enthusiastically jumping on the bandwagon. I learned a great deal during the many hours I spent talking to Urs in Bonn, in various cafés around Zürich and, occasionally, at the University. Without his mentorship, guidance and the nLab, it is safe to say this thesis would never have been completed.

I would like to express my thanks to all of the people whose mathematical trajectories have interfered with my own over these last years: Ivan Contreras, Yaël Frégier, Dmitry Roytenberg, Ricardo Campos, Santosh Kandel, Giovanni Canepa, and Nima Moshayedi. Special thanks to Michele Schiavina, for long conversations about nothing and everything; to Pavel Mnëv for his great company and mathematical wisdom, not to mention his tolerance of my *supercazzola*; to Konstantin Wernli for being my token Swiss friend and for successfully coaxing me out of my shell from time to time. Special thanks also to Jonathan Lorand for the *Zauberlehrlingsstunden*, and to Alessandro Valentino for constantly reminding me that mathematics is not just the science of being technically correct, but can and should be a form of art in its own right. Alessandro always made sure that the amount of time I spent gazing into the Abyss was not too long, but not too short either.

I am grateful for the opportunities that I have had to spend time with Sir Michael Atiyah and the late Andrew Ranicki during my trips to Edinburgh. During many fascinating hours spent talking with Sir Michael, I have acquired a deep appreciation for the long view in all things mathematical.

My thanks go to Stefan Schwede and Brooke Shipley for their help with certain niggling technical issues. I also owe a substantial debt to Daniel Quillen, whose mathematical influence is pervasive throughout my thesis; at last count, I have used his name over 400 times.

On a personal level, I am deeply grateful for the support my diasporic family. Having family spread across the world in Australia, Switzerland and the US means that no matter the time zone, there is always someone to talk to about the vagaries of life. It's also a great excuse to travel and unwind, whether closer to home or farther afield.

Thanks to my friends here in Zürich for constantly reminding me that life exists beyond mathematical toil. Special thanks go to Maarten Flink for his irrepressible dutchness and for appointing me custodian of his Swiss friends while he travels the world; to Sophie Calabretto for being hands-down the best housemate one could wish for, and for opening the door to cinephilia; and to Satik and Lizarc for their help in studying the subtle art of brunch. I would also like to especially thank Thomas Geldmacher, Vic Peters, and Kerstin and Lars Doerner for filling my weeknights with wine and *Seafall*.

My deepest thanks of all go to my wife Lydia. She knows why.

Contents

Preface

Parametrised spectra are objects of key significance in homotopy theory, embodying a subtle mixture of both stable and unstable homotopy theory. Many classical results in algebraic topology are implicitly statements about parametrised spectra; for example, the Eilenberg–Moore spectral sequences are parametrised versions of the Künneth Theorems. Parametrised spectra are precisely the objects that describe twisted cohomology theories, and are central to the study of pushforward operations in generalised cohomology.

A useful heuristic is that parametrised spectra are homotopical counterparts of vector bundles, much as ordinary spectra are homotopy-theoretic versions of abelian groups and vector spaces. Understanding vector bundles as collections of linear fibre spaces smoothly glued to one another along a base manifold, by this analogy parametrised spectra are collections of stable homotopy types that are homotopically glued together along a base space. Vector bundle operations such as direct sum and tensor product have homotopical counterparts in the fibrewise wedge sum and smash product respectively. Moreover, a pullback functor on parametrised spectra analogous to the pullback of vector bundles allows us to move from one parametrised context to another. Parametrised stable homotopy theory owes much of its versatility and usefulness to these pullback functors, their left and right adjoints, and the compatibility of these base change operations with fibrewise sums and smash products.

However, despite their descriptive power and ubiquity within homotopy theory, parametrised spectra have languished in relative obscurity. A primary reason for this is the necessary complexity of the rigorous foundations set down by May and Sigurdsson in [MS06]. A more recent approach in terms of $(\infty, 1)$-categories [And+14] (see also [Lur17, Chapter 7]) helps illuminate many of the conceptual features of the theory, but its level of abstraction does not provide a good framework for carrying out computations.

This thesis is the author's attempt develop a framework for twisted differential cohomology in terms of sheaves of parametrised spectra. Twisted differential cohomology has slowly emerged in recent decades as the proper mathematical language for dealing with fully non-perturbative quantum physics. Many key mathematical structures in fundamental physics, such as gauge field bundles and the background B-field of string theory, are naturally equivalent to classes in nonabelian differential cohomology. Classical (or, rather, *pre-quantum*) Lagrangian field theory is completely encoded by a configuration space of fields and an action functional. Using these data as input, quantum expectation values are obtained by computing weighted integrals over configuration space, with weight specified by the action functional. The data of classical Lagrangian field theory is captured by nonabelian differential cohomology; action functionals are connections on higher gerbes over the smooth moduli ∞-stack of fields. In this language, quantisation proceeds by passing from higher gerbes with connection to a stable coefficient theory, hence to a differential cohomology theory such as differential K-theory, and then computing pushforwards over the moduli ∞-stack of fields. Since cohomological pushforwards are often obstructed (this is the source of many "quantum anomalies"), we are forced to work with twisted differential cohomology.

In order to have any hope of realising this vague picture of non-perturbative quantisation, it is necessary to have a tractable and robust mathematical framework for twisted differential cohomology at hand. One of the main accomplishments of this thesis is to provide such a model in terms of sheaves of parametrised spectra. To do this, and in order to better interface with the work of Schreiber [Sch17] on nonabelian differential cohomology, it proved necessary to develop simplicial models for the homotopy theory of parametrised spectra. The simplicial models developed here fall into a conceptual middle ground between the technically-involved approach of May–Sigurdsson and the streamlined abstract $(\infty, 1)$-categorical approach; our simplicial models allow us to build model categories for twisted differential cohomology upon which to base calculations and future explorations, but at the same time do not fully capture certain features such as fibrewise smash products.

A separate topic studied in this thesis is the issue of rationalisation. It is a well-known fact of classical homotopy theory that discarding torsion makes the whole story drastically simpler. Working rationally, we can identify rational homotopy types with algebraic data; passing from a homotopy type to its rationalisation is akin to taking an infinitesimal approximation or Lie differentiation. In the stable setting, working rationally identifies spectra with graded rational vector spaces. A major accomplishment of this thesis is the combination of stable and unstable rational homotopy theory. We show that the identification of the (unstable) rational homotopy category with algebra lifts naturally to categories of representations. Whereas the rational homotopy type of a simply connected space X is encoded by a dg Lie algebra $\mathfrak{l}_X$ or dg coalgebra C_X, a parametrised X-spectrum is, rationally, precisely an unbounded $\mathfrak{l}_X$-representation or unbounded C_X-comodule respectively.

The identification of rational parametrised stable homotopy theory with algebra obtained in this thesis is extremely useful. From the perspective of the algebraic topologist, it provides an important tool for calculations in twisted cohomology in the torsion-free approximation and provides a conceptual home for the homotopy theory of dg modules over Sullivan algebras, as studied by Roig and collaborators [Roi94a; RSA00]. Our rationalisation results also serve to further amplify the necessity of parametrised stable homotopy theory in the mathematical physicist's lingua franca. For example, the results of this thesis provide an interpretation of the Lie-theoretic calculations of [FSS17a; FSS17b] as a rational-homotopy-theoretic derivation of D-brane charges in twisted K-theory from first principles.

This thesis consists of three chapters and two supplementary appendices. We now provide an overview of what we do in each of these parts. The introductory sections of each chapter provide more specific details.

Chapter 1. In the first chapter of this thesis we construct various simplicial model categories of parametrised spectra. Working with simplicial objects has the advantage that all of the model categories we construct are locally presentable, cofibrantly generated and left proper, so are amenable to left Bousfield localisation. For each simplicial set X, we construct model categories $\mathrm{Sp}_X^{\mathbb{N}}$ and Sp_X^{Σ} of sequential and symmetric spectra parametrised by X. For each map of parameter spaces $f\colon X \to Y$ we establish base change Quillen adjunctions $(f_! \dashv f^*)\colon \mathrm{Sp}_X \to \mathrm{Sp}_Y$ in both the sequential and symmetric settings. Using the Grothendieck construction we obtain model categories $\mathrm{Sp}_{\mathrm{sSet}}^{\mathbb{N}}$ and $\mathrm{Sp}_{\mathrm{sSet}}^{\Sigma}$ for the global homotopy theory of parametrised spectra. We show that that both of these model categories are left proper and combinatorial, and are Quillen equivalent to one another. We also show that $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ is a symmetric monoidal model category with respect to the external smash product of parametrised spectra.

The final part of Chapter 1 applies our work to twisted differential cohomology. Using $(\infty, 1)$-categorical arguments, we show that twisted differential cohomology theories are homotopy sheaves of parametrised spectra on the site of smooth manifolds. Having ascertained the abstract picture, we use our simplicial model categories $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ and $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ to construct models for the homotopy theory of twisted differential cohomology. We characterise fibrant objects in these model and discuss descent-type spectral sequences for computing twisted differential cohomology on a manifold equipped with a given good open cover.

Chapter 2. The second chapter is the longest and most technically involved of the three. This chapter is devoted to studying the rational homotopy theory of parametrised spectra. For a 1-connected space X, we lift Quillen's rational homotopy theory equivalences to equivalences between the homotopy theories of:

(a) rational spectra parametrised by X;

(b) stable rational homotopy representations of the loop group ΩX;

(c) unbounded dg representations of a strict dg model $\mathfrak{l}_X$ of the Whitehead Lie algebra of X; and

(d) unbounded dg comodules over a strict dg coalgebra model C_X of the rational chains of X.

These equivalences show that the rational homotopy theory of parametrised spectra over 1-connected spaces is completely algebraic. We also show that the equivalences (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c)$\Leftrightarrow$(d) satisfy a weak naturality property with respect to the parameter space X, a particular consequence of which is that the homotopy theories (a), (b), (c) and (d) depend only on the rational homotopy type of X up to equivalence. Moreover, we show that (b) and (c) are presented by monoidal model categories and that the equivalence (b)$\Leftrightarrow$(c) is strongly monoidal: the fibrewise smash product of parametrised spectra is therefore represented rationally by the derived tensor product of dg Lie representations. The proof is rather long, involving a large number of steps, and requires the symmetric parametrised spectra of Chapter 1.

We conclude Chapter 2 with a treatment of rational parametrised spectra that builds upon Sullivan's dual approach to rational homotopy theory. By stabilising the Sullivan–de Rham adjunction we obtain Quillen adjunctions between model categories of parametrised spectra and dg modules over dg algebras. This establishes a direct link between parametrised spectra and algebra, from which we obtain twisted cohomology theories associated to any cofibrant dg module over a Sullivan algebra. As a further illustration of the conceptual link between parametrised spectra and vector bundles, we prove a homotopical analogue of the Serre–Swan Theorem; for a Sullivan algebra A of finite type, we demonstrate an equivalence between the derived category of perfect A-modules and a full subcategory of finitely-presented objects inside the rational homotopy category of spectra parametrised by the spatial realisation $\mathcal{S}(A)$ of A. We show that in the 1-connected case, this equivalence extends to all bounded-below A-modules of finite cohomological type.

Chapter 3. The final chapter of this thesis explores the link between algebra and twisted differential cohomology. Building upon the results of Chapters 1 and 2, we show that cofibrant dg modules over real cofibrant cDGAs give rise to fibrant sheaves of parametrised spectra, and hence to twisted differential cohomology theories. The construction is motivated by the path method of higher Lie integration, which we show is one half of a Quillen adjunction linking smooth ∞-stacks with algebra. This adjunction is a smooth version of the Sullivan–de Rham adjunction and stabilises to yield an adjunction between sheaves of parametrised spectra and categories of dg modules.

We conclude the final chapter with a summary of our work and by amplifying the conceptual link between algebraically-presented smooth parametrised spectra and higher Lie theory. We also discuss some natural questions arising from our work which are very important for future explorations of twisted differential cohomology using our methods.

Appendix A. A large part of this thesis uses model categories to present homotopy theories. In Appendix A, we briefly summarise some of the model-category-theoretic tools that are frequently used. Specifically, we recall model structure transfer theorems, the sequential and symmetric stabilisation machines, and the Grothendieck construction for model categories.

Appendix B. In this appendix, we record some classical results on model categories of simplicial spaces acted on by a simplicial group. We recall specific results that are required in the arguments of Chapter 2.

Chapter 1

Parametrised Spectra

In homotopical algebra, it is commonly held that having several different models for a particular homotopy theory provides several different toolboxes for attacking problems within that theory. In line with this principle, this chapter is devoted to the construction of various simplicial model categories that present several different aspects of the stable homotopy theory of parametrised spectra. The models we construct complement May and Sigurdsson's treatment of parametrised stable homotopy theory in terms of k-spaces [MS06] and the $(\infty, 1)$-categorical approach of [And+14].

The bulk of our work in this chapter concerns setting up model categories of simplicial objects, establishing their formal properties, and their interrelationships. We subsequently apply these results to twisted differential cohomology (in §1.3 and Chapter 3) and parametrised rational stable homotopy theory (in Chapter 2). We begin in §1.1 in the unstable setting of parametrised spaces. We construct model categories for the local theory, working over a fixed parameter space, and show that maps between parameter spaces give rise to base change Quillen adjunctions. We assemble a global model category, which considers all possible parameter spaces at once, from the totality of local models using a Grothendieck construction for model categories [HP15].

In §1.2, we pass to stable homotopy theory using the stabilisation machinery of [Hov01]. We construct local model categories of parametrised sequential and symmetric spectra for a fixed parameter space. As in the unstable setting, for each map between parameter spaces we exhibit base change Quillen adjunctions between model categories of parametrised sequential (respectively, symmetric) spectra. Using these base change adjunctions, we construct sequential and symmetric models for the global homotopy theory of parametrised spectra and show that these global models are Quillen equivalent, so both present the global homotopy theory of parametrised spectra. We show that both global models are left proper and combinatorial, technical features which are crucial to our subsequent work in §1.3. Finally, we show that the global model category of parametrised symmetric spectra is a monoidal model category with respect to the external smash product.

The final part of this chapter applies our model categories to the study of twisted differential cohomology theories. In §1.3, we use $(\infty, 1)$-categorical arguments to show that twisted differential cohomology theories are represented by homotopy sheaves of parametrised spectra on the site of smooth manifolds (or, rather, the dense subsite of Cartesian spaces). Using the global models of §1.2 we construct model categories of homotopy sheaves of parametrised spectra, presenting the global homotopy theory of twisted differential cohomology theories. The construction relies crucially on left properness and combinatoriality as we implement homotopical descent via a left Bousfield localisation. We include a characterisation of fibrant objects and conclude with a brief discussion of descent spectral sequences in twisted differential cohomology.

1.1 Retractive Spaces

We begin our study of simplicial models of parametrised spectra with a discussion of retractive spaces. For a space X, a retractive space over X is simply a surjection $p\colon E \to X$ equipped with a section. By taking homotopy fibres, these data determine a sort of nonabelian local coefficient system on X

$$\mathrm{hofib}_{(-)}E\colon \Pi X \longrightarrow \mathcal{H}_*$$

which takes values in pointed homotopy types. A retractive space over X thus describes a sort of *X-twisted nonabelian cohomology theory*; this is the unstable analogue of X-twisted cohomology.

In this section, we develop simplicial model categories R_X describing the homotopy theory of retractive spaces for a fixed base space X. For each map of spaces $f\colon X \to Y$ we obtain homotopically well-behaved pushforward and pullback functors $f_!\colon R_X \to R_Y$ and $f^*\colon R_Y \to R_X$. These base change functors allow us to glue together the R_X for varying X to obtain a new *global model category* R_{sSet} of retractive spaces, whose homotopy category describes twisted nonabelian cohomology theories with all possible twists. The global model category R_{sSet} is equipped with a homotopically well-behaved external smash product which describes pairings on twisted nonabelian cohomology. Moreover, we show that R_{sSet} is left proper and combinatorial—these technical conditions are crucial for our construction of model categories for twisted differential cohomology in §1.3.

1.1.1 Categories of Retractive Spaces

In this section we recall the basic properties of categories of retractive spaces in the simplicial setting. These results are more or less standard, however we take care to emphasise certain features, such as the enrichment over pointed simplicial sets, that are crucial to our treatment of parametrised spectra.

Definition 1.1.1. For a simplicial set X, the category of *retractive spaces over X* is the under-over-category

$$R_X := \left(\mathrm{sSet}_{/X}\right)^{X/} \cong \left(\mathrm{sSet}^{X/}\right)_{/X}.$$

A retractive space over X is thus a triple (Y, i, r), with $r\colon Y \to X$ a surjection with section $i\colon X \to Y$. A morphism of retractive spaces $(Y, i, r) \to (Y', i', r')$ over X is the data of a commuting diagram of simplicial sets

$$\begin{tikzcd} X \arrow[r, "i'"] \arrow[d, "i"'] & Y' \arrow[d, "r'"] \\ Y \arrow[r, "r"'] \arrow[ur, "\psi"] & X, \end{tikzcd}$$

namely, a map $\psi\colon Y \to Y'$ commuting with the section and retraction maps. We commonly abuse notation by writing Y as shorthand for the data $(Y, i, r) \in R_X$, especially when a base space X is understood.

Lemma 1.1.2. *For each simplicial set X, the category R_X is locally presentable.*

Proof. The category of simplicial sets is locally presentable [Bor94, 5.2.2b]. Locally presentable categories are stable under forming under- and over-categories [AR94, Proposition 1.57]. □

Recall that the category $\mathrm{sSet}_* = R_*$ of pointed simplicial sets is a closed symmetric monoidal category with respect to the smash product $\wedge$. For pointed simplicial sets $k\colon * \to K$ and $l\colon * \to L$, the smash product is the pointed simplicial set defined as the pushout

$$\begin{tikzcd} K \vee L \arrow[r] \arrow[d] & K \times L \arrow[d] \\ * \arrow[r] & K \wedge L, \end{tikzcd}$$

where $K \vee L = K \coprod_* L$ and the top horizontal map is the obvious one determined by k and l.

Lemma 1.1.3. *For each $X \in \mathrm{sSet}$ the category R_X is enriched, tensored and powered over $(\mathrm{sSet}_*, \wedge)$.*

Proof. This result is elementary, however we include a proof for completeness. For objects $(Y, i_Y, r_Y), (Z, i_Z, r_Z)$ of R_X we have the usual simplicial mapping space

$$\mathrm{Map}(Y, Z)\colon [n] \mapsto \mathrm{sSet}(\Delta[n] \times Y, Z).$$

The simplicial enrichment of R_X is obtained from these mapping spaces by restricting to the subcomplexes of maps respecting the section and retraction maps. Define $\mathrm{Map}_{R_X}(Y, Z)$ to be the limit of the diagram of simplicial sets

$$\begin{tikzcd} * \arrow[dr, "\dashv r_Y"] & & \mathrm{Map}(Y,Z) \arrow[dl, "\mathrm{Map}(Y, r_Z)"'] \arrow[dr, "\mathrm{Map}(i_Y, Z)"] & & * \arrow[dl, "\dashv i_Z"'] \\ & \mathrm{Map}(Y, X) & & \mathrm{Map}(X, Z) & \end{tikzcd}$$

where $\dashv r_Y$ and $\dashv i_Z$ are the exponential adjuncts of $r_Y\colon Y \to X$ and $i_Z\colon X \to Z$. Composition and identity morphisms for the simplicial mapping spaces of R_X are induced from the sSet-enrichment of sSet. The category R_X is pointed with zero object $(X, \mathrm{id}_X, \mathrm{id}_X)$; for any $Y, Z \in R_X$ the simplicial mapping space $\mathrm{Map}_{R_X}(Y, Z)$ is thus pointed by the zero map

$$0_X\colon Y \xrightarrow{r_Y} X \xrightarrow{i_Z} Z.$$

For any triple of objects $Y, Z, W \in R_X$ the composition map on simplicial mapping spaces fits into a commuting diagram

$$\begin{tikzcd} \mathrm{Map}_{R_X}(Z, W) \vee \mathrm{Map}_{R_X}(Y, Z) \arrow[r] \arrow[d] & \mathrm{Map}_{R_X}(Z, W) \times \mathrm{Map}_{R_X}(Y, Z) \arrow[d, "\circ_{Y,Z,W}"] \\ * \arrow[r, "\dashv 0_X"] & \mathrm{Map}_{R_X}(Y, W) \end{tikzcd}$$

since post- or pre-composing any map in R_X with 0_X yields 0_X. The composition map therefore descends to a map of pointed simplicial sets

$$\overline{\circ}_{Y,Z,W}\colon \mathrm{Map}_{R_X}(Z, W) \wedge \mathrm{Map}_{R_X}(Y, Z) \longrightarrow \mathrm{Map}_{R_X}(Y, W),$$

giving rise to a $(\mathrm{sSet}_*, \wedge)$-enrichment on R_X.

For a simplicial set K and retractive space Y over X, the tensoring $K \otimes_X Y$ is obtained as the retractive space over X exhibited as the bottom row of the diagram of pushout squares of simplicial sets

$$\begin{tikzcd} K \times X \arrow[r] \arrow[d] & K \times Y \arrow[r] \arrow[d] & K \times X \arrow[d] \\ X \arrow[r] & K \otimes_X Y \arrow[r] & X. \end{tikzcd}$$

This construction determines a bifunctor $(-) \otimes_X (-) \colon \mathrm{sSet} \times R_X \to R_X$ which, by universality of colimits in sSet, preserves colimits in each argument.

For retractive spaces $Y, Z \in R_X$ elements of the hom-set $R_X(\Delta[n] \otimes_X Y, Z)$ naturally correspond to commuting diagrams of simplicial sets

$$\begin{tikzcd} \Delta[n] \times X \arrow[r] \arrow[d] & \Delta[n] \times Y \arrow[r] \arrow[d] & \Delta[n] \times X \arrow[d] \\ X \arrow[r, "i_Z"] & Z \arrow[r, "r_Z"] & X. \end{tikzcd}$$

In turn, such diagrams naturally correspond to commuting diagrams

$$\begin{tikzcd} & & \Delta[n] \arrow[lld, bend right] \arrow[d] \arrow[rrd, bend left] & & \\ * \arrow[dr, "\dashv r_Y"] & & \mathrm{Map}(Y,Z) \arrow[dl, "\mathrm{Map}(Y{,}r_Z)"'] \arrow[dr, "\mathrm{Map}(i_Y{,}Z)"] & & * \arrow[dl, "\dashv i_Z"'] \\ & \mathrm{Map}(Y,X) & & \mathrm{Map}(X,Z) & \end{tikzcd}$$

and hence with n-simplices of the simplicial mapping space $\mathrm{Map}_{R_X}(Y, Z)$. Since the functors

$$R_X\big((-) \otimes_X Y, Z\big) \quad \mathrm{sSet}\big(-, \mathrm{Map}_{R_X}(Y, Z)\big) \colon \mathrm{sSet}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

both preserve limits and are naturally isomorphic when restricted to $\Delta \to \mathrm{sSet}^{\mathrm{op}}$, they are naturally isomorphic. In particular we have natural isomorphisms

$$R_X(K \otimes_X Y, Z) \cong \mathrm{sSet}\big(K, \mathrm{Map}_{R_X}(Y, Z)\big)$$

for all $K \in \mathrm{sSet}$ and $Y, Z \in R_X$, which shows that $(-) \otimes_X (-)$ does indeed define a sSet-tensoring.

For $K \in \mathrm{sSet}$ and $Y \in R_X$ the powering $Y^K \in R_X$ is obtained as the retractive space over X determined by the top row of the diagram of pullback squares

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y^K_X \arrow[r] \arrow[d] & X \arrow[d] \\ X^K \arrow[r] & Y^K \arrow[r] & X^K, \end{tikzcd}$$

where X^K, say, is the usual exponential object in sSet. By dualising the above argument for the tensoring we show that this assignment determines a sSet-powering on R_X. $\square$

For a pointed simplicial set $K \in \mathrm{sSet}_*$ and retractive space $Y \in R_X$ we define $K \owedge_X Y$ as the pushout

$$\begin{tikzcd} * \owedge_X Y \cong Y \arrow[r] \arrow[d] & K \otimes_X Y \arrow[d] \\ X \arrow[r] & K \owedge_X Y \end{tikzcd} \tag{1.1}$$

in R_X or, equivalently, as the retractive space over X determined as the colimit of the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[dr] \arrow[dd] & \\ K \times X \arrow[rr] \arrow[dr] & & K \times Y \\ & X & \end{tikzcd} \tag{1.2}$$

of simplicial sets (the equivalence is demonstrated as Lemma 1.1.6 below). For a pointed simplicial set of the form $K \cong L \coprod *$ we find that $K \owedge_X Y \cong L \otimes_X Y$ for all $Y \in R_X$ and from (1.1) it is easy to see that we have natural isomorphisms $\mathrm{sSet}_*(K, \mathrm{Map}_{R_X}(A, B)) \cong R_X(K \owedge_X A, B)$ so that $\owedge_X$ indeed defines a sSet_*-tensoring. We define a sSet_*-powering by setting $K \pitchfork_X Y$ to be the pullback of the diagram

$$X \longrightarrow Y \longleftarrow Y_X^K$$

in R_X—this is seen to define a powering via a dual argument. In conclusion, we have proven

Lemma 1.1.4. *For each $X \in \mathrm{sSet}$, the category R_X is enriched, tensored and powered over the monoidal category $(\mathrm{sSet}_*, \wedge)$ of pointed simplicial spaces equipped with the smash product.*

We conclude this section with a recollection of the base change adjunctions and some of their basic properties. A map of simplicial sets $f\colon X \to Y$ induces a functor $f^*\colon R_Y \to R_X$ via pullback along f. The base change functor f^* is determined on objects by sending $Z \in R_Y$ to the retractive space over X determined by the top row of the diagram of pullback squares

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & f^*Z \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "i_Z"] & Z \arrow[r, "r_Z"] & Y. \end{tikzcd}$$

The functor f^* acts on morphisms in the obvious way.

Lemma 1.1.5. *For each map of simplicial sets $f\colon X \to Y$, there is a triple of sSet_*-enriched adjunctions*

$$(f_! \dashv f^* \dashv f_*)\colon\ R_X \underset{\underset{\perp}{\longrightarrow}}{\overset{\overset{\longrightarrow}{\perp}}{\longleftarrow}} R_Y$$

on categories of retractive spaces.

Proof. The total left adjoint $f_!$ may be defined directly in terms of pushouts as follows. To an object $A \in R_X$ the functor $f_!$ assigns the retractive space over Y given by the bottom row of the diagram of pushout squares

$$
\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"'] & A \arrow[r, "r"] \arrow[d] & X \arrow[d, "f"] \\
Y \arrow[r] & f_!A \arrow[r] & Y.
\end{tikzcd}
$$

The functor $f_!$ is determined on morphisms in the obvious way. The sets $R_X(A, f^*B)$ and $R_Y(f_!A, B)$ are both naturally isomorphic to the set of commuting diagrams of simplicial sets

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & A \arrow[r] \arrow[d, "\psi"] & X \arrow[d, "f"] \\
Y \arrow[r] & B \arrow[r] & Y,
\end{tikzcd}
$$

which establishes the $(f_! \dashv f^*)$-adjunction at the level of hom-sets.

To see that $(f_! \dashv f^*)$ is an enriched adjunction it is sufficient to check that $f_!$ preserves sSet_*-tensors up to natural isomorphism. This condition guarantees natural isomorphisms $R_Y(K \oslash_Y f_!A, B) \cong R_X(K \oslash_X A, f^*B)$ for all $K \in \mathrm{sSet}_*$ from which it follows that there is are natural isomorphisms $\mathrm{Map}_{R_Y}(f_!A, B) \cong \mathrm{Map}_{R_X}(A, f^*B)$ on enriched hom-sets. The sSet_*-enrichment of, say, $f_!$ is then witnessed by the composite

$$\mathrm{Map}_{R_X}(A, B) \longrightarrow \mathrm{Map}_{R_X}(A, f^*f_!B) \xrightarrow{\cong} \mathrm{Map}_{R_Y}(f_!A, f_!B),$$

where the first morphism is induced by the $(f_! \dashv f^*)$-unit. Using the counit maps $\mathrm{Map}_{R_X}(A, B) \oslash_X A \to B$ for $A, B \in R_X$ we show that the maps on simplicial mapping spaces defined above respect the composition morphisms.

To see that $f_!$ preserves sSet_*-tensors up to natural isomorphism, we first take $A \in R_X$ and $K \in \mathrm{sSet}$. The underlying simplicial set of $f_!(K \otimes_X A)$ is the iterated pushout

$$
\begin{tikzcd}
K \times X \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
K \times A \arrow[r] & K \otimes_X A \arrow[r] & f_!(K \otimes_X A).
\end{tikzcd}
$$

On the other hand, $K \otimes_Y f_!A$ has underyling simplicial set given by the iterated pushout

$$
\begin{tikzcd}
K \times X \arrow[r, "\mathrm{id}_K \times f"] \arrow[d] & K \times Y \arrow[r] \arrow[d] & Y \arrow[d] \\
K \times A \arrow[r] & K \times f_!A \arrow[r] & K \otimes_Y f_!A,
\end{tikzcd}
$$

where the left-most square is a pushout since colimits in sSet are universal. This shows that $f_!(K \otimes_X A)$ and $K \otimes_Y f_!A$ are both colimits of $K \times A \leftarrow K \times X \to Y$, so are naturally isomorphic. Since $f_!$ preserves colimits and $K \oslash_X A$ is defined from $K \otimes_X A$ via a pushout diagram it follows that $f_!$ preserves sSet_*-tensors up to natural isomorphism.

We now consider the $(f^* \dashv f_*)$-adjunction. The colimit of a small diagram $D\colon \mathcal{I} \to R_Y$ is computed as the colimit of the augmented diagram $D^{\triangleleft}\colon \mathcal{I}^{\triangleleft} \to \mathrm{sSet}$, where $\mathcal{I}^{\triangleleft} := \{*\} \star \mathcal{I}$ is the categorical join, and $D^{\triangleleft}$ is given by

$$D^{\triangleleft}(i) := \begin{cases} Y & \text{for } i = * \text{ the cone point} \\ D(i) & \text{for } i \in \mathcal{I}, \end{cases}$$

where the unique morphism $* \to j$ from the cone point is sent to the section map $i_{D(j)}$ of $D(j) \in R_Y$ and all other morphisms are as for D. Universality of colimits in sSet implies that there is an isomorphism

$$f^* \left(\operatorname*{colim}_{i \in \mathcal{I}^{\triangleleft}} D^{\triangleleft}(i) \right) \cong \operatorname*{colim}_{i \in \mathcal{I}^{\triangleleft}} f^* D^{\triangleleft}(i),$$

where, since $f^* D^{\triangleleft}(*) = f^* Y = X$, the right hand side coincides with the colimit of $(f^* \circ D) \colon \mathcal{I} \to R_X$. We conclude that f^* preserves colimits and so, by Lemma 1.1.2 and the Adjoint Functor Theorem, admits a right adjoint f_*. For all $K \in \mathrm{sSet}_*$ and $A \in R_Y$ universality of colimits in sSet directly implies that $f^*(K \oslash_Y A) \cong K \oslash_X f^* A$ so that $(f^* \dashv f_*)$ is a sSet_*-adjunction by the above argument. $\square$

The last result in this section verifies that the two definitions of sSet_*-tensoring on R_X do coincide up to natural isomorphism. The result is elementary and is included as a timely sanity check.

Lemma 1.1.6. *The underlying simplicial sets of the colimit diagrams* (1.1) *and* (1.2) *coincide.*

Proof. Unravelling the definition of the sSet-tensoring, the underlying simplicial sets of (1.1) and (1.2) can both be expressed as the colimit of the diagram

$$\begin{tikzcd} & & X \arrow[dl] \arrow[dr] & & \\ & K \times X \arrow[dl] \arrow[dr] & & Y \arrow[dl] \arrow[dr] & \\ X & & K \times Y & & X \end{tikzcd}$$

of simplicial sets, using that $X \to X \times K \to X$ and $X \to Y \to X$ are the identity. $\square$

1.1.2 Model Structures and Base Change

The categories of retractive spaces inherit model structures from sSet in the obvious way. A morphism of retractive spaces over X

$$\begin{tikzcd} X \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] \arrow[ur, "\psi"] & X \end{tikzcd}$$

is a weak equivalence, fibration or cofibration precisely if ψ is such when regarded as a morphism in the Kan model structure [Hov99, §1.1 & Ch. 3]. The induced model structure on R_X inherits various useful properties from the Kan model structure:

Lemma 1.1.7. *For each $X \in \mathrm{sSet}$, the model category R_X is combinatorial and left proper, with*

- *generating cofibrations given by the set of morphisms*

$$\mathcal{I}^X_{\mathrm{Kan}} := \left\{ \begin{tikzcd} X \arrow[r] \arrow[d] & \Delta[n] \coprod X \arrow[d] \\ \partial\Delta[n] \coprod X \arrow[r] \arrow[ur] & X \end{tikzcd} \right\}$$

  obtained by taking the coproduct with X of boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$ of simplices in X, and

- *generating acyclic cofibrations given by the set of morphisms*

$$\mathcal{J}^X_{\mathrm{Kan}} := \left\{ \begin{tikzcd} X \arrow[r] \arrow[d] & \Delta[n] \coprod X \arrow[d] \\ \Lambda^n_k \coprod X \arrow[r] \arrow[ur] & X \end{tikzcd} \right\}$$

  obtained by taking the coproduct with X of horn inclusions $h^n_k \colon \Lambda^n_k \hookrightarrow \Delta[n]$ of simplices in X.

Proof. The category R_X is locally presentable by Lemma 1.1.2. Properness together and cofibrant generation (with respect to the specified sets of generating cofibrations and acyclic cofibrations) are inherited from the Kan model structure [Hir05]. □

Recall that sSet$_*$ is a monoidal model category with respect to the smash product [Hov99, Corollary 4.2.10]. With respect to the sSet$_*$-enrichments established in the previous section, the categories of retractive spaces become sSet$_*$-model categories:

Lemma 1.1.8. *For each $X \in$ sSet, the category of retractive spaces R_X is a sSet$_*$-model category.*

Proof. It is sufficient to show that the sSet$_*$-tensoring $(-) \oslash_X (-) \colon \mathrm{sSet}_* \times R_X \to R_X$ is a Quillen bifunctor, that is for $i \colon K \to L$ and $j \colon A \to B$ cofibrations in sSet$_*$ and R_X, the pushout-product

$$i \Box j \colon L \oslash_X A \coprod_{K \oslash_X A} K \oslash_X B \longrightarrow L \oslash_X B$$

is a cofibration, which is acyclic if at least one of i, j is.

For a pointed simplicial set of the form $K \cong L \coprod *$ and retractive space of the form $A \cong Y \coprod X$ for some morphism $Y \to X$ we compute $K \oslash_X A \cong (L \times Y) \coprod X$ equipped with the induced projection maps, with section given by the coprojection map $X \to (L \times Y) \coprod X$. For any map of simplicial sets $K \to L$ and diagram

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & X & \end{tikzcd}$$

of (non-retractive!) spaces over X, we therefore find that the pushout-product for $\oslash_X$ of $K \coprod * \to L \coprod *$ with $A \coprod X \to B \coprod X$ is the map of retractive spaces

$$\left((L \times A) \coprod_{K \times A} (K \times B) \right) \coprod X \longrightarrow (L \times B) \coprod X$$

over X. By Lemma 1.1.7 and the fact that $(-)\times(-)\colon \mathrm{sSet}\times\mathrm{sSet}\to\mathrm{sSet}$ is a Quillen bifunctor, the set of pushout-products $\mathcal{I}^*_{\mathrm{Kan}}\,\square\,\mathcal{I}^X_{\mathrm{Kan}}$ consists of cofibrations and the sets $\mathcal{J}^*_{\mathrm{Kan}}\,\square\,\mathcal{I}^X_{\mathrm{Kan}}$ and $\mathcal{I}^*_{\mathrm{Kan}}\,\square\,\mathcal{J}^X_{\mathrm{Kan}}$ both consist of acyclic cofibrations. The result now follows by cofibrant generation of $\mathrm{sSet}_* = R_*$ and R_X (Lemma A.1.2). $\square$

Remark 1.1.9. Setting $X = *$ in the Lemma, together with the observation that the smash unit $S^0\in\mathrm{sSet}_*$ is cofibrant, gives another proof that $(\mathrm{sSet}_*,\wedge)$ is a symmetric monoidal model category.

Remark 1.1.10. For retractive spaces $A,B\in R_X$ we can define their *internal smash product* as the retractive space $A\wedge_X B$ over X determined by the colimit of the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & B \arrow[dr] \arrow[dd] & \\
A \arrow[rr] \arrow[dr] & & A\times_X B \\
& X &
\end{tikzcd}$$

of simplicial sets. The internal smash product furnishes R_X with the structure of a closed symmetric monoidal category and it can be shown that the pullback functors $f^*\colon R_Y\to R_X$ are strongly closed monoidal. However unless $X = *$ the internal smash product on R_X does *not* make $(R_X,\wedge_X)$ into a symmetric monoidal model category; the issue at heart here is the fact that the assignment $(A,B)\mapsto A\times_X B$ only preserves weak equivalences if $X = *$. For this reason we do not further pursue internal smash products in this work.

Lemma 1.1.11. *For any map of simplicial sets $f\colon X\to Y$, the base change adjunction $(f_!\dashv f^*)\colon R_X\to R_Y$ is a* sSet_**-enriched Quillen adjunction. Moreover*

(i) *if f is a weak equivalence then $(f_!\dashv f^*)$ is a* sSet_**-Quillen equivalence;*

(ii) *if f is a fibration then the base change adjunction $(f^*\dashv f_*)\colon R_Y\to R_X$ is a* sSet_**-enriched Quillen adjunction; and*

(iii) *if f is an acyclic fibration then $(f_!\dashv f^*)$ and $(f^*\dashv f_*)$ are* sSet_**-Quillen equivalences.*

Proof. Lemmas 1.1.5 and 1.1.8 dispense with the issue of sSet_*-enrichment so we can argue on the underlying model categories. Arguing using the generating sets of cofibrations and acyclic cofibrations from Lemma 1.1.7, it is easy to see that $f_!(\mathcal{I}^X_{\mathrm{Kan}})\subset\mathcal{I}^Y_{\mathrm{Kan}}$ and $f_!(\mathcal{J}^X_{\mathrm{Kan}})\subset\mathcal{J}^Y_{\mathrm{Kan}}$, so that the adjunction $(f_!\dashv f^*)$ is Quillen. A formal argument also shows that f^* preserves monomorphisms and so $f^*\colon R_Y\to R_X$ preserves cofibrations.

Let $f\colon X\to Y$ be a weak equivalence of simplicial sets. Suppose that $A\in R_X$ is cofibrant, $B\in R_Y$ is fibrant and that we have a map $\psi\colon f_!A\to B$ in R_Y. The map ψ and its $(f_!\dashv f^*)$-adjunct $\psi^\vee$ fit into a commuting diagram of simplicial sets

$$\begin{tikzcd}
X \arrow[rr, tail] \arrow[dd, "f"'] & & A \arrow[rr] \arrow[dl] \arrow[dr, "\psi^\vee"] & & X \arrow[dd, "f"] \\
& f_!A \arrow[dr, "\psi"] & & f^*B \arrow[ur] \arrow[dl] & \\
Y \arrow[rr] \arrow[ur] & & B \arrow[rr, two heads] & & Y,
\end{tikzcd}$$

in which the top left triangle is a pushout diagram and the bottom right triangle is a pullback diagram. Cofibrations and fibrations as indicated are due to cofibrancy and fibrancy of A and B respectively. Properness of the Kan model structure implies that the maps $A \to f_!A$ and $f^*B \to B$ in this diagram are weak equivalences. By the 2-out-of-3 property, it now follows that ψ is a weak equivalence if and only if $\psi^\vee$ is a weak equivalence. This shows that $(f_! \dashv f^*)$ is a Quillen equivalence, establishing (i).

For (ii), if $f\colon X \to Y$ is a fibration then pulling back along f preserves weak equivalences by right properness of the Kan model structure. We have already established that f^* preserves cofibrations and so the adjunction $(f^* \dashv f_*)$ is Quillen.

For (iii), if $f\colon X \to Y$ is an acyclic fibration it remains only to show that $(f^* \dashv f_*)$ is a Quillen equivalence. Since f^* is left and right Quillen it preserves all weak equivalences which implies that the derived functors $\mathbb{R}f^* \cong \mathbb{L}f^*$ are isomorphic. Since $(\mathbb{L}f_! \dashv \mathbb{R}f^*)$ is an adjoint equivalence of categories we conclude by essential uniqueness of adjoints that $\mathbb{R}f_* \cong \mathbb{L}f_!$ is an equivalence of categories. □

1.1.3 The Global Model Category and External Smash Products

In this section we introduce a global model category of retractive simplicial sets and establish some key properties. In the previous sections we considered retractive spaces over a fixed base space $X \in \mathrm{sSet}$ and the base change adjunctions allow us to view the assignment $X \mapsto R_X$ as a pseudofunctor

$$\begin{aligned} R_{(-)}\colon \mathrm{sSet} &\longrightarrow \mathbf{Adj} \\ X &\longmapsto R_X \\ (f\colon X \to Y) &\longmapsto \big((f_! \dashv f^*)\colon R_X \to R_Y\big) \end{aligned}$$

(with notation as in §A.4). The Grothendieck construction of this pseudofunctor is the *global category of retractive spaces*. Recall that the Grothendieck construction of $R_{(-)}$ is the category $\int_{\mathrm{sSet}} R_{(-)}$ with

- objects given by pairs (X, A) with $X \in \mathrm{sSet}$ and $A \in R_X$; and
- morphisms given by maps of pairs $(f, \psi)\colon (X, A) \to (Y, B)$ where $f\colon X \to Y$ is a morphism in sSet and $\psi\colon f_!A \to B$ is a morphism in R_Y.

Using the Grothendieck construction for model categories [HP15] (recalled in §A.4) we show that the global category R_{sSet} is equipped with a natural model structure. We begin by collecting some useful properties of the global category.

Construction 1.1.12. Denote by $\iota_{[0<2]}\colon \Delta[1] \to \partial\Delta[2]$ the map of simplicial sets that picks out the 1-simplex $[0] \to [2]$. Applying the realisation functor sSet $\to$ Cat we obtain a functor $\iota_{[0<2]}\colon \Delta[1] \to \partial\Delta[2]$. Let $R_{\mathrm{sSet}} \hookrightarrow \mathrm{Fun}(\partial\Delta[2], \mathrm{sSet})$ denote the full subcategory of functors $\xi\colon \partial\Delta[2] \to \mathrm{sSet}$ such that $\xi \circ \iota_{[0<2]} = (\mathrm{id}_X\colon X \to X)$ for some simplicial set X. This construction comes equipped with a natural functor base: $R_{\mathrm{sSet}} \to \mathrm{sSet}$.

Lemma 1.1.13. *There is a canonical isomorphism of categories* $R_{\mathrm{sSet}} \cong \int_{\mathrm{sSet}} R_{(-)}$ *that respects the projection functors to* sSet.

Proof. On objects, the isomorphism sends the retractive space $X \to A \to X$ to the object (X, A) of the Grothendieck construction. A morphism of retractive spaces from $X \to A \to X$ to $Y \to B \to Y$ is the data of a commuting diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & A \arrow[r] \arrow[d, "\overline{\psi}"'] & X \arrow[d, "f"] \\ Y \arrow[r] & B \arrow[r] & Y \end{tikzcd}$$

which is equivalent to the map of pairs $(X, A) \to (Y, B)$ given by $f\colon X \to Y$ and the pushout map $\psi\colon f_! A \to B$ induced by $\overline{\psi}$. It is clear that the isomorphism of categories so defined respects the projection functors to sSet. □

Remark 1.1.14. In light of the Lemma we are free to refer to objects of the global model category either as retractive spaces $(X \to A \to X)$ or as pairs (X, A) with $X \in \mathrm{sSet}$ and R_X. In the sequel we freely employ both notations without further comment.

Lemma 1.1.15. *The category R_{sSet} is locally presentable.*

Proof. Since sSet is locally presentable, $\mathrm{Fun}(\mathcal{C}, \mathrm{sSet})$ is locally presentable for any small category $\mathcal{C}$ [AR94, Proposition 1.53]. Let $\Delta_{\mathrm{sSet}} \hookrightarrow \mathrm{Fun}(\Delta[1], \mathrm{sSet})$ be the accessibly embedded full subcategory on the identity morphisms $(\mathrm{id}_X\colon X \to X)$ for $X \in \mathrm{sSet}$. According to Construction 1.1.12, R_{sSet} is the inverse image of Δ_{sSet} by the colimit-preserving pullback functor

$$\imath^*_{[0<2]}\colon \mathrm{Fun}(\partial\Delta[2], \mathrm{sSet}) \longrightarrow \mathrm{Fun}(\Delta[1], \mathrm{sSet})$$

of locally presentable categories and so is accessible by [Lur09, Corollary A.2.6.5]. Since the fully-faithful functor $R_{\mathrm{sSet}} \hookrightarrow \mathrm{Fun}(\partial\Delta[2], \mathrm{sSet})$ creates limits and colimits, the assertion follows. □

Lemma 1.1.16. *For each $X \in \mathrm{sSet}$ there is a natural faithful functor $\imath_X\colon R_X \to R_{\mathrm{sSet}}$ preserving limits and colimits. In particular $\imath_X$ has both a left and a right adjoint.*

Proof. The functor $\imath_X$ is determined on objects by sending $A \in R_X$ to the object $(X \to A \to X) \in R_{\mathrm{sSet}}$ and is extended to morphisms in the obvious fashion. The functor $\imath_X$ preserves all limits and colimits since these are computed objectwise in R_{sSet} (see the proof of Lemma 1.1.15). The existence of left and right adjoints to $\imath_X$ is guaranteed by the Adjoint Functor Theorem since R_{sSet} is locally presentable. □

Lemma 1.1.17. *The projection functor* $\mathrm{base}\colon R_{\mathrm{sSet}} \to \mathrm{sSet}$ *has a two-sided adjoint* $0_{(-)}$.

Proof. Let $0_X \in R_X$ denote the zero object. The assignment $X \mapsto (X, 0_X)$ determines a functor $0_{(-)}\colon \mathrm{sSet} \to R_{\mathrm{sSet}}$ and it is easy to see that maps of pairs $(X, 0_X) \to (Y, B)$ and $(X, A) \to (Y, 0_Y)$ are both in natural correspondence with morphisms $X \to Y$ in sSet, which establishes the triple $(0_{(-)} \dashv \mathrm{base} \dashv 0_{(-)})$ at the level of hom-sets. □

There is another important functor in the global context that forgets the section map of a retractive space. The functor $U\colon R_{\mathrm{sSet}} \to \mathrm{Fun}(\Delta[1], \mathrm{sSet})$ is determined by the assignment

$$\left(\begin{tikzcd} X \arrow[r, "i_A"] \arrow[d, "f"'] & A \arrow[r, "r_A"] \arrow[d, "\overline{\psi}"'] & X \arrow[d, "f"] \\ Y \arrow[r, "i_B"] & B \arrow[r, "r_B"] & Y \end{tikzcd}\right) \longmapsto \left(\begin{tikzcd} A \arrow[r, "r_A"] \arrow[d, "\overline{\psi}"'] & X \arrow[d, "f"] \\ B \arrow[r, "r_B"] & Y \end{tikzcd}\right),$$

where the horizontal arrows are understood to determine objects and the vertical arrows to determine morphisms in the respective categories.

Lemma 1.1.18. *The functor $U\colon R_{\mathrm{sSet}} \to \mathrm{Fun}(\Delta[1], \mathrm{sSet})$ has a left adjoint $(-)_+^{(-)}$ which sends $(A \to X)$ to the retractive space $(X \to A \coprod X \to X)$.*

Proof. For each simplicial set X there is an adjunction

$$\begin{tikzcd} \mathrm{sSet}_{/X} \arrow[r, shift left, "(-)_+^X"] \arrow[r, phantom, "\perp"] & R_X, \arrow[l, shift left, "U"] \end{tikzcd}$$

in which the left adjoint sends $(A \to X)$ to the retractive space $(X \to A \coprod X \to X)$. For each map of simplicial sets $f\colon X \to Y$ there is a natural isomorphism of functors $f_!((A \to X)_+^X) \cong (f_!(A \to X))_+^Y\colon \mathrm{sSet}_{/X} \to R_Y$, which defines a pseudonatural transformation between the pseudofunctors $\mathrm{sSet}_{/(-)}, R_{(-)}\colon \mathrm{sSet} \to \mathbf{Adj}$. Via the Grothendieck construction we obtain an adjunction

$$\begin{tikzcd} \mathrm{Fun}(\Delta[1], \mathrm{sSet}) \arrow[r, shift left, "(-)_+^{(-)}"] \arrow[r, phantom, "\perp"] & R_{\mathrm{sSet}} \arrow[l, shift left, "U"] \end{tikzcd}$$

in which both adjoints commute with the projection functors to sSet. □

We now turn to the issue of the model structure on the global category R_{sSet}. By Lemma 1.1.11 the assignment $X \mapsto R_X$ determines a pseudofunctor $R_{(-)}\colon \mathrm{sSet} \to \mathbf{Mod}$. The prerequisites for the existence of the integral model structure on the Grothendieck construction $\int_{\mathrm{sSet}} R_X \cong R_{\mathrm{sSet}}$ are seen to be met by the following

Lemma 1.1.19. *The pseudofunctor $R_{(-)}\colon \mathrm{sSet} \to \mathbf{Mod}$ is proper and relative.*

Proof. This is a formal consequence of Lemma 1.1.11; indeed relativeness is immediately implied from that result.

For any simplicial set X, all objects of R_X are cofibrant. Ken Brown's Lemma ([Hov99, Lemma 1.1.12]) thus implies that the pushforward functor $f_!\colon R_X \to R_Y$ preserves weak equivalences for any map $f\colon X \to Y$. In particular, the pseudofunctor $R_{(-)}$ is left proper.

If $f\colon X \to Y$ is an acyclic fibration then Lemma 1.1.11 implies that $f^*\colon R_Y \to R_X$ is both a right and a left Quillen functor and so preserves all weak equivalences. This establishes right properness. □

Theorem 1.1.20. *The category R_{sSet} is equipped with the* global model structure *for which a map of pairs $(f, \psi)\colon (X, A) \to (Y, B)$ is*

- *a weak equivalence if $f\colon X \to Y$ and $\psi\colon f_!A \to B$ are weak equivalences in* sSet *and R_Y respectively;*
- *a cofibration if $f\colon X \to Y$ and $\psi\colon f_!A \to B$ are cofibrations in* sSet *and R_Y respectively; and*
- *a fibration if $f\colon X \to Y$ and $\psi^\vee\colon A \to f^*B$ are fibrations in* sSet *and R_X respectively, where $\psi^\vee$ is the $(f_! \dashv f^*)$-adjunct of ψ.*

Proof. The integral model structure on $\int_{\mathrm{sSet}} R_{(-)} \cong R_{\mathrm{sSet}}$ exists by Lemma 1.1.19 and Theorem A.4.6. Note that no cofibrant replacements are necessary in the characterisation of weak equivalences since all objects of R_X are cofibrant for all X. □

Corollary 1.1.21. *The adjunctions* $(\mathrm{base} \dashv 0_{(-)})$ *and* $(0_{(-)} \dashv \mathrm{base})$ *are Quillen.*

Corollary 1.1.22. *For any simplicial set* X *the inclusion functor* $\imath_X \colon R_X \to R_{\mathrm{sSet}}$ *is left and right Quillen.*

Corollary 1.1.23. *The adjunction* $((-)_+^{(-)} \dashv U)$ *is Quillen with respect to both the injective and projective model structures on* $\mathrm{Fun}(\Delta[1], \mathrm{sSet})$.

Lemma 1.1.24. *The global model structure on* R_{sSet} *is left proper and combinatorial.*

Proof. We first prove left properness. Consider a pushout diagram in R_{sSet}

$$\begin{tikzcd}
(X, A) \arrow[r, tail, "{(f,\varphi)}"] \arrow[d, "{(g,\psi)}"', "\wr"] & (Y, B) \arrow[d, "{(g',\psi')}"] \\
(Z, C) \arrow[r, "{(f',\varphi')}"] & (P, Q)
\end{tikzcd}$$

in which (f, φ) is a cofibration and (g, ψ) is a weak equivalence. The simplicial set P is the pushout of the span $Z \leftarrow X \to Y$ and $Q \in R_P$ is the pushout

$$\begin{tikzcd}
g'_! f_! A \cong f'_! g_! A \arrow[r, "g'_!(\varphi)"] \arrow[d, "f'_!(\psi)"'] & g'_! B \arrow[d, "\psi'"] \\
f'_! C \arrow[r, "\varphi'"] & Q.
\end{tikzcd}$$

Left properness of the Kan model structure implies that $g' \colon Y \to P$ is a weak equivalence. By hypothesis $\psi \colon f_! A \to B$ is a cofibration in R_Y and $\psi \colon g_! A \to C$ is a weak equivalence in R_C. As observed in the proof of Lemma 1.1.19 the pushforward functors preserve cofibrations and weak equivalences so that $g'_!(\phi)$ and $f'_!(\psi)$ are respectively a cofibration and weak equivalence in R_P. Since R_P is left proper the above pushout diagram implies that ψ' is a weak equivalence so that the map of pairs $(g', \psi') \colon (Y, B) \to (P, Q)$ is a weak equivalence for the global model structure. This establishes left properness.

To see that the global model structure is combinatorial it is sufficient to establish cofibrant generation since R_{sSet} is is locally presentable by Lemma 1.1.15. The small object argument automatically holds in any locally presentable category so cofibrant generation is guaranteed if we can find sets $\mathcal{I}^{\mathrm{ret}}$ and $\mathcal{J}^{\mathrm{ret}}$ such that $\mathcal{F}\mathrm{ib} \cap \mathcal{W} = \mathrm{rlp}(\mathcal{I}^{\mathrm{ret}})$ and $\mathcal{F}\mathrm{ib} = \mathrm{rlp}(\mathcal{J}^{\mathrm{ret}})$. As a candidate set of generating cofibrations we take the union of the sets

- $\mathrm{const}(\mathcal{I}_{\mathrm{Kan}})$ of morphisms obtained by applying $0_{(-)}$ to $\mathcal{I}_{\mathrm{Kan}}$; and
- $\mathcal{I}^+_{\mathrm{Kan}}$ of morphisms

$$\begin{tikzcd}
\Delta[n] \arrow[r, "q_2"] \arrow[d, "q_2"'] & \Delta[n] \coprod \Delta[n] \arrow[d, "\mathrm{id}+\mathrm{id}"] \\
\partial\Delta[n] \coprod \Delta[n] \arrow[ur, "i_n \coprod \mathrm{id}"] \arrow[r, "i_n+\mathrm{id}"'] & \Delta[n]
\end{tikzcd}$$

  for $n \in \mathbb{N}$, where $i_n \colon \partial\Delta[n] \to \Delta[n]$ is the boundary inclusion and q_2 is the coprojection onto the second summand.

Since $\mathcal{I}^{\mathrm{ret}}$ is a set of cofibrations, it follows that $\mathcal{F}\mathrm{ib} \cap \mathcal{W} \subset \mathrm{rlp}(\mathcal{I}^{\mathrm{ret}})$. To show the reverse inclusion, suppose that $(f, \psi) \colon (X, A) \to (Y, B)$ is a morphism in R_{sSet} with the right lifting property with respect to $\mathcal{I}^{\mathrm{ret}}$. The right lifting property with respect to $\mathrm{const}(\mathcal{I}_{\mathrm{Kan}})$ implies, via the adjunction of Lemma 1.1.17, that the map of base spaces $f\colon X \to Y$ is an acyclic fibration. Since $f\colon X \to Y$ is an acyclic fibration, right properness of the pseudofunctor $R_{(-)}$ implies that $\psi\colon f_!A \to B$ is a weak equivalence precisely if its adjunct $\psi^\vee\colon A \to f^*B$ is. It therefore remains show that $\psi^\vee\colon A \to f^*B$ is an acyclic fibration. by Lemma 1.1.7 $\psi^\vee$ is an acyclic fibration if and only if it has the right lifting property with respect to all maps in $\mathcal{I}^X_{\mathrm{Kan}}$, which are all of the form

$$\left(\begin{tikzcd} X \arrow[r] \arrow[d] & \Delta[n]\coprod X \arrow[d, "\sigma+\mathrm{id}_X"] \\ \partial\Delta[n]\coprod X \arrow[ur, "i_n\coprod \mathrm{id}_X"] \arrow[r, "\sigma|_{\partial\Delta[n]}+\mathrm{id}_X"'] & X \end{tikzcd}\right) = \sigma_! \left(\begin{tikzcd} \Delta[n] \arrow[r] \arrow[d] & \Delta[n]\coprod \Delta[n] \arrow[d, "\mathrm{id}+\mathrm{id}"] \\ \partial\Delta[n]\coprod \Delta[n] \arrow[ur, "i_n\coprod \mathrm{id}"] \arrow[r, "i_n+\mathrm{id}"'] & \Delta[n] \end{tikzcd}\right)$$

for $\sigma\colon \Delta[n] \to X$ an n-simplex in X and $i_n\colon \partial\Delta[n] \to \Delta[n]$ the boundary inclusion, so that all maps in $\mathrm{I}^X_{\mathrm{Kan}}$ are obtained via base change from maps in $\mathcal{I}^+_{\mathrm{Kan}}$. Liftings of the diagram

$$\begin{tikzcd} \partial\Delta[n]\coprod X \arrow[r, "\alpha"] \arrow[d, "i_n\coprod \mathrm{id}_X"'] & A \arrow[d, "\psi^\vee"] \\ \Delta[n]\coprod X \arrow[r, "\beta"] & f^*B \end{tikzcd}$$

in R_X are equivalent to liftings of the diagram

$$\begin{tikzcd} (\Delta[n], \partial\Delta[n]\coprod\Delta[n]) \arrow[r, "(\sigma,\alpha)"] \arrow[d] & (X, A) \arrow[d, "(f,\psi)"] \\ (\Delta[n], \Delta[n]\coprod\Delta[n]) \arrow[r, "(f\sigma,\beta^\vee)"] & (X, B) \end{tikzcd}$$

in R_{sSet}. Since the latter diagrams admit lifts by hypothesis, we conclude that $\psi^\vee$ is an acyclic fibration and therefore that $\mathcal{F}\mathrm{ib}\cap\mathcal{W} = \mathrm{rlp}(\mathcal{I}^{\mathrm{ret}})$.

An similar argument shows that the fibrations in R_{sSet} are characterised by the right lifting property with respect to the set $\mathcal{J}^{\mathrm{ret}} := \mathrm{const}(\mathcal{J}_{\mathrm{Kan}}) \cup \mathcal{J}^+_{\mathrm{Kan}}$, where $\mathcal{J}^+_{\mathrm{Kan}}$ is defined similarly to $\mathcal{I}^+_{\mathrm{Kan}}$ by taking the horn inclusions $h^n_k\colon \Lambda^n_k \to \Delta[n]$ instead of the boundary inclusions. □

We conclude this section by defining external smash products of retractive spaces. The external smash product makes R_{sSet} into a symmetric monoidal model category, in contrast with the monoidal categories $(R_X, \wedge_X)$ for fixed X (see Remark 1.1.10).

Definition 1.1.25. For objects $(X, A), (Y, B) \in R_{\mathrm{sSet}}$ we define the *external smash product* as the colimit $A \barwedge B$ of the diagram of simplicial sets

$$\begin{tikzcd} X\times Y \arrow[r] \arrow[d] & X\times B \arrow[dr] \arrow[dd] & \\ A\times Y \arrow[rr] \arrow[dr] & & A\times B \\ & X\times Y. & \end{tikzcd}$$

The section and retraction maps of A and B natrually equip $A \overline{\wedge} B$ with the structure of a retractive space over $X \times Y$ so that this assignment gives rise to a bifunctor

$$\begin{aligned}(-)\overline{\wedge}(-)\colon R_{\mathrm{sSet}} \times R_{\mathrm{sSet}} &\longrightarrow R_{\mathrm{sSet}} \\ \big((X,A),(Y,B)\big) &\longmapsto (X \times Y, A \overline{\wedge} B)\end{aligned}$$

covering the cartesian product on sSet.

Lemma 1.1.26. *The external smash product defines a closed symmetric monoidal structure on R_{sSet}. The projection functor* base: $R_{\mathrm{sSet}} \to$ sSet *is strongly closed monoidal.*

Proof. The definition of the external smash product is manifestly symmetric. The monoidal unit is $S^0 = (* \to * \coprod * \to *)$ and associators for the external smash product are obtained using the universality of colimits in sSet together with the fact that iterated colimits commute. Universality of colimits in sSet also implies that $(-)\overline{\wedge}(-)$ preserves colimits in each argument. Therefore, for any $(X,A) \in R_{\mathrm{sSet}}$ the endofunctor

$$A \overline{\wedge} (-)\colon R_{\mathrm{sSet}} \longrightarrow R_{\mathrm{sSet}}$$

preserves colimits so that the Adjoint Functor Theorem implies the existence of a right adjoint $\overline{F}(A,-)$ to $A \overline{\wedge} (-)$ so that the monoidal structure is closed.

It was already remarked in Definition 1.1.25 that base: $R_{\mathrm{sSet}} \to$ sSet is a strong monoidal functor. For objects (X,A) and (Y,B) of R_{sSet} and $Z \in$ sSet, maps of simplicial sets

$$Z \longrightarrow \mathrm{base}\big(\overline{F}(A,B)\big)$$

are naturally equivalent to maps $0_{X\times Z} \cong A \overline{\wedge} 0_Z \to (Y,B)$ in R_{sSet}. This shows that $\mathrm{base}\big(\overline{F}(A,-)\big)$ is right adjoint to $0_{X\times(-)}$. But the functor $0_{X\times(-)}$ is left adjoint to $\mathrm{base}(-)^X$ so by essential uniqueness of adjoints there is a natural isomorphism $\mathrm{base}\big(\overline{F}(A,-)\big) \cong \mathrm{base}(-)^X$ which proves the assertion. □

Theorem 1.1.27. *The global model structure on R_{sSet} is a symmetric monoidal model category with respect to the external smash product.*

Proof. In Lemma 1.1.26 we showed that the external smash product bifunctor $\overline{\wedge}$ defines a closed symmetric monoidal structure and so determines an adjunction of two variables. The monoidal unit S^0 is cofibrant so we need only verify the pushout-product axiom for $\overline{\wedge}$.

Recall the sets of morphisms $\mathcal{I}^{\mathrm{ret}}$ and $\mathcal{J}^{\mathrm{ret}}$ from (the proof of) Lemma 1.1.24. We consider the set of pushout-products $\mathcal{I}^{\mathrm{ret}} \,\square\, \mathcal{I}^{\mathrm{ret}}$. For $i,j \in \mathcal{I}^{\mathrm{ret}}$ there are two cases to consider:

(i) At least one of i,j is in $\mathrm{const}(\mathcal{I}_{\mathrm{Kan}})$. Since the external smash product of $0_X = (X \to X \to X)$ with any $(Y \to A \to Y)$ is $0_{X\times Y}$ we have

$$i \,\square\, j = 0_{(-)}\big(\mathrm{base}(i) \boxtimes \mathrm{base}(j)\big),$$

where $\boxtimes$ denotes the pushout-product for the cartesian product on sSet. Since $0_{(-)}$ is left Quillen and $\mathrm{base}(i) \boxtimes \mathrm{base}(j)$ is a cofibration (since the cartesian product on sSet satisfies the pushout-product axiom) it follows that that $i \,\square\, j$ is a cofibration in this case.

(ii) Both of i and j are in $\mathcal{I}^+_{\mathrm{Kan}}$. In this case fix

$$i = \left(\begin{tikzcd} \Delta[k] \arrow[r] \arrow[d] & \Delta[k] \coprod \Delta[k] \arrow[d, "\mathrm{id}+\mathrm{id}"] \\ \partial\Delta[k] \coprod \Delta[k] \arrow[ur, "i_k \coprod \mathrm{id}"] \arrow[r, "i_k+\mathrm{id}"'] & \Delta[k] \end{tikzcd}\right), \quad j = \left(\begin{tikzcd} \Delta[l] \arrow[r] \arrow[d] & \Delta[l] \coprod \Delta[l] \arrow[d, "\mathrm{id}+\mathrm{id}"] \\ \partial\Delta[l] \coprod \Delta[l] \arrow[ur, "i_l \coprod \mathrm{id}"] \arrow[r, "i_l+\mathrm{id}"'] & \Delta[l] \end{tikzcd}\right).$$

Calculating the external smash products, the pushout-product of i and j is obtained by applying the left Quillen free basepoint functor

$$(-)_+^{\Delta[k]\times\Delta[l]} \colon \mathrm{sSet}_{/\Delta[k]\times\Delta[l]} \longrightarrow R_{\Delta[k]\times\Delta[l]}$$

to the morphism marked $\overline{i \,\square\, j}$ in the diagram

$$\begin{tikzcd} \partial\Delta[k] \times \partial\Delta[l] \arrow[r, tail] \arrow[d, tail] & \partial\Delta[k] \times \Delta[l] \arrow[d, tail] \arrow[dr, tail] & \\ \Delta[k] \times \partial\Delta[l] \arrow[r, tail] & P \arrow[r, "\overline{i \,\square\, j}"] & \Delta[k] \times \Delta[l], \end{tikzcd}$$

where P denotes the pushout of the left hand square, cofibrations are as marked and the structure maps to $\Delta[k] \times \Delta[l]$ are the obvious ones. The map $\overline{i \,\square\, j}$ is a cofibration by the pushout-product axiom for the cartesian product on sSet so that $i \,\square\, j$ is a cofibration in this case.

We now turn to the set of pushout-products $\mathcal{I}^{\mathrm{ret}} \,\square\, \mathcal{J}^{\mathrm{ret}}$. As before, there are two cases to consider:

(a) If $i \in \mathrm{const}(\mathcal{I}_{\mathrm{Kan}})$ or $j \in \mathrm{const}(\mathcal{J}_{\mathrm{Kan}})$ then $i \,\square\, j$ is an acyclic cofibration by essentially the same argument as case (i) above. In fact, in both of the cases $i \in \mathrm{const}(\mathcal{I}_{\mathrm{Kan}})$, $j \in \mathcal{J}^+_{\mathrm{Kan}}$ and $i \in \mathcal{I}^+_{\mathrm{Kan}}$, $j \in \mathrm{const}(\mathcal{J}_{\mathrm{Kan}})$ the pushout-product is an isomorphism since either $\mathrm{base}(i)$ or $\mathrm{base}(j)$ is constant.

(b) If $i \in \mathcal{I}^+_{\mathrm{Kan}}$ and $j \in \mathcal{J}^+_{\mathrm{Kan}}$. Fixing

$$i = \left(\begin{tikzcd} \Delta[k] \arrow[r] \arrow[d] & \Delta[k] \coprod \Delta[k] \arrow[d, "\mathrm{id}+\mathrm{id}"] \\ \partial\Delta[k] \coprod \Delta[k] \arrow[ur, "i_k \coprod \mathrm{id}"] \arrow[r, "i_k+\mathrm{id}"'] & \Delta[k] \end{tikzcd}\right), \quad j = \left(\begin{tikzcd} \Delta[l] \arrow[r] \arrow[d] & \Delta[l] \coprod \Delta[l] \arrow[d, "\mathrm{id}+\mathrm{id}"] \\ \Lambda^l_p \coprod \Delta[l] \arrow[ur, "h^l_p \coprod \mathrm{id}"] \arrow[r, "h^l_p+\mathrm{id}"'] & \Delta[l] \end{tikzcd}\right)$$

for concreteness, we once again compute that the pushout-product is obtained from a morphism $\overline{i \,\square\, j}$ in $\mathrm{sSet}_{/\Delta[k]\times\Delta[l]}$ by applying the left Quillen functor $(-)_+^{\Delta[k]\times\Delta[l]}$ to

$$\begin{tikzcd} \partial\Delta[k] \times \Lambda^l_p \arrow[r, tail] \arrow[d, tail, "\wr"'] & \partial\Delta[k] \times \Delta[l] \arrow[d, tail, "\wr"'] \arrow[dr, tail, "\sim"] & \\ \Delta[k] \times \Lambda^l_p \arrow[r, tail] & P \arrow[r, "\overline{i \,\square\, j}"] & \Delta[k] \times \Delta[l]. \end{tikzcd}$$

By the 2-out-of-3 axiom, the morphism $\overline{i \square j}$ is a weak equivalence, which proves this case.

We have verified that the sets of pushout-products $\mathcal{I}^{\mathrm{ret}} \square \mathcal{I}^{\mathrm{ret}}$ and $\mathcal{I}^{\mathrm{ret}} \square \mathcal{J}^{\mathrm{ret}}$ consist of cofibrations and acyclic cofibrations respectively. By symmetry, Lemma 1.1.24 and cofibrant generation, the pushout-product axiom for $\overline{\wedge}$ is satisfied. □

Corollary 1.1.28. *The global model structure on* R_{sSet} *is* sSet_**-enriched.*

Proof. The inclusion functor $\mathrm{sSet}_* = R_* \hookrightarrow R_{\mathrm{sSet}}$ is a strong monoidal left Quillen functor. Defining the sSet_*-tensoring by $K \overline{\otimes} (X, A) := K \overline{\wedge} A$, the assertion follows easily from the Theorem. □

Finally, we record some results relating the external smash products to the internal smash products of Remark 1.1.10. They are proven by unwinding the definitions and comparing colimit diagrams, using universality of colimits in sSet.

Lemma 1.1.29. *Let* $X \in \mathrm{sSet}$ *with* $\Delta_X \colon X \to X \times X$ *the diagonal map. For* $A, B \in R_X$ *there is an isomorphism*

$$\Delta_X^*(A \overline{\wedge} B) \xrightarrow{\cong} A \wedge_X B$$

of objects of R_{sSet} *covering* id_X *in* sSet *which is natural in* A *and* B*.*

Lemma 1.1.30. *For* $X, Y \in \mathrm{sSet}$ *let* $\mathrm{pr}_1 \colon X \times Y \to X$ *and* $\mathrm{pr}_2 \colon X \times Y \to Y$ *denote the projection maps. For* $A \in R_X$ *and* $B \in R_Y$ *there is an isomorphism*

$$A \overline{\wedge} B \xrightarrow{\cong} \mathrm{pr}_1^* A \wedge_{X \times Y} \mathrm{pr}_2^* B$$

of objects of R_{sSet} *which covers* $\mathrm{id}_{X \times Y}$ *and is natural in* A *and* B*.*

Remark 1.1.31. For any $X \in \mathrm{sSet}$ the sSet_*-tensoring on R_X coincides with the external smash product, in the sense that for $K \in \mathrm{sSet}_*$ and $A \in R_X$ there are natural isomorphisms

$$K \otimes_X A \cong K \overline{\wedge} A \cong X^* K \wedge_X A$$

between objects of R_X, where $X \colon X \to *$ is the terminal map (so $X^* = X \times K$ with the induced structure as a retractive space over X). The first isomorphism is seen by comparing (1.2) with Definition 1.1.25 and the second is immediate from Lemma 1.1.30. In what follows, we switch between these three equivalent desriptions of the sSet_*-tensoring without further comment.

1.2 Stabilisation

In this section, we pass from the unstable theory of parametrised retractive spaces just discussed to the stable theory of parametrised spectra. As in the unstable case, we consider both the local and global settings. In the local setting the objects of study are families of spectra parametrised by a fixed base space X, where morphisms are constrained to cover the identity on the base. By taking (stable) homotopy fibres, such families give rise to local coefficient systems of stable homotopy types

$$\Pi X \longrightarrow \mathcal{S}p,$$

so represent cohomology theories twisted by X. In the global setting, the fixed base space restriction is removed so that we obtain a homotopy theory capturing twisted cohomology theories with all possible twists.

Passage from unstable to stable homotopy theory is implemented by stabilisation machines. Our work in §1.1 shows that both the local and global unstable theories are robust enough to apply two such machines: *sequential* and *symmetric* stabilisation. In this section, we study the sequential and symmetric stabilisations of both the local and global theories of retractive spaces and discuss various interrelationships between them. A recapitulation of the important properties of the stabilisation machines that we employ is given in §A.3.

1.2.1 Local Stabilisation

For a fixed base space $X \in \mathrm{sSet}$ we have seen that the category R_X of retractive spaces over X supports a model structure with a variety of useful properties. In particular, the sSet_*-enrichment gives rise to Quillen endofunctors that model suspension and looping at the level of the homotopy category. Recall that we fix $S^1 := \Delta[1]/\partial\Delta[1]$ regarded as a pointed simplicial set in the unique way.

Lemma 1.2.1. *For any $X \in \mathrm{sSet}$, the sSet_*-tensoring $\Sigma_X := S^1 \oslash_X (-)$ determines a Quillen adjunction*

$$\begin{tikzcd} R_X \arrow[r, shift left=2, "\Sigma_X"] \arrow[r, phantom, "\perp"] & R_X \arrow[l, shift left=2, "\Omega_X"] \end{tikzcd}$$

in which Σ_X models suspension on $Ho(R_X)$.

Proof. The endofunctor Σ_X is left adjoint to the endofunctor Ω_X which sends $A \in R_X$ to its sSet_*-powering with S^1. Since S^1 is cofibrant in sSet_*, the pushout-product axiom implies that $\Sigma_X = S^1 \oslash_X (-)$ preserves cofibrations and acyclic cofibrations so that the adjunction $(\Sigma_X \dashv \Omega_X)$ is Quillen.

Note that any object $A \in R_X$ is cofibrant, and that S^1 can also be expressed as the cokernel of the map of pointed simplicial sets $\partial\Delta[1]_+ \hookrightarrow \Delta[1]_+$. Moreover, $\partial\Delta[1]_+ \oslash_X A = A \vee_X A$ and $\Delta[1]_+ \oslash_X A = A \otimes_X \Delta[1]$ is a good cylinder object for A. Taking the cokernel of the map $\partial\Delta[1]_+ \oslash_X A \to \Delta[1]_+ \oslash_X A$ therefore yields the suspension of A in $Ho(R_X)$ (as defined in [Qui67]). But this cokernel coincides with $\Sigma_X A$ so the result is proven. □

Lemma 1.2.2. *For any map of simplicial sets $f\colon X \to Y$ there are natural isomorphisms of functors $f_!\Sigma_X \cong \Sigma_Y f_!$ and $f^*\Sigma_Y \cong \Sigma_X f^*$.*

Proof. This follows at once from Lemma 1.1.5. □

Remark 1.2.3. The compatibilities between the base change functors and fibrewise suspension functors of the Lemma are special cases of some general compatibility relations between base change functors and fibrewise smash products. In general, if $B \in R_X$ and $A, C \in R_Y$ with $f\colon X \to Y$ a map of base spaces one can show that

$$f_!(f^*A \wedge_X B) \cong A \wedge_Y f_!B \quad \text{and} \quad f^*\big(A \wedge_Y C\big) \cong f^*A \wedge_X f^*C$$

[MS06, Proposition 2.2.2]. The only non-formal ingredient used to prove these compatibility relations is the fact that colimits commute with pullbacks in sSet. The Lemma is recovered by observing that the functors $S^1 \oslash_X (-)$ and $X^*S^1 \wedge_X (-)$ are naturally isomorphic, where $X\colon X \to *$ is the terminal map.

Local Sequential Spectra. In our first approach to local parametrised spectra, we apply the sequential stabilisation machine to the endofunctor Σ_X on R_X for a fixed base space $X \in \mathrm{sSet}$. Results in this paragraph that are stated without proof are particular instances of general results recalled in §A.3.1.

Definition 1.2.4. A *sequential X-spectrum* is a sequence $A = \{A_n\}_{n\in\mathbb{N}}$ of retractive spaces over X equipped with maps $\sigma_n^A\colon \Sigma_X A_n \to A_{n+1}$ for each $n \in \mathbb{N}$. A morphism of sequential X-spectra $f\colon A \to B$ is a sequence of maps $f_n\colon A_n \to B_n$ compatible with the structure maps in the sense that for each n the diagram

$$\begin{tikzcd} \Sigma_X A_n \arrow[r, "\Sigma_X f_n"] \arrow[d, "\sigma_n^A"'] & \Sigma_X B_n \arrow[d, "\sigma_n^B"] \\ A_{n+1} \arrow[r, "f_{n+1}"] & B_{n+1} \end{tikzcd}$$

commutes. The category of sequential X-spectra is denoted $\mathrm{Sp}_X^{\mathbb{N}}$.

Remark 1.2.5. For each $X \in \mathrm{sSet}$ the category $\mathrm{Sp}_X^{\mathbb{N}}$ is locally presentable and limits and colimits of sequential X-spectra are computed termwise in R_X.

For each $n \in \mathbb{N}$ there is a free-forgetful adjunction $(F_n^X \dashv \mathrm{Ev}_n)\colon R_X \to \mathrm{Sp}_X^{\mathbb{N}}$. The right adjoint $\mathrm{Ev}_n\colon A \mapsto A_n$ extracts the n-th term of a sequential spectrum, and the left adjoint F_n^X sends a retractive space $Y \in R_X$ to the sequential X-spectrum freely generated by Y at level n:

$$F_n^X(Y)_m := \begin{cases} 0_X & \text{for } m < n \\ \Sigma_X^{m-n} Y & \text{otherwise,} \end{cases}$$

which is equipped with the obvious structure maps.

The category of sequential X-spectra inherits a projective model structure from R_X. With respect to the *projective model structure* on $\mathrm{Sp}_X^{\mathbb{N}}$, a morphism $f\colon A \to B$ of sequential X-spectra is

- a weak equivalence if it is a *level weak equivalence*; namely each $f_n\colon A_n \to B_n$ is a weak equivalence in R_X;
- a fibration if it is a *level fibration*; namely each $f_n\colon A_n \to B_n$ is a fibration: and
- a cofibration if it has the left lifting property with respect to all level acyclic fibrations.

The projective model structure is left proper and combinatorial, with generating sets of cofibrations and acyclic cofibrations given respectively by

$$\mathcal{I}_{\mathbb{N}}^X := \bigcup_{\mathbb{N}} F_n^X(\mathcal{I}_{\mathrm{Kan}}^X) \quad \text{and} \quad \mathcal{J}_{\mathbb{N}}^X := \bigcup_{\mathbb{N}} F_n^X(\mathcal{J}_{\mathrm{Kan}}^X),$$

with $\mathcal{I}_{\mathrm{Kan}}^X$ and $\mathcal{J}_{\mathrm{Kan}}^X$ as per Lemma 1.1.7. The projective model structure, which we denote $\mathrm{PSp}_X^{\mathbb{N}}$ when model structures are to be understood, inherits sSet_*-enrichment from R_X. The sSet_*-tensoring is defined by setting $(K \oslash_X A)_n := K \oslash_X A_n$ equipped with structure maps induced from the structure maps of A via the natural isomorphisms $\Sigma_X(K \oslash_X A_n) \cong K \oslash_X \Sigma_X A_n$. Note that for $K = S^1$ the structure maps of $S^1 \oslash_X A$ differ from those of $\Sigma_X A$ by the twist map $S^1 \wedge S^1 \to S^1 \wedge S^1$ which interchanges the smash factors.

Stabilisation proper—by which we mean passing to a model category in which Σ_X is a Quillen equivalence—is implemented by left Bousfield localisation of the projective model structure. The set of morphisms to be localised is

$$\mathbf{S}^X_{\mathbb{N}} := \left\{ \zeta^X_n(C) \colon F^X_{n+1}(\Sigma_X C) \longrightarrow F^X_n(C) \right\},$$

where $\zeta^X_n(A)$ is adjoint to the identity $\Sigma_X A \to \Sigma_X A = \mathrm{Ev}_{n+1} F^X_n(A)$, $n \in \mathbb{N}$, and C ranges over the domains and codomains of morphisms in $\mathcal{J}^X_{\mathrm{Kan}}$. The *stable model structure* on sequential X-spectra is the left Bousfield localisation

$$\mathrm{Sp}^{\mathbb{N}}_X := L_{\mathbf{S}^X_{\mathbb{N}}} \mathrm{PSp}^{\mathbb{N}}_X,$$

where it is to be understood that $\mathrm{Sp}^{\mathbb{N}}_X$ always denotes the stable model structure. The cofibrations, fibrations and weak equivalences of $\mathrm{Sp}^{\mathbb{N}}_X$ are called *stable cofibrations*, *stable fibrations* and *stable weak equivalences* respectively; note that the classes of stable cofibrations and projective cofibrations coincide. We summarise some of the important features of the stable model structure on sequential X-spectra:

(i) $\mathrm{Sp}^{\mathbb{N}}_X$ is a left proper combinatorial sSet_*-model category.

(ii) The functor Σ_X prolongs to a Quillen equivalence on $\mathrm{Sp}^{\mathbb{N}}_X$. Note, however, that this is *not* the suspension functor, which is instead given by the sSet_*-tensoring $S^1 \oslash_X (-)$ and incorporates the twist map $S^1 \wedge S^1 \to S^1 \wedge S^1$.

(iii) Stably fibrant objects are precisely the *fibrant Ω_X-spectra*: sequential X-spectra A such that each $A_n \in R_X$ is fibrant and the adjoint structure maps

$$\sigma^\vee_n \colon A_n \longrightarrow \Omega_X A_{n+1}$$

are weak equivalences in R_X for all $n \in \mathbb{N}$.

(iv) Stable weak equivalences between fibrant Ω_X-spectra are precisely the levelwise weak equivalences between these objects.

(v) For any $Y \in R_X$ and $n \in \mathbb{N}$ the map $\zeta^X_n(Y) \colon F^X_{n+1}(\Sigma_X Y) \to F^X_n(Y)$ is a stable weak equivalence.

Compatibility of base change with suspension (Lemmas 1.1.5 and 1.2.2) and Corollary A.3.16 together imply

Lemma 1.2.6. *For any map of simplicial sets $f \colon X \to Y$ there is a sSet_*-enriched Quillen adjunction*

$$\begin{tikzcd} \mathrm{Sp}^{\mathbb{N}}_X \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\mathbb{N}}_Y \arrow[l, shift left, "f^*"] \end{tikzcd}$$

in which both $f_!$ and f^* are defined by levelwise application of the base change functors $(f_! \dashv f^*) \colon R_X \to R_Y$. Moreover

(i) *if f is a weak equivalence then $(f_! \dashv f^*)$ is a sSet_*-Quillen equivalence;*

(ii) *if f is a fibration then there is a prolonged base change Quillen adjunction*

$$\begin{tikzcd} \mathrm{Sp}^{\mathbb{N}}_Y \arrow[r, shift left, "f^*"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\mathbb{N}}_X \arrow[l, shift left, "f_*"] \end{tikzcd}$$

respecting sSet_**-enrichments; and*

(iii) *if f is an acyclic fibration then the prolonged base change adjunctions $(f_! \dashv f^*)$ and $(f^* \dashv f_*)$ are* sSet_**-Quillen equivalences.*

Remark 1.2.7. The assertions of Lemma 1.2.6 also hold for the projective model structures.

For (unparametrised) sequential spectra it is a classical fact that stable weak equivalences are detected by stable homotopy groups. That is, a map of sequential spectra $f\colon A \to B$ is a stable weak equivalence in $\mathrm{Sp}_*^{\mathbb{N}}$ precisely if the induced map on stable homotopy groups is an isomorphism. Recall that the stable homotopy groups of a sequential spectrum A are defined as

$$\pi_n^{\mathrm{st}}(A) := \operatorname*{colim}_k \pi_{n+k}(A_k), \quad n \in \mathbb{Z}, \tag{1.3}$$

where the colimit is over the maps $\pi_{n+k}(A_k) \to \pi_{n+k+1}(\Sigma A_{k+1}) \to \pi_{n+k+1}(A_{n+k+1})$ induced by the spectrum structure maps. In the case that A is a fibrant Ω-spectrum then we also have

$$\pi_n(A_k) = \pi_{n-k}^{\mathrm{st}}(A) \quad \text{for all } n,k \in \mathbb{N}. \tag{1.4}$$

In the parametrised case the situation is subtler.

Definition 1.2.8. A morphism $f\colon A \to B$ of sequential X-spectra is a *fibrewise stable equivalence* if $x^*(\mathcal{R}f)$ is a stable weak equivalence of spectra for all points $x\colon * \to X$, where $\mathcal{R}$ is any fibrant replacement functor on $\mathrm{Sp}_X^{\mathbb{N}}$.

Remark 1.2.9. Since $x^*\colon \mathrm{Sp}_X^{\mathbb{N}} \to \mathrm{Sp}^{\mathbb{N}}$ is a right Quillen functor, Ken Brown's Lemma implies that the defintion of fibrewise stable equivalences is independent of the choice of of fibrant replacement functor $\mathcal{R}$.

Lemma 1.2.10. *A morphism $f\colon A \to B$ of sequential X-spectra is a stable weak equivalence if and only if it is a fibrewise stable equivalence.*

Proof. It is obvious that stable weak equivalences are fibrewise stable equivalences. Conversely, suppose that $f\colon A \to B$ is a fibrewise stable equivalence of sequential X-spectra. Let $\{x_i\}$ be a collection of points indexed by the path components of X, so that $[x_i] = i \in \pi_0(X)$.

By the 2-out-of-3 property applied to the naturality diagram for the fibrant replacement functor $\mathcal{R}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\wr"'] & B \arrow[d, "\wr"] \\ \mathcal{R}(A) \arrow[r, "\mathcal{R}(f)"] & \mathcal{R}(B) \end{tikzcd}$$

we find that f is a stable weak equivalence precisely if $\mathcal{R}f$ is. Since $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are fibrant Ω_X-spectra, $\mathcal{R}(f)$ is a stable weak equivalence precisely if it is a levelwise weak equivalence. The structure maps $\mathcal{R}(A)_n \to X$ and $\mathcal{R}(B)_n \to X$ are fibrations for all $n \in \mathbb{N}$, so choosing $a_i \in \mathcal{R}(A)_n$ in the fibre over x_i and setting $b_i = \mathcal{R}(f)_n(a_i)$ the map $\mathcal{R}(f)_n\colon \mathcal{R}(A)_n \to \mathcal{R}(B)_n$ in R_X induces a morphism of long exact sequences of homotopy groups

$$\begin{tikzcd} \cdots \arrow[r] & \pi_k(x_i^*\mathcal{R}(A)_n, a_i) \arrow[r] \arrow[d] & \pi_k(\mathcal{R}(A)_n, a_i) \arrow[r] \arrow[d] & \pi_k(X, x_i) \arrow[r] \arrow[d, equal] & \cdots \\ \cdots \arrow[r] & \pi_k(x_i^*\mathcal{R}(B)_n, b_i) \arrow[r] & \pi_k(\mathcal{R}(B)_n, b_i) \arrow[r] & \pi_k(X, x_i) \arrow[r] & \cdots \end{tikzcd}$$

Since $x_i^*(\mathcal{R}(f))$ is by hypothesis a stable weak equivalence of fibrant Ω-spectra, the maps

$$\pi_k(x_i^*\mathcal{R}(A)_n, a_i) \longrightarrow \pi_k(x_i^*\mathcal{R}(B)_n, b_i)$$

are isomorphisms for all $n, k \in \mathbb{N}$ (compare (1.4)). By the Five Lemma, the maps

$$\pi_k(\mathcal{R}(A)_n, a_i) \longrightarrow \pi_k(x_i^*\mathcal{R}(B)_n, b_i)$$

are isomorphisms for all $n, k \in \mathbb{N}$, and since this is true for each x_i it follows that $\mathcal{R}(f)_n\colon \mathcal{R}(A)_n \to \mathcal{R}(B)_n$ is a weak equivalence. As n was arbitrary, $\mathcal{R}(f)$ is a levelwise weak equivalence and so the assertion is proven. □

Remark 1.2.11. The proof above implicitly uses a stabilised version of the classical fact that for a map of spaces $Y \to X$ and points $x_0, x_1\colon * \to X$ in the same path component, the homotopy fibres $\mathrm{hofib}_{x_0}(A)$ and $\mathrm{hofib}_{x_1}(A)$ are weakly equivalent. In particular if X is connected then a map of sequential X-spectra $f\colon A \to B$ is a stable equivalence precisely if $x^*(\mathcal{R}f)$ is a stable weak equivalence for *some* point $x\colon * \to X$.

With this characterisation of stable weak equivalences in hand we are able to prove that $\mathrm{Sp}_X^{\mathbb{N}}$ is a stable model category, meaning that the suspension endofunctor $S^1 \oslash_X (-)$ is a Quillen equivalence. Conceptually at least this is obvious, and indeed is the whole point of stabilisation, however the presence of the twist maps complicates matters (a further discussion of these issues is in [Hov01, §10]).

Lemma 1.2.12. *For any* $X \in \mathrm{sSet}$ *the model category* $\mathrm{Sp}_X^{\mathbb{N}}$ *is stable.*

Proof. We suppose without loss of generality that X is path-connected; indeed, if $|\pi_0(X)| > 1$ we simply apply our argument to each path component of X separately. We may further suppose that X is reduced by taking a fibrant replacement $X \to X'$, choosing some $x\colon * \to X'$ and passing to the 1st Eilenberg subcomplex of X' at x, defined as the pullback

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & X' \arrow[d] \\ * \arrow[r, "x"] & \mathrm{cosk}_0 X'. \end{tikzcd}$$

The maps $X'' \to X' \leftarrow X$ are weak equivalences [May67, Theorem 8.4] so that the induced base change adjunctions

$$\mathrm{Sp}_{X''}^{\mathbb{N}} \longrightarrow \mathrm{Sp}_{X'}^{\mathbb{N}} \longleftarrow \mathrm{Sp}_{X}^{\mathbb{N}}$$

are Quillen equivalences by Lemma 1.2.6 (only the left Quillen functors are shown). Since the base change functors commute with the sSet_*-tensoring, it follows from the 2-out-of-3 property that $S^1 \oslash_X (-)$ is a Quillen equivalence of $\mathrm{Sp}_X^{\mathbb{N}}$ precisely if $S^1 \oslash_{X''} (-)$ is a Quillen equivalence of $\mathrm{Sp}_{X''}^{\mathbb{N}}$.

Suppose now that X is a reduced simplicial set, and let $\pi_X\colon \mathbb{P}X \to X$ denote Kan's simplicial path fibration (recalled in Construction B.1.9). Since π_X is a fibration the pullback $\pi_X^*\colon \mathrm{Sp}_X^{\mathbb{N}} \to \mathrm{Sp}_{\mathbb{P}X}^{\mathbb{N}}$ is both right and left Quillen and so preserves any stable equivalence whatsoever. Let $\iota\colon * \to \mathbb{P}X$ represent the contant path at $x \in X$, so that $\pi_X \circ \iota = x\colon * \to X$, and let $\rho\colon \mathbb{P}X \to *$ be the terminal map. Since $\mathbb{P}X$ is weakly contractible, ι and ρ are weak equivalences. By Lemma 1.2.6 we have a diagram of left sSet$_*$-Quillen functors

$$\begin{tikzcd} \mathrm{Sp}^{\mathbb{N}} \arrow[r, "\iota_!"] & \mathrm{Sp}^{\mathbb{N}}_{\mathbb{P}X} \arrow[r, "\rho_!"] & \mathrm{Sp}^{\mathbb{N}} \\ & \mathrm{Sp}^{\mathbb{N}}_X \arrow[u, "\pi_X^*"] & \end{tikzcd}$$

in which the horizontal functors are Quillen equivalences. Let $A \in \mathrm{Sp}^{\mathbb{N}}_X$ and take a cofibrant $A^c \to A$ and a fibrant $A \to A^f$ replacement of A. Then $\pi_X^* A^c \to \pi_X^* A$ and $\pi_X^* A \to \pi_X^* A^f$ are stable equivalences. Let H^f be a fibrant model for $\rho_! \pi_X^* A^c$ in $\mathrm{Sp}^{\mathbb{N}}$. Since $(\rho_! \dashv \rho^*)$ is a Quillen equivalence the derived unit map $\eta \colon \pi_X^* A^c \to \rho^* H^f$ is a stable equivalence. Factor η into an acyclic cofibration followed by a fibration, so that we have a diagram of stable equivalences

$$\begin{tikzcd} & \pi_X^* A^c \arrow[dl, bend right] \arrow[d, tail] \arrow[dr, bend left, "\eta"] & \\ \pi_X^* A^f & (\pi_X^* A^c)^f \arrow[l] \arrow[r] & \rho^* H^f \end{tikzcd}$$

in $\mathrm{Sp}^{\mathbb{N}}_{\mathbb{P}X}$ in which the bottom row consists of fibrant objects, and the bottom left morphism exists by the lifting axioms and is a stable equivalence by the 2-out-of-3 property. Applying the right Quillen functor ι^* now gives a diagram of stable equivalences of spectra

$$\iota^* \pi_X^* A^f \longleftarrow \iota^*(\pi_X^* A^c)^f \longrightarrow \iota^* \rho^* H^f = H^f \longleftarrow \rho_! \pi_X^* A^c. \tag{1.5}$$

In particular, this shows that the stable homotopy groups of $\rho_! \pi_X^* A^c$ are the stable homotopy groups of the homotopy fibre of A at $x \colon * \to X$.

Since $\rho_!$ and π_X^* preserve sSet$_*$-tensors we have $\rho_! \pi_X^*(S^1 \oslash_X A^c) \cong S^1 \oslash \rho_! \pi_X^* A^c$. It is a classical fact that the suspension functor $S^1 \oslash (-)$ on $\mathrm{Sp}^{\mathbb{N}}$ has the effect of shifing stable homotopy groups up by one $\pi_*^{\mathrm{st}}(\Sigma P) \cong \pi_{*-1}^{\mathrm{st}}(P)$ so that $S^1 \oslash_X (-)$ has the effect of shifting the fibrewise homotopy groups up by one. This is what we would hope, so we have here a useful sanity check. Similarly, $S^1 \pitchfork (\iota^* \pi_X^* A^f) \cong \iota^* \pi_X^*(S^1 \pitchfork_X A^f)$ so that the sSet$_*$-powering with S^1 shifts fibrewise stable homotopy groups down by one. Putting this all together, if $(S^1 \oslash_X A^c)^f$ denotes a fibrant replacement for $S^1 \oslash_X A^c$ in $\mathrm{Sp}^{\mathbb{N}}_X$ we have that the derived unit map

$$A^c \longrightarrow S^1 \pitchfork_X \left(S^1 \oslash_X A^c\right) \longrightarrow S^1 \pitchfork_X \left(S^1 \oslash_X A^c\right)^f \tag{1.6}$$

is a stable homotopy equivalence on the homotopy fibre at x. Lemma 1.2.10 (and the observation made in Remark 1.2.11) implies that (1.6) is a stable equivalence in $\mathrm{Sp}^{\mathbb{N}}_X$. The right Quillen functor $S^1 \pitchfork_X (-) \colon \mathrm{Sp}^{\mathbb{N}}_X \to \mathrm{Sp}^{\mathbb{N}}_X$ preserves and reflects stable weak equivalences between fibrant objects—this is now easily checked either by using the characterisation given above or directly as per the proof of Lemma 1.2.10. We thus have that $(S^1 \oslash_X (-) \dashv S^1 \pitchfork_X (-))$ is a Quillen equivalence on $\mathrm{Sp}^{\mathbb{N}}_X$ by Lemma A.1.7. The sSet$_*$-tensoring $S^1 \oslash_X (-)$ models the suspension on $Ho(\mathrm{Sp}^{\mathbb{N}}_X)$ so the proof is complete. □

Remark 1.2.13. We might hope for a more direct proof of stability of $\mathrm{Sp}^{\mathbb{N}}_X$ considering that the functors Σ_X and $S^1 \oslash_X (-)$ only differ by the twist map $S^1 \wedge S^1 \to S^1 \wedge S^1$.

The approach of comparing the twisted and untwisted suspension functors is considered in [Hov01, §10], where it is shown that their derived functors are isomorphic if S^1 is *symmetric* in the sense that there is a simplicial homotopy

$$S^1 \wedge S^1 \wedge S^1 \wedge I \longrightarrow S^1 \wedge S^1 \wedge S^1$$

between the identity map on $S^1 \wedge S^1 \wedge S^1$ and the cyclic permutation of smash factors, where I is a cylinder object of S^0. While it is certainly true that the cyclic permutation is homotopic to the identity (both are self-maps of the 3-sphere of index 1), the author was not able to find a simplicial homotopy of the prescribed form which witnesses this fact.

Local Symmetric Spectra. The second approach to local parametrised spectra that we consider uses the symmetric stabilisation machine. Working with symmetric spectra gives us a slightly richer structure than in the sequential case. This extra structure is used in our discussion of the rational theory of parametrised spectra in Chapter 2, §2.3. As for the sequential case, many of the structural results in this paragraph follow from general considerations of the symmetric stabilisation machine as detailed in §A.3.2. Recall our convention that $S^n := (S^1)^{\wedge n}$ is the n-fold smash product of the simplicial circle $S^1 = \Delta[1]/\partial\Delta[1]$.

Definition 1.2.14. A *symmetric X-spectrum A* is the data of

- a sequence $\{A_n\}_{n\in\mathbb{N}}$ of retractive spaces over X such that A_n has a (left) action of the permutation group Σ_n; and
- a Σ_n-equivariant map $\sigma_n^A \colon S^1 \oslash_X A_n \to A_{n+1}$ for each $n \in \mathbb{N}$

such that the composite map

$$S^p \oslash_X A_q \xrightarrow{S^{p-1}\oslash_X \sigma_q} S^{p-1} \oslash_X A_{q+1} \xrightarrow{S^{p-2}\oslash_X \sigma_{q+1}} \cdots \xrightarrow{\sigma_{q+p-1}} A_{p+q}$$

is $(\Sigma_p \times \Sigma_q)$-equivariant for all $p, q \geq 0$. A morphism of symmetric X-spectra $f \colon A \to B$ is the data of Σ_n-equivariant maps $f_n \colon A_n \to B_n$ in R_X which respect the spectrum structure maps in the sense that the diagram of Σ_{n+1}-equivariant maps of retractive spaces over X

$$\begin{tikzcd}
S^1 \oslash_X A_n \arrow[r, "S^1\oslash_X f_n"] \arrow[d, "\sigma_n^A"'] & S^1 \oslash_X B_n \arrow[d, "\sigma_n^B"] \\
A_{n+1} \arrow[r, "f_{n+1}"] & B_{n+1}
\end{tikzcd}$$

commutes for all $n \in \mathbb{N}$. The category of symmetric X-spectra is denoted Sp_X^Σ.

Remark 1.2.15. Let $\mathbf{\Sigma}$ be the category with objects $n \in \mathbb{N}$, morphisms

$$\mathbf{\Sigma}(n, m) := \begin{cases} \Sigma_n & \text{if } n = m \\ \varnothing & \text{otherwise} \end{cases}$$

and composition defined by left multiplication. The assignment $(n, m) \mapsto n + m$ determines a symmetric monoidal structure on $\mathbf{\Sigma}$ so that the sSet_*-tensoring on R_X gives rise to a $\mathrm{Fun}(\mathbf{\Sigma}, \mathrm{sSet}_*)$-tensoring on $\mathrm{Fun}(\mathbf{\Sigma}, R_X)$ via Day convolution. In the coend notation, the tensoring of $K \in \mathrm{Fun}(\mathbf{\Sigma}, \mathrm{sSet}_*)$ with $A \in \mathrm{Fun}(\mathbf{\Sigma}, R_X)$ has n-th term

$$K \owedge_{\mathrm{Day}} A \colon n \longmapsto \int^{p,q \in \mathbf{\Sigma}} \big(\mathbf{\Sigma}(p+q,n) \otimes K_p\big) \owedge_X A_q,$$

which is equipped with the induced left Σ_n-action. The symmetric monoidal functor $\mathbf{\Sigma} \to \mathrm{sSet}_*$ which sends $n \mapsto S^n$ determines a commutative monoid $\mathbb{S}$ for the Day convolution tensor product on symmetric sequences $\mathrm{Fun}(\mathbf{\Sigma}, \mathrm{sSet}_*)$. The category of symmetric X-spectra can be equivalently defined as the category of (left) $\mathbb{S}$-modules in $\mathrm{Fun}(\mathbf{\Sigma}, R_X)$. Consequently there is a free-forgetful adjunction

$$\begin{tikzcd} \mathrm{Fun}(\mathbf{\Sigma}, R_X) \arrow[r, shift left, "\mathbb{S} \owedge_{\mathrm{Day}} (-)"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\Sigma}_X \arrow[l, shift left] \end{tikzcd}$$

in which the right adjoint forgets the structure maps. The category Sp^{Σ}_X is locally presentable and is enriched, tensored and powered over the symmetric monoidal category Sp^{Σ} of symmetric spectra. Explicitly, the Sp^{Σ}-tensoring on Sp^{Σ}_X is given by

$$P \owedge^{\Sigma}_X A = \operatorname{colim} \left(\begin{tikzcd} P \owedge_{\mathrm{Day}} \mathbb{S} \owedge_{\mathrm{Day}} A \arrow[r, shift left, "\rho_P \owedge A"] \arrow[r, shift right, "P \owedge \rho_A"'] & P \owedge_{\mathrm{Day}} A \end{tikzcd} \right)$$

where ρ_P and ρ_A are respectively the $\mathbb{S}$-module structure maps of $P \in \mathrm{Sp}^{\Sigma}$ and $A \in \mathrm{Sp}^{\Sigma}_X$.

Remark 1.2.16. There is a free-forgetful adjunction $(\mathrm{F}^X_n \dashv \mathrm{Ev}_n) \colon R_X \to \mathrm{Sp}^{\Sigma}_X$ for each $n \in \mathbb{N}$. As in the sequential case, the right adjoint $\mathrm{Ev}_n \colon A \mapsto A_n$ extracts the n-th term of a symmetric spectrum (and forgets the Σ_n-action). Because of the symmetric group actions, the free functors F^X_n are more subtle than their sequential counterparts F^X_n and arise in the following way. For fixed $n \in \mathbb{N}$ the functor $\mathrm{Fun}(\mathbf{\Sigma}, R_X) \to R_X$ which sends a symmetric sequence to its n-th term has left adjoint

$$A \longmapsto \mathrm{s}_n(A)_m := \begin{cases} \Sigma_n \otimes_X A & \text{if } m = n \\ 0_X & \text{otherwise.} \end{cases}$$

The functor F^X_n is the composite $\mathbb{S} \owedge_{\mathrm{Day}} (\mathrm{s}_n(-)) \colon R_X \to \mathrm{Sp}^{\Sigma}_X$; explicitly it sends the retractive space A over X to the symmetric X-spectrum with m-th term

$$\mathrm{F}^X_n(A)_m := \begin{cases} \Sigma_m \otimes_{\Sigma_{m-n}} \big(S^{m-n} \owedge_X A\big) & \text{for } m \geq n \\ 0_X & \text{otherwise} \end{cases}$$

with structure maps either the identity or the zero map depending on m and n.

The category Sp^{Σ}_X of symmetric X-spectra inherits a *projective model structure* from R_X, with respect to which a morphism $f \colon A \to B$ is

- a weak equivalence if it is a level weak equivalence;
- a fibration if it is a level fibration; and
- a cofibration if it has the left lifting property with respect to all level acyclic fibrations.

The projective model structure on symmetric X-spectra, which we denote PSp^{Σ}_X, is a left proper combinatorial PSp^{Σ}-model category with respect to the enrichment of

Remark 1.2.15. Generating sets of cofibrations and acyclic cofibrations are given by

$$\mathcal{I}^X_\Sigma := \bigcup_{\mathbb{N}} \mathrm{F}^X_n(\mathcal{I}^X_{\mathrm{Kan}}) \quad \text{and} \quad \mathcal{J}^X_\Sigma := \bigcup_{\mathbb{N}} \mathrm{F}^X_n(\mathcal{J}^X_{\mathrm{Kan}}),$$

respectively.

The projective model structure is not yet stable. In the case of symmetric spectra, stabilisation is achieving by left Bousfield localisation of the projective model structure at the set of morphisms

$$\mathbf{S}^X_\Sigma := \left\{ \zeta^X_n(C) \colon \mathrm{F}^X_{n+1}(S^1 \oslash_X C) \longrightarrow \mathrm{F}^X_n(C) \right\},$$

where $\zeta^X_n(A)$ is the adjunct of the map

$$S^1 \oslash_X A \longrightarrow \Sigma_{n+1} \otimes_X \left(S^1 \oslash_X A\right) = \mathrm{Ev}_{n+1}\mathrm{F}^X_n(A)$$

determined by tensoring with the unit map $* \to \Sigma_{n+1}$, $n \in \mathbb{N}$, and C ranges over the domains and codomains of generating cofibrations in $\mathcal{I}^X_{\mathrm{Kan}}$. The *stable model structure* on symmetric X-spectra is the left Bousfield localisation

$$\mathrm{Sp}^\Sigma_X := L_{\mathbf{S}^X_\Sigma}\mathrm{PSp}^\Sigma_X.$$

The cofibrations, fibrations and weak equivalences of Sp^Σ_X are called *stable cofibrations*, *stable fibrations* and *stable weak equivalences* of symmetric X-spectra respectively. As in the sequential case, the stable cofibrations and projective cofibrations coincide. We list some important properties of the stable model structure on symmetric X-spectra:

(i) Sp^Σ_X is a left proper combinatorial Sp^Σ-model category.

(ii) Sp^Σ_X is a stable model category, with suspension modelled by the tensoring $\Sigma_X \cong S^1 \oslash^\Sigma_X (-)$.

(iii) Stably fibrant objects are precisely the *fibrant Ω_X-spectra*: symmetric X-spectra A such that each $A_n \in R_X$ is fibrant and the adjoint structure maps

$$\sigma^\vee_n \colon A_n \longrightarrow S^1 \pitchfork_X A_{n+1}$$

are weak equivalences in R_X for all $n \in \mathbb{N}$.

(iv) Stable weak equivalences between fibrant Ω_X-spectra are precisely the levelwise weak equivalences between these objects.

(v) For any $Y \in R_X$ and $n \in \mathbb{N}$ the map $\zeta^X_n(Y) \colon \mathrm{F}^X_{n+1}(S^1 \oslash_X Y) \to \mathrm{F}^X_n(Y)$ is a stable weak equivalence.

After applying the symmetric stabilisation machine, the sSet_*-enriched base change functors on categories of retractive spaces imply the following

Lemma 1.2.17. *For any map of simplicial sets $f \colon X \to Y$ there is a* Sp^Σ*-enriched Quillen adjunction*

$$\begin{tikzcd} \mathrm{Sp}^\Sigma_X \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^\Sigma_Y \arrow[l, shift left, "f^*"] \end{tikzcd}$$

in which both $f_!$ and f^* are determined on underlying symmetric sequences by objectwise application of the base change functors $(f_! \dashv f^*)\colon R_X \to R_Y$. Moreover

(i) *if f is a weak equivalence then $(f_! \dashv f^*)$ is a Sp^Σ-Quillen equivalence;*

(ii) *if f is a fibration then there is a prolonged base change Quillen adjunction*

$$\begin{tikzcd} \mathrm{Sp}^\Sigma_Y \arrow[r, shift left, "f^*"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^\Sigma_X \arrow[l, shift left, "f_*"] \end{tikzcd}$$

respecting Sp^Σ-enrichments; and

(iii) *if f is an acyclic fibration then the prolonged base change adjunctions $(f_! \dashv f^*)$ and $(f^* \dashv f_*)$ are Sp^Σ-Quillen equivalences.*

Remark 1.2.18. The assertions of Lemma 1.2.17 also hold when projective model structures are substituted for stable model structures.

By forgetting the levelwise actions of the symmetric groups we obtain a functor $\mathrm{Sp}^\Sigma_X \to \mathrm{Sp}^\mathbb{N}_X$. In the unparametrised case, it is a peculiar fact that the forgetful functor $\mathrm{Sp}^\Sigma \to \mathrm{Sp}^\mathbb{N}$ reflects but does not preserve stable weak equivalences. The standard example is $\zeta_0(S^0)\colon \mathrm{F}_1(S^1) \to \mathrm{F}_0(S^0)$; this is a stable equivalence of symmetric spectra, but the induced map on π_0^{st} is the inclusion of a summand in a countable sum $\mathbb{Z} \hookrightarrow \bigoplus_\mathbb{N} \mathbb{Z}$ and so is certainly not a stable homotopy equivalence. The root cause of this issue is the fact that the quotients Σ_p/Σ_{p-q} do not become highly connected as $p \to \infty$.

Definition 1.2.19. The *naïve stable homotopy groups* of a symmetric spectrum P are defined as $\overline{\pi}^{\mathrm{st}}_*(P) := \pi^{\mathrm{st}}_*(P)$, the stable homotopy groups of the underlying sequential spectrum. Fixing a fibrant replacement functor $\mathcal{R}$ on Sp^Σ, the *true stable homotopy groups* of a symmetric spectrum P are $\boldsymbol{\pi}^{\mathrm{st}}_*(P) := \pi^{\mathrm{st}}_*(\mathcal{R}P)$, the naïve stable homotopy groups of the fibrant replacement $\mathcal{R}P$ of P.

Lemma 1.2.20. *The definition of true stable homotopy groups is independent of the choice of fibrant replacement functor $\mathcal{R}$ up to isomorphism. A morphism of symmetric spectra $f\colon P \to Q$ is a stable equivalence if and only if it induces an isomorphism on true stable homotopy groups.*

Definition 1.2.21. A morphism $f\colon A \to B$ of symmetric X-spectra is a *fibrewise stable equivalence* if $x^*(\mathcal{R}f)$ is a stable weak equivalence of spectra for all points $x\colon * \to X$, where $\mathcal{R}$ is any fibrant replacement functor on Sp^Σ_X.

Lemma 1.2.22. *A morphism $f\colon A \to B$ of symmetric X-spectra is a stable weak equivalences if and only if is a fibrewise stable equivalence.*

Proof. This is the symmetric version of Lemma 1.2.10; essentially the same proof applies. □

Comparison of Stabilisation Machines. We conclude our treatment of local model categories of parametrised spectra by providing a comparison result between the sequential and symmetric models. Our comparison result, modelled on [Hov01, Theorem 10.1], rests on the ideas that stabilisation is idempotent (Lemmas A.3.13 and A.3.27) and that the sequential and symmetric stabilisation machines commute in an appropriate sense.

Theorem 1.2.23. *For each $X \in \mathrm{sSet}$ there is a zig-zag of sSet_*-Quillen equivalences*

$$\begin{tikzcd} \mathrm{Sp}^{\Sigma}_X \arrow[r, shift left, "L^{\Sigma}_X"] \arrow[r, phantom, "\perp"] & \mathcal{D}_X \arrow[l, shift left, "R^{\Sigma}_X"] \arrow[r, shift right, "R^{\mathbb{N}}_X"'] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\mathbb{N}}_X \arrow[l, shift right, "L^{\mathbb{N}}_X"'] \end{tikzcd}$$

inducing isomorphisms $Ho(\mathrm{Sp}^{\Sigma}_X)\big(\mathrm{F}^X_0(A), \mathrm{F}^X_0(B)\big) \cong Ho(\mathrm{Sp}^{\mathbb{N}}_X)\big(F^X_0(A), F^X_0(B)\big)$ for all $A, B \in R_X$. For each morphism of simplicial sets $f\colon X \to Y$ there is a diagram of left Quillen functors

$$\begin{tikzcd} \mathrm{Sp}^{\Sigma}_X \arrow[r, "L^{\Sigma}_X"] \arrow[d, shift right, "f_!"'] & \mathcal{D}_X \arrow[d, shift right, "f_!"'] & \mathrm{Sp}^{\mathbb{N}}_X \arrow[l, "L^{\Sigma}_X"'] \arrow[d, shift right, "f_!"'] \\ \mathrm{Sp}^{\Sigma}_Y \arrow[r, "L^{\Sigma}_Y"] \arrow[u, shift right, "f^*"'] & \mathcal{D}_Y \arrow[u, shift right, "f^*"'] & \mathrm{Sp}^{\mathbb{N}}_Y \arrow[l, "L^{\Sigma}_Y"'] \arrow[u, shift right, "f^*"'] \end{tikzcd}$$

which commutes up to natural isomorphism. If $f\colon X \to Y$ is a fibration then there are also left Quillen functors f^*, marked in grey in the diagram above, such that the resultant diagram of left Quillen functors commutes up to natural isomorphism.

Proof. For a fixed base $X \in \mathrm{sSet}$ let $\mathcal{D}_X := \mathrm{Sp}^{\mathbb{N}}(\mathrm{Sp}^{\Sigma}_X, \Sigma_X)$ be the stable model category of sequential spectra in Sp^{Σ}_X with respect to the suspension endofunctor Σ_X (which is given by the sSet_*-tensoring with S^1). Objects of $\mathcal{D}_X$ are collections $\{A_{m,n}\}_{m,n\in\mathbb{N}}$ of objects in R_X equipped with structure maps so that

- $A_{m,\bullet}$ is a symmetric X-spectrum for all m;
- $A_{\bullet,n}$ is a sequential X-spectrum for all n; and
- for all $m, n \in \mathbb{N}$ the structure maps $\Sigma_X A_{m,\bullet} \to A_{m+1,\bullet}$ and $\Sigma_X A_{\bullet,n} \to A_{\bullet,n+1}$ are maps of symmetric X-spectra and sequential X-spectra respectively.

The morphisms in $\mathcal{D}_X$ are collections of morphisms $\{f_{m,n}\colon A_{m,n} \to B_{m,n}\}_{m,n\in\mathbb{N}}$ in R_X respecting the structure maps. The category $\mathrm{Sp}^{\mathbb{N}}(\mathrm{Sp}^{\Sigma}_X, \Sigma_X)$ has a *double projective model structure* in which weak equivalences and morphisms are levelwise. The stable model structure is obtained from this doubly projective model structure by taking left Bousfield localisation at the set of morphisms

$$\mathbf{S}^X_{\mathcal{D}} := \Big\{F^X_{m+1,n}(\Sigma_X C) \longrightarrow F^X_{m,n}(C)\Big\} \cup \Big\{F^X_{m,n+1}(\Sigma_X C) \longrightarrow F^X_{m,n}(C)\Big\},$$

where $F^X_{m,n}\colon R_X \to \mathrm{Sp}^{\mathbb{N}}(\mathrm{Sp}^{\Sigma}_X, \Sigma_X)$ is left adjoint to $\mathrm{Ev}_{m,n}\colon \{A_{p,q}\} \mapsto A_{m,n}$, $m, n \in \mathbb{N}$ and C ranges over domains and codomains of generating cofibrations in $\mathcal{I}^X_{\mathrm{Kan}}$.

From the description of objects and morphisms it is easy to see that there is a canonical isomorphism of categories $\mathrm{Sp}^{\mathbb{N}}(\mathrm{Sp}^{\Sigma}_X, \Sigma_X) \cong \mathrm{Sp}^{\Sigma}(\mathrm{Sp}^{\mathbb{N}}_X, S^1 \oslash_X (-))$. On objects, we can understand this as interchanging indices $A_{m,n} \leftrightarrow A_{n,m}$. This isomorphism respects the doubly projective model structures and therefore also the stable model structures. As remarked in [Hov01, p. 116], this is the sense in which the symmetric and sequential stabilisation machines commute.

The zig-zag is obtained by the free-forgetful sSet_*-functors

$$\begin{tikzcd} \mathrm{Sp}^{\Sigma}_X \arrow[r, shift left, "L^{\Sigma}_X := F_0"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\mathbb{N}}(\mathrm{Sp}^{\Sigma}_X, \Sigma_X) = \mathcal{D}_X \cong \mathrm{Sp}^{\Sigma}(\mathrm{Sp}^{\mathbb{N}}_X, \Sigma_X) \arrow[l, shift left, "R^{\Sigma}_X := \mathrm{Ev}_0"] \arrow[r, shift right, "R^{\mathbb{N}}_X := \mathrm{Ev}_0"'] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\mathbb{N}}_X. \arrow[l, shift right, "L^{\mathbb{N}}_X := \mathrm{F}_0"'] \end{tikzcd}$$

Since Sp^{Σ}_X and $\mathrm{Sp}^{\mathbb{N}}_X$ are stable, the adjunctions $(L^{\Sigma}_X \dashv R^{\Sigma}_X)$ and $(L^{\mathbb{N}}_X \dashv R^{\mathbb{N}}_X)$ are Quillen equivalences by Lemmas A.3.13 and A.3.27. The composite functors

$$R_X \xrightarrow{\mathrm{F}^X_0} \mathrm{Sp}^{\Sigma}_X \xrightarrow{\mathrm{F}_0} \mathcal{D}_X \quad \text{and} \quad R_X \xrightarrow{F^X_0} \mathrm{Sp}^{\mathbb{N}}_X \xrightarrow{\mathrm{F}_0} \mathrm{Sp}^{\mathbb{N}}(\mathrm{Sp}^{\Sigma}_X, \Sigma_X) \cong \mathcal{D}_X$$

are both naturally isomorphic to $F_{0,0} \colon R_X \to \mathcal{D}_X$, whence

$$Ho(\mathrm{Sp}^{\Sigma}_X)\big(\mathrm{F}^X_0(A), \mathrm{F}^X_0(B)\big) \cong Ho(\mathcal{D}_X)\big(F^X_{0,0}(A), F^X_{0,0}(B)\big) \cong Ho(\mathrm{Sp}^{\mathbb{N}}_X)\big(F^X_0(A), F^X_0(B)\big)$$

and the first part of the Theorem is proven.

For $f \colon X \to Y$ any map of simplicial sets, Lemmas 1.2.6 and 1.2.17 together with Corollary A.3.16 and Theorem A.3.28 imply that there is a sSet$_*$-Quillen adjunction $(f_! \dashv f^*) \colon \mathcal{D}_X \to \mathcal{D}_Y$ in which $f_!$ and f^* are determined on objects by levelwise application of the functors $(f_! \dashv f^*) \colon R_X \to R_Y$. The base change diagram is seen to commute up to natural isomorphism since $f_!$ preserves sSet$_*$ tensors for both symmetric and sequential spectra. If $f \colon X \to Y$ is a fibration then we have the claimed sSet$_*$-Quillen adjunction $(f^* \dashv f_*) \colon \mathcal{D}_Y \to \mathcal{D}_X$ by the same argument. □

Corollary 1.2.24. *For any point $x \colon * \to X$ and retractive space $A \in R_X$ there is a natural isomorphism between*

(i) *the stable homotopy groups of the homotopy fibre* $\mathrm{hofib}_x(A)$ *of A at x;*

(ii) *the stable homotopy groups of the homotopy fibre of the free sequential X-spectrum $F^X_0(A) \in \mathrm{Sp}^{\mathbb{N}}_X$ at x; and*

(iii) *the true stable homotopy groups of the homotopy fibre of the free symmetric X-spectrum $\mathrm{F}^X_0(A) \in \mathrm{Sp}^{\Sigma}_X$ at x.*

Proof. Let S^0_x be the retractive space over X

$$\big(X \longrightarrow * \textstyle\coprod X \xrightarrow{x+\mathrm{id}} X\big) = x_!(* \longrightarrow * \textstyle\coprod * \longrightarrow *),$$

where, as usual, the section maps are coprojections onto the second summand. For $n \geq 0$ we set $S^n_x = \Sigma^n_X S^0_x \cong x_! \Sigma^n S^0 \cong x_! S^n$, which is the result of attaching S^n by its basepoint to X at x. Then for $k \geq 0$

$$\begin{aligned}
\pi^{\mathrm{st}}_k(\mathrm{hofib}_x F^X_0(A)) &:= Ho(\mathrm{Sp}^{\mathbb{N}})\big(F_0 S^k, x^* F^X_0(A)\big) \\
&\cong Ho(\mathrm{Sp}^{\mathbb{N}}_X)\big(F^X_0 S^k_x, F^X_0(A)\big) \\
&\cong Ho(\mathcal{D}_X)\big(F^X_{0,0} S^k_x, F^X_{0,0}(A)\big) \\
&\cong Ho(\mathrm{Sp}^{\Sigma}_X)\big(\mathrm{F}^X_0 S^k_x, \mathrm{F}^X_0(A)\big) \\
&\cong Ho(\mathrm{Sp}^{\Sigma})\big(\mathrm{F}_0 S^k, x^* \mathrm{F}^X_0(A)\big) \\
&=: \boldsymbol{\pi}^{\mathrm{st}}_k(\mathrm{hofib}_x \mathrm{F}^X_0(A)),
\end{aligned}$$

where we have used the Theorem and the compatibility relations $x_! \circ F_0 \cong F^X_0 \circ x_!$ and $x_! \circ \mathrm{F}_0 \cong \mathrm{F}^X_0 \circ x_!$. This shows the natural isomorphism between (ii) and (iii) when $k \geq 0$. We also have

$$\pi^{\mathrm{st}}_k(\mathrm{hofib}_x(A)) := Ho(\mathrm{Sp}^{\mathbb{N}})\big(F_0 S^k, F_0(x^* A)\big)$$

and since $F_0 \circ x^* \cong x^* \circ F^X_0$ this shows the natural isomorphism between (i) and (ii) for $k \geq 0$. The $k < 0$ case is shown similarly, using $\pi^{\mathrm{st}}_k(P) \cong Ho(\mathrm{Sp}^{\mathbb{N}})\big(F_0 S^0, \Sigma^k P\big)$

together with the compatibility relations between the suspension, free spectrum, and base change functors. □

1.2.2 Global Sequential Stabilisation

We now turn to the first of our two models for the global homotopy theory of parametrised spectra. This model—the *global sequential stabilisation*—is constructed in one of two equivalent ways:

(1) by applying the Grothendieck construction to the pseudofunctor $X \mapsto \mathrm{Sp}^{\mathbb{N}}_X$; and

(2) by considering sequential spectra in the global category R_{sSet} with respect to the endofunctor $S^1 \overline{\otimes} (-)$ and restricting to those spectra with constant base space sequences.

We use both approaches; in light of its general formulation, (1) gives the most direct path to establishing the model structure, whereas (2) allows us to establish that the integral model structure is combinatorial.

Lemma 1.2.25. *The pseudofunctor*

$$\begin{aligned} \mathrm{Sp}^{\mathbb{N}}_{(-)} \colon \mathrm{sSet} &\longrightarrow \mathbf{Mod} \\ X &\longmapsto \mathrm{Sp}^{\mathbb{N}}_X \\ (f \colon X \to Y) &\longmapsto \big((f_! \dashv f^*) \colon \mathrm{Sp}^{\mathbb{N}}_X \to \mathrm{Sp}^{\mathbb{N}}_Y\big). \end{aligned}$$

is proper and relative.

Proof. Relativeness and right properness of $\mathrm{Sp}^{\mathbb{N}}_{(-)}$ are immediate consequences of Lemma 1.2.6 (compare with the proof of Lemma 1.1.19).

For left properness, fix a map of simplicial sets $f \colon X \to Y$. In the proof of Lemma 1.1.19 we saw that the pushforward functor $f_! \colon R_X \to R_Y$ preserves weak equivalences, so that the prolongation $f_! \colon \mathrm{PSp}^{\mathbb{N}}_X \to \mathrm{PSp}^{\mathbb{N}}_Y$ to the projective model structure on sequential spectra preserves level equivalences. Let $\mathcal{R}$ be a fibrant replacement functor on $\mathrm{Sp}^{\mathbb{N}}_X$ such that the comparison maps $A \to \mathcal{R}(A)$ are acyclic cofibrations. Consider the naturality diagram for the fibrant replacement functor

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, hook, "\wr"'] & B \arrow[d, hook, "\wr"] \\ \mathcal{R}(A) \arrow[r, "\mathcal{R}(\psi)"] & \mathcal{R}(B). \end{tikzcd}$$

If ψ is a stable equivalence, then $\mathcal{R}(\psi)$ is a levelwise weak equivalence of fibrant Ω_X-spectra. Applying the left Quillen functor $f_!$, the vertical and bottom horizontal morphisms are sent to stable equivalences in $\mathrm{Sp}^{\mathbb{N}}_Y$. By the 2-out-of-3 property, $f_!(\psi)$ is also a stable equivalence. Thus the pushforward map $f_!$ preserves stable equivalences for any f whatsoever, in particular this shows left properness. □

Remark 1.2.26. The above proof also shows that the pseudofunctor $X \mapsto \mathrm{PSp}^{\mathbb{N}}_X$, which instead takes the projective model structure on parametrised spectra, is proper and relative.

We write

$$\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}} := \int_{\mathrm{sSet}} \mathrm{Sp}^{\mathbb{N}}_{(-)}$$

for the Grothendieck construction of the pseudofunctor $\mathrm{Sp}^{\mathbb{N}}_{(-)} \colon \mathrm{sSet} \to \mathbf{Adj}$. As in the local case there is a projective and a stable model structure on the global category:

Theorem 1.2.27. *The category $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ is equipped with the* global stable model structure *for which a map of pairs $(f, \psi) \colon (X, A) \to (Y, B)$ is*

- *a weak equivalence if $f \colon X \to Y$ and $\psi \colon f_! A \to B$ are weak equivalences in* sSet *and $\mathrm{Sp}^{\mathbb{N}}_{X}$ respectively;*
- *a cofibration if $f \colon X \to Y$ and $\psi \colon f_! A \to B$ are cofibrations in* sSet *and $\mathrm{Sp}^{\mathbb{N}}_{Y}$ respectively; and*
- *a fibration if $f \colon X \to Y$ and $\psi^\vee \colon A \to f^* B$ are fibrations in* sSet *and $\mathrm{Sp}^{\mathbb{N}}_{X}$ respectively, where $\psi^\vee$ is the $(f_! \dashv f^*)$-adjunct of ψ.*

The global projective model structure*, denoted $\mathrm{PSp}^{\mathbb{N}}_{\mathrm{sSet}}$, is characterised by substituting $\mathrm{Sp}^{\mathbb{N}}_{(-)} \mapsto \mathrm{PSp}^{\mathbb{N}}_{(-)}$ in the characterisations of fibrations, cofibrations and weak equivalences given above. There is a Quillen adjunction*

$$\begin{tikzcd} \mathrm{PSp}^{\mathbb{N}}_{\mathrm{sSet}} \arrow[r, shift left, "\mathrm{id}"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}} \arrow[l, shift left, "\mathrm{id}"] \end{tikzcd} \tag{1.7}$$

between the global projective and global stable model structures.

Proof. The integral model structures of the pseudofunctors $\mathrm{PSp}^{\mathbb{N}}_{(-)}$ and $\mathrm{Sp}^{\mathbb{N}}_{(-)}$ exist by Lemma 1.2.25, Remark 1.2.26 and Theorem A.4.6. No cofibrant replacements are necessary in the characterisation of weak equivalences since the pushforward functors $f_!$ preserve weak equivalences in both the projective and stable settings.

The identity functor $\mathrm{PSp}^{\mathbb{N}}_{\mathrm{sSet}} \to \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ is left Quillen since projective weak equivalences are stable weak equivalences and the classes of projective and stable cofibrations coincide. □

Let $f \colon X \to Y$ be a morphism of simplicial sets. Since the base change functors $f_!$ and f^* preserve sSet_*-tensors we have natural isomorphisms $f_! F^X_n \cong F^Y_n f_!$ for each $n \in \mathbb{N}$. Since the prolonged adjunction $(f_! \dashv f^*) \colon \mathrm{Sp}^{\mathbb{N}}_X \to \mathrm{Sp}^{\mathbb{N}}_Y$ is defined by levelwise application of $(f_! \dashv f^*) \colon R_X \to R_Y$ we conclude that the assignment

$$X \longmapsto \left(\begin{tikzcd} R_X \arrow[r, shift left, "F^X_n"] \arrow[r, phantom, "\perp"] & \mathrm{PSp}^{\mathbb{N}}_X \arrow[l, shift left, "\mathrm{Ev}_n"] \end{tikzcd} \right)$$

is a pseudonatural transformation $R_{(-)} \Rightarrow \mathrm{PSp}^{\mathbb{N}}_{(-)}$ of pseudofunctors $\mathrm{sSet} \to \mathbf{Mod}$ for all n. Applying the Grothendieck construction for model categories yields

Lemma 1.2.28. *For each n there is a Quillen adjunction*

$$\begin{tikzcd} R_{\mathrm{sSet}} \arrow[r, shift left, "F^{(-)}_n"] \arrow[r, phantom, "\perp"] & \mathrm{PSp}^{\mathbb{N}}_{\mathrm{sSet}} \arrow[l, shift left, "\mathrm{Ev}_n"] \end{tikzcd}$$

which preserves the projection functors to sSet*. Composing with the adjunction (1.7) yields free-forgetful adjunctions for the global stable model structure.*

Similarly, sending $X \in \mathrm{sSet}$ to the zero X-spectrum 0_X determines a functor $0_{(-)}\colon \mathrm{sSet} \to \mathrm{PSp}^{\mathbb{N}}_{\mathrm{sSet}}$.

Lemma 1.2.29. *The functor* $\mathrm{base}\colon \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}} \to \mathrm{sSet}$ *has a two-sided adjoint* $0_{(-)}$ *which sends a space to its corresponding parametrised zero spectrum. The adjunctions* $(0_{(-)} \dashv \mathrm{base})$ *and* $(\mathrm{base} \dashv 0_{(-)})$ *are Quillen for the global projective and global stable model structures.*

We now turn to our second construction of the global category of sequential parametrised spectra. This second construction allows us to deduce a number of useful properties of the global projective and global stable model structures. We first dispense with some preliminaries.

Lemma 1.2.30. *The* relative suspension *endofunctor* $\Sigma_{(-)} := S^1 \,\overline{\wedge}\, (-)\colon R_{\mathrm{sSet}} \to R_{\mathrm{sSet}}$ *is left Quillen.*

Proof. Since $S^1 \in R_{\mathrm{sSet}}$ is cofibrant, this is immediate from the pushout-product axiom for the external smash product verified in Theorem 1.1.27. □

Remark 1.2.31. We now have a possible conflict of notations: for $(X, A) \in R_{\mathrm{sSet}}$ we can consider either the relative suspension $\Sigma_{(-)}(X, A)$ or $\Sigma_X A := S^1 \oslash_X A$. Since $S^1 \oslash_X (-) = X^* S^1 \wedge_X (-)$ these objects coincide by Lemma 1.1.30.

Remark 1.2.32. A *parametrised sequential spectrum* is a sequence $\{(X_n, A_n)\}_{n\in\mathbb{N}}$ of objects of R_{sSet} such that $X_n = \mathrm{base}(X_n, A_n)$ is the constant sequence on some X, which is moreover equipped with a collection of structure maps $\sigma_n\colon \Sigma_X A_n \to A_{n+1}$ for each $n \in \mathbb{N}$, with $\Sigma_X A_n \equiv \Sigma_{(-)}(X, A_n)$. Parametrised sequential spectra are therefore naturally equivalent to pairs (X, A) with $X \in \mathrm{sSet}$ and $A \in \mathrm{Sp}^{\mathbb{N}}_X$.

A morphism of parametrised sequential spectra $(X, A) \to (Y, B)$ is the data of a collection of commutative diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & A_n \arrow[r] \arrow[d, "\overline{\psi}_n"'] & Y \arrow[d, "f"] \\ Y \arrow[r] & B \arrow[r] & Y \end{tikzcd}$$

in sSet for each $n \in \mathbb{N}$ such that the diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \Sigma_X A_n \arrow[r, "\sigma^A_n"] \arrow[d, "\Sigma_f \overline{\psi}_n"'] & A_{n+1} \arrow[r] \arrow[d, "\overline{\psi}_{n+1}"] & X \arrow[d, "f"] \\ Y \arrow[r] & \Sigma_Y B_n \arrow[r, "\sigma^B_n"] & B_{n+1} \arrow[r] & Y \end{tikzcd}$$

commute for each n, where $\Sigma_f \overline{\psi}_n\colon \Sigma_X A_n \to \Sigma_Y B_n$ is the morphism $S^1 \,\overline{\wedge}\, (f, \overline{\psi}_n)$. Since $f_! \Sigma_X \cong \Sigma_Y f_!$ morphisms of parametrised sequential spectra are seen to be equivalent to morphisms of pairs $(X, A) \to (Y, B)$ in $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$, giving an alternative description for objects and morphisms of the Grothendieck construction (compare with Lemma 1.1.13).

Lemma 1.2.33. $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ *is locally presentable.*

Proof. Recall the notation of Construction 1.1.12; we argue similarly to Lemma 1.1.15. The functor categories $\mathrm{Fun}(\mathbb{N} \times \partial\Delta[2], \mathrm{sSet})$ and $\mathrm{Fun}(\mathbb{N} \times \Delta[1], \mathrm{sSet})$ are locally presentable and the pullback functor

$$(\mathrm{id} \times \imath_{[0<2]})^*\colon \mathrm{Fun}(\mathbb{N} \times \partial\Delta[2], \mathrm{sSet}) \longrightarrow \mathrm{Fun}(\mathbb{N} \times \Delta[1], \mathrm{sSet})$$

induced by $\iota_{[0<2]}\colon \Delta[1] \to \partial\Delta[2]$ is accessible. Let $\Delta_{\mathrm{sSet}} \hookrightarrow \mathrm{Fun}(\mathbb{N} \times \Delta[1], \mathrm{sSet})$ be the full subcategory on those functors which are constant on $(\mathrm{id}_X\colon X \to X)$ for some $X \in \mathrm{sSet}$. Since limits and colimits in functor categories are computed objectwise, Δ_{sSet} is an accessibly embedded subcategory. Let

$$\mathrm{RetSeq}_{\mathrm{sSet}} := \left((\mathrm{id} \times \iota_{[0<2]})^*\right)^{-1}(\Delta_{\mathrm{sSet}})$$

be the inverse image, which is accessible by [Lur09, Corollary A.2.6.5]. Limits and colimits in $\mathrm{RetSeq}_{\mathrm{sSet}}$ are created in $\mathrm{Fun}(\mathbb{N} \times \partial\Delta[2], \mathrm{sSet})$ so that $\mathrm{RetSeq}_{\mathrm{sSet}}$ is moreover locally presentable.

We now define an accessible monad T^{sp} on $\mathrm{RetSeq}_{\mathrm{sSet}}$ whose category of algebras is isomorphic to $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$. To an object $\{(X, A_n)\}_{n\in\mathbb{N}}$ in $\mathrm{RetSeq}_{\mathrm{sSet}}$ the monad T^{sp} assigns the sequence of retractive spaces over X whose n-th term is the colimit of the diagram of simplicial sets

$$\begin{tikzcd}[column sep=small]
 & & & X \arrow[dlll] \arrow[dll] \arrow[d] \arrow[drr] \arrow[drrr] & & & \\
\Sigma^n_X A_0 & \Sigma^{n-1}_X A_1 & \cdots & \Sigma^{n-i}_X A_i & \cdots & \Sigma_X A_{n-1} & A_n
\end{tikzcd}$$

equipped with the induced structure as a retractive space over X. In light of Remark 1.2.32 this assignment is extended to morphisms in the obvious way. The monad multiplication on T^{sp} is determined at level n by the folding maps

$$\underbrace{\Sigma^{n-i}_X A_i \coprod_X \cdots \coprod_X \Sigma^{n-i}_X A_i}_{(n+1-i)\text{ summands}} \longrightarrow \Sigma^{n-i}_X A_i,$$

and the monad unit is determined at level n by the coprojection $A_n \to T^{\mathrm{sp}}(A)_n$. It is straightforward (but tedious) to verify the monad axioms. Since T^{sp} is defined in terms of colimits and the relative suspension functor $\Sigma_{(-)}$, which is a left adjoint, the monad T^{sp} is accessible. Moreover, limits and colimits in $T^{\mathrm{sp}}-\mathrm{Alg}$ are created in $\mathrm{RetSeq}_{\mathrm{sSet}}$. The category $T^{\mathrm{sp}}-\mathrm{Alg}$ is locally presentable by [Bor94, Theorem 5.5.9].

We conclude the proof by exhibiting $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ as an accessibly embedded full subcategory of $T^{\mathrm{sp}}-\mathrm{Alg}$. This is sufficient as $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ has all colimits, as is shown directly. Let $\{(X, A_n)\}_{n\in\mathbb{N}}$ be a T^{sp}-algebra, so that by precomposing the T^{sp}-algebra structure map at level $n+1$ with the coprojections $\Sigma^{n+1-i}_X A_i \to T^{\mathrm{sp}}(A)_{n+1}$ we obtain morphisms in R_{sSet}

$$\rho^i_{n+1}\colon \Sigma^{n-i}_X A_i \longrightarrow A_{n+1}$$

for each $0 \le i \le n+1$. Write $\overline{\sigma}_n := \rho^n_{n+1}\colon \Sigma_X A_n \to A_{n+1}$. The T^{sp}-algebra axioms imply that ρ^{n+1}_{n+1} is the identity and that ρ^i_{n+1} is equal to the composite

$$\Sigma^i_X A_{n+1-i} \xrightarrow{\Sigma^{i-1}_X(\overline{\sigma}_{n+1-i})} \Sigma^{i-1}_X A_{n-i} \xrightarrow{\Sigma^{i-2}_X(\overline{\sigma}_{n-i})} \cdots \xrightarrow{\overline{\sigma}_n} A_{n+1}$$

for all $0 \le i \le n$. Such objects are not quite sequential parametrised spectra since the maps $\overline{\sigma}_n$ may not cover the identity on X. However, this is the only difference. To remedy this small issue, let $\beta\colon T^{\mathrm{sp}}-\mathrm{Alg} \to \mathrm{Fun}(\mathbb{N} \times \Delta[1], \mathrm{sSet})$ be the functor defined on objects by sending the T^{sp}-algebra $\{(X, A_n)\}_{n\in\mathbb{N}}$ to the functor

$$n \longmapsto \left(\mathrm{base}(\overline{\sigma}_n)\colon X \to X\right).$$

The functor β preserves colimits, so is an accessible functor between locally presentable categories. Letting $\Delta_{\mathrm{sSet}} \hookrightarrow \mathrm{Fun}(\mathbb{N} \times \Delta[1], \mathrm{sSet})$ be the full accessibly embedded subcategory on constant sequences of the form $n \mapsto (\mathrm{id}_X : X \to X)$, we find that $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ is canonically isomorphic to the inverse image $(\beta)^{-1}(\Delta_{\mathrm{sSet}})$ and so is accessible by [Lur09, Corollary A.2.6.5]. This completes the proof, since $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ has all small colimits. □

Theorem 1.2.34. *The global projective and global stable model structures on* $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ *are left proper combinatorial* sSet_**-model structures.*

Proof. Left properness is inherited from the left properness of sSet and $\mathrm{PSp}^{\mathbb{N}}_X$ and $\mathrm{Sp}^{\mathbb{N}}_X$ (for all $X \in \mathrm{sSet}$) as in the proof of Lemma 1.1.24.

To show that $\mathrm{PSp}^{\mathbb{N}}_{\mathrm{sSet}}$ is combinatorial, it is sufficient to show that it is cofibrantly generated by Lemma 1.2.33. The small object argument automatically holds in a locally presentable category so cofibrant generation of $\mathrm{PSp}^{\mathbb{N}}_{\mathrm{sSet}}$ follows if we can find sets of $\mathcal{I}^{\mathbb{N}}$ and $\mathcal{J}^{\mathbb{N}}$ such that $\mathcal{F}\mathrm{ib} \cap \mathcal{W} = \mathrm{rlp}(\mathcal{I}^{\mathbb{N}})$ and $\mathcal{F}\mathrm{ib} = \mathrm{rlp}(\mathcal{J}^{\mathbb{N}})$. The candidate set of generating cofibrations $\mathcal{I}^{\mathbb{N}}$ is the union of the sets

- $\mathcal{I}^0_{\mathrm{Kan}}$ obtained by applying the left Quillen functor $0_{(-)} : \mathrm{sSet} \to \mathrm{PSp}^{\mathbb{N}}_{\mathrm{sSet}}$ to $\mathcal{I}_{\mathrm{Kan}}$; and
- $F_n(\mathcal{I}^+_{\mathrm{Kan}})$ obtained by applying the left Quillen functor $F^{(-)}_n$ to the set $\mathcal{I}^+_{\mathrm{Kan}}$ of cofibrations from the proof of Lemma 1.1.24, for each $n \in \mathbb{N}$.

The set $\mathcal{J}^{\mathbb{N}}$ is defined similarly replacing all $\mathcal{I}$'s by $\mathcal{J}$'s. The argument of Lemma 1.1.24 carries over to show that $\mathcal{I}^{\mathbb{N}}$ and $\mathcal{J}^{\mathbb{N}}$ are generating sets of cofibrations and acyclic cofibrations respectively. The central idea is that, for all $X \in \mathrm{sSet}$, generating cofibrations and acyclic cofibrations for $\mathrm{PSp}^{\mathbb{N}}_X$ are all obtained from maps in $\mathcal{I}^{\mathbb{N}}$ and $\mathcal{J}^{\mathbb{N}}$ respectively by pushforwards along simplex inclusions $\Delta[k] \to X$.

Having shown that $\mathrm{PSp}^{\mathbb{N}}_{\mathrm{sSet}}$ is combinatorial and left proper, we argue the same holds for $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ by identifying the adjunction (1.7) as a left Bousfield localisation. Recall that for $A \in R_X$ and any $n \in \mathbb{N}$ there is a map of sequential X-spectra $\zeta^X_n(A) : F^X_{n+1}(\Sigma_X A) \to F^X_n(A)$. For each $k \in \mathbb{N}$ consider the retractive spaces

$$\partial_k := \Big(\Delta[k] \to \partial\Delta[k] \coprod \Delta[k] \to \Delta[k]\Big), \quad \Delta_k := \Big(\Delta[k] \to \Delta[k] \coprod \Delta[k] \to \Delta[k]\Big), \tag{1.8}$$

with structure maps given by coprojections, boundary inclusions and identity maps. Consider the set of cofibrations

$$\mathbf{S}^{\mathrm{glob}}_{\mathbb{N}} := \left\{ \zeta^{\Delta[k]}_n(\partial_k), \;\; \zeta^{\Delta[k]}_n(\Delta_k) \right\}_{n,k \in \mathbb{N}}.$$

Since $\mathrm{PSp}^{\mathbb{N}}_{\mathrm{sSet}}$ is combinatorial and left proper, the left Bousfield localisation

$$\overline{\mathrm{Sp}}^{\mathbb{N}}_{\mathrm{sSet}} := L_{\mathbf{S}^{\mathrm{glob}}_{\mathbb{N}}} \mathrm{PSp}^{\mathbb{N}}_{\mathrm{sSet}}$$

exists and is a left proper combinatorial model category [Lur09, Proposition A.3.7.3]. We show that the localised model structure coincides with the global stable model structure, for which we need the following

Claim: the fibrant objects of $\overline{\mathrm{Sp}}^{\mathbb{N}}_{\mathrm{sSet}}$ are precisely the pairs (X, P) for which X is a Kan complex and P is a fibrant Ω_X-spectrum.

Proof of Claim. The fibrant object of $\overline{\mathrm{Sp}}^{\mathbb{N}}_{\mathrm{sSet}}$ are precisely the $\mathbf{S}^{\mathrm{glob}}_{\mathbb{N}}$-local fibrant objects of $\mathrm{PSp}^{\mathbb{N}}_{\mathrm{sSet}}$. The fibrant objects in the global projective model structure are those pairs (X, P) for which X is a Kan complex and P is a levelwise fibrant sequential X-spectrum.

By definition, a projectively fibrant object (X, P) is $\mathbf{S}^{\mathrm{glob}}_{\mathbb{N}}$-local precisely if the map of homotopy function complexes

$$\mathrm{map}_{\mathrm{PSp}^{\mathbb{N}}_{\mathrm{sSet}}}\Big(F_n^{\Delta[k]}(A), (X, P)\Big) \longrightarrow \mathrm{map}_{\mathrm{PSp}^{\mathbb{N}}_{\mathrm{sSet}}}\Big(F_{n+1}^{\Delta[k]}(\Sigma_{\Delta[k]} A), (X, P)\Big)$$

induced by $\zeta_n^{\Delta[k]}(A)$ is a weak equivalence for all $\zeta_n^{\Delta[k]}(A) \in \mathbf{S}^{\mathrm{glob}}_{\mathbb{N}}$. Using the Quillen adjunctions of Lemma 1.2.28 and that relative suspension is left Quillen, this latter condition is equivalent to the condition that the morphism of homotopy function complexes induced by the adjunct structure map $(X, P_n) \to (X, \Omega_X P_{n+1})$

$$\mathrm{map}_{R_{\mathrm{sSet}}}\Big((\Delta[k], A), (X, P_n)\Big) \longrightarrow \mathrm{map}_{R_{\mathrm{sSet}}}\Big((\Delta[k]), A), (X, \Omega_X P_{n+1})\Big) \tag{1.9}$$

is a weak equivalence for each n. We conclude that $\mathbf{S}^{\mathrm{glob}}_{\mathbb{N}}$-locality of a projectively fibrant object (X, P) holds precisely precisely if the maps of homotopy function complexes induced by the structure maps $P_n \to \Omega_X P_{n+1}$

$$\mathrm{map}_{R_{\mathrm{sSet}}}\Big(K, (X, P_n)\Big) \longrightarrow \mathrm{map}_{R_{\mathrm{sSet}}}\Big(K, (X, \Omega_X P_{n+1})\Big)$$

are weak equivalences for all n and for all domains and codomains K of morphisms in the set $\mathcal{I}^{\mathrm{ret}}$ of generating cofibrations of R_{sSet}. Indeed, to see this we recall that $\mathcal{I}^{\mathrm{ret}} := \mathrm{const}(\mathcal{I}_{\mathrm{Kan}}) \cup \mathcal{I}^{+}_{\mathrm{Kan}}$—(1.9) dispenses with the domains and codomains of morphisms in $\mathcal{I}^{+}_{\mathrm{Kan}}$ and mapping a domain or codomain of a morphism in $\mathrm{const}(\mathcal{I}_{\mathrm{Kan}})$ into $(X, P_n) \to (X, \Omega_X P_{n+1})$ yields the identity on homotopy function complexes since $P_n \to \Omega_X P_{n+1}$ covers the identity on X. By Lemma A.1.5 we deduce that (X, P) is a fibrant object of the local model structure precisely if $(X, P_n) \to (X, \Omega_X P_{n+1})$ is a weak equivalence in R_{sSet} for all n, that is, if $P \in R_X$ is a fibrant Ω_X-spectrum. □

With this characterisation of fibrant objects in $\overline{\mathrm{Sp}}^{\mathbb{N}}_{\mathrm{sSet}}$ in hand, observe that

- the classes of cofibrations in $\overline{\mathrm{Sp}}^{\mathbb{N}}_{\mathrm{sSet}}$, $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$, and $\mathrm{PSp}^{\mathbb{N}}_{\mathrm{sSet}}$ coincide; and
- the fibrant objects of $\overline{\mathrm{Sp}}^{\mathbb{N}}_{\mathrm{sSet}}$ and $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ are precisely those pairs (X, P) for which X is a Kan complex and $P \in \mathrm{Sp}^{\mathbb{N}}_{X}$ is a fibrant Ω_X-spectrum.

In any model category $\mathcal{M}$, a morphism $\chi \colon A \to B$ is a weak equivalence precisely if the induced map of homotopy function complexes $\mathrm{map}_{\mathcal{M}}(f, X)$ is a weak equivalence for all fibrant $X \in \mathcal{M}$ [Hir03, Theorem 17.7.7]. The fibrant objects of the localised and global stable model structures coincide, and in both cases for a fibrant (X, P) the homotopy function complexes can be computed by taking Reedy cosimplicial coframings of cofibrant objects. The cofibrant objects of $\overline{\mathrm{Sp}}^{\mathbb{N}}_{\mathrm{sSet}}$ and $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ coincide, and Reedy cosimplicial coframes can be constructed in both cases by using the global projective model structure. It follows that the homotopy function complexes of the localised and global stable model structures can be modelled by the same simplicial sets. In particular the classes of weak equivalences coincide. Since any two of the classes of fibrations, cofibrations and weak equivalences determines the third, we conclude that $\overline{\mathrm{Sp}}^{\mathbb{N}}_{\mathrm{sSet}} = \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ as model categories, so that the global stable model structure is left proper and combinatorial.

We finally turn to the sSet_*-enrichment. At the level of the underlying categories there is a sSet_*-tensoring defined via the external smash product according to the formula

$$K \overline{\oslash} (X, A)_n := (X, K \overline{\wedge} A_n) \cong (X, K \oslash_X A_n)$$

which inherits sequential spectrum structure maps from (X, A) via the natural isomorphisms $K \oslash_X \Sigma_X A \cong \Sigma_X(K \oslash_X A)$ for $A \in R_X$. This assignment preserves colimits in each argument and so, by local presentability and the Adjoint Functor Theorem, defines an adjunction of two variables

$$(-) \overline{\oslash} (-) \colon \mathrm{sSet}_* \times \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}} \longrightarrow \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}.$$

For $(i \colon K \to L)$ and $(g \colon (X, A) \to (Y, B))$ morphisms in sSet_* and R_{sSet} we compute the pushout-product

$$i \,\square\, F_n^{(-)}(g) \cong F_n^{(-)}(i \,\square\, g)$$

for any $n \in \mathbb{N}$. The pushout-product axiom for the sSet_* tensoring on $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ then follows from cofibrant generation, arguing on the sets $\mathcal{I}^*_{\mathrm{Kan}}$, $\mathcal{J}^*_{\mathrm{Kan}}$, $\mathcal{I}^{\mathbb{N}}$ and $\mathcal{J}^{\mathbb{N}}$. This shows that $\mathrm{PSp}^{\mathbb{N}}_{\mathrm{sSet}}$ is a sSet_*-model category.

To show that $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ is a sSet_*-model category we argue as follows. For $i \colon K \to L$ a cofibration in sSet_*, the functors $K \overline{\oslash} (-)$ and $L \overline{\oslash} (-)$ are left Quillen for the global projective model structure. For any morphism of the form $\zeta^X_n(A)$ for some $n \in \mathbb{N}$, $X \in \mathrm{sSet}$ and $A \in R_X$ we have a natural isomorphism

$$K \overline{\oslash} \zeta^X_n(A) \cong \zeta^X_n(K \oslash_X A)$$

and similarly for $K \mapsto L$. It follows that $K \overline{\oslash} (-)$ and $L \overline{\oslash} (-)$ send maps in $\mathbf{S}^{\mathrm{glob}}_{\mathbb{N}}$ to stable weak equivalences and so descend to left Quillen endofunctors of the global stable model category. For $j \colon (X, A) \to (Y, B)$ a stable weak equivalence we therefore have a diagram with weak equivalences and cofibrations as marked

$$\begin{tikzcd}
K \overline{\oslash} (X, A) \arrow[r, tail] \arrow[d, tail, "\wr"'] & L \overline{\oslash} (X, A) \arrow[d, tail, "\wr"'] \arrow[dr, tail, "\sim"] & \\
K \overline{\oslash} (Y, B) \arrow[r, tail] & P \arrow[r, tail, "i \,\square\, j"] & K \overline{\oslash} (Y, B),
\end{tikzcd}$$

where P is the pushout. By the 2-out-of-3 property we conclude that $i \,\square\, j$ is a stable weak equivalence, which completes the proof of the pushout-product axiom for the sSet_*-enrichment of the global stable model structure on sequential parametrised spectra. □

Corollary 1.2.35. *For each* $X \in \mathrm{sSet}$ *there is a faithful* sSet_**-enriched inclusion functor* $\imath_X \colon \mathrm{Sp}^{\mathbb{N}}_X \to \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ *which is left and right Quillen.*

Proof. The functor $\imath_X$ is determined on objects by the assignment $A \mapsto (X, A)$, from which it is clear that $\imath_X$ is faithful and preserves limits, colimits and sSet_*-tensors. From the characterisation of fibrations, cofibrations and weak equivalences for the global stable model strucutre given in Theorem 1.2.27 it is clear that $\imath_X$ preserves each of these classes of maps. That $\imath_X$ is a left and right Quillen functor now follows from the Adjoint Functor Theorem, since $\mathrm{Sp}^{\mathbb{N}}_X$ and $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ are locally presentable. □

Construction 1.2.36. Using our models for the global homotopy categories of retractive spaces and sequential parametrised spectra, we can assign to each $X \in \mathrm{sSet}$ its *cotangent complex*, which is a morphism $\mathbb{L}_X \to X$ of simplicial sets. Composing the Quillen adjunctions of Lemmas 1.1.18 and 1.2.28

$$\begin{tikzcd} \mathrm{Fun}(\Delta[1], \mathrm{sSet}) \arrow[r, shift left, "(-)_+^{(-)}"] \arrow[r, phantom, "\perp"] & R_{\mathrm{sSet}} \arrow[l, shift left, "U"] \arrow[r, shift left, "F_0^{(-)}"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}} \arrow[l, shift left, "\mathrm{Ev}_0"] \end{tikzcd}$$

the cotangent complex of X is obtained via the derived unit of the adjunction at the cofibrant object $(\mathrm{id}_X : X \to X) \in \mathrm{Fun}(\Delta[1], \mathrm{sSet})$:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\mathrm{id}_X"'] & \mathbb{L}'_X \arrow[d] \\ X \arrow[r, "f"] & X', \end{tikzcd}$$

namely we set $\mathbb{L}_X$ to be the pullback of $\mathbb{L}'_X$ along f. By definition, the right hand vertical morphism in this diagram is obtained as the zeroth space of a fibrant replacement $(X', \mathbb{F}_X)$ of $(X, F_0^X(X \coprod X))$. In particular there is a weak equivalence $f : X \to X'$ of simplicial sets (as in the diagram above) and a diagram of stable weak equivalences

$$F_0^{X'}(X' \coprod X') \longleftarrow F_0^{X'}(X \coprod X') \cong f_! F_0^X(X \coprod X) \longrightarrow \mathbb{F}_X$$

of parametrised X'-spectra. We can compute the stable homotopy groups of the homotopy fibres of $\mathbb{F}_X$, following the proof of Lemma 1.2.12. Fixing $x : * \to X$, we can restrict to the path component of $f(x) \in X'$, which we can without loss of generality take to be reduced. With these caveats in mind, writing $\pi_{X'} : \mathbb{P}X' \to X'$ for Kan's simplicial path fibration and $\rho : \mathbb{P}X' \to *$ for the terminal map the stable homotopy groups of the homotopy fibre of $\mathbb{F}_X$ at $f(x)$ are then equal to the stable homotopy groups of

$$\rho_! \pi^*_{X'}\left(F_0^{X'}(X' \coprod X')\right) \cong F_0^*(\mathbb{P}X' \coprod *),$$

which is seen to be stably weak equivalent to the sphere spectrum $\mathbb{S} \cong F_0(S^0)$ via F_0 applied to the weak equivalence of pointed spaces $\mathbb{P}X' \coprod * \to * \coprod * = S^0$.

Since $\mathbb{L}_{X'}$ is the zeroth space of the fibrant $\Omega_{X'}$-spectrum $\mathbb{F}_X$, the map $\mathbb{L}_{X'} \to X'$ is a fibration. Our argument above identifies the homotopy fibre of $\mathbb{L}_{X'}$ at $f(x)$ with the infinite loop space $\Omega^\infty \Sigma^\infty S^0$. The upshot is that the cotangent complex $\mathbb{L}_X \to X$ is the fibration obtained by replacing each of the (homotopy) fibres of the identity map $X \to X$ with the sphere spectrum $\mathbb{S}$. An alternative way that we could have arrived at the same conclusion is by using Corollary 1.2.24.

We also note that the free X-spectrum $F_0^X(X \coprod X)$ is the trivial $\mathbb{S}$-bundle over X, since

$$F_0^X(X \coprod X) \cong X^* F_0^*(* \coprod *) = X^* \mathbb{S}$$

by universality of colimits in sSet, where here $X : X \to *$ denotes the terminal map. Applying the pushforward $X_! : \mathrm{Sp}^{\mathbb{N}}_X \to \mathrm{Sp}^{\mathbb{N}}$ yields

$$X_! F_0^X(X \coprod X) \cong F_0^*(X \coprod *) \cong \Sigma^\infty_+ X$$

which is the suspension spectrum of X.

Example 1.2.37. Consider a pointed simplicial set $x\colon * \to X$, which determines the retractive space $(X \to * \coprod X \to X)$ over X. Applying the free X-spectrum functor F_0^X yields the sequential X-spectrum $F_0^X(* \coprod X)$, whose n-th term is the wedge sum $S^n \vee_x X$. We argue as in the proof of Lemma 1.2.12, and as in that case we restrict to the path component of x, which we can take to be reduced. Writing $\pi_X\colon \mathbb{P}X \to X$ for Kan's simplicial path fibration and $\rho\colon \mathbb{P}X \to *$ for the terminal map, the stable homotopy groups of the homotopy fibre of $F_0^X(* \coprod X)$ at x are the same as those of

$$\rho_! \pi_X^* \left(F_0^X\left(* \coprod X \right) \right) \cong \rho_! F_0^{\mathbb{P}X}(\mathbb{G}X \coprod \mathbb{P}X) \cong F_0^*(\mathbb{G}X \coprod *),$$

where we have used that the pullback of the cospan $\mathbb{P}X \to X \leftarrow *$ is $\mathbb{G}X$, Kan's model of the simplicial loop group of X at x. Thus, we have shown that the homotopy fibre of the X-spectrum $F_0^X(* \coprod X)$ at x is $\mathbb{S}[\mathbb{G}X] \cong \Sigma^\infty_+ \mathbb{G}X$, the suspension spectrum of the loop group of X at x. Note that this is in perfect agreement with Corollary 1.2.24.

1.2.3 Global Symmetric Stabilisation

The global sequential stabilisation construction has one deficiency, in that the external smash product on R_{sSet} does not extend to a symmetric monoidal structure on $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$. To remedy this we introduce another model for the global homotopy theory of parametrised spectra—the *global symmetric stabilisation*. As in the previous section our approach combines the Grothendieck construction for model categories with a direct construction, which allows us to deduce a number of useful properties along the way.

Construction 1.2.38. Define SymSeq as the full subcategory of $\mathrm{Fun}(\mathbf{\Sigma}, R_{\mathrm{sSet}})$ on symmetric sequences $P\colon n \mapsto P_n$ for which the sequence $n \mapsto \mathrm{base}(P_n)$ of base spaces is constant on some $X \in \mathrm{sSet}$ (with trivial Σ_n-action at level n). The category SymSeq is the inverse image of the accessibly embedded subcategory $\Delta_{\mathrm{sSet}} \hookrightarrow \mathrm{Fun}(\mathbf{\Sigma}, \mathrm{sSet})$ of constant sequences under the functor

$$\mathrm{Fun}(\mathbf{\Sigma}, R_{\mathrm{sSet}}) \xrightarrow{\mathrm{base}} \mathrm{Fun}(\mathbf{\Sigma}, \mathrm{sSet}).$$

Since this latter functor preserves limits and colimits and the functor categories in question are locally presentable, SymSeq is accessible by [Lur09, Corollary A.2.6.5]. Since the inclusion $\mathrm{SymSeq} \hookrightarrow \mathrm{Fun}(\mathbf{\Sigma}, R_{\mathrm{sSet}})$ creates limits and colimits, it follows that SymSeq is in fact locally presentable.

Lemma 1.2.39. *The Day convolution product on* $\mathrm{Fun}(\mathbf{\Sigma}, R_{\mathrm{sSet}})$ *restricts to a symmetric monoidal structure on* SymSeq.

Proof. Since the monoidal unit for Day convolution

$$n \longmapsto \begin{cases} S^0 & \text{for } n = 0 \\ 0_* & \text{otherwise} \end{cases}$$

is in SymSeq (the base space sequence is constant on $*$) it suffices to show that for $P, Q \in \mathrm{SymSeq}$ the sequence $\mathrm{base}(P \otimes_{\mathrm{Day}} Q)$ of base spaces is constant. Let X and Y be the base spaces of the symmetric sequences P and Q respectively. The coend formula for Day convolution reads

$$P \otimes_{\mathrm{Day}} Q \colon n \longmapsto \int^{(p,q)\in\mathbf{\Sigma}} \mathbf{\Sigma}(p+q,n)_+ \mathbin{\overline{\wedge}} (P_p \mathbin{\overline{\wedge}} Q_q),$$

with $\overline{\wedge}$ the external smash product on R_{sSet} and $\mathbf{\Sigma}(p+q,n)_+ := \mathbf{\Sigma}(p+q,n) \coprod *$. Applying base yields

$$\mathrm{base}(P \otimes_{\mathrm{Day}} Q) : n \longmapsto \int^{(p,q)\in\mathbf{\Sigma}} * \times X \times Y \cong X \times Y$$

as required. □

The symmetric monoidal functor $\mathbf{\Sigma} \to R_{\mathrm{sSet}}$ which sends $n \mapsto (*, S^n)$ determines a commutative monoid $\mathbb{S}$ for the Day convolution product which lies in SymSeq.

Definition 1.2.40. Let $\mathbb{S}-$Mod be the category of $\mathbb{S}$-modules in SymSeq with respect to the Day convolution product. A pairing $\mathbb{S} \otimes_{\mathrm{Day}} P \to P$ is equivalent to the data of a collection of $(\Sigma_p \times \Sigma_q)$-equivariant maps

$$m_{p,q} \colon S^p \mathbin{\overline{\wedge}} P_q \longrightarrow P_{p+q}.$$

The maps $m_{p,q}$ are completely determined by the maps $\sigma_n := m_{1,n} \colon S^1 \mathbin{\overline{\wedge}} P_n \to P_{n+1}$ in the sense that $m_{p,q}$ is equal to the composite

$$S^p \mathbin{\overline{\wedge}} P_q \xrightarrow{S^{p-1} \overline{\wedge} \sigma_q} S^{p-1} \mathbin{\overline{\wedge}} P_{q+1} \xrightarrow{S^{p-2} \overline{\wedge} \sigma_{q+1}} \cdots \xrightarrow{\sigma_{q+p-1}} P_{p+q}.$$

for all $p, q \in \mathbb{N}$. The *global category of parametrised symmetric spectra* is the full subcategory $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}} \hookrightarrow \mathbb{S}-$Mod on those $\mathbb{S}$-modules for which the maps

$$\mathrm{base}(\sigma_n) \colon * \times X \cong X \longrightarrow X$$

are the identified with the identity on $X = \mathrm{base}(P)$ via the canonical isomorphism $* \times X \cong X$ (so that $\mathrm{base}(m_{p,q}) \cong \mathrm{id}_X$ for all p and q). The global category of parametrised symmetric spectra inherits a projection functor base: $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}} \to \mathrm{sSet}$.

Since $\mathbb{S}$ is a commutative monoid, the category $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ inherits a *(symmetric) external smash product*, a symmetric monoidal structure determined on objects by

$$P \mathbin{\overline{\wedge}}^{\Sigma} Q := \mathrm{colim} \left(P \otimes_{\mathrm{Day}} \mathbb{S} \otimes_{\mathrm{Day}} Q \underset{P \otimes \rho_Q}{\overset{\rho_P \otimes Q}{\rightrightarrows}} P \otimes_{\mathrm{Day}} Q \right),$$

where ρ_P, ρ_Q are the $\mathbb{S}$-actions on P and Q respectively. Applying base to the defining colimit diagram shows that $P \mathbin{\overline{\wedge}}^{\Sigma} Q$ does indeed lie in $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$. The external smash product covers the cartesian product on sSet.

Lemma 1.2.41. $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ *is locally presentable and the external smash product preserves colimits in each argument.*

Proof. The Day convolution product on SymSeq is easily seen to preserve colimits in both arguments. This implies that the monad $\mathbb{S} \otimes_{\mathrm{Day}} (-)$ on SymSeq is preserves all colimits, and $\mathbb{S}-$Mod is locally presentable. Moreover, this shows that colimits (and limits) in $\mathbb{S}-$Mod are created by the forgetful functor to SymSeq.

Let $\beta\colon \mathbb{S}{-}\mathrm{Mod} \to \mathrm{Fun}(\mathbf{\Sigma}\times\Delta[1], \mathrm{sSet})$ be the functor determined on objects by sending the $\mathbb{S}$-module P to the functor

$$n \longmapsto \mathrm{base}\big(S^1 \barwedge P_n \to P_{n+1}\big).$$

The functor β preserves colimits, so is an accessible functor between locally presentable categories. Letting $\Delta_{\mathrm{sSet}} \hookrightarrow \mathrm{Fun}(\mathbf{\Sigma}\times\Delta[1], \mathrm{sSet})$ be the full accessibly embedded subcategory on constant functors of the form $n \mapsto (\mathrm{id}_X\colon X \to X)$ we have that $\mathrm{Sp}^\Sigma_{\mathrm{sSet}}$ is the inverse image $(\beta)^{-1}(\Delta_{\mathrm{sSet}})$ and so is accessible. A straightforward check shows that the composite functor

$$\mathrm{Sp}^\Sigma_{\mathrm{sSet}} \hookrightarrow \mathbb{S}{-}\mathrm{Mod} \to \mathrm{SymSeq} \to \mathrm{Fun}(\mathbf{\Sigma}, R_{\mathrm{sSet}})$$

creates limits and colimits and so $\mathrm{Sp}^\Sigma_{\mathrm{sSet}}$ is locally presentable.

From the above argument it is clear that the external smash product on $\mathbb{S}{-}\mathrm{Mod}$ preserves colimits in each variable, so the assertion now follows. □

The results of §1.2.1 show that assignment of a simplicial set X to the category of symmetric X-spectra determines a pseudfunctor

$$\begin{aligned} \mathrm{Sp}^\Sigma_{(-)}\colon \mathrm{sSet} &\longrightarrow \mathbf{Mod} \\ X &\longmapsto \mathrm{Sp}^\Sigma_X \\ (f\colon X \to Y) &\longmapsto \big((f_! \dashv f^*)\colon \mathrm{Sp}^\Sigma_X \to \mathrm{Sp}^\Sigma_Y\big). \end{aligned}$$

Forgetting the model structures for the moment, we have the

Lemma 1.2.42. *There is a canonical isomorphism of categories between $\mathrm{Sp}^\Sigma_{\mathrm{sSet}}$ and the Grothendieck construction $\int_{\mathrm{sSet}} \mathrm{Sp}^\Sigma_{(-)}$ respecting the projection functors to* sSet*.*

Proof. Unwinding the definitions, an object $P \in \mathrm{Sp}^\Sigma_{\mathrm{sSet}}$ is precisely the data of a symmetric sequence of retractive spaces over $X = \mathrm{base}(P)$ together with the data of a Σ_n-equivariant morphism

$$\sigma_n\colon S^1 \barwedge P_n \longrightarrow P_{n+1}$$

covering $\mathrm{id}_X\colon X \to X$ for each $n \in \mathbb{N}$ such that the composite

$$S^p \barwedge P_q \xrightarrow{S^{p-1}\barwedge \sigma_q} S^{p-1}\barwedge P_{q+1} \xrightarrow{S^{p-2}\barwedge\sigma_{q+1}} \cdots \xrightarrow{\sigma_{q+p-1}} P_{p+q}$$

is $(\Sigma_p \times \Sigma_q)$-equivariant. In light of Remark 1.1.31, this is precisely the data of a symmetric X-spectrum so that we have a canonical isomorphism of the collections of objects of $\mathrm{Sp}^\Sigma_{\mathrm{sSet}}$ and $\int_{\mathrm{sSet}} \mathrm{Sp}^\Sigma_{(-)}$. Further unravelling the definitions, a morphism $\Psi\colon P \to Q$ in $\mathrm{Sp}^\Sigma_{\mathrm{sSet}}$ is to be equivalent to the data of

- a morphism $f = \mathrm{base}(\Psi)\colon X \to Y$ of base spaces; together with
- for each $n \in \mathbb{N}$, a Σ_n-equivariant map $(f, \psi_n)\colon (X, P_n) \to (Y, Q_n)$ covering f such that the diagram

$$\begin{tikzcd} f_!(S^1 \barwedge P_n) \cong S^1 \barwedge f_! P_n \arrow[r, "f_!(\sigma_n^P) = \sigma_n^{f_! P}"] \arrow[d, "S^1 \barwedge \psi_n"'] & f_! P_{n+1} \arrow[d, "\psi_{n+1}"] \\ Q_n \arrow[r, "\sigma_n^Q"] & Q_{n+1} \end{tikzcd}$$

commutes for each n.

Such data are equivalent to morphisms $(f,\psi)\colon (X,P)\to(Y,Q)$ in the Grothendieck construction $\int_{\mathrm{sSet}}\mathrm{Sp}^{\Sigma}_{(-)}$. □

As in the global sequential setting, the category $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ can be equipped with a global projective and global stable model structure:

Theorem 1.2.43. *The category* $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ *is equipped with the* global stable model structure *for which a map of pairs* $(f,\psi)\colon (X,A)\to(Y,B)$ *is*

- *a weak equivalence if* $f\colon X\to Y$ *and* $\psi\colon f_!A\to B$ *are weak equivalences in* sSet *and* Sp^{Σ}_{X} *respectively;*
- *a cofibration if* $f\colon X\to Y$ *and* $\psi\colon f_!A\to B$ *are cofibrations in* sSet *and* Sp^{Σ}_{Y} *respectively; and*
- *a fibration if* $f\colon X\to Y$ *and* $\psi^{\vee}\colon A\to f^*B$ *are fibrations in* sSet *and* Sp^{Σ}_{X} *respectively, where* $\psi^{\vee}$ *is the* $(f_!\dashv f^*)$*-adjunct of* ψ.

The global projective model structure, *denoted* $\mathrm{PSp}^{\Sigma}_{\mathrm{sSet}}$, *is characterised by substituting* $\mathrm{Sp}^{\Sigma}_{(-)}\mapsto\mathrm{PSp}^{\Sigma}_{(-)}$ *in the characterisations of fibrations, cofibrations and weak equivalences given above. There is a Quillen adjunction*

$$\begin{tikzcd} \mathrm{PSp}^{\Sigma}_{\mathrm{sSet}} \arrow[r, shift left, "\mathrm{id}"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\Sigma}_{\mathrm{sSet}} \arrow[l, shift left, "\mathrm{id}"] \end{tikzcd} \tag{1.10}$$

between the global projective and global stable model structures.

Proof. The result is proven by adapting the arguments of §1.2.2 to the symmetric setting. The proof of Lemma 1.2.25 carries over almost verbatim to the symmetric setting to show that the pseudofunctors

$$\mathrm{PSp}^{\Sigma}_{(-)},\ \mathrm{Sp}^{\Sigma}_{(-)}\colon \mathrm{sSet}\longrightarrow \mathbf{Mod}$$

are proper and relative, and that the pushforwards $f_!$ preserve levelwise and stable weak equivalences of parametrised symmetric spectra. The Grothendieck construction for model categories then implies the existence of the model structures as claimed. □

Lemma 1.2.44. *For each n there is a Quillen adjunction*

$$\begin{tikzcd} R_{\mathrm{sSet}} \arrow[r, shift left, "\mathrm{F}_n^{(-)}"] \arrow[r, phantom, "\perp"] & \mathrm{PSp}^{\Sigma}_{\mathrm{sSet}} \arrow[l, shift left, "\mathrm{Ev}_n"] \end{tikzcd}$$

which preserves the projection functors to sSet*. Composing with the adjunction* (1.10) *yields free-forgetful adjunctions for the global stable model structure.*

Proof. This is the symmetric analogue of Lemma 1.2.28, proven in essentially the same way. □

Lemma 1.2.45. *The functor* base: $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}\to\mathrm{sSet}$ *has a two-sided adjoint* $0_{(-)}$ *which sends a space to its corresponding parametrised zero spectrum. The adjunctions* $(0_{(-)}\dashv\mathrm{base})$ *and* $(\mathrm{base}\dashv 0_{(-)})$ *are Quillen for the global projective and global stable model structures.*

Theorem 1.2.46. *The global projective and global stable model structures on $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ are left proper and combinatorial.*

Proof. The model structures $\mathrm{PSp}^{\Sigma}_{\mathrm{sSet}}$ and $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ are left proper by the same argument as in the proof of Theorem 1.2.34. With Lemma 1.2.41 as essential input, adapting the proof of Theorem 1.2.34 also shows that $\mathrm{PSp}^{\Sigma}_{\mathrm{sSet}}$ is combinatorial. A set of generating cofibrations $\mathcal{I}^{\Sigma}$ is the given as the union of the sets

- $\mathcal{I}^{0}_{\mathrm{Kan}}$ obtained by applying the (left Quillen) functor $0_{(-)} \colon \mathrm{sSet} \to \mathrm{PSp}^{\Sigma}_{\mathrm{sSet}}$ to $\mathcal{I}_{\mathrm{Kan}}$; and
- $\mathrm{F}_n(\mathcal{I}^{+}_{\mathrm{Kan}})$ obtained by applying the (left Quillen) functor $\mathrm{F}^{(-)}_n$ to the set $\mathcal{I}^{+}_{\mathrm{Kan}}$ of cofibrations from the proof of Lemma 1.1.24, for each $n \in \mathbb{N}$.

A set $\mathcal{J}^{\Sigma}$ of generating acyclic cofibrations is defined similarly replacing all $\mathcal{I}$'s by $\mathcal{J}$'s. Combinatoriality of $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ follows by identifying the global stable model structure with the left Bousfield localisation

$$\overline{\mathrm{Sp}}^{\Sigma}_{\mathrm{sSet}} := L_{\mathbf{S}^{\mathrm{glob}}_{\Sigma}} \mathrm{PSp}^{\Sigma}_{\mathrm{sSet}}$$

of $\mathrm{PSp}^{\Sigma}_{\mathrm{sSet}}$ at the set of morphisms

$$\mathbf{S}^{\mathrm{glob}}_{\Sigma} := \Big\{ \xi^{\Delta[k]}_n(\partial_k),\ \ \xi^{\Delta[k]}_n(\Delta_k) \Big\}_{n,k \in \mathbb{N}}$$

with ∂_k and Δ_k as in (1.8). The argument is, once again, essentially the same as the proof of Theorem 1.2.34. □

Corollary 1.2.47. *For any $(X, A) \in \mathcal{R}_{\mathrm{sSet}}$ and $n \in \mathbb{N}$ the comparison morphism*

$$(\mathrm{id}, \xi^X_n(A)) \colon \big(X, \mathrm{F}^X_{n+1}(S^1 \barwedge A)\big) \longrightarrow \big(X, \mathrm{F}^X_n(A)\big)$$

is a stable weak equivalence in $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$.

The raison d'être for symmetric stabilisation is that it extends monoidal model structures to the stable context. In the case of the local model categories Sp^{Σ}_X for fixed X we used this to show that the stable model structure is Sp^{Σ}-enriched. In the global setting, our adapted version of the symmetric stabilisation machine makes $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ a symmetric monoidal model category with respect to the external smash product.

Lemma 1.2.48. *For objects $(X, A), (Y, B) \in \mathcal{R}_{\mathrm{sSet}}$ and $m, n \in \mathbb{N}$ there is a natural isomorphism*

$$\mathrm{F}^X_n(A) \barwedge^{\Sigma} \mathrm{F}^Y_m(B) \cong \mathrm{F}^{X \times Y}_{n+m}(A \barwedge B)$$

of parametrised symmetric spectra covering the identity on $X \times Y$.

Proof. Recall from Remark 1.2.16 that $\mathrm{F}^X_n(A)$ is the symmetric X-spectrum with p-th term

$$\mathrm{F}^X_n(A)_p := \begin{cases} \Sigma_p \otimes_{\Sigma_{p-n}} (S^{p-n} \barwedge A) & \text{for } p \geq n \\ 0_X & \text{otherwise.} \end{cases}$$

In levels $k < n + m$ we thus have that

$$\Big(\mathrm{F}^X_n(A) \barwedge^{\Sigma} \mathrm{F}^Y_m(B)\Big)_k \cong 0_{X \times Y} \cong \Big(\mathrm{F}^{X \times Y}_{n+m}(A \barwedge B)\Big)_k.$$

In levels $k \geq n+m$ the k-th term of $\mathrm{F}^X_n(A) \barwedge^{\Sigma} \mathrm{F}^Y_m(B)$ is the coequaliser of the $\mathbb{S}$-module structure maps

$$\begin{tikzcd} \int^{(p,q,r)\in\mathbf{\Sigma}} \mathbf{\Sigma}(p+q+r,k)_+ \barwedge \left(\mathrm{F}^X_n(A)_p \barwedge S^q \barwedge \mathrm{F}^Y_m(B)_r\right) \arrow[d, shift left] \arrow[d, shift right] \\ \int^{(a,b)\in\mathbf{\Sigma}} \mathbf{\Sigma}(a+b,k)_+ \barwedge \left(\mathrm{F}^X_n(A)_a \barwedge \mathrm{F}^Y_m(B)_b\right), \end{tikzcd}$$

which we compute as

$$\Sigma_k \otimes_{\Sigma_{k-m-n}} \left(S^{k-m-n} \barwedge (A \barwedge B)\right),$$

the k-th term of $\mathrm{F}^{X\times Y}_{n+m}(A \barwedge B)$. □

Corollary 1.2.49. $\mathrm{F}^{(-)}_0 \colon R_{\mathrm{sSet}} \to \mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ *is a symmetric monoidal functor.*

Theorem 1.2.50. $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ *is a symmetric monoidal model category with respect to the external smash product.*

Proof. We first remark that the external smash product on $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ is a closed symmetric monoidal structure. This can be seen by either constructing the internal hom objects explicitly or by using Lemma 1.2.41 and the Adjoint Functor Theorem.

The monoidal unit is $\mathbb{S} \cong \mathrm{F}^*_0(S^0)$, which is cofibrant, so we need only verify the pushout-product axiom for the external smash product bifunctor. Recalling the generating sets $\mathcal{I}^{\Sigma}$ and $\mathcal{J}^{\Sigma}$ of Theorem 1.2.46, by Lemma 1.2.41 and Theorem 1.1.27 we have that the sets of pushout-products $\mathcal{I}^{\Sigma} \square \mathcal{I}^{\Sigma}$ and $\mathcal{I}^{\Sigma} \square \mathcal{J}^{\Sigma}$ consist of cofibrations and acyclic cofibrations respectively. By symmetry and cofibrant generation we conclude that the global projective model structure is symmetric monoidal.

To show that the global stable model structure is a monoidal model category with respect to the external smash product, by cofibrant generation it is sufficient to show that $i \square s$ is a stable weak equivalence for $i \in \mathcal{I}^{\Sigma}$ a generating cofibration and s a stable acyclic cofibration. For $i \in \mathcal{I}^0_{\mathrm{Kan}}$ this is clear, since smashing with a parametrised zero spectrum yields a zero spectrum and so the pushout-product $i \square s$ is the functor $0_{(-)}$ applied to the pushout-product on the base spaces. We may therefore suppose that

$$i = \mathrm{F}^{\Delta[k]}_n \left(\begin{tikzcd} \Delta[k] \arrow[r] \arrow[d] & \Delta[k] \coprod \Delta[k] \arrow[d, "\mathrm{id}+\mathrm{id}"] \\ \partial\Delta[k] \coprod \Delta[k] \arrow[ur, "i_k \coprod \mathrm{id}"] \arrow[r, "i_k+\mathrm{id}"'] & \Delta[k] \end{tikzcd} \right)$$

for some $k, n \geq 0$. To ease notation, write $i = \mathrm{F}^{\Delta[k]}_n(\partial_k) \to \mathrm{F}^{\Delta[k]}_n(\Delta_k)$. The functors

$$\mathrm{F}^{\Delta[k]}_n(\partial_k) \barwedge^{\Sigma} (-) \quad \text{and} \quad \mathrm{F}^{\Delta[k]}_n(\Delta_k) \barwedge^{\Sigma} (-)$$

are left Quillen for the global projective model structure. By Lemma 1.2.48 we have

$$\mathrm{F}^X_n(A) \barwedge \xi^Y_m(B) \cong \xi^{X\times Y}_{n+m}(A \barwedge B).$$

Applying this to the set of morphisms $\mathbf{S}_{\Sigma}^{\text{glob}}$, Corollary 1.2.47 implies that the functors

$$\mathrm{F}_n^{\Delta[k]}(\partial_k) \overline{\wedge}^{\Sigma} (-) \quad \text{and} \quad \mathrm{F}_n^{\Delta[k]}(\Delta_k) \overline{\wedge}^{\Sigma} (-)$$

are left Quillen for the global stable model structure. If $s\colon P \to Q$ is now any stably acyclic cofibration we have a diagram

$$\begin{tikzcd}
\mathrm{F}_n^{\Delta[k]}(\partial_k) \overline{\wedge}^{\Sigma} P \arrow[r, tail] \arrow[d, tail, "\wr"'] & \mathrm{F}_n^{\Delta[k]}(\Delta_k) \overline{\wedge}^{\Sigma} P \arrow[d, tail, "\wr"'] \arrow[dr, tail, "\sim"] & \\
\mathrm{F}_n^{\Delta[k]}(\partial_k) \overline{\wedge}^{\Sigma} Q \arrow[r, tail] & R \arrow[r, tail, "i \,\square\, s"] & \mathrm{F}_n^{\Delta[k]}(\Delta_k) \overline{\wedge}^{\Sigma} Q,
\end{tikzcd}$$

where R is the pushout and we have cofibrations and stable equivalences as marked. By the 2-out-of-3 property $i \,\square\, s$ is a stable weak equivalence, as required. □

Corollary 1.2.51. *The global stable model structure on* $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ *is a* Sp^{Σ}*-model category.*

Proof. There is a faithful inclusion functor $\imath_*\colon \mathrm{Sp}^{\Sigma} \to \mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ determined on objects by $P \mapsto (*, P)$. The functor $\imath_*$ preserves fibrations, cofibrations and stable equivalences as well as limits and colimits, so by the Adjoint Functor Theorem is a right and left Quillen functor. It is not too hard to see that $\imath_*$ is a strong monoidal functor, allowing us to define a Sp^{Σ}-tensoring on $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ by

$$(P, (X, A)) \longmapsto (X, \imath_*(P) \overline{\wedge} A).$$

By the Theorem this makes $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ into a Sp^{Σ}-model category. □

Corollary 1.2.52. *For each* $X \in \mathrm{sSet}$ *there is a faithful* Sp^{Σ}*-enriched inclusion functor* $\imath_X\colon \mathrm{Sp}^{\Sigma}_X \to \mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ *which is left and right Quillen.*

Proof. The proof is essentially the same as the previous result, where $\imath_X$ is defined on objects by $A \mapsto (X, A)$. Remark 1.1.31 guarantees that $\imath_X$ preserves Sp^{Σ}-tensors. □

1.2.4 Comparison of Global Models

In this section we prove that our two models for the global homotopy theory of parametrised spectra—namely, $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ and $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$—are Quillen equivalent. We have already seen in Theorem 1.2.23 that this is true for a fixed base space $X \in \mathrm{sSet}$; the global argument bootstraps this result using the Grothendieck construction for model categories. For this we require the following

Lemma 1.2.53. *The assignment*

$$X \longmapsto \mathcal{D}_X := \mathrm{Sp}^{\mathbb{N}}(\mathrm{Sp}^{\Sigma}_X, \Sigma_X)$$

determines a pseudofunctor $\mathcal{D}_{(-)}\colon \mathrm{sSet} \to \mathbf{Mod}$ *which is relative and proper.*

Proof. That the assignment $X \mapsto \mathcal{D}_X$ defines a pseudofunctor $\mathrm{sSet} \to \mathbf{Mod}$ is shown in the proof of Theorem 1.2.23, which also shows that $\mathcal{D}_{(-)}$ is relative and right proper (the latter since if $f\colon X \to Y$ is a fibration then $f^*\colon \mathcal{D}_Y \to \mathcal{D}_X$ is left and right Quillen.

For left properness we observe that the fibrant objects in $\mathcal{D}_X$ are those collections $\mathbb{A} := \{A_{m,n}\}_{m,n \in \mathbb{N}}$ such that

- $A_{m,n} \in R_X$ is fibrant for all m, n;
- the symmetric X-spectrum $A_{m,\bullet}$ is a fibrant Ω_X-spectrum for all m; and
- the sequential X-spectrum $A_{\bullet,n}$ is a fibrant Ω_X-spectrum for all n.

This is shown using the fact that the stable model structure on $\mathcal{D}_X$ is obtained by left Bousfield localisation of the "double projective model structure" (in which weak equivalences and fibrations are levelwise) at the set of morphisms

$$\mathbf{S}^X_{\mathcal{D}} := \left\{ F^X_{m+1,n}(\Sigma_X C) \longrightarrow F^X_{m,n}(C) \right\} \cup \left\{ F^X_{m,n+1}(\Sigma_X C) \longrightarrow F^X_{m,n}(C) \right\},$$

where $m, n \in \mathbb{N}$ and C ranges over domains and codomains of morphisms in $\mathcal{J}^X_{\mathrm{Kan}}$. The weak equivalences between stably fibrant objects are precisely the levelwise weak equivalences. Let $f\colon X \to Y$ be any map of simplicial sets and suppose that $\Psi\colon \mathbb{A} \to \mathbb{B}$ is a stable weak equivalence in $\mathcal{D}_X$. Let $\mathcal{R}$ be a fibrant replacement functor for $\mathcal{D}_X$ for which the comparison maps $\mathbb{A} \to \mathcal{R}(\mathbb{A})$ are acyclic cofibrations. Then applying $f_!\colon \mathcal{D}_X \to \mathcal{D}_Y$ to the naturality square

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "\Psi"] \arrow[d, tail, "\wr"'] & \mathbb{B} \arrow[d, tail, "\wr"] \\
\mathcal{R}(\mathbb{A}) \arrow[r, "\mathcal{R}(\Psi)"] & \mathcal{R}(\mathbb{B})
\end{tikzcd}$$

we have that the vertical and bottom horizontal maps are sent to stable weak equivalences in $\mathcal{D}_Y$ as $f_!$ is a left Quillen functor which preserves levelwise weak equivalences (recall that the latter statement is true since $f_!\colon R_X \to R_Y$ preserves weak equivalences). By the 2-out-of-3 property we have that $f_!(\Psi)$ is a stable weak equivalence. In particular, $\mathcal{D}_{(-)}$ is left proper. □

Remark 1.2.54. The proof also shows that the assertion of the Lemma holds for the pseudofunctor $\mathcal{D}^{\mathrm{dproj}}_{(-)}$ that instead takes the doubly projective model structure.

By Theorem 1.2.23 there are pseudonatural Quillen equivalences

$$\mathrm{Sp}^{\Sigma}_{(-)} \Longrightarrow \mathcal{D}_{(-)} \quad \text{and} \quad \mathrm{Sp}^{\mathbb{N}}_{(-)} \Longrightarrow \mathcal{D}_{(-)}$$

of pseudofunctors $\mathrm{sSet} \to \mathbf{Mod}$. Together with Lemma 1.2.53, the Grothendieck construction for model categories (Theorem A.4.8) applied to these pseudonatural Quillen equivalences completes the proof of

Theorem 1.2.55. *There is a zig-zag of left Quillen equivalences*

$$\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}} \longrightarrow \int_{\mathrm{sSet}} \mathcal{D}_{(-)} \longleftarrow \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$$

commuting with the projection functors to sSet.

Remark 1.2.56. By suitably adapting the arguments of §1.2.2 and §1.2.3 we show that the Grothendieck construction $\int_{\mathrm{sSet}} \mathcal{D}_{(-)}$ is locally presentable. Arguing as in Theorems 1.2.34 and 1.2.46, we then show that the integral stable model structure on $\int_{\mathrm{sSet}} \mathcal{D}_{(-)}$ is left proper and combinatorial by

(1) showing this is true explicitly for the integral (double projective) model structure $\int_{\mathrm{sSet}} \mathcal{D}^{\mathrm{dproj}}_{(-)}$; then

(2) showing that it is true for the integral stable model structure by proving that this is the left Bousfield localisation of $\int_{\mathrm{sSet}} \mathcal{D}^{\mathrm{dproj}}_{(-)}$ at the set $\mathbf{S}^{\mathrm{glob}}_{\mathrm{double}}$ of morphisms

$$\left\{ F^{\Delta[k]}_{m+1,n}(\Sigma_{\Delta[k]}\Delta_k) \to F^{\Delta[k]}_{m,n}(\Delta_k) \right\} \cup \left\{ F^{\Delta[k]}_{m,n+1}(\Sigma_{\Delta[k]}\Delta_k) \to F^{\Delta[k]}_{m,n}(\Delta_k) \right\}$$
$$\cup \left\{ F^{\Delta[k]}_{m+1,n}(\Sigma_{\Delta[k]}\partial_k) \to F^{\Delta[k]}_{m,n}(\partial_k) \right\} \cup \left\{ F^{\Delta[k]}_{m,n+1}(\Sigma_{\Delta[k]}\partial_k) \to F^{\Delta[k]}_{m,n}(\partial_k) \right\}$$

with ∂_k and Δ_k as in (1.8).

The upshot is that the statement of Theorem 1.2.55 is a zig-zag of Quillen equivalences between *left proper combinatorial* model categories. This is used in the next section when we wish to compare model structures on model categories of presheaves localised at Čech covers.

1.3 Twisted Differential Cohomology

Our goal in this section is to extend our models for the global homotopy theory of parametrised spectra to the setting of *twisted differential cohomology*. Differential cohomology, first considered in the guise of the "differential characters" of [CS85], is a refined notion of cohomology for smooth manifolds which captures additional geometric information.

A differential cohomology theory $\widehat{\mathcal{E}}$ assigns to a smooth manifold an abelian group of differential cohomology classes $M \mapsto \widehat{\mathcal{E}}^*(M)$ which obeys some axioms [HS05; BS12]. Each differential cohomology theory has a *curvature map* $\theta\colon \widehat{\mathcal{E}} \to \flat_{\mathrm{dR}}\widehat{\mathcal{E}}$ which projects out geometric information and a *shape map* $\widehat{\mathcal{E}} \to \mathcal{E}$ to an underlying (generalised) cohomology theory which projects out topological information. The curvature and shape maps are related, in that for any smooth manifold M there is a commuting diagram

$$\begin{tikzcd} & \flat_{\mathrm{dR}}\widehat{\mathcal{E}}^*(M) \arrow[dr] & \\ \widehat{\mathcal{E}}^*(M) \arrow[ur, "\theta"] \arrow[dr] & & \underline{\mathcal{E}}^*(M) \\ & \mathcal{E}^*(M) \arrow[ur, "\mathrm{ch}"'] & \end{tikzcd} \tag{1.11}$$

in which $\underline{\mathcal{E}}$ is another generalised cohomology theory which should be understood as a sort of rationalisation of $\mathcal{E}$; the map $\mathrm{ch}\colon \mathcal{E} \to \underline{\mathcal{E}}$ is a sort of generalised Chern character. A differential cohomology theory $\widehat{\mathcal{E}}$ should be thought of as an extension of the generalised cohomology theory $\mathcal{E}$ by some geometric information encoded by $\flat_{\mathrm{dR}}\widehat{\mathcal{E}}$. The basic example is Deligne cohomology for smooth manifolds, for which the fracture square takes the form

$$\begin{tikzcd}
 & \Omega^{n+1}_{\mathrm{cl}}(M) \arrow[dr] & \\
H^{n+1}_{\mathrm{conn}}(M;\mathbb{Z}) \arrow[ur] \arrow[dr] & & H^{n+1}(M;\mathbb{R}) \\
 & H^{n+1}(M;\mathbb{Z}). \arrow[ur] &
\end{tikzcd}$$

In this example, the curvature map takes a circle n-gerbe with connection, represented by a class in $H^{n+1}_{\mathrm{conn}}(M;\mathbb{Z})$, to its curvature $(n+1)$-form. The shape map computes the underlying integral $(n+1)$-cohomology class of the n-gerbe (for a 2-gerbe this is the Dixmier–Douady class), and commutativity of the above diagram witnesses the fact that the cohomology class of the curvature form encodes the torsion-free part of the underlying integral class of the n-gerbe. It is extremely useful to view the general picture through a similar lens, so that a differential cohomology theory $\widehat{\mathcal{E}}$ should be thought of as assigning higher bundles with connective data, whose underlying characteristic classes are controlled by the cohomology theory $\mathcal{E}$.

Differential cohomology theories are encoded by homotopy sheaves of spectra on the site of smooth manifolds [BNV16]. This makes the study of differential cohomology theories amenable to the techniques of axiomatic homotopy theory. Similarly, the nonabelian theory of smooth ∞-groupoids is completely captured by homotopy sheaves of spaces on the site of smooth manifolds (see [Sch17] for a comprehensive survey). In summary we have identifications of:

- smooth ∞-groupoids with homotopy sheaves of spaces;
- differential cohomology theories with homotopy sheaves of spectra; and
- twisted cohomology theories with parametrised spectra.

These identifications suggest that we should model twisted differential cohomology theories as homotopy sheaves of parametrised spectra. The main results of §§1.2.2–1.2.3 provide models for the global homotopy theory of parametrised spectra that are robust enough for us to pursue this approach.

1.3.1 Abstractly: The Smooth Tangent ∞-Topos

Before delving into the model category-theoretic details we outline the details of our construction at the level of $(\infty, 1)$-categories. Our treatment is rather terse and is mainly intended to paint the broad picture and to motivate our concrete construction in terms of model categories. We begin by recalling some of the fundamentals of higher differential geometry. Throughout this section, $\mathcal{S}$ and $\mathcal{S}\mathrm{p}$ denote the $(\infty, 1)$-categories of spaces and spectra respectively ([Lur09; Lur17]).

Definition 1.3.1. Let CartSp be the category whose

- objects are the smooth Cartesian spaces $\mathbb{R}^n$ for $n \geq 0$;
- morphisms are smooth maps of manifolds $\mathbb{R}^n \to \mathbb{R}^m$.

A *good open cover* of a manifold M is a smooth open cover $\mathcal{U} = \{U_i \to M\}_{i \in \mathcal{I}}$ such that each non-empty finite intersection of the U_i is diffeomorphic to a Cartesian space. It is a classical result that any paracompact manifold M admits a good open cover, and that the collection of good open covers is cofinal in the collection of all smooth open covers. The category CartSp has a Grothendieck topology τ determined by the coverage that sends $\mathbb{R}^n$ to the set of its good open covers.

Remark 1.3.2. (CartSp, τ) is a dense subsite of the site Diff of paracompact smooth manifolds equipped with the Grothendieck topology generated by the good open coverage. It follows that categories of sheaves over these sites are equivalent.

Definition 1.3.3. The *smooth* $(\infty,1)$*-topos* is the $(\infty,1)$-topos of $(\infty,1)$-sheaves

$$\mathbf{H} := \mathcal{S}\mathrm{hv}_\tau(\mathrm{CartSp}) \hookrightarrow \mathcal{F}\mathrm{un}(\mathrm{CartSp}^{\mathrm{op}}, \mathcal{S})$$

on the site (CartSp, τ) of Cartesian spaces.

Remark 1.3.4. The smooth ∞-topos $\mathbf{H}$ is a *cohesive* $(\infty,1)$*-topos*, lending $\mathbf{H}$ the interpretation as a mathematical universe which combines differential geometry with homotopy theory in a consistent manner. There is a fully-faithful embedding of the category of (paracompact) smooth manifolds $\mathrm{Man} \hookrightarrow \mathbf{H}$ inside the 0-truncated objects which preserves open covers and transverse fibre products. Diffeological spaces also embed into $\mathbf{H}$ in a similar fashion, so $\mathbf{H}$ provides a context for (infinite-dimensional) differential geometry. One aspect of cohesion is the *shape functor*

$$\Pi\colon \mathbf{H} \longrightarrow \mathcal{S},$$

which sends a smooth ∞-groupoid to its underlying homotopy type. The shape $\Pi(M)$ of a (paracompact) smooth manifold M is precisely the homotopy type of the underlying topological space. There are very many other important consequences of cohesion, which are explored in detail in [Sch17]. We do not dwell on them further here, except to amplify that the differential cohomology fracture squares discussed above are formal consequences of cohesion in the stable setting.

Definition 1.3.5. Let $\mathcal{X}$ be an $(\infty,1)$-topos and $\mathcal{C}$ a presentable $(\infty,1)$-category. A $\mathcal{C}$*-sheaf* on $\mathcal{X}$ is a functor $\mathcal{O}\colon \mathcal{X}^{\mathrm{op}} \to \mathcal{C}$ which preserves all small (∞-)limits. We write

$$\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \hookrightarrow \mathcal{F}\mathrm{un}(\mathcal{X}^{\mathrm{op}}, \mathcal{C})$$

for the full sub-$(\infty,1)$-category spanned by $\mathcal{C}$-sheaves on $\mathcal{X}$.

Lemma 1.3.6. *Let $\mathcal{X} = \mathcal{S}\mathrm{hv}_\tau(C)$ be the $(\infty,1)$-topos of $(\infty,1)$-sheaves on a site (C,τ). Then there are equivalences between the $(\infty,1)$-categories*

(i) $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{p}}(\mathcal{X})$ *of sheaves of spectra on $\mathcal{X}$;*

(ii) $\mathcal{S}\mathrm{tab}(\mathcal{X})$*, the stabilisation of $\mathcal{X}$; and*

(iii) $\mathcal{S}\mathrm{hv}_\tau(C; \mathcal{S}\mathrm{p})$ *of $\mathcal{S}\mathrm{p}$-sheaves on (C,τ).*

Sketch of Proof. This is essentially proven as [Lur11, Remarks 1.2 & 1.3]; we sketch the argument here without diving too far into the details. The infinite loop space functor $\Omega^\infty\colon \mathcal{S}\mathrm{p} \to \mathcal{S}$ preserves limits and so induces a functor

$$\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{p}}(\mathcal{X}) \longrightarrow \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}),$$

where $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X})$ is the $(\infty,1)$-category of (∞-)limit-preserving functors $\mathcal{X}^{\mathrm{op}} \to \mathcal{S}$. The Yoneda embedding induces a canonical equivalence $\mathcal{X} \cong \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X})$. For an $(\infty,1)$-category $\mathcal{C}$ with finite (∞-)limits, the stabilisation $\mathcal{S}\mathrm{tab}(\mathcal{C})$ of $\mathcal{C}$ is the stable-$(\infty,1)$-category obtained as the ∞-limit of the tower of $(\infty,1)$-categories

$$\cdots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*,$$

where $\mathcal{C}_*$ is the $(\infty,1)$-category of pointed objects and Ω is the loop space functor. It follows that $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{p}}(\mathcal{X})$ is the ∞-limit of the tower

$$\cdots \xrightarrow{\Omega} \mathcal{S}\mathrm{hv}_{\mathcal{S}_*}(\mathcal{X}) \xrightarrow{\Omega} \mathcal{S}\mathrm{hv}_{\mathcal{S}_*}(\mathcal{X})$$

which, via the equivalence $\mathcal{X}_* \cong \mathcal{S}\mathrm{hv}_{\mathcal{S}_*}(\mathcal{X})$, is equivalent to $\mathcal{S}\mathrm{tab}(\mathcal{X})$. This shows the equivalence between (i) and (ii).

The equivalence between (i) and (iii) uses the fact that $\mathcal{X} = \mathcal{S}\mathrm{hv}_\tau(C)$ exhibits $\mathcal{X}$ as the τ-local ∞-cocompletion of C. From this it follows that a functor $\mathcal{X}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ preserves limits precisely if its restriction along the Yoneda embedding $C^{\mathrm{op}} \hookrightarrow \mathcal{X}^{\mathrm{op}}$ is τ-local. □

Example 1.3.7. By the Lemma, $\mathcal{S}\mathrm{tab}(\mathbf{H})$ is equivalently the $(\infty,1)$-category of $(\infty,1)$-sheaves of spectra on CartSp and so presents the homotopy theory of (untwisted) differential cohomology theories.

Construction 1.3.8. We review the construction of the *tangent* $(\infty,1)$*-category* [Lur17, §7.3]. For $\mathcal{X}$ an $(\infty,1)$-topos (or, more generally, a presentable $(\infty,1)$-category) there is a presentable fibration $\mathrm{cod}\colon \mathcal{F}\mathrm{un}(\Delta[1],\mathcal{X}) \to \mathcal{X}$ which sends a morphism in $\mathcal{X}$ to its codomain: $(f\colon X \to Y) \mapsto Y$. Via the Grothendieck–Lurie construction, this fibration is identified with the ∞-functor $\mathcal{X}_{/(-)}\colon \mathcal{X}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$

$$\begin{aligned} X &\longmapsto \mathcal{X}_{/X} \\ (f\colon X \to Y) &\longmapsto (f^*\colon \mathcal{X}_{/Y} \to \mathcal{X}_{/X}). \end{aligned}$$

For any $f\colon X \to Y$ the pullback functor $f^*\colon \mathcal{X}_{/Y} \to \mathcal{X}_{/X}$ has a left and right adjoint; in particular it preserves finite (∞-)limits. We can therefore apply the stabilisation construction objectwise to $\mathcal{X}_{/(-)}$ to obtain an ∞-functor $\mathcal{S}\mathrm{tab}(\mathcal{X}_{/(-)})\colon \mathcal{X}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$:

$$\begin{aligned} X &\longmapsto \mathcal{S}\mathrm{tab}(\mathcal{X}_{/X}) \\ (f\colon X \to Y) &\longmapsto (f^*\colon \mathcal{S}\mathrm{tab}(\mathcal{X}_{/Y}) \to \mathcal{S}\mathrm{tab}(\mathcal{X}_{/X})). \end{aligned}$$

The stable base change functors f^* fit into a triple of adjunctions $(f_! \dashv f^* \dashv f_*)$ between $\mathcal{S}\mathrm{tab}(\mathcal{X}_{/X})$ and $\mathcal{S}\mathrm{tab}(\mathcal{X}_{/Y})$). Via the Grothendieck construction for $(\infty,1)$-categories ([Lur09, §3.2]), this ∞-functor defines a presentable fibration

$$\mathrm{base}\colon T\mathcal{X} \longrightarrow \mathcal{X},$$

which is the *tangent* $(\infty,1)$*-category of* $\mathcal{X}$. The $(\infty,1)$-category $T\mathcal{X}$ is thus the result of gluing together the stable ∞-categories $\mathcal{S}\mathrm{tab}(\mathcal{X}_{/X})$ along $\mathcal{X}$. With our hypotheses on $\mathcal{X}$, the $(\infty,1)$-category $T\mathcal{X}$ is presentable.

Remark 1.3.9. The $(\infty,1)$-category $T\mathcal{X}$ also inherits an ∞-functor

$$\mathrm{dom}\colon T\mathcal{X} \longrightarrow \mathcal{X}$$

from the domain projection $\mathrm{dom}\colon \mathcal{F}\mathrm{un}(\Delta[1], \mathcal{X}) \to \mathcal{X}$. This functor admits a left adjoint $\mathbb{L}\colon \mathcal{X} \to T\mathcal{X}$ called the *cotangent complex*, which is a section of $\mathrm{base}\colon T\mathcal{X} \to \mathcal{X}$.

Example 1.3.10. For $\mathcal{S}$ the $(\infty,1)$-category of spaces, the tangent $(\infty,1)$-category $T\mathcal{S}$ describes the global homotopy theory of twisted cohomology theories. Indeed, for a fixed space $B \in \mathcal{S}$, the fibre $T_B\mathcal{S}$ of $T\mathcal{S} \to \mathcal{S}$ at B is $\mathcal{S}\mathrm{tab}(\mathcal{S}_{/B}) \cong \mathcal{S}\mathrm{p}_B$, the stable $(\infty,1)$-category of families of spectra parametrised by B.

The external smash product makes $T\mathcal{S}$ into a closed symmetric monoidal $(\infty,1)$-category such that dom becomes a closed symmetric monoidal ∞-functor. For spaces X, B and a B-spectrum P, the internal hom $\overline{F}(\mathbb{L}X, P)$ is a spectrum parametrised by the mapping space $\mathcal{S}(X,B)$. Fixing a (nonabelian) cocycle $\tau\colon X \to B$, the homotopy fibre of the parametrised function spectrum $\overline{F}(\mathbb{L}X, P)$ at $\tau \in \mathcal{S}(X,B)$ is the spectrum

$$\mathrm{hofib}_\tau \overline{F}(\mathbb{L}X, P) \cong X_* F_X(\mathbb{L}X, \tau^* P),$$

where F_X is the internal hom with respect to the relative smash product on X-spectra and $X_*\colon \mathcal{S}\mathrm{p}_X \to \mathcal{S}\mathrm{p}$ sends an X-spectrum to its spectrum of global sections. Since the cotangent complex $\mathbb{L}X \cong X^*\mathbb{S}$ is the trivial $\mathbb{S}$-bundle over X in this setting, the stable homotopy groups of $\mathrm{hofib}_\tau \overline{F}(\mathbb{L}X, P)$ compute the τ-twisted P-cohomology of X. In the case that $B = *$, we recover the usual P-cohomology of X by using the collapse functor $X_!\colon \mathcal{S}\mathrm{p}_X \to \mathcal{S}\mathrm{p}$ which sends $\mathbb{L}X \mapsto \Sigma^\infty_+ X$.

According to a general yoga, the hom-spaces of an $(\infty,1)$-topos $\mathcal{X}$ encode intrinsic nonabelian cohomology. This is by direct analogy with the case that $\mathcal{X} = \mathcal{S}$ is the $(\infty,1)$-topos of spaces, where the homotopy groups $\pi_n\mathcal{S}(X,K)$ are by definition the cohomology groups of X with (nonabelian) K-coefficients. Returning to the general setting, the stabilisation $\mathcal{S}\mathrm{tab}(\mathcal{X})$ inherits a closed symmetric monoidal structure via the *smash product*, which is a stabilised version of the categorical product (this is spelled out in [Lur17, §6.2.4], for example), and there is an adjunction

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift left=1.5ex, "\Sigma^\infty_+"] \arrow[r, phantom, "\perp"] & \mathcal{S}\mathrm{tab}(\mathcal{X}) \arrow[l, shift left=1.5ex, "\Omega^\infty"] \end{tikzcd}$$

such that Σ^∞_+ is a symmetric monoidal ∞-functor. In the $\mathcal{X} = \mathcal{S}$ case these are the familiar smash product and suspension spectrum and infinite loop space ∞-functors. The stabilisation $\mathcal{S}\mathrm{tab}(\mathcal{X})$ is naturally a Sp-enriched $(\infty,1)$-category, and for a stable object $A \in \mathcal{S}\mathrm{tab}(\mathcal{X})$ and $X \in \mathcal{X}$, the stable homotopy groups of the hom-spectrum $[\Sigma^\infty_+ X, A]_\mathcal{X}$ compute the intrinsic (abelian) A-cohomology groups of X. In light of this, the tangent $(\infty,1)$-category $T\mathcal{X}$ encodes intrinsic *twisted* cohomology groups (compare Example 1.3.10).

Remark 1.3.11. By Example 1.3.7 the intrinsic cohomology of $\mathbf{H}$ *is* (nonabelian) differential cohomology. The tangent $(\infty,1)$-category $T\mathbf{H}$ thus encodes *twisted* differential cohomology, for all possible choices of differential twist.

An explicit construction of the tangent $(\infty,1)$-category $T\mathcal{X}$ proceeds via excisive functors [Lur17, Proposition 7.3.1.10]. Writing $\mathcal{S}^{\mathrm{fin}}_*$ for the $(\infty,1)$-category of finite pointed spaces, a functor $F\colon \mathcal{S}^{\mathrm{fin}}_* \to \mathcal{X}$ is *excisive* if

- F takes $*$ to the terminal object of $\mathcal{X}$; and
- F sends ∞-pushouts in $\mathcal{S}^{\mathrm{fin}}_*$ to ∞-pullbacks in $\mathcal{X}$.

Write $\mathrm{Exc}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{X}) \hookrightarrow \mathcal{F}\mathrm{un}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{X})$ for the full sub-$(\infty,1)$-category spanned by excisive functors. By [Lur17, Proposition 1.4.2.24] there is a canonical natural equivalence of $(\infty,1)$-categories $\mathrm{Exc}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{X}) \cong \mathcal{S}\mathrm{tab}(\mathcal{X})$. Via this equivalence, the tangent $(\infty,1)$-category $T\mathcal{X}$ is seen to be equivalent to the full sub-$(\infty,1)$-category of the fibre product of $(\infty,1)$-categories

$$\mathcal{F}\mathrm{un}(\mathcal{S}_*^{\mathrm{fin}} \times \Delta[1], \mathcal{X}) \xrightarrow{\mathrm{cod}} \mathcal{F}\mathrm{un}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{X}) \xleftarrow{\mathrm{const}} \mathcal{X}$$

spanned by excisive functors $\mathcal{S}_*^{\mathrm{fin}} \to \mathcal{X}_{/X} \cong \mathrm{cod}^{-1}(X)$ for some $X \in \mathcal{X}$.

Lemma 1.3.12. *For an $(\infty,1)$-topos $\mathcal{X}$, there is an equivalence of $(\infty,1)$-categories between $T\mathcal{X}$ and $\mathcal{S}\mathrm{hv}_{T\mathcal{S}}(\mathcal{X})$, where $T\mathcal{S}$ is the tangent $(\infty,1)$-category of spaces.*

Proof. The Yoneda embedding implies an equivalence $\mathcal{X} \cong \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X})$ so that we have an equivalence

$$\mathcal{F}\mathrm{un}(\mathcal{S}_*^{\mathrm{fin}} \times \Delta[1], \mathcal{X}) \prod_{\mathcal{F}\mathrm{un}\left(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{X}\right)} \mathcal{X} \cong \mathcal{F}\mathrm{un}(\mathcal{S}_*^{\mathrm{fin}} \times \Delta[1], \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X})) \prod_{\mathcal{F}\mathrm{un}\left(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X})\right)} \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}).$$

Via the natural equivalences $\mathcal{F}\mathrm{un}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{F}\mathrm{un}(\mathcal{D}, \mathcal{E}))$, the right-hand side is identified with the full sub-$(\infty,1)$-category of limit-preserving functors

$$\mathcal{X}^{\mathrm{op}} \longrightarrow \mathcal{F}\mathrm{un}(\mathcal{S}_*^{\mathrm{fin}} \times \Delta[1], \mathcal{S}) \prod_{\mathcal{F}\mathrm{un}\left(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}\right)} \mathcal{S}.$$

Under the composite equivalence $T\mathcal{X}$ is identified with limit-preserving functors $\mathcal{X}^{\mathrm{op}} \to T\mathcal{S}$. Explicitly, an excisive functor $F\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{X}_{/X}$ is identified with the sheaf of excisive functors

$$Y \longmapsto \left(F_Y\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}_{/\mathcal{X}(Y,X)}\right)$$

where $F_Y(K) = \mathcal{X}(Y, F(K))$ for $K \in \mathcal{S}_*^{\mathrm{fin}}$. □

Remark 1.3.13. The proof of the Lemma uses the following fact. Let

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{q'} & \mathcal{X} \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{Y}' & \xrightarrow{q} & \mathcal{Y} \end{array}$$

be a homotopy cartesian diagram of $(\infty,1)$-categories (in the Joyal model structure). Suppose that for $K \in \mathrm{sSet}$, $\mathcal{X}$ and $\mathcal{Y}'$ admit limits for all diagrams indexed by K and p, q preserve limits of diagrams indexed by K. Then

(i) a diagram $f^{\triangleleft}\colon K^{\triangleleft} \to \mathcal{X}'$ is a limit of $f = f^{\triangleleft}|_K\colon K \to \mathcal{X}'$ if and only if $p \circ f^{\triangleleft}$ and $q \circ f^{\triangleleft}$ are limit diagrams so that p and q preserve K-indexed limits; and

(ii) every diagram $f\colon K \to \mathcal{X}'$ has a limit.

The dual statement for colimits is proven as [Lur09, Lemma 5.4.5.5]; the statement for limits is proven in similar fashion.

Corollary 1.3.14. *Let $\mathcal{X} = \mathcal{S}\mathrm{hv}_\tau(C)$ be the $(\infty,1)$-topos of $(\infty,1)$-sheaves on a site (C,τ). Then $T\mathcal{X}$ is equivalent to the $(\infty,1)$-category $\mathcal{S}\mathrm{hv}_\tau(C; T\mathcal{S})$ of $T\mathcal{S}$-valued sheaves on (C,τ).*

Proof. This follows immediately from the Lemma, since $\mathcal{X}$ is by definition the τ-local ∞-cocompletion of C, so that $\mathcal{X}^{\mathrm{op}} \to T\mathcal{S}$ preserves limits precisely if the restriction along the Yoneda embedding $C^{\mathrm{op}} \hookrightarrow \mathcal{X}^{\mathrm{op}}$ is τ-local. □

We observed in Remark 1.3.11 that $T\mathbf{H}$ is the $(\infty,1)$-category which encodes the global homotopy theory of twisted differential cohomology. Since $\mathbf{H} = \mathcal{S}\mathrm{hv}_\tau(\mathrm{CartSp})$, Corollary 1.3.14 allows us to model $T\mathbb{H}$ via $T\mathcal{S}$-valued sheaves on (CartSp,τ). In the next section we show that our work in §1.2.2 and §1.2.3 allows us to produce model categories presenting the $(\infty,1)$-category $T\mathbf{H} \cong \mathcal{S}\mathrm{hv}_\tau(\mathrm{CartSp}; T\mathcal{S})$.

1.3.2 Concretely: Sheaves of Parametrised Spectra

In this section, we use our models $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ and $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ for the global homotopy theory of parametrised spectra to build a model category presenting the tangent $(\infty,1)$-category $T\mathbf{H}$. According to Corollary 1.3.14 we must find a model presentation of $\mathcal{S}\mathrm{hv}_\tau(\mathrm{CartSp}; T\mathcal{S})$. We first observe that $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ and $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ are presentations of $T\mathcal{S}$:

Remark 1.3.15. Let $\mathcal{M}$ be a model category and $\mathcal{F}\colon \mathcal{M} \to \mathbf{Mod}$ a proper relative pseudofunctor. These data give rise to a map of $(\infty,1)$-categories

$$\mathcal{F}^\circ\colon \mathcal{M}^\circ \longrightarrow \mathcal{C}\mathrm{at},$$

where $\mathcal{C}\mathrm{at}$ is the $(\infty,1)$-category of $(\infty,1)$-categories and $\mathcal{M}^\circ$ is the $(\infty,1)$-category presented by the model category $\mathcal{M}$. Applying the Grothendieck–Lurie construction to $\mathcal{F}^\circ$ gives coCartesian fibration $\int_{\mathcal{M}^\circ} \mathcal{F}^\circ \to \mathcal{M}^\circ$. There is a natural equivalence of $(\infty,1)$-categories

$$\begin{tikzcd} \Big(\int_{\mathcal{M}} \mathcal{F}\Big)^\circ \arrow[rr, "\cong"] \arrow[dr] & & \int_{\mathcal{M}^\circ} \mathcal{F}^\circ \arrow[dl] \\ & \mathcal{M}^\circ & \end{tikzcd}$$

over $\mathcal{M}^\circ$ so that the model category-theoretic Grothendieck construction presents the Grothendieck–Lurie construction for $(\infty,1)$-categories (see also Remark A.4.7).

For any simplicial set X, the model categories $\mathrm{Sp}^{\mathbb{N}}_X$ and Sp^{Σ}_X present the stabilisation $\mathcal{S}\mathrm{tab}(\mathcal{S}_{/X})$. Comparing with Construction 1.3.8, we find that there are natural equivalences of $(\infty,1)$-categories

$$\Big(\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}\Big)^\circ \cong \Big(\int_{\mathrm{sSet}} \mathrm{Sp}^{\mathbb{N}}_{(-)}\Big)^\circ \cong \int_{\mathcal{S}} \mathcal{S}\mathrm{tab}(\mathcal{S}_{/(-)}) \cong \Big(\int_{\mathrm{sSet}} \mathrm{Sp}^{\Sigma}_{(-)}\Big)^\circ \cong \Big(\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}\Big)^\circ,$$

so that $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ and $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ both present $T\mathcal{S}$. Moreover, the external smash product on $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ presents $T\mathcal{S}$ as a symmetric monoidal $(\infty,1)$-category.

Let us momentarily fix $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ as our model for $T\mathcal{S}$. Since $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ is combinatorial, the category of $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$-valued presheaves on $\mathcal{CS}$ can be equipped with either the injective or projective model structures. These model structures are Quillen equivalent and model the $(\infty,1)$-category of functors $\mathrm{CartSp}^{\mathrm{op}} \to T\mathcal{S}$

$$\Big(\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}})_{\mathrm{proj/inj}}\Big)^\circ \cong \mathcal{F}\mathrm{un}(\mathrm{CartSp}^{\mathrm{op}}, T\mathcal{S}). \tag{1.12}$$

We fix the projective model structure on $\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}})$, for which weak equivalences and fibrations are objectwise. In order to pass to *sheaves* of parametrised spectra, we need the following

Definition 1.3.16. For $U \in \mathrm{CartSp}$ and $\mathcal{U} = \{U_i \to U\}_{i\in\mathcal{I}}$ a good open cover, for a $(k+1)$-tuple $(i_0, \ldots, i_k)$ of indices in $\mathcal{I}$ write $U_{i_0\cdots i_k} := U_{i_0} \cap \cdots \cap U_{i_k}$ for the intersection in U. Since $\mathcal{U}$ is a good open cover, each $U_{i_0\cdots i_k}$ is either empty or diffeomorphic to U. The *Čech nerve* of $\mathcal{U}$ is the simplicial presheaf on CartSp

$$\check{C}(\mathcal{U})\colon [k] \longmapsto \coprod_{i_0,\ldots,i_k\in\mathcal{I}} U_{i_0\cdots i_k},$$

with simplicial structure maps induced by inclusions and restrictions. Note that $\check{C}(\mathcal{U})$ is in each simplicial degree a coproduct of representable functors by the goodness hypothesis. The covering maps $U_i \to U$ induce a map of simplicial presheaves $\check{C}(\mathcal{U}) \to U$. Let

$$\mathcal{S}_{\check{C}} := \{\check{C}(\mathcal{U}) \to U \mid U \in \mathrm{CartSp}\}$$

denote the set of morphisms of simplicial presheaves defined by Čech nerves of good open covers of Cartesian spaces.

Remark 1.3.17. A model category presentation of $\mathbf{H} := \mathcal{S}\mathrm{hv}_\tau(\mathrm{CartSp})$ is given as the left Bousfield localisation

$$\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet})_{\mathrm{proj,loc}} := L_{\mathcal{S}_{\check{C}}}\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}))_{\mathrm{proj}}$$

of the projective model structure on simplicial presheaves at the set $\mathcal{S}_{\check{C}}$ of Čech nerves. The key point that makes this model presentation work is the fact that sSet is left proper and combinatorial; these features are inherited by the projective model structure on simplicial presheaves and are required to implement the left Bousfield localisation.

The fibrant objects for the local projective model structure are the *simplicial homotopy sheaves on* CartSp (or *smooth* ∞*-stacks*); those functors $F\colon \mathrm{CartSp}^{\mathrm{op}} \to \mathrm{sSet}$ such that

- the simplicial set $F(U)$ is a Kan complex for all $U \in \mathrm{CartSp}$; and
- F satisfies Čech descent; so that for any Cartesian space U and good open cover $\mathcal{U} = \{U_i \to U\}_{i\in\mathcal{I}}$, the morphism induced by the Čech nerve

  $$F(U) \longrightarrow \mathrm{holim}\left(\prod_{i_0} F(U_{i_0}) \rightrightarrows \prod_{i_0,i_1} F(U_{i_0 i_1}) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots\right)$$

  is a weak equivalence.

If only the first condition holds, F is fibrant for the projective model structure and is called a *simplicial homotopy presheaf on* CartSp (or *smooth* ∞*-prestack*).

In general, the cofibrant objects for the local projective model structure are more difficult to characterise. However, we can say that a simplicial presheaf X which is a degreewise coproduct of representables whose degenerate cells split off is cofibrant for the local projective model structure [Dug01c, §9]. In particular, let M be a paracompact manifold which defines a simplicial presheaf via the assignment

$$\underline{M}\colon U \longmapsto \mathrm{Man}(U, M).$$

If $\mathcal{U}$ is a good open cover for M, then the Čech nerve morphism $\check{C}(\mathcal{U}) \to \underline{M}$ is a cofibrant resolution for the local projective model structure.

We lift the set $\mathcal{S}_{\check{C}}$ of Čech nerve maps to parametrised spectra in the following fashion. The functor $\Delta \colon \mathrm{sSet} \to \mathrm{Fun}(\Delta[1], \mathrm{sSet})$ which sends $X \mapsto (\mathrm{id}_X \colon X \to X)$ has a right adjoint $\mathrm{ev}_0 \colon (Z \to Y) \mapsto Z$. The adjunction $(\Delta \dashv \mathrm{ev}_0)$ is Quillen for both the projective and injective model structures on $\mathrm{Fun}(\Delta[1], \mathrm{sSet})$. In §1.2.2 we showed that there is a composite Quillen adjunction

$$\begin{tikzcd} \mathrm{Fun}(\Delta[1], \mathrm{sSet}) \arrow[r, shift left, "(-)_+^{(-)}"] \arrow[r, phantom, "\perp"] & R_{\mathrm{sSet}} \arrow[l, shift left, "U"] \arrow[r, shift left, "F_n^{(-)}"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}} \arrow[l, shift left, "\mathrm{Ev}_n"] \end{tikzcd}$$

for any $n \in \mathbb{N}$, so that there is a family of composite Quillen adjunctions

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, shift left, "\overline{\Sigma}^{\infty-n}_{(-)}"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}} . \arrow[l, shift left, "\overline{\Omega}^{\infty-n}_{(-)}"] \end{tikzcd} \tag{1.13}$$

Note that for all $n \in \mathbb{N}$ the composite $\mathrm{base} \circ \overline{\Sigma}^{\infty-n}_{(-)}$ is the identity functor on sSet. For a parametrised spectrum $(X, A) \in \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ the right adjoint $\overline{\Omega}^{\infty-n}_{(-)}$ sends

$$(X, A) \longmapsto A_n,$$

the total space of the n-th term of the sequential X-spectrum A.

Definition 1.3.18. Define the set of morphisms $T\mathcal{S}_{\check{C}}$ in $\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}})$ as the union of the sets

- $\bigcup_{n \in \mathbb{N}} \overline{\Sigma}^{\infty-n}_{(-)}(\mathcal{S}_{\check{C}})$ obtained by applying the functors $\overline{\Sigma}^{\infty-n}_{(-)}$ to the set $\mathcal{S}_{\check{C}}$ of Čech nerves; and
- $\mathcal{S}^0_{\check{C}}$ obtained by applying $0_{(-)} \colon \mathrm{sSet} \to \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ to the set $\mathcal{S}_{\check{C}}$ of Čech nerves.

We refer to $T\mathcal{S}_{\check{C}}$ as the set of *spectral Čech nerves*.

The projective model structure on the functor category $\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}})$ is left proper and combinatorial; properties which are inherited from $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$. We therefore define the local projective model structure as the left Bousfield localisation

$$\Big(\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}})\Big)_{\mathrm{proj,loc}} := L_{T\mathcal{S}_{\check{C}}} \Big(\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}})\Big)_{\mathrm{proj}}$$

at the set of stabilised Čech nerves $T\mathcal{S}_{\check{C}}$. In light of (1.12), the local projective model structure presents a localisation of the $(\infty, 1)$-category of $T\mathcal{S}$-valued presheaves on CartSp. The following result identifies the fibrant objects for the local projective model structure with the homotopy sheaves of parametrised spectra on CartSp, so that we have equivalences

$$\Big(\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}})_{\mathrm{proj,loc}}\Big)^{\circ} \cong \mathcal{S}\mathrm{hv}_{\tau}(\mathrm{CartSp}; T\mathcal{S}) \cong T\mathbf{H}. \tag{1.14}$$

as $(\infty, 1)$-categories.

Lemma 1.3.19. *A functor $F \in \mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}})$ is fibrant for the local projective model structure precisely if*

(i) *$F(U)$ is a fibrant object of $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ for all $U \in \mathrm{CartSp}$; and*

(ii) *F satisfies Čech descent; for any $U \in \mathrm{CartSp}$ and good open cover $\mathcal{U} = \{U_i \to U\}_{i \in \mathcal{I}}$, the morphism induced by the Čech nerve*

$$F(U) \longrightarrow \operatorname{holim} \left(\prod_{i_0} F(U_{i_0}) \rightrightarrows \prod_{i_0, i_1} F(U_{i_0 i_1}) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots \right)$$

is a weak equivalence of parametrised sequential spectra.

Proof. By general properties of left Bousfield localisation, the fibrant objects for the local projective model strucutre are precisely the $T\mathcal{S}_{\check{C}}$-local projectively fibrant objects. An object is projectively fibrant if and only if (i) is satisfied, so the true heart of the proof lies in showing that Čech descent is equivalent to being $T\mathcal{S}_{\check{C}}$-local.

We now show that for a projectively fibrant functor F, condition (ii) is equivalent to $T\mathcal{S}_{\check{C}}$-locality. A fundamental ingredient in our argument is the fact that every simplicial presheaf is the homotopy colimit over its simplices, so that

$$\check{C}(\mathcal{U}) \cong \operatorname*{hocolim}_{[n]} \Big([n] \mapsto \check{C}(\mathcal{U})_n \Big) . \tag{1.15}$$

[DHI04, Remark 2.1]. Let us fix a good open cover $\mathcal{U} = \{U_i \to U\}_{i \in \mathcal{I}}$ for some Cartesian space U, so that the Čech nerve $\mathfrak{n}_{\mathcal{U}} \colon \check{C}(\mathcal{U}) \to U$ determines a subset of morphisms in $T\mathcal{S}_{\check{C}}$

$$T\mathcal{U} := \Big\{ 0_{(-)}(\mathfrak{n}_{\mathcal{U}}),\ \overline{\Sigma}^{\infty}_{(-)}(\mathfrak{n}_{\mathcal{U}}),\ \overline{\Sigma}^{\infty-1}_{(-)}(\mathfrak{n}_{\mathcal{U}}),\ \ldots \Big\}$$

by applying the functors $0_{(-)}$ and $\overline{\Sigma}^{\infty-n}_{(-)}$ for each $n \in \mathbb{N}$. We show that F is $T\mathcal{U}$-local precisely if it satisfies descent with respect to $\mathcal{U}$; since $T\mathcal{S}_{\check{C}}$ is the union over the $T\mathcal{U}$ this proves the assertion.

Let us write the projectively fibrant $F \colon \mathrm{CartSp}^{\mathrm{op}} \to \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ as

$$F \colon U \longmapsto (X(U), A(U)),$$

so that $A(U)$ is a fibrant $\Omega_{X(U)}$-spectrum. For our fixed Čech nerve morphism $\mathfrak{n}_{\mathcal{U}} \colon \check{C}(\mathcal{U}) \to U$ and projectively fibrant F, the map of homotopy function complexes

$$\big(0_{(-)}(\mathfrak{n}_{\mathcal{U}})\big)^* \colon \operatorname{map}\big(0_{(-)}(U), F\big) \longrightarrow \operatorname{map}\big(0_{(-)}(\check{C}(\mathcal{U}), F\big)$$

is by adjunction equivalent to

$$\operatorname{map}\big(U, \operatorname{base}(F)\big) \longrightarrow \operatorname{map}\big(\check{C}(\mathcal{U}), \operatorname{base}(F)\big).$$

By (1.15), this latter map of derived hom-spaces is equivalent to the map of Kan complexes

$$X(U) \longrightarrow \operatorname{holim} \left(\prod_{i_0} X(U_{i_0}) \rightrightarrows \prod_{i_0, i_1} X(U_{i_0 i_1}) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots \right) . \tag{1.16}$$

Similarly, for each $n \in \mathbb{N}$ the map of homotopy function complexes

$$\Big(\overline{\Sigma}^{\infty-n}_{(-)}(\mathfrak{n}_{\mathcal{U}})\Big)^* \colon \operatorname{map}\big(\overline{\Sigma}^{\infty-n}_{(-)}(U), F\big) \longrightarrow \operatorname{map}\Big(\overline{\Sigma}^{\infty-n}_{(-)}(\check{C}(\mathcal{U})), F\Big)$$

is by adjunction and (1.15) equivalent to the map of Kan complexes

$$A_n(U) \longrightarrow \operatorname{holim}\left(\prod_{i_0} A_n(U_{i_0}) \rightrightarrows \prod_{i_0,i_1} A_n(U_{i_0 i_1}) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots \right). \tag{1.17}$$

From these equivalences it follows that F is $T\mathfrak{U}$-local precisely if (1.16) and (1.17) are weak equivalences for all $n \in \mathbb{N}$, which is the case precisely if F satisfies descent with respect to $\mathfrak{U}$. $\square$

Remark 1.3.20. The local projective model structure on functors $\mathrm{CartSp}^{\mathrm{op}} \to \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ comes equipped with a number of useful Quillen adjunctions. For instance, the projection functor $\mathrm{base}\colon \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}} \to \mathrm{sSet}$ induces a left and right Quillen

$$\mathrm{Fun}\big(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}\big)_{\mathrm{proj,loc}} \longrightarrow \mathrm{Fun}\big(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}\big)_{\mathrm{proj,loc}},$$

with two-sided adjoint induced by $0_{(-)}\colon \mathrm{sSet} \to \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$. At the level of $(\infty,1)$-categories, these functors present $\mathrm{base}\colon T\mathbf{H} \to \mathbf{H}$ and $0_{(-)}\colon \mathbf{H} \to T\mathbf{H}$. The adjunctions of (1.13) prolong to Quillen adjunctions

$$\begin{tikzcd} \mathrm{Fun}\big(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}\big)_{\mathrm{proj,loc}} \arrow[r, shift left=1.5ex, "\overline{\Sigma}^{\infty-n}_{(-)}"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}\big(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}\big)_{\mathrm{proj,loc}} . \arrow[l, shift left=1.5ex, "\overline{\Omega}^{\infty-n}_{(-)}"] \end{tikzcd}$$

At the level of $(\infty,1)$-categories, $\overline{\Sigma}^{\infty}_{(-)}$ presents the cotangent complex $\mathbb{L}\colon \mathbf{H} \to T\mathbf{H}$.

Remark 1.3.21. We shall refer to a fibrant object for the local projective model structure on functors $\mathrm{CartSp}^{\mathrm{op}} \to \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ as a *twisted differential cohomology theory*. For a twisted differential cohomology theory F, the functor $X := \mathrm{base}(F)$ is a smooth ∞-stack, so that F is an *X-twisted differential cohomology theory*.

The results of this section carry over for $\mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ replaced by $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$. Namely, the category of functors $\mathrm{CartSp}^{\mathrm{op}} \to \mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ supports a local projective model structure, obtained by left Bousfield localisation of the projective model structure at the set of spectral Čech nerves[1]. The following result is the symmetric analogue of Lemma 1.3.19 and has the same proof.

Lemma 1.3.22. *A functor $F \in \mathrm{Fun}\big(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}\big)$ is fibrant for the local projective model structure precisely if*

(i) *$F(U)$ is a fibrant object of $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ for all $U \in \mathrm{CartSp}$; and*

(ii) *F satisfies Čech descent; for any $U \in \mathrm{CartSp}$ and good open cover $\mathfrak{U} = \{U_i \to U\}_{i \in \mathfrak{I}}$, the morphism induced by the Čech nerve*

$$F(U) \longrightarrow \operatorname{holim}\left(\prod_{i_0} F(U_{i_0}) \rightrightarrows \prod_{i_0,i_1} F(U_{i_0 i_1}) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots \right)$$

is a weak equivalence of parametrised symmetric spectra.

[1] which, in the symmetric setting, is defined using the functors $\mathrm{F}^{(-)}_n$ instead of the $F^{(-)}_n$

Remark 1.3.23. Similarly to the sequential case, there is an equivalence of $(\infty, 1)$-categories

$$\left(\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Sp}^{\Sigma}_{\mathrm{sSet}})_{\mathrm{proj,loc}}\right)^{\circ} \cong \mathcal{S}\mathrm{hv}_{\tau}(\mathrm{CartSp}; T\mathcal{S}) \cong T\mathbf{H}.$$

This equivalence can be seen directly at the level of model categories. Theorem 1.2.55 implies left Quillen equivalences

$$\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}})_{\mathrm{proj}} \longrightarrow \mathrm{Fun}\big(\mathrm{CartSp}^{\mathrm{op}}, \textstyle\int_{\mathrm{sSet}} \mathcal{D}_{(-)}\big)_{\mathrm{proj}} \longleftarrow \mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Sp}^{\Sigma}_{\mathrm{sSet}})_{\mathrm{proj}}$$

on projective model structures. By Remark 1.2.56 the integral model structure on $\int_{\mathrm{sSet}} \mathcal{D}_{(-)}$ is left proper and combinatorial, properties which are inherited by the projective model structure on $\mathrm{Fun}\big(\mathrm{CartSp}^{\mathrm{op}}, \int_{\mathrm{sSet}} \mathcal{D}_{(-)}\big)$. Taking appropriate left Bousfield localisations we obtain a zig-zag of Quillen equivalences between the local projective model structures on $\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Sp}^{\Sigma}_{\mathrm{sSet}})$ and $\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}})$.

Remark 1.3.24. $\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Sp}^{\Sigma}_{\mathrm{sSet}})$ inherits a closed symmetric monoidal structure from the external smash product on $\mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ via

$$(F \,\overline{\wedge}\, G) \colon U \longmapsto F(U) \,\overline{\wedge}\, G(U).$$

Since CartSp has finite products, a standard argument shows that the projective model structure is a symmetric monoidal model category with respect to $\overline{\wedge}$. Using Lemma 1.3.22 it can be shown that $\overline{\wedge}$ descends to the local projective model structure, presenting $T\mathbf{H}$ as a symmetric monoidal $(\infty, 1)$-category. Since cofibrant objects are difficult to characterise in general and our focus is on twisted differential cohomology as presented by fibrant objects, we do not pursue this further here.

We conclude this section with a discussion of how twisted differential cohomology arises in our model category presentation, focussing on the sequential case for the sake of clarity. We also include a brief discussion of computational techniques. Let $F \colon U \mapsto (X(U), A(U))$ be a fibrant object for the local projective model structure on functors $\mathrm{CartSp}^{\mathrm{op}} \to \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}$ and let M be a paracompact manifold with a good open cover $\mathcal{U}$. Using (1.16) and (1.17) we define

$$X(M) := \mathrm{holim}\left(\prod_{i_0} X(U_{i_0}) \rightrightarrows \prod_{i_0, i_1} X(U_{i_0 i_1}) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \cdots \right) \tag{1.18}$$

and

$$A_n(M) := \mathrm{holim}\left(\prod_{i_0} A_n(U_{i_0}) \rightrightarrows \prod_{i_0, i_1} A_n(U_{i_0 i_1}) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \cdots \right) \tag{1.19}$$

for each $n \in \mathbb{N}$, noting that $A_n(M)$ is naturally a fibrant retractive space over $X(M)$. By passing to refinements, the homotopy types of $X(M)$ and $A_n(M)$ are seen to be independent of the choice of good open cover $\mathcal{U}$ by Lemma 1.3.19. Since each of the $A(U)$ is a fibrant $\Omega_{X(U)}$-spectrum, the retractive spaces $A_n(M)$ assemble into a fibrant $\Omega_{X(M)}$-spectrum $A(M)$.

Definition 1.3.25. Let $F \colon U \mapsto (X(U), A(U))$ be a fibrant object for the local projective model structure and let M be a paracompact manifold with a good open cover $\mathcal{U}$. The pair $(X(M), A(M))$ represents the *X-twisted differential A-cohomology of M*. A *differential twist* is a map $\tau \colon * \to X(M)$; this is an explicit cocycle representative for a class in nonabelian differential cohomology $[\tau] \in \pi_0 X(M) \cong \pi_0 \mathbf{H}(\underline{M}, X)$. For a fixed differential twist τ, the abelian groups

$$A^{\tau+\bullet}(M) := \pi^{\mathrm{st}}_{-\bullet}(\tau^* A(M)),$$

are the *τ-twisted differential A-cohomology groups of M.*

Remark 1.3.26. This terminology is probably not the best, as for a connected space X a parametrised X-spectrum A is exhibits a twisting of the *(homotopy) fibre* of A by X.

Remark 1.3.27. In the setting of Definition 1.3.25, the spectrum $\tau^* A(M)$ is a fibrant Ω-spectrum, so that

$$A^{\tau+n}(M) = \pi_{k-n}(\tau^* A_k(M))$$

for all n and $k \geq 0$ such that $k - n \geq 0$.

Example 1.3.28. Let $\mathcal{E}$ be a fibrant Ω-spectrum representing a generalised cohomology theory $\mathcal{E}^*$. The assignment

$$\underline{\mathcal{E}} \colon U \longmapsto \mathcal{E}$$

defines a fibrant object of the local projective model structure. There are no twists since $\mathrm{base}(\underline{\mathcal{E}}) = *$ is the constant simplicial sheaf on the terminal object. Evaluating on Cartesian spaces we have $\pi^{\mathrm{st}}_{-*}(\underline{\mathcal{E}}(U)) = \pi^{\mathrm{st}}_{-*}\mathcal{E} = \mathcal{E}^*(*)$, the $\mathcal{E}$-cohomology of the point. For a paracompact manifold M the differential cohomology $\underline{\mathcal{E}}^*(M) \cong \mathcal{E}^*(M)$ is the $\mathcal{E}$-cohomology of the underlying topological space of M (using the Nerve Theorem [Bor48], as in [Sch17, Proposition 6.3.2]).

Remark 1.3.29. Let $F \colon U \mapsto (X(U), A(U))$ be a fibrant object for the local projective model structure and let M be a paracompact manifold with a good open cover $\mathcal{U}$. By Remark 1.3.17 and (1.15), we can write $\underline{M} \colon \mathsf{CartSp}^{\mathrm{op}} \to \mathsf{sSet}$ as the homotopy colimit of the simplicial object

$$\check{C}(\mathcal{U}) = \left(\cdots \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \coprod_{i_0, i_1} U_{i_0 i_1} \rightrightarrows \coprod_{i_0} U_{i_0} \right),$$

which is Reedy cofibrant when regarded as a simplicial object of $\mathrm{Fun}(\mathsf{CartSp}^{\mathrm{op}}, \mathsf{sSet})$. The cosimplicial spaces

$$\prod_{i_0} X(U_{i_0}) \rightrightarrows \prod_{i_0, i_1} X(U_{i_0 i_1}) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots \quad \text{and} \quad \prod_{i_0} A_n(U_{i_0}) \rightrightarrows \prod_{i_0, i_1} A_n(U_{i_0 i_1}) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots$$

are Reedy fibrant, so that there are (not necessarily convergent) Bousfield–Kan type spectral sequences

$$E_2^{p,q} = \pi^p \pi_q X(U_\bullet) \Longrightarrow \pi_{q-p} X(M)$$

and

$$E_2^{p,q} = \pi^p \pi_q A_n(U_\bullet) \Longrightarrow \pi_{q-p} A_n(M)$$

for each $n \in \mathbb{N}$ [GJ09, VII Proposition 7.7]. In the case that the cover $\mathcal{U}$ is finite, these spectral sequences converge [GJ09, VII Corollary 6.21].

A choice of twist $\tau \colon * \to X(M)$ determines twists $\tau_{i_0, \dots, i_n} \colon * \to X(U_{i_0 \cdots i_n})$ for each $(n+1)$-tuple of indices $i_0, \dots, i_n \in \mathcal{I}$ by composing with the restriction maps. Each $A_n(U_{i_0 \cdots i_n})$ is a fibrant retractive space over $X(U_{i_0 \cdots i_n})$, so taking homotopy fibres at the $\tau_{i_0,\dots,i_n}$ we have

$$\tau^* A_n(M) \cong \operatorname{holim}\left(\prod_{i_0} \tau_{i_0}^* A_n(U_{i_0}) \rightrightarrows \prod_{i_0,i_1} \tau_{i_0 i_1}^* A_n(U_{i_0 i_1}) \overrightarrow{\underrightarrow{\longrightarrow}} \cdots \right).$$

By Remark 1.3.27, we have the corresponding spectral sequence of Bousfield–Kan type

$$E_2^{p,q} = \pi^p \pi_q\big(\tau_\bullet^* A_n(U_\bullet)\big) \Longrightarrow \pi_{q-p}(\tau^* A_n(M)) \cong A^{\tau+n+p-q}(M),$$

which converge if the cover $\mathcal{U}$ is finite. This is essentially a recapitulation of the *Atiyah–Hirzebruch spectral sequence for twisted differential spectra* [GS17, Theorem 18].

Chapter 2

Rational Parametrised Spectra

An important maxim in algebraic topology is that ignoring torsion phenomena results in a drastic simplification of homotopy theory. This state of affairs is reflected in the work of Quillen and Sullivan, in which the rational homotopy category of 1-connected spaces is completely described in terms of purely algebraic data. Working rationally also results in a drastic simplification of stable homotopy theory; it is a classical fact that the rational stable homotopy category is equivalent to the category of $\mathbb{Z}$-graded rational vector spaces.

In this chapter we study the rational homotopy theory of parametrised spectra. Loosely interpreting a parametrised X-spectrum as a stable homotopy type "on which X acts", stable and unstable rational homotopy theory together suggest that in the torsion free approximation, a parametrised X-spectrum ought to be described by some algebraic data upon which an algebraic incarnation of the rational homotopy type of X acts. The bulk of this chapter is occupied with the proof of our main result, Theorem 2.6.1, in which this vague picture is borne out.

We begin in §2.1 with a short survey of the classical approaches to rational homotopy theory due to Quillen and Sullivan. Quillen's approach identifies the rational homotopy type of a 1-connected space X with either a rational dg Lie algebra $\mathfrak{l}_X$ or a rational dg coalgebra C_X. Sullivan's approach identifies a full subcategory of objects in the rational homotopy category of finite type with connective rational cochain algebras with dimensionwise finite cohomology groups.

In §2.2 we clarify the sense in which a space X "acts" on a parametrised X-spectrum. We establish an equivalence of homotopy theories between parametrised X-spectra and ΩX-module spectra for a connected space X; this is the stable version of the equivalence between G-spaces and spaces over BG. In §2.3 we use this equivalence to pass to rational homotopy theory. A rational homotopy equivalence of parametrised X-spectra is a map of parametrised spectra inducing a rational homotopy equivalence on all stable homotopy fibres. For a reduced simplicial set X, we show that the the rational homotopy theory of parametrised X-spectra is equivalent to the stable homotopy theory of rational representations of the Kan simplicial loop group $\mathbb{G}X$.

Changing tack slightly, §2.4 is concerned with stable homotopy categories of Lie representations. Over $\mathbb{Q}$, the normalisation functor sends a simplicial Lie algebra $\mathfrak{g}$ to a dg Lie algebra $N(\mathfrak{g})$. We show that there is a strong monoidal equivalence between the stable homotopy categories of $\mathfrak{g}$-representations on the one hand and of $N(\mathfrak{g})$-representations on the other. In §2.5, we prove a Koszul duality result which gives an equivalence between derived categories of unbounded dg Lie representations and unbounded dg comodules. The argument relies on spectral sequences which fail to converge for unbounded (co)modules, which forces us to work with a notion of weak equivalence for unbounded dg comodules that is different from quasi-isomorphism (Definition 2.5.24) but that coincides with quasi-isomorphism for bounded-below dg comodules. This is a well-known subtlety of unbounded Koszul duality [Pos11].

The whole of §2.6 is devoted to proving Theorem 2.6.1. This result says that for a 1-connected space X, there are equivalences between:

- the rational homotopy category of parametised X-spectra;
- the homotopy category of unbounded dg representations of any dg Lie model $\mathfrak{l}_X$ of X; and
- the homotopy category of unbouded dg comodules over any dg coalgebra model C_X of X.

We show that these equivalences depend on the rational homotopy type of X in a natural way. These equivalences explicitly exhibit the speculated algebraic presentations of the rational homotopy theory of parametrised spectra. Moreover, we show that the fibrewise smash product of parametrised X-spectra is modelled rationally by the derived tensor product of dg Lie representations. The proof of Theorem 2.6.1 is quite intricate and combines the results of §§2.3–2.5 with aspects of Quillen's work in [Qui69].

Finally, in §2.7 we conclude with a dual approach to the rational homotopy theory of parametrised spectra. We show that Sullivan's PL de Rham theory lifts to modules and parametrised spectra; specifically that unbounded DG modules give rise to parametrised spectra in a manner essentially dual to Theorem 2.6.1. The main result of this final section is Theorem 2.7.34, a sort of "homotopical Sullivan–de Rham–Serre–Swan Theorem" that provides an equivalence between certain finitely-presented rational parametrised spectra and perfect DG modules. In the 1-connected case, we prove a strengthened version of this result as Theorem 2.7.42.

2.1 Rational Homotopy Theory ...

In this section, we provide a lightning review of rational homotopy theory à la Quillen and Sullivan. Along the way, we recall specific results needed in our treatment of the rational homotopy theory of parametrised spectra.

Rational homotopy theory is a simplification of homotopy theory in which all torsion phenomena are systematically ignored. Whereas the homotopy category is obtained by localising the category of spaces at weak homotopy equivalences, the rational homotopy category is defined by localising at the *rational* homotopy equivalences: those maps of spaces inducing isomorphisms on homotopy groups modulo torsion subgroups. A primary motivation for rational homotopy theory comes from Serre's mod-$\mathcal{C}$ theory, which shows that the homotopy groups of spheres are very simple once torsion is ignored. This means that the rational homotopy groups of finite CW complexes are vastly easier to compute. Another consequence of Serre's mod-$\mathcal{C}$ theory is the mod-$\mathcal{C}$ Whitehead Homology Theorem, a particular instance of which is the following

Lemma 2.1.1. *Let $f\colon X \to Y$ be a map of 1-connected pointed simplicial sets. The following are equivalent:*

(i) $\pi_* f\colon \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}$ *is an isomorphism;*

(ii) $f_\bullet\colon H_\bullet(X;\mathbb{Q}) \to H_\bullet(Y;\mathbb{Q})$ *is an isomorphism.*

Proof. [Spa95, p. 512]. □

Corollary 2.1.2. *Let $\psi\colon K \to G$ be a map of connected simplicial groups. The following are equivalent:*

(i) $\pi_*\psi\colon \pi_*(K)\otimes\mathbb{Q} \to \pi_*(G)\otimes\mathbb{Q}$ *is an isomorphism;*

(ii) $\psi_\bullet\colon H_\bullet(K;\mathbb{Q}) \to H_\bullet(G;\mathbb{Q})$ *is an isomorphism.*

Proof. Using Kan's construction $\overline{W}$ of the classifying space (recalled in Appendix B), we have a commuting diagram of homotopy fibre sequences

$$\begin{tikzcd} K \arrow[r] \arrow[d, "\psi"'] & WK \arrow[r] \arrow[d, "W\psi"] & \overline{W}K \arrow[d, "\overline{W}\psi"] \\ G \arrow[r] & WG \arrow[r] & \overline{W}G, \end{tikzcd}$$

in which WK and WG are weakly contractible. As $\mathbb{Q}$ is flat over $\mathbb{Z}$, the long exact sequences of homotopy groups imply that ψ is a rational homotopy equivalence precisely if $\overline{W}\psi$ is a rational homotopy equivalence. By the Lemma, $\overline{W}\psi$ is a rational homotopy equivalence precisely if it is a rational homology equivalence. The Comparison Theorem for spectral sequences applied to the Leray–Serre spectral sequences

$$E^2_{p,q} = H_p(\overline{W}K;\mathbb{Q}) \otimes H_q(K;\mathbb{Q}) \Longrightarrow H_{p+q}(WK;\mathbb{Q})$$

then implies that $\overline{W}\psi$ is a rational homology isomorphism precisely if ψ is, which proves the assertion. □

2.1.1 ... à la Quillen

An important invariant of rational homotopy type is the graded Lie algebra structure on the shifted rational homotopy groups $\pi_*^{\mathbb{Q}}X := \pi_{*+1}X \otimes \mathbb{Q}$ induced by the Whitehead product

$$\pi_{p+1}X \times \pi_{q+1}X \longrightarrow \pi_{p+q+1}X.$$

For 1-connected spaces, Lemma 2.1.1 implies that the rational homology coalgebra is also a rational homotopy invariant. In Quillen's approach to rational homotopy theory, these invariants are used to characterise the rational homotopy category. More specifically, Quillen proves that the rational homotopy category of 1-connected spaces is equivalent to the homotopy category of reduced dg Lie algebras over $\mathbb{Q}$ and also to the homotopy category of 2-reduced dg cocommutative coalgebras over $\mathbb{Q}$:

Theorem 2.1.3 ([Qui69]). *There are equivalences of categories*

$$Ho_{\mathbb{Q}}\big(\mathrm{sSet}_{\geq 2}\big) \xrightarrow{\lambda} Ho\big(\mathrm{dgLie}_{\geq 1}\big) \xrightarrow{\mathcal{C}} Ho\big(\mathrm{dgCoalg}_{\geq 2}\big)$$

together with natural isomorphisms $\pi_*^{\mathbb{Q}}X \cong H_\bullet(\lambda X)$ *and* $H_\bullet(X;\mathbb{Q}) \cong \mathcal{C}(\lambda(X))$.

The following categories are used in Quillen's rational homotopy theorem:

- $\mathrm{sSet}_{\geq 2} \hookrightarrow \mathrm{sSet}$ is the full subcategory of 2-*reduced simplicial sets*, namely those simplicial sets with precisely one 0-simplex and one 1-simplex. The rational homotopy category $Ho_{\mathbb{Q}}(\mathrm{sSet}_{\geq 2})$ is obtained from $\mathrm{sSet}_{\geq 2}$ by localising at the class of rational homotopy equivalences.

- $\mathrm{dgLie}_{\geq 1}$ is the category of *(1-)reduced dg Lie algebras over* $\mathbb{Q}$, so that $L_n = 0$ for $n < 1$. The homotopy category $Ho(\mathrm{dgLie}_{\geq 1})$ is obtained from $\mathrm{dgLie}_{\geq 1}$ by localising at the class of quasi-isomorphisms.

- $\mathrm{dgCoalg}_{\geq 2}$ is the category of 2-*reduced dg coalgebras over* $\mathbb{Q}$. For C a (strictly coassociative and strictly cocommutative) dg coalgebra with counit $\epsilon\colon C \to \mathbb{Q}$ and $\overline{C} := \ker(\epsilon)$, C is 2-reduced if $\overline{C}_n = 0$ for all $n < 2$. The homotopy category $Ho(\mathrm{dgCoalg}_{\geq 2})$ is obtained from $\mathrm{dgCoalg}_{\geq 2}$ by localising at the class of quasi-isomorphisms.

For an object c in a category $\mathcal{C}$ with weak equivalences, we write $[c] \in Ho(\mathcal{C})$ for the corresponding homotopy type, where $Ho(\mathcal{C})$ is the localisation of $\mathcal{C}$ at the class of weak equivalences. For a 1-connected simplicial set X we define

- a *Lie model of* X to be a reduced dg Lie algebra $\mathfrak{l}_X$ such that $[\mathfrak{l}_X] \cong \lambda([X])$; and
- a *coalgebra model of* X to be a 2-reduced dg coalgebra C_X such that $[C_X] \cong \mathcal{C}\lambda([X])$.

The upshot of Theorem 2.1.3 is that the rational homotopy type of a 1-connected space X is completely encoded by algebraic data; namely in a Lie or coalgebra model for X.

Quillen's proof of Theorem 2.1.3 is rather complicated and is the main reason he invented the theory of model categories [Qui69, p. 206]. Quillen constructs a sequence of model categories and Quillen equivalences between them:

$$\begin{tikzcd} \mathrm{sSet}_{\geq 2,\mathbb{Q}} \arrow[r, shift left, "G"] \arrow[r, phantom, "\perp"] & \mathrm{sGrp}_{\geq 1,\mathbb{Q}} \arrow[l, shift left, "\overline{W}"] \arrow[r, shift left, "\widehat{\mathbb{Q}}[-]"] \arrow[r, phantom, "\perp"] & \mathrm{sCHopf}_{\geq 1} \arrow[l, shift left, "\mathcal{G}"] \arrow[r, shift right, "\mathcal{P}"'] \arrow[r, phantom, "\perp"] & \mathrm{sLie}_{\geq 1} \arrow[l, shift right, "\widehat{\mathcal{U}}"'] \arrow[r, shift right, "N"'] \arrow[r, phantom, "\perp"] & \mathrm{dgLie}_{\geq 1} \arrow[l, shift right, "N^*"'] \arrow[r, shift right, "\mathcal{C}"'] \arrow[r, phantom, "\perp"] & \mathrm{dgCoalg}_{\geq 2}. \arrow[l, shift right, "\mathcal{L}"'] \end{tikzcd}$$

The derived adjunctions determine a sequence of adjoint equivalences on homotopy categories, proving the equivalences of Theorem 2.1.3. We summarise the model categories used by Quillen. We often specify only two out of three of the classes of cofibrations, fibrations and weak equivalences since knowledge of any two of these classes already determines the third.

(i) $\mathrm{sSet}_{\geq 2,\mathbb{Q}}$ is the category of 2-reduced simplicial sets. Cofibrations are precisely the monomorphisms and weak equivalences are the rational homotopy equivalences.

(ii) $\mathrm{sGrp}_{\geq 1,\mathbb{Q}}$ is the category of 1-reduced simplicial groups. The cofibrations are retracts of free simplicial group maps and weak equivalences are rational homotopy equivalences of the underlying simplicial sets.

(iii) $\mathrm{sCHopf}_{\geq 1}$ is the category of reduced complete simplicial Hopf algebras over $\mathbb{Q}$. The functor $\mathcal{P}\colon \mathrm{sCHopf}_{\geq 1} \to \mathrm{sLie}_{\geq 1}$ which computes simplicial Lie algebras of primitives creates fibrations and weak equivalences.

(iv) $\mathrm{sLie}_{\geq 1}$ is the category of reduced simplicial Lie algebras over $\mathbb{Q}$. The fibrations and weak equivalences are created by the forgetful functor $\mathrm{sLie}_{\geq 1} \to \mathrm{sSet}_{\geq 1}$ which sends a a reduced simplicial Lie algebra to its underlying reduced simplicial set.

(v) $\mathrm{dgLie}_{\geq 1}$ is the category of reduced dg Lie algebras over $\mathbb{Q}$. Weak equivalences are the quasi-isomorphisms and fibrations are those maps which are surjective in degrees > 1.

(vi) $\mathrm{dgCoalg}_{\geq 2}$ is the category of 2-reduced dg coalgebras over $\mathbb{Q}$. Cofibrations are the injective maps and weak equivalences are the quasi-isomorphisms.

The adjunctions appearing in Quillen's proof are

($\mathbb{G} \dashv \overline{W}$): this is the restriction of Kan's adjunction (recalled in Appendix B) to 2-reduced simplicial sets. The functor $\mathbb{G}$ sends a reduced simplicial set to its Kan simplicial loop group, and $\overline{W}$ sends a simplicial group to its classifying space.

($\widehat{\mathbb{Q}} \dashv \mathcal{G}$): the functor $\widehat{\mathbb{Q}}[-]$ sends a reduced simplicial group G to the rational simplicial group ring $\mathbb{Q}[G]$ which is then completed at the augmentation ideal. The right adjoint $\mathcal{G}$ computes the simplicial group of group-like elements.

($\widehat{\mathcal{U}} \dashv \mathcal{P}$): the functor $\widehat{\mathcal{U}}$ sends a reduced simplicial Lie algebra $\mathfrak{g}$ to the completion of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ at the augmentation ideal.

($N^* \dashv N$): for a simplicial Lie algebra $\mathfrak{g}$, computing the normalised chain complex yields a dg Lie algebra $N\mathfrak{g}$ with Lie bracket defined via the shuffle map. If ($\Gamma \dashv N$) is the rational Dold–Kan correspondence (recalled in Remark 2.4.1), the left adjoint N^* sends the free dg Lie algebra generated by the chain complex V to the free simplicial Lie algebra generated by the simplicial $\mathbb{Q}$-vector space$\Gamma(V)$,
.

($\mathcal{L} \dashv \mathcal{C}$): the functor $\mathcal{L}$ sends a 2-reduced dg coalgebra to the dg Lie algebra of primitives of the cobar construction ΩC of C. The right adjoint $\mathcal{C}$ sends a dg Lie algebra to its Chevalley–Eilenberg coalgebra.

Remark 2.1.4. The adjunctions ($\widehat{\mathbb{Q}}[-] \dashv \mathcal{G}$) and ($\widehat{\mathcal{U}} \dashv \mathcal{P}$) together implement a sort of homotopical Lie theory. This is one way to understand why *completed* Hopf algebras are used: they are analogues of formal Lie groups. Another crucial reason that Quillen works with completed simplicial Hopf algebras is that the category of complete Hopf algebras over $\mathbb{Q}$ has a projective generator [Qui69, Appendix A], which is used to establish the model structure on $\mathrm{sCHopf}_{\geq 1}$.

Another important ingredient in Quillen's proof which we need is the following result, based on work of Curtis:

Theorem 2.1.5 ([Qui69]). *If G is a connected free simplicial group, then the unit of the ($\widehat{\mathbb{Q}}[-] \dashv \mathcal{G}$)-adjunction induces an isomorphism*

$$\pi_*(G) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \pi_*(\mathcal{G}\widehat{\mathbb{Q}}[G]).$$

If $\mathfrak{g}$ is a connected free simplicial Lie algebra, the ($\widehat{\mathcal{U}} \dashv \mathcal{P}$)-unit induces an isomorphism

$$\pi_*(\mathfrak{g}) \longrightarrow \pi_*(\mathcal{P}\widehat{\mathcal{U}}(\mathfrak{g})).$$

If R is a connected free simplicial augmented associative algebra, then completion at the augmentation ideal $R \to \widehat{R}$ is a weak equivalence.

2.1.2 ... à la Sullivan

Sullivan's approach to rational homotopy works quite differently, by identifying a rational homotopy type with a strict DG representative of its rational cohomology algebra. This is approach is very much dual to Quillen's, and taking duals (from homology to cohomology) requires some finiteness conditions in order to get an equivalence of homotopy theories (Theorem 2.1.10). Our reference for Sullivan's rational homotopy theory is the monograph [BG76].

Sullivan's approach to rational homotopy theory starts by assigning to each simplicial set X a connective commutative differential graded cochain algebra (cDGA) over $\mathbb{Q}$. The cohomology of the cDGA $\mathcal{O}(X)$ so obtained is naturally isomorphic to the rational cohomology algebra $H^\bullet(X;\mathbb{Q})$. To the combinatorial n-simplex $\Delta[n]$ we assign the rational cDGA $\mathcal{O}(\Delta[n])^\bullet \equiv \nabla_n^\bullet$ which has generators $t_0,\dots,t_n$ in degree 0 and $dt_0,\dots,dt_n$ in degree 1 subject to the relations

$$\sum_{i=0}^{n} t_i = 1 \quad \text{and} \quad \sum_{i=0}^{n} dt_i = 0.$$

The differential on $\nabla_n^\bullet$ is such that $d(t_i) = dt_i$ and $d(dt_i) = 0$ for all $0 \le i \le n$. The assignment $[n] \mapsto \nabla_n^\bullet$ determines a simplicial cDGA with face and degeneracy maps defined on generators by

$$\partial_i(t_k) = \begin{cases} t_{k-1} & i < k \\ 0 & i = k \\ 0 & i > k \end{cases} \quad \text{and} \quad s_i(t_k) = \begin{cases} t_{k+1} & i < k \\ t_k + t_{k+1} & i = k \\ t_k & i > k. \end{cases}$$

For X a simplicial set, the cDGA $\mathcal{O}(X)$ is defined as the hom-space

$$\mathcal{O}(X) := \mathrm{sSet}\big(X, \nabla^\bullet\big),$$

which inherits the structure of a rational cDGA. Equivalently, the functor $\mathcal{O}$ is obtained as the left Kan extension of $\nabla\colon \Delta \to \mathrm{DGAlg}^{\mathrm{op}}$ along the Yoneda embedding $\Delta \to \mathrm{sSet}$. Here and elsewhere DGAlg denotes the category of connective rational cDGAs. The functor $\mathcal{O}$ admits a right adjoint

$$\mathcal{S}\colon A \longmapsto \big([n] \mapsto \mathrm{DGAlg}(A, \nabla_n^\bullet)\big),$$

which we call the *spatial realisation functor*. The resulting adjunction

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, shift left, "\mathcal{O}"] \arrow[r, phantom, "\perp"] & \mathrm{DGAlg}^{\mathrm{op}} \arrow[l, shift left, "\mathcal{S}"] \end{tikzcd}$$

is called the *Sullivan–de Rham adjunction*, which reflects the fact that $\mathcal{O}$ implements rational PL de Rham theory.

Remark 2.1.6. The category DGAlg has a *projective model structure*, for which a map $f\colon A \to B$ of cDGAs is

- a weak equivalence if the underlying map of cochain complexes is a quasi-isomorphism;
- a fibration if it is a degreewise epimorphism, i.e. a surjective map; and
- a cofibration if it has the left lifting property with respect to all acyclic fibrations.

The projective model structure is cofibrantly generated, with generating cofibrations $\mathcal{I}_{\mathrm{cDGA}} := \{i_n\colon S_n \to D_n\}_{n\ge1} \cup \{i_0\colon S_0 \to \mathbb{Q}\} \cup \{i_0'\colon \mathbb{Q} \to S_0\}$ and generating acyclic cofibrations $\mathcal{J}_{\mathrm{cDGA}} := \{j_n\colon \mathbb{Q} \to D_n\}_{n\ge1}$. The cDGAs S_n and D_n are the sphere and disk algebras, so that S_n is freely generated by a closed element a of degree n and D_n has generators b and c of degrees $n-1$ and n respectively with differential $db = c$.

For $i > 0$ the sphere inclusions $i_n\colon S_n \to D_n$ send $a \mapsto c$, i_0 sends $a \mapsto 0$, and i_0' and the j_n are the unit maps.

One way to establish the existence of the projective model structure on DGAlg is via the free-forgetful adjunction

$$\begin{tikzcd} Ch^+ \arrow[r, shift left, "\mathrm{Sym}"] \arrow[r, phantom, "\perp"] & \mathrm{cDGA} \arrow[l, shift left] \end{tikzcd}$$

using the Right Transfer Theorem A.1.3. Here we equip the category Ch^+ of connective cochain complexes over $\mathbb{Q}$ with its projective model structure, for which the weak equivalences and fibrations are the the quasi-isomorphisms and surjective maps respectively. The model structure on Ch^+ is cofibrantly generated by sets $\mathcal{I}^+$ and $\mathcal{J}^+$ such that $\mathrm{Sym}(\mathcal{I}^+) = \mathcal{I}_{\mathrm{cDGA}}$ and $\mathrm{Sym}(\mathcal{J}^+) = \mathcal{J}_{\mathrm{cDGA}}$.

The Sullivan-de Rham adjunction is a Quillen adjunction such that $\mathcal{O}$ preserves finite sSet-tensors; that is, for K a finite simplicial set and $X \in$ sSet there is a natural isomorphism of cDGAs $\mathcal{O}(K \otimes X) \cong \mathcal{O}(K) \otimes \mathcal{O}(X)$ [BG76, Lemma 5.2]. In particular, any cofibrant cDGA A is sent to a Kan complex by the spatial realisation functor $\mathcal{S}$.

Remark 2.1.7. For an augmented cDGA $\epsilon\colon A \to \mathbb{Q}$, write $\overline{A} := \ker(\epsilon)$ for the augmentation ideal. The *complex of indecomposables* is defined as $QA := \overline{A}/\overline{A} \cdot \overline{A}$ and the *cohomotopy groups* of A are defined by $\pi^*(A) := H^\bullet(QA)$. If A is cofibrant the Kan complex $\mathcal{S}(A)$ is pointed by $\mathcal{S}(\epsilon)\colon * \to \mathcal{S}(A)$. If A is moreover homologically connected (Definition 2.1.9 below) then the Kan complex $\mathcal{S}(A)$ is connected and we have natural isomorphisms of sets

$$\pi_n(\mathcal{S}(A), \mathcal{S}(\epsilon)) \cong \mathrm{Hom}_{\mathbb{Q}}(\pi^n(A), \mathbb{Q})$$

for all $n \geq 1$, which are group isomorphisms for $n > 1$ [BG76, Proposition 8.13].

Remark 2.1.8. Minimality is an important notion in Sullivan's rational homotopy theory. A map of cDGAs $A \to B$ is *minimal* if the underlying map of graded algebras is an inclusion of the form $A \hookrightarrow A \otimes \mathrm{Sym}(V)$ for some non-negatively graded $\mathbb{Q}$-vector space V such that

- there is a well-ordered set $\mathcal{I}$ indexing a basis $\{v_i\}_{i \in \mathcal{I}}$ of V;
- $i < j$ implies that $|v_i| \leq |v_j|$; and
- the differential on $B \cong A \otimes \mathrm{Sym}(V)$ is such that

  $$dv_i \in A \otimes \mathrm{Sym}(V_{<i}),$$

  where $V_{<i} = \langle v_j \mid j < i \rangle$.

An algebra A is *minimal* if the unit map $\mathbb{Q} \to A$ is minimal; this implies that any minimal algebra has a canonical augmentation induced by taking Sym of the zero map $V \to 0$. Minimal maps are cofibrations for the projective model structure, so minimal algebras are cofibrant. Any map of cDGAs $f\colon A \to B$ admits a factorisation

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & B \\ & B' \arrow[ur, "\psi"'] & \end{tikzcd}$$

in which $i\colon A \to B'$ is minimal and $\psi\colon B' \to B$ is a quasi-isomorphism. Minimal factorisations of this form are unique up to isomorphism so that in particular any cDGA A admits a minimal model which is unique up to isomorphism.

Definition 2.1.9. A connected simplicial set X is *nilpotent* if $\pi_1(X)$ is a nilpotent group and $\pi_n(X)$ is a nilpotent $\pi_1(X)$-module for all $n \geq 2$. A nilpotent space is *rational* if $\pi_n(X)$ is uniquely divisible for all $n \geq 1$, and is of *finite rational type* if the rational vector spaces $H_1(X;\mathbb{Q})$ and $\pi_n(X) \otimes \mathbb{Q}$ are finite dimensional for all $n \geq 2$.

A cDGA A is *homologically connected* if the unit map induces an isomorphism $\mathbb{Q} \cong H^0(A)$. A homologically connected cDGA A is of *finite type* if M^p is finite-dimensional for all $p > 1$, where $M \to A$ is a minimal resolution of A.

Theorem 2.1.10. *The derived Sullivan–de Rham adjunction*

$$\begin{tikzcd} Ho(\mathrm{sSet}) \arrow[r, shift left, "\mathcal{O}"] \arrow[r, phantom, "\perp"] & Ho(\mathrm{DGAlg})^{\mathrm{op}} \arrow[l, shift left, "\mathbb{R}\mathcal{S}"] \end{tikzcd}$$

restricts to an equivalence of categories

$$\begin{tikzcd} Ho(\mathrm{sSet})_{\mathrm{nil},\mathbb{Q}} \arrow[r, shift left, "\mathcal{O}"] \arrow[r, phantom, "\perp"] & Ho(\mathrm{DGAlg})^{\mathrm{op}}_{\mathrm{ft}} \arrow[l, shift left, "\mathbb{R}\mathcal{S}"] \end{tikzcd}$$

where $Ho(\mathrm{sSet})_{\mathrm{nil},\mathbb{Q}}$ is the full subcategory spanned by rational nilpotent spaces of finite type and $Ho(\mathrm{DGAlg})_{\mathrm{ft}}$ is the full subcategory of spanned by homologically connected cDGAs of finite type.

Proof. We sketch the proof given in [BG76, Ch. 10]. The main point is to observe that X is a rational nilpotent space of finite type precisely if the terminal map $X_n \to *$ from the n-th Postnikov section can be factored as the composite of finitely many principal $K(\mathbb{Q},p)$-fibrations for $p \geq 1$. For $p \geq 1$, the functor $\mathcal{O}$ sends a $K(\mathbb{Q},p)$ to a cDGA which is weakly equivalent to the sphere algebra S_p. Conversely, the spatial realisation functor $\mathcal{S}$ sends S_p to a $K(\mathbb{Q},p)$.

Next, we use the Eilenberg–Moore Theorem (or, rather, the strict version of [BG76, §3.1]) to show that if A is a homologically connected cofibrant cDGA of finite type and the counit $A \to \mathcal{O}\mathcal{S}(A)$ is a weak equivalence, then for any $p > 1$ and pushout square

$$\begin{tikzcd} S_p \arrow[r] \arrow[d] & A \arrow[d] \\ D_p \arrow[r] & B, \end{tikzcd} \tag{2.1}$$

the counit at the cofibrant cDGA B is also a weak equivalence. Supposing that A is a homologically connected minimal algebra of finite type, we can work over the "Postnikov sections" of A, where the n-th Postnikov section A_n is the subalgebra of A generated by the elements a of degree $|a| \leq n$ and their differentials. By hypothesis, A_n can be obtained from $\mathbb{Q}$ by finitely many pushouts of the form (2.1). Since any cDGA has a minimal resolution, by working over Postnikov sections we conclude that the derived Sullivan–de Rham counit is a natural isomorphism when restricted to $Ho(\mathrm{DGAlg})^{\mathrm{op}}_{\mathrm{ft}}$.

Conversely, a similar argument using the Eilenberg–Moore Theorem shows that if X is a connected Kan complex with finite dimensional rational cohomology groups for which the derived unit $X \to \mathbb{R}\mathcal{S}(\mathcal{O}X)$ is a weak equivalence, then for any $p > 1$

and pullback diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & PK(\mathbb{Q}, p) \arrow[d, "\pi"] \\ X \arrow[r] & K(\mathbb{Q}, p) \end{tikzcd} \tag{2.2}$$

with π a path fibration, the derived unit $Y \to \mathbb{R}\mathcal{S}(\mathcal{O}Y)$ is also a weak equivalence. Since the n-th Postnikov section of a nilpotent rational space of finite type can be obtained from $*$ in finitely many steps of the form used to obtain Y from X in (2.2), by working over Postnikov sections we deduce that the derived Sullivan–de Rham unit is a natural isomorphism when restricted to $Ho(\mathrm{sSet})_{\mathrm{nil},\mathbb{Q}}$. □

2.2 Parametrised Spectra and Loop Space Modules

We begin our study of the rational homotopy theory of parametrised spectra with a careful treatment of the relationship between a parametrised spectrum and its homotopy fibre spectra. In Lemma 1.2.10 (and Lemma 1.2.22) we saw that stable weak equivalences of parametrised spectra are detected at homotopy fibres. Thus for any space X there is an embedding

$$\mathrm{hofib}\colon Ho(\mathrm{Sp}_X) \hookrightarrow \prod_{\pi_0(X)} Ho(\mathrm{Sp}), \tag{2.3}$$

which we interpret as saying that a parametrised X-spectrum is determined by its collection of homotopy fibre spectra over the components of X.

To identify the subcategory spanned by the image of hofib, recall the unstable situation. For a connected space X and map $P \to X$, the homotopy fibre $\mathrm{hofib}(P)$ is equipped with a (homotopy) ΩX-action. It is classical that there is a weak equivalence $X \cong B\Omega X$, and the assignment $(P \to X) \mapsto \mathrm{hofib}(P)$ is one half of an equivalence of homotopy theories between spaces over X and homotopy ΩX-spaces. In the stable setting this becomes an equivalence between the homotopy theory of parametrised X-spectra and ΩX-module spectra, identifying the image of hofib in (2.3) for a connected base space. This section is devoted to a careful formulation and proof of these facts in the simplicial setting.

2.2.1 Spaces Over X and $\mathbb{G}X$-Spaces

In this section, we prove a Quillen equivalence modelling the equivalence of homotopy theories between spaces over a connected space X and ΩX-spaces. Throughout this section we shall suppose that X is a reduced simplicial set, so that $X_0 = *$, and $\mathbb{G}X$ denotes the Kan simplicial loop group of X. We make extensive use of Kan's simplicial path fibration $\pi_X\colon \mathbb{P}X \to X$ (see Construction B.1.9).

Theorem 2.2.1. *For X a reduced simplicial set, there is a* sSet*-Quillen equivalence*

$$\begin{tikzcd} \mathrm{sSet}_{/X} \arrow[r, shift left, "\beta_X"] \arrow[r, phantom, "\perp"] & \mathbb{G}X\text{–}\mathrm{sSet}. \arrow[l, shift left, "\gamma_X"] \end{tikzcd}$$

Proof. The desired Quillen equivalence is constructed as a composite of adjunctions

$$(\beta_X \dashv \gamma_X)\colon\ \mathrm{sSet}_{/X} \underset{\pi^X_!}{\overset{\pi^*_X}{\rightleftarrows}} \mathbb{G}X{-}\mathrm{sSet}_{/\mathbb{P}X} \underset{\mathbb{P}X^*}{\overset{\mathbb{P}X_!}{\rightleftarrows}} \mathbb{G}X{-}\mathrm{sSet}.$$

To define the first adjunction, for $(Y \to X) \in \mathrm{sSet}_{/X}$ let $\pi^*_X Y$ be the pullback

$$\begin{tikzcd} \pi^*_X Y \arrow[r] \arrow[d] & \mathbb{P}X \arrow[d, "\pi_X"] \\ Y \arrow[r] & X \end{tikzcd}$$

equipped with the induced map to $\mathbb{P}X$. Regarding X and Y as trivial $\mathbb{G}X$-spaces, this is a pullback in $\mathbb{G}X{-}\mathrm{sSet}$ defining a functor $\pi^*_X\colon \mathrm{sSet}_{/X} \to \mathbb{G}X{-}\mathrm{sSet}_{/\mathbb{P}X}$. The functor π^*_X is an equivalence of categories, with inverse equivalence given by the functor

$$\pi^X_!\colon (P \to \mathbb{P}X) \longmapsto (P/\mathbb{G}X \to \mathbb{P}X/\mathbb{G}X \cong X).$$

Indeed, for any $Y \to X$ the pullback $\pi^*_X Y$ is a free $\mathbb{G}X$-space and there is a natural isomorphism $\pi^*_X Y/Y \cong Y$ so that $\pi^X_! \pi^*_X$ is naturally isomorphic to the identity functor. On the other hand, for $P \to \mathbb{P}X$ a map of $\mathbb{G}X$-spaces there is a natural commuting diagram of $\mathbb{G}X$-spaces

$$\begin{tikzcd} P \arrow[r] \arrow[dr] & \pi^*_X \pi^X_! P \arrow[r] \arrow[d] & \mathbb{P}X \arrow[d, "\pi_X"] \\ & P/\mathbb{G}X \arrow[r] & X. \end{tikzcd}$$

Since $\pi^*_X \pi^X_! P$ is a free $\mathbb{G}X$-space and the quotient of $P \to \pi^*_X \pi^X_! P$ by $\mathbb{G}X$ is the identity, this map is an isomorphism by Lemma B.1.3. It is a classical fact of category theory that we can promote this equivalence of categories to an adjoint equivalence $(\pi^*_X \dashv \pi^X_!)$. Since sSet is right proper and π_X is a fibration the equivalence π^*_X preserves weak equivalences. Generating cofibrations of $\mathrm{sSet}_{/X}$ are of the form

$$\begin{tikzcd} \partial\Delta[n] \arrow[rr, "i_n"] \arrow[dr, "\sigma|_{\partial\Delta[n]}"'] & & \Delta[n] \arrow[dl, "\sigma"] \\ & X & \end{tikzcd}$$

for $i_n\colon \partial\Delta[n] \hookrightarrow \Delta[n]$ the standard boundary inclusion. Since $\mathbb{P}X$ is a free $\mathbb{G}X$-space, applying the pullback functor to such a morphism yields the generating cofibration

$$\begin{tikzcd} \mathbb{G}X \times \partial\Delta[n] \arrow[rr, "\mathbb{G}X \times i_n"] \arrow[dr] & & \mathbb{G}X \times \Delta[n] \arrow[dl] \\ & \mathbb{P}X. & \end{tikzcd}$$

The functor π^*_X thus preserves cofibrations and weak equivalences, so that the adjoint equivalence $(\pi^*_X \dashv \pi^X_!)$ is a Quillen equivalence. As to the sSet-enrichment, it suffices to show that π^*_X preserves sSet-tensors up to isomorphism. For $K \in \mathrm{sSet}$ and $(Y \to X) \in \mathrm{sSet}_{/X}$, the sSet-tensoring is given by $(K \times Y \to X)$. Applying π^*_X sends this to $K \times \pi_X^* Y \to \mathbb{P}X$, which coincides with the sSet-tensoring on $\mathbb{G}X{-}\mathrm{sSet}_{/\mathbb{P}X}$.

We now turn to the second equivalence. Writing $\mathbb{P}X\colon \mathbb{P}X \to *$ for the terminal morphism, let $\mathbb{P}X_!$ be the base change functor sending $(P \to \mathbb{P}X)$ to the composite map of $\mathbb{G}X$-spaces $(P \to \mathbb{P}X \to *)$. The functor $\mathbb{P}X_!$ has right adjoint $\mathbb{P}X^*$ which sends $Q \mapsto \mathbb{P}X \times Q$ equipped with the diagonal $\mathbb{G}X$-action, as is easily checked. The functor $\mathbb{P}X$ preserves and reflects weak equivalences and fibrations and $\mathbb{P}X_!$ preserve sSet-tensors, so that $(\mathbb{P}X_! \dashv \mathbb{P}X^*)$ is a sSet-Quillen adjunction. It is immediate that $\mathbb{P}X_!$ preserves sSet-tensors, so that the adjunction is sSet-enriched. Finally, for any $P \in \mathbb{G}X{-}\mathrm{sSet}_{/\mathbb{P}X}$ the unit map $P \to \mathbb{P}X^*\mathbb{P}X_!(P) \cong \mathbb{P}X \times P$ is a weak equivalence since $\mathbb{P}X$ is weakly contractible. It follows that $(\mathbb{P}X_! \dashv \mathbb{P}X^*)$ is a sSet-Quillen equivalence. $\square$

Remark 2.2.2. In the setting of the Theorem, the functor β_X sends a space over X to its homotopy fibre equipped with its natural $\mathbb{G}X$-action. We already used a stable version of this in the proof of Lemma 1.2.12, where we did not keep track of the $\Omega X \cong \mathbb{G}X$-actions.

We shall also need to understand how the equivalences of Theorem 2.2.1 behave with respect to base change functors. For this we shall need the following

Lemma 2.2.3. *For $\phi\colon G \to H$ a morphism of simplicial groups, there is a* sSet*-Quillen adjunction*

$$\begin{tikzcd} G{-}\mathrm{sSet} \arrow[r, shift left=1.5ex, "\phi_!"] \arrow[r, phantom, "\perp"] & H{-}\mathrm{sSet} \arrow[l, shift left=1.5ex, "\phi^*"] \end{tikzcd}$$

which is moreover a Quillen equivalence if ϕ is a weak equivalence.

Proof. The restriction functor $\phi^*\colon H{-}\mathrm{sSet} \to G{-}\mathrm{sSet}$ is defined in the obvious way, namely by setting $\phi^* X := X$ with G-action obtained via the composite

$$G \times X \xrightarrow{\phi \times \mathrm{id}_X} H \times X \xrightarrow{\rho} X,$$

for ρ the H-action on X. The restriction functor ϕ^* preserves limits and colimits, so by the Adjoint Functor Theorem has a left adjoint $\phi_!$ and a right adjoint ϕ_*. Explicitly, the left adjoint $\phi_!$ sends a G-space X to the space with n-simplices

$$\big(H \times_G X\big)_n := \big(H_n \times X_n\big)\Big/ (h \cdot \phi(g), x) \sim (h, g \cdot x)$$

with its inherited left H-action. For any simplicial group H, the forgetful functor $H{-}\mathrm{sSet} \to \mathrm{sSet}$ creates weak equivalences and fibrations so that $(\phi_! \dashv \phi^*)$ is Quillen. From the explicit expression for $\phi_!$ above it is easy to see that

$$\phi_!(K \times M) \cong K \times \phi_!(M)$$

for all $K \in \mathrm{sSet}$ and $M \in G{-}\mathrm{sSet}$, so that $\phi_!$ preserves sSet-tensors.

Finally, suppose that ϕ is a weak equivalence. Since ϕ^* preserves and reflects weak equivalences, to show that $(\phi_! \dashv \phi^*)$ is a Quillen equivalence it is sufficient to show that the unit $X \to \phi^*\phi_! X$ is a weak equivalence for any cofibrant G-space X. For this, we first observe that since X is cofibrant, so too is $H \times X$ regarded as a G-space with action $g \cdot (h, x) := (h \cdot \phi(g)^{-1}, g \cdot x)$ since it is levelwise free. Letting $\varrho\colon G \to *$ denote the terminal map of simplicial groups, Ken Brown's Lemma implies that $\varrho_!(G \times X) \to \varrho_!(H \times X)$ is a weak equivalence since $G \times X$ and $H \times X$

are cofibrant G-spaces. But $\varrho_!(G \times X) \cong X$ and $\varrho_!(H \times X) \cong \phi_!(X)$, from which the assertion follows. $\square$

Remark 2.2.4. In the context of the Lemma, the functor ϕ^* also has a right adjoint $\phi_*\colon G\text{−sSet} \to H\text{−sSet}$ which sends a G-space to the *coinduced* H-space. However, the adjunction $(\phi^* \dashv \phi_*)$ is generally not homotopically well-behaved. An easy counterexample arises is the map of free (simplicial) groups $\varphi\colon F_2 \to F_1$ obtained by identifying generators. Under φ^*, the cofibration $\varnothing \to F_1$ is sent to the morphism of F_2-spaces $\varnothing \to F_1$. But F_1 is not a free F_2-space, so φ^* cannot be a left Quillen functor.

A standard remedy is to replace $\phi\colon G \to H$ by a cofibration $\phi'\colon G \to H'$ of simplicial groups. It is not too hard to show that $(\phi')^*$ is then a left Quillen functor but we will not pursue this further.

Recall that for a morphism of simplicial sets $f\colon X \to Y$ there is a sSet-Quillen adjunction

$$\begin{tikzcd} \mathrm{sSet}_{/X} \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] & \mathrm{sSet}_{/Y} \arrow[l, shift left, "f^*"] \end{tikzcd}$$

in which $f_!$ sends $(K \to X)$ to $(K \to X \xrightarrow{f} Y)$ and f^* takes pullbacks along f—this the unpointed version of the base change adjunction of Lemma 1.1.11. The behaviour of the Quillen equivalences $(\beta_X \dashv \gamma_X)$ with respect to base change is described by the following

Lemma 2.2.5. *Let $f\colon X \to Y$ be a morphism of reduced simplicial sets. Then there is a natural isomorphism $\mathbb{G}(f)_!\beta_X \cong \beta_Y f_!$, and hence a diagram of left Quillen functors*

$$\begin{tikzcd} \mathrm{sSet}_{/X} \arrow[r, "f_!"] \arrow[d, "\beta_X"'] & \mathrm{sSet}_{/Y} \arrow[d, "\beta_Y"] \\ \mathbb{G}X\text{−sSet} \arrow[r, "\mathbb{G}f_!"] & \mathbb{G}Y\text{−sSet} \end{tikzcd}$$

which commutes up to natural isomorphism.

Proof. We verify this directly. Applying $\beta_Y f_!$ to $(K \to X) \in \mathrm{sSet}_{/X}$ gives the $\mathbb{G}Y$-space obtained as the pullback

$$\begin{tikzcd} \pi_Y^* K \arrow[rr] \arrow[d] & & \mathbb{P}Y \arrow[d, "\pi_Y"] \\ K \arrow[r] & X \arrow[r, "f"] & Y, \end{tikzcd}$$

forgetting the map to $\mathbb{P}Y$. Naturality of the fibration $\pi_X\colon \mathbb{P}X \to X$ in X implies that there is a morphism of $\mathbb{G}X$-spaces $\pi_X^* K \to \mathbb{G}f^*(\pi_Y^* K)$ commuting with the projection maps to K. The $\mathbb{G}X$-space $\pi_X^* K$ is cofibrant with $\pi_X^* K/\mathbb{G}X \cong K$, so that $\mathbb{G}f_!(\pi_X^* K)$ is cofibrant with $\mathbb{G}f_!(\pi_X^* K)/\mathbb{G}Y = K$. The $(\mathbb{G}f_! \dashv \mathbb{G}f^*)$-adjunct of $\pi_X^* K \to \mathbb{G}f^*(\pi_Y^* K)$ is therefore a map of $\mathbb{G}Y$-spaces $\mathbb{G}f_!(\pi_X^* K) \to \pi_Y^* K$ whose quotient by $\mathbb{G}Y$ is the identity on K. Since $\pi_Y^* K$ is cofibrant the map $\mathbb{G}f_!(\pi_X^* K) \to \pi_Y^* K$ is an isomorphism by Lemma B.1.3. This is the component of our sought-after natural isomorphism $\mathbb{G}f_!\beta_X \cong \beta_Y f_!$ at $(K \to X)$. $\square$

2.2.2 Stabilisation

Building on Theorem 2.2.1, in this section we prove an equivalence of homotopy theories between spectra parametrised by a connected space X and ΩX-module spectra. The result is Theorem 2.2.13, which we prove by passing to pointed objects and then applying the symmetric stabilisation machine. As in the previous section, X generically denotes a reduced simplicial set.

Remark 2.2.6. Let $\mathcal{M}$ be a sSet-model category with terminal object $*$. The category of pointed objects $\mathcal{M}_*$ has a canonical sSet$_*$-model structure inherited from $\mathcal{M}$ [Hov99, Proposition 4.2.19]. For a pointed simplicial set $(k\colon * \to K)$ the sSet$_*$-tensoring with $(m\colon * \to M) \in \mathcal{M}_*$ is defined as the pushout

$$\begin{tikzcd} K \otimes * \coprod M \arrow[r, "K\otimes m + k \otimes M"] \arrow[d] & K \otimes M \arrow[d] \\ * \arrow[r] & K \circledwedge M \end{tikzcd}$$

in $\mathcal{M}$, where $\otimes$ denotes the sSet-tensoring. Note that for $\mathcal{M} = \mathrm{sSet}_{/X}$, the induced sSet$_*$-model structure on $R_X \cong (\mathrm{sSet}_{/X})^{X/}$ coincides with that of Lemma 1.1.8.

Remark 2.2.7. The functor $(-)_+\colon \mathrm{sSet} \to \mathrm{sSet}_*$ which freely adjoints a basepoint is strong monoidal, so that $(K \times L)_+ \cong K_+ \wedge L_+$. Consequently, any simplicial monoid M is sent to a monoid in sSet under $M \mapsto M_+$. We write $M_+{-}\mathrm{Mod}_{\mathrm{un}}$ for the category of (left) M_+-modules in sSet$_*$. The subscript "un" indicates that we are working in the unstable setting.

Unravelling the definitions, an M_+-module is seen to be equivalent to the data of a pointed simplicial set $k\colon * \to M$ equipped with a left M-action $M \times K \to K$ which fixes k. This determines a canonical isomorphism of categories $M_+{-}\mathrm{Mod}_{\mathrm{un}} \cong M{-}\mathrm{sSet}_*$. Under this equivalence, the sSet$_*$-model structure on $M{-}\mathrm{sSet}_*$ obtained via Remark 2.2.6 is easily seen to coindice with the model structure on $M_+{-}\mathrm{Mod}_{\mathrm{un}}$ arising via the free-forgetful adjunction

$$\begin{tikzcd} \mathrm{sSet}_* \arrow[r, shift left, "M_+\wedge(-)"] \arrow[r, phantom, "\perp"] & M_+{-}\mathrm{Mod}_{\mathrm{un}}. \arrow[l, shift left] \end{tikzcd}$$

Weak equivalences and fibrations are detected on underlying simplicial sets. The model structure on $M_+{-}\mathrm{Mod}_{\mathrm{un}}$ is left proper and combinatorial: sets of generating cofibrations and acyclic cofibrations are given by $\mathcal{I}^M_+ := \{M_+ \wedge i_+ \mid i \in \mathcal{I}_{\mathrm{Kan}}\}$ and $\mathcal{J}^M_+ := \{M_+ \wedge j_+ \mid j \in \mathcal{J}_{\mathrm{Kan}}\}$ respectively; left properness follows from the facts that the forgetful functor $M_+{-}\mathrm{Mod}_{\mathrm{un}} \to \mathrm{sSet}_*$ preserves colimits and that the forgetful functor sends $\mathcal{I}^M_+$ to cofibrations in sSet$_*$. $M{-}$sSet is naturally a sSet-model category, where the tensoring of $K \in \mathrm{sSet}$ with $N \in M{-}\mathrm{sSet}$ is defined by the product $K \times N$ equipped with the M-action obtained by regarding K as having trivial M-action. By Remark 2.2.6, $M_+{-}\mathrm{Mod}_{\mathrm{un}}$ is therefore a sSet$_*$-model category. Explicitly, the tensoring sends $(K, N) \mapsto K \wedge N$ equipped with the induced M_+-action.

Lemma 2.2.8. *For $\phi\colon G \to H$ a morphism of simplicial groups, there is a* sSet$_*$*-Quillen adjunction*

$$\begin{tikzcd} G_+{-}\mathrm{Mod}_{\mathrm{un}} \arrow[r, shift left, "\phi_!"] \arrow[r, phantom, "\perp"] & H_+{-}\mathrm{Mod}_{\mathrm{un}}. \arrow[l, shift left, "\phi^*"] \end{tikzcd}$$

If ϕ is a Quillen equivalence then $(\phi_! \dashv \phi^*)$ is a Quillen equivalence.

Proof. The restriction functor $\phi^*\colon H{-}\mathrm{sSet} \to G{-}\mathrm{sSet}$ preserves the terminal object and so determines a functor $\phi^*\colon H{-}\mathrm{Mod}_{\mathrm{un}} \to G{-}\mathrm{Mod}_{\mathrm{un}}$ which preserves limits, colimits, fibrations and weak equivalences. The left adjoint $\phi_!$ is the usual "extension of scalars" functor; explicitly

$$M \longmapsto \phi_!(M) := \operatorname{colim}\left(H_+ \wedge G_+ \wedge M \underset{\rho_H \wedge \mathrm{id}}{\overset{\mathrm{id}\wedge\rho_M}{\rightrightarrows}} H_+ \wedge M \right)$$

equipped with the induced left H_+-action. In the above colimit diagram, ρ_M is the G_+-action on M and ρ_H is the right G_+-action on H_+ induced by ϕ_+. From the explicit description of $\phi_!$ and the fact that $K \wedge (-)$ preserves colimits we find that $\phi_!$ preserves sSet_*-tensors.

Suppose that ϕ is a weak equivalence of simplicial groups. As the right adjoint ϕ^* preserves and reflects weak equivalences and fibrations, it is sufficient to check that the unit map $M \to \phi^*\phi_! M$ is a weak equivalence for all cofibrant G_+-modules M. Let us first suppose that $M = X_+$ is in the image of the left Quillen functor $(-)_+\colon G{-}\mathrm{sSet} \to G_+{-}\mathrm{Mod}_{\mathrm{un}}$, in which case the unit

$$X_+ \longrightarrow \operatorname{colim}\left((H \times G \times X)_+ \rightrightarrows (H \times X)_+ \right) = (\phi_! X)_+$$

is a weak equivalence by Lemma 2.2.3. The set of cofibrant generators $\mathcal{I}_+^G$ of Remark 2.2.7 lies in the image of $(-)_+\colon G{-}\mathrm{sSet} \to G_+{-}\mathrm{Mod}_{\mathrm{un}}$ so that $M \to \phi^*\phi_! M$ is a weak equivalence for any domain or codomain M of a morphism in $\mathcal{I}_+^G$.

Suppose that $M \in G_+{-}\mathrm{Mod}^{\geq 0}$ is such that $M \to \phi^*\phi_! M$ is a weak equivalence. Consider the pushout diagram

$$\begin{tikzcd}
(G \times \partial\Delta[n])_+ \cong G_+ \wedge \partial\Delta[n]_+ \arrow[r] \arrow[d, hook] & M \arrow[d] \\
(G \times \Delta[n])_+ \cong G_+ \wedge \Delta[n]_+ \arrow[r] & N,
\end{tikzcd}$$

exhibiting N as a homotopy pushout by left properness of $G_+{-}\mathrm{Mod}_{\mathrm{un}}$ (the forgetful functor $G_+{-}\mathrm{Mod}_{\mathrm{un}} \to \mathrm{sSet}_*$ preserves cofibrations and creates colimits and weak equivalences). Applying $\phi_!$ yields a pushout diagram

$$\begin{tikzcd}
(H \times \partial\Delta[n])_+ \arrow[r] \arrow[d, hook] & \phi_! M \arrow[d] \\
(H \times \Delta[n])_+ \arrow[r] & \phi_! N,
\end{tikzcd}$$

exhibiting $\phi_! N$ as a homotopy pushout. Since ϕ^* does not change the underlying simplicial sets it follows that $N \to \phi^*\phi_! N$ is a weak equivalence.

Let $\mathcal{C}$ be the class of objects M in $G_+{-}\mathrm{Mod}_{\mathrm{un}}$ for which the unit $M \to \phi^*\phi_! M$ is a weak equivalence. It is clear that $\mathcal{C}$ is closed under retracts since the class of weak equivalences has this property. We have shown that $\mathcal{C}$ contains the domains and codomains of the generating cofibrations, and is moreover closed under forming pushouts of generating cofibrations. The forgetful functor $U\colon G_+{-}\mathrm{Mod}_{\mathrm{un}} \to \mathrm{sSet}_*$ creates colimits so that the domains and codomains of $\mathcal{I}_+^G$ are compact objects. By (the proof of) [Dug01a, Proposition 7.3] this implies that filtered colimits in $G_+-\mathrm{Mod}_{\mathrm{un}}$ preserve weak equivalences and hence that $\mathcal{C}$ is closed under transfinite composition. We have therefore shown that $\mathcal{C}$ contains all cofibrant objects from which the assertion now follows. □

Theorem 2.2.9. *For X a reduced simplicial set, there is a* sSet_**-Quillen equivalence*

$$\begin{tikzcd} R_X \arrow[r, shift left, "\beta_X^+"] \arrow[r, phantom, "\perp"] & \mathbb{G}X_+-\mathrm{Mod}_{\mathrm{un}}. \arrow[l, shift left, "\gamma_X^+"] \end{tikzcd}$$

For $f\colon X \to Y$ a morphism of reduced simplicial sets, there is a diagram of left sSet_**-Quillen functors*

$$\begin{tikzcd} R_X \arrow[r, "f_!"] \arrow[d, "\beta_X^+"'] & R_Y \arrow[d, "\beta_Y^+"] \\ \mathbb{G}X_+-\mathrm{Mod}_{\mathrm{un}} \arrow[r, "\mathbb{G}f_!"] & \mathbb{G}Y_+-\mathrm{Mod}_{\mathrm{un}} \end{tikzcd}$$

which commutes up to natural isomorphism.

Proof. Throughout the proof we use the isomorphism $\mathbb{G}X-\mathrm{sSet}_* \cong \mathbb{G}X_+-\mathrm{Mod}_{\mathrm{un}}$ of Remark 2.2.7 with impunity. The adjoint equivalence $(\pi_X^* \dashv \pi_!^X)$ from the proof of Theorem 2.2.1 extends to an adjoint equivalence

$$(\pi_X^* \dashv \pi_!^X)\colon R_X \longrightarrow \left(\mathbb{G}X-\mathrm{sSet}_{/\mathbb{P}X}\right)^{\mathbb{P}X/}.$$

The latter adjunction intertwines the model structures, so is a Quillen equivalence. The underlying functor of $\pi_X^*\colon R_X \to \left(\mathbb{G}X-\mathrm{sSet}_{/\mathbb{P}X}\right)^{\mathbb{P}X/}$, obtained by forgetting the $\mathbb{G}X$-action, coincides with the base change functor $\pi_X^*\colon R_X \to R_{\mathbb{P}X}$ on retractive spaces.

The adjunction $(\beta_X^+ \dashv \gamma_X^+)$ is obtained as the composite

$$\begin{tikzcd} R_X \arrow[r, shift left, "\pi_X^*"] \arrow[r, phantom, "\perp"] & \left(\mathbb{G}X-\mathrm{sSet}_{/\mathbb{P}X}\right)^{\mathbb{P}X/} \arrow[l, shift left, "\pi_!^X"] \arrow[r, shift left, "\mathbb{P}X_!"] \arrow[r, phantom, "\perp"] & \mathbb{G}X_+-\mathrm{Mod}_{\mathrm{un}}, \arrow[l, shift left, "\mathbb{P}X^*"] \end{tikzcd}$$

where $\mathbb{P}X_!$ sends $(\mathbb{P}X \to A \to \mathbb{P}X)$ to the pushout of $A \leftarrow \mathbb{P}X \to *$ regarded as an object of $\mathbb{G}X_+-\mathrm{Mod}_{\mathrm{un}}$ in the obvious way. The right adjoint $\mathbb{P}X^*$ sends a pointed $\mathbb{G}X$-space B to the product $\mathbb{P}X \times B$ with the obvious structure maps. Recall that the forgetful functor $\mathbb{G}X-\mathrm{sSet} \to \mathrm{sSet}$ preserves limits and colimits, so that after forgetting $\mathbb{G}X$-actions the composite $\beta_X^+ = \mathbb{P}X_! \circ \pi_X^*$ coincides with the composite base change functor $\mathbb{P}X_! \circ \pi_X^*\colon R_X \to R_{\mathbb{P}X} \to \mathrm{sSet}_*$. By Lemma 1.1.11 and Remark 2.2.7 it follows that β_X^+ preserves sSet_*-tensors up to isomorphism.

We show that the adjunction $(\mathbb{P}X_! \dashv \mathbb{P}X^*)\colon \left(\mathbb{G}X-\mathrm{sSet}_{/\mathbb{P}X}\right)^{\mathbb{P}X/} \to \mathbb{G}X_+-\mathrm{Mod}_{\mathrm{un}}$ is a Quillen equivalence, when then implies that $(\beta_X^+ \dashv \gamma_X^+)$ is a sSet_*-Quillen equivalence by the above. The right ajoint $\mathbb{P}X^*$ preserves and reflects weak equivalences and fibrations, so it is sufficient to check that the unit $P \to \mathbb{P}X^*\mathbb{P}X_!(P)$ is a weak equivalence for all cofibrant P. For this, we note that for any $P \in \left(\mathbb{G}X-\mathrm{sSet}_{/\mathbb{P}X}\right)^{\mathbb{P}X/}$ the structure map $\mathbb{P}X \to P$ is a monomorphism and hence a cofibration for the Kan model structure. Left properness of the Kan model structure implies that the bottom horizontal morphism in the defining pushout diagram

$$\begin{tikzcd} \mathbb{P}X \arrow[r, "\sim"] \arrow[d, tail] & * \arrow[d] \\ P \arrow[r] & \mathbb{P}X_!P \end{tikzcd}$$

is a weak equivalence. As $\mathbb{P}X$ is weakly contractible, the map of $\mathbb{G}X$-spaces

$$P \longrightarrow \mathbb{P}X^*\mathbb{P}X_!(P) \cong \mathbb{P}X \times \mathbb{P}X_!P$$

is a weak equivalence for any P. In particular, $(\mathbb{P}X_! \dashv \mathbb{P}X^*)$ is a Quillen equivalence.

Finally, let $f\colon X \to Y$ be a map of reduced simplicial sets. By inspection, the functor γ_X^+ is naturally isomorphic to the functor $\mathbb{G}X_+{-}\mathrm{Mod}_{\mathrm{un}} \cong \mathbb{G}X{-}\mathrm{sSet}_* \to R_X$ which sends

$$(* \to A) \longmapsto \big(\gamma_X(*) = X \to \gamma_X(A) \to X\big)$$

and similarly for γ_Y^+. For a pointed $\mathbb{G}Y$-space $(* \to B)$, Lemma 2.2.5 (and essential uniqueness of adjoints) implies that there are natural isomorphisms

$$(\gamma_X^+\mathbb{G}f^*)(* \to B) \cong \left(\begin{tikzcd}[column sep=small] X \arrow[r] \arrow[dr] & \gamma_X(\mathbb{G}f^*(B)) \cong f^*\gamma_Y(Q) \arrow[d] \\ & X \end{tikzcd}\right) \cong (f^*\gamma_Y^+)(* \to B).$$

Invoking essential uniqueness of adjoints completes the proof. □

Remark 2.2.10. Observe that both β_X^+ and γ_X^+ preserve sSet_*-tensors.

Definition 2.2.11. Recall that the free symmetric spectrum functor $\mathrm{F}_0\colon \mathrm{sSet}_* \to \mathrm{Sp}^\Sigma$ is symmetric monoidal. For a simplicial group G, the assignment $G \mapsto \mathrm{F}_0(G_+)$ thus defines a symmetric ring spectrum. The *category of stable G_+-modules* $G_+{-}\mathrm{Mod}$ is defined to be the category of left $\mathrm{F}_0(G_+)$-modules in Sp^Σ.

Remark 2.2.12. For a symmetric spectrum P, the smash product $\mathrm{F}_0G_+ \wedge P$ has n-th term $(\mathrm{F}_0G_+ \wedge P)_n \cong G_+ \wedge P_n$ (beware that "$\wedge$" has different meanings on each side of this equation). Due to this we can identity $G_+{-}\mathrm{Mod}$ with symmetric spectrum objects in $G_+{-}\mathrm{Mod}_{\mathrm{un}}$ with respect to the usual suspension functor:

$$\mathrm{Sp}^\Sigma\big(G_+{-}\mathrm{Mod}_{\mathrm{un}}; \Sigma\big) \cong G_+{-}\mathrm{Mod}.$$

In particular we have a family of adjunctions

$$\begin{tikzcd} G_+{-}\mathrm{Mod}_{\mathrm{un}} \arrow[r, shift left, "\mathrm{F}_n"] \arrow[r, phantom, "\perp"] & G_+{-}\mathrm{Mod} \arrow[l, shift left, "\mathrm{Ev}_n"] \end{tikzcd}$$

for each $n \in \mathbb{N}$, where the right adjoint sends a symmetric spectrum of unstable G_+-modules to its n-th term. We also have a free-forgetful monadic adjunction

$$\begin{tikzcd} \mathrm{Sp}^\Sigma \arrow[r, shift left, "\mathrm{F}_0(G_+)\wedge(-)"] \arrow[r, phantom, "\perp"] & G_+{-}\mathrm{Mod}, \arrow[l, shift left] \end{tikzcd}$$

which shows that $G_+{-}\mathrm{Mod}$ is locally presentable. The model structure on $G_+{-}\mathrm{Mod}$ coincides with that transferred by the free-forgetful adjunction since the classes of fibrations and weak equivalences coincide.

Theorem 2.2.13. *For X a reduced simplicial set, there is a* Sp^Σ*-Quillen equivalence*

$$\begin{tikzcd} \mathrm{Sp}^\Sigma_X \arrow[r, shift left, "\beta^+_X"] \arrow[r, phantom, "\perp"] & \mathbb{G}X_+{-}\mathrm{Mod}. \arrow[l, shift left, "\gamma^+_X"] \end{tikzcd}$$

For $f\colon X \to Y$ a morphism of reduced simplicial sets, there is a diagram of left Sp^Σ*-Quillen functors*

$$\begin{tikzcd} \mathrm{Sp}^\Sigma_X \arrow[r, "f_!"] \arrow[d, "\beta^+_X"'] & \mathrm{Sp}^\Sigma_Y \arrow[d, "\beta^+_Y"] \\ \mathbb{G}X_+{-}\mathrm{Mod} \arrow[r, "\mathbb{G}f_!"] & \mathbb{G}Y_+{-}\mathrm{Mod} \end{tikzcd}$$

commuting up to natural isomorphism.

Proof. The result follows at once by applying the symmetric stabilisation machine (Theorem A.3.28) to the adjunctions of Theorem 2.2.9, using the isomorphism of categories of Remark 2.2.12. □

Remark 2.2.14. $\gamma^+_X\colon \mathbb{G}X_+{-}\mathrm{Mod} \to \mathrm{Sp}^\Sigma_X$ also preserves Sp^Σ-tensors by Remark 2.2.10.

Remark 2.2.15. Observe that $\mathbb{S}[\mathbb{G}X_+]$ is a weak compact generator of $Ho(\mathbb{G}X_+{-}\mathrm{Mod})$ (as is readily seen from the free/forgetful adjunction). By direct computation we have

$$\beta^+_X \mathrm{F}^X_0(\ast \coprod X) \cong \mathbb{S}[\mathbb{G}X]$$

so by Theorem 2.2.13 it follows that $\mathrm{F}^X_0(\ast \coprod X)$ is a weak compact generator of $Ho(\mathrm{Sp}^\Sigma_X)$. Under the equivalence of homotopy categories supplied by Theorem 1.2.23, the symmetric X-spectrum $\mathrm{F}^X_0(\ast \coprod X)$ is identified with the sequential X-spectrum $F^X_0(\ast \coprod X)$ of Example 1.2.37. Recall that the latter sequential spectrum has n-th space $S^n \vee_x X$ where $x\colon \ast \to X$ is the basepoint.

For connected (but not necessarily reduced) simplicial sets X, we therefore find that $x_!\mathbb{S}$ is a weak compact generator of $Ho(\mathrm{Sp}_X)$ for any $x\colon \ast \to X$ (in either the sequential or symmetric setting).

2.2.3 Monoidal Structures

In this section, we show that the equivalences of Theorems 2.2.9 and 2.2.13 induce symmetric monoidal structures on the homotopy categories of X-spaces and X-spectra for connected X. We prove that $\mathbb{G}X{-}\mathrm{sSet}$ and $\mathbb{G}X_+{-}\mathrm{Mod}$ are symmetric monoidal model categories, despite the fact that $\mathrm{sSet}_{/X}$ and Sp^Σ_X are not (unless $X = \ast$). The monoidal structures are then transferred along the equivalences of homotopy categories presented by the Quillen equivalences of Theorems 2.2.9 and 2.2.13.

Lemma 2.2.16. *For G a simplicial group, the product of G-spaces (equipped with the diagonal action) defines a symmetric monoidal model structure on $G{-}\mathrm{sSet}$.*

Proof. This is proven as Lemma B.2.2. □

As before, we pass to the stable setting by first taking pointed objects and then applying the symmetric stabilisation machine.

Remark 2.2.17. Let $\mathcal{M}$ be a monoidal model category with respect to $\otimes$, and write $*$ for the terminal object. The category $\mathcal{M}_*$ becomes a monoidal category, with monoidal structure defined as the pushout

$$\begin{tikzcd}
M \otimes * \coprod * \otimes N \arrow[r, "M\otimes n + m \otimes N"] \arrow[d] & M \otimes N \arrow[d] \\
* \arrow[r] & M \oslash N
\end{tikzcd}$$

for pointed objects $m\colon * \to M$ and $n\colon * \to M$. This makes $\mathcal{M}_*$ a monoidal model category [Hov99, Proposition 4.2.9].

According to Remark 2.2.17, $G_+{-}\mathrm{Mod}_{\mathrm{un}}$ becomes a symmetric monoidal model category with respect to the monoidal product

$$(M, N) \longmapsto M \wedge N,$$

where G_+ acts on $M \wedge N$ via

$$G_+ \wedge M \wedge N \xrightarrow{\Delta_+ \wedge M \wedge N} G_+ \wedge G_+ \wedge M \wedge N \cong G_+ \wedge M \wedge G_+ \wedge N \xrightarrow{\rho_M \wedge \rho_N} M \wedge N,$$

where $\Delta\colon G \to G \times G$ is the diagonal map and ρ_M, ρ_N are the G_+-actions on M and N respectively. Applying the symmetric stabilisation machine, we have

Lemma 2.2.18. *For G a simplicial group, $G_+{-}\mathrm{Mod}$ is a symmetric monoidal Sp^Σ-model category with respect to the smash product equipped with the diagonal $\mathrm{F}_0(G_+)$-action.*

We conclude this section with a comparison of these monoidal structures on $\mathbb{G}X{-}\mathrm{sSet}$ and $\mathbb{G}X_+{-}\mathrm{Mod}$ with the fibre product $\times_X$ and internal smash product $\wedge_X$, ignoring homotopy-theoretic issues.

Lemma 2.2.19. *For X a reduced simplicial set, the right adjoint functor*

$$\gamma_X\colon \mathbb{G}X{-}\mathrm{sSet} \longrightarrow \mathrm{sSet}_{/X}$$

is strong monoidal.

Proof. This is immediate since the monoidal structure in both cases is the categorical product. $\square$

Recall that R_X is a symmetric monoidal category with respect to the internal smash product $\wedge_X$, though this monoidal structure does not behave well with respect to the model structure (Remark 1.1.10). For a reduced simplicial set X, the functor γ_X^+ is the composite

$$\mathbb{G}X_+{-}\mathrm{Mod}_{\mathrm{un}} \xrightarrow{\mathbb{P}X^*} \left(\mathbb{G}X{-}\mathrm{sSet}_{/\mathbb{P}X}\right)^{\mathbb{P}X/} \xrightarrow{\pi_!^X} R_X.$$

As the functors $\pi_!^X$ and $\mathbb{P}X^*$ both preserve colimits, we conclude that γ_X^+ sends the smash product $\wedge$ to the internal smash product $\wedge_X$. This completes the proof of the

Lemma 2.2.20. *For X a reduced simplicial set, the right adjoint functor*

$$\gamma_X^+\colon \mathbb{G}X_+{-}\mathrm{Mod}_{\mathrm{un}} \longrightarrow R_X$$

is strong monoidal with respect to $\wedge$ *and* $\wedge_X$.

The functor $\gamma_X^+ \colon \mathbb{G}X_+\text{-Mod} \to \mathrm{Sp}_X^\Sigma$ is defined by levelwise application of the functor γ_X^+ of the Lemma, which is strong monoidal and preserves colimits. We immediately obtain the following

Lemma 2.2.21. *For* X *a reduced simplicial set, the right adjoint functor*

$$\gamma_X^+ \colon \mathbb{G}X_+\text{-Mod} \longrightarrow \mathrm{Sp}_X^\Sigma$$

is strong monoidal.

Remark 2.2.22. The upshot of our work in this section is that although the internal smash product on symmetric X-spectra does not define a monoidal structure on the homotopy category, we can obtain a monoidal structure by passing to homotopy fibres. In the sequel, we shall see that this monoidal structure has a particularly nice algebraic description after rationalisation.

2.3 Rational Parametrised Spectra

We are now in a position to tackle the rational homotopy theory of parametrised spectra. In the previous section, we identified spectra parametrised by a connected space X with their homotopy fibre ΩX-module spectra. This identification forms the basis for our approach to the rational homotopy theory: a map of X-spectra $f \colon P \to Q$ is a *rational homotopy equivalence* if induced map of homotopy fibre spectra is a rational homotopy equivalence (Definition 2.3.6). In this section, we study the rational homotopy theory of spectra parametrised by connected spaces and connect this to rational homotopy representations of the loop group.

2.3.1 Rationalisation is Smashing

As a warm-up to our treatment of the rational homotopy theory of parametrised spectra we first treat the unparametrised case. We present a particular incarnation of the fact that passage to the rational stable homotopy category is a *smashing localisation*, so that rationalisation is implemented by smashing with a particular spectrum $M \mapsto M \wedge^{\mathbb{L}} \mathcal{H}\mathbb{Q}$.

Remark 2.3.1. The assignment $S \mapsto \mathbb{Q}[S]$ which sends a set S to the free $\mathbb{Q}$-vector space generated by S extends to a free-forgetful adjunction between categories of simplicial objects

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, shift left, "{\mathbb{Q}[-]}"] \arrow[r, phantom, "\perp"] & \mathrm{sVect}_{\mathbb{Q}} . \arrow[l, shift left, "U"] \end{tikzcd} \tag{2.4}$$

For a simplicial set K, the homotopy groups $\pi_*(\mathbb{Q}[K]) \cong H_\bullet(K;\mathbb{Q})$ so that after passing to homotopy groups, the unit of the adjunction is the rational Hurewicz map $\pi_*(K) \to H_\bullet(K;\mathbb{Q})$. In particular, the left adjoint $\mathbb{Q}[-]$ preserves weak equivalences. It is standard that $\mathrm{sVect}_{\mathbb{Q}}$ has a model structure with respect to which the fibrations and weak equivalences are created by the forgetful functor U. One way to see this is to transfer the Kan model structure to $\mathrm{sVect}_{\mathbb{Q}}$ via $\mathbb{Q}[-]$ since the requirements of the Right Transfer Theorem are satisfied in this case (as can be checked, for example, by using the normalisation functor, recalled in Remark 2.4.1). In particular, $\mathrm{sVect}_{\mathbb{Q}}$ is a proper combinatorial model category.

There is a closed symmetric monoidal structure on $\mathrm{sVect}_{\mathbb{Q}}$ defined by taking levelwise tensor products of simplicial $\mathbb{Q}$-vector spaces. This makes $\mathrm{sVect}_{\mathbb{Q}}$ a symmetric monoidal model category.

Finally, the free-forgetful adjunction of (2.4) factors as

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, shift left, "(-)_+"] \arrow[r, phantom, "\perp"] & \mathrm{sSet}_* \arrow[l, shift left] \arrow[r, shift left, "\widetilde{\mathbb{Q}}[-]"] \arrow[r, phantom, "\perp"] & \mathrm{sVect}_{\mathbb{Q}}, \arrow[l, shift left, "U_0"] \end{tikzcd}$$

where U_0 regards the underlying simplicial set of a simplicial $\mathbb{Q}$-vector space as pointed by the zero element, and for a pointed simplicial set $k\colon * \to K$ we set

$$\widetilde{\mathbb{Q}}[K]_n := \mathbb{Q}[K_n \setminus s_0(k)] \cong \mathbb{Q}[K_n]/\mathbb{Q}[s_0(k)],$$

where $s_0\colon K_0 \to K_n$ is the (unique) iterated degeneracy map. Note that for a pointed simplicial set K, the homotopy groups $\pi_*(\widetilde{\mathbb{Q}}[K]) \cong \widetilde{H}_\bullet(K;\mathbb{Q})$ compute the reduced rational homology groups of K.

Lemma 2.3.2. *The functor $\widetilde{\mathbb{Q}}[-]$ is strong symmetric monoidal with respect to $\wedge$ and $\otimes_{\mathbb{Q}}$.*

Proof. It is easy to see that $\mathbb{Q}[-]$ is strong symmetric monoidal with respect to $\times$ and $\otimes_{\mathbb{Q}}$; the levelwise isomorphisms

$$\begin{aligned} \mathbb{Q}[X \times Y]_n &\longrightarrow \mathbb{Q}[X]_n \otimes_{\mathbb{Q}} \mathbb{Q}[Y_n] \\ \sum_i a_i \cdot (x_i, y_i) &\longmapsto \sum_i a_i \cdot (x_i \otimes y_i) \end{aligned}$$

witness the strong monoidal structure. For a pointed simplicial set $k\colon * \to K$, we have $\mathbb{Q}[K] \cong \mathbb{Q} \oplus \widetilde{\mathbb{Q}}[K]$, from which it follows that $\widetilde{\mathbb{Q}}[K \wedge L] \cong \widetilde{\mathbb{Q}}[K] \otimes_{\mathbb{Q}} \widetilde{\mathbb{Q}}[L]$. Since $\widetilde{\mathbb{Q}}[S^0] = \mathbb{Q}$, the result follows. □

Construction 2.3.3. The *symmetric Eilenberg–Mac Lane spectrum* of $\mathbb{Q}$ is the symmetric spectrum $\mathcal{H}\mathbb{Q}$ with underlying symmetric sequence

$$n \longmapsto \mathcal{H}\mathbb{Q}_n := \widetilde{\mathbb{Q}}[S^n],$$

with the Σ_n-action induced by permuting smash factors. The spectrum structure maps are the composites

$$S^1 \wedge \mathcal{H}\mathbb{Q}_n \longrightarrow \widetilde{\mathbb{Q}}[S^1] \otimes_{\mathbb{Q}} \widetilde{\mathbb{Q}}[\mathcal{H}\mathbb{Q}_n] \longrightarrow \widetilde{\mathbb{Q}}[S^1] \otimes_{\mathbb{Q}} \mathcal{H}\mathbb{Q}_n \cong \mathcal{H}\mathbb{Q}_{n+1},$$

where the first map is the $(\widetilde{\mathbb{Q}}[-] \dashv U_0)$-unit. Via the maps $S^n \wedge S^m \to S^{n+m}$ and Lemma 2.3.2, $\mathcal{H}\mathbb{Q}$ is a commutative symmetric ring spectrum. The symmetric spectrum $\mathcal{H}\mathbb{Q}$ is level fibrant and each $\mathcal{H}\mathbb{Q}_n$ is a $K(n;\mathbb{Q})$, so that $\mathcal{H}\mathbb{Q}$ is a fibrant Ω-spectrum with (true) stable homotopy groups

$$\pi_k^{\mathrm{st}}(\mathcal{H}\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

We fix, once and for all, a cofibrant replacement $\mathrm{H}\mathbb{Q}$ for $\mathcal{H}\mathbb{Q}$ in the model category of commutative symmetric ring spectra using the main result of [Shi04]. In particular, $\mathrm{H}\mathbb{Q}$ is a cofibrant symmetric spectrum and the map $\mathrm{H}\mathbb{Q} \to \mathcal{H}\mathbb{Q}$ is a stable weak equivalence.

Recall that we write $\wedge^{\mathbb{L}}$ for the derived smash product

$$Ho(\mathrm{Sp}^{\Sigma}) \times Ho(\mathrm{Sp}^{\Sigma}) \longrightarrow Ho(\mathrm{Sp}^{\Sigma}),$$

obtained as the derived functor of the smash product on symmetric spectra. For symmetric spectra P, Q, there is a pairing on stable homotopy groups

$$\pi_k^{\mathrm{st}}(P) \otimes \pi_l^{\mathrm{st}}(Q) \longrightarrow \pi_{k+l}^{\mathrm{st}}(P \wedge^{\mathbb{L}} Q)$$

(for symmetric spectra, this pairing is described in [Sch12, Ch. II (4.13)]). Using this pairing we can present the passage to rational homotopy theory by taking smash products with $\mathcal{H}\mathbb{Q}$:

Lemma 2.3.4. *For any symmetric spectrum P, the pairing on true stable homotopy groups*

$$\pi_k^{\mathrm{st}}(P) \otimes \pi_0^{\mathrm{st}}(\mathcal{H}\mathbb{Q}) \cong \pi_k^{\mathrm{st}}(P) \otimes \mathbb{Q} \longrightarrow \pi_k^{\mathrm{st}}(P \wedge^{\mathbb{L}} \mathcal{H}\mathbb{Q})$$

is an isomorphism for all $k \in \mathbb{Z}$.

Proof. This is a classical result and our proof is essentially the standard one. Let $\mathcal{S}_{\mathbb{Q}} \hookrightarrow Ho(\mathrm{Sp}^{\Sigma})$ be the full subcategory on those $[P]$ for which the pairing on stable homotopy groups $\pi_*^{\mathrm{st}}(P) \otimes \mathbb{Q} \to \pi_*^{\mathrm{st}}(P \wedge^{\mathbb{L}} \mathcal{H}\mathbb{Q})$ is an isomorphism.

As Sp^{Σ} is a stable model category, $Ho(\mathrm{Sp}^{\Sigma})$ is triangulated [Hov99, Ch. 7]. By the Five Lemma, $\mathcal{S}_{\mathbb{Q}}$ is a triangulated subcategory. The functors $(-) \wedge^{\mathbb{L}} \mathcal{H}\mathbb{Q}$ and π_k^{st} commute with small homotopy coproducts, from which it follows that $\mathcal{S}_{\mathbb{Q}}$ is also localising.

The sphere spectrum $\mathbb{S}$ defines a weak compact generator of $Ho(\mathrm{Sp}^{\Sigma})$ (this is seen, for example, by using the criterion of [SS03b, Lemma 2.2.1]), so to complete the proof it is sufficient to show that $\mathbb{S}$ is in $\mathcal{S}_{\mathbb{Q}}$. By Serre's calculation of the homotopy groups of spheres, $\pi_k^{\mathrm{st}}(\mathbb{S}) \otimes \mathbb{Q} \cong \pi_k^{\mathrm{st}}(\mathcal{H}\mathbb{Q})$ which completes the proof. □

The precise sense in which rationalisation is a smashing localisation is captured by fact that there is a model category of $\mathcal{H}\mathbb{Q}$-module spectra presenting the localisation of the stable homotopy category at rational equivalences. At this stage, we make the (largely cosmetic) choice to work with *right* $\mathcal{H}\mathbb{Q}$-module spectra, as it slightly simplifies our treatment of the parametrised case.

Lemma 2.3.5. *The free-forgetful adjunction*

$$\begin{tikzcd} \mathrm{Sp}^{\Sigma} \arrow[r, shift left, "(-)\wedge\mathcal{H}\mathbb{Q}"] \arrow[r, phantom, "\perp"] & \mathrm{Mod}-\mathcal{H}\mathbb{Q} \arrow[l, shift left] \end{tikzcd}$$

equips $\mathrm{Mod}-\mathcal{H}\mathbb{Q}$ *with a combinatorial* Sp^{Σ}*-model structure. The derived adjunction*

$$\begin{tikzcd} Ho(\mathrm{Sp}^{\Sigma}) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & Ho(\mathrm{Mod}-\mathcal{H}\mathbb{Q}) \arrow[l, hook, shift left] \end{tikzcd}$$

exhibits the localisation of $Ho(\mathrm{Sp}^{\Sigma})$ at rational stable homotopy equivalences.

Proof. The monoidal model category Sp^{Σ} satisfies the monoid axiom [HSS00, §5.4], so for *any* symmetric ring spectrum R the free-forgetful adjunction

$$\begin{tikzcd} \mathrm{Sp}^{\Sigma} \arrow[r, shift left, "(-)\wedge R"] \arrow[r, phantom, "\perp"] & \mathrm{Mod}-R \arrow[l, shift left] \end{tikzcd}$$

equips Mod$-R$ with a transferred Sp^Σ-model structure, in which fibrations and weak equivalences are created by the forgetful functor. The pushout-product axiom for the Sp^Σ-tensoring is verified directly on generating cofibrations and acyclic cofibrations. For $\mathrm{H}\mathbb{Q} \to \mathcal{H}\mathbb{Q}$ as in Construction 2.3.3, we therefore have a sequence of Quillen adjunctions

$$\begin{tikzcd} \mathrm{Sp}^\Sigma \arrow[r, shift left, "(-)\wedge \mathrm{H}\mathbb{Q}"] \arrow[r, phantom, "\perp"] & \mathrm{Mod}-\mathrm{H}\mathbb{Q} \arrow[l, shift left] \arrow[r, shift left, "(-)\wedge_{\mathrm{H}\mathbb{Q}}\mathcal{H}\mathbb{Q}"] \arrow[r, phantom, "\perp"] & \mathrm{Mod}-\mathcal{H}\mathbb{Q}. \arrow[l, shift left] \end{tikzcd}$$

By [HSS00, Lemma 5.4.4], for any cofibrant $M \in \mathrm{Mod}-\mathrm{H}\mathbb{Q}$ the functor $M \wedge_{\mathrm{H}\mathbb{Q}} (-)$ preserves stable equivalences. For any $N \in \mathrm{Mod}-\mathcal{H}\mathbb{Q}$ we have a diagram of symmetric spectra arising from the second adjunction above:

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d] & N \\ M \wedge_{\mathrm{H}\mathbb{Q}} N, \arrow[ur, "\psi^\vee"'] & \end{tikzcd}$$

where the vertical arrow is the unit of the adjunction. For M cofibrant, the vertical arrow is

$$M \wedge_{\mathrm{H}\mathbb{Q}} \mathrm{H}\mathbb{Q} \cong M \longrightarrow M \wedge_{\mathrm{H}\mathbb{Q}} \mathcal{H}\mathbb{Q},$$

which we have established is a stable weak equivalence. By the 2-out-of-3 property, the map ψ is a stable equivalence if and only if its adjunct $\psi^\vee$ is. In particular, $(-) \wedge_{\mathrm{H}\mathbb{Q}} \mathcal{H}\mathbb{Q}\colon \mathrm{Mod}-\mathrm{H}\mathbb{Q} \to \mathrm{Mod}-\mathcal{H}\mathbb{Q}$ is a left Quillen equivalence.

It suffices therefore to work with Mod$-\mathrm{H}\mathbb{Q}$ from now on. Smashing with a cofibrant symmetric spectrum preserves stable weak equivalences (by the pushout-product axiom), from which it follows that $P \wedge \mathrm{H}\mathbb{Q}$ represents $P \wedge^{\mathbb{L}} \mathcal{H}\mathbb{Q}$ for any symmetric spectrum P. By Lemma 2.3.4 we now have that a map of symmetric spectra $f\colon A \to B$ is sent to a stable weak equivalence by $(-) \wedge \mathrm{H}\mathbb{Q}$ precisely if f is a $(\pi_*^{\mathrm{st}} \otimes \mathbb{Q})$-isomorphism. Similarly, the $\mathrm{H}\mathbb{Q}$-action on an object $M \in \mathrm{Mod}-\mathrm{H}\mathbb{Q}$ naturally furnishes $\pi_*^{\mathrm{st}}(M)$ with the structure of a graded $\mathbb{Q}$-vector space. For a fibrant $M \in \mathrm{Mod}-\mathrm{H}\mathbb{Q}$ the derived counit of the free-forgetful adjunction is represented by

$$M^{\mathrm{cof}} \wedge \mathrm{H}\mathbb{Q} \longrightarrow M \wedge \mathrm{H}\mathbb{Q} \longrightarrow M,$$

which is a stable weak equivalence by the above. The derived counit of the free-forgetful adjunction is therefore a natural isomorphism, so we have a reflective localisation

$$\begin{tikzcd} Ho(\mathrm{Sp}^\Sigma) \arrow[r, shift left, hook] \arrow[r, phantom, "\perp"] & Ho(\mathrm{Mod}-\mathrm{H}\mathbb{Q}) \cong Ho(\mathrm{Mod}-\mathcal{H}\mathbb{Q}). \arrow[l, shift left] \end{tikzcd}$$

By the above, $Ho(\mathrm{Mod}-\mathrm{H}\mathbb{Q}) \cong Ho(\mathrm{Mod}-\mathcal{H}\mathbb{Q})$ is the localisation of $Ho(\mathrm{Sp}^\Sigma)$ at rational homotopy equivalences. □

2.3.2 Rational Parametrised Spectra

In this section we study the rational homotopy theory of spectra parametrised by a connected space X. We saw previously that, in this setting, an X-spectrum P is completely encoded by its homotopy fibre ΩX-module spectrum. The main thrust of our work in this section is that this identification persists after smashing (fibrewise) with the Eilenberg–Mac Lane spectrum of $\mathbb{Q}$, and so gives us a way of identifying the rational homotopy theory of X-parametrised spectra.

To motivate the following definition, recall that a morphism of symmetric X-spectra $f\colon A \to B$ is a stable weak equivalence if and only if it is a stable homotopy equivalence on homotopy fibres (Lemma 1.2.22).

Definition 2.3.6. A morphism $f\colon A \to B$ of (symmetric) X-spectra is a *rational homotopy equivalence* if $x^*(\mathcal{R}f)$ is a rational stable weak equivalence for all points $x\colon * \to X$, where $\mathcal{R}$ is any fibrant replacement functor on Sp^{Σ}_X.

Remark 2.3.7. As before, the choice of fibrant replacement functor $\mathcal{R}$ is immaterial. Of course we do not need to test homotopy fibres at *all* points, merely one point in each path component of the base space X (as in Remark 1.2.11).

The ideal parametrised version of Lemma 2.3.5 would be the statement that taking the internal smash product with the trivial $\mathcal{H}\mathbb{Q}$-module spectrum over X exhibits the rationalisation of $Ho(\mathrm{Sp}^{\Sigma}_X)$. In trying to produce such a statement we are confronted with two difficulties: firstly that the relative smash product is not homotopically well-behaved unless $X = *$, and secondly that we must take fibrant replacements before we can compute homotopy fibres. When X is connected, we circumvent these problems by

- in the first instance by observing that Sp^{Σ}_X has Sp^{Σ}-tensors which *are* homotopically well-behaved, as well as sufficient for our purposes; and
- in the second instance by replacing X by a reduced simplicial set and then by passing to homotopy fibres via $\beta^+_X\colon \mathrm{Sp}^{\Sigma}_X \to \mathbb{G}X_+\mathrm{-Mod}$.

We start by setting up a model theory of parametrised $\mathcal{H}\mathbb{Q}$-module spectra. Recall our fixed choice of cofibrant resolution $\mathrm{H}\mathbb{Q} \to \mathcal{H}\mathbb{Q}$ (Construction 2.3.3). For any $X \in \mathrm{sSet}$ we obtain a monad $T_X = (-) \oslash_X \mathrm{H}\mathbb{Q}$ on Sp^{Σ}_X for any X by regarding the tensoring functor

$$\mathrm{H}\mathbb{Q} \oslash_X (-)\colon \mathrm{Sp}^{\Sigma}_X \longrightarrow \mathrm{Sp}^{\Sigma}_X$$

as equipped with the *twisted* monad multiplication map

$$\mathrm{H}\mathbb{Q} \oslash \mathrm{H}\mathbb{Q} \oslash (-) \cong (\mathrm{H}\mathbb{Q} \wedge \mathrm{H}\mathbb{Q}) \oslash (-) \xrightarrow{\tau} (\mathrm{H}\mathbb{Q} \wedge \mathrm{H}\mathbb{Q}) \oslash (-) \xrightarrow{\mu \oslash (-)} \mathrm{H}\mathbb{Q} \oslash (-),$$

where $\tau\colon \mathrm{H}\mathbb{Q} \wedge \mathrm{H}\mathbb{Q} \to \mathrm{H}\mathbb{Q} \wedge \mathrm{H}\mathbb{Q}$ is the isomorphism interchanging smash factors and μ is the multiplication map. The monadic unit is obtained by tensoring with the unit map $\mathbb{S} \to \mathrm{H}\mathbb{Q}$ as usual. As $\mathrm{H}\mathbb{Q}$ is a commutative symmetric ring spectrum, we need not be so pedantic about taking the twisted multiplication; however this amplifies that we are interested in parametrised *right* $\mathrm{H}\mathbb{Q}$-module spectra.

Definition 2.3.8. The *category of (right)* $\mathrm{H}\mathbb{Q}$*-module* X*-spectra* is the category of algebras $\mathrm{Mod{-}H\mathbb{Q}}_X := T_X\mathrm{-Alg}$ of algebras for the monad $T_X = (-) \oslash_X \mathrm{H}\mathbb{Q}$ on Sp^{Σ}_X.

As $(-) \oslash_X \mathrm{H}\mathbb{Q}$ preserves colimits, $\mathrm{Mod{-}H\mathbb{Q}}_X$ is locally presentable and limits and colimits are preserved by the forgetful functor $\mathrm{Mod{-}H\mathbb{Q}}_X \to \mathrm{Sp}^{\Sigma}_X$. As $\mathrm{H}\mathbb{Q}$ is cofibrant, the underlying functor of the monad T_X is left Quillen by the pushout-product axiom for the Sp^{Σ}-tensoring on Sp^{Σ}_X, so that the free-forgetful functor

$$\begin{tikzcd} \mathrm{Sp}^{\Sigma}_X \arrow[r, shift left, "(-)\oslash_X \mathrm{H}\mathbb{Q}"] \arrow[r, phantom, "\perp"] & \mathrm{Mod{-}H\mathbb{Q}}_X \arrow[l, shift left] \end{tikzcd}$$

equips $\mathrm{Mod}-\mathrm{H}\mathbb{Q}_X$ with a combinatorial Sp^Σ-model structure in which the fibrations and weak equivalences are created by the forgetful functor. Moreover, observe that since $\mathrm{H}\mathbb{Q}$ is cofibrant the pushout-product axiom shows that each of the transferred generating cofibrations in $\mathcal{I}^X_\Sigma \otimes_X \mathrm{H}\mathbb{Q}$ is sent to a cofibration by the forgetful functor, so that $\mathrm{Mod}-\mathrm{H}\mathbb{Q}_X \to \mathrm{Sp}^\Sigma_X$ preserves cofibrations (with notation as in §1.2.1).

Lemma 2.3.9. *For $f\colon X \to Y$ a map of simplicial sets, there is a diagram of left Sp^Σ-Quillen functors*

$$\begin{tikzcd} \mathrm{Sp}^\Sigma_X \arrow[r, "f_!"] \arrow[d, "(-)\otimes_X \mathrm{H}\mathbb{Q}"'] & \mathrm{Sp}^\Sigma_Y \arrow[d, "(-)\otimes_Y \mathrm{H}\mathbb{Q}"] \\ \mathrm{Mod}-\mathrm{H}\mathbb{Q}_X \arrow[r, "f_!"] & \mathrm{Mod}-\mathrm{H}\mathbb{Q}_Y \end{tikzcd}$$

commuting up to natural isomorphism, where $f_!\colon \mathrm{Mod}-\mathrm{H}\mathbb{Q}_X \to \mathrm{Mod}-\mathrm{H}\mathbb{Q}_Y$ is given by $f_!\colon \mathrm{Sp}^\Sigma_X \to \mathrm{Sp}^\Sigma_Y$ on underlying objects. If f is a weak equivalence then the pushforward functor $f_!$ on fibrewise $\mathrm{H}\mathbb{Q}$-modules is a left Quillen equivalence.

Proof. The follows immediately from the above discussion and Lemma 1.2.17: the adjoint functors $(f_! \dashv f^*)\colon \mathrm{Mod}-\mathrm{H}\mathbb{Q}_X \to \mathrm{Mod}-\mathrm{H}\mathbb{Q}_Y$ coincide with the adjoint functors $(f_! \dashv f^*)\colon \mathrm{Sp}^\Sigma_X \to \mathrm{Sp}^\Sigma_Y$ on underlying objects since both functors in the latter adjunction preserve Sp^Σ-tensors.

We spell out the proof of the last part, using the fact that $\mathrm{Mod}-\mathrm{H}\mathbb{Q}_X \to \mathrm{Sp}^\Sigma_X$ preserves cofibrations. Suppose that $A \in \mathrm{Mod}-\mathrm{H}\mathbb{Q}_X$ and $B \in \mathrm{Mod}-\mathrm{H}\mathbb{Q}_Y$ are respectively cofibrant and fibrant. Then A and B are cofibrant and fibrant objects in Sp^Σ_X and Sp^Σ_Y respectively. For a morphism $\psi\colon f_!A \to B$ in $\mathrm{Mod}-\mathrm{H}\mathbb{Q}_Y$ the following are equivalent

(i) ψ is a weak equivalence in $\mathrm{Mod}-\mathrm{H}\mathbb{Q}_Y$;

(ii) the underlying morphism of ψ is a stable weak equivalence in Sp^Σ_Y;

(iii) the underlying morphism of the adjoint map $\psi^\vee$ is a stable weak equivalence in Sp^Σ_X;

(iv) the adjoint morphism $\psi^\vee$ is a weak equivalence in $\mathrm{Mod}-\mathrm{H}\mathbb{Q}_X$.

The equivalence (i)⇔(ii) is by definition, (ii)⇔(iii) is since $(f_! \dashv f^*)\colon \mathrm{Sp}^\Sigma_X \to \mathrm{Sp}^\Sigma_Y$ is a Quillen equivalence by Lemma 1.2.17, and (iii)⇔(iv) is once again by definition since the underlying map of X-spectra of $\psi^\vee\colon A \to f^*B$ coincides with the adjoint of the underlying map of $\psi\colon f_!A \to B$. This proves the assertion. □

We now turn to the other side of the story, as it were, and study rationalisation of G_+-module spectra for a simplicial group G. Since $M \mapsto M \wedge \mathrm{H}\mathbb{Q}$ implements rationalisation in the unparametrised setting, it is natural to expect that the rational homotopy theory of G_+-module spectra is described by $(G_+, \mathrm{H}\mathbb{Q})$-bimodule spectra.

Definition 2.3.10. Let G be a simplicial group G and R a symmetric ring spectrum. We define the category $(G_+, R)-\mathrm{Bimod}$ of *(symmetric) (G_+, R)-bimodule spectra* as either

- the category of algebras over the monad $\mathrm{F}_0(G_+) \wedge (-)$ on $\mathrm{Mod}-R$; or
- the category of algebras over the monad $(-) \wedge R$ on $G_+-\mathrm{Mod}$,

noting that these categories are all canonically isomorphic.

Remark 2.3.11. The category $(G_+, R)-\mathrm{Bimod}$ is locally presentable, with limits and colimits preserved by the forgetful functor to Sp^Σ. This follows from the facts that the monads $\mathrm{F}_0(G_+) \wedge (-)$ and $(-) \wedge R$ on Sp^Σ both preserve colimits and Sp^Σ is locally presentable.

Lemma 2.3.12. *The free-forgetful adjunction*

$$G_+-\mathrm{Mod} \underset{\perp}{\overset{(-)\wedge \mathrm{H}\mathbb{Q}}{\rightleftarrows}} (G_+, \mathrm{H}\mathbb{Q})-\mathrm{Bimod}$$

produces a combinatorial Sp^Σ*-model structure on* $(G_+, \mathrm{H}\mathbb{Q})-\mathrm{Bimod}$*. The derived adjunction exhibits the localisation of* $Ho(\mathrm{G}_+-\mathrm{Mod})$ *at rational stable homotopy equivalences.*

Proof. The model structure is produced using the Right Transfer Theorem A.1.3. We observed in Remark 2.2.12 that the model structure on $G_+-\mathrm{Mod}$ coincides with the model structure transferred via the free-forgetful adjunction

$$\mathrm{Sp}^\Sigma \underset{\perp}{\overset{\mathrm{F}_0(G_+)\wedge(-)}{\rightleftarrows}} G_+-\mathrm{Mod},$$

so that $\mathcal{I}_\Sigma^{G_+} := \mathrm{F}_0(G_+) \wedge \mathcal{I}_\Sigma$ is a set of generating cofibrations for $G_+-\mathrm{Mod}$. The monoid axiom for symmetric spectra ([HSS00, §5.4]) implies that maps in

$$(\mathcal{J}_\Sigma^{G_+} \wedge \mathrm{H}\mathbb{Q})-\mathrm{cell}$$

are stable weak equivalences so that the Right Transfer Theorem applies. The pushout-product axiom for the Sp^Σ-tensoring is checked directly on generating sets of cofibrations and acyclic cofibrations. In fact, this holds for $\mathrm{H}\mathbb{Q}$ replaced by any symmetric ring spectrum R.

The forgetful functor $G_+-\mathrm{Mod} \to \mathrm{Sp}^\Sigma$ preserves colimits and, since $\mathrm{F}_0(G_+)$ is cofibrant, the generating cofibrations of $G_+-\mathrm{Mod}$ are cofibrations in Sp^Σ by the pushout-product axiom. The forgetful functor $G_+-\mathrm{Mod} \to \mathrm{Sp}^\Sigma$ therefore preserves cofibrations. We can now argue as in Lemma 2.3.5; a morphism $f\colon M \to N$ in $G_+-\mathrm{Mod}$ is sent to a stable weak equivalence in $(G_+, \mathrm{H}\mathbb{Q})-\mathrm{Bimod}$ precisely if the underlying morphism of symmetric spectra is a $(\pi_*^{\mathrm{st}} \otimes \mathbb{Q})$-isomorphism. By the above, the same argument as Lemma 2.3.5 also shows that the derived counit of the free-forgetful adjunction is a stable weak equivalence, so that the derived adjunction

$$Ho(G_+-\mathrm{Mod}) \underset{\perp}{\rightleftarrows} Ho\big((G_+, \mathrm{H}\mathbb{Q})-\mathrm{Bimod}\big)$$

exhibits localisation at the rational homotopy equivalences. □

Corollary 2.3.13. *For any simplicial group G, there is a* Sp^Σ*-Quillen equivalence*

$$(G_+, \mathrm{H}\mathbb{Q})-\mathrm{Bimod} \underset{\perp}{\overset{(-)\wedge_{\mathrm{H}\mathbb{Q}}\mathcal{H}\mathbb{Q}}{\rightleftarrows}} (G_+, \mathcal{H}\mathbb{Q})-\mathrm{Bimod}$$

induced by the stable equivalence of commutative symmetric ring spectra $\mathrm{H}\mathbb{Q} \to \mathcal{H}\mathbb{Q}$*.*

Proof. By the proof of the Lemma, $(G_+, \mathcal{H}\mathbb{Q})-\mathrm{Bimod}$ is a combinatorial Sp^Σ-model category. We also have a factorisation

$$G_+-\mathrm{Mod} \underset{\perp}{\overset{(-)\wedge \mathrm{H}\mathbb{Q}}{\rightleftarrows}} (G_+, \mathrm{H}\mathbb{Q})-\mathrm{Bimod} \underset{\perp}{\overset{(-)\wedge_{\mathrm{H}\mathbb{Q}}\mathcal{H}\mathbb{Q}}{\rightleftarrows}} (G_+, \mathcal{H}\mathbb{Q})-\mathrm{Bimod}.$$

of the free-forgetful adjunction for $(G_+, \mathcal{H}\mathbb{Q})$-bimodule spectra. Note that all of the right adjoint functors above preserve and reflect weak equivalences and fibrations and the left adjoints all preserve Sp^Σ-tensors. In the proof of the Lemma above we established that the forgetful functor $G_+-\mathrm{Mod} \to \mathrm{Sp}^\Sigma$ preserves cofibrations, and the forgetful functor $(G_+, \mathrm{H}\mathbb{Q})-\mathrm{Bimod} \to \mathrm{Mod}-\mathrm{H}\mathbb{Q}$ inherits this property by essentially the same argument. By [HSS00, Lemma 5.4.4], $M \wedge_{\mathrm{H}\mathbb{Q}} (-)$ preserves stable weak equivalences for any cofibrant $M \in \mathrm{Mod}-\mathrm{H}\mathbb{Q}$.

Let $M \in (G_+, \mathrm{H}\mathbb{Q})-\mathrm{Bimod}$ be cofibrant and $N \in (G_+, \mathcal{H}\mathbb{Q})-\mathrm{Bimod}$ arbitrary, with a map of $(G_+, \mathrm{H}\mathbb{Q})$-bimodule spectra $\psi\colon M \to N$. The underlying morphisms of ψ and its adjunct $\psi^\vee$ fit into a commuting diagram

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d] & N \\ M \wedge_{\mathrm{H}\mathbb{Q}} N, \arrow[ur, "\psi^\vee"'] & \end{tikzcd}$$

where the vertical arrow is

$$M \wedge_{\mathrm{H}\mathbb{Q}} \mathrm{H}\mathbb{Q} \cong M \longrightarrow M \wedge_{\mathrm{H}\mathbb{Q}} \mathcal{H}\mathbb{Q}$$

which is a weak equivalence since M is cofibrant as an object of $\mathrm{Mod}-\mathrm{H}\mathbb{Q}$. The 2-out-of-3 property of stable weak equivalences now implies the desired Quillen equivalence. ∎

Corollary 2.3.14. *For any morphism of simplicial groups $\phi\colon G \to H$ there is a diagram of left Sp^Σ-Quillen functors*

$$\begin{tikzcd} G_+-\mathrm{Mod} \arrow[r, "(-)\wedge \mathrm{H}\mathbb{Q}"] \arrow[d, "\phi_!"'] & (G_+, \mathrm{H}\mathbb{Q})-\mathrm{Mod} \arrow[r, "(-)\wedge_{\mathrm{H}\mathbb{Q}} \mathcal{H}\mathbb{Q}"] \arrow[d, "\phi_!"'] & (G_+, \mathcal{H}\mathbb{Q})-\mathrm{Mod} \arrow[d, "\phi_!"] \\ H_+-\mathrm{Mod} \arrow[r, "(-)\wedge \mathrm{H}\mathbb{Q}"] & (H_+, \mathrm{H}\mathbb{Q})-\mathrm{Mod} \arrow[r, "(-)\wedge_{\mathrm{H}\mathbb{Q}} \mathcal{H}\mathbb{Q}"] & (H_+, \mathcal{H}\mathbb{Q})-\mathrm{Mod} \end{tikzcd}$$

commuting up to natural isomorphism. If ϕ is either

- *a weak equivalence of simplicial groups; or*
- *a rational homotopy equivalence of connected simplicial groups*

then $\phi_!\colon (G_+, \mathrm{H}\mathbb{Q})-\mathrm{Bimod} \to (H_+, \mathrm{H}\mathbb{Q})-\mathrm{Bimod}$ is a left Quillen equivalence.

Proof. The existence of the diagram follows immediately from the Lemma and by applying the symmetric stabilisation machine to the adjunction of Lemma 2.2.8.

If $\phi\colon G \to H$ is a weak equivalence of simplicial groups, then by passing Lemma 2.2.8 through the symmetric stabilisation machine we obtain a Quillen equivalence $(\phi_! \dashv \phi^*)\colon G_+-\mathrm{Mod} \to H_+-\mathrm{Mod}$. Since $\mathrm{H}\mathbb{Q}$ is cofibrant, from the pushout-product axiom we deduce that the forgetful functor $(G_+, \mathrm{H}\mathbb{Q})-\mathrm{Bimod} \to G_+-\mathrm{Mod}$ preserves cofibrations. Arguing as in Lemma 2.3.9, $(\phi_! \dashv \phi^*)$ is a Quillen equivalence on model categories of bimodules.

Finally, suppose that $\phi\colon G \to H$ is a rational homotopy equivalence of connected simplicial groups. By Corollary 2.1.2, ϕ is equivalently a rational *homology* isomorphism. Since the right Quillen functor $\phi^*\colon (H_+, \mathrm{H}\mathbb{Q})-\mathrm{Bimod} \to (G_+, \mathrm{H}\mathbb{Q})-\mathrm{Bimod}$ preserves and reflects stable weak equivalences, to complete the proof it is sufficient to show that the derived unit of the adjunction is a natural isomorphism. For this we use the fact that $(G_+, H\mathbb{Q})-\mathrm{Bimod}$ is a stable model category (by appealing directly to true stable homotopy groups, for instance), so that its homotopy category is triangulated. Using the free-forgetful adjunctions and the fact that $\mathbb{S}$ is a weak compact generator of the stable homotopy category, we find that the object $F_0(G_+) \wedge H\mathbb{Q}$ determines a weak compact generator of the homotopy category of bimodules.

Let $\mathcal{E} \hookrightarrow Ho\big((G_+, H\mathbb{Q})-\mathrm{Bimod}\big)$ be the full subcategory on those objects for which the derived unit the $(\phi_! \dashv \phi^*)$-adjunction is an isomorphism. We see at once that $\mathcal{E}$ is a localising subcategory. The component of the derived unit at the weak compact generator $F_0(G_+) \wedge H\mathbb{Q}$ is presented by the morphism of spectra

$$F_0(\phi_+) \wedge H\mathbb{Q} \colon F_0(G_+) \wedge H\mathbb{Q} \longrightarrow F_0(H_+) \wedge H\mathbb{Q},$$

which on homotopy groups is the map $\widetilde{H}_\bullet(G;\mathbb{Q}) \to \widetilde{H}_\bullet(H;\mathbb{Q})$. By hypothesis, this is an isomorphism so that $\mathcal{E}$ contains the weak compact generator. Thus, $\mathcal{E}$ is the whole homotopy category and the assertion is proven. $\square$

Having dispensed with all the necessary preliminaries, we now prove a rational version of Theorem 2.2.13 for reduced simplicial sets. The underlying idea of this result is a synthesis of the previously established facts that $\beta_X^+ \colon \mathrm{Sp}_X^\Sigma \to \mathbb{G}X_+-\mathrm{Mod}$ computes the homotopy fibre of an X-spectrum and that smashing with $H\mathbb{Q}$ implements rationalisation.

Theorem 2.3.15. *For X a reduced simplicial set, there is a Sp^Σ-Quillen equivalence*

$$\begin{tikzcd} \mathrm{Mod}-H\mathbb{Q}_X \arrow[r, shift left, "\beta_X^+"] \arrow[r, phantom, "\perp"] & (\mathbb{G}X_+, H\mathbb{Q})-\mathrm{Bimod}. \arrow[l, shift left, "\gamma_X^+"] \end{tikzcd}$$

For $f \colon X \to Y$ a morphism of reduced simplicial sets, there is a diagram of left Sp^Σ-Quillen functors

$$\begin{tikzcd} \mathrm{Mod}-H\mathbb{Q}_X \arrow[r, "f_!"] \arrow[d, "\beta_X^+"'] & \mathrm{Mod}-H\mathbb{Q}_Y \arrow[d, "\beta_Y^+"] \\ (\mathbb{G}X_+, H\mathbb{Q})-\mathrm{Bimod} \arrow[r, "\mathbb{G}f_!"] & (\mathbb{G}Y_+, H\mathbb{Q})-\mathrm{Bimod} \end{tikzcd}$$

commuting up to natural isomorphism.

Proof. We first observe that the adjunction of Theorem 2.2.13 prolongs to categories of right $H\mathbb{Q}$-modules since β_X^+ and γ_X^+ preserve Sp^Σ-tensors. Note that the second part of the theorem regarding commutativity of the base change diagram follows at once as soon as we know that the prolonged adjunction on right $H\mathbb{Q}$-modules is Quillen.

We turn to the problem of demonstrating that the adjunction

$$\begin{tikzcd} \mathrm{Mod}-H\mathbb{Q}_X \arrow[r, shift left, "\beta_X^+"] \arrow[r, phantom, "\perp"] & (\mathbb{G}X_+, H\mathbb{Q})-\mathrm{Bimod}. \arrow[l, shift left, "\gamma_X^+"] \end{tikzcd}$$

is a Quillen equivalence. This adjunction is Quillen since the right adjoint preserves fibrations and acyclic fibrations. We have previously observed that cofibrancy of $H\mathbb{Q}$ implies that the forgetful functor $\mathrm{Mod}-H\mathbb{Q}_X \to \mathrm{Sp}_X^\Sigma$ preserves cofibrations. For $P \in \mathrm{Mod}-H\mathbb{Q}_X$ cofibrant and $M \in (G_+, H\mathbb{Q})-\mathrm{Bimod}$ fibrant, by Theorem 2.2.13 and the above we have that a morphism $\psi \colon \beta_X^+(P) \to M$ is a weak equivalence of $(G_+, H\mathbb{Q})$-bimodule spectra precisely if its adjunct $\psi^\vee\colon P \to \gamma_X^+(M)$ is a weak equivalence in $\mathrm{Mod}{-}H\mathbb{Q}_X$ (similarly to the proof of Lemma 2.3.9). This shows that the prolonged adjunction is a Quillen equivalence and so completes the proof. □

Remark 2.3.16. The upshot of the Theorem (and of Lemma 2.3.12) is that, for a connected space X, the model category $\mathrm{Mod}{-}H\mathbb{Q}_X$ does indeed present the rational homotopy of parametrised spectra in the sense of Definition 2.3.6. Our proof of this fact followed a rather circuitous path through the homotopy theory of loop space module spectra. We expect that there is a more direct proof, however we prefer the one we have given here as it is the first step toward our algebraic characterisation of rational parametrised spectra.

Corollary 2.3.17. *For $f\colon X \to Y$ a rational homotopy equivalence of simply connected simplicial sets, the base change adjunction*

$$\begin{tikzcd} \mathrm{Mod}{-}H\mathbb{Q}_X \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] & \mathrm{Mod}{-}H\mathbb{Q}_Y \arrow[l, shift left, "f^*"] \end{tikzcd}$$

is a Quillen equivalence.

Proof. By Lemma 2.3.9 we may without loss of generality suppose that X and Y are reduced. Since $f\colon X \to Y$ is a rational homotopy equivalence of simply connected simplicial sets, the map on Kan simplicial loop groups $\mathbb{G}f\colon \mathbb{G}X \to \mathbb{G}Y$ is a rational homology equivalence (Corollary 2.1.2). In the commuting diagram of the Theorem, we have that the vertical and bottom horizontal functors are Quillen equivalences by Corollary 2.3.14. The assertion follows from the 2-out-of-3 property of Quillen equivalences. □

2.3.3 Strictifying Rational Homotopy Representations

Let X be a connected simplicial set. In the last section we proved an equivalence between the rational homotopy theory of X-spectra and the homotopy theory of certain bimodule spectra. The first goal of this section is to show that $H\mathbb{Q}$-module spectra can be strictified to spectrum objects in simplicial $\mathbb{Q}$-vector spaces, following the strictification arguments of [Shi07]. We then extend this to show that for any simplicial group G, there is a strictification functor taking $(G_+, H\mathbb{Q})$-bimodule spectra to spectrum objects in rational G-representations inducing an equivalence of homotopy theories. Throughout this section, all tensor products are over $\mathbb{Q}$.

Remark 2.3.18. We showed in Lemma 2.3.2 that the functor $\widetilde{\mathbb{Q}}[-]\colon \mathrm{sSet}_* \to \mathrm{sVect}_\mathbb{Q}$ is a strong symmetric monoidal left Quillen functor. We can use this to define a sSet_*-tensoring on $\mathrm{sVect}_\mathbb{Q}$ via the bifunctor

$$\widetilde{\mathbb{Q}}[-] \otimes (-)\colon \mathrm{sSet}_* \times \mathrm{sVect}_\mathbb{Q} \longrightarrow \mathrm{sVect}_\mathbb{Q},$$

which makes $\mathrm{sVect}_\mathbb{Q}$ a sSet_*-model category. A consequence of this is that the endofunctor $\Sigma_\mathbb{Q} = \widetilde{\mathbb{Q}}[S^1] \otimes (-)$ models suspension on the homotopy category $Ho(\mathrm{sVect}_\mathbb{Q})$.

Applying the symmetric stabilisation machine to the adjunction $(\widetilde{\mathbb{Q}}[-] \dashv U_0)$ of Remark 2.3.1, we obtain a Sp^Σ-Quillen adjunction

$$\begin{tikzcd} \mathrm{Sp}^\Sigma \arrow[r, shift left, "{\widetilde{\mathbb{Q}}[-]}"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}) \arrow[l, shift left, "U_0"] \end{tikzcd} := \mathrm{Sp}^\Sigma\big(\mathrm{sVect}_\mathbb{Q}; \Sigma_\mathbb{Q}\big). \tag{2.5}$$

The tensor product of simplicial $\mathbb{Q}$-vector spaces prolongs to a symmetric monoidal model structure on $\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})$ with respect to which the prolonged functor $\widetilde{\mathbb{Q}}[-]$ is strong symmetric monoidal. The functor $\Sigma_\mathbb{Q}$ prolongs to a Quillen equivalence on $\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})$, which is therefore a stable model category (Lemma A.3.26). As U_0 preserves filtered (homotopy) colimits, we find that $\widetilde{\mathbb{Q}}[\mathbb{S}]$ defines a weak compact generator of the homotopy category (since it satisfies the criterion of [SS03b, Lemma 2.2.1] by adjointness). Finally, observe that we have a family of adjunctions

$$\begin{tikzcd} \mathrm{sVect}_\mathbb{Q} \arrow[r, shift left, "\mathrm{F}_n"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}) \arrow[l, shift left, "\mathrm{Ev}_n"] \end{tikzcd}$$

for each $n \in \mathbb{N}$, in which the right adjoint sends a symmetric spectrum of simplicial $\mathbb{Q}$-vector spaces to its n-th term.

Lemma 2.3.19. *The stable model structure on $\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})$ coincides with the model structure transferred from Sp^Σ via $\widetilde{\mathbb{Q}}[-]$.*

Proof. This is the rational version of [Shi07, Proposition 4.1]. Define a morphism $f\colon M \to N$ in $\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})$ to be a *provisional fibration* or *provisional weak equivalence* if $U_0(f)$ is a fibration or weak equivalence in Sp^Σ—these are respectively the fibrations and weak equivalences of the transferred model structure if it exists. Indeed, the usual transfer yoga immediately shows that the transferred model structure satisfies all of the model category axioms, except for possibly the existence of a functorial factorisation of morphisms into acyclic cofibrations followed by fibrations. The provisionally fibrant objects are then the objects M such that $U_0(M)$ is an Ω-spectrum (automatically levelwise fibrant), and the provisional acyclic fibrations are precisely the levelwise acyclic fibrations. The acyclic fibrations for the stable model structure are also the levelwise acyclic fibrations, so that the transferred cofibrations coincide with the stable cofibrations.

In any model category $\mathcal{M}$ whatsoever, a map $f\colon A \to B$ is a weak equivalence precisely if the induced map of homotopy function complexes $\mathrm{map}_\mathcal{M}(f, C)$ is a weak equivalence for all fibrant objects C. In the case at hand, the provisionally fibrant and stably fibrant objects coincide, as do the cofibrations. Homotopy function complexes can be built using cosimplicial framings, which can be computed in the projective model structures; namely, using only projective cofibrations and levelwise equivalences (the latter of which are certainly provisional equivalences). It follows that the homotopy function complexes (with codomain already fibrant) can be represented by the same simplicial sets for both the stable and transferred model structures. In particular, this implies that the classes of weak equivalences coincide. Since the classes of cofibrations and weak equivalences are identical, so too are the fibrations—in particular this proves that the transferred model structure *is* a model structure. □

Corollary 2.3.20. *The forgetful functor $U_0\colon \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}) \to \mathrm{Sp}^\Sigma$ preserves and reflects stable fibrations and stable weak equivalences.*

Remark 2.3.21. Under the adjunction of (2.5), the sphere spectrum $\mathbb{S} = \mathrm{F}_0(S^0)$ is sent to $\widetilde{\mathbb{Q}}[\mathbb{S}] = \mathcal{H}\mathbb{Q}$. If V is a symmetric spectrum in simplicial $\mathbb{Q}$-vector spaces, its underlying symmetric spectrum is a (right) $\mathcal{H}\mathbb{Q}$-module is a canonical way: explicitly, the $\mathcal{H}\mathbb{Q}$-action is the composite

$$V \wedge \mathcal{H}\mathbb{Q} \xrightarrow{\eta_{V\wedge\mathcal{H}\mathbb{Q}}} \widetilde{\mathbb{Q}}[V \wedge \mathcal{H}\mathbb{Q}] \xrightarrow{\cong} \widetilde{\mathbb{Q}}[V] \otimes \widetilde{\mathbb{Q}}[\mathcal{H}\mathbb{Q}] \xrightarrow{\epsilon_V \otimes \epsilon_{\widetilde{\mathbb{Q}}[\mathbb{S}]}} V \otimes \widetilde{\mathbb{Q}}[\mathbb{S}] \xrightarrow{\cong} V,$$

where η and ϵ are respectively the unit and counit of the spectral $(\widetilde{\mathbb{Q}}[-] \dashv U_0)$-adjunction. Therefore, we have a factorisation

$$\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}) \longrightarrow \mathrm{Mod}\!-\!\mathcal{H}\mathbb{Q} \longrightarrow \mathrm{Sp}^\Sigma$$

in which each functor preserves and reflects weak equivalences and fibrations.

Lemma 2.3.22. *There is a strong symmetric monoidal* Sp^Σ*-Quillen equivalence*

$$\begin{tikzcd} \mathrm{Mod}\!-\!\mathcal{H}\mathbb{Q} \arrow[r, shift left, "\mathrm{str}"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}) \arrow[l, shift left, "U"] \end{tikzcd}$$

of stable symmetric monoidal model categories.

Proof. The right adjoint U is the functor appearing in the factorisation of Remark 2.3.21. The left adjoint str is described explicitly as follows: for M a (right) $\mathcal{H}\mathbb{Q}$-module spectrum, applying $\widetilde{\mathbb{Q}}[-]$ produces a (right) $\widetilde{\mathbb{Q}}[\mathcal{H}\mathbb{Q}]$-module $\widetilde{\mathbb{Q}}[M]$. The counit of the $(\widetilde{\mathbb{Q}}[-] \dashv U_0)$-adjunction produces a map of commutative monoids $\varepsilon\colon \widetilde{\mathbb{Q}}[\mathcal{H}\mathbb{Q}] \to \widetilde{\mathbb{Q}}[\mathbb{S}] = \mathcal{H}\mathbb{Q}$ in $\mathrm{sVect}_\mathbb{Q}$, so we set

$$\mathrm{str}(M) := \widetilde{\mathbb{Q}}[M] \bigotimes_{\widetilde{\mathbb{Q}}[\mathcal{H}\mathbb{Q}]} \widetilde{\mathbb{Q}}[\mathbb{S}].$$

Observe that str is the composite

$$\mathrm{Mod}\!-\!\mathcal{H}\mathbb{Q} \xrightarrow{\widetilde{\mathbb{Q}}[-]} \mathrm{Mod}\!-\!\widetilde{\mathbb{Q}}[\mathcal{H}\mathbb{Q}] \xrightarrow{\varepsilon_!} \mathrm{Mod}\!-\!\widetilde{\mathbb{Q}}[\mathbb{S}] \cong \widetilde{\mathbb{Q}}[\mathbb{S}]\!-\!\mathrm{Mod} = \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}),$$

which shows that $(\mathrm{str} \dashv U)$. This adjunction is Quillen since U preserves weak equivalences and fibrations, and str is strong symmetric monoidal since $\widetilde{\mathbb{Q}}[-]$ and $\varepsilon_!$ are. This latter fact also implies that str preserves Sp^Σ-tensors, since these can be expressed in terms of the monoidal structures on both sides of the adjunction.

To see that $(\mathrm{str} \dashv U)$ is a Quillen equivalence, observe that U preserves and reflects weak equivalences. It is therefore sufficient to check that the (underived) unit $M \to U\mathrm{str}(M)$ is a weak equivalence for all cofibrant $M \in \mathrm{Mod}\!-\!\mathcal{H}\mathbb{Q}$. For this, let $\mathcal{E} \hookrightarrow Ho(\mathrm{Mod}\!-\!\mathcal{H}\mathbb{Q})$ be the full subcategory on objects for which the derived unit is an isomorphism. A direct calculation shows that $\mathcal{H}\mathbb{Q}$ defines an object of $\mathcal{E}$. Since the derived functors $\mathbb{R}U$ and $\mathbb{L}\mathrm{str}$ are exact, $\mathcal{E}$ is a triangulated subcategory. At the level of the homotopy category, $\mathcal{H}\mathbb{Q}$ corepresents π_0^{st} on (right) $\mathcal{H}\mathbb{Q}$-modules so that $\mathcal{H}\mathbb{Q}$ defines a weak compact generator of $Ho(\mathrm{Mod}\!-\!\mathcal{H}\mathbb{Q})$ ([SS03b, Lemma 2.2.1]). For any set $\mathcal{I}$ and collection of objects $\{[M_i]\}_{i\in\mathcal{I}} \subset \mathcal{E}$ we have

$$\begin{aligned}
\left[\mathcal{H}\mathbb{Q}, \mathbb{R}U \circ \mathbb{L}\mathrm{str}\Big(\bigoplus_{i\in\mathcal{I}} M_i\Big)\right]_{Ho(\mathrm{Mod}-\mathcal{H}\mathbb{Q})} &\cong \left[\mathcal{H}\mathbb{Q}, \mathbb{L}\mathrm{str}\Big(\bigoplus_{i\in\mathcal{I}} M_i\Big)\right]_{Ho\left(\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})\right)} \\
&\cong \left[\mathcal{H}\mathbb{Q}, \bigoplus_{i\in\mathcal{I}} \mathbb{L}\mathrm{str}(M_i)\right]_{Ho\left(\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})\right)} \\
&\cong \bigoplus_{i\in\mathcal{I}} [\mathcal{H}\mathbb{Q}, \mathbb{L}\mathrm{str}(M_i)]_{Ho\left(\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})\right)} \\
&\cong \bigoplus_{i\in\mathcal{I}} [\mathcal{H}\mathbb{Q}, \mathbb{R}U \circ \mathbb{L}\mathrm{str}(M_i)]_{Ho(\mathrm{Mod}-\mathcal{H}\mathbb{Q})},
\end{aligned}$$

where in the first and fourth lines we have used $\mathrm{str}(\mathcal{H}\mathbb{Q}) = \mathcal{H}\mathbb{Q}$, and the third line uses that $\mathcal{H}\mathbb{Q}$ is a compact object in $Ho(\mathrm{Sp}^\Sigma(s\mathbb{Q}{-}\mathrm{Mod}))$. Since $\mathbb{R}U$ and $\mathbb{L}\mathrm{str}$ are exact and $[\mathcal{H}\mathbb{Q}, -]_{Ho(\mathrm{Mod}-\mathcal{H}\mathbb{Q})} \cong \pi_0^{\mathrm{st}}(U(-))$ it follows that the subcategory $\mathcal{E}$ is localising. Since $\mathcal{E}$ contains a weak compact generator we have $\mathcal{E} = Ho(\mathrm{Mod}{-}\mathcal{H}\mathbb{Q})$, which completes the proof of the Quillen equivalence. □

We now turn to the issue of strictifying $(G_+, \mathcal{H}\mathbb{Q})$-bimodule spectra for a simplicial group G. Our argument hinges on the fact that the strictification functor of Lemma 2.3.22 is strong symmetric monoidal, so that $\mathrm{F}_0(G_+)$-actions on right $\mathcal{H}\mathbb{Q}$-module spectra become actions of the rational group ring $\mathbb{Q}[G]$ on spectra of simplicial $\mathbb{Q}$-vector spaces.

Remark 2.3.23. Fix a simplicial group G. To fix notation, we write $\mathbb{Q}[G]{-}\mathrm{Mod}_{\geq 0}$ for the category of (left) $\mathbb{Q}[G]$-modules in $\mathrm{sVect}_\mathbb{Q}$, noting that this is equivalently the category of G-representations in simplicial $\mathbb{Q}$-vector spaces. The monad $\mathbb{Q}[G] \otimes (-)$ is a left Quillen functor, so its category of algebras $\mathbb{Q}[G]{-}\mathrm{Mod}_{\geq 0}$ becomes a combinatorial model category by a routine application of the Right Transfer Theorem; weak equivalences and fibrations are created by the forgetful functor to $\mathrm{sVect}_\mathbb{Q}$. The model category $\mathbb{Q}[G]{-}\mathrm{Mod}_{\geq 0}$ inherits a sSet_*-tensoring from that on $\mathrm{sVect}_\mathbb{Q}$ which makes it a sSet_*-model category. Explicitly, the tensoring is $(K, V) \mapsto \mathbb{Q}[K] \otimes V$, where $\mathbb{Q}[G]$ acts trivially on $\mathbb{Q}[K]$ (which is to say, via the counit $\mathbb{Q}[G] \to \mathbb{Q}$). The pushout-product axiom for the Sp^Σ-tensoring is easily verified on generating cofibrations and acyclic cofibrations and so holds in general by cofibrant generation.

Sending $G \mapsto \widetilde{\mathbb{Q}}[G_+] = \mathbb{Q}[G] \mapsto \mathrm{F}_0(\mathbb{Q}[G])$ defines a monoid in $\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})$, and we write $\mathbb{Q}[G]{-}\mathrm{Mod}$ for the category of (left) $\mathrm{F}_0(\mathbb{Q}[G])$-modules in $\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})$. Similarly to Remark 2.2.12, there is a canonical isomorphism of categories

$$\mathrm{Sp}^\Sigma(\mathbb{Q}[G]{-}\mathrm{Mod}_{\geq 0}; \Sigma_\mathbb{Q}) \cong \mathbb{Q}[G]{-}\mathrm{Mod},$$

so that the symmetric stabilisation machine equips $\mathbb{Q}[G]{-}\mathrm{Mod}$ with a combinatorial stable model structure.

Lemma 2.3.24. *The model structure transferred by the adjunction*

$$\begin{tikzcd} \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}) \arrow[r, shift left, "\mathrm{F}_0(\mathbb{Q}[G])\otimes(-)"] \arrow[r, phantom, "\perp"] & \mathbb{Q}[G]{-}\mathrm{Mod} \arrow[l, shift left] \end{tikzcd}$$

coincides with the stable model structure of Remark 2.3.23.

Proof. The proof of Lemma 2.3.19 applies. □

Theorem 2.3.25. *For any simplicial group G, there is a Sp^Σ-Quillen equivalence*

$$\begin{tikzcd} (G_+, \mathcal{H}\mathbb{Q}){-}\mathrm{Bimod} \arrow[r, shift left, "\mathrm{str}_G"] \arrow[r, phantom, "\perp"] & \mathbb{Q}[G]{-}\mathrm{Mod}. \arrow[l, shift left, "U_G"] \end{tikzcd}$$

Proof. This is the G-equivariant version of Lemma 2.3.22. The strictification functor $\mathrm{str}\colon \mathrm{Mod}{-}\mathcal{H}\mathbb{Q} \to \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})$ is strong monoidal and we have a natural isomorphism $\widetilde{\mathbb{Q}}[\mathrm{F}_0(-)] \cong \mathrm{F}_0 \circ \widetilde{\mathbb{Q}}[-]$ of functors $\mathrm{sSet}_* \to \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})$. Therefore, for any right $\mathcal{H}\mathbb{Q}$-module M we have

$$\mathrm{str}(\mathrm{F}_0(G_+) \wedge M) \cong \Big(\mathrm{F}_0(\mathbb{Q}[G]) \otimes_\mathbb{Q} \widetilde{\mathbb{Q}}[M]\Big) \bigotimes_{\widetilde{\mathbb{Q}}[\mathcal{H}\mathbb{Q}]} \widetilde{\mathbb{Q}}[\mathbb{S}] \cong \mathrm{F}_0(\mathbb{Q}[G]) \otimes \mathrm{str}(M),$$

from which it easily follows that str sends $(G_+, \mathcal{H}\mathbb{Q})$-bimodule spectra to $\mathrm{F}_0(\mathbb{Q}[G])$-modules in $\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})$. The sought-after left adjoint is thus $\mathrm{str}_G = \mathrm{str}$. Similarly, the right adjoint functor U_G is determined on underlying objects by the forgetful functor of Lemma 2.3.22. Indeed, for a $\mathbb{Q}[G]$-module V, the right $\mathcal{H}\mathbb{Q}$-module $U(V)$ inherits a left $\mathrm{F}_0(G_+)$-action via the composite map

$$\mathrm{F}_0(G_+) \wedge V \xrightarrow{\eta_{\mathrm{F}_0(G_+)\wedge V}} \widetilde{\mathbb{Q}}[\mathrm{F}_0(G_+) \wedge V] \cong \mathrm{F}_0(\mathbb{Q}[G]) \otimes \widetilde{\mathbb{Q}}[V] \xrightarrow{\mathrm{id}\otimes\epsilon_V} \mathrm{F}_0(\mathbb{Q}[G]) \otimes V \xrightarrow{\rho_V} V,$$

where η and ϵ are are respectively the unit and counit of the spectral $(\widetilde{\mathbb{Q}}[-] \dashv U_0)$-adjunction, and ρ_V is the left $\mathbb{Q}[G]$-action on V. Note that str_G preserves Sp^Σ-tensors since str does.

The adjunction $(\mathrm{str}_G \dashv U_G)$ is Quillen as U_G preserves and reflects weak equivalences and fibrations (which are created by the forgetful functors on both sides of the adjunction). Observe that since $\mathrm{F}_0(G_+)$ is cofibrant, by the pushout-product axiom the forgetful functor $(G_+, \mathcal{H}\mathbb{Q})-\mathrm{Bimod} \to \mathrm{Mod}-\mathcal{H}\mathbb{Q}$ preserves cofibrations. If M is a cofibrant $(G_+, \mathcal{H}\mathbb{Q})$-bimodule, the underlying $\mathcal{H}\mathbb{Q}$-module is thus cofibrant and so by the proof of Lemma 2.3.22 the (underived) unit

$$\eta_M \colon M \longrightarrow U_G\mathrm{str}_G(M) \cong U\mathrm{str}(M)$$

is a stable weak equivalence. For any $V \in \mathbb{Q}[G]-\mathrm{Mod}$ whatsoever, the underlying map of a morphism $\psi \colon M \to U_G(N)$ of $(G_+, \mathcal{H}\mathbb{Q})$-bimodule spectra and the underlying map of its adjunct $\psi^\vee \colon \mathrm{str}_G(M) \to V$ fit into a diagram

$$\begin{tikzcd}
M \arrow[r, "\psi"] \arrow[d, "\eta_M"'] & U_G(V) \\
U_G(\mathrm{str}_G(M)) \arrow[ur, "U_G(\psi^\vee)"'] &
\end{tikzcd}$$

so that ψ is a weak equivalence precisely if $U_G(\psi^\vee)$ is. But U_G preserves and reflects weak equivalences (Remark 2.3.21) so that $(\mathrm{str}_G \dashv U_G)$ is a Quillen equivalence. □

Since the strictification functor of Lemma 2.3.22 preserves Sp^Σ-modules, it is easy to see that the strictification functors of the Theorem behave well with respect to varying the simplicial group G:

Lemma 2.3.26. *For any morphism of simplicial groups $\phi \colon G \to H$ there is a diagram of left Sp^Σ-Quillen functors*

$$\begin{tikzcd}
(G_+, \mathcal{H}\mathbb{Q})-\mathrm{Bimod} \arrow[r, "\phi_!"] \arrow[d, "\mathrm{str}_G"'] & (H_+, \mathcal{H}\mathbb{Q})-\mathrm{Bimod} \arrow[d, "\mathrm{str}_H"] \\
\mathbb{Q}[G]-\mathrm{Mod} \arrow[r, "\mathbb{Q}[\phi]_!"] & \mathbb{Q}[H]-\mathrm{Mod}
\end{tikzcd}$$

commuting up to natural isomorphism.

Proof. The bottom horizontal Quillen adjunction is simply the base change adjunction corresponding to the map $\mathbb{Q}[\phi] \colon \mathbb{Q}[G] \to \mathbb{Q}[H]$ of monoids in $\mathrm{sVect}_\mathbb{Q}$ (and its stabilisation). Accordingly, the right adjoint $\mathbb{Q}[\phi]^* \colon \mathbb{Q}[H]-\mathrm{Mod} \to \mathbb{Q}[G]-\mathrm{Mod}$ is the identity on underlying objects, and so preserves and reflects weak equivalences and fibrations. The desired commuting diagram is easily checked on the right adjoints, where we have $U_G \mathrm{Q}[\phi]^* \cong \phi^* U_H$. Essential uniqueness of adjoints completes the proof. □

2.3.4 Monoidal Structures

In this section we show that Theorem 2.3.15 gives rise to symmetric monoidal structures on the rational homotopy categories of parametrised spectra, further to our work in §2.2.3. We also show that the strictification equivalences of Theorem 2.3.25 are suitably monoidal, which we use later to obtain a simple algebraic characterisation of the smash product of rational parametrised spectra.

Remark 2.3.27. For R a commutative symmetric ring spectrum, the category of (right) R-modules $\mathrm{Mod}{-}R$ is a symmetric monoidal model category with respect to the relative smash product defined as

$$M \wedge_R N := \operatorname{colim}\left(M \wedge R \wedge N \underset{M \wedge \rho_N}{\overset{\rho_M \wedge N}{\rightrightarrows}} M \wedge N \right)$$

([SS00, §4]). If $\phi\colon R \to S$ is a stable weak equivalence of commutative symmetric ring spectra, then the corresponding adjunction $(\phi_! \dashv \phi^*)\colon \mathrm{Mod}{-}R \to \mathrm{Mod}{-}S$ is a strong symmetric monoidal Quillen equivalence; the standard argument for extension of scalars along a map of commutative monoids implies $\phi_!(M \wedge_R N) \cong \phi_! M \wedge_S \phi_! N$ for all $M, N \in \mathrm{Mod}{-}R$.

Lemma 2.3.28. *For any simplicial group G and cofibrant commutative symmetric ring spectrum R, the category of (G_+, R)-bimodule spectra is a symmetric monoidal model category with respect to*

$$(M, N) \longmapsto M \wedge_R N,$$

equipped with the diagonal $\mathrm{F}_0(G_+)$-action.

Proof. We note that the purported monoidal product does indeed define a symmetric monoidal structure, using cocommutativity and coassociativity of the diagonal map $G \to G \times G$. The closed monoidal structure is inferred from the Adjoint Functor Theorem since $M \wedge_R (-)$ preserves colimits for all M (alternatively, the internal homs can be constructed directly using the group inversion map $i\colon G \to G$). The model structure exists by the proof of Lemma 2.3.12, and has generating cofibrations and acyclic cofibrations

$$\mathrm{F}_0(G_+) \wedge \mathcal{I}_\Sigma \wedge R \quad \text{and} \quad \mathrm{F}_0(G_+) \wedge \mathcal{J}_\Sigma \wedge R$$

respectively. In particular, the generating cofibrations are all of the form

$$i_n(G; R)\colon \mathrm{F}_0\big((G \times \partial\Delta[n])_+\big) \wedge R \longrightarrow \mathrm{F}_0(G_+) \wedge \cong \mathrm{F}_0\big((G \times \Delta[n])_+\big) \wedge R$$

for some $n \in \mathbb{N}$. Fixing m and n, via the usual description of the non-degenerate simplices of $\Delta[m] \times \Delta[n]$ in terms of (m, n)-shuffles, we write

$$(\partial\Delta[n] \times \Delta[m]) \coprod_{(\partial\Delta[n] \times \partial\Delta[m])} (\Delta[n] \times \partial\Delta[m]) \longrightarrow \Delta[m] \times \Delta[n]$$

as a finite sequence of attachments of simplices $\Delta[k]$ along their boundaries $\partial\Delta[k]$. Using Lemma B.2.1, we find that the pushout-product $i_m(G; R) \square i_n(G; R)$ is a finite sequence of pushouts along cofibrations of the form

$$\mathrm{F}_0\big((G \times \Delta_G \times \partial\Delta[k])_+\big) \wedge R \longrightarrow \mathrm{F}_0\big((G \times \Delta_G \times \Delta[k])_+\big) \wedge R,$$

and hence is a cofibration. By cofibrant generation, it follows that the pushout-product of any two cofibrations is a cofibration. The remainder of the pushout-product axiom is proven using that the forgetful functor $(G_+, R)-\mathrm{Bimod} \to \mathrm{Mod}-R$ preserves cofibrations (by cofibrancy of $\mathrm{F}_0(G_+)$ and the pushout-product axiom of the smash product in Sp^Σ). If $i\colon K \to L$ is now a cofibration between cofibrant bimodule spectra and $j\colon M \to N$ is an acyclic cofibration, by the pushout-product axiom for $\wedge_R$ on $\mathrm{Mod}-R$ and the above observation, the functors $K \wedge_R (-)$ and $L \wedge_R (-)$ are left Quillen on categories of bimodule spectra. Computing the pushout-product, we have a diagram of bimodule spectra

$$\begin{tikzcd}
K \wedge_R M \arrow[r] \arrow[d, tail, "\wr"'] & L \wedge_R M \arrow[d, tail, "\wr"'] \arrow[dr, tail, "\sim"] & \\
K \wedge_R N \arrow[r] & P \arrow[r, "i \square j"] & L \wedge_R N
\end{tikzcd}$$

with acyclic cofibrations as marked, where P denotes the pushout. By the 2-out-of-3 property it follows that the pushout-product $i \square j$ is a stable weak equivalence of bimodule spectra. In particular, this shows that the set of pushout-products $(\mathrm{F}_0(G_+) \wedge \mathcal{I}_\Sigma \wedge R) \square (\mathrm{F}_0(G_+) \wedge \mathcal{J}_\Sigma \wedge R)$ consists of acyclic cofibrations, so by cofibrant generation and symmetry the full pushout-product axiom follows.

To complete the proof we verify the unit axiom. The G-space $WG \to *$ is a cofibrant resolution of $* \in \mathrm{G-sSet}$ which gives rise to a cofibrant resolution

$$\mathrm{F}_0(WG_+) \wedge R \longrightarrow R$$

of the monoidal unit in $(G_+, R)-\mathrm{Bimod}$. Since the underlying spectrum of R is cofibrant, the forgetful functor $\mathrm{Mod}-R \to \mathrm{Sp}^\Sigma$ preserves cofibrations. Hence for any cofibrant bimodule spectrum M, the comparison map

$$(\mathrm{F}_0(WG_+) \wedge R) \wedge_R M \cong \mathrm{F}_0(WG_+) \wedge M \longrightarrow M$$

is a stable weak equivalence, since $\mathrm{F}_0(WG_+) \to F_0(S^0)$ is such and smashing with a cofibrant symmetric spectrum preserves stable weak equivalences [HSS00, Corollary 5.3.10]. □

Remark 2.3.29. If the commutative symmetric ring spectrum R is not cofibrant our proof of the unit axiom fails. We have not found a proof of the unit axiom that works more generally, though this is the only point where our argument becomes unstuck.

For our chosen cofibrant model $\mathrm{H}\mathbb{Q}$ of the Eilenberg–Mac Lane spectrum of $\mathbb{Q}$, we conclude that $(G_+, \mathrm{H}\mathbb{Q})-\mathrm{Bimod}$ is a symmetric monoidal model category. This presents a symmetric monoidal model structure on the homotopy category. In the case that $G = \mathbb{G}X$ for some reduced simplicial set X, Theorem 2.3.15 allows us to transfer this symmetric monoidal structure to the rational homotopy category of parametrised X-spectra.

For a simplicial group G, the category $\mathbb{Q}[G]-\mathrm{Mod}$ of strict rational homotopy representations of G is also a symmetric monoidal model category, as we now show. The monoidal structure is obtained via the monoidal structure on $\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})$, which is equipped with a left $\mathbb{Q}[G]$-action via the coproduct $\mathbb{Q}[G] \to \mathbb{Q}[G] \otimes \mathbb{Q}[G]$. In what follows, all tensor products are over $\mathbb{Q}$ and our convention is such that all Hopf algebras are cocommutative. We need the following

Lemma 2.3.30. *Let H be a Hopf algebra over $\mathbb{Q}$. The tensor product of free H-modules is free.*

Proof. We first observe that $H \otimes H$ is a left Hopf module over H, with left H-action given by
$$h \cdot (a \otimes b) := \sum h_{(0)} a \otimes h_{(1)} b$$
and left H-coaction given by
$$\kappa \colon (a \otimes b) \longmapsto \sum a_{(0)} \otimes a_{(1)} \otimes b \in H \otimes (H \otimes H),$$
in the Sweedler notation. The H-action and H-coaction are compatible since
$$\kappa\big(h \cdot (a \otimes b)\big) = \sum (h_{(0)} a)_{(0)} \otimes (h_{(0)} a)_{(0)} \otimes h_{(1)} b = \\ \sum h_{(0)} a_{(0)} \otimes h_{(1)} a_{(1)} \otimes h_{(2)} b = h \cdot \kappa(a \otimes b)$$
by coassociativity of the coproduct on H. If
$$(H \otimes H)_H := \{v \in H \otimes H \mid \kappa(v) = 1 \otimes v\}$$
is the H-covariant submodule of $H \otimes H$, the Fundamental Theorem of Hopf Modules [LS69, Proposition 1] now asserts that the morphism
$$\begin{aligned} H \otimes (H \otimes H)_H &\longrightarrow H \otimes H \\ h \otimes v &\longmapsto h \cdot v \end{aligned}$$
is an isomorphism of left H-modules. For free H-modules $H \otimes V$ and $H \otimes W$, we have that $(H \otimes V) \otimes (H \otimes W) \cong H \otimes \big((H \otimes H)_H \otimes V \otimes W\big)$ is free. □

Remark 2.3.31. A similar argument shows that
$$H^{\otimes(n)} \cong H \otimes \big(H^{\otimes n}\big)_H$$
for all $n \geq 1$, where $H^{\otimes(n)}$ is a left Hopf module over H with respect to the usual H-action and H-coaction $\Delta \otimes H^{\otimes(n-1)} \colon H^{\otimes n} \to H \otimes H^{\otimes n}$.

Corollary 2.3.32. *Let H be a simplicial Hopf algebra over $\mathbb{Q}$. Then the tensor product of free H-modules is free.*

Proof. Apply the Lemma in each simplicial degree. □

Lemma 2.3.33. *For H a simplicial Hopf algebra over $\mathbb{Q}$, the category $H{-}\mathrm{Mod}_{\geq 0}$ of left H-modules in $\mathrm{sVect}_{\mathbb{Q}}$ is a combinatorial symmetric monodial monoidal sSet_*-model category.*

Proof. The underlying object of H is cofibrant in $\mathrm{sVect}_{\mathbb{Q}}$ (all simplicial $\mathbb{Q}$-vector spaces are free), so the combinatorial sSet_*-model structure is obtained exactly as in Remark 2.3.23. Generating sets of cofibrations and acyclic cofibrations are supplied by
$$\mathcal{I}^H_{\geq 0} := \{H \otimes \mathbb{Q}[i_n] \mid i_n \in \mathcal{I}_{\mathrm{Kan}}\} \quad \text{and} \quad \mathcal{J}^H_{\geq 0} := \{H \otimes \mathbb{Q}[h^n_k] \mid h^n_k \in \mathcal{J}_{\mathrm{Kan}}\}$$
respectively.

We define the monoidal structure on $H-\mathrm{Mod}_{\geq 0}$ by sending left H-modules V and W to $V \otimes W$ equipped with the left H-action

$$H \otimes (V \otimes W) \xrightarrow{\Delta\otimes\mathrm{id}} H \otimes H \otimes V \otimes W \cong H \otimes V \otimes H \otimes W \xrightarrow{\rho_V\otimes\rho_W} V \otimes W,$$

with ρ_V, ρ_W the H-actions on V and W respectively. Coassociativity and cocommutativity of Δ imply that this is indeed a symmetric monoidal structure. Internal homs are inferred by the Adjoint Functor Theorem, using that $V \otimes (-)$ preserves colimits for all V (alternatively, the internal homs can be constructed directly using the antipode $S\colon H \to H$).

The pushout-product axiom for the monoidal structure is proven directly on the sets $\mathcal{I}^H_{\geq 0}$ and $\mathcal{J}^H_{\geq 0}$ as in Lemma 2.3.28, using Corollary 2.3.32 as essential input. Cofibrant generation then implies that the pushout-product axiom holds in general.

To complete the proof, let $\mathbb{Q}^H \to \mathbb{Q}$ be a cofibrant resolution of the monoidal unit in $H-\mathrm{Mod}_{\geq 0}$. For any H-module V, the comparison map $\mathbb{Q}^H \otimes V \to V$ is a weak equivalence, since $(-) \otimes V$ preserves all weak equivalences of simplicial $\mathbb{Q}$-vector spaces. This verifies the unit axiom and thereby completes the proof. □

Remark 2.3.34. It will prove useful to have an explicit cofibrant replacement of the monoidal unit $\mathbb{Q}$ as a H-module. One way of constructing such a resolution is via the bar construction, which we briefly summarise (a standard textbook reference for this material is [Wei94, §8.6]). The free-forgetful adjunction

$$\begin{tikzcd} \mathrm{sVect}_{\mathbb{Q}} \arrow[r, shift left, "H\otimes(-)"] \arrow[r, phantom, "\perp"] & H-\mathrm{Mod}_{\geq 0} \arrow[l, shift left, "U"] \end{tikzcd}$$

induces a comonad $\perp := H \otimes U(-)$ on $H-\mathrm{Mod}_{\geq 0}$. By iterated application of this comonad to $\mathbb{Q}$ we obtain an augmented simplicial object

$$\underline{\mathbb{Q}}^H_* := \Big(\cdots \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} H \otimes H \begin{smallmatrix}\longrightarrow\\ \longrightarrow\end{smallmatrix} H \Big) \xrightarrow{\epsilon} \mathbb{Q},$$

in $H-\mathrm{Mod}_{\geq 0}$, whose n-th term is $H^{\otimes(n+1)}$ with face maps defined using the counit of $\perp$ and degeneracies using the comultiplication. On underlying simplicial $\mathbb{Q}$-vector spaces, $\underline{\mathbb{Q}}^H_* \to \mathbb{Q}$ is (right) contractible via the the unit map $f_{-1} = \eta\colon \mathbb{Q} \to H$ and

$$\begin{aligned} f_n\colon H^{\otimes(n+1)} &\longrightarrow H^{\otimes(n+2)} \\ a^0 \otimes \cdots \otimes a^n &\longmapsto a^0 \otimes \cdots \otimes a^n \otimes 1. \end{aligned}$$

This contraction witnesses the fact that $\mathbb{Q}^H := \mathrm{diag}(\underline{\mathbb{Q}}^H_*) \to \mathbb{Q}$ is a weak equivalence of simplicial $\mathbb{Q}$-vector spaces (as is proven, for instance, using the Eilenberg–Zilber Theorem). Observe that $\mathbb{Q}^H$ can be expressed as the colimit of the countable sequence

$$X_{-1} = 0 \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \cdots,$$

where X_n is obtained from X_{n-1} as a pushout of the form

$$\begin{tikzcd} \mathbb{Q}[\partial\Delta[n]] \otimes H^{\otimes(n+1)} \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ \mathbb{Q}[\Delta[n]] \otimes H^{\otimes(n+1)} \arrow[r] & X_n. \end{tikzcd}$$

By Remark 2.3.31 (or rather, its simplicial version) and the pushout-product axiom for the sSet_*-tensoring, the right vertical morphisms are all cofibrations in $H\text{−Mod}_{\geq 0}$, so that $\mathbb{Q}^H \to \mathbb{Q}$ is indeed a cofibrant resolution.

Applying the symmetric stabilisation machine, the general results of §A.3.2 (and the argument of Remark 2.3.23) imply

Lemma 2.3.35. *For H a simplicial Hopf algebra over $\mathbb{Q}$, the category*

$$H\text{−Mod} \cong \mathrm{Sp}^{\Sigma}(H\text{−Mod}_{\geq 0}; \Sigma_{\mathbb{Q}})$$

of $\mathrm{F}_0(H)$-modules in $\mathrm{Sp}^{\Sigma}(\mathrm{sVect}_{\mathbb{Q}})$ is a symmetric monoidal Sp^{Σ}-model category.

For a map of simplicial Hopf algebras $\phi\colon H \to K$ the extension of scalars functor on spectral modules is generally only oplax monoidal. In the case that ϕ is a weak equivalence, however, this is still sufficient to induce a strong symmetric monoidal equivalence of derived categories. To prove this, we once more make use of the weak monoidal Quillen adjunctions.

Theorem 2.3.36. *Let $\phi\colon H \to K$ be a weak equivalence of simplicial Hopf algebras over $\mathbb{Q}$. Then restriction and extension of scalars*

$$\begin{tikzcd} H\text{−Mod} \arrow[r, shift left, "\phi_!"] \arrow[r, phantom, "\perp"] & K\text{−Mod} \arrow[l, shift left, "\phi^*"] \end{tikzcd}$$

defines a weak monoidal Sp^{Σ}-Quillen equivalence.

Proof. The right adjoint ϕ^* is the identity on underlying objects and so preserves and reflects weak equivalences and fibrations. The left adjoint may be described explicitly as sending $M \in H\text{−Mod}$ to the $\mathrm{F}_0(K)$-module with underlying symmetric spectrum

$$\phi_!(M) := \operatorname{colim}\left(\mathrm{F}_0(K) \otimes \mathrm{F}_0(H) \otimes M \underset{\rho_K \otimes \mathrm{id}}{\overset{\mathrm{id}\otimes\rho_M}{\rightrightarrows}} \mathrm{F}_0(K) \otimes M \right),$$

where ρ_M is the $\mathrm{F}_0(H)$-action on M and ρ_K is the right $\mathrm{F}_0(H)$-action on $\mathrm{F}_0(K)$. To establish that $(\phi_! \dashv \phi^*)$ is a Quillen equivalence, it is therefore sufficient to show that the derived unit is a natural isomorphism. For this, observe that the model category $H\text{−Mod}$ is stable and the triangulated homotopy category has weak compact generator presented by $\mathrm{F}_0(H)$. The underlying objects of H and K are cofibrant in $\mathrm{sVect}_{\mathbb{Q}}$, so that the component of the derived unit $\mathrm{F}_0(H) \to \mathrm{F}_0(K)$ is a stable weak equivalence by Ken Brown's Lemma. As in the proof of Lemma 2.3.22, it follows that the derived unit is a natural isomorphism which implies the desired Quillen equivalence. Extension of scalars preserves Sp^{Σ}-tensors by a direct calculation, using that the monoidal structure on $\mathrm{Sp}^{\Sigma}(\mathrm{sVect}_{\mathbb{Q}})$ preserves colimits in each argument.

The restriction of scalars functor ϕ^* is strong symmetric monoidal so that $\phi_!$ is naturally oplax monoidal. To complete the proof, we must show

(i) for all cofibrant $M, N \in H\text{−Mod}$, the oplax structure map

$$\Lambda_{M,N}\colon \phi_!(M \otimes N) \longrightarrow \phi_!(M) \otimes \phi_!(N)$$

is a weak equivalence; and

(ii) for some (and hence any) cofibrant replacement $\mathbb{Q}^H_{\mathrm{cof}} \to \mathrm{F}_0(\mathbb{Q})$ of the monoidal unit in $H-\mathrm{Mod}$, the composite $\phi_!(\mathbb{Q}^H_{\mathrm{cof}}) \to \phi_!(\mathrm{F}_0(\mathbb{Q})) \to \mathrm{F}_0(\mathbb{Q})$ is a stable weak equivalence in $K-\mathrm{Mod}$.

We prove (i) in a series of steps. Firstly, we consider the case that

$$M = \mathrm{F}_0(H) \otimes \mathrm{F}_m(\mathbb{Q}[K]) \cong \mathrm{F}_m\big(H \otimes \mathbb{Q}[K]\big) \ \text{ and } \ N = \mathrm{F}_0(H) \otimes \mathrm{F}_n(\mathbb{Q}[L]) \cong \mathrm{F}_n\big(H \otimes \mathbb{Q}[L]\big)$$

for $m, n \in \mathbb{N}$ and simplicial sets K, L. Then we have

$$M \otimes N \cong \mathrm{F}_{n+m}\big(H \otimes H \otimes \mathbb{Q}[K \times L]\big) \cong \mathrm{F}_{n+m}\big(H \otimes (H \otimes H)_H \otimes \mathbb{Q}[K \times L]\big)$$

with $(H \otimes H)_H$ the H-covariant submodule (as in Lemma 2.3.30). Since $\phi\colon H \to K$ is a weak equivalence and $\otimes$ preserves weak equivalences in both arguments, we have a diagram of weak equivalences

$$\begin{tikzcd} H \otimes H \cong H \otimes (H \otimes H)_H \arrow[r, "\phi\otimes\phi"] \arrow[dr, "\phi\otimes\mathrm{id}"'] & K \otimes K \cong K \otimes (K \otimes K)_K \\ & K \otimes (H \otimes H)_H \arrow[u] \end{tikzcd} \tag{2.6}$$

by the 2-out-of-3 property. Putting this all together, we find that the oplax structure map $\Lambda_{M,N}$ is isomorphic to

$$\mathrm{F}_{n+m}\big(K \otimes (H \otimes H)_H \otimes \mathbb{Q}[K \times L]\big) \longrightarrow \mathrm{F}_{n+m}\big(K \otimes K \otimes \mathbb{Q}[K \times L]\big)$$

and so is a weak equivalence by Ken Brown's Lemma (all simplicial $\mathbb{Q}$-vector spaces are free, hence cofibrant). Let us now fix $M = \mathrm{F}_n(H \otimes \mathbb{Q}[K])$ for some $n \in \mathbb{N}$ and $K \in \mathrm{sSet}$. Let ${}_M\mathcal{C}$ be the class of objects B of $H-\mathrm{Mod}$ for which the oplax structure map $\Lambda_{M,B}$ is a weak equivalence. The class ${}_M\mathcal{C}$ is closed under retracts, and by the above argument also contains all domains and codomains of the set of generating cofibrations $\cup_{n\in\mathbb{N}}\mathrm{F}_n(\mathcal{I}^H_{\geq 0})$ of $H-\mathrm{Mod}$. Let $i\colon A \to B$ be a generating cofibration and $C \in {}_M\mathcal{C}$ cofibrant, and consider the pushout diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"'] & C \arrow[d] \\ B \arrow[r] & P. \end{tikzcd}$$

The oplax structure map Λ gives rise to a map of pushout diagrams

$$\left(\begin{tikzcd} \phi_!(M \otimes A) \arrow[r] \arrow[d, tail] & \phi_!(M \otimes C) \arrow[d] \\ \phi_!(M \otimes B) \arrow[r] & \phi_!(M \otimes P) \end{tikzcd}\right) \xrightarrow{\Lambda_{M,-}} \left(\begin{tikzcd} \phi_!(M) \otimes \phi_!(A) \arrow[r] \arrow[d, tail] & \phi_!(M) \otimes \phi_!(C) \arrow[d] \\ \phi_!(M) \otimes \phi_!(B) \arrow[r] & \phi_!(M) \otimes \phi_!(P) \end{tikzcd}\right)$$

in which the components at the top left, bottom left and top right are weak equivalences since $A, B, C \in {}_M\mathcal{M}$. Note that these diagrams have cofibrations as marked due to the pushout-product axiom and the fact that $\phi_!$ is left Quillen. Moreover, all objects in both diagrams are cofibrant so that $\Lambda_{M,P}$ is a weak equivalence by the Cube Lemma ([Hov99, Lemma 5.2.6]). Let us now suppose that $B \in {}_M\mathcal{C}$ is a cofibrant, and

$B \to C$ is a transfinite composition

$$B = B(0) \longrightarrow B(1) \longrightarrow \cdots \longrightarrow B(\kappa) = C$$

of pushouts of generating cofibrations. We argue that $C \in {}_M\mathcal{C}$ by transfinite induction. Suppose that $B(\alpha)$ is in ${}_M\mathcal{C}$ for all $\alpha < \alpha_0 \leq \kappa$. Then

- if $\alpha_0 = \alpha + 1$ is a successor ordinal then $B(\alpha_0) \in {}_M\mathcal{C}$ by the above; and
- if α_0 is a limit ordinal, then for $\alpha < \alpha_0$ we have a morphism of sequences of cofibrations

$$\begin{tikzcd}
\cdots \arrow[r, tail] & \phi_!(M \otimes B(\alpha)) \arrow[r, tail] \arrow[d, "\Lambda_{M,B(\alpha)}"'] & \phi_!(M \otimes B(\alpha+1)) \arrow[r, tail] \arrow[d, "\Lambda_{M,B(\alpha+1)}"] & \cdots \\
\cdots \arrow[r, tail] & \phi_!(M) \otimes \phi_!(B(\alpha)) \arrow[r, tail] & \phi_!(M) \otimes \phi_!(B(\alpha+1)) \arrow[r, tail] & \cdots
\end{tikzcd}$$

  in which each of the vertical morphisms is a weak equivalence by the inductive hypothesis. The domains and codomains of the generating cofibrations are all compact objects, so that taking colimits implies that $\Lambda_{M,B(\alpha_0)}$ is a weak equivalence.

We conclude that ${}_M\mathcal{C}$ contains all cofibrant objects.

For a cofibrant object $A \in H\mathrm{-Mod}$, let $\mathcal{C}_A$ be the class of objects B for which $\Lambda_{B,A} \colon \phi_!(B \otimes A) \to \phi_!(B) \otimes \phi_!(A)$ is a weak equivalence. We have seen that $\mathcal{C}_A$ contains the domains and codomains of the set of generating cofibrations $\cup_{n\in\mathbb{N}} \mathrm{F}_n(\mathcal{I}^H_{\geq 0})$. Repeating the above argument shows that $\mathcal{C}_A$ contains all cofibrant objects. This completes the proof of (i).

To prove (ii), we use the explicit cofibrant resolutions of Remark 2.3.34. For this, we note that $\mathbb{Q}^H \to \mathbb{Q}$ is sent to a cofibrant resolution $\mathrm{F}_0(\mathbb{Q}^H) \to \mathrm{F}_0(\mathbb{Q})$ under the free functor $\mathrm{F}_0 \colon H\mathrm{-Mod}_{\geq 0} \to H\mathrm{-Mod}$ and similarly for K. Recall the presentation of $\mathbb{Q}^H$ given in Remark 2.3.34 as the colimit of a countable sequence

$$X^H_{-1} \longrightarrow X^H_0 \longrightarrow X^H_1 \longrightarrow \cdots$$

of cell attachments. Applying $\phi_! \colon H\mathrm{-Mod}_{\geq 0} \to K\mathrm{-Mod}_{\geq 0}$ to the defining pushout diagram for X_n gives a pushout diagram

$$\begin{tikzcd}
\mathbb{Q}[\partial\Delta[n]] \otimes \phi_!(H^{\otimes(n+1)}) \arrow[r] \arrow[d, hook] & \phi_!(X^H_{n-1}) \arrow[d] \\
\mathbb{Q}[\Delta[n]] \otimes \phi_!(H^{\otimes(n+1)}) \arrow[r] & \phi_!(X^H_n)
\end{tikzcd} \tag{2.7}$$

in which all objects are cofibrant and the left vertical arrow is a cofibration as marked. Since $H^{\otimes(n+1)} \cong H \otimes (H^{\otimes(n+1)})_H$ by Remark 2.3.31, $\phi_!(H^{\otimes(n+1)}) \cong K \otimes (H^{\otimes(n+1)})_H$ and thus there is a sequence of weak equivalences

$$H^{\otimes(n+1)} \longrightarrow \phi_!(H^{\otimes(n+1)}) \longrightarrow K^{\otimes(n+1)},$$

similarly to (2.6). Using the Cube Lemma applied to each of the diagrams (2.7), a straightforward inductive argument produces a compatible family of weak equivalences $\phi_!(X^H_n) \to X^K_n$ for each $n \in \mathbb{N}$. Taking the colimit over n results in a weak equivalence $\phi_!(\mathbb{Q}^H) \to \mathbb{Q}^K$, and the commuting diagram

$$\begin{tikzcd} \phi_!(\mathbb{Q}^H) \arrow[r, "\sim"] \arrow[d] & \mathbb{Q}^K \arrow[d, "\wr"] \\ \phi_!(\mathbb{Q}) \arrow[r] & \mathbb{Q} \end{tikzcd}$$

shows that the composite $\phi_!(\mathbb{Q}^H) \to \phi_!(\mathbb{Q}) \to \mathbb{Q}$ is a weak equivalence on underlying simplicial $\mathbb{Q}$-vector spaces. Applying $\mathrm{F}_0\colon K\mathrm{-Mod}_{\geq 0} \to K\mathrm{-Mod}$ we obtain

$$\mathrm{F}_0(\phi_!(\mathbb{Q}^H)) \cong \phi_!(\mathrm{F}_0(\mathbb{Q}^H)) \longrightarrow \mathrm{F}_0(\mathbb{Q}). \tag{2.8}$$

Since the free K-module spectrum functor F_0 is simply $\mathrm{F}_0\colon \mathrm{sVect}_{\mathbb{Q}} \to \mathrm{Sp}^{\Sigma}(\mathrm{sVect}_{\mathbb{Q}})$ on underlying objects and $\mathbb{Q}^H \to \mathbb{Q}$ is a weak equivalence of (necessarily cofibrant) objects in $\mathrm{sVect}_{\mathbb{Q}}$, the map (2.8) is a stable weak equivalence by Ken Brown's Lemma. This proves (ii) and completes the proof. □

Fixing a simplicial group G, we saw in Corollary 2.3.13 that the stable weak equivalence $\mathrm{H}\mathbb{Q} \to \mathcal{H}\mathbb{Q}$ induces a Sp^{Σ}-Quillen equivalence

$$\begin{tikzcd} (G_+, \mathrm{H}\mathbb{Q})\mathrm{-Bimod} \arrow[r, shift left, "(-)\wedge_{\mathrm{H}\mathbb{Q}}\mathcal{H}\mathbb{Q}"] \arrow[r, phantom, "\perp"] & (G_+, \mathcal{H}\mathbb{Q})\mathrm{-Bimod}. \arrow[l, shift left] \end{tikzcd}$$

By Remark 2.3.27 the left adjoint is strong symmetric monoidal (though we have not proven that $(G_+, \mathcal{H}\mathbb{Q})\mathrm{-Bimod}$ is a monoidal *model* category). By Lemma 2.3.22, the strictification functor $\mathrm{str}\colon \mathrm{Mod}\mathrm{-}\mathcal{H}\mathbb{Q} \to \mathrm{Sp}^{\Sigma}(\mathrm{sVect}_{\mathbb{Q}})$ is strong symmetric monoidal, so that the composite

$$(G_+, \mathrm{H}\mathbb{Q})\mathrm{-Bimod} \longrightarrow (G_+, \mathcal{H}\mathbb{Q})\mathrm{-Bimod} \longrightarrow \mathbb{Q}[G]\mathrm{-Mod}$$

is also strong symmetric monoidal. Combining Corollary 2.3.13, Theorem 2.3.25 and Lemmas 2.3.28 and 2.3.35 proves most of the

Theorem 2.3.37. *For any simplicial group G, there is a strong symmetric monoidal* Sp^{Σ}*-Quillen equivalence*

$$\begin{tikzcd} (G_+, \mathrm{H}\mathbb{Q})\mathrm{-Bimod} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbb{Q}[G]\mathrm{-Mod}. \arrow[l, shift left] \end{tikzcd}$$

Proof. The only thing left to check is that the left Quillen functor satisfies the unit axiom ([Hov99, Definition 4.2.6 (2)]). We saw in the proof of Lemma 2.3.28 that $\mathrm{F}_0(WG_+) \wedge \mathrm{H}\mathbb{Q}$ is a cofibrant resolution of the unit $\mathrm{H}\mathbb{Q}$ in the monoidal model category of $(G_+, \mathrm{H}\mathbb{Q})$-bimodule spectra. Under the left adjoint this is sent to

$$\mathrm{F}_0(WG_+) \wedge \mathrm{H}\mathbb{Q} \longmapsto \mathrm{F}_0(WG_+) \wedge \mathcal{H}\mathbb{Q} \longmapsto \widetilde{\mathbb{Q}}[\mathrm{F}_0(WG_+)] \cong \mathrm{F}_0(\mathbb{Q}[WG]).$$

Since $WG \to *$ is a weak equivalence of (necessarily cofibrant) simplicial sets, Ken Brown's Lemma implies that $\mathrm{F}_0(\mathbb{Q}[WG]) \to \mathrm{F}_0(\mathbb{Q}[*]) = \mathrm{F}_0(\mathbb{Q})$ is a stable weak equivalence which completes the proof. □

2.4 Rational Homotopy Lie Representations

This section is devoted to studying stable model categories of modules over rational Lie algebras in the simplicial and differential graded settings. In §2.6 we use these results to build a bridge between the rational homotopy theory of parametrised spectra on the one hand and the theory of homotopy Lie representations on the other.

A key part of Quillen's approach to rational homotopy concerns the passage from simplicial to dg Lie algebras using the normalisation functor $N\colon \mathrm{sVect}_{\mathbb{Q}} \to ch_+$. In this section, we stabilise the normalisation functor to obtain a zig-zag of monoidal Quillen equivalences between $\mathrm{Sp}^{\Sigma}(\mathrm{sVect}_{\mathbb{Q}})$ and the model category ch of unbounded rational chain complexes. Using these monoidal equivalences, we then lift Quillen's equivalence between simplicial and dg Lie algebras to their derived categories of representations.

2.4.1 The Stable Dold–Kan Correspondence

Following work of Shipley in [Shi07] (which deals with strictification over $\mathbb{Z}$), in this section we promote the rational Dold–Kan correspondence

$$\begin{tikzcd} \mathrm{sVect}_{\mathbb{Q}} \arrow[r, shift left, "N"] \arrow[r, phantom, "\perp/\top"] & ch_+ \arrow[l, shift left, "\Gamma"] \end{tikzcd}$$

to a zig-zag of (weakly) monoidal left Quillen equivalences between the stable model categories

$$\mathrm{Sp}^{\Sigma}(\mathrm{sVect}_{\mathbb{Q}}) \xleftarrow{\sim} \mathrm{Sp}^{\Sigma}(ch_+) \xrightarrow{\sim} ch.$$

In this section and henceforth, ch_+ and ch always denote categories of *rational* chain complexes, with tensor products over $\mathbb{Q}$ unless stated otherwise. The subscript "+" indicates *connective* chain complexes, for which $M_k = 0$ for $k < 0$.

Remark 2.4.1. We recall the classical Dold–Kan correspondence [GJ09, III §2]. For a simplicial $\mathbb{Q}$-vector space V, its *normalised chain complex* is the connective chain complex with

$$NV_n := \bigcap_{i=1}^{n} \ker(d_i),$$

the joint kernel of the simplicial face maps $d_i\colon NV_n \to NV_{n-1}$ for $0 < i \leq n$, equipped with differential $\partial = d_0\colon NV_n \to NV_{n-1}$ defined as the restriction of the remaining face map. This assignment on objects defines the *normalisation functor* $N\colon \mathrm{sVect}_{\mathbb{Q}} \to ch_+$, which is an equivalence of categories. The inverse equivalence $\Gamma\colon ch_+ \to \mathrm{sVect}_{\mathbb{Q}}$ sends the chain complex M to the simplicial $\mathbb{Q}$-vector space

$$\Gamma(M)_n := ch_+\big(N_\bullet(\Delta[n]), M\big)$$

where $N_\bullet(\Delta[n]) := N(\mathbb{Q}[\Delta[n]])$ is the normalised rational chain complex of the combinatorial n-simplex. As usual, the equivalence of categories determined by N and Γ can be regarded as an adjoint equivalence with either N or Γ as the left adjoint.

Remark 2.4.2. The category ch_+ is a proper combinatorial model category with respect to the *projective model structure*, for which

- weak equivalences are the quasi-isomorphisms; those maps of chain complexes inducing isomorphisms in homology;
- fibrations are the maps which are epimorphisms in each positive degree; and
- cofibrations are the degreewise split monomorphisms with degreewise projective cokernel (the latter condition being automatic over $\mathbb{Q}$).

Sets of generating cofibrations and acyclic cofibrations are provided by the sets of inclusions

$$\mathcal{I}_+ := \{S^{n-1} \to D^n \mid n > 0\} \cup \{0 \to S^0\} \quad \text{and} \quad \mathcal{J}_+ := \{0 \to D^n \mid n > 0\}$$

respectively, where

$$S^n := \Big[\cdots \to 0 \to \underset{n}{\mathbb{Q}} \to 0 \to \cdots\Big], \qquad D^n := \Big[\cdots \to 0 \to \underset{n}{\mathbb{Q}} \xrightarrow{\mathrm{id}} \underset{n-1}{\mathbb{Q}} \to 0 \to \cdots\Big]$$

are the "sphere" and "disk" complexes. With respect to the tensor product of chain complexes ch_+ becomes a symmetric monoidal model category, as is easily checked directly on $\mathcal{I}_+$ and $\mathcal{J}_+$. A standard reference is [Hov99, Ch. 4].

For a simplicial $\mathbb{Q}$-vector space V we have $\pi_*(V) \cong H_\bullet(NV)$, so that N and Γ preserve weak equivalences. It is also not difficult to show that N and Γ preserve cofibrations, so that the Dold–Kan correspondences $(N \dashv \Gamma)$ and $(\Gamma \dashv N)$ are Quillen equivalences.

In the setting of the Dold–Kan correspondence, the normalisation functor is lax symmetric monoidal. This is witnessed by the *shuffle map*: for simplicial $\mathbb{Q}$-vector spaces V and W and $v \in V_p$, $w \in W_q$, the *shuffle product* of v and w is the element

$$\nabla(v \otimes w) := \sum_{(\mu,\nu)} \mathrm{sgn}(\mu,\nu) \cdot (s_\nu v \otimes s_\mu w) \in V_{p+q} \otimes W_{p+q} = (V \otimes W)_{p+q},$$

where the sum is over (p,q)-shuffles $(\mu,\nu) = (\mu_1, \dots \mu_p, \nu_1, \dots, \nu_q)$ and the map s_μ, for instance, is the iterated degeneracy $s_{\mu_p} \circ \cdots \circ s_{\mu_1}$. The shuffle product restricts to a natural transformation

$$\nabla_{V,W} \colon NV \otimes NW \longrightarrow N(V \otimes W)$$

which makes N lax symmetric monoidal. The shuffle map $\nabla_{V,W}$ is a natural quasi-isomorphism [May67, §29], and by adjointness induces a natural weak equivalence

$$\widetilde{\nabla}_{M,N} \colon \Gamma(M \otimes N) \longrightarrow \Gamma(M) \otimes \Gamma(N)$$

for all $M, N \in ch_+$. Accordingly, the Dold–Kan correspondence, regarded as a Quillen equivalence $(\Gamma \dashv N)$, defines a symmetric monoidal equivalence of categories on homotopy categories, despite the fact the left adjoint Γ is only symmetric oplax monoidal at the level of model categories. This is the ur-example of a *weak monoidal Quillen equivalence* ([SS03a, Definition 3.6]). In summary, we have

Lemma 2.4.3. *The Dold–Kan correspondence*

$$\begin{tikzcd} ch_+ \arrow[r, shift left, "\Gamma"] \arrow[r, phantom, "\perp"] & \mathrm{sVect}_{\mathbb{Q}} \arrow[l, shift left, "N"] \end{tikzcd}$$

is a weak monoidal Quillen equivalence.

Corollary 2.4.4. *For all $V, W \in \mathrm{sVect}_{\mathbb{Q}}$, the shuffle map induces a lax closed structure map*

$$\Xi_{V,W} \colon N([V,W]) \longrightarrow \big[N(V), N(W)\big],$$

which is a natural weak equivalence.

Proof. For $M \in ch_+$ and $V, W \in \mathrm{sVect}_{\mathbb{Q}}$ there are natural isomorphisms

$$\begin{aligned} ch_+\big(M, N([V,W])\big) &\cong \mathrm{sVect}_{\mathbb{Q}}\big(\Gamma(M), [V,W]\big) \\ &\cong \mathrm{sVect}_{\mathbb{Q}}\big(V \otimes \Gamma(M), W\big) \end{aligned}$$

and

$$\begin{aligned} ch_+\big(M, [N(V), N(W)]\big) &\cong ch_+\big(N(V) \otimes M, N(W)\big) \\ &\cong \mathrm{sVect}_{\mathbb{Q}}\big(\Gamma\big(N(V) \otimes M\big), W\big). \end{aligned}$$

The structure map $\Xi_{V,W} \colon N([V,W]) \to [N(V), N(W)]$ is defined as the map corresponding to

$$\xi_{V,M} \colon \Gamma\big(N(V) \otimes M\big) \xrightarrow{\widetilde{\nabla}_{N(V),M}} \Gamma(N(V)) \otimes \Gamma(M) \xrightarrow{\epsilon \otimes \mathrm{id}} V \otimes \Gamma(M)$$

under these natural identifications. That Ξ exhibits N as a lax closed functor is a standard consequence of the fact that N is lax monoidal. The maps $\xi_{V,M}$ are weak equivalences for all V and M by the Lemma; we use this to show that $\Xi_{V,W}$ is a weak equivalence by arguing on homotopy function complexes. Indeed, M, V and W are all fibrant–cofibrant objects in their respective model categories, so that $N([V,W])$ and $[N(V), N(W)]$ are fibrant by the monoidal model category axioms. Let M^* be a Reedy cosimplicial framing of M, so that the maps ξ induce a weak equivalence of cosimplicial framings

$$\Gamma\big(N(V) \otimes M^*\big) \longrightarrow V \otimes \Gamma(M^*)$$

of $\Gamma(N(V) \otimes M)$ and $V \otimes \Gamma(M)$ as both functors N and Γ preserve cofibrations, weak equivalences and colimits. The induced map of homotopy function complexes

$$\mathrm{map}_{\mathrm{sVect}_{\mathbb{Q}}}\big(V \otimes \Gamma(M^*), W\big) \longrightarrow \mathrm{map}_{\mathrm{sVect}_{\mathbb{Q}}}\big(\Gamma\big(N(V) \otimes M^*\big), W\big)$$

is then a weak equivalence. This map is the adjunct of the map determined by $\Xi_{V,W}$ on homotopy function complexes. In any model category $\mathcal{M}$ whatsoever, a map f is a weak equivalence precisely if the induced map of homotopy function complexes $\mathrm{map}_{\mathcal{M}}(X, f)$ is a weak equivalence for all fibrant $X \in \mathcal{M}$ [Hir03, Theorem 17.7.7]. By adjointness, we conclude that $\Xi_{[V,W]}$ is a natural weak equivalence. □

Our goal is stabilisation, for which we need the following

Lemma 2.4.5. *Write $\mathbb{Q}[-1]$ for the rational chain complex which is $\mathbb{Q}$ concentrated in degree 1. Then the assignment $M \mapsto M[-1] := M \otimes \mathbb{Q}[-1]$ models suspension on $Ho(ch_+)$.*

Proof. This is well-known, but we include a proof for the sake of completeness. Let $I_\bullet := N(\mathbb{Q}[\Delta[1]])$ be the normalised rational chains on the 1-simplex; explicitly

$$I_\bullet = \Big[\cdots \longrightarrow 0 \longrightarrow \underset{1}{\mathbb{Q}} \xrightarrow{(\mathrm{id}, -\mathrm{id})} \underset{0}{\mathbb{Q} \oplus \mathbb{Q}} \Big].$$

Then for any $M \in ch_+$ the tensor product $M \otimes I_\bullet$ is a very good cylinder object for M. Consequently, suspension is modelled by the pushout of the span

$$M \otimes I_\bullet \longleftarrow M \oplus M \longrightarrow 0,$$

which coincides with $M \otimes \mathbb{Q}[-1]$. □

Writing s for the left Quillen endofunctor $M \mapsto M[-1]$ of ch_+, the symmetric stabilisation machine produces a category of symmetric spectra of connective chain complexes

$$\mathrm{Sp}^\Sigma(ch_+) := \mathrm{Sp}^\Sigma(ch_+; s).$$

By construction, $\mathrm{Sp}^\Sigma(ch_+)$ is the category of (left) modules over the commutative monoid

$$\mathrm{Sym}(\mathbb{Q}[-1])\colon n \longmapsto \mathbb{Q}[-1]^{\otimes n} = \mathbb{Q}[-n]$$

for the Day convolution tensor product on the category of symmetric sequences $\mathrm{Fun}(\boldsymbol{\Sigma}, ch_+)$. By the general properties of the symmetric stabilisation machine (§A.3.2) $\mathrm{Sp}^\Sigma(ch_+)$ is a left proper combinatorial symmetric monoidal model category. The functor s prolongs to a left Quillen equivalence on $\mathrm{Sp}^\Sigma(ch_+)$ modelling suspension on the homotopy category, so that $\mathrm{Sp}^\Sigma(ch_+)$ is a stable model category. The first half of the stable Dold–Kan correspondence is the following

Theorem 2.4.6. *There is a weak monoidal Quillen equivalence*

$$\begin{tikzcd} \mathrm{Sp}^\Sigma(ch_+) \arrow[r, shift left, "N_!"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}) \arrow[l, shift left, "N^*"] \end{tikzcd}$$

induced by the Dold–Kan correspondence.

Proof. This is the rational version of [Shi07, Proposition 4.4]. First, we describe the right adjoint N^*. The normalisation functor $N\colon \mathrm{sVect}_\mathbb{Q} \to ch_+$ is lax symmetric monoidal, so that the commutative monoid $\widetilde{\mathbb{Q}}[\mathbb{S}] = \widetilde{\mathbb{Q}}[\mathrm{F}_0(S^0)]$ in $\mathrm{Fun}(\boldsymbol{\Sigma}, \mathrm{sVect}_\mathbb{Q})$ is sent to a commutative monoid $\mathcal{N}[\mathbb{S}]$ in $\mathrm{Fun}(\boldsymbol{\Sigma}, ch_+)$ by levelwise application of N. Likewise, normalisation sends left $\widetilde{\mathbb{Q}}[\mathbb{S}]$-modules in $\mathrm{Fun}(\boldsymbol{\Sigma}, \mathrm{sVect}_\mathbb{Q})$ are sent to left $\mathcal{N}[\mathbb{S}]$-modules in $\mathrm{Fun}(\boldsymbol{\Sigma}, ch_+)$.

The isomorphism $\mathbb{Q}[-1] \cong N(\widetilde{\mathbb{Q}}[S^1])$ induces a map of commutative monoids $\zeta\colon \mathrm{Sym}(\mathbb{Q}[-1]) \to \mathcal{N}[\mathbb{S}]$ in $\mathrm{Fun}(\boldsymbol{\Sigma}, ch_+)$; at level n this is the iterated shuffle map

$$\mathbb{Q}[-n] = \mathbb{Q}[-1]^{\otimes n} \longrightarrow N(\widetilde{\mathbb{Q}}[S^n])$$

so that ζ is a level weak equivalence, and hence a stable weak equivalence when regarded as a map in $\mathrm{Sp}^\Sigma(ch_+)$. In Lemma A.3.31 we verify that Quillen invariance holds in $\mathrm{Sp}^\Sigma(ch_+)$, so that extension and restriction of scalars along ζ is a Quillen equivalence $(\zeta_! \dashv \zeta^*)\colon \mathrm{Sym}(\mathbb{Q}[-1])-\mathrm{Mod} = \mathrm{Sp}^\Sigma(ch_+) \to \mathcal{N}[\mathbb{S}]-\mathrm{Mod}$.

The sought-after right adjoint functor N^* is defined as the composite functor

$$N^*\colon \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}) \xrightarrow{N} \mathcal{N}[\mathbb{S}]-\mathrm{Mod} \xrightarrow{\zeta^*} \mathrm{Sp}^\Sigma(ch_+).$$

Observe that $N\colon \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}) \to \mathcal{N}[\mathbb{S}]-\mathrm{Mod}$ preserves limits and filtered colimits and the domain and codomain are locally presentable, so N admits a left adjoint $\overline{N}$ by the Adjoint Functor Theorem. The left adjoint of N^* is the composite functor $N_! := \overline{N} \circ \zeta_!$.

We now show that $(\overline{N} \dashv N)\colon \mathcal{N}[\mathbb{S}]-\mathrm{Mod} \to \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})$ is a Quillen adjunction. By [Dug01b, Corollary A.2] is it sufficient to show that N preserves fibrations between fibrant objects as well as acyclic fibrations. The stably acyclic fibrations in $\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})$ coincide with the level acyclic fibrations, which are sent to level acyclic fibrations by the prolonged normalisation functor $N\colon \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}) \to \mathcal{N}[\mathbb{S}]-\mathrm{Mod}$. As any simplicial $\mathbb{Q}$-vector space is necessarily fibrant, the stably fibrant objects in $\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})$ are precisely those symmetric spectra V in $\mathrm{sVect}_\mathbb{Q}$ with the property that for each $n \in \mathbb{N}$ the adjunct of the structure map

$$\sigma_n \colon \widetilde{\mathbb{Q}}[S^1] \otimes V_n \longrightarrow V_{n+1}$$

is a weak equivalence. By general properties of left Bousfield localisation, fibrations between stably fibrant objects in $\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})$ are precisely the levelwise fibrations between such objects. Since $N(\widetilde{\mathbb{Q}}[S^1]) \cong \mathbb{Q}[-1]$, Corollary 2.4.4 implies that the prolonged normalisation functor sends stably fibrant objects to stably fibrant objects. The prolonged normalisation functor also preserves level fibrations, from which it follows that $N\colon \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}) \to \mathcal{N}[\mathbb{S}]-\mathrm{Mod}$ preserves fibrations between fibrant objects. The conclude that the adjunction $(\overline{N} \dashv N)$ is Quillen.

To show that $(N_! \dashv N^*)$ is a Quillen equivalence, it is sufficient to show that (i) N^* preserves and reflects weak equivalences between fibrant objects and (ii) that the derived unit is a natural isomorphism (Lemma A.1.7). For the first condition, we argue on $N\colon \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}) \to \mathcal{N}[\mathbb{S}]-\mathrm{Mod}$ since ζ^* preserves and reflects weak equivalences. The weak equivalences between fibrant objects in $\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})$ are precisely the levelwise weak equivalences, which are preserved and reflected by the prolonged normalisation functor since $N\colon \mathrm{sVect}_\mathbb{Q} \to ch_+$ has this property. To verify (ii) we argue as follows. The derived adjunction $(\mathbb{L}N_! \dashv \mathbb{R}N^*)$ is an adjoint pair of exact functors between triangulated categories since $\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})$ and $\mathrm{Sp}^\Sigma(ch_+)$ are stable. The full subcategory $\mathcal{E} \hookrightarrow Ho(\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}))$ on those objects for which the derived unit is an isomorphism is thus a localising triangulated subcategory. The object $\mathrm{Sym}(\mathbb{Q}[-1])$ presents a weak compact generator of $Ho(\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}))$, and the component of the derived unit at the generator is presented by

$$\zeta\colon \mathrm{Sym}(\mathbb{Q}[-1]) \longrightarrow N^* N_!(\mathrm{Sym}(\mathbb{Q}[-1])) = \mathcal{N}[\mathbb{S}],$$

since N^* preserves stable weak equivalences (Lemma 2.4.7 below). Since ζ is a stable weak equivalence, $\mathcal{E}$ contains a weak compact generator and so must be the whole homotopy category. Thus, condition (ii) is satisfied and the adjunction $(N_! \dashv N^*)$ is a Quillen equivalence.

Finally, we show that the Quillen equivalence $(N_! \dashv N^*)$ is weak monoidal. For this, we observe that the right adjoint N^* is lax monoidal since N and ζ^* are. The cofibrant object $\mathrm{Sym}(\mathbb{Q}[-1])$ defines a weak compact generator of $Ho(\mathrm{Sp}^\Sigma(ch_+))$, and the lax structure map of N^* on the monoidal units $N_!(\mathrm{Sym}(\mathbb{Q}[-1])) \to \widetilde{\mathbb{Q}}[\mathbb{S}]$ is an isomorphism, which completes the proof. $\square$

Lemma 2.4.7. *The functor $N^*\colon \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}) \to \mathrm{Sp}^\Sigma(ch_+)$ preserves and reflects stable equivalences.*

Proof. As in [HSS00, p. 3.4.9], we can choose a set of generating acyclic cofibrations $\mathcal{J}^{\mathrm{st}}_\Sigma$ for the stable model structure on Sp^Σ such that for all $j \in \mathcal{J}$ either

(i) j is a levelwise weak equivalence; or

(ii) j is level weakly equivalent to a map of the form $\zeta_n\colon \mathrm{F}_{n+1}(S^1 \wedge K) \to \mathrm{F}_n(K)$ for some pointed simplicial set K.

By Lemma 2.3.19, $\widetilde{\mathbb{Q}}[\mathcal{J}^{\mathrm{st}}_\Sigma]$ is then a set of generating acyclic cofibrations for the stable model structure on $\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})$. Observe that for all $j \in \widetilde{\mathbb{Q}}[\mathcal{J}^{\mathrm{st}}_\Sigma]$ either

(a) j is a levelwise weak equivalence; or

(b) j is level weakly equivalent some $\widetilde{\mathbb{Q}}[\zeta_n(K)]\colon \widetilde{\mathbb{Q}}[\mathrm{F}_{n+1}(S^1 \wedge K)] \to \widetilde{\mathbb{Q}}[\mathrm{F}_n(K)]$.

The prolonged normalisation functor $N\colon \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}) \to \mathcal{N}[\mathbb{S}]-\mathrm{Mod}$ preserves levelwise weak equivalences and colimits, so preserves stably acyclic cofibrations once we know that morphisms of the form (b) are sent to stable weak equivalences in $\mathrm{Sp}^\Sigma(ch_+)$. The k-th term of the underlying symmetric sequence of $N(\widetilde{\mathbb{Q}}[\mathrm{F}_n(K)])$ is

$$(\Sigma_k)_+ \wedge_{(\Sigma_{k-n})_+} N\big(\widetilde{\mathbb{Q}}[S^{k-n} \wedge K]\big)$$

(compare Remark 1.2.16). For any $K, L \in \mathrm{sSet}_*$, the shuffle map

$$N(\widetilde{\mathbb{Q}}[K]) \otimes N(\widetilde{\mathbb{Q}}[L]) \longrightarrow N\big(\widetilde{\mathbb{Q}}[K] \otimes \widetilde{\mathbb{Q}}[L]\big) \cong N\big(\widetilde{\mathbb{Q}}[K \wedge L]\big)$$

is a weak equivalence, so combining these facts the horizontal morphisms in the commutative diagram in $\mathrm{Sp}^\Sigma(ch_+)$

$$\begin{tikzcd}
\mathcal{N}[\mathbb{S}] \bigotimes_{\mathrm{Sym}(\mathbb{Q}[-1])} \Big(\mathrm{F}^{ch}_{n+1}\big[N\big(\widetilde{\mathbb{Q}}[S^1] \otimes \widetilde{\mathbb{Q}}[K]\big)\big]\Big) \arrow[r] \arrow[d] & N\big(\widetilde{\mathbb{Q}}[\mathrm{F}_{n+1}(S^1 \wedge K)]\big) \arrow[d] \\
\mathcal{N}[\mathbb{S}] \bigotimes_{\mathrm{Sym}(\mathbb{Q}[-1])} \mathrm{F}^{ch}_n\big(N(\widetilde{\mathbb{Q}}[K])\big) \arrow[r] & N\big(\widetilde{\mathbb{Q}}[\mathrm{F}_n(K)]\big)
\end{tikzcd}$$

are levelwise weak equivalences for all $n \geq 0$ and $K \in \mathrm{sSet}_*$. Here we have written $\mathrm{F}^{ch}_n\colon ch_+ \to \mathrm{Sp}^\Sigma(ch_+)$ for the free symmetric spectrum functor which is left adjoint to $\mathrm{Ev}_n\colon \mathrm{Sp}^\Sigma(ch_+) \to ch_+$, in order to distinguish from the case for $\mathrm{sVect}_\mathbb{Q}$. The left vertical morphism in the above diagram is a stable weak equivalence between cofibrant objects, since $\mathrm{F}^{ch}_{n+1}(\mathbb{Q}[-1] \otimes N(\widetilde{\mathbb{Q}}[K])) \to \mathrm{F}^{ch}_n(N(\widetilde{\mathbb{Q}}[K]))$ is a stable weak equivalence in $\mathrm{Sp}^\Sigma(ch_+)$ and $\zeta_!$ is left Quillen. By the 2-out-of-3 property $N(\widetilde{\mathbb{Q}}[\xi_n(K)]$ is a stable weak equivalence. In the proof the Theorem 2.4.6 above we showed that the prolonged normalisation functor N preserves stably acyclic fibrations, so we conclude that N preserves all stable weak equivalences. As $\zeta^*\colon \mathcal{N}[\mathbb{S}]-\mathrm{Mod} \to \mathrm{Sp}^\Sigma(ch_+)$ preserves and reflects stable weak equivalences, it follows that $N^* = \zeta^* \circ N$ preserves stable weak equivalences.

To complete the proof we show that $N\colon \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}) \to \mathcal{N}[\mathbb{S}]-\mathrm{Mod}$ reflects stable weak equivalences. Indeed, suppose that $f\colon V \to W$ is a map in $\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})$ which is sent to a stable weak equivalence in $\mathcal{N}[\mathbb{S}]-\mathrm{Mod}$. Fixing a fibrant replacement functor $\mathcal{R}$ on $\mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q})$, $\mathcal{R}(f)$ is thus sent to a stable weak equivalence between stably fibrant objects in $\mathcal{N}[\mathbb{S}]-\mathrm{Mod}$. Since the stable weak equivalences between stably fibrant objects are precisely the levelwise weak equivalences, and these are reflected by $N\colon \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}) \to \mathcal{N}[\mathbb{S}]-\mathrm{Mod}$, it follows that $\mathcal{R}(f)$ is a levelwise and hence stable weak equivalence. The assertion now follows from the naturality diagram

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "\wr"'] & W \arrow[d, "\wr"] \\
\mathcal{R}(V) \arrow[r, "\mathcal{R}(f)"] & \mathcal{R}(W)
\end{tikzcd}$$

and the 2-out-of-3 property of stable weak equivalences. $\square$

Having made the passage from spectral simplicial $\mathbb{Q}$-vector spaces to spectral connective chain complexes, we turn to the second half of the stable Dold–Kan correspondence. This works by assembling spectral connective chain complexes into unbounded chain complexes, and conversely by disassembling unbounded chain complexes into spectrum objects.

Construction 2.4.8. The *assembly complex* of $M \in \mathrm{Sp}^\Sigma(ch_+)$ is the chain complex $\mathcal{A}(M) \in ch$ constructed as follows. For $m \geq n \geq 0$, the iterated symmetric spectrum structure maps

$$\sigma_{n,m} \colon M_n[n-m] \longrightarrow M_{n+1}[n-m+1] \longrightarrow \cdots \longrightarrow M_m$$

are adjoint to maps $\sigma^\vee_{n,m} \colon M_n \to M_m[m-n]$ of generically non-connective chain complexes. Let $\mathfrak{Fin}_{\mathrm{in}}$ be the skeleton of the category of finite sets with injective maps; objects are ordered sets $\langle n \rangle := \{1, \ldots, n\}$ for $n \geq 1$ and $\langle 0 \rangle := \varnothing$. From M we construct a functor $\overline{M} \colon \mathfrak{Fin}_{\mathrm{in}} \to ch$ which sends $\langle n \rangle \mapsto M_n[n]$ and is determined on morphisms by

- sending the inclusion $\langle n \rangle \hookrightarrow \langle m \rangle$ of the first $n \leq m$ elements to the map of chain complexes $\sigma^\vee_{n,m}[m] \colon M_n[n] \to M_m[m]$; and
- sending the automorphism $f \in \mathfrak{Fin}_{\mathrm{in}}(\langle n \rangle, \langle n \rangle) = \Sigma_n$ to the automorphism $(-1)^{\mathrm{sgn}(f)}\rho(f)$ of $M_n[n]$, where ρ is the Σ_n-action on M_n.

The assembly complex is the colimit of this functor:

$$\mathcal{A}(M) := \operatorname*{colim}_{\mathfrak{Fin}_{\mathrm{in}}} \overline{M},$$

and this construction gives rise to the *assembly functor* $\mathcal{A} \colon \mathrm{Sp}^\Sigma(ch_+) \to ch$.

Conversely, we decompose an unbounded chain complex into a symmetric spectrum object. Recall that the inclusion $ch_+ \hookrightarrow ch$ has right adjoint

$$\Big[\cdots \to M_1 \xrightarrow{\partial_1} M_0 \xrightarrow{\partial_0} M_{-1} \to \cdots \Big] \xmapsto{\mathrm{cn}_0} \Big[\cdots \to M_1 \xrightarrow{\partial_1} \ker(\partial_0) \to 0 \to \cdots \Big].$$

The functor cn_0 is the *connective cover*; the counit $\mathrm{cn}_0 M \to M$ induces homology isomorphisms $H_k(\mathrm{cn}_0 M) \cong H_k(M)$ for all $k \geq 0$. For $M \in ch$ an unbounded chain complex, its *disassembly (symmetric) spectrum* has underlying symmetric sequence $\mathcal{D}(M)_n := \mathrm{cn}_0(M[-n])$ with Σ_n acting by the sign representation (equivalently, by permuting the first n tensor factors in $M[-n] = \mathbb{Q}[-1]^{\otimes n} \otimes M$). For each $n \geq 0$, the adjoint of the connective cover map $\mathrm{cn}_0(M[-n]) \to M[-n] \cong \mathbb{Q}[1] \otimes M[-(n+1)]$ factors as

$$\mathbb{Q}[-1] \otimes \mathrm{cn}_0(M[-n]) \xrightarrow{\sigma^M_n} \mathrm{cn}_0(M[-(n+1)]) \longrightarrow M[-(n+1)].$$

The maps $\sigma^M_n \colon \mathbb{Q}[-1] \otimes \mathcal{D}(M)_n \to \mathcal{D}(M)_{n+1}$ are Σ_n-equivariant, so $\mathcal{D}(M)$ is a symmetric spectrum. This defines the *disassembly functor* $\mathcal{D} \colon ch \to \mathrm{Sp}^\Sigma(ch_+)$.

Lemma 2.4.9. *Assembly and disassembly functors are adjoint:* $(\mathcal{A} \dashv \mathcal{D}) \colon \mathrm{Sp}^\Sigma(ch_+) \to ch$.

Proof. By construction, $\mathcal{A}$ factros as the composite

$$\mathrm{Sp}^\Sigma(ch_+) \xrightarrow{\overline{(-)}} \mathrm{Fun}(\mathfrak{Fin}_{\mathrm{in}}, ch) \xrightarrow{\mathrm{colim}} ch.$$

The colimit functor admits a right adjoint $\Delta \colon ch \to \mathrm{Fun}(\mathfrak{Fin}_{\mathrm{in}}, ch)$ sending $M \in ch$ to the constant functor on M. The functor $\overline{(-)}$ admits a right adjoint $\mathfrak{sp}$, which to

$A\colon \mathfrak{Fin}_{\mathrm{in}} \to ch$ assigns the symmetric spectrum with n-th term

$$\mathfrak{sp}(A)_n := \mathrm{cn}_0\big(A\langle n\rangle[-n]\big),$$

with Σ_n-action given by the sign representation. Spectrum structure maps are defined via adjuncts of the morphisms

$$\mathrm{cn}_0\big(A\langle n\rangle[-n]\big) \longrightarrow A\langle n\rangle[-n] \xrightarrow{A(\iota_n)[-n]} A\langle n+1\rangle,$$

with $\iota_n\colon \langle n\rangle \hookrightarrow \langle n+1\rangle$ the inclusion of the first n elements. For $M \in \mathrm{Sp}^\Sigma(ch_+)$ and functor $A\colon \mathfrak{Fin}_{\mathrm{in}} \to ch$, a natural transformation $\overline{M} \to A$ determines and is completely determined by the family of Σ_n-equivariant maps $\eta_{\langle n\rangle}\colon M_n[n] \to A\langle n\rangle$ for $n \geq 0$. The $\eta_{\langle n\rangle}$ are adjoint to maps $M_n \to \mathrm{cn}_0(A\langle n\rangle[-n])$ which define a map of symmetric spectra $M \to \mathfrak{sp}(A)$. Since $\mathcal{D} = \mathfrak{sp} \circ \Delta$, this shows that $(\mathcal{A} \dashv \mathcal{D})$ at the level of homsets. $\square$

Lemma 2.4.10. *The assembly functor $\mathcal{A}\colon \mathrm{Sp}^\Sigma(ch_+) \to ch$ is strong symmetric monoidal.*

Proof. Firstly, we observe that $\mathrm{Ev}_0 \circ \mathcal{D}\colon ch \to ch_+$ coincides with the connective cover functor cn_0. By essential uniqueness of adjoints we have $\mathcal{A}(\mathrm{Sym}(\mathbb{Q}[-1]) \cong \mathbb{Q}$, so that $\mathcal{A}$ preserves the monoidal units.

We construct a natural isomorphism $\Lambda\colon \mathcal{A}(-) \otimes \mathcal{A}(-) \Rightarrow \mathcal{A}((-) \otimes (-))$ which witnesses that $\mathcal{A}$ is a strong monoidal functor. For $M, N \in \mathrm{Sp}^\Sigma(ch_+)$, we have

$$\mathcal{A}(M) \otimes \mathcal{A}(N) \cong \operatorname*{colim}_{\langle m\rangle, \langle n\rangle \in \mathfrak{Fin}_{\mathrm{in}}} \big(X_m \otimes Y_n\big)[-(m+n)]$$

since $\otimes$ commutes with colimits. On the other hand,

$$\overline{M \otimes N}\,\langle m+n\rangle = (M \otimes N)_{m+n}[-(m+n)]$$

so that we have maps of chain complexes $M_m \otimes N_n \to (M \otimes N)_{m+n}$ (beware! the "$\otimes$" symbol in the codomain is the smash product of spectra, whereas in the domain it is the usual tensor product of chain complexes). Writing $+\colon \mathfrak{Fin}_{\mathrm{in}} \times \mathfrak{Fin}_{\mathrm{in}} \to \mathfrak{Fin}_{\mathrm{in}}$ for the addition functor $(\langle m\rangle, \langle n\rangle) \mapsto \langle m+n\rangle$, the maps obtained above induce a natural transformation $\overline{M} \otimes \overline{N} \to (+)^*\overline{M \otimes N}$. Taking colimits now gives $\Lambda_{M,N}\colon \mathcal{A}(M) \otimes \mathcal{A}(N) \to \mathcal{A}(M \otimes N)$. A straightforward argument shows that Λ is a symmetric monoidal structure on $\mathcal{A}$.

To see that Λ is a natural isomorphism, we first argue on free symmetric spectra. For $V \in ch_+$ and $n \geq 0$

$$\overline{\mathrm{F}_n(V)}\,\langle k\rangle \cong \coprod_{\mathfrak{Fin}_{\mathrm{in}}(\langle n\rangle, \langle k\rangle)} V[n] \cong \mathcal{Y}_{\langle n\rangle}(V[n])\langle k\rangle,$$

where $\mathcal{Y}_{\langle n\rangle}\colon ch \to \mathrm{Fun}(\mathfrak{Fin}_{\mathrm{in}}, ch)$ is left adjoint to evaluation at $\langle n\rangle$. Since $\mathrm{colim} \circ \mathcal{Y}_{\langle n\rangle}$ is left adjoint to $\mathrm{ev}_{\langle n\rangle} \circ \Delta = \mathrm{id}_{ch}$, there is a natural isomorphsim $\mathrm{colim} \circ \mathcal{Y}_{\langle n\rangle} \cong \mathrm{id}_{ch}$. In particular, $\mathcal{A}(\mathrm{F}_n(V)) \cong V[n]$ so that $\Lambda_{\mathrm{F}_m(V), \mathrm{F}_m(W)}$ is isomorphic to

$$V[n] \otimes W[m] \longrightarrow (V \otimes W)[m+n],$$

and so is itself an isomorphism. Any $M \in \mathrm{Sp}^\Sigma(ch_+)$ is colimit of free spectra $\mathrm{F}_n(V)$ for $n \geq 0$ and $V \in ch_+$ and the functors $\mathcal{A}(-) \otimes \mathcal{A}(-)$ and $\mathcal{A}((-) \otimes (-))$ preserve colimits in both arguments, so it follows that Λ is a natural isomorphism. $\square$

Remark 2.4.11. Recall the *projective model structure* on the category ch of unbounded chain complexes, where

- weak equivalences are the quasi-isomorphisms;
- fibrations are the maps which are epimorphisms in each degree; and
- cofibrations are characterised by the left lifting property with respect to acyclic fibrations.

Sets of generating cofibrations and acyclic cofibrations are provided by the sets of inclusions $\mathcal{I} := \{S^{n-1} \to D^n \mid n \in \mathbb{N}\}$ and $\mathcal{J} := \{0 \to D^n \mid n \in \mathbb{N}\}$ respectively. The projective model structure is proper and combinatorial and the tensor product makes ch a symmetric monoidal model category (a standard reference is [Hov99]). With respect to the projective model structures

$$\begin{tikzcd} ch_+ \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & ch \arrow[l, shift left, "\mathrm{cn}_0"] \end{tikzcd}$$

is a strong monoidal Quillen adjunction. Both adjoints preserve weak equivalences.

Remark 2.4.12. An alternative model structure on ch is the *injective model structure*, which has weak equivalences and cofibrations given by quasi-isomorphisms and monomorphisms respectively [Hov99, Theorem 2.3.13]. As we are working over $\mathbb{Q}$ (or, more generally, over any field k), the projective and injective model structures coincide. Indeed, the projective cofibrations are the degreewise split monomorphisms with cofibrant cokernel ([Hov99, Proposition 2.3.9]). As we are working over $\mathbb{Q}$, all monomorphisms are degreewise split and all objects are projectively cofibrant, since any chain complex is degreewise free and so can be written as a colimit over cell attachments of the form $S^{n-1} \to D^n$. Hence, the classes of projective and injective cofibrations coincide.

Theorem 2.4.13. *The adjunction*

$$\begin{tikzcd} \mathrm{Sp}^{\Sigma}(ch_+) \arrow[r, shift left, "\mathcal{A}"] \arrow[r, phantom, "\perp"] & ch \arrow[l, shift left, "\mathcal{D}"] \end{tikzcd}$$

is a strong monoidal Quillen equivalence.

Proof. Note that $\mathrm{Sym}(\mathbb{Q}[-1])$ is cofibrant, so the unit axiom is automatic once we know that $(\mathcal{A} \dashv \mathcal{D})$ is a Quillen adjunction.

It remains to show that $(\mathcal{A} \dashv \mathcal{D})$ is a Quillen equivalence. For any $A \in ch$, its disassembly spectrum $\mathcal{D}(A)$ is a stably fibrant symmetric spectrum: indeed, the adjoint structure maps are isomorphisms. The connective cover functor cn_0 is right Quillen so that $\mathcal{D}$ sends fibrations in weak equivalences in ch to levelwise fibrations and levelwise weak equivalences in $\mathrm{Sp}^{\Sigma}(ch_+)$. The stable fibrations and stable weak equivalences between stably fibrant symmetric spectra are precisely the levelwise such maps so that $\mathcal{D}$ is a right Quillen functor. More is true: $\mathcal{D}$ also reflects weak equivalences. Indeed, if $f\colon A \to B$ is a map in ch such that $\mathcal{D}(f)$ is a stable weak equivalence, then $\mathcal{D}(f)$ is necessarily a levelwise weak equivalence. But $H_n(\mathcal{D}(A)_k) = H_{n-k}(A)$ for all $n, k \geq 0$, so that $f\colon A \to B$ is a quasi-isomorphism.

We show that $(\mathcal{A} \dashv \mathcal{D})$ is a Quillen equivalence by proving that the derived unit is a natural isomorphism. Let $\mathcal{E} \hookrightarrow Ho(\mathrm{Sp}^{\Sigma}(ch_+)$ be the full (localising, triangulated) subcategory on those objects for which the derived unit is an isomorphism. The symmetric spectrum $\mathrm{Sym}(\mathbb{Q}[-1])$ presents a weak compact generator of $Ho(\mathrm{Sp}^{\Sigma}(ch_+)$; the component of the derived unit at $\mathrm{Sym}(\mathbb{Q}[-1])$ is presented by the isomorphism

$$\mathrm{Sym}(\mathbb{Q}[-1]) \longrightarrow \mathcal{D}\big(\mathcal{A}(\mathrm{Sym}(\mathbb{Q}[-1]))\big) \cong \mathcal{D}(\mathbb{Q}) = \mathrm{Sym}(\mathbb{Q}[-1]).$$

The localising subcategory $\mathcal{E}$ thus contains the weak compact generator and so is the whole homotopy category. □

2.4.2 Lie Representations: Simplicial versus Differential Graded

Quillen established an equivalence of homotopy theories between reduced rational Lie algebras in the simplicial and dg contexts. Our present goal is to lift this equivalence to derived categories of rational homotopy Lie representations. As usual, everything in this section is over $\mathbb{Q}$.

Recall that if $\mathfrak{g}$ is a simplicial Lie algebra, its normalised chain complex $N(\mathfrak{g})$ is a differential graded Lie algebra with respect to the bracket

$$[\![-,-]\!]\colon N(\mathfrak{g}) \otimes N(\mathfrak{g}) \xrightarrow{\nabla} N(\mathfrak{g}\otimes\mathfrak{g}) \xrightarrow{N([-,-])} N(\mathfrak{g}).$$

The shuffle map also gives a way of comparing the universal enveloping algebras of $\mathfrak{g}$ and its normalisation $N(\mathfrak{g})$:

Lemma 2.4.14. *For $\mathfrak{g} \in \mathrm{sLie}$, there is a natural weak equivalence $\chi_{\mathfrak{g}}\colon \mathcal{U}(N(\mathfrak{g})) \to N(\mathcal{U}(\mathfrak{g}))$ commuting with multiplication in the sense that the diagram*

$$\begin{tikzcd}
\mathcal{U}(N(\mathfrak{g})) \otimes \mathcal{U}(N(\mathfrak{g})) \arrow[r] \arrow[d, "\chi_{\mathfrak{g}}\otimes\chi_{\mathfrak{g}}"'] & \mathcal{U}(N(\mathfrak{g})) \arrow[d, "\chi_{\mathfrak{g}}"] \\
N(\mathcal{U}(\mathfrak{g})) \otimes N(\mathcal{U}(\mathfrak{g})) \arrow[r] & N(\mathcal{U}(\mathfrak{g}))
\end{tikzcd}$$

commutes, where the horizontal maps are induced by the shuffle map and the multiplication operations in $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(N(\mathfrak{g}))$.

Proof. For any $V \in \mathrm{sVect}_{\mathbb{Q}}$, iterated shuffle maps define a natural weak equivalence $\eta_V\colon \mathrm{T}(N(V)) \to N(\mathrm{T}(V))$, where T denotes the free tensor algebra. For $V = \mathfrak{g}$ a simplicial Lie algebra, the component $\eta_{\mathfrak{g}}$ carries the two-sided tensor ideal

$$\mathcal{I}_{N(\mathfrak{g})} := \langle a\otimes b - (-1)^{|a||b|} b\otimes a - [\![a,b]\!] \mid a,b \in N(\mathfrak{g})\rangle$$

into the image by N of the two-sided tensor ideal

$$\mathcal{I}_{\mathfrak{g}} := \langle x\otimes y - y \otimes -[x,y] \mid x,y \in \mathfrak{g}\rangle.$$

The map $\eta_{\mathfrak{g}}$ thus induces a map

$$\chi_{\mathfrak{g}}\colon \mathcal{U}(N(\mathfrak{g})) \cong \mathrm{T}(N(\mathfrak{g}))/\mathcal{I}_{N(\mathfrak{g})} \longrightarrow N(\mathrm{T}(\mathfrak{g}))/N(\mathcal{I}_{\mathfrak{g}}) \cong N(\mathcal{U}(\mathfrak{g})),$$

which inherits the stated "commutativity with multiplication" property from the lax monoidal structure of N. The construction of $\chi_{\mathfrak{g}}$ is manifestly natural in $\mathfrak{g}$.

For $\mathfrak{h}$ a dg Lie algebra with $\imath\colon \mathfrak{h} \to \mathcal{U}(\mathfrak{h})$ the canonical inclusion, the assignment

$$(x_1,\dots,x_n) \longmapsto \frac{1}{n!}\sum_{\sigma\in\Sigma_n} \pm\imath(x_{\sigma(1)})\cdots\imath(x_{\sigma(n)})$$

determines an isomorphism of chain complexes $\mathrm{PBW}_{\mathfrak{h}}\colon \mathrm{Sym}(\mathfrak{h}) \to \mathcal{U}(\mathfrak{h})$ by the Poincaré–Birkhoff–Witt Theorem ("$\pm$" denotes the Koszul sign). As we are working rationally, the quotient $\mathrm{T(V)} \to \mathrm{Sym}(V)$ admits a natural section; in the dg case, for instance, this section is

$$(x_1, \dots, x_n) \longmapsto \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \pm x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

The natural weak equivalence $\eta_{\mathfrak{g}}$ descends to a map $\eta'_{\mathfrak{g}}\colon \mathrm{Sym}(N(\mathfrak{g})) \to N(\mathrm{Sym}(\mathfrak{g}))$. The natural sections of $\mathrm{T} \to \mathrm{Sym}$ exhibit $\eta'_{\mathfrak{g}}$ as a retract of $\eta_{\mathfrak{g}}$, so that $\eta'_{\mathfrak{g}}$ too is a weak equivalence. Inspecting the above formulae for the PBW maps, we find that there is a commuting diagram

$$\begin{tikzcd}
\mathrm{Sym}(N(\mathfrak{g})) \arrow[r] \arrow[d, "\eta'_{\mathfrak{g}}"'] \arrow[rr, bend left, "\mathrm{PBW}_{N(\mathfrak{g})}"] & \mathrm{T}(N(\mathfrak{g})) \arrow[r] \arrow[d, "\eta_{\mathfrak{g}}"'] & \mathcal{U}(N(\mathfrak{g})) \arrow[d, "\chi_{\mathfrak{g}}"] \\
N(\mathrm{Sym}(\mathfrak{g})) \arrow[r] \arrow[rr, bend right, "N(\mathrm{PBW}_{\mathfrak{g}})"'] & N(\mathrm{T}(\mathfrak{g})) \arrow[r] & N(\mathcal{U}(\mathfrak{g}))
\end{tikzcd}$$

from which it follows that $\chi_{\mathfrak{g}}$ is a weak equivalence. □

Remark 2.4.15. For a (simplicial or dg) Lie algebra $\mathfrak{g}$, a $\mathfrak{g}$-representation is precisely a left $\mathcal{U}(\mathfrak{g})$-module. For $\mathfrak{g}$ a simplicial Lie algebra, we write $\mathfrak{g}{-}\mathrm{Rep}_{\geq 0} := \mathcal{U}(\mathfrak{g}){-}\mathrm{Mod}_{\geq 0}$ and $\mathfrak{g}{-}\mathrm{Rep} := \mathcal{U}(\mathfrak{g}){-}\mathrm{Mod}$.

For $\mathfrak{h}$ a dg Lie algebra, we similarly set $\mathfrak{h}{-}\mathrm{Rep}_+$ to be the category of left $\mathcal{U}(\mathfrak{h})$-modules in ch_+. Since $\mathcal{U}(\mathfrak{h}) \otimes (-)\colon ch_+ \to ch_+$ is a left Quillen functor, the free-forgetful adjunction

$$\begin{tikzcd}
ch_+ \arrow[r, shift left, "\mathcal{U}(\mathfrak{h})\otimes(-)"] \arrow[r, phantom, "\perp"] & \mathfrak{h}{-}\mathrm{Rep}_+ \arrow[l, shift left]
\end{tikzcd}$$

equips $\mathfrak{h}{-}\mathrm{Rep}_+$ with a proper combinatorial model structure. Weak equivalences fibrations in $\mathfrak{h}{-}\mathrm{Rep}_+$ are created by the forgetful functor to ch_+. The left Quillen endofunctor $s\colon V \mapsto V[-1]$ models suspension on $Ho(\mathfrak{h}{-}\mathrm{Rep}_+)$.

We consider two models for the derived category of $\mathfrak{h}$-representations. The first is $\mathfrak{h}{-}\mathrm{Rep}$, defined as the category of left $\mathcal{U}(\mathfrak{h})$-modules in ch. Similarly to $\mathfrak{h}{-}\mathrm{Rep}_+$, $\mathfrak{h}{-}\mathrm{Rep}$ is a proper combinatorial model category with $s\colon V \mapsto V[-1]$ modelling suspension on the homotopy category. The endofunctor $s\colon \mathfrak{h}{-}\mathrm{Rep} \to \mathfrak{h}{-}\mathrm{Rep}$ is a Quillen equivalence with right adjoint $V \mapsto V[1]$. In particular, $\mathfrak{h}{-}\mathrm{Rep}$ is a stable model category. The $\mathfrak{h}$-representation $\mathcal{U}(\mathfrak{h})$ presents a weak compact generator of the derived category $Ho(\mathfrak{h}{-}\mathrm{Rep})$.

Our second candidate for the stable model category of $\mathfrak{h}$-representations is the symmetric stabilisation

$$\mathfrak{h}{-}\mathrm{Rep}^{\Sigma} := \mathrm{Sp}^{\Sigma}(\mathfrak{h}{-}\mathrm{Rep}_+; s)$$

equipped with its stable model structure. Similarly to Remark 2.3.23, $\mathfrak{h}{-}\mathrm{Rep}^{\Sigma}$ is isomorphic to the category of left $\mathrm{F}_0(\mathcal{U}(\mathfrak{h}))$-modules in $\mathrm{Sp}^{\Sigma}(ch_+)$. This isomorphism identifies the stable model structure with the model structure transferred via the free-forgetful adjunction

$$\begin{tikzcd} \mathrm{Sp}^\Sigma(ch_+) \arrow[r, shift left, "\mathrm{F}_0(\mathcal{U}(\mathfrak{h}))\otimes(-)"] \arrow[r, phantom, "\perp"] & \mathfrak{h}-\mathrm{Rep}^\Sigma \arrow[l, shift left] \end{tikzcd}$$

similarly to Lemma 2.3.19. The model category $\mathfrak{h}-\mathrm{Rep}^\Sigma$ is stable and $\mathrm{F}_0(\mathcal{U}(\mathfrak{h}))$ presents a weak compact generator of the derived category $Ho(\mathfrak{h}-\mathrm{Rep}^\Sigma)$.

Having dispensed with all the preliminaries, we come to the main result of this section:

Theorem 2.4.16. *For any $\mathfrak{g} \in \mathrm{sLie}$, there is a zig-zag of Quillen equivalences*

$$\begin{tikzcd} \mathfrak{g}-\mathrm{Rep} \arrow[r, shift right] \arrow[r, phantom, "\perp"] & N(\mathfrak{g})-\mathrm{Rep}^\Sigma \arrow[l, shift right] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & N(\mathfrak{g})-\mathrm{Rep} \arrow[l, shift left] \end{tikzcd}$$

inducing an equivalence of categories between $Ho(\mathfrak{g}-\mathrm{Rep})$ and $Ho(N(\mathfrak{g})-\mathrm{Rep})$.

Proof. We begin with the left-hand adjunction. Normalisation $N\colon \mathrm{sVect}_\mathbb{Q} \to ch_+$ is lax monoidal, so for M a left $\mathcal{U}(\mathfrak{g})$-module in $\mathrm{sVect}_\mathbb{Q}$ the map

$$\mathcal{U}(N(\mathfrak{g})) \otimes N(M) \xrightarrow{\chi_\mathfrak{g}\otimes\mathrm{id}} N(\mathcal{U}(\mathfrak{g})) \otimes N(M) \xrightarrow{\nabla} N(\mathcal{U}(\mathfrak{g}) \otimes M) \xrightarrow{N(\rho_M)} N(M)$$

equips $N(M)$ with the structue of a left $\mathcal{U}(N(\mathfrak{g}))$-module by Lemma 2.4.14. Since $\mathfrak{g}-\mathrm{Rep} \cong \mathrm{Sp}^\Sigma(\mathfrak{g}-\mathrm{Rep}_{\geq 0})$ and $N(\mathfrak{h})-\mathrm{Rep}^\Sigma = \mathrm{Sp}^\Sigma(N(\mathfrak{g})-\mathrm{Rep}_+)$ and the functor $N^*\colon \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}) \to \mathrm{Sp}^\Sigma(ch_+)$ is given by levelwise application of $N\colon \mathrm{sVect}_\mathbb{Q} \to ch_+$, we deduce that there is a factorisation

$$\begin{tikzcd} \mathfrak{g}-\mathrm{Rep} \arrow[r] \arrow[d, "\exists N^*_\mathfrak{g}"'] & \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}) \arrow[d, "N^*"] \\ N(\mathfrak{g})-\mathrm{Rep}^\Sigma \arrow[r] & \mathrm{Sp}^\Sigma(ch_+) \end{tikzcd}$$

where the horizontal arrows are the forgetful functors. The monads $\mathrm{F}_0(\mathcal{U}(\mathfrak{g})) \otimes (-)$ and $\mathrm{F}_0(\mathcal{U}(N(\mathfrak{g}))) \otimes (-)$ preserve colimits, so that $\mathfrak{g}-\mathrm{Rep}$ and $N(\mathfrak{g})-\mathrm{Rep}^\Sigma$ are locally presentable [Bor94, Theorem 5.5.9], with limits and colimits preserved by the respective forgetful functors. In particular, $N^*_\mathfrak{g}$ preserves filtered colimits and so admits a left adjoint $N^\mathfrak{g}_!$ by the Adjoint Functor Theorem.

By Theorem 2.4.6 and Lemma 2.4.7, $N^*_\mathfrak{g}$ is right Quillen and moreover preserves and reflects stable weak equivalences. To show that $(N^\mathfrak{g}_! \dashv N^*_\mathfrak{g})$ is a Quillen equivalence we show that the derived unit is a natural isomorphism using our standard argument. For this, we need only show the component of the derived unit at $\mathrm{F}_0(\mathcal{U}(N(\mathfrak{g}))$ is a weak equivalence since $\mathrm{F}_0(\mathcal{U}(N(\mathfrak{g}))$ presents a weak compact generator of the homotopy category. The derived unit at $\mathrm{F}_0(\mathcal{U}(N(\mathfrak{g}))$ is presented by the map

$$\mathrm{F}_0(\mathcal{U}(N(\mathfrak{g})) \longrightarrow N^*_\mathfrak{g} N^\mathfrak{g}_!(\mathrm{F}_0(\mathcal{U}(N(\mathfrak{g}))) \cong N^*_\mathfrak{g}(\mathrm{F}_0(\mathcal{U}(\mathfrak{g})) \cong \mathrm{F}_0(N(\mathcal{U}(\mathfrak{g})))$$

obtained by applying the left Quillen functor $\mathrm{F}_0\colon ch_+ \to \mathrm{Sp}^\Sigma(ch_+)$ to the weak equivalence $\chi_\mathfrak{g}\colon \mathcal{U}(N\mathfrak{g})) \to N(\mathcal{U}(\mathfrak{g}))$ of Lemma 2.4.14, and so is a weak equivalence by Ken Brown's Lemma.

The right-hand adjunction is slightly simpler. We write $\mathfrak{n} := N(\mathfrak{g})$ to ease the notation slightly. In the proof of Lemma 2.4.10 we showed that the assembly functor

$\mathcal{A}\colon \mathrm{Sp}^{\Sigma}(ch_+) \to ch$ is strong symmetric monoidal and sends $\mathrm{F}_n(M) \mapsto M[n]$ for all $M \in ch_+$ and $n \geq 0$. Consequently, $\mathcal{A}$ sends any left $\mathrm{F}_0(\mathcal{U}(\mathfrak{n})$-module spectrum M to a left $\mathcal{U}(\mathfrak{n}))$-module $\mathcal{A}(M)$. Let us write $\mathcal{A}_{\mathfrak{n}}\colon \mathfrak{n}-\mathrm{Rep}^{\Sigma} \to \mathfrak{n}-\mathrm{Rep}$ for the functor so obtained. The functor $\mathcal{A}_{\mathfrak{n}}$ has right adjoint $\mathcal{D}_{\mathfrak{n}}$ coinciding with $\mathcal{D}\colon ch \to \mathrm{Sp}^{\Sigma}(ch_+)$ on underlying objects—this is due to the fact if M is a (left) module over a *connective* dg algebra A, then the connective cover $\mathrm{cn}_0 M$ inherits a left A-module structure (as is easily checked directly). Since weak equivalences and fibrations in $\mathfrak{n}-\mathrm{Rep}^{\Sigma}$ and $\mathfrak{n}-\mathrm{Rep}$ are created by the respective forgetful functors, the adjunction $(\mathcal{A}_{\mathfrak{n}} \dashv \mathcal{D}_{\mathfrak{n}})$ is Quillen.

In the proof of Theorem 2.4.13 we showed that $\mathcal{D}$ preserves and reflects weak equivalences. To conclude that $(\mathcal{A}_{\mathfrak{n}} \dashv \mathcal{D}_{\mathfrak{n}})$ is a Quillen equivalence by our standard argument, we need only check that the derived unit at $\mathcal{U}(\mathfrak{n})$ is an isomorphism since this object presents a weak compact generator of the homotopy category. But the derived unit at $\mathcal{U}(\mathfrak{n})$ is presented by the ordinary unit, which is an isomorphism in this case. □

Remark 2.4.17. In the above proof, the right adjoint functors are determined by $N^*\colon \mathrm{Sp}^{\Sigma}(\mathrm{sVect}_{\mathbb{Q}}) \to \mathrm{Sp}^{\Sigma}(ch_+)$ and $\mathcal{D}\colon ch \to \mathrm{Sp}^{\Sigma}(ch_+)$ on underlying objects. If $f\colon \mathfrak{g} \to \mathfrak{h}$ is a map of simplicial Lie algebras, the naturality clause of Lemma 2.4.14 therefore implies that we have a diagram of right adjoint functors

$$\begin{tikzcd}
\mathfrak{g}-\mathrm{Rep} \arrow[r] & N(\mathfrak{g})-\mathrm{Rep}^{\Sigma} & N(\mathfrak{g})-\mathrm{Rep} \arrow[l] \\
\mathfrak{h}-\mathrm{Rep} \arrow[r] \arrow[u, "\mathcal{U}(f)^*"] & N(\mathfrak{h})-\mathrm{Rep}^{\Sigma} \arrow[u, "\mathcal{U}(N(f))^*"] & N(\mathfrak{h})-\mathrm{Rep} \arrow[l] \arrow[u, "\mathcal{U}(N(f))^*"']
\end{tikzcd}$$

which commutes up to natural isomorphism, where the vertical arrows are restriction of scalars.

2.4.3 Monoidal Structures and Base Change

In this section we show that the zig-zag of Quillen equivalences of Theorem 2.4.16 descends to a zig-zag of strong monoidal equivalences of homotopy categories. In §2.3.4 we proved that the category of spectral representations $\mathfrak{g}-\mathrm{Rep} = \mathcal{U}(\mathfrak{g})-\mathrm{Rep}$ is a monoidal model category for any simplicial Lie algebra $\mathfrak{g}$. Our first goal is to extend this to the stable model categories of representations $\mathfrak{h}-\mathrm{Rep}^{\Sigma}$ and $\mathfrak{h}-\mathrm{Rep}$ for $\mathfrak{h}$ a dg Lie algebra. For this we shall need the following

Lemma 2.4.18 (Fundamental Theorem of dg Hopf Modules)**.** *Let H be a dg Hopf algebra and M a left Hopf module over H. Then M is free.*

Proof. The argument is a dg version of the proof of [LS69, Proposition 1]. If κ is the H-coaction on M, let $M_H := \{m \mid \kappa(m) = 1 \otimes m\}$ be the complex of H-*coinvariants* of M. The map $H \otimes M_H \to M$ sending $h \otimes m \mapsto h \cdot m$ has inverse

$$m \longmapsto \sum m_{(0)} \otimes S(m_{(1)}) \cdot m_{(2)},$$

where S is the antipode. The isomorphism of complexes $H \otimes M_H \cong M$ respects the H-action and H-coaction, so that M is free. □

Corollary 2.4.19. *Let H a dg Hopf algebra. The tensor product of free H-modules is free.*

Proof. This follows from the Lemma, since $H \otimes H \cong H \otimes (H \otimes H)_H$ is free. □

Lemma 2.4.20. *For H a connective dg Hopf algebra, the tensor product of left H-modules induces symmetric monoidal model structures on the categories*

(1) $H{-}\mathrm{Mod}_+$ *of connective left H-modules;*

(2) $H{-}\mathrm{Mod}$ *of unbounded left H-modules; and*

(3) $H{-}\mathrm{Mod}^\Sigma := \mathrm{Sp}^\Sigma(H{-}\mathrm{Mod}_+; s)$ *of symmetric spectra of connective H-modules.*

Proof. The model structures are obtained by transfer, as in Remark 2.4.15; we focus only on the monoidal structures. Note that the monoidal structures are closed symmetric in each case by our standard arguments.

The proofs of (1) and (2) are essentially the same, so we only provide the argument for (2). The sphere and disk inclusions $\mathcal{I}$ and $\mathcal{J}$ of Remark 2.4.11 furnish generating cofibrations and acyclic cofibrations for ch, which by transfer become generating sets $\mathcal{I}^H := \{H \otimes i \mid i \in \mathcal{I}\}$ and $\mathcal{J}^H := \{H \otimes i \mid i \in \mathcal{J}\}$ for the model structure on $H{-}\mathrm{Mod}$. Arguing as in Lemmas 2.3.28 and 2.3.33, we verify the pushout-product axiom on the sets $\mathcal{I}^H$ and $\mathcal{J}^H$ using Corollary 2.4.19. The full pushout-product axiom holds by cofibrant generation. Noting that H is cofibrant when regarded as an object of ch, the forgetful functor $H{-}\mathrm{Mod} \to ch$ preserves cofibrations by the pushout-product axiom for $\otimes$ on ch. It follows that for cofibrant $M \in H{-}\mathrm{Mod}$, the tensor product $(-) \otimes M$ preserves weak equivalences between cofibrant chain complexes. In particular, if $\mathbb{Q}^H \to \mathbb{Q}$ is a cofibrant resolution of the monoidal unit in $H{-}\mathrm{Mod}$ then $\mathbb{Q}^H \otimes M \to \mathbb{Q} \otimes M = M$ is a weak equivalence for all cofibrant M, verifying the unit axiom.

Applying the symmetric stabilisation machine to (1) proves (3). $\square$

We now come to the main result of this section:

Theorem 2.4.21. *For any $\mathfrak{g} \in \mathrm{sLie}$, there is a zig-zag of left Quillen equivalences*

$$\mathfrak{g}{-}\mathrm{Rep} \xleftarrow{\ 1\ } N(\mathfrak{g}){-}\mathrm{Rep}^\Sigma \xrightarrow{\ 2\ } N(\mathfrak{g}){-}\mathrm{Rep}$$

in which 1 is weak monoidal and 2 is strong monoidal.

Proof. We deal with 2 first: in Theorem 2.4.16, we showed that the left Quillen equivalence $\mathcal{A}_{N(\mathfrak{g})} \colon N(\mathfrak{g}){-}\mathrm{Rep}^\Sigma \to N(\mathfrak{g}){-}\mathrm{Rep}$ coincides with $\mathcal{A} \colon \mathrm{Sp}^\Sigma(ch_+) \to ch$ on underlying objects, so is strong monoidal by Lemma 2.4.10.

For 1, recall that the right Quillen functor $N^*_{\mathfrak{g}}$ is given on underlying objects by the lax monoidal functor $N^* \colon \mathrm{Sp}^\Sigma(\mathrm{sVect}_\mathbb{Q}) \to \mathrm{Sp}^\Sigma(ch_+)$. The left adjoint $N^{\mathfrak{g}}_!$ is therefore oplax monoidal by an adjunction argument, with lax monoidal structure written as

$$\Lambda_{A,B} \colon N^{\mathfrak{g}}_!(A \otimes B) \longrightarrow N^{\mathfrak{g}}_!(A) \otimes N^{\mathfrak{g}}_!(B)$$

for $A, B \in N(\mathfrak{g}){-}\mathrm{Rep}^\Sigma$. Writing $\mathfrak{n} := N(\mathfrak{g})$ for brevity, for each $n \in \mathbb{N}$ we have a diagram of right adjoint functors commuting up to natural isomorphism:

$$\begin{tikzcd}
\mathfrak{g}{-}\mathrm{Rep} \arrow[r, "\mathrm{Ev}_n"] \arrow[d, "N^*_{\mathfrak{g}}"'] & \mathfrak{g}{-}\mathrm{Rep}_{\geq 0} \arrow[r] \arrow[d, "N"'] & \mathrm{sVect}_\mathbb{Q} \arrow[d, "N"] \\
\mathfrak{n}{-}\mathrm{Rep}^\Sigma \arrow[r, "\mathrm{Ev}_n"] & \mathfrak{n}{-}\mathrm{Rep}_+ \arrow[r] & ch_+ .
\end{tikzcd}$$

Essential uniqueness of adjoints implies $N_!^{\mathfrak{g}}(\mathrm{F}_n(\mathcal{U}(\mathfrak{n})\otimes V))\cong \mathrm{F}_n(\mathcal{U}(\mathfrak{g})\otimes\Gamma(V))$ for $V\in ch_+$, $n\in\mathbb{N}$. Take cofibrant objects of $\mathfrak{n}-\mathrm{Rep}^\Sigma$ of the form $A=\mathrm{F}_m(\mathcal{U}(\mathfrak{n})\otimes M)$ and $B=\mathrm{F}_n(\mathcal{U}(\mathfrak{n})\otimes N)$ for $m,n\in\mathbb{N}$ and $M,N\in ch_+$, so that

$$A\otimes B\cong \mathrm{F}_{m+n}\Big(\mathcal{U}(\mathfrak{n})\otimes\big((\mathcal{U}(\mathfrak{n})\otimes\mathcal{U}(\mathfrak{n}))_{\mathcal{U}(\mathfrak{n})}\otimes M\otimes N\big)\Big)$$

by Corollary 2.4.19. The oplax structure map $\Lambda_{A,B}$ is then isomorphic to a map

$$\mathrm{F}_{m+n}\Big(\mathcal{U}(\mathfrak{g})\otimes\Gamma\big((\mathcal{U}(\mathfrak{n})\otimes\mathcal{U}(\mathfrak{n}))_{\mathcal{U}(\mathfrak{n})}\otimes M\otimes N\big)\Big)\longrightarrow \mathrm{F}_{m+n}\big(\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})\otimes\Gamma(M)\otimes\Gamma(N)\big)$$

in the image of $\mathrm{F}_{m+n}\colon\mathfrak{g}-\mathrm{Rep}_{\geq 0}\to\mathfrak{g}-\mathrm{Rep}$. By Ken Brown's Lemma it is sufficient to show that the map of simplicial $\mathbb{Q}$-vector spaces

$$\mathcal{U}(\mathfrak{g})\otimes\Gamma\big((\mathcal{U}(\mathfrak{n})\otimes\mathcal{U}(\mathfrak{n}))_{\mathcal{U}(\mathfrak{n})}\otimes M\otimes N\big)\longrightarrow\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})\otimes\Gamma(M)\otimes\Gamma(N)\tag{2.9}$$

is a weak equivalence. Passing to normalised chain complexes, via the shuffle map the map (2.9) is weakly equivalent to the horizontal morphisms in the commuting diagram

$$\begin{tikzcd}
N(\mathcal{U}(\mathfrak{g}))\otimes(\mathcal{U}(\mathfrak{n})\otimes\mathcal{U}(\mathfrak{n}))_{\mathcal{U}(\mathfrak{n})}\otimes M\otimes N \arrow[r] \arrow[d, "\chi_{\mathfrak{g}}\otimes\mathrm{id}"'] & N(\mathcal{U}(\mathfrak{g}))\otimes N(\mathcal{U}(\mathfrak{g}))\otimes M\otimes N \arrow[d, "\chi_{\mathfrak{g}}\otimes\chi_{\mathfrak{g}}\otimes\mathrm{id}"] \\
\mathcal{U}(\mathfrak{n})\otimes(\mathcal{U}(\mathfrak{n})\otimes\mathcal{U}(\mathfrak{n}))_{\mathcal{U}(\mathfrak{n})}\otimes M\otimes N \arrow[r] & \mathcal{U}(\mathfrak{n})\otimes\mathcal{U}(\mathfrak{n})\otimes M\otimes N.
\end{tikzcd}$$

But the bottom horizontal map is an isomorphism by Corollary 2.4.19, so that the image of (2.9) by the normalisation functor N is a weak equivalence. Since N reflects weak equivalences, we have that (2.9) and hence $\Lambda_{A,B}$ are weak equivalences.

We have verified $\Lambda_{A,B}$ is a weak equivalence whenever A and B are domains or codomains of generating cofibrations for $\mathfrak{n}-\mathrm{Rep}^\Sigma$. The argument of Theorem 2.3.36 now shows that $\Lambda_{A,B}$ is a weak equivalence for all cofibrant A and B.

To complete the proof, we must show that if $\mathbb{Q}^{\mathfrak{n}}\to\mathrm{F}_0(\mathbb{Q})$ is some cofibrant replacement for the monoidal unit in $\mathfrak{n}-\mathrm{Rep}^\Sigma$, then $N_!^{\mathfrak{g}}(\mathbb{Q}^{\mathfrak{n}})\to\mathrm{F}_0(\mathbb{Q})$ is a stable weak equivalence. For this, we shall need the dg analogue of Remark 2.3.34, which provides a cofibrant replacement for $\mathbb{Q}\in\mathfrak{n}-\mathrm{Rep}_+$ as the colimit of the countable sequence

$$X_0^{\mathfrak{n}}=\mathcal{U}(\mathfrak{n})\longrightarrow X_1^{\mathfrak{n}}\longrightarrow\cdots,$$

in which $X_n^{\mathfrak{n}}$ is obtained from $X_{n-1}^{\mathfrak{n}}$ as a pushout

$$\begin{tikzcd}
N(\mathbb{Q}[\partial\Delta[n]])\otimes\mathcal{U}(\mathfrak{n})^{\otimes(n+1)} \arrow[r] \arrow[d] & X_{n-1}^{\mathfrak{n}} \arrow[d] \\
N(\mathbb{Q}[\Delta[n]])\otimes\mathcal{U}(\mathfrak{n})^{\otimes(n+1)} \arrow[r] & X_n^{\mathfrak{n}}.
\end{tikzcd}$$

Applying $N_!^{\mathfrak{g}} \colon \mathfrak{n}-\mathrm{Rep}_+ \to \mathfrak{g}-\mathrm{Rep}_{\geq 0}$ to each of these pushout diagrams yields a pushout diagram

$$\begin{tikzcd} \mathcal{U}(\mathfrak{g}) \otimes \Gamma\left(N(\mathbb{Q}[\partial\Delta[n]]) \otimes \mathcal{U}(\mathfrak{n})^{\otimes(n+1)}_{\mathcal{U}(\mathfrak{n})}\right) \arrow[r] \arrow[d] & N_!^{\mathfrak{g}}(X^{\mathfrak{n}}_{n-1}) \arrow[d] \\ \mathcal{U}(\mathfrak{g}) \otimes \Gamma\left(N(\mathbb{Q}[\partial\Delta[n]]) \otimes \mathcal{U}(\mathfrak{n})^{\otimes(n+1)}_{\mathcal{U}(\mathfrak{n})}\right) \arrow[r] & N_!^{\mathfrak{g}}(X^{\mathfrak{n}}_{n}). \end{tikzcd}$$

Employing the same argument that we used to show that (2.9) is a weak equivalence, we have weak equivalences

$$\mathcal{U}(\mathfrak{g}) \otimes \Gamma\left(N(\mathbb{Q}[K]) \otimes \mathcal{U}(\mathfrak{n})^{\otimes(n+1)}_{\mathcal{U}(\mathfrak{n})}\right) \longrightarrow \mathbb{Q}[K] \otimes \mathcal{U}(\mathfrak{g})^{\otimes(n+1)}$$

for all $K \in \mathrm{sSet}$ and $n \in \mathbb{N}$. As in the proof of Theorem 2.3.36, these maps and the Cube Lemma give rise to a weak equivalence $N_!^{\mathfrak{g}}(X_k^{\mathfrak{n}}) \to X_k^{\mathfrak{g}}$. Taking colimits over $k \in \mathbb{N}$ we obtain a weak equivalence

$$\operatorname*{colim}_k N_!^{\mathfrak{g}}(X_k^{\mathfrak{n}}) \longrightarrow \mathbb{Q}^{\mathcal{U}(\mathfrak{g})}, \tag{2.10}$$

with $\mathbb{Q}^{\mathcal{U}(\mathfrak{g})} \to \mathbb{Q}$ as in Remark 2.3.34. By Ken Brown's Lemma, the morphisms $\mathrm{F}_0(\mathrm{colim}_k X_k^{\mathfrak{n}}) \to \mathrm{F}_0(\mathbb{Q})$ and $\mathrm{F}_0(\mathbb{Q}^{\mathcal{U}(\mathfrak{g})} \to \mathrm{F}_0(\mathbb{Q})$ are cofibrant resolutions of the units in $\mathfrak{n}-\mathrm{Rep}^{\Sigma}$ and $\mathfrak{g}-\mathrm{Rep}$ respectively. Applying $\mathrm{F}_0 \colon \mathfrak{g}-\mathrm{Rep}_{\geq 0} \to \mathfrak{g}-\mathrm{Rep}$ to (2.10) now completes the proof. □

We conclude this section by recording a monoidal base change result for model categories of unbounded Lie representations. This result is the dg analogue of Theorem 2.3.36.

Theorem 2.4.22. *Let $f \colon \mathfrak{h} \to \mathfrak{k}$ be a weak equivalence of connective dg Lie algebras. Then there is a weak monoidal Quillen equivalence*

$$\begin{tikzcd} \mathfrak{h}-\mathrm{Rep} \arrow[r, shift left, "\mathcal{U}(f)_!"] \arrow[r, phantom, "\perp"] & \mathfrak{k}-\mathrm{Rep} \arrow[l, shift left, "\mathcal{U}(f)^*"] \end{tikzcd}$$

on model categories of unbounded representations.

Proof. The weak equivalence f induces a weak equivalence $\mathcal{U}(f) \colon \mathcal{U}(\mathfrak{h}) \to \mathcal{U}(\mathfrak{k})$ on universal enveloping algebras by Lemma 2.4.23. Restriction and extension of scalars $(\mathcal{U}(f)_! \dashv \mathcal{U}(f)^*)$ is seen to be a weak monoidal Quillen equivalence by transplanting the proof of Theorem 2.3.36 to the dg setting. □

Lemma 2.4.23. *A weak equivalence $f \colon \mathfrak{g} \to \mathfrak{h}$ of dg Lie algebras induces a weak equivalence $\mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})$.*

Proof. For a chain complex V over $\mathbb{Q}$, the Künneth theorem implies that there is a natural isomorphism $\mathrm{T}(H_\bullet(V)) \to H_\bullet(\mathrm{T}V)$, where T denotes the tensor algebra functor. As we are working over $\mathbb{Q}$, the quotient map $\mathrm{T}V \to \mathrm{Sym}(V)$ admits a section, so that the retract diagram

$$\begin{tikzcd}
\mathrm{Sym}(H_\bullet(V)) \arrow[r] \arrow[d] & \mathrm{T}(H_\bullet(V)) \arrow[r] \arrow[d, "\wr"] & \mathrm{Sym}(H_\bullet(V)) \arrow[d] \\
H_\bullet(\mathrm{Sym}(V)) \arrow[r] & H_\bullet(\mathrm{T}V) \arrow[r] & H_\bullet(\mathrm{Sym}(V))
\end{tikzcd}$$

implies that $\mathrm{Sym}(H_\bullet(V)) \to H_\bullet(\mathrm{Sym}(V))$ is an isomorphism. By the PBW Theorem for dg Lie algebras, we have a commuting diagram

$$\begin{tikzcd}
 & \mathrm{Sym}(H_\bullet(\mathfrak{h})) \arrow[rr] \arrow[dd] & & \mathcal{U}(H_\bullet(\mathfrak{h})) \arrow[dd] \\
\mathrm{Sym}(H_\bullet(\mathfrak{g})) \arrow[ur, "\sim"] \arrow[rr] \arrow[dd] & & \mathcal{U}(H_\bullet(\mathfrak{g})) \arrow[ur, "\sim"] \arrow[dd] & \\
 & H_\bullet(\mathrm{Sym}(\mathfrak{h})) \arrow[rr] & & H_\bullet(\mathcal{U}(\mathfrak{h})) \\
H_\bullet(\mathrm{Sym}(\mathfrak{g})) \arrow[ur] \arrow[rr] & & H_\bullet(\mathcal{U}(\mathfrak{g})) \arrow[ur] &
\end{tikzcd}$$

in which the horizontal morphisms are isomorphisms of graded vector spaces. The assertion follows. $\square$

2.5 Koszul Duality

The final step in Quillen's chain of rational homotopy theory adjunctions identifies the homotopy theory of reduced rational dg Lie algebras with the homotopy theory of 2-reduced dg coalgebras. In this section we prove an associated Koszul duality result, which identifies Lie representations with comodules over the corresponding coalgebra. While this sort of result is by no means new, we were unable to find a precise result of the kind needed for our discussion of rational parametrised spectra in the literature. Our treatment is similar to that of [Pos11].

2.5.1 Twisting Modules and Comodules

Throughout this section, C denotes a cocommutative dg coalgebra over $\mathbb{Q}$ whose underlying chain complex is connective. The comultiplication and counit maps are denoted by Δ and ε, and we shall often make use of the Sweedler notation.

Construction 2.5.1. Let $\eta\colon \mathbb{Q} \to C$ be a coaugmentation of the coalgebra C. Write $\overline{C} := \mathrm{coker}(\eta) \cong \ker(\varepsilon)$ and $\pi\colon C \to \overline{C}$ for the quotient map. The *cobar construction* of C is the augmented dg algebra ΩC with underlying graded algebra the tensor algebra $\mathrm{T}(\overline{C}[1])$, equipped with differential defined on generators $tc \equiv \pi c[1]$ by

$$d(tc) := -t(dc) - \sum (-1)^{|c_{(0)}|} tc_{(0)} \otimes tc_{(1)}.$$

The augmentation $\Omega C \to \mathbb{Q}$ is determined by sending the generators $tc \mapsto 0$. The cobar construction ΩC is connective precisely if C is reduced.

Remark 2.5.2. For a dg coalgebra C, the *universal twisting chain* $t\colon C \rightsquigarrow \Omega C$ is the degree -1 map of *graded* $\mathbb{Q}$-vector spaces which sends $c \mapsto tc$. It is the algebraic analogue of the twisting function of Construction B.1.4.

More generally, for A a dg algebra, a *twisting chain* $\tau\colon C \rightsquigarrow A$ is a degree-(-1) map of graded $\mathbb{Q}$-vector spaces such that

$$d(\tau(c)) = -\tau(dc) - \sum(-1)^{|c_{(0)}|}\tau(c_{(0)}) \cdot \tau(c_{(1)}).$$

There is a natural isomorphism between $\mathrm{Twist}(C, A) := \{C \rightsquigarrow A\}$ and the set of maps of dg algebras $\Omega C \to A$. Under this isomorphism, the universal twisting chain corresponds to the identity map on ΩC.

Construction 2.5.3. Let N be a left C-comodule with C-coaction $\rho\colon N \to C \otimes N$. For a twisting chain $\tau\colon C \rightsquigarrow A$, the *$\tau$-twisted extension* of N is the left A-module $A \otimes_\tau N$ with underlying graded vector space $A \otimes N$ equipped with differential defined on homogeneous elements by

$$d(a \otimes n) := da \otimes n + (-1)^{|a|} a \otimes dn - \sum(-1)^{|a|} a \cdot \tau(n_{(0)}) \otimes n_{(1)},$$

where $\rho(n) = \sum n_{(0)} \otimes n_{(1)} \in C \otimes N$ in the Sweedler notation. The induced left A-action respects the differentials, so that forming twisted extensions gives rise to a functor $A \otimes_\tau (-)\colon C\mathrm{-Comod} \to A\mathrm{-Mod}$.

Conversely, for M a left A-module we construct its *τ-twisted extension* as the left C-comodule $C \otimes_\tau M$ which is $C \otimes M$ equipped with the differential

$$d(c \otimes m) := dc \otimes m + (-1)^{|c|} c \otimes dm + (-1)^{|c|} \sum(-1)^{|c_{(0)}|} c_{(0)} \otimes \tau(c_{(1)}) \cdot m.$$

The induced C-coaction on $C \otimes_\tau M$ respects the differentials, so that we have a functor $C \otimes_\tau (-)\colon A\mathrm{-Mod} \to C\mathrm{-Comod}$.

Remark 2.5.4. In the case that τ is the universal twisting chain $t\colon C \rightsquigarrow \Omega C$ and C is reduced, forming t-twisted extensions preserves connectivity. For $k \in \mathbb{Z}$, write $C\mathrm{-Comod}_{\geq k}$ and $\Omega C\mathrm{-Mod}_{\geq k}$ respectively for the full subcategories of C-comodules and ΩC-modules whose underlying complexes are concentrated in degrees $[k, \infty]$. Then we have $C \otimes_t (-)\colon \Omega C\mathrm{-Mod}_{\geq k} \to C\mathrm{-Comod}_{\geq k}$ since C is connective and conversely $\Omega C \otimes_t (-)\colon C\mathrm{-Comod}_{\geq k} \to \Omega C\mathrm{-Mod}_{\geq k}$ since ΩC is connective when C is reduced.

Lemma 2.5.5. *For any twisting chain $t\colon C \rightsquigarrow A$, there is an adjunction*

$$\begin{tikzcd} C\mathrm{-Comod} \arrow[r, shift left, "A\otimes_\tau(-)"] \arrow[r, phantom, "\perp"] & A\mathrm{-Mod}. \arrow[l, shift left, "C\otimes_\tau(-)"] \end{tikzcd}$$

Moreover, when C is reduced and $\tau = t$ is the universal twisting chain, this adjunction restricts to adjunctions between full subcategories of k-connected objects for all $k \in \mathbb{Z}$ by Remark 2.5.4.

Proof. For $N \in C\mathrm{-Comod}$ and $M \in A\mathrm{-Mod}$, a *modular twisting chain* $N \rightsquigarrow M$ is a map of graded vector spaces $\theta\colon N \to M$ such that

$$d\theta(n) = \theta(dn) - \sum \tau(n_{(0)}) \cdot \theta(n_{(1)}).$$

Write $\mathrm{ModTwist}(N, M)$ for the vector space of modular twisting functions $N \rightsquigarrow M$, so that we have a functor

$$\mathrm{ModTwist}(-,-)\colon C\mathrm{-Comod}^{\mathrm{op}} \times A\mathrm{-Mod} \longrightarrow \mathrm{Vect}_{\mathbb{Q}}.$$

For $f\colon A \otimes_\tau N \to M$ a map of A-modules, let $\theta(f)$ be the composite

$$N \xrightarrow{\eta\otimes\mathrm{id}} A \otimes N \xrightarrow{f} M,$$

where $\eta\colon \mathbb{Q} \to A$ is the unit. The map of graded vector spaces $\theta(f)$ is a modular twisting function. Conversely, given $\theta\colon N \rightsquigarrow M$ we define a map of A-modules $f(\theta)\colon A \otimes_\tau N \to M$ by

$$f(\theta)\colon a \otimes n \longrightarrow a \cdot \theta(n).$$

The map $f(\theta)$ respects the differentials, and the assignments $\theta \mapsto f(\theta)$ and $f \mapsto \theta(f)$ are inverse to one another. We conclude that there is a natural isomorphism

$$A\mathrm{-Mod}\big(A \otimes_\tau N, M\big) \cong \mathrm{ModTwist}(N, M).$$

On the other hand, forgetting the differentials $C \otimes_\tau M = C \otimes M$ is the cofree C-comodule cogenerated by M so that any map of C-comodules $g\colon N \to C \otimes_\tau M$ is necessarily of the form

$$n \longmapsto \sum n_{(0)} \otimes \theta(g)(n_{(1)})$$

for some map of graded vector spaces $\theta(g)\colon N \to M$. A straightforward check shows that g respects the differentials precisely if $\theta(g)$ is a modular twisting function, so that we have a natural isomorphism

$$\mathrm{ModTwist}(N, M) \cong C\mathrm{-Comod}\big(N, C \otimes_\tau M\big).$$

This establishes the desired adjunction at the level of homsets. $\square$

The remainder of this section is devoted to proving various relationships between the homology groups of modules, comodules and their twisted extensions. We restrict to the case that C is 2-reduced, for which our spectral sequences have particularly nice E^2-pages.

Lemma 2.5.6. *Let C be 2-reduced, A a connective dg algebra and $\tau\colon C \rightsquigarrow A$ a twisting chain. For M a left A-module whose underlying chain complex is bounded below, there is a convergent spectral sequence*

$$E^2_{p,q} = H_p(C) \otimes H_q(M) \Longrightarrow H_{p+q}(C \otimes_\tau M).$$

Proof. Consider the filtration $\mathcal{F}_p C := \bigoplus_{n \le p} C_n$ which induces the differential filtration $\mathcal{F}_p\overline{M} := \mathcal{F}_p C \otimes M$ of $C \otimes_\tau M$. Since $\tau(1) = 0$ by connectivity of A and C is 2-reduced, in the formula for the differential on $C \otimes_\tau M$

$$d(c \otimes m) := dc \otimes m + (-1)^{|c|} c \otimes dm + (-1)^{|c|} \sum (-1)^{|c_{(0)}|} c_{(0)} \otimes \tau(c_{(1)}) \cdot m$$

the last summand is a sum over terms $c_i \otimes m_i$ for which $|c_i| \le |c| - 2$. If $|c| = p$ we therefore have

$$d(c \otimes m) - dc \otimes m - (-)^{|c|} c \otimes dm \in \mathcal{F}_{p-2}\overline{M}.$$

We compute spectral sequence of the filtration $\mathcal{F}_\bullet\overline{M}$, for which $E^1_{p,q} \cong C_p \otimes H_q(M)$ and $E^2_{p,q} \cong H_p(C) \otimes H_q(M)$. As M is bounded below, the spectral sequence is bounded and hence convergent. $\square$

Lemma 2.5.7. *Let C be 2-reduced, A a connective dg algebra and $\tau\colon C \rightsquigarrow A$ a twisting chain. For N a left C-comodule whose underlying chain complex is bounded below, there is a convergent spectral sequence*

$$E^2_{p,q} = H_p(A) \otimes H_q(N) \Longrightarrow H_{p+q}(A \otimes_\tau N).$$

Proof. Filter N by $\mathcal{F}_pN := \bigoplus_{n\leq p} N_n$, which induces the differential filtration

$$\mathcal{F}_p\overline{N} := A \otimes \mathcal{F}_pN$$

of $A \otimes_\tau N$. According to the formula for the differential

$$d(a \otimes n) := da \otimes n + (-1)^{|a|} a \otimes n - \sum (-1)^{|a|} a \cdot \tau(n_{(0)}) \otimes n_{(1)},$$

if $|n| = p$ then the last summand $\sum a \cdot \tau(n_{(0)}) \otimes n_{(1)} \in \mathcal{F}_{p-2}\overline{N}$ as $\tau(1) = 0$ due to connectivity of A and C is 2-reduced. Computing the spectral sequence of the filtration, we therefore have $E^1_{p,q} \cong H_p(A) \otimes N_q$ and $E^2_{p,q} \cong H_p(A) \otimes H_q(N)$. By our assumptions on A and N, the spectral sequence is bounded and hence convergent. $\square$

In the case that $\tau = t\colon C \rightsquigarrow \Omega C$ is the universal twisting chain, we shall also need the following results pertaining to the unit and counit of the twisting adjunction between C-comodules and ΩC-modules:

Lemma 2.5.8. *The unit map $N \to C \otimes_t \Omega C \otimes_t N$ is a weak equivalence for all C-comodules N.*

Proof. Explicitly, the unit sends $n \mapsto \sum n_{(0)} \otimes 1 \otimes n_{(1)}$, with $n \mapsto \sum n_{(0)} \otimes n_{(1)}$ the C-coaction. For $\pi\colon C \to \overline{C}$ as in Construction 2.5.1, consider the endomorphism Þ of $C \otimes_t \Omega C \otimes_t N$ defined on homogeneous elements by

$$Þ\colon c \otimes (tc_1 \otimes \cdots \otimes tc_k) \otimes n \longmapsto \begin{cases} 0 & \text{for } k = 0 \\ (-1)^{|c|}\varepsilon(c)\pi(c_1) \otimes (tc_2 \otimes \cdots \otimes tc_k) \otimes n & \text{otherwise.} \end{cases}$$

An explicit calculation shows that $Þd + dÞ = \mathrm{id} - þ$, where

$$þ\colon c \otimes (tc_1 \otimes \cdots \otimes tc_k) \otimes n \longmapsto \begin{cases} \sum \varepsilon(c) n_{(0)} \otimes 1 \otimes n_{(1)} & \text{for } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The unit $N \to C \otimes_t \Omega C \otimes_t N$ maps isomorphically onto the image of þ, from which the result follows. $\square$

Lemma 2.5.9. *Let C be 2-reduced. Then for any $k \in \mathbb{Z}$ and k-connected ΩC-module M, the counit $\Omega C \otimes_t C \otimes_t M \to M$ is a weak equivalence.*

Proof. We first remark that the counit map $\varrho_M\colon \Omega C \otimes_t C \otimes_t M \to M$ is given explicitly on homogeneous elements by

$$(tc_1 \otimes \cdots \otimes tc_k) \otimes c \otimes m \longmapsto tc_1 \cdots tc_k \cdot \varepsilon(c)m.$$

The cobar construction ΩC coincides with the universal enveloping algebra $\mathcal{U}(\mathcal{L}(C))$ of the reduced dg Lie algebra $\mathcal{L}(C)$ that Quillen's rational homotopy theory assigns to the 2-reduced coalgebra C [Qui69, Remark B.6.6]. To any (reduced) dg Lie algebra L, Quillen also assigns a dg coalgebra $\mathcal{C}(L) \cong \mathrm{Sym}(L[-1])$ which comes equipped with a universal twisting chain $t_L \colon \mathcal{C}(L) \rightsquigarrow \mathcal{U}(L)$

$$t_L \colon x_1[-1] \cdots x_q[-1] \longmapsto \begin{cases} x_1 & \text{for } q = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Writing $\mathfrak{L} := \mathcal{L}(C)$, by Lemma 2.5.5 we therefore have an adjunction

$$\begin{tikzcd} \mathcal{C}(\mathfrak{L})\text{−Comod} \arrow[r, shift left, "\Omega C \otimes_{t_{\mathfrak{L}}} (-)"] \arrow[r, phantom, "\perp"] & \Omega C\text{−Mod} \arrow[l, shift left, "\mathcal{C}(\mathfrak{L}) \otimes_{t_{\mathfrak{L}}} (-)"] \end{tikzcd}$$

and hence a counit map $\varpi_M \colon \Omega C \otimes_{t_{\mathfrak{L}}} \mathcal{C}(\mathfrak{L}) \otimes_{t_{\mathfrak{L}}} M \to M$ for any ΩC-module M. Writing elements of $\mathfrak{L}$ as x, we write elements of $\mathcal{C}(\mathfrak{L})$ as $X^K := x_{i_1}[-1] \cdots x_{i_k}[-1]$. The counit map ϖ_M then sends $x \otimes X^K \otimes m \mapsto x \cdot \varepsilon(X^K) m$, and by unravelling the definitions the differential on $\Omega C \otimes_{t_{\mathfrak{L}}} \mathcal{C}(\mathfrak{L}) \otimes_{t_{\mathfrak{L}}} M$ is seen to act on homogeneous elements by

$$\begin{aligned} d(x \otimes X^K \otimes m) = dx \otimes X^K \otimes m - (-1)^{|x|} \textstyle\sum x \cdot t_{\mathfrak{L}}(X^K_{(0)}) \otimes X^K_{(1)} \otimes M \\ + (-1)^{|x|} x \otimes dX^K \otimes m + (-1)^{|x|+|X^K|} x \otimes X^K \otimes dm \\ - \textstyle\sum (-1)^{|x|+|X^K_{(0)}|} x \otimes X^K_{(0)} \otimes t_{\mathfrak{L}}(X^K_{(1)}) \cdot m. \end{aligned}$$

The assignment $\sigma_M \colon m \mapsto 1 \otimes 1 \otimes m$ is thus a chain map giving a section of the counit map ϖ_M, so that $H_\bullet(M)$ is a retract of $H_\bullet(\Omega C \otimes_{t_{\mathfrak{L}}} \mathcal{C}(\mathfrak{L}) \otimes_{t_{\mathfrak{L}}} M)$.

Fix an ΩC-module M whose underlying chain complex is bounded below. The filtration $\overline{\mathcal{F}}_p := \bigoplus_{n \leq p} (\Omega C \otimes_{t_{\mathfrak{L}}} \mathcal{C}(\mathfrak{L}))_n$ of $\Omega C \otimes_{t_{\mathfrak{L}}} \mathcal{C}(\mathfrak{L})$ induces a differential filtration $\mathcal{F}_p := \overline{\mathcal{F}}_p \otimes M$ of $\Omega C \otimes_{t_{\mathfrak{L}}} \mathcal{C}(\mathfrak{L}) \otimes_{t_{\mathfrak{L}}} M$ and for $x \otimes X^K \otimes m \in \mathcal{F}_p$ we have

$$d(x \otimes X^K \otimes m) - d_{\mathfrak{L}}(x \otimes X^K) \otimes m - (-1)^{|x|+|X^K|} x \otimes X^K \otimes dm \in \mathcal{F}_{p-2}$$

since $\mathcal{C}(\mathfrak{L})$ is 2-reduced and $t_{\mathfrak{L}}(1) = 0$. In this expression $d_{\mathfrak{L}}$ is the $t_{\mathfrak{L}}$-twisted differential on $\Omega C \otimes_{t_{\mathfrak{L}}} \mathcal{C}(\mathfrak{L})$. Computing the spectral sequence of this filtration, we have

$$E^1_{p,q} \cong (\Omega C \otimes_{t_{\mathfrak{L}}} \mathcal{C}(\mathfrak{L}))_p \otimes H_q(M) \quad \text{and} \quad E^2_{p,q} \cong H_p(\Omega C \otimes_{t_{\mathfrak{L}}} \mathcal{C}(\mathfrak{L})) \otimes H_q(M).$$

As M is bounded below by hypothesis and $\Omega C \otimes_{\mathfrak{L}} \mathcal{C}(\mathfrak{L})$ is connective, the spectral sequence is bounded and hence convergent:

$$E^2_{p,q} \cong H_p(\Omega C \otimes_{t_{\mathfrak{L}}} \mathcal{C}(\mathfrak{L})) \otimes H_q(M) \Longrightarrow H_{p+q}(\Omega C \otimes_{t_{\mathfrak{L}}} \mathcal{C}(\mathfrak{L}) \otimes_{t_{\mathfrak{L}}} M).$$

The twisted extension $\Omega C \otimes_{t_{\mathfrak{L}}} \mathcal{C}(\mathfrak{L})$ is acyclic (it is the universal envloping algebra on the acyclic dg Lie algebra $\mathrm{cone}(\mathfrak{L})$, compare [Qui69, p. 291]) from which we conclude that ϖ_M is a quasi-isomorphism.

The $(\mathcal{L} \dashv \mathcal{C})$-unit is a natural weak equivalence by [Qui69, Theorem B.7.5]. The unit $C \to \mathcal{C}(\mathfrak{L}) := \mathcal{C}(\mathcal{L}(C))$ is compatible with the filtrations of Lemma 2.5.6, so induces a morphism of spectral sequences

$$\begin{tikzcd} H_p(C)\otimes H_q(M) \arrow[r, Rightarrow] \arrow[d] & H_{p+q}(C\otimes_t M) \arrow[d] \\ H_p(\mathcal{C}(\mathfrak{L}))\otimes H_q(M) \arrow[r, Rightarrow] & H_{p+q}(\mathcal{C}(\mathfrak{L})\otimes_{t_{\mathfrak{L}}} M). \end{tikzcd}$$

The morphism is a quasi-isomorphism on the E^2 page, so $C\otimes_t M \to \mathcal{C}(\mathfrak{L})\otimes_{t_{\mathfrak{L}}} M$ is a quasi-isomorphism by Zeeman's Comparison Theorem [Zee57]. Similarly, we have a morphism of comodule spectral sequences (Lemma 2.5.7)

$$\begin{tikzcd} H_p(\Omega C)\otimes H_q(C\otimes_t M) \arrow[r, Rightarrow] \arrow[d] & H_{p+q}(\Omega C\otimes_t C\otimes_t M) \arrow[d] \\ H_p(\Omega C)\otimes H_q(\mathcal{C}(\mathfrak{L})\otimes_{t_{\mathfrak{L}}} M) \arrow[r, Rightarrow] & H_{p+q}(\Omega C\otimes_{t_{\mathfrak{L}}} \mathcal{C}(\mathfrak{L})\otimes_{t_{\mathfrak{L}}} M) \end{tikzcd}$$

which is a quasi-isomorphism on the E^2 page by the above so that the Comparison Theorem once more implies that $\Omega C\otimes_t C\otimes_t M \to \Omega C\otimes_{t_{\mathfrak{L}}} \mathcal{C}(\mathfrak{L})\otimes_{t_{\mathfrak{L}}} M$ is a weak equivalence. Finally, we observe that the composite

$$\Omega C\otimes_t C\otimes_t M \longrightarrow \Omega C\otimes_{t_{\mathfrak{L}}} \mathcal{C}(\mathfrak{L})\otimes_{t_{\mathfrak{L}}} M \xrightarrow{\varpi_M} M$$

coincides with the counit ϱ_M, which is therefore a quasi-isomorphism. □

Finally, by working stage-by-stage on the "complicial Whitehead tower" we can remove the connectivity requirement in the last result:

Lemma 2.5.10. *Let C be 2-reduced. Then the counit $\Omega C\otimes_t C\otimes_t M \to M$ is a weak equivalence for any ΩC-module M.*

Proof. For $k\in\mathbb{Z}$ let $\mathrm{cn}_k\colon ch\to ch_{\geq k}$ be the right adjoint to the inclusion $ch_{\geq k}\hookrightarrow ch$ of k-connective chain complexes, so that $\mathrm{cn}_k M\to M$ is a homology isomorphism in degrees $n\geq k$ (Remark 2.5.11). The underlying chain complex of ΩC is connective as C is 2-reduced, which implies that for any $M\in \Omega C-\mathrm{Mod}$ and $k\in\mathbb{Z}$ the k-connective cover $\mathrm{cn}_k M$ inherits a ΩC-module structure. Moreover, we have $M\cong \mathrm{colim}_k\,\mathrm{cn}_k M$ as ΩC-modules. The endofunctor $\Omega C\otimes_t C\otimes_t (-)$ of $\Omega C-\mathrm{Mod}$ preserves filtered colimits; the class of weak equivalences in $\Omega C-\mathrm{Mod}$ is also closed under filtered colimits. By Lemma 2.5.9 the counit map

$$\Omega C\otimes_t C\otimes_t M \cong \operatorname*{colim}_{k\to-\infty} \Omega C\otimes_t C\otimes_t \mathrm{cn}_k M \longrightarrow \operatorname*{colim}_{k\to-\infty} \mathrm{cn}_k M \cong M$$

is therefore a weak equivalence. □

2.5.2 Model Koszul Duality

In this section, we use the results of the previous section to prove a Quillen equivalence

$$\begin{tikzcd} C-\mathrm{Comod} \arrow[r, shift left, "\Omega C\otimes_t(-)"] \arrow[r, phantom, "\perp"] & \Omega C-\mathrm{Mod} \arrow[l, shift left, "C\otimes_t(-)"] \end{tikzcd}$$

for any 2-reduced rational dg coalgebra C over $\mathbb{Q}$. For bounded below modules and comodules, we use the spectral sequences of Lemmas 2.5.6 and 2.5.7 to show that both adjoint functors preserve quasi-isomorphisms. In the unbounded case, however, our spectral sequences generally fail to converge. This failure is a well-known issue of Koszul duality, which is handled in [Pos11] by working with "derived categories of the second kind". Our unbounded Koszul duality result is essentially a recapitulation of [Pos11, §8.4] following a different argument.

Along the way, we shall prove some Koszul duality results for bounded below (co)modules, for which we shall need the following

Remark 2.5.11. For any $k \in \mathbb{Z}$, the category $ch_{\geq k}$ of chain complexes concentrated in degrees $[k, \infty]$ has a *projective model structure* for which

- weak equivalences are the quasi-isomorphisms;
- fibrations are the epimorphisms in degrees $n > k$; and
- cofibrations are characterised by the left lifting property with respect to acyclic fibrations.

As was the case for connective chain complexes (Remark 2.4.2), the model structure is combinatorial and proper, with generating sets

$$\mathcal{I}_{\geq k} := \{S^{n-1} \to D^n \mid n > k\} \cup \{0 \to S^k\} \quad \text{and} \quad \mathcal{J}_{\geq k} := \{0 \to D^n \mid n > k\}$$

of cofibrations and acyclic cofibrations respectively. For all $k \in \mathbb{Z}$ there is a Quillen adjunction

$$\begin{tikzcd} ch_{\geq k} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & ch \arrow[l, shift left, "\mathrm{cn}_k"] \end{tikzcd}$$

where cn_k is the *k-connective cover* functor

$$\left[\cdots \to M_{k+1} \xrightarrow{\partial_{k+1}} M_k \xrightarrow{\partial_k} M_{k-1} \to \cdots\right] \xmapsto{\mathrm{cn}_k} \left[\cdots \to M_{k+1} \xrightarrow{\partial_{k+1}} \ker(\partial_k) \to 0 \to \cdots\right].$$

The counit $\mathrm{cn}_k M \to M$ is a homology isomorphism in degrees $n \geq k$.

Lemma 2.5.12. *For any $k \in \mathbb{Z} \cup \{-\infty\}$, the model category $ch_{\geq k}$ is a Quillen module for the monoidal model category ch_+.*

Proof. For A connective and B k-connective, the tensor product $A \otimes B$ is k-connective so that we have a bifunctor

$$(-) \otimes (-) \colon ch_+ \times ch_{\geq k} \longrightarrow ch_{\geq k}$$

which defines a ch_+-module structure on $ch_{\geq k}$. This bifunctor preserves colimits in both variables, so defines an adjunction of two variables by the Adjoint Functor Theorem.

For any $k \in \mathbb{Z} \cup \{-\infty\}$, the set of pushout-products $\mathcal{I}_+ \square \mathcal{I}_{\geq k}$ consists of degree-wise monomorphisms, hence cofibrations. Similarly, a straightforward check shows that the sets of pushout-products $\mathcal{J}_+ \square \mathcal{I}_{\geq k}$ and $\mathcal{I}_+ \square \mathcal{J}_{\geq k}$ consist of acyclic monomorphisms. By cofibrant generation, the pushout-product axiom follows in general. As the monoidal unit $\mathbb{Q} \in ch_+$ is cofibrant, this shows that $ch_{\geq k}$ is a Quillen module over ch_+. $\square$

Corollary 2.5.13. *For a connective dg algebra A and $k \in \mathbb{Z} \cup \{-\infty\}$, the free-forgetful adjunction*

$$\begin{tikzcd} ch_{\geq k} \arrow[r, shift left, "A\otimes(-)"] \arrow[r, phantom, "\perp"] & A\text{–}\mathrm{Mod}_{\geq k} \arrow[l, shift left] \end{tikzcd}$$

induces a combinatorial model structure on the category $A\text{--}\mathrm{Mod}_{\geq k}$ of k-connective A-modules.

Proof. The monad $A \otimes (-)$ on $ch_{\geq k}$ preserves colimits, so that $A\text{--}\mathrm{Mod}_{\geq k}$ is locally presentable [Bor94, Theorem 5.5.9], and all limits and colimits are preserved by the forgetful functor to $ch_{\geq k}$. Thus, since the underlying chain complex of A is cofibrant the Lemma shows that the hypotheses of the Right Transfer Theorem A.1.3 are met. □

Remark 2.5.14. For a connective dg algebra A and $k \in \mathbb{Z}$, we have already seen (for instance in the proof of Lemma 2.5.10) that the k-connective cover of a left A-module M inherits an A-module structure. We thus have an adjunction

$$\begin{tikzcd} A\text{--}\mathrm{Mod}_{\geq k} \arrow[r, hook, shift left=1.5ex] \arrow[r, phantom, "\perp"] & A\text{--}\mathrm{Mod} \arrow[l, shift left=1.5ex, "\mathrm{cn}_k"] \end{tikzcd}$$

in which cn_k preserves weak equivalences and fibrations, as these are created in both the domain and codomain by the forgetful functors. We shall see that there is also a connective cover Quillen adjunction for comodules, but this is more subtle than the modular case.

Remark 2.5.15. For C a connective dg coalgebra, the comonad $C \otimes (-)$ on ch preserves connectivity. Accordingly, for all $k \in \mathbb{Z} \cup \{-\infty\}$ we obtain forgetful-cofree adjunctions:

$$\begin{tikzcd} C\text{--}\mathrm{Comod}_{\geq k} \arrow[r, shift left=1.5ex, "U"] \arrow[r, phantom, "\perp"] & ch_{\geq k}. \arrow[l, shift left=1.5ex, "C\otimes(-)"] \end{tikzcd}$$

Recall that the cofree C-comodule cogenerated by N is simply $C \otimes N$ equipped with C-coaction $\Delta \otimes \mathrm{id}\colon C \otimes N \to C \otimes C \otimes N$. As $ch_{\geq k}$ is locally presentable and the comonad $C \otimes (-)$ preserves colimits, its category of coalgebras $C\text{--}\mathrm{Comod}_{\geq k}$ is locally presentable by [CR14, Proposition A.1]. As usual for (co)monadic adjunctions, the forgetful functor U creates colimits and U-split equalisers

For any $k \in \mathbb{Z} \cup \{-\infty\}$, the comonad $C \otimes (-)\colon ch_{\geq k} \to ch_{\geq k}$ also preserves finite limits since $C \otimes (-)$ is an exact functor. It follows that the forgetful functor $C\text{--}\mathrm{Comod}_{\geq k} \to ch_{\geq k}$ creates finite limits for all $k \in \mathbb{Z} \cup \{-\infty\}$.

Lemma 2.5.16. *Let C be a connective dg coalgebra. For all finite k, the inclusion functor $C\text{--}\mathrm{Comod}_{\geq k} \hookrightarrow C\text{--}\mathrm{Comod}$ admits a right adjoint cn_k^C. The diagram of right adjoints*

$$\begin{tikzcd} ch \arrow[r, "\mathrm{cn}_k"] \arrow[d, "C\otimes(-)"'] & ch_{\geq k} \arrow[d, "C\otimes(-)"] \\ C\text{--}\mathrm{Comod} \arrow[r, "\mathrm{cn}_k^C"] & C\text{--}\mathrm{Comod}_{\geq k} \end{tikzcd}$$

commutes up to natural isomorphism.

Proof. The existence of the right adjoint follows immediately from the Adjoint Functor Theorem and Remark 2.5.15. The diagram in the statement of the Lemma commutes up to natural isomorphism by arguing on left adjoints and then invoking essential uniqueness of adjoints.

Explicitly, the right adjoint cn_k sends the C-comodule $(N, \rho\colon N \to C \otimes N)$ to the equaliser

$$C \otimes \mathrm{cn}_k N \underset{(\Delta\otimes\mathrm{id})_k}{\overset{C\otimes\mathrm{cn}_k(\rho)}{\rightrightarrows}} C \otimes \mathrm{cn}_k(C \otimes N)$$

in $C\text{-}\mathrm{Comod}_{\geq k}$, where $(\Delta \otimes \mathrm{id})_k$ is the composite

$$C \otimes \mathrm{cn}_k N \xrightarrow{\Delta\otimes\mathrm{id}} C \otimes C \otimes \mathrm{cn}_k N \longrightarrow C \otimes \mathrm{cn}_k(C \otimes N),$$

with $C \otimes \mathrm{cn}_k N \hookrightarrow \mathrm{cn}_k(C \otimes N)$ the obvious inclusion (this uses connectivity of C). That this prescription describes the right adjoint cn_k^C is a straightforward consequence of the fact that any comodule (N, ρ) is the equaliser of cofree comodules:

$$C \otimes N \underset{\Delta\otimes\mathrm{id}}{\overset{C\otimes\rho}{\rightrightarrows}} C \otimes C \otimes N$$

which is a standard result for (co)algebras over (co)monads. $\square$

Lemma 2.5.17. *Let C be a connective dg coalgebra and (N, ρ) a C-comodule. There is a sequence of natural monomorphisms*

$$\cdots \longrightarrow \mathrm{cn}_k^C N \longrightarrow \mathrm{cn}_{k-1}^C N \longrightarrow \cdots \longrightarrow N$$

such that $N \cong \operatorname*{colim}_{k\to-\infty} \mathrm{cn}_k^C N$.

Proof. The monomorphisms $\mathrm{cn}_k M \to \mathrm{cn}_{k-1} M \to M$ for $M \in ch$ and $k \in \mathbb{Z}$ induce a diagram of cofree C-comodules

$$\begin{tikzcd}
\cdots \arrow[r] & C \otimes \mathrm{cn}_k N \arrow[r] \arrow[d, shift right, "(\Delta\otimes\mathrm{id})_k"'] \arrow[d, shift left, "C\otimes\mathrm{cn}_k(\rho)"] & C \otimes \mathrm{cn}_{k-1} N \arrow[r] \arrow[d, shift right, "(\Delta\otimes\mathrm{id})_{k-1}"'] \arrow[d, shift left, "C\otimes\mathrm{cn}_{k-1}(\rho)"] & \cdots \arrow[r] & C \otimes N \arrow[d, shift right, "\Delta\otimes\mathrm{id}"'] \arrow[d, shift left, "C\otimes\rho"] \\
\cdots \arrow[r] & C \otimes \mathrm{cn}_k(C \otimes N) \arrow[r] & C \otimes \mathrm{cn}_{k-1}(C \otimes N) \arrow[r] & \cdots \arrow[r] & C \otimes C \otimes N.
\end{tikzcd}$$

Since $C \otimes (-)$ is exact, each of the horizontal morphisms is a monomorphism so that taking limits in the vertical direction gives a sequence of monomorphisms

$$\cdots \longrightarrow \mathrm{cn}_k^C N \longrightarrow \mathrm{cn}_{k-1}^C N \longrightarrow \cdots \longrightarrow N$$

(by Remark 2.5.15 finite limits of C-comodules are created by the forgetful functor). Forgetting the right-most terms in the above diagram and taking colimits in the horizontal direction we obtain the diagram

$$C \otimes N \underset{\Delta\otimes\mathrm{id}}{\overset{C\otimes\rho}{\rightrightarrows}} C \otimes C \otimes N$$

whose limit is N by comonadicity. Since filtered colimits of chain complexes commute with finite limits, it follows that the comparison morphism

$$\operatorname*{colim}_{k\to-\infty} \mathrm{cn}_k^C N \longrightarrow N$$

is an isomorphism of C-comodules. Naturality is manifest in the argument. $\square$

Lemma 2.5.18. *Let C be a connective dg coalgebra. For all $k \in \mathbb{Z} \cup \{-\infty\}$ the forgetful-cofree adjunction*

$$\begin{tikzcd} C\text{-Comod}_{\geq k} \arrow[r, shift left, "U"] \arrow[r, phantom, "\perp"] & ch_{\geq k} \arrow[l, shift left, "C\otimes(-)"] \end{tikzcd}$$

induces a combinatorial model structure on C-Comod.

Proof. A map $f\colon N \to M$ in C-Comod$_{\geq k}$ is a *provisional cofibration* or *provisional weak equivalence* if $U(f)$ is a cofibration or weak equivalence in $ch_{\geq k}$ respectively. Thus, the provisional cofibrations are precisely the monomorphisms and the provisional weak equivalences are the quasi-isomorphisms.

Since C-Comod$_{\geq k}$ is locally presentable and all objects are provisionally cofibrant, to apply the Left Transfer Theorem A.1.4 it is sufficient to establish the existence of good cylinder objects for all comodules N. Tensoring with $I_\bullet := N(\mathbb{Q}[\Delta[1]])$ does the trick (compare Lemma 2.4.5). □

Corollary 2.5.19. *Let C be a connective dg coalgebra. For all $k \in \mathbb{Z}$ the adjunction*

$$\begin{tikzcd} C\text{-Comod}_{\geq k} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & C\text{-Comod} \arrow[l, shift left, "\mathrm{cn}^C_k"] \end{tikzcd}$$

is Quillen.

Proof. The inclusion functor preserves injections and quasi-isomorphisms, so is left Quillen. □

We are now in a position to prove a Koszul duality result for bounded below (co)modules:

Lemma 2.5.20. *Let C be a 2-reduced dg coalgebra and take any $k \in \mathbb{Z}$. The adjunction*

$$\begin{tikzcd} C\text{-Comod}_{\geq k} \arrow[r, shift left, "\Omega C\otimes_t(-)"] \arrow[r, phantom, "\perp"] & \Omega C\text{-Mod}_{\geq k} \arrow[l, shift left, "C\otimes_t(-)"] \end{tikzcd}$$

given by forming t-twisted extensions on k-connective (co)modules is a Quillen equivalence.

Proof. The spectral sequences of Lemmas 2.5.6 and 2.5.7 and the Spectral Sequence Comparison Theorem imply that $C \otimes_t (-)$ and $\Omega C \otimes_t (-)$ preserve weak equivalences between k-connective (co)modules for all finite k. We show that $\Omega C \otimes_t (-)$ preserves cofibrations: left $f\colon N \to M$ be an injective map of C-comodules and left $g\colon A \to B$ be an acyclic fibration of ΩC-modules, and suppose we have a commutative diagram of ΩC-modules

$$\begin{tikzcd} \Omega C \otimes_t N \arrow[r] \arrow[d, "\Omega C\otimes_t f"'] & A \arrow[d, "g"] \\ \Omega C \otimes_t M \arrow[r] & B. \end{tikzcd}$$

Passing to the underlying chain complexes, $\Omega C \otimes_t N \to \Omega C \otimes_t M$ is a monomorphism and hence a cofibration, so that there is a lift $\alpha\colon \Omega C \otimes_t M \to A$ making the resulting diagram in $ch_{\geq k}$ commute. Defining $\alpha'\colon \Omega C \otimes_t M \to A$ by

$$\alpha'\colon (tc_1 \otimes \cdots \otimes tc_k) \otimes m \longmapsto tc_1 \cdots tc_k \cdot \alpha(m)$$

defines a lifting of the above diagram of ΩC-modules. Thus, $\Omega C \otimes_t N \to \Omega C \otimes_t M$ is a cofibration in $\Omega C\mathrm{-Mod}_{\geq k}$, so the adjunction is Quillen.

Consider $N \in C\mathrm{-Comod}_{\geq k}$ and $M \in \Omega C\mathrm{-Mod}_{\geq k}$, which are necessarily cofibrant and fibrant respectively. If $f\colon N \to C \otimes_t M$ is a weak equivalence, then its adjunct

$$f^\vee \colon \Omega C \otimes_t N \xrightarrow{\Omega C \otimes : t f} \Omega C \otimes_t C \otimes_t M \longrightarrow M$$

is a weak equivalence by Lemma 2.5.10 since $\Omega C \otimes_t (-)$ preserves weak equivalences between bounded below comodules. Similarly, if $g\colon \Omega C \otimes_t N \to M$ is a weak equivalence, then its adjunct

$$g^\vee \colon N \longrightarrow C \otimes_t \Omega C \otimes_t N \xrightarrow{C \otimes_t g} C \otimes_t M$$

is a weak equivalence by Lemma 2.5.8. This proves the Quillen equivalence. □

Remark 2.5.21. The proof of the Lemma also shows that $\Omega C \otimes_t (-)$ sends injective maps of *unbounded* C-comodules to cofibrations in $\Omega C\mathrm{-Mod}$.

We now move toward our unbounded Koszul duality result. In order to obtain an equivalence of homotopy theories in the unbounded setting, we are forced to slightly modify our notion of weak equivalence between unbounded comodules. The underlying reason for this is that we cannot show that $\Omega C \otimes_t (-)$ preserves quasi-isomorphisms in general. However, we have the following

Lemma 2.5.22. *Let C be a 2-reduced dg coalgebra. Then $C \otimes_t (-)$ preserves weak equivalences.*

Proof. Let $f\colon A \to B$ be a weak equivalence of ΩC-modules. Then by Remark 2.5.14, $\mathrm{cn}_k(f)\colon \mathrm{cn}_k A \to \mathrm{cn}_k B$ is a weak equivalence of ΩC-modules for all $k \in \mathbb{Z}$. The functor $C \otimes_t (-)\colon \Omega C\mathrm{-Mod} \to C\mathrm{-Comod}$ preserves filtered colimits and the class of weak equivalences in $C\mathrm{-Comod}$ is closed under filtered colimits, so that

$$C \otimes_t A \cong \operatorname*{colim}_{k \to -\infty} C \otimes_t \mathrm{cn}_k A \longrightarrow C \otimes_t B \cong \operatorname*{colim}_{k \to -\infty} C \otimes_t \mathrm{cn}_k B$$

is a weak equivalence. □

Corollary 2.5.23. *Let C a 2-reduced dg coalgebra and $f\colon N \to M$ a map of C-comodules. If $\Omega C \otimes_t f$ is a quasi-isomorphism, then so too is f.*

Proof. Suppose f is such that $\Omega C \otimes_t f$ is a quasi-isomorphism of ΩC-modules. By the Lemma we then have that the right vertical morphism in the counit naturality square

$$\begin{tikzcd} N \arrow[r] \arrow[d, "f"'] & C \otimes_t \Omega C \otimes_t N \arrow[d, "C \otimes_t \Omega C \otimes_t f"] \\ M \arrow[r] & C \otimes_t \Omega C \otimes_t M \end{tikzcd}$$

is a quasi-isomorphism. But the horizontal counit maps are quasi-isomorphisms by Lemma 2.5.10, so the result follows by the 2-out-of-3 property. □

Definition 2.5.24. For C a 2-reduced dg coalgebra and $t\colon C \rightsquigarrow \Omega C$ its universal twisting chain, a map $f\colon N \to M$ is a *t-equivalence* if $\Omega C \otimes_t f$ is a weak equivalence.

Remark 2.5.25. We have already seen that t-equivalences are quasi-isomorphisms. We expect that the converse fails, though have not found an explicit counterexample in this context. As a partial converse, however, note that the proof of Lemma 2.5.20 shows that the t-equivalences of bounded below C-comodules coincide with quasi-isomorphisms.

Lemma 2.5.26. *For C a 2-reduced dg coalgebra, the adjunction*

$$\begin{tikzcd} C\text{−Comod} \arrow[r, shift left, "\Omega C\otimes_t(-)"] \arrow[r, phantom, "\perp"] & \Omega C\text{−Mod} \arrow[l, shift left, "C\otimes_t(-)"] \end{tikzcd}$$

induces a combinatorial model structure on C−Comod, *which we denote* C−Comod$^{\mathrm{II}}$ *and call the* model structure of the second kind *on C-comodules. With respect to the model structure of the second kind, the adjunction given by forming t-twisted extensions is a Quillen equivalence.*

Proof. By Remark 2.5.21 all injective maps of C-comodules are sent to cofibrations by $\Omega C \otimes_t (-)$. If we define a map of C-comodules $f : N \to M$ to be a *t-cofibration* if $\Omega C \otimes_t f$ is a cofibration, every C-comodule is t-cofibrant. Noting that C−Comod is locally presentable, in order to apply the Left Transfer Theorem A.1.4 it is sufficient to show that good cylinder objects exist. For this, we simply tensor the C-comodule N with $I_\bullet := N(\mathbb{Q}[\Delta[1]])$; there is a natural diagram of C-comodules

$$N \oplus N \longrightarrow N \otimes I_\bullet \longrightarrow N,$$

in which the first arrow is a monomorphism and hence a t-cofibration. Moreover, $\Omega C \otimes_t (N \otimes I_\bullet) \cong (\Omega C \otimes_t N) \otimes I_\bullet$ shows that $N \otimes I_\bullet \to N$ is a t-equivalence so that $N \otimes I_\bullet$ is a good cylinder object.

We thus have a Quillen adjunction

$$\begin{tikzcd} C\text{−Comod}^{\mathrm{II}} \arrow[r, shift left, "\Omega C\otimes_t(-)"] \arrow[r, phantom, "\perp"] & \Omega C\text{−Mod} \arrow[l, shift left, "C\otimes_t(-)"] \end{tikzcd}$$

in which $\Omega C \otimes_t (-)$ creates weak equivalences. To conclude the Quillen equivalence, it is sufficient to show that the counit map $\Omega C \otimes_t C \otimes_t M \to M$ is a weak equivalence for all $M \in \Omega C$−Mod (necessarily fibrant). But this was already proven as Lemma 2.5.10. □

To conclude this section, we record a sort of base change result for the twisting adjunctions.

Lemma 2.5.27. *Let $\psi : C \to D$ be a morphism of 2-reduced dg coalgebras. Then there is a diagram of left Quillen adjunctions*

$$\begin{tikzcd} C\text{−Comod}^{\mathrm{II}} \arrow[r, shift left, "\Omega C\otimes_t(-)"] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\psi_!"] \arrow[d, phantom, "\vdash"] & \Omega C\text{−Mod} \arrow[l, shift left, "C\otimes_t(-)"] \arrow[d, shift left, "\Omega\psi_!"] \arrow[d, phantom, "\vdash"] \\ D\text{−Comod}^{\mathrm{II}} \arrow[r, shift left, "\Omega D\otimes_t(-)"] \arrow[r, phantom, "\perp"] \arrow[u, shift left, "\psi^*"] & \Omega D\text{−Mod} \arrow[l, shift left, "D\otimes_t(-)"] \arrow[u, shift left, "\Omega\psi^*"] \end{tikzcd}$$

commuting up to natural isomorphism. If ψ is a weak equivalence then all of the above adjunctions are Quillen equivalences.

Proof. The left adjoint $\psi_!\colon C-\mathrm{Comod} \to D-\mathrm{Comod}$ sends the C-comodule (N,ρ) to N regarded as a D-comodule via the coaction

$$N \xrightarrow{\rho} C\otimes N \xrightarrow{\psi\otimes\mathrm{id}} D\otimes N.$$

The functor $\psi_!$ preserves colimits, so has a right adjoint ψ^* by the Adjoint Functor Theorem. Explicitly, the right adjoint ψ^* sends the D-comodule (K,ϱ) to the equaliser of the diagram

$$\begin{tikzcd} C\otimes K \arrow[r, shift left, "C\otimes\varrho"] \arrow[r, shift right, "(\mathrm{id}\otimes\psi\otimes\mathrm{id})(\Delta\otimes\mathrm{id})"'] & C\otimes D\otimes K \end{tikzcd}$$

in $C-\mathrm{Comod}$.

The cobar construction is functorial, so associated to the coalgebra map $\psi_!$ is a map of dg algebras $\Omega\psi\colon \Omega C \to \Omega D$ which intertwines the universal twisting chains $t_C\colon C \rightsquigarrow \Omega C$ and $t_D\colon D \rightsquigarrow \Omega D$, i.e. $t_D\circ\psi = \Omega\psi\circ t_C\colon C\rightsquigarrow \Omega D$. From this we deduce a natural isomorphism $\Omega D\otimes_{t_D}(\psi_! N)\cong \Omega\psi_!(\Omega C\otimes_{t_C} N)$, so we have a diagram of left adjoint functors

$$\begin{tikzcd} C-\mathrm{Comod} \arrow[r, "\Omega C\otimes_t(-)"] \arrow[d, "\psi_!"'] & \Omega C-\mathrm{Mod} \arrow[d, "\Omega\psi_!"] \\ D-\mathrm{Comod} \arrow[r, "\Omega D\otimes_t(-)"] & \Omega D-\mathrm{Mod} \end{tikzcd}$$

which commutes up to natural isomorphism. Restriction and extension of scalars determine a Quillen adjunction $(\Omega\psi_! \dashv \Omega\psi^*)\colon \Omega C-\mathrm{Mod}^{\mathrm{II}} \to \Omega D-\mathrm{Mod}^{\mathrm{II}}$ (the right adjoint preserves and reflects weak equivalences and fibrations), so Ken Brown's Lemma and the above diagram together imply that $\psi_!$ preserves t-cofibrations and t-equivalences. We conclude that $(\psi_!\dashv\psi^*)$ is a Quillen adjunction.

Suppose that ψ is a weak equivalence of 2-reduced dg coalgebras. As rational homotopy theory functor $\mathcal{L}\colon \mathrm{dgCoalg}_{\geq 2}\to \mathrm{dgLie}_{\geq 1}$ is left Quillen and all objects of $\mathrm{dgCoalg}_{\geq 2}$ are cofibrant, Ken Brown's Lemma implies that $\mathcal{L}(\psi)\colon \mathcal{L}(C)\to\mathcal{L}(D)$ is a weak equivalence of dg Lie algebras. By Lemma 2.4.23, the map on universal enveloping algebras $\Omega\psi\colon \Omega C\cong \mathcal{U}(\mathcal{L}(C))\to \Omega D\cong\mathcal{U}(\mathcal{L}(D))$ is a weak equivalence of dg Hopf algebras. By Theorem 2.4.22 the adjunction $(\Omega\psi_!\dashv\Omega\psi^*)$ is a Quillen equivalence, so that all of the adjunctions in the diagram

$$\begin{tikzcd} C-\mathrm{Comod}^{\mathrm{II}} \arrow[r, shift left, "\Omega C\otimes_t(-)"] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\psi_!"] & \Omega C-\mathrm{Mod} \arrow[l, shift left, "C\otimes_t(-)"] \arrow[d, shift left, "\Omega\psi_!"] \\ D-\mathrm{Comod}^{\mathrm{II}} \arrow[u, shift left, "\psi^*"] \arrow[u, phantom, "\vdash"] \arrow[r, shift left, "\Omega D\otimes_t(-)"] \arrow[r, phantom, "\perp"] & \Omega D-\mathrm{Mod} \arrow[l, shift left, "D\otimes_t(-)"] \arrow[u, shift left, "\Omega\psi^*"] \arrow[u, phantom, "\vdash"] \end{tikzcd}$$

are Quillen equivalences by Lemma 2.5.26 and the 2-out-of-3 property. □

2.6 Equivalences of Stable Rational Homotopy Theories

The entirety of this section is devoted to proving the main result of this chapter. Quillen's approach to rational homotopy theory shows that the rational homotopy type of a simply-connected space X is completely encoded by algebraic data; through a dg Lie model $\mathfrak{l}_X$ for the Whitehead product or, equivalently, as a strict dg coalgebra model C_X for the rational chains. Our main theorem lifts this identification to categories of representations:

Theorem 2.6.1. *Let X be a 1-connected simplicial set and let $\mathfrak{l}_X$ and C_X be a Lie model and a dg coalgebra model for X respectively. There is a zig-zag of Quillen equivalences between the model categories*

(i) $\mathrm{Mod}{-}\mathrm{H}\mathbb{Q}_X$ *of (right)* $\mathrm{H}\mathbb{Q}$*-module spectra parametrised by X;*

(ii) $\mathbb{Q}[\mathbb{G}X]{-}\mathrm{Mod}$ *of stable representations of the rational homology Hopf algebra of the loop group;*

(iii) $\mathfrak{l}_X{-}\mathrm{Rep}$ *of unbounded $\mathfrak{l}_X$-representations; and*

(iv) $C_X{-}\mathrm{Comod}^{\mathrm{II}}$ *of unbounded C_X-comodules, equipped with the model structure of the second kind.*

The zig-zag of Quillen equivalences is weakly natural in X and the above model categories are independent of the rational homotopy type of X up to Quillen equivalence.

We clarify what is meant by "weak naturality" during the course of the proof. This is a sort of naturality up-to-homotopy property, necessitated by the fact that morphisms in the homotopy category do not have canonical lifts. Of course, these strictness issues disappear when working directly with the homotopy categories.

Another consequence of the proof of Theorem 2.6.1 is that the fibrewise smash product of rational parametrised spectra is modelled by the derived tensor product of Lie representations. The proof of the following result appears at the very end of this section:

Corollary 2.6.2. *Let X be a 1-connected simplicial set and $\mathfrak{l}_X$ be a Lie model for X. The equivalence of categories $\lambda_X\colon Ho_{\mathbb{Q}}(\mathrm{Sp}_X) \to Ho(\mathfrak{l}_X{-}\mathrm{Rep})$ is strong symmetric monoidal.*

Proof of Theorem 2.6.1. Let X be a 1-connected simplicial set, with $\mathfrak{l}_X$ and C_X a Lie and coalgebra model for X respectively. The proof is a synthesis of many of the results we have recorded in this chapter so far and involves many steps. To make the proof more navigable, we start with a schematic diagram outlining the structure of the argument

$$\begin{tikzcd}
\mathrm{Mod}{-}\mathrm{H}\mathbb{Q}_{X'} \arrow[r, "2"] \arrow[d, leftrightarrow, "1"'] & \widehat{\mathbb{Q}}[\mathbb{G}X]{-}\mathrm{Mod} & N(\mathfrak{g}_X){-}\mathrm{Rep}^{\Sigma} \arrow[l, "3"'] \arrow[r, "4"] & N(\mathfrak{g}_X){-}\mathrm{Rep} \\
\mathrm{Mod}{-}\mathrm{H}\mathbb{Q}_X & & & \overline{N(\mathfrak{g}_X)}{-}\mathrm{Rep} \arrow[u, "5'"'] \arrow[d, "5''"] \\
& C_X{-}\mathrm{Comod} \arrow[r, "6"] & \mathcal{L}C_X{-}\mathrm{Rep} \arrow[r, "7"] & \mathfrak{l}_X{-}\mathrm{Rep}
\end{tikzcd}$$

in which each arrow represents a left Quillen equivalence. This schematic contains some new notation (X', $\mathfrak{g}_X$ and so on) which is explained over the course of the proof.

① **- Passing to 2-reduced models:** Fix a fibrant replacement $X \to X'$ and choose a basepoint x of X'. Set $X'' = E_2(X', x)$ to be the 2nd Eilenberg subcomplex of X' a x, recalling that this is defined as the pullback

$$\begin{tikzcd} X'' \arrow[r, "\imath"] \arrow[d] & X' \arrow[d] \\ * \arrow[r, "x"] & \mathrm{cosk}_1 X'. \end{tikzcd}$$

The simplicial set X'' is then 2-reduced and the map $X'' \to X'$ is a weak equivalence [May67, Theorem 8.4]. Recall that we had fixed a cofibrant replacement $\mathrm{H}\mathbb{Q} \to \mathcal{H}\mathbb{Q}$ of commutative symmetric ring spectra (Construction 2.3.3); by Lemma 2.3.9 we have a diagram of left Quillen functors

$$\begin{tikzcd} \mathrm{Sp}^{\Sigma}_{X''} \arrow[r] \arrow[d, "(-)\oslash_{X''}\mathrm{H}\mathbb{Q}"'] & \mathrm{Sp}^{\Sigma}_{X'} \arrow[d, "(-)\oslash_{X'}\mathrm{H}\mathbb{Q}"'] & \mathrm{Sp}^{\Sigma}_{X} \arrow[l] \arrow[d, "(-)\oslash_{X}\mathrm{H}\mathbb{Q}"] \\ \mathrm{Mod}{-}\mathrm{H}\mathbb{Q}_{X''} \arrow[r] & \mathrm{Mod}{-}\mathrm{H}\mathbb{Q}_{X'} & \mathrm{Mod}{-}\mathrm{H}\mathbb{Q}_{X} \arrow[l] \end{tikzcd}$$

in which the horizontal functors are left Quillen equivalences.

② **- Toward rational representations of the loop group:** We can now take X to be 2-reduced without loss of generality so that we can define the Kan simplicial loop group $\mathbb{G}X$, which is itself reduced. The sought-after left Quillen equivalence is the composite of left Quillen equivalences

$$\begin{tikzcd} (\mathbb{G}X_+, \mathrm{H}\mathbb{Q}){-}\mathrm{Bimod} \arrow[r, "(-)\wedge_{\mathrm{H}\mathbb{Q}}\mathcal{H}\mathbb{Q}"] & (\mathbb{G}X_+, \mathcal{H}\mathbb{Q}){-}\mathrm{Bimod} \arrow[r, "\mathrm{str}_{\mathbb{G}X}"] & \mathbb{Q}[\mathbb{G}X]{-}\mathrm{Mod} \arrow[d, "\kappa[\mathbb{G}X]_!"] \\ \mathrm{Mod}{-}\mathrm{H}\mathbb{Q}_X \arrow[u, "\beta^+_X"] & & \widehat{\mathbb{Q}}[\mathbb{G}X]{-}\mathrm{Mod}. \end{tikzcd}$$

Working from bottom left to bottom right, the functors are those of Theorem 2.3.15, Corollary 2.3.13, Theorem 2.3.25 and Theorem 2.3.36 respectively. The last result applies since $\mathbb{G}[X]$ is a free reduced simplicial group, so $\mathbb{Q}[\mathbb{G}X]$ is a free reduced simplicial algebra and this in turn implies that completion at the augmentation ideal $\kappa[\mathbb{G}X] \colon \mathbb{Q}[\mathbb{G}X] \to \widehat{\mathbb{Q}}[\mathbb{G}X]$ is a weak equivalence by Theorem 2.1.5.

Remark 2.6.3. The functors $(-) \wedge_{\mathrm{H}\mathbb{Q}} \mathcal{H}\mathbb{Q}$, $\mathrm{str}_{\mathbb{G}X}$ and $\kappa[\mathbb{G}X]_!$ above are all (weak) monoidal Quillen equivalences, so induce strong symmetric monoidal equivalences on homotopy categories.

③ **- Comparing with simplicial Lie representations:** The next stage of the proof makes uses of Quillen's rational homotopy theory equivalences

$$\begin{tikzcd} \mathrm{sGrp}_{\geq 1,\mathbb{Q}} \arrow[r, shift left, "\widehat{\mathbb{Q}}[-]"] \arrow[r, phantom, "\perp"] & \mathrm{sCHopf}_{\geq 1} \arrow[l, shift left, "\mathcal{G}"] \arrow[r, shift right, "\mathcal{P}"'] \arrow[r, phantom, "\perp"] & \mathrm{sLie}_{\geq 1}, \arrow[l, shift right, "\widehat{\mathcal{U}}"'] \end{tikzcd}$$

which implement a sort of formal Lie theory for rational homotopy types. Let $\mathfrak{g}_X$ be a cofibrant replacement for the reduced simplicial Lie algebra $\mathcal{P}\widehat{\mathbb{Q}}[\mathbb{G}X]$ in $\mathrm{sLie}_{\geq 1}$; $\mathfrak{g}_X$ is necessarily free. The weak equivalence $\mathfrak{g}_X \to \mathcal{P}\widehat{\mathbb{Q}}[\mathbb{G}X]$ is the $(\widehat{\mathcal{U}} \dashv \mathcal{P})$-adjunct of a morphism

$$\kappa_X \colon \widehat{\mathcal{U}}(\mathfrak{g}_X) \longrightarrow \widehat{\mathbb{Q}}[\mathbb{G}X]$$

of reduced complete simplicial Hopf algebras. Since $\mathbb{G}X$ and $\mathfrak{g}_X$ are free, the complete simplicial Hopf algebras $\widehat{\mathbb{Q}}[\mathbb{G}X]$ and $\widehat{\mathcal{U}}(\mathfrak{g}_X)$ are also free [Qui69, Example A.2.11]. This is crucial for the proof of the following

Lemma 2.6.4. *The map $\kappa_X \colon \widehat{\mathcal{U}}(\mathfrak{g}_X) \to \widehat{\mathbb{Q}}[\mathbb{G}X]$ induces an isomorphism on homotopy groups.*

Proof. We shall first show that $\mathcal{P}(\kappa_X)$ is a weak equivalence of simplicial Lie algebras. Indeed, by construction we have a factorisation

$$\begin{tikzcd}
\widehat{\mathcal{U}}(\mathfrak{g}_X) \arrow[rr, "\kappa_X"] \arrow[dr] & & \widehat{\mathbb{Q}}[\mathbb{G}X] \\
& \widehat{\mathcal{U}}(\mathcal{P}(\widehat{\mathbb{Q}}[\mathbb{G}X])), \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

where ϵ is the $(\widehat{\mathcal{U}} \dashv \mathcal{P})$-counit. Applying the functor $\mathcal{P}$ yields the diagram

$$\begin{tikzcd}
\mathcal{P}\widehat{\mathcal{U}}(\mathfrak{g}_X) \arrow[r] \arrow[rr, bend left, "\mathcal{P}(\kappa_X)"] & \mathcal{P}\widehat{\mathcal{U}}\mathcal{P}(\widehat{\mathbb{Q}}[\mathbb{G}X]) \arrow[r, "\mathcal{P}(\epsilon)"] & \mathcal{P}(\widehat{\mathbb{Q}}[\mathbb{G}X]) \\
\mathfrak{g}_X \arrow[u, "\eta"] \arrow[r, "\kappa_X^\vee"] & \mathcal{P}(\widehat{\mathbb{Q}}[\mathbb{G}X]) \arrow[u, "\eta"] \arrow[ur, equal] &
\end{tikzcd}$$

where the maps marked η are components of the $(\widehat{\mathcal{U}} \dashv \mathcal{P})$-unit and the identity is simply the triangle identity of the adjunction. Since $\kappa_X^\vee$ is a weak equivalence by construction and the unit map $\mathfrak{g}_X \to \mathcal{P}\widehat{\mathcal{U}}(\mathfrak{g}_X)$ is a weak equivalence by Theorem 2.1.5, it follows from the 2-out-of-3 property that $\mathcal{P}(\kappa_X)$ is a weak equivalence also.

Quillen proved [Qui69, Proposition I.6.9] that for $\psi \colon A \to B$ a map of free objects in $\mathrm{sCHopf}_{\geq 1}$, the following are equivalent:

- $\mathcal{P}(\psi)$ is a weak equivalence; and
- $\mathrm{gr}_1 \psi$ is a weak equivalence,

where, for instance, $\mathrm{gr}_1 A = \overline{A}/\overline{A}^2$ with $\overline{A}$ the augmentation ideal. The associated graded algebra $\mathrm{gr}A$ is generated by $\mathrm{gr}_1 A$[1], so by $\mathrm{gr}_p A \cong \mathrm{T}^p \mathrm{gr}_1 A$ is the p-th tensor power of $\mathrm{gr}_1 A$ and similarly for B. Consider the filtration on A given by powers of the augmentation ideal, which has associated spectral sequence

$$E^1_{p,q} = \pi_{p+q}(\mathrm{gr}_p A) \Longrightarrow \pi_{p+q}(A).$$

Since A is reduced we have $\pi_k(\mathrm{gr}_p A) \cong H_k(\mathrm{T}^p N(\mathrm{gr}_1 A)) = 0$ for $k < p$, so the spectral sequence is concentrated in the first quadrant and hence converges. Let $\psi \colon A \to B$ be such that $\mathcal{P}(\psi)$ is a weak equivalence. Then the induced map $\mathrm{gr}_p \psi$ is a weak equivalence for all $p \geq 1$. Feeding this into the above spectral sequences we obtain that ψ is a π_*-isomorphism by convergence.

[1]this is one of the conditions of completeness, compare [Qui69, §A.1].

Applying this to the problem at hand, we have a map of free complete reduced simplicial Hopf algebras $\kappa_X \colon \widehat{\mathcal{U}}(\mathfrak{g}_X)$ and $\widehat{\mathbb{Q}}[\mathbb{G}X]$ such that the induced map on Lie algebras of primitives $\mathcal{P}(\kappa_X)$ is a weak equivalence. By the argument above we conclude that κ_X induces an isomorphism on homotopy groups. □

Remark 2.6.5. By Zeeman's Comparison Theorem [Zee57], the converse of our above argument holds, namely for a map $\psi \colon A \to B$ of free objects in $\mathrm{sCHopf}_{\geq 1}$ the following are equivalent:

- $\mathcal{P}(\psi)$ is a π_*-isomorphism on Lie algebras of primitive elements; and
- ψ is a π_*-isomorphism.

For any 1-connected space X, its rational homotopy type is encoded by the simplicial group $\mathcal{G}A$ of group-like elements of some free object A in $\mathrm{sCHopf}_{\geq 1}$; in particular $\pi_*(\mathcal{G}A) \otimes \mathbb{Q} \cong \pi_*(\Omega X) \otimes \mathbb{Q}$. On the other hand, A models the rational homology of ΩX: $\pi_*(A) \cong H_\bullet(\Omega X; \mathbb{Q})$. Since A is complete, the exponential map $\mathcal{P}A \to \mathcal{G}A$ is an isomorphism of simplicial sets [Qui69, Proposition A.2.8]. The above equivalence of conditions on ψ is an algebraic incarnation of Corollary 2.1.2.

The left Quillen equivalence of ③ is the composite of left Quillen equivalences

$$N(\mathfrak{g}_X)-\mathrm{Rep}^{\Sigma} \longrightarrow \mathfrak{g}_X-\mathrm{Rep} = \mathcal{U}(\mathfrak{g}_X)-\mathrm{Mod} \longrightarrow \widehat{\mathcal{U}}(\mathfrak{g}_X)-\mathrm{Mod} \longrightarrow \widehat{\mathbb{Q}}[\mathbb{G}X]-\mathrm{Mod}.$$

From left to right, the first left Quillen equivalence is from Theorem 2.4.16 and the second and third are from Theorem 2.3.36. The hypotheses of Theorem 2.3.36 are met since, in the first instance, freeness of $\mathfrak{g}_X$ implies that $\mathcal{U}(\mathfrak{g}_X) \to \widehat{\mathcal{U}}(\mathfrak{g}_X)$ is a weak equivalence (Theorem 2.1.5) and, in the second instance, by Lemma 2.6.4.

Remark 2.6.6. Each of the Quillen equivalences that make up step ③ is (weak) monoidal.

④ **- Assembling spectra of connective dg Lie representations:** By Theorem 2.4.16 there is a strong symmetric monoidal Quillen equivalence

$$\begin{tikzcd} N(\mathfrak{g}_X)-\mathrm{Rep}^{\Sigma} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & N(\mathfrak{g}_X)-\mathrm{Rep} \arrow[l, shift left] \end{tikzcd}$$

relative spectra in connective Lie representations with unbounded representations.

⑤ **- Comparing Lie models:** We must now compare the representation categories of two Lie models. Since $N(\mathfrak{g}_X)$ and $\mathfrak{l}_X$ are both Lie models for X, they are isomorphic in the homotopy category $Ho(\mathrm{dgLie}_{\geq 1})$: since all dg Lie algebras are fibrant, this means that we can find cofibrant replacements

$$\overline{N(\mathfrak{g}_X)} \longrightarrow N(\mathfrak{g}_X) \quad \text{and} \quad \overline{\mathfrak{l}_X} \longrightarrow \mathfrak{l}_X$$

together with an isomorphism $[\overline{N(\mathfrak{g}_X)}] \to [\overline{\mathfrak{l}_X}]$ in $Ho(\mathrm{dgLie}_{\geq 1})$. In particular, by choosing a representative of the latter map we obtain a diagram of weak equivalences of dg Lie algebras

$$N(\mathfrak{g}_X) \longleftarrow \overline{N(\mathfrak{g}_X)} \longrightarrow \overline{\mathfrak{l}_X} \longrightarrow \mathfrak{l}_X.$$

Applying Theorem 2.4.22 produces the weak monoidal Quillen equivalences ⑤′ and ⑤″.

⑥ **- Koszul duality:** The coalgebra model C_X of X is 2-reduced, and the bar construction $\Omega C_X = \mathcal{U}(\mathcal{L}C_X)$ coincides with the universal enveloping algebra of $\mathcal{L}C_X$, where $\mathcal{L}\colon \mathrm{dgCoalg}_{\geq 2} \to \mathrm{dgLie}_{\geq 1}$ is Quillen's rational homotopy theory functor (see [Qui69, Remark B.6.6]). The Quillen equivalence ⑥ is obtained from Lemma 2.5.26.

⑦ **- Comparing coalgebra and Lie models:** The final adjunction compares the two Lie models $\mathcal{L}(C_X)$ and $\mathfrak{l}_X$ of X. All objects of $\mathrm{dgLie}_{\geq 1}$ are fibrant and $\mathcal{L}(C_X)$ is cofibrant, so that we can find a weak equivalence $\mathcal{L}(C_X) \to \mathfrak{l}_X$ (as both are Lie models for X, similarly to ⑤). Theorem 2.4.22 now applies.

Weak naturality: Let $f\colon X \to Y$ be a map of 1-connected spaces, with $\mathfrak{l}_f\colon \mathfrak{l}_X \to \mathfrak{l}_Y$ and $C_f\colon C_X \to C_Y$ be respectively a Lie and coalgebraic model of f. This means that $[f] \cong \lambda([f])$ and $[C_f] \cong \mathcal{C}\lambda([f])$ in $Ho(\mathrm{dgLie}_{\geq 1})$ and $Ho(\mathrm{dgCoalg}_{\geq 2})$ respectively, via the equivalences λ and $\mathcal{C}$ of Theorem 2.1.3. We verify naturality of each of the Quillen equivalences in the composite in turn:

① We can take X, Y to be Kan complexes without loss of generality. Indeed, fixing a fibrant replacement functor on sSet and applying the argument below allows us to reduce to this case. Fix a basepoint x of X and let Y have basepoint $y = f(x)$. The construction of Eilenberg subcomplexes is functorial for pointed spaces, so we have a commuting diagram of simplicial sets

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\iota_X"'] & Y' \arrow[d, "\iota_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and hence a diagram of left Quillen functors

$$\begin{tikzcd} \mathrm{Mod}{-}\mathbb{H}\mathbb{Q}_{X'} \arrow[r, "(\iota_X)_!"] \arrow[d, "f'_!"'] & \mathrm{Mod}{-}\mathbb{H}\mathbb{Q}_X \arrow[d, "f_!"] \\ \mathrm{Mod}{-}\mathbb{H}\mathbb{Q}_{Y'} \arrow[r, "(\iota_Y)_!"] & \mathrm{Mod}{-}\mathbb{H}\mathbb{Q}_Y \end{tikzcd}$$

commuting up to natural isomorphism (Lemma 2.3.9), in which the horizontal functors are left Quillen equivalences.

② From this point on, we may suppose without loss of generality that $f\colon X \to Y$ is a map of 2-reduced simplicial sets, so that we obtain a map $\mathbb{G}f\colon \mathbb{G}X \to \mathbb{G}Y$ of simplicial loop groups. We have a diagram of left Quillen functors which commutes up to natural isomorphism:

$$\begin{tikzcd} \mathrm{Mod}{-}\mathcal{H}\mathbb{Q}_X \arrow[r, "\beta_X^+"] \arrow[d, "f_!"'] & (\mathbb{G}X_+, \mathcal{H}\mathbb{Q}){-}\mathrm{Bimod} \arrow[r, "\mathrm{str}_{\mathbb{G}X}"] \arrow[d, "\mathbb{G}f_!"'] & \mathbb{Q}[\mathbb{G}X]{-}\mathrm{Mod} \arrow[r, "{\kappa[\mathbb{G}X]_!}"] \arrow[d, "{\mathbb{Q}[\mathbb{G}f]_!}"] & \widehat{\mathbb{Q}}[\mathbb{G}X]{-}\mathrm{Mod} \arrow[d, "{\widehat{\mathbb{Q}}[\mathbb{G}f]_!}"] \\ \mathrm{Mod}{-}\mathcal{H}\mathbb{Q}_Y \arrow[r, "\beta_Y^+"] & (\mathbb{G}Y_+, \mathcal{H}\mathbb{Q}){-}\mathrm{Bimod} \arrow[r, "\mathrm{str}_{\mathbb{G}Y}"] & \mathbb{Q}[\mathbb{G}Y]{-}\mathrm{Mod} \arrow[r, "{\kappa[\mathbb{G}Y]_!}"] & \widehat{\mathbb{Q}}[\mathbb{G}Y]{-}\mathrm{Mod}. \end{tikzcd}$$

The left-most square commutes up to natural isomorphism by Theorem 2.3.15 and Corollary 2.3.14. The middle square commutes up to natural isomorphism by Lemma 2.3.26. The final square commutes up to natural isomorphism since we have a commuting diagram

$$\begin{tikzcd}
\mathbb{Q}[\mathbb{G}X] \arrow[r, "\kappa[\mathbb{G}X]"] \arrow[d, "\mathbb{Q}[\mathbb{G}f]"'] & \widehat{\mathbb{Q}}[\mathbb{G}X] \arrow[d, "\widehat{\mathbb{Q}}[\mathbb{G}f]"] \\
\mathbb{Q}[\mathbb{G}Y] \arrow[r, "\kappa[\mathbb{G}Y]"] & \widehat{\mathbb{Q}}[\mathbb{G}Y]
\end{tikzcd}$$

of simplicial Hopf algebras. In this diagram, $\widehat{\mathbb{Q}}[\mathbb{G}f]$ is the map of completions induced by the fact that $\mathbb{Q}[\mathbb{G}f]$ carries the augmentation ideal of $\mathbb{Q}[\mathbb{G}X]$ into that of $\mathbb{Q}[\mathbb{G}Y]$.

③ We regard the morphism $\mathcal{P}\widehat{\mathbb{Q}}[\mathbb{G}f]\colon \mathcal{P}\widehat{\mathbb{Q}}[\mathbb{G}X] \to \mathcal{P}\widehat{\mathbb{Q}}[\mathbb{G}Y]$ as an object of the functor category $\mathrm{Fun}(\Delta[1], \mathrm{sLie}_{\geq 1})$ equipped with the Reedy model structure (which coincides with the injective model structure in this case). Taking a cofibrant replacement in the Reedy model structure

$$\begin{tikzcd}
\mathfrak{g}_X \arrow[r, "\kappa_X"] \arrow[d, "\mathfrak{g}_f"'] & \mathcal{P}\widehat{\mathbb{Q}}[\mathbb{G}X] \arrow[d, "\mathcal{P}\widehat{\mathbb{Q}}[\mathbb{G}f]"] \\
\mathfrak{g}_Y \arrow[r, "\kappa_Y"] & \mathcal{P}\widehat{\mathbb{Q}}[\mathbb{G}Y]
\end{tikzcd}$$

yields a map of cofibrant (namely, free) simplicial Lie algebras $\mathfrak{g}_f\colon \mathfrak{g}_X \to \mathfrak{g}_Y$. The induced map of universal enveloping algebras gives rise to commuting diagrams

$$\begin{tikzcd}
\mathcal{U}(\mathfrak{g}_X) \arrow[r] \arrow[d, "\mathcal{U}(\mathfrak{g}_f)"'] & \widehat{\mathcal{U}}(\mathfrak{g}_X) \arrow[r] \arrow[d, "\widehat{\mathcal{U}}(\mathfrak{g}_f)"'] & \widehat{\mathbb{Q}}[\mathbb{G}X] \arrow[d, "\widehat{\mathbb{Q}}[\mathbb{G}f]"] \\
\mathcal{U}(\mathfrak{g}_Y) \arrow[r] & \widehat{\mathcal{U}}(\mathfrak{g}_Y) \arrow[r] & \widehat{\mathbb{Q}}[\mathbb{G}Y]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{U}(N(\mathfrak{g}_X)) \arrow[r, "\chi_{\mathfrak{g}_X}"] \arrow[d, "\mathcal{U}(N(\mathfrak{g}_f))"'] & N(\mathcal{U}(\mathfrak{g}_X)) \arrow[d, "N(\mathcal{U}(\mathfrak{g}_f))"] \\
\mathcal{U}(N(\mathfrak{g}_Y)) \arrow[r, "\chi_{\mathfrak{g}_Y}"] & N(\mathcal{U}(\mathfrak{g}_Y)),
\end{tikzcd}$$

the latter by Lemma 2.4.14. Consequently, we have a commuting diagram of left Quillen functors

$$\begin{tikzcd}
N(\mathfrak{g}_X)\text{–}\mathrm{Rep}^\Sigma \arrow[r] \arrow[d, "N(\mathfrak{g}_f)_!"'] & \mathcal{U}(\mathfrak{g}_X)\text{–}\mathrm{Mod} \arrow[r] \arrow[d, "\mathcal{U}(\mathfrak{g}_f)_!"'] & \widehat{\mathcal{U}}(\mathfrak{g}_X)\text{–}\mathrm{Mod} \arrow[r] \arrow[d, "\widehat{\mathcal{U}}(\mathfrak{g}_f)_!"] & \widehat{\mathbb{Q}}[\mathbb{G}X]\text{–}\mathrm{Mod} \arrow[d, "\widehat{\mathbb{Q}}[\mathbb{G}f]"] \\
N(\mathfrak{g}_Y)\text{–}\mathrm{Rep}^\Sigma \arrow[r] & \mathcal{U}(\mathfrak{g}_Y)\text{–}\mathrm{Mod} \arrow[r] & \widehat{\mathcal{U}}(\mathfrak{g}_Y)\text{–}\mathrm{Mod} \arrow[r] & \widehat{\mathbb{Q}}[\mathbb{G}Y]\text{–}\mathrm{Mod}
\end{tikzcd}$$

in which all horizontal arrows are left Quillen equivalences. The left-most square in the above diagram commutes up to isomorphism by Remark 2.4.17 and essential uniqueness of adjoints.

④ By essential uniqueness of adjoints and Remark 2.4.17, we have a diagram of left Quillen functors

$$\begin{tikzcd}
N(\mathfrak{g}_X)\text{–}\mathrm{Rep}^\Sigma \arrow[r] \arrow[d, "\mathcal{U}(N(\mathfrak{g}_f))_!"'] & N(\mathfrak{g}_X)\text{–}\mathrm{Rep} \arrow[d, "\mathcal{U}(N(\mathfrak{g}_f))_!"] \\
N(\mathfrak{g}_Y)\text{–}\mathrm{Rep}^\Sigma \arrow[r] & N(\mathfrak{g}_Y)\text{–}\mathrm{Rep}
\end{tikzcd}$$

which commutes up to natural isomorphism.

⑤ It is at this stage that the "weak" epithet really becomes necessary since we must compare the morphisms $N(\mathfrak{g}_f)\colon N(\mathfrak{g}_X) \to N(\mathfrak{g}_Y)$ and $\mathfrak{l}_f\colon \mathfrak{l}_X \to \mathfrak{l}_Y$ in the homotopy category. By hypothesis $N(\mathfrak{g}_f)$ and $\mathfrak{l}_f$ represent the same map in $Ho(\mathrm{dgLie}_{\geq 1})$, which means that we can find diagrams of dg Lie algebras

$$\begin{tikzcd} \overline{N(\mathfrak{g}_X)} \arrow[r] \arrow[d, "\overline{\mathfrak{g}_f}"'] & N(\mathfrak{g}_X) \arrow[d, "N(\mathfrak{g}_f)"] \\ \overline{N(\mathfrak{g}_Y)} \arrow[r] & N(\mathfrak{g}_Y) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \overline{\mathfrak{l}_X} \arrow[r] \arrow[d, "\overline{\mathfrak{l}_f}"'] & \mathfrak{l}_X \arrow[d, "\mathfrak{l}_f"] \\ \overline{\mathfrak{l}_X} \arrow[r] & \mathfrak{l}_Y \end{tikzcd}$$

in which the left hand side of each diagram is a morphism between cofibrant objects of $\mathrm{dgLie}_{\geq 1}$ and all horizontal morphisms are weak equivalences. Since $\overline{\mathfrak{g}_f}$ and $\overline{\mathfrak{l}_f}$ are morphisms of fibrant-cofibrant objects representing the same map the homotopy category, we can find weak equivalences

$$\overline{N(\mathfrak{g}_X)} \xrightarrow{\xi_X} \overline{\mathfrak{l}_X} \quad \text{and} \quad \overline{N(\mathfrak{g}_Y)} \xrightarrow{\xi_Y} \overline{\mathfrak{l}_Y}$$

such that the composites $\xi_Y \circ \overline{\mathfrak{g}_f}$ and $\overline{\mathfrak{l}_f} \circ \xi_X$ are homotopic. Choosing an explicit right homotopy $\Xi\colon \overline{N(\mathfrak{g}_X)} \to \mathcal{P}\overline{\mathfrak{l}_Y}$ between $\xi_Y \circ \overline{\mathfrak{g}_f}$ and $\overline{\mathfrak{l}_f} \circ \xi_X$, we have a commuting diagram in $\mathrm{dgLie}_{\geq 1}$

$$\begin{tikzcd} N(\mathfrak{g}_X) \arrow[d, "N(\mathfrak{g}_f)"'] & & \overline{N(\mathfrak{g}_X)} \arrow[ll] \arrow[rr, "\xi_X"] \arrow[dl, "\overline{\mathfrak{g}_f}"'] \arrow[dr, "\Xi"] & & \overline{\mathfrak{l}_X} \arrow[r] \arrow[d, "\overline{\mathfrak{l}_f}"] & \mathfrak{l}_X \arrow[d, "\mathfrak{l}_f"] \\ N(\mathfrak{g}_Y) & \overline{N(\mathfrak{g}_Y)} \arrow[l] \arrow[r, "\xi_Y"] & \overline{\mathfrak{l}_Y} & \mathcal{P}\overline{\mathfrak{l}_Y} \arrow[l] \arrow[r] & \overline{\mathfrak{l}_Y} \arrow[r] & \mathfrak{l}_Y \end{tikzcd}$$

in which all horizontal morphisms are weak equivalences. By Theorem 2.4.22 we have a diagram of left Quillen functors

$$\begin{tikzcd} N(\mathfrak{g}_X)\mathrm{-Rep} \arrow[dd] & & & \overline{N(\mathfrak{g}_X)}\mathrm{-Rep} \arrow[lll, "\sim"'] \arrow[rr, "\sim"] \arrow[dl] \arrow[d] & & \overline{\mathfrak{l}_X}\mathrm{-Rep} \arrow[r, "\sim"] \arrow[dd] & \mathfrak{l}_X\mathrm{-Rep} \arrow[dd] \\ & & \overline{N(\mathfrak{g}_Y)}\mathrm{-Rep} \arrow[dll, "\sim"'] \arrow[d, "\sim"'] & \mathcal{P}\overline{\mathfrak{l}_Y}\mathrm{-Rep} \arrow[dl, "\sim"'] \arrow[drr, "\sim"] & & & \\ N(\mathfrak{g}_Y)\mathrm{-Rep} & & \overline{\mathfrak{l}_Y}\mathrm{-Rep} & & & \overline{\mathfrak{l}_Y}\mathrm{-Rep} \arrow[r, "\sim"] & \mathfrak{l}_Y\mathrm{-Rep} \end{tikzcd}$$

which commutes up to natural isomorphism, where each arrow marked with "$\sim$" represents a left Quillen equivalence.

⑥ For a 2-reduced coalgebra C have an identification $\Omega C \cong \mathcal{U}(\mathcal{L}C)$. Lemma 2.5.27 applied to the given coalgebraic model $C_f\colon C_X \to C_Y$ of f yields a diagram of left Quillen functors

$$\begin{tikzcd} C_X\mathrm{-Comod} \arrow[r] \arrow[d, "(C_f)_!"'] & \Omega C_X\mathrm{-Mod} \cong \mathcal{L}C_X\mathrm{-Rep} \arrow[d, "(\Omega C_f)_!"] \\ C_Y\mathrm{-Comod} \arrow[r] & \Omega C_Y\mathrm{-Mod} \cong \mathcal{L}C_Y\mathrm{-Rep} \end{tikzcd}$$

commuting up to natural isomorphism.

⑦ Finally, we compare the morphisms $\mathcal{L}C_f$ and $\mathfrak{l}_f$, which are both Lie models of f. The argument is the same as it was for ⑤. In particular, it is not generally the case that $\mathcal{L}C_f$ fits into a strictly commuting diagram with any cofibrant replacement of $\mathfrak{l}_X$; this is another stage at which we are forced to weaken our notion of naturality at the model category level.

Independence of rational homotopy type: To complete the proof of Theorem 2.6.1, we verify that for $f\colon X \to Y$ a rational homotopy equivalence of 2-reduced simplicial sets (with $\mathfrak{l}_f$ and C_f respectively a Lie and coalgebraic model for f) each of the base change adjunctions appearing in the proof of weak naturality is a Quillen equivalence. This is now an immediate consequence of Corollary 2.3.17 and the 2-out-of-3 property of Quillen equivalences. □

Proof of Corollary 2.6.2. In §2.3.4 we used the Quillen equivalence

$$(\beta^+_X \dashv \gamma^+_X)\colon \mathrm{Mod}{-}H\mathbb{Q}_X \longrightarrow (\mathbb{G}X_+, H\mathbb{Q}){-}\mathrm{Bimod}$$

to model the smash product on the rational homotopy category of X-spectra via the smash product of their homotopy fibre spectra equipped with the diagonal action of the simplicial loop group $\mathbb{G}X$. In the proof of Theorem 2.6.1 above, we remarked that each of the Quillen equivalences ③, ④, ⑤′, ⑤″ and ⑦ is (weakly) monoidal, so induces a strong symmetric monoidal equivalence on homotopy categories. □

2.7 The Dual Picture

In the final section of this chapter, we study parametrised spectra via Sullivan's PL de Rham theory. In contrast to the algebraic characterisation of parametrised spectra given by Theorem 2.6.1, whose proof is rather complex with numerous intermediate steps, the dual picture is rather straightforward and builds directly on the Sullivan–de Rham adjunction. We prove a Quillen adjunction

$$\begin{tikzcd} \mathrm{Sp}^{\mathbb{N}}_{\mathcal{S}(A)} \arrow[r, shift left=1.5ex, "\mathcal{M}_A"] \arrow[r, phantom, "\perp"] & A{-}\mathrm{Mod}^{\mathrm{op}} \arrow[l, shift left=1.5ex, "\mathcal{P}_A"] \end{tikzcd} \tag{2.11}$$

relating modules over a cDGA A with spectra parametrised by the rational space $\mathcal{S}(A)$. When A is cofibrant and homologically connected, it represents the rational cohomology of the space $\mathcal{S}(A)$, so that (2.11) is dual to our work in §2.6 characterising rational parametrised spectra via comodules over the homology coalgebra. Since we are working dually, however, the adjunctions (2.11) do not fully capture the rational homotopy theory of spectra parametrised by $\mathcal{S}(A)$. Nevertheless, we obtain a sort of homotopical Serre–Swan theorem which characterises a full subcategory of finitely-presented objects of the rational homotopy category $Ho_{\mathbb{Q}}(\mathrm{Sp}_{\mathcal{S}(A)})$ in terms of perfect A-modules (Theorem 2.7.34). When we assume 1-connectedness, we obtain a sharper result (Theorem 2.7.42) characterising a much larger full subcategory of finitely-presented objects.

Throughout this section we work with rational *cochain* complexes, as is reflected in the notation: "DG" rather than "dg", "cDGA" for commutative DG algebras and so on. For $A \in \mathrm{DGAlg}$, we write

$$\mathrm{DGAlg}_{/\!/ A} := (\mathrm{DGAlg}^{A/})_{/A}$$

for the category of augmented A-algebras.

2.7.1 Slicing and Stabilising

The Sullivan–de Rham adjunction has the useful property that the rational cochain functor $\mathcal{O}\colon \mathrm{sSet} \to \mathrm{DGAlg}^{\mathrm{op}}$ preserves finite sSet-tensors. It is ultimately this property which allows us to relate parametrised spectra to modules via a modification of the sequential stabilisation machine.

Recall that for $\mathcal{M}$ a model category and any object $x \in \mathcal{M}$, the slice categories $\mathcal{M}^{x/}$ and $\mathcal{M}_{/x}$ inherit model structures for which weak equivalences, cofibrations and fibrations are created by the forgetful functors to $\mathcal{M}$. When $\mathcal{M}$ is a sSet-enriched model category with finite sSet-tensors (and powers), the slice categories $\mathcal{M}^{x/}$ and $\mathcal{M}_{/x}$ also inherit these properties [Qui67, Proposition II.2.6].

Lemma 2.7.1. *For any $A \in \mathrm{DGAlg}$ there is a sliced Quillen adjunction*

$$\begin{tikzcd} R_{\mathcal{S}(A)} \arrow[r, shift left, "\mathcal{O}_A"] \arrow[r, phantom, "\perp"] & (\mathrm{DGAlg}_{/\!/ A})^{\mathrm{op}} \arrow[l, shift left, "\mathcal{S}_A"] \end{tikzcd}$$

in which the left adjoint preserves finite sSet*-tensors.*

Proof. The existence of the adjunction on categories of retractive objects follows from a standard argument: the right adjoint sends

$$\mathcal{S}_A\colon (A \leftarrow B \leftarrow A) \longmapsto (\mathcal{S}(A) \to \mathcal{S}(B) \to \mathcal{S}(A)),$$

whilst the left adjoint sends the retractive space $X \in R_{\mathcal{S}(A)}$ to the augmented A-algebra determined by the top row of the diagram of pullback squares in DGAlg

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\varepsilon_A"'] & \mathcal{O}_A(X) \arrow[r] \arrow[d] & A \arrow[d, "\varepsilon_A"] \\ \mathcal{O}(\mathcal{S}(A)) \arrow[r] & \mathcal{O}(X) \arrow[r] & \mathcal{O}(\mathcal{S}(A)), \end{tikzcd}$$

where $\varepsilon_A\colon A \to \mathcal{O}(\mathcal{S}(A))$ is (the opposite of) the $(\mathcal{O} \dashv \mathcal{S})$-counit. Recall that for a simplicial set K and retractive space $X \in R_{\mathcal{S}(A)}$, the tensoring is defined by the pushout diagram of simplicial sets

$$\begin{tikzcd} K \times \mathcal{S}(A) \arrow[r] \arrow[d] & K \times X \arrow[d] \\ \mathcal{S}(A) \arrow[r] & K \otimes_{\mathcal{S}(A)} X. \end{tikzcd}$$

Since $\mathcal{O}$ preserves finite sSet-tensors and colimits, for any finite simplicial set K we have a pullback diagram

$$\begin{tikzcd} \mathcal{O}\big(K \otimes_{\mathcal{S}(A)} X\big) \arrow[r] \arrow[d] & \mathcal{O}(K) \otimes \mathcal{O}(X) \arrow[d] \\ \mathcal{O}(\mathcal{S}(A)) \arrow[r] & \mathcal{O}(K) \otimes \mathcal{O}(\mathcal{S}(A)) \end{tikzcd}$$

Further pulling back along $\varepsilon_A \colon A \to \mathcal{O}(\mathcal{S}(A))$ gives $\mathcal{O}_A(K \otimes_{\mathcal{S}(A)} X)$.

On the other hand, the tensoring of K with $\mathcal{O}_A(X)$ in $(\mathrm{DGAlg}_{/\!/A})^{\mathrm{op}}$ is represented by the top left of the iterated pullback diagram

$$\begin{tikzcd} K \otimes^A \mathcal{O}_A(X) \arrow[r] \arrow[d] & \mathcal{O}(K) \otimes \mathcal{O}_A(X) \arrow[r] \arrow[d] & \mathcal{O}(K) \otimes \mathcal{O}(X) \arrow[d] \\ A \arrow[r] & \mathcal{O}(K) \otimes A \arrow[r] & \mathcal{O}(K) \otimes \mathcal{O}(\mathcal{S}(A)) \end{tikzcd}$$

in DGAlg. The right-hand square is a pullback since $\mathcal{O}(K) \otimes (-) \colon ch_+ \to ch_+$ is exact and limits in DGAlg are created by the forgetful functor to ch_+. The bottom horizontal composite in the above diagram coincides with the composite

$$A \xrightarrow{\varepsilon_A} \mathcal{O}(\mathcal{S}(A)) \longrightarrow \mathcal{O}(K) \otimes \mathcal{O}(\mathcal{S}(A)),$$

so that $K \otimes^A \mathcal{O}_A(X) \cong \mathcal{O}_A(K \otimes_{\mathcal{S}(A)} X)$ for finite K by the pasting law of pullbacks.

Finally, the adjunction $(\mathcal{O}_A \dashv \mathcal{S}_A)$ is Quillen since $\mathcal{S}_A$ preserves fibrations and acyclic fibrations. ☐

Lemma 2.7.2. *For a map $f \colon A \to B$ of cDGAs, there is a diagram of left Quillen functors*

$$\begin{tikzcd} R_{\mathcal{S}(B)} \arrow[r, "\mathcal{O}_B"] \arrow[d, "\mathcal{S}(f)_!"'] & \big(\mathrm{DGAlg}_{/\!/B}\big)^{\mathrm{op}} \arrow[d, "f^*"] \\ R_{\mathcal{S}(A)} \arrow[r, "\mathcal{O}_A"] & \big(\mathrm{DGAlg}_{/\!/A}\big)^{\mathrm{op}} \end{tikzcd}$$

commuting up to natural isomorphism.

Proof. The map of cDGAs f induces a functor

$$f^* \colon \mathrm{DGAlg}_{/\!/B} \longrightarrow \mathrm{DGAlg}_{/\!/A}$$

via pullback, with left adjoint

$$\begin{aligned} f_! \colon \mathrm{DGAlg}_{/\!/A} &\longrightarrow \mathrm{DGAlg}_{/\!/B} \\ C &\longmapsto B \bigotimes_A C. \end{aligned}$$

The functor f^* is easily seen to be right Quillen, so is left Quillen when regarded as a functor between opposite categories.

Finally, consider a retractive space $X \in R_{\mathcal{S}(B)}$. Unwinding the definitions, we find that the algebras $\mathcal{O}_A(\mathcal{S}(f)_!(X))$ and $f^*\mathcal{O}_B(X)$ are both obtained as the pullback of the cospan

$$\mathcal{O}(X) \longrightarrow \mathcal{O}(\mathcal{S}(B)) \longleftarrow A,$$

so that we have a natural isomorphism $f^* \circ \mathcal{O}_B \cong \mathcal{O}_A \circ \mathcal{S}(f)_!$ as required. □

Remark 2.7.3. The left Quillen functor $f^* \colon (\mathrm{DGAlg}_{//B})^{\mathrm{op}} \to (\mathrm{DGAlg}_{//A})^{\mathrm{op}}$ preserves finite sSet-tensors. Indeed, for K a finite simplicial set and C an augmented B-algebra, both $f^*(K \otimes^B C)$ and $K \otimes^A f^*C$ are pullbacks of the cospan

$$A \longrightarrow \mathcal{O}(K) \otimes B \longleftarrow \mathcal{O}(K) \otimes C$$

since $\mathcal{O}(K) \otimes (-)$ preserves pullbacks.

Remark 2.7.4. For any cDGA A, the finite sSet-tensoring on $(\mathrm{DGAlg}_{//A})^{\mathrm{op}}$ induces a left Quillen suspension endofunctor Σ_A as follows. For an augmented A-algebra B, the object $\Sigma_A B \in \mathrm{DGAlg}^{A/}_{A/}$ is the pullback algebra

$$\begin{tikzcd} \Sigma_A B \arrow[r] \arrow[d] & \mathcal{O}(\Delta[1]) \otimes^A B \arrow[d] \\ A \arrow[r] & \mathcal{O}(\partial\Delta[1]) \otimes^A B. \end{tikzcd}$$

The endofunctor Σ_A on $\mathrm{DGAlg}_{//A}$ so defined preserves fibrations and acyclic fibrations by the sSet-enrichment of the model structure. The endofunctor Σ_A also preserves limits and filtered colimits so has a left adjoint Ω_A by the Adjoint Functor Theorem. As a left Quillen endofunctor of the opposite category, Σ_A models suspension on $Ho(\mathrm{DGAlg}_{//A})^{\mathrm{op}}$.

Corollary 2.7.5. *For any $A \in \mathrm{DGAlg}$, the functor $\mathcal{O}_A \colon R_{\mathcal{S}(A)} \to (\mathrm{DGAlg}_{//A})^{\mathrm{op}}$ preserves suspension endofunctors; there is a natural isomorphism $\mathcal{O}_A \circ \Sigma_{\mathcal{S}(A)} \cong \Sigma_A \circ \mathcal{O}_A$.*

Proof. This follows immediately from Lemma 2.7.1: $\mathcal{O}_A$ preserves finite sSet-tensors and colimits. □

For any $A \in \mathrm{DGAlg}$, the relative adjunction of Lemma 2.7.1 induces an adjunction on categories of sequential spectrum objects

$$\begin{tikzcd} \mathrm{Sp}^{\mathbb{N}}(R_{\mathcal{S}(A)}; \Sigma_{\mathcal{S}(A)}) \arrow[r, shift left, "\mathcal{O}_A"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\mathbb{N}}((\mathrm{DGAlg}_{//A})^{\mathrm{op}}; \Sigma_A) \arrow[l, shift left, "\mathcal{S}_A"] \end{tikzcd}$$

by levelwise application of the adjoint functors $(\mathcal{O}_A \dashv \mathcal{S}_A)$ (Lemma A.3.14). We cannot quite employ the sequential stabilisation machine as $\mathrm{DGAlg}_{//A}$ is not combinatorial: it is neither locally presentable nor cofibrantly generated.

Definition 2.7.6. Fix $A \in \mathrm{DGAlg}$. A *sequential cospectrum* of augmented A-algebras is a sequence $\{B_n\}_{n\in\mathbb{N}} \subset \mathrm{DGAlg}_{//A}$ together with structure maps $\sigma_n \colon B_{n+1} \to \Sigma_A B_n$ for all $n \in \mathbb{N}$. Morphisms of cospectra are sequences of maps $\{f_n \colon B_n \to C_n\}_{n\in\mathbb{N}}$ respecting the structure maps, to wit the diagram

$$\begin{tikzcd} B_{n+1} \arrow[r, "\sigma^B_n"] \arrow[d, "f_{n+1}"'] & \Sigma_A B_n \arrow[d, "\Sigma_A f_n"] \\ C_{n+1} \arrow[r, "\sigma^C_n"] & C_n \end{tikzcd}$$

commutes for all n. We write $\mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A)$ for the category of sequential cospectra of A-algebras.

Remark 2.7.7. There is a canonical isomorphism of categories

$$\mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A)^{\mathrm{op}} \cong \mathrm{Sp}^{\mathbb{N}}\big((\mathrm{DGAlg}_{/\!/A})^{\mathrm{op}}; \Sigma_A\big).$$

For a map of cDGAs $f\colon A \to B$, we have an adjunction on categories of cospectra

$$\begin{tikzcd} \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A) \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] & \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(B) \arrow[l, shift left, "f^*"] \end{tikzcd}$$

given by levelwise application of the adjunction $(f_! \dashv f^*)\colon \mathrm{DGAlg}_{/\!/A} \to \mathrm{DGAlg}_{/\!/B}$ by Remark 2.7.3. The diagram of left Quillen functors of Lemma 2.7.2 gives rise to a diagram of left adjoint functors

$$\begin{tikzcd} \mathrm{Sp}^{\mathbb{N}}_{\mathcal{S}(B)} \arrow[r, "\mathcal{O}_B"] \arrow[d, "\mathcal{S}(f)_!"'] & \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(B)^{\mathrm{op}} \arrow[d, "f^*"] \\ \mathrm{Sp}^{\mathbb{N}}_{\mathcal{S}(A)} \arrow[r, "\mathcal{O}_A"] & \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A)^{\mathrm{op}} \end{tikzcd}$$

which commutes up to natural isomorphism.

Lemma 2.7.8. *For any $A \in$ DGAlg, the category of sequential cospectra $\mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A)$ is a combinatorial model category with respect to the* injective model structure, *for which the cofibrations and weak equivalences are levelwise.*

Proof. We first show that $\mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A)$ is locally presentable. We identity the category of sequential cospectra with the category of coalgebras over an accessible comonoad on $\mathrm{Fun}(\mathbb{N}, \mathrm{DGAlg}_{/\!/A})$. Since the latter category is locally presentable, this shows local presentability of $\mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A)$ by [CR14, Proposition A.1].

Let $\mathbb{L}$ be the comonad on the category of sequences $\mathrm{Fun}(\mathbb{N}, \mathrm{DGAlg}_{/\!/A})$ which sends the sequence $\{B_n\}_{n\in\mathbb{N}}$ to the sequence with n-th term

$$\mathbb{L}B_n := \prod_{i=0}^{n} \Sigma_A^{n-i}(B_i).$$

The comultiplication and counit are determined by diagonal maps and projections, in an essentially dual way to the argument of Lemma 1.2.33. Filtered colimits commute with finite limits in $\mathrm{DGAlg}_{/\!/A}$ and Σ_A is a left adjoint, so that the comonad $\mathbb{L}$ is accessible. There is an isomorphism of categories $\mathbb{L}-\mathrm{Coalg} \cong \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A)$, from which we conclude local presentability.

The injective model structure is obtained by applying the Left Transfer Theorem A.1.4 to the comonadic adjunction

$$\begin{tikzcd} \mathrm{Fun}(\mathbb{N}, \mathrm{DGAlg}_{/\!/A}) \arrow[r, shift right, "\mathbb{L}"'] \arrow[r, phantom, "\perp"] & \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A), \arrow[l, shift right] \end{tikzcd}$$

where $\mathrm{Fun}(\mathbb{N}, \mathrm{DGAlg}_{/\!/A})$ is equipped with the level model structure, which is combinatorial. To verify the conditions of the Left Transfer Theorem, we first construct a cofibrant replacement functor $\mathcal{Q}$ on $\mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A)$ as follows. For a cospectrum B, we take a functorial factorisation of the initial map into a cofibration followed by an acyclic fibration

$$A \rightarrowtail \mathcal{Q}B_0 \twoheadrightarrow B_0.$$

Let us suppose that for all $0 \le k \le n$ we have commuting diagrams in $\mathrm{DGAlg}_{//A}$

$$A \rightarrowtail \mathcal{Q}B_k \twoheadrightarrow B_k \quad \text{and} \quad \begin{tikzcd} \mathcal{Q}B_k \arrow[r, two heads, "\sim"] \arrow[d] & B_k \arrow[d] \\ \Sigma_A(\mathcal{Q}B_{k-1}) \arrow[r, two heads, "\sim"] & \Sigma_A B_{k-1} \end{tikzcd}$$

which are functorial in B. Note that the bottom horizontal morphism in the right-hand diagram is indeed an acyclic fibration since Σ_A is right Quillen on $\mathrm{DGAlg}_{//A}$. The next stage $\mathcal{Q}B_{n+1}$ is defined first by taking the pullback

$$\begin{tikzcd} \mathcal{Q}'B_{n+1} \arrow[r, two heads, "\sim"] \arrow[d] & B_{n+1} \arrow[d] \\ \Sigma_A(\mathcal{Q}B_n) \arrow[r, two heads, "\sim"] & \Sigma_A B_n \end{tikzcd}$$

followed by the funtorial factorisation

$$A \rightarrowtail \mathcal{Q}B_{n+1} \xtwoheadrightarrow{\sim} \mathcal{Q}'B_{n+1}.$$

By induction, we obtain a level cofibrant cospectrum $\mathcal{Q}B$ together with a level weak equivalence $\mathcal{Q}B \to B$.

It remains to establish the existence of good cylinder objects

$$B \coprod B \rightarrowtail \mathrm{Cyl}(B) \xrightarrow{\sim} B$$

for all cospectra B. For this we also employ an inductive argument over the level n, noting that colimits of cospectra are created by the forgetful functor (as are limits, in fact). At level zero, we take a functorial factorisation of $B_0 \coprod B_0 \to B_0$:

$$B_0 \coprod B_0 \rightarrowtail \mathrm{Cyl}(B)_0 \xtwoheadrightarrow{\sim} B_0.$$

Suppose that for all $0 \le k \le n$ we have factorisations

$$B_k \coprod B_k \rightarrowtail \mathrm{Cyl}(B)_k \xtwoheadrightarrow{\sim} B_k$$

together with commuting diagrams

$$\begin{tikzcd} B_k \coprod B_k \arrow[r, tail] \arrow[d] & \mathrm{Cyl}(B)_k \arrow[r, two heads, "\sim"] \arrow[d] & B_k \arrow[d] \\ \Sigma_A B_{k-1} \coprod \Sigma_A B_{k-1} \arrow[r] & \Sigma_A \mathrm{Cyl}(B)_{k-1} \arrow[r, two heads, "\sim"] & \Sigma_A B_{k-1} \end{tikzcd}$$

which are functorial in B. We then construct the diagram

$$\begin{tikzcd} B_{n+1} \coprod B_{n+1} \arrow[r, tail] \arrow[d] & \mathrm{Cyl}(B)_{n+1} \arrow[r, two heads, "\sim"] & P_{n+1} \arrow[r, two heads, "\sim"] \arrow[d] & B_{n+1} \arrow[d] \\ \Sigma_A B_n \coprod \Sigma_A B_n \arrow[r] & \Sigma_A(B_n \coprod B_n) \arrow[r] & \Sigma_A \mathrm{Cyl}(B)_n \arrow[r, two heads, "\sim"] & \Sigma_A B_n \end{tikzcd}$$

in which the right-hand square is a pullback and the three horizontal terms on the top left constitute a functorial factorisation of $B_{n+1} \coprod B_{n+1} \to P_{n+1}$ into a cofibration followed by an acyclic fibration. By induction we obtain the desired good cylinder objects (which are in fact functorial). This completes the proof. □

Corollary 2.7.9. *For each $n \in \mathbb{N}$ there is a Quillen adjunction*

$$\begin{tikzcd} \mathrm{DGAlg}_{//A} \arrow[r, shift right, "F^n"'] \arrow[r, phantom, "\perp"] & \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A)_{\mathrm{inj}} \arrow[l, shift right, "\mathrm{Ev}_n"'] \end{tikzcd}$$

in which the left adjoint evaluates the n-th term of a sequential cospectrum.

Proof. The functor $\mathrm{Fun}(\mathbb{N}, \mathrm{DGAlg}_{//A}) \to \mathrm{DGAlg}_{//A}$ which evaluates the n-th term of a sequence has a two-sided adjoint ι_n which sends the augmented A-algebra B to the sequence with n-th term B and all other terms given by the zero object. The composite adjunction

$$\begin{tikzcd} \mathrm{DGAlg}_{//A} \arrow[r, shift right, "\iota_n"'] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(\mathbb{N}, \mathrm{DGAlg}_{//A}) \arrow[l, shift right] \arrow[r, shift right, "L"'] \arrow[r, phantom, "\perp"] & \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A) \arrow[l, shift right] \end{tikzcd}$$

is Quillen for the injective model structure on cospectra. □

Corollary 2.7.10. *For any morphism of cDGAs $f\colon A \to B$ there is a diagram of left Quillen functors*

$$\begin{tikzcd} (\mathrm{Sp}^{\mathbb{N}}_{\mathcal{S}(B)})_{\mathrm{proj}} \arrow[r, "\mathcal{O}_B"] \arrow[d, "\mathcal{S}(f)_!"'] & \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(B)^{\mathrm{op}}_{\mathrm{inj}} \arrow[d, "f^*"] \\ (\mathrm{Sp}^{\mathbb{N}}_{\mathcal{S}(A)})_{\mathrm{proj}} \arrow[r, "\mathcal{O}_A"] & \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A)^{\mathrm{op}}_{\mathrm{inj}} \end{tikzcd}$$

which commutes up to natural isomorphism.

Proof. The diagram is from Remark 2.7.7. For any cDGA A, the prolonged functor $\mathcal{S}_A\colon \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A)^{\mathrm{op}} \to \mathrm{Sp}^{\mathbb{N}}_{\mathcal{S}(A)}$ is right Quillen since it is given levelwise by the right Quillen functor $\mathcal{S}_A$ of Lemma 2.7.1.

Similarly, the functor $f_!\colon \mathrm{DGAlg}_{//A} \to \mathrm{DGAlg}_{//B}$ is left Quillen so gives rise to a left Quillen functor on injective model categories of sequential cospectra. □

2.7.2 Algebras and Modules

In this section, we pass from cospectral augmented algebras to unbounded DG modules. The argument works via an intermediate model category of module cospectra, and uses a key relationship between different models for the suspension: one defined via sSet-tensors and the other by shifts.

Fix $A \in \mathrm{DGAlg}$ and for an augmented A-algebra $B \in \mathrm{DGAlg}_{//A}$ write $\varepsilon\colon B \to A$ for the augmentation map. The augmentation ideal $\mathrm{aug}_A(B) := \ker(\varepsilon)$ is a connective A-module in a natural way; a direct way to see this is via the splitting of chain complexes $B \cong A \oplus \mathrm{aug}_A(B)$ induced by the unit map $A \to B$. Taking augmentation ideals is functorial

$$\mathrm{aug}_A\colon \mathrm{DGAlg}_{//A} \longrightarrow A\text{–}\mathrm{Mod}^+,$$

and the functor aug_A has left adjoint given by sending a connective DG A-module M to the free symmetric A-algebra it generates: $M \mapsto \mathrm{Sym}_A(M)$. The augmentation of $\mathrm{Sym}_A(M)$ is the natural one obtained by applying Sym_A to the zero map $M \to 0$.

Remark 2.7.11. For any $A \in \mathrm{DGAlg}$ the categories $A{-}\mathrm{Mod}^+$ and $A{-}\mathrm{Mod}$ of connective, respectively unbounded, DG A-modules are locally presentable. This is a result of the fact that $A{-}\mathrm{Mod}^+$ and $A{-}\mathrm{Mod}$ are the categories of algebras over the colimit-preserving monad $A \otimes (-)$ on the locally presentable categories Ch^+ and Ch of connective and unbounded rational cochain complexes respectively. Recall that Ch^+ and Ch are both combinatorial model categories and in both categories a map $f\colon M \to N$ of cochain complexes is

- a weak equivalence if it is a quasi-isomorphism; and
- a fibration if it is degreewise epic.

There are free-forgetful adjunctions

$$\begin{tikzcd} Ch^{(+)} \arrow[r, shift left, "A\otimes(-)"] \arrow[r, phantom, "\perp"] & A{-}\mathrm{Mod}^{(+)} \arrow[l, shift left] \end{tikzcd}$$

which equip $A{-}\mathrm{Mod}$ and $A{-}\mathrm{Mod}^+$ with combinatorial model structures (the evident cochain version of Corollary 2.5.13 holds). In either case, weak equivalences and fibrations of A-modules are detected on underlying cochain complexes.

The inclusion of connective cochain complexes $Ch^+ \hookrightarrow Ch$ is a right Quillen functor with left adjoint

$$\Big[\cdots \to M_{-1} \xrightarrow{d_{-1}} M_0 \xrightarrow{d_0} M_1 \to \cdots\Big] \xmapsto{\mathrm{cn}^0} \Big[\cdots \to 0 \longrightarrow \mathrm{coker}(d_{-1}) \xrightarrow{d_0'} M_1 \to \cdots\Big],$$

where $d_0'\colon \mathrm{coker}(d_{-1}) = M_0/\mathrm{im}(d_{-1}) \to M_1$ is the map induced by d_0. The unit of the adjunction $M \to \mathrm{cn}^0(M)$ is a cohomology isomorphism in dimensions $n \geq 0$.

For any cDGA A, the fully-faithful functor $\iota_A\colon A{-}\mathrm{Mod}^+ \to A{-}\mathrm{Mod}$ preserves limits, filtered colimits, fibrations and weak equivalences. The Adjoint Functor Theorem implies that there is a left adjoint $\mathrm{cn}^0_A\colon A{-}\mathrm{Mod} \to A{-}\mathrm{Mod}^+$; the resulting adjunction is Quillen. Via monadicity we can find an explicit description of cn^0_A, but we do not record it here since it is nevertheless rather opaque. For a free A-module $M = A \otimes N$ we have $\mathrm{cn}^0_A M \cong A \otimes \mathrm{cn}^0 N$ by uniqueness of adjoints.

More generally, for any $k \in \mathbb{Z}$ the category of *k-connective* A-modules $A{-}\mathrm{Mod}^{\geq k}$ carries a locally presentable model structure with respect to which there is a Quillen adjunction

$$\begin{tikzcd} A{-}\mathrm{Mod} \arrow[r, shift left, "\mathrm{cn}^k_A"] \arrow[r, phantom, "\perp"] & A{-}\mathrm{Mod}^{\geq k}. \arrow[l, shift left] \end{tikzcd}$$

Recall that an A-module M is said to be *k-connective* (or $(k-1)$-*connected*) if $M_n = 0$ for $n < k$.

Lemma 2.7.12. *The adjunction*

$$\begin{tikzcd} A{-}\mathrm{Mod}^+ \arrow[r, shift left, "\mathrm{Sym}_A"] \arrow[r, phantom, "\perp"] & \mathrm{DGAlg}_{/\!/A} \arrow[l, shift left, "\mathrm{aug}_A"] \end{tikzcd}$$

is Quillen.

Proof. For any augmented A-algebra B, the augmentation and unit induce a splitting $B \cong A \oplus \mathrm{aug}_A(B)$ of the underlying cochain complex. It follows at once that aug_A preserves weak equivalences and fibrations, so is a right Quillen functor. □

Corollary 2.7.13. *For any morphism of cDGAs $f\colon A \to B$, there is a diagram of left Quillen functors*

$$\begin{tikzcd} A\text{--}\mathrm{Mod}^+ \arrow[r, "\mathrm{Sym}_A"] \arrow[d, "f_!"'] & \mathrm{DGAlg}_{//A} \arrow[d, "f_!"] \\ B\text{--}\mathrm{Mod}^+ \arrow[r, "\mathrm{Sym}_B"] & \mathrm{DGAlg}_{//B} \end{tikzcd}$$

which commutes up to natural isomorphism.

Proof. Recall that f produces an adjunction $(f_! \dashv f^*)\colon A\text{--}\mathrm{Mod}^+ \to B\text{--}\mathrm{Mod}^+$ via extension and restriction of scalars. Restriction of scalars is the identity on underlying objects, so is right Quillen.

We argue that there is a natural isomorphism of composite right adjoint functors $\mathrm{aug}_A \circ f^* \cong f^* \circ \mathrm{aug}_B\colon \mathrm{DGAlg}_{//B} \to A\text{--}\mathrm{Mod}^+$, which is sufficient to prove the assertion by essential uniqueness of adjoints. Indeed, for C an augmented B-algebra the fact that limits in DGAlg are created by the forgetful functor to Ch^+ implies that the underlying cochain complex of the pullback algebra f^*C splits as $f^*C \cong A \oplus \mathrm{aug}_B(C)$. □

Our goal is to pass to the stable setting, for which we shall need to have a model for the suspension functor on $(A\text{--}\mathrm{Mod}^+)^{\mathrm{op}}$. Similarly to Lemma 2.4.5, we find that shifting does the trick. Note that we are working dually: shifting a chain complex up by one models suspension, whereas for cochain complexes this procedure models the loop space.

Lemma 2.7.14. *The endofunctor on $A\text{--}\mathrm{Mod}^+$ determined on objects by*

$$\ell\colon M \longmapsto M[-1] := M \otimes \mathbb{Q}[-1]$$

is right Quillen and models the loop space on $Ho(A\text{--}\mathrm{Mod}^+)$.

Proof. Tensoring an A-module M with the connective cochain complex

$$I^\bullet = \Big[\underset{0}{\mathbb{Q} \oplus \mathbb{Q}} \xrightarrow{(a,b)\mapsto a-b} \underset{1}{\mathbb{Q}} \longrightarrow 0 \longrightarrow \cdots \Big]$$

furnishes a good path object for M. Since M is necessarily fibrant, the loop space of M in $Ho(A\text{--}\mathrm{Mod}^+)$ is modelled by the pullback of the cospan

$$0 \longrightarrow M \oplus M \longleftarrow M \otimes I^\bullet,$$

which coincides with $M[-1]$. The endofunctor ℓ preserves fibrations and weak equivalences, and has left adjoint $\varsigma\colon N \mapsto \mathrm{cn}^0_A N[1]$ so is indeed right Quillen. □

Remark 2.7.15. The looping endofunctors ℓ commute with restriction of scalars by inspection.

In §2.7.1 we discussed suspension endofunctors on the categories $(\mathrm{DGAlg}_{//A})^{\mathrm{op}}$ defined via sSet-tensors. On the other hand, Lemma 2.7.14 characterises the suspension endofunctor on $(A\text{--}\mathrm{Mod}^+)^{\mathrm{op}}$ differently, by shifting the underlying cochain complexes. The following result shows that the $(\mathrm{Sym}_A \dashv \mathrm{aug}_A)$-adjunctions preserve suspension up to homotopy:

Lemma 2.7.16. *For any augmented A-algebra B, there are natural weak equivalences*

$$\eta_B\colon \mathrm{aug}_A(\Sigma_A B) \longrightarrow (\mathrm{aug}_A B)[-1] \quad \textit{and} \quad \tau_B\colon (\mathrm{aug}_A B)[-1] \longrightarrow \mathrm{aug}_A(\Sigma_A B),$$

and, moreover, $\eta_B \circ \tau_B$ is the identity.

Proof. For $B \in \mathrm{DGAlg}_{//A}$ and K a (finite) simplicial set, we have a splitting of rational cochain complexes

$$K \otimes^A B \cong A \oplus \big(\mathcal{O}(K) \otimes \mathrm{aug}_A(B)\big).$$

Recalling the construction of Σ_A (Remark 2.7.4) and using that aug_A is right Quillen, we have a pullback diagram

$$\begin{tikzcd}
\mathrm{aug}_A(\Sigma_A B) \arrow[r] \arrow[d] & \mathcal{O}(\Delta[1]) \otimes \mathrm{aug}_A B \arrow[d, two heads] \\
0 \arrow[r] & \mathcal{O}(\partial\Delta[1]) \otimes \mathrm{aug}_A B \cong \mathrm{aug}_A B \oplus \mathrm{aug}_A B
\end{tikzcd}$$

of connective A-modules, where the right-hand vertical morphism is a fibration as indicated. Since $A{-}\mathrm{Mod}^+$ is right proper, the above pullback diagram is also a homotopy pullback.

A key ingredient in Sullivan's approach to rational homotopy theory is the fact that for any simplicial set K, the natural map $\int\colon \mathcal{O}(K) \to N^\bullet(K)$ to normalised rational simplicial cochains induced by the integration pairing is a quasi-isomorphism [BG76, Theorem 2.2]. Via these comparison maps, we obtain a commuting diagram of cochain complexes

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \mathcal{O}(\partial\Delta[1]) \arrow[d] & \mathcal{O}(\Delta[1]) \arrow[l, two heads] \arrow[d] \\
0 \arrow[r] & N^\bullet(\partial\Delta[1]) \otimes \mathrm{aug}_A B & N^\bullet(\Delta[1]), \arrow[l, two heads]
\end{tikzcd}$$

in which all vertical morphisms are weak equivalences. Since Ch^+ is right proper, the pullbacks of the above horizontal cospans both compute homotopy pullbacks. Consequently, the induced map of pullbacks $P \to \mathbb{Q}[-1]$ is a weak equivalence. Moreover, if $\mathbb{Q}[-1] \to \mathcal{O}(\Delta[1])$ is the map $a \mapsto a \cdot dt_1$ then the commuting diagram

$$\begin{tikzcd}
\mathbb{Q}[-1] \arrow[r] \arrow[d] & \mathcal{O}(\Delta[1]) \arrow[d] \\
0 \arrow[r] & \mathcal{O}(\partial\Delta[1])
\end{tikzcd}$$

implies a map $\mathbb{Q}[-1] \to P$ which is a section of $P \to \mathbb{Q}[-1]$ and is therefore a weak equivalence by the 2-out-of-3 property. Tensoring with rational cochain complexes is exact so that $\mathrm{aug}_A(\Sigma_A B) \cong P \otimes \mathrm{aug}_A B$. By the Künneth Theorem, tensoring the weak equivalences $P \to \mathbb{Q}[-1]$ and $\mathbb{Q}[-1] \to P$ with $\mathrm{aug}_A B$ yields the sought-after weak equivalences η_B and τ_B respectively. Naturality of η and τ is manifest in their construction. $\square$

The relationship between the suspension endofunctors we have just proven allows us to compare algebraic and modular cospectrum objects:

Definition 2.7.17. Fix $A \in \mathrm{DGAlg}$. A *sequential cospectrum* of connective A-modules is a sequence $\{M_n\}_{n\in\mathbb{N}} \subset A\mathrm{-Mod}^+$ together with structure maps $\sigma_n \colon M_{n+1} \to \ell M_n$ for all $n \in \mathbb{N}$. Morphisms of cospectra are sequences of maps $\{f_n \colon M_n \to N_n\}_{n\in\mathbb{N}}$ respecting the structure maps, so that the diagram

$$\begin{tikzcd} M_{n+1} \arrow[r, "\sigma_n^M"] \arrow[d, "f_{n+1}"'] & \ell M_n \arrow[d, "\ell f_n"] \\ N_{n+1} \arrow[r, "\sigma_n^N"] & N_n \end{tikzcd}$$

commute for all n. We write $\mathrm{CoSp}^{\mathbb{N}}_{\mathrm{mod}}(A)$ for the category of sequential cospectra of connective A-modules.

The proofs of Lemma 2.7.8 and Corollary 2.7.9 carry over from algebras to modules almost verbatim. We obtain the following

Lemma 2.7.18. *For any $A \in \mathrm{DGAlg}$, the category of sequential cospectra $\mathrm{CoSp}^{\mathbb{N}}_{\mathrm{mod}}(A)$ is a combinatorial model category with respect to the* injective model structure, *for which the cofibrations and weak equivalences are levelwise.*

Corollary 2.7.19. *For each $n \in \mathbb{N}$ there is a Quillen adjunction*

$$\begin{tikzcd} A\mathrm{-Mod}^+ \arrow[r, shift right, "F^n"'] \arrow[r, phantom, "\perp"] & \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{mod}}(A)_{\mathrm{inj}} \arrow[l, shift right, "\mathrm{Ev}_n"'] \end{tikzcd}$$

in which the left adjoint evaluates the n-th term of a sequential cospectrum.

Lemma 2.7.20. *For any $A \in \mathrm{DGAlg}$, there is a Quillen adjunction*

$$\begin{tikzcd} \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{mod}}(A) \arrow[r, shift left, "\mathrm{Sym}_A"] \arrow[r, phantom, "\perp"] & \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A) \arrow[l, shift left, "\mathrm{aug}_A^\infty"] \end{tikzcd}$$

in which the left adjoint is given by levelwise application of $\mathrm{Sym}_A \colon A\mathrm{-Mod}^+ \to \mathrm{DGAlg}_{/\!/A}$.

Proof. Recall that we write ς and Ω_A for the left adjoints of $\ell \colon A\mathrm{-Mod}^+ \to A\mathrm{-Mod}^+$ and $\Sigma_A \colon \mathrm{DGAlg}_{/\!/A} \to \mathrm{DGAlg}_{/\!/A}$ respectively. Let $\tau^\vee \colon \Omega_A \circ \mathrm{Sym}_A \Rightarrow \mathrm{Sym}_A \circ \varsigma$ be the natural transformation dual to $\tau \colon \ell \circ \mathrm{aug}_A \Rightarrow \mathrm{aug}_A \circ \Sigma_A$ (Lemma 2.7.16). Recall that this means the components of $\tau^\vee$ are determined by the diagram of hom-sets

$$\begin{tikzcd} A\mathrm{-Mod}^+(M, (\mathrm{aug}_A B)[-1]) \arrow[r, "(\tau_B)_*"] \arrow[d, "\cong"'] & A\mathrm{-Mod}^+(M, \mathrm{aug}_A(\Sigma_A B)) \arrow[d, "\cong"] \\ \mathrm{DGAlg}_{/\!/A}(\mathrm{Sym}_A(\varsigma M), B) \arrow[r, "(\tau_M^\vee)^*"] & \mathrm{DGAlg}_{/\!/A}(\Omega_A(\mathrm{Sym}_A M), B). \end{tikzcd}$$

For a cospectrum $M = \{M_n, \sigma_n\}_{n\in\mathbb{N}}$ of connective A-modules, we define a cospectrum of augmented A-algebras $(\mathrm{Sym}_A M)_n := \mathrm{Sym}_A(M_n)$ equipped with structure maps adjoint to

$$\Omega_A(\mathrm{Sym}_A M_{n+1}) \xrightarrow{\tau^\vee_{M_{n+1}}} \mathrm{Sym}_A(\varsigma M_{n+1}) \xrightarrow{\mathrm{Sym}_A \sigma_n^\vee} \mathrm{Sym}_A M_n,$$

with $\sigma_n^\vee\colon \varsigma M_{n+1} \to M_n$ the adjunct of $\sigma_n\colon M_{n+1} \to \ell M_n$. This assignment determines a functor

$$\mathrm{Sym}_A\colon \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{mod}}(A) \longrightarrow \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A).$$

Since colimits of cospectra are computed objectwise, it follows that Sym_A preserves colimits and so admits a right adjoint aug^∞_A by the Adjoint Functor Theorem.

Explicitly, the right adjoint aug^∞_A sends $B = \{B_n\}_{n\in\mathbb{N}}$ to the cospectrum of connective A-modules with n-th defined as the equaliser of the diagram

$$\prod_{i=0}^{n}(\mathrm{aug}_A B_i)[i-n] \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \prod_{i=0}^{n}\prod_{j=0}^{i}(\mathrm{aug}_A(\Sigma_A^{i-j}B_j))[i-n].$$

In this diagram, α is obtained by taking the iterated structure maps $B_i \to \Sigma_A^{i-j}B_j$, applying aug_A and then shifting; β is obtained via the iterates

$$\mathrm{aug}_A B_i[i-n] \xrightarrow{\tau} \mathrm{aug}_A(\Sigma_A B_i)[i+1-n] \xrightarrow{\tau} \cdots \xrightarrow{\tau} \mathrm{aug}_A(\Sigma_A^j B_i)[j+i-n],$$

with τ the natural weak equivalence of Lemma 2.7.16.

Finally, we observe that $\mathrm{Sym}_A\colon \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{mod}}(A) \to \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A)$ preserves levelwise cofibrations and weak equivalences, so is a left Quillen functor. □

Lemma 2.7.21. *For any morphism of cDGAs $f\colon A \to B$, there is a diagram of left Quillen functors*

$$\begin{tikzcd} \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{mod}}(A) \arrow[r, "\mathrm{Sym}_A"] \arrow[d, "f_!"'] & \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A) \arrow[d, "f_!"] \\ \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{mod}}(B) \arrow[r, "\mathrm{Sym}_B"] & \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(B) \end{tikzcd}$$

which commutes up to natural isomorphism.

Proof. Similary to Remark 2.7.7, the adjunction $(f_! \dashv f^*)\colon A\mathrm{-Mod}^+ \to B\mathrm{-Mod}^+$ prolongs to an adjunction

$$(f_! \dashv f^*)\colon \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{mod}}(A) \longrightarrow \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{mod}}(B)$$

by levelwise application (since f^* commutes with ℓ by Remark 2.7.15). The prolonged adjunction is Quillen for the injective model structures since the prolonged left adjoint $f_!$ preserves levelwise cofibrations and weak equivalence.

By Corollary 2.7.13, the diagram of prolonged functors commutes up to natural isomorphism. □

We conclude this section with a comparison of modular cospectra with unbounded modules.

Construction 2.7.22. For $M \in \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{mod}}(A)$ a cospectrum of connective A-modules, its *assembly complex* is the unbounded A-module $\mathcal{A}(M)$ obtained as the limit over the sequence

$$\cdots \longrightarrow M_{n+1}[n+1] \xrightarrow{\sigma_n[n+1]} M_n[n] \xrightarrow{\sigma_{n-1}[n]} M_{n-1}[n-1] \longrightarrow \cdots \longrightarrow M_0,$$

where $\sigma_n\colon M_{n+1} \to M_n[-1]$ are the cospectrum structure maps. This assignment determines the *assembly functor* $\mathcal{A}\colon \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{mod}}(A) \to A\mathrm{-Mod}$.

The assembly functor has a left adjoint $\mathcal{D}$, which sends an unbounded A-module M to its *disassembly cospectrum*. For $M \in A\text{−Mod}$, the cospectrum $\mathcal{D}(M)$ has n-th term $\mathcal{D}(M)_n := \mathrm{cn}^0_A(N[-n])$. The cospectrum structure maps are obtained via the factorisation

$$\begin{tikzcd} N[-(n+1)] = (N[-n])[-1] \arrow[rr] \arrow[dr] & & (\mathrm{cn}^0_A(N[-n]))[-1] \\ & \mathrm{cn}^0_A(N[-(n+1)]), \arrow[ur] & \end{tikzcd}$$

using that $\mathrm{cn}^0_A(N[-n])[-1]$ is a connective A-module.

Lemma 2.7.23. *The adjunction*

$$\begin{tikzcd} A\text{−Mod} \arrow[r, shift left, "\mathcal{D}"] \arrow[r, phantom, "\perp"] & \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{mod}}(A) \arrow[l, shift left, "\mathcal{A}"] \end{tikzcd}$$

is Quillen.

Proof. The truncation functor $\mathrm{cn}^0_A \colon A\text{−Mod} \to A\text{−Mod}^+$ is left Quillen and the endofunctor $M \mapsto M[-1]$ of $A\text{−Mod}$ preserves weak equivalences and cofibrations. It follows that $\mathcal{D}$ sends cofibrations and weak equivalences to levelwise such maps, hence is a left Quillen functor. □

Lemma 2.7.24. *For any morphism of cDGAs $f \colon A \to B$, there is a diagram of left Quillen functors*

$$\begin{tikzcd} A\text{−Mod} \arrow[r, "\mathcal{D}"] \arrow[d, "f_!"'] & \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{mod}}(A) \arrow[d, "f_!"] \\ B\text{−Mod} \arrow[r, "\mathcal{D}"] & \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{mod}}(B) \end{tikzcd}$$

which commutes up to natural isomorphism.

Proof. This is easy to see on right adjoints: inspecting Construction 2.7.22 we have a natural isomorphism $f^* \circ \mathcal{A} \cong \mathcal{A} \circ f^*$ since restriction of scalars preserves limits and shifts. □

2.7.3 Parametrised Spectra and Modules

The results of the previous two sections combine to produce an adjunction

$$\begin{tikzcd} \mathrm{Sp}^{\mathbb{N}}_{\mathcal{S}(A)} \arrow[r, shift left, "\mathcal{M}_A"] \arrow[r, phantom, "\perp"] & A\text{−Mod}^{\mathrm{op}} \arrow[l, shift left, "\mathcal{P}_A"] \end{tikzcd}$$

for any $A \in \mathrm{DGAlg}$. The adjunction is obtained as the composite of the adjunction of Corollary 2.7.10 with (the opposites of) the adjunctions of Lemmas 2.7.20 and 2.7.23, and is Quillen for the *projective* model structure on sequential $\mathcal{S}(A)$-spectra. Our first goal is to show that this Quillen adjunction descends to the stable model structure:

Lemma 2.7.25. *For any $A \in \mathrm{DGAlg}$, there is a Quillen adjunction*

$$\begin{tikzcd} \left(\mathrm{Sp}^{\mathbb{N}}_{\mathcal{S}(A)}\right)_{\mathrm{stab}} \arrow[r, shift left, "\mathcal{M}_A"] \arrow[r, phantom, "\perp"] & A\text{−Mod}^{\mathrm{op}} \arrow[l, shift left, "\mathcal{P}_A"] \end{tikzcd}$$

of stable model categories.

Proof. Recall that the stable model structure on $\mathrm{Sp}^{\mathbb{N}}_{\mathcal{S}(A)}$ is obtained by taking the left Bousfield localisation of the projective model structure at the set of morpshisms

$$\mathbf{S}^{\mathcal{S}(A)}_{\mathbb{N}} = \{\zeta^{\mathcal{S}(A)}_n(C)\colon F^{\mathcal{S}(A)}_{n+1}(\Sigma_{\mathcal{S}(A)}C) \to F^{\mathcal{S}(A)}_n(C)\},$$

where C ranges over the domains and codomains of morphisms in $\mathcal{J}^{\mathcal{S}(A)}_{\mathrm{Kan}}$. To establish descent to the stable model structure, it is thus sufficient to show that any morphism of parametrised spectra of the form $\zeta_n(K)$ for some retractive space K is sent to a weak equivalence of A-modules (we omit the superscript "$\mathcal{S}(A)$" for the remainder of the proof).

For each $n \in \mathbb{N}$, the functor $L_n\colon A{-}\mathrm{Mod}^+ \to A{-}\mathrm{Mod}$ which sends $M \mapsto M[n]$ is *right* Quillen, with left adjoint $R_n\colon N \mapsto \mathrm{cn}^0_A(N[-n])$. We have a diagram of left Quillen functors in which each square commutes up to natural isomorphism:

$$\begin{tikzcd}
\mathrm{Sp}^{\mathbb{N}}_{\mathcal{S}(A)} \arrow[r, "\mathcal{O}_A"] & \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A)^{\mathrm{op}} \arrow[r, "\mathrm{aug}^{\infty}_A"] & \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{mod}}(A)^{\mathrm{op}} \arrow[r, "A"] & A{-}\mathrm{Mod}^{\mathrm{op}} \\
R_{\mathcal{S}(A)} \arrow[u, "F_n"] \arrow[r, "\mathcal{O}_A"] & (\mathrm{DGAlg}_{/\!/A})^{\mathrm{op}} \arrow[u, "F^n"] \arrow[r, "\mathrm{aug}_A"] & (A{-}\mathrm{Mod}^+)^{\mathrm{op}} \arrow[u, "F^n"] \arrow[r, equal] & (A{-}\mathrm{Mod}^+)^{\mathrm{op}}. \arrow[u, "L_n"']
\end{tikzcd}$$

Applying $\mathcal{M}_A$ to $F_n(C)$ thus yields the A-module $(\mathrm{aug}_A\mathcal{O}_A(C))[n]$. By Corollary 2.7.5 we find that

$$\mathcal{M}_A\big(F_{n+1}(\Sigma_{\mathcal{S}(A)}C)\big) \cong \big(\mathrm{aug}_A(\Sigma_A\mathcal{O}_AC)\big)[n+1],$$

and using the explicit description for aug^{∞}_A given in the proof of Lemma 2.7.20 we find that

$$\mathcal{M}_A(\zeta_n(C)) \cong \tau_{\mathcal{O}_A(C)}[n+1]\colon (\mathrm{aug}_A\mathcal{O}_A(C))[n] \longrightarrow \big(\mathrm{aug}_A(\Sigma_A\mathcal{O}_AC\big)[n+1],$$

which is a weak equivalence by Lemma 2.7.16. □

Corollary 2.7.26. *For any morphism of cDGAs $f\colon A \to B$, there is a diagram of left Quillen functors*

$$\begin{tikzcd}
\mathrm{Sp}^{\mathbb{N}}_{\mathcal{S}(B)} \arrow[r, "\mathcal{M}_B"] \arrow[d, "\mathcal{S}(f)_!"'] & B{-}\mathrm{Mod}^{\mathrm{op}} \arrow[d, "f^*"] \\
\mathrm{Sp}^{\mathbb{N}}_{\mathcal{S}(A)} \arrow[r, "\mathcal{M}_A"] & A{-}\mathrm{Mod}^{\mathrm{op}}
\end{tikzcd}$$

which commutes up to natural isomorphism.

Proof. This follows from the Lemma above, Remark 2.7.7 and Lemmas 2.7.21 and 2.7.24. □

Remark 2.7.27. In the proof of Lemma 2.7.25 above, we saw that for any retractive space $Y \in R_{\mathcal{S}(A)}$ and $n \in \mathbb{N}$, the A-module associated to the fibrewise suspension spectrum $F_nY \in \mathrm{Sp}^{\mathbb{N}}_{\mathcal{S}(A)}$ is $\mathrm{aug}_A(\mathcal{O}_A(Y))[n]$. If Y is a retractive space of the form $Y = (\mathcal{S}(A) \to X \coprod \mathcal{S}(A) \to \mathcal{S}(A))$, then we moreover have $\mathcal{M}_A(F_n(Y)) \cong \mathcal{O}(X)$, regarded as an A-module via the algebra map

$$A \longrightarrow \mathcal{O}\mathcal{S}(A) \longrightarrow \mathcal{O}(X),$$

the latter map being the result of applying $\mathcal{O}$ to $X \to \mathcal{S}(A)$. In particular, in this case $\mathcal{M}_A(F_0(Y))$ simply exhibits the rational cohomology action of $\mathcal{S}(A)$ on X.

In summary, we have that

$$\Sigma^{\infty-n}_{\mathcal{S}(A)}\colon Ho(R_{\mathcal{S}(A)}) \longrightarrow Ho(\mathrm{Sp}^{\mathbb{N}}_{\mathcal{S}(A)})$$

corresponds to taking shifts of augmentation ideals of augmented A-algebras. By essential uniqueness of adjoints,

$$\Omega^{\infty-n}_{\mathcal{S}(A)}\colon Ho(\mathrm{Sp}^{\mathbb{N}}_{\mathcal{S}(A)}) \longrightarrow Ho(R_{\mathcal{S}(A)})$$

corresponds to sending the A-module M to

$$\mathrm{Sym}_A(\mathrm{cn}^0_A(M[-n])),$$

the free A-algebra on the connective cover of the n-fold suspension of M.

Example 2.7.28. Let A be a cofibrant connected cDGA of finite type. The functor $\mathcal{M}_A$ sends the trivial $\mathbb{S}$-bundle over $\mathcal{S}(A)$

$$\mathbb{S}_{\mathcal{S}(A)} := F_0^{\mathcal{S}(A)}(\mathcal{S}(A) \coprod \mathcal{S}(A)) \cong \mathcal{S}(A)^*\mathbb{S}$$

(compare Construction 1.2.36) to $\mathcal{O}(\mathcal{S}(A))$, regarded as an A-module via the counit of the Sullivan–de Rham adjunction $\varepsilon_A\colon A \to \mathcal{O}(\mathcal{S}(A))$. Under the above hypotheses on A, the Sullivan–de Rham theorem implies that ε_A is a weak equivalence of A-modules.

Remark 2.7.29. There is a theory of *minimal* DG modules over a rational cDGA, used by Roig and collaborators to student the rational homotopy theory of local coefficient systems on rational spaces [Roi94a; RSA00]. For a cDGA A and A-module M, a *minimal extension* of M is an inclusion of A-modules $\imath\colon M \to M \oplus (A \otimes V)$ for some graded vector space V such that

- there is a well-ordered set $\mathcal{J}$ indexing a basis $\{v_i\}_{i\in\mathcal{J}}$;
- $i < j$ implies $|v_i| \leq |v_j|$; and
- the differential on $M \oplus (A \otimes V)$ is such that

  $$dv_i \in M \oplus (A \otimes V_{<i})$$

  where $V_{<i} = \langle v_j \mid j < i\rangle$.

An A-module M is *minimal* if it is a minimal extension of the trivial A-module. Any cohomologically bounded-below A-module admits a bounded-below minimal model which is unique up to isomorphism (by a straightforward adaptation of the results of [RSA00, §1.3]). When $A = \mathbb{Q}$, a minimal model of the differential graded vector space V is given by the graded vector space $H^\bullet(V)$.

As minimal DG modules are necessarily cofibrant, Lemma 2.7.25 realises Roig's theory as providing minimal algebraic presentations of parametrised rational homotopy types.

The adjunction of Lemma 2.7.25 descends to an adjunction relating the homotopy theory of parametrised spectra with the homotopy theory of modules. In order to better understand some of the implications of this relationship, we first analyse the case that $A = \mathbb{Q}$, regarded as a cDGA in the obvious fashion.

Lemma 2.7.30. *For any cofibrant sequential spectrum P, $\mathcal{M}_{\mathbb{Q}}(P)$ computes the rational cohomology of P.*

Proof. Let P^i denote the sequential spectrum such that

$$P^i_n = \begin{cases} P_n & \text{for } n \leq i \\ \Sigma^{n-i} P_i & \text{for } n \geq i, \end{cases}$$

equipped with the obvious structure maps (so that each P^i is "ultimately a suspension spectrum"). Then since P is cofibrant, so too is each P^i and we have a sequence of cofibrations

$$\cdots \rightarrowtail P^i \rightarrowtail P^{i+1} \rightarrowtail \cdots$$

such that $\mathrm{colim}_i P^i \cong \mathrm{hocolim}_i P^i \cong P$. Note that for each i there is a stable weak equivalence $P^i \to F_i P_i \cong \Sigma^{\infty - i} P_i$, so since $\mathcal{M}_{\mathbb{Q}}$ preserves stable weak equivalences between cofibrant sequential spectra we have a quasi-isomorphism

$$\mathcal{M}_{\mathbb{Q}}(P^i) \longleftarrow \mathcal{M}_{\mathbb{Q}}(\Sigma^{\infty - i} P_i).$$

As $\mathcal{M}_{\mathbb{Q}}$ is left Quillen, each of the maps $\mathcal{M}_{\mathbb{Q}}(P^{i+1}) \to \mathcal{M}_{\mathbb{Q}}(P^i)$ is a surjection, and $\mathcal{M}_{\mathbb{Q}}(P) \cong \lim_i \mathcal{M}_{\mathbb{Q}}(P^i)$. We therefore have a Milnor exact sequence

$$0 \longrightarrow \lim^1_i H^{k-1}(\mathcal{M}_{\mathbb{Q}}(\Sigma^{\infty - i} P_i)) \longrightarrow H^k(\mathcal{M}_{\mathbb{Q}}(P)) \longrightarrow \lim_i H^k(\mathcal{M}_{\mathbb{Q}}(\Sigma^{\infty - i} P_i)) \longrightarrow 0$$

[Wei94, Theorem 3.5.8]. Observe that we have isomorphisms

$$\mathcal{M}_{\mathbb{Q}}(\Sigma^{\infty - i} P_i) \cong (\mathrm{aug}_{\mathbb{Q}} \mathcal{O}_{\mathbb{Q}}(P_i))[i] = (\mathrm{aug}_{\mathbb{Q}} \mathcal{O}(P_i))[i] \tag{2.12}$$

for each i.

On the other hand, the rational cohomology of a cofibrant sequential spectrum P is computed as the cohomology of the cochain complex

$$\varprojlim \widetilde{N}^\bullet(P_n)[n],$$

where $\widetilde{N}^\bullet(P_n) = \mathrm{Hom}_{\mathbb{Q}}(\widetilde{N}_\bullet(P_n), \mathbb{Q})$ are the normalised reduced rational cochains of P_n, and the limit is over the sequence of maps dual to

$$\begin{array}{c} \widetilde{N}_\bullet(P_n)[n] \cong N\widetilde{\mathbb{Q}}[S^1] \otimes N_\bullet(P_n)[n+1] \\ \Big\downarrow \nabla \\ N\big(\widetilde{\mathbb{Q}}[S^1] \otimes \widetilde{\mathbb{Q}}[P_n]\big)[n+1] \cong \widetilde{N}_\bullet(S^1 \wedge P_n)[n+1] \longrightarrow \widetilde{N}(P_{n+1})[n+1] \end{array}$$

where ∇ is the shuffle map. For a pointed simplicial set K, the natural weak equivalence $\mathcal{O}(K) \to N^\bullet(K)$ induced by the integration pairing restricts to a natural weak equivalence $\mathrm{aug}_{\mathbb{Q}} \mathcal{O}(K) \to \widetilde{N}^\bullet(K)$. Cofibrancy of P implies that each of the maps $\Sigma P_n \to P_{n+1}$ is an injection and hence that the induced sequence of maps on normalised reduced rational cochains is a sequence of surjections. Hence we have a morphism of Milnor exact sequences

$$\begin{tikzcd}
\lim^1_i H^{k-1}(\mathcal{M}_{\mathbb{Q}}(\Sigma^{\infty-i}P_i)) \arrow[r] \arrow[d, "\alpha"'] & H^k(\mathcal{M}_{\mathbb{Q}}(P)) \arrow[r] \arrow[d] & \lim_i H^k(\mathcal{M}_{\mathbb{Q}}(\Sigma^{\infty-i}P_i)) \arrow[d, "\beta"] \\
\lim^1_i \widetilde{H}^{k+i-1}(P_i;\mathbb{Q}) \arrow[r] & H^k(P;\mathbb{Q}) = H^k(\varprojlim \widetilde{N}^\bullet(P_i)[i]) \arrow[r] & \lim_i \widetilde{H}^{k+i}(P_i;\mathbb{Q})
\end{tikzcd}$$

induced by the isomorphisms (2.12) and the natural maps $\int'_K\colon \mathrm{aug}_{\mathbb{Q}}\mathcal{O}(K) \to \widetilde{N}^\bullet(K)$ (we have omitted the zero terms on either side in order to fit the diagram into the page!). Since the maps $\int'_K$ are quasi-isomorphisms for all K, the morphisms α and β above are isomorphisms. The assertion now follows from the Five Lemma. □

Corollary 2.7.31. *The functor $\mathcal{M}_{\mathbb{Q}}$ sends stable rational homotopy equivalences of cofibrant sequential spectra to quasi-isomorphisms.*

Proof. Recall that for a sequential spectrum P we have

$$\pi_*^{\mathrm{st}}(P) \otimes \mathbb{Q} \cong \pi_k^{\mathrm{st}}(H\mathbb{Q} \wedge P) \equiv H_*(P;\mathbb{Q})$$

(for symmetric spectra this Lemma 2.3.4; for sequential spectra this is proven using essentially the same argument on the stable homotopy category). The rational cohomology of a spectrum P is

$$H^\bullet(P;\mathbb{Q}) = \pi_{-\bullet}^{\mathrm{st}}[H\mathbb{Q} \wedge P, H\mathbb{Q}]_{H\mathbb{Q}},$$

where $[-,-]_{H\mathbb{Q}}$ is the $H\mathbb{Q}$-module mapping spectrum. If $P \to Q$ is a stable rational homotopy equivalence then $H\mathbb{Q} \wedge P \to H\mathbb{Q} \wedge Q$ is a stable homotopy equivalence, so that $H^\bullet(Q;\mathbb{Q}) \to H^\bullet(P;\mathbb{Q})$ is an isomorphism.

By the Lemma, if P is cofibrant then $\mathcal{M}_{\mathbb{Q}}(P)$ computes rational cohomology. The assertion follows. □

Remark 2.7.32. The last two results show that for any spectrum P,

$$H^\bullet(\mathcal{M}_{\mathbb{Q}}(P)) \cong (\pi_\bullet^{\mathrm{st}}(P) \otimes \mathbb{Q})^\vee$$

is the graded dual of the rational homotopy graded vector space. Combined with Remark 2.7.29, we have in particular that $V \cong (\pi_\bullet^{\mathrm{st}}(P) \otimes \mathbb{Q})^\vee$, where V is a minimal model of $\mathcal{M}_{\mathbb{Q}}(P)$.

We now turn to the general case. Before we state and prove the main result of this section, we record some preliminary structural results.

Remark 2.7.33. For any cDGA A, the unit map $\eta\colon \mathbb{Q} \to A$ is sent to the terminal map $\mathcal{S}(A) \to *$ on spaces, so that we have a diagram of left Quillen functors

$$\begin{tikzcd}
\mathrm{Sp}^{\mathbb{N}}_{\mathcal{S}(A)} \arrow[r, "\mathcal{M}_A"] \arrow[d, "\mathcal{S}(A)_!"'] & A\!-\!\mathrm{Mod}^{\mathrm{op}} \arrow[d, "\eta^*"] \\
\mathrm{Sp}^{\mathbb{N}} \arrow[r, "\mathcal{M}_{\mathbb{Q}}"] & Ch^{\mathrm{op}}
\end{tikzcd} \tag{2.13}$$

commuting up to natural isomorphism. At the level of homotopy categories, the collapse functor

$$\mathcal{S}(A)_!\colon Ho(\mathrm{Sp}_{\mathcal{S}(A)}) \longrightarrow Ho(\mathrm{Sp})$$

sends fibrewise rational stable weak equivalences to rational stable weak equivalences[2]. By Corollary 2.7.31, we therefore find that if $P \to Q$ is a fibrewise rational stable weak equivalence of $\mathcal{S}(A)$-spectra, the map of cochain complexes

$$\mathcal{M}_{\mathbb{Q}}(\mathcal{S}(A)_! P) \cong \mathcal{M}_A P \longleftarrow \mathcal{M}_{\mathbb{Q}}(\mathcal{S}(A)_! Q) \cong \mathcal{M}_A Q$$

is a quasi-isomorphism.

To fix the terminology and notation, for any connected space X and DG algebra A of finite type, we write

- $\mathrm{Perf}(A) \hookrightarrow Ho(A\text{–}\mathrm{Mod})$ for the smallest full subcategory containing A and closed under homotopy pushouts, homotopy pullbacks and retracts. Objects of $\mathrm{Perf}(A)$ are *perfect A-modules* in the usual sense.
- $Ho(A\text{–}\mathrm{Mod})_{\mathrm{ft,bbl}} \hookrightarrow Ho(A\text{–}\mathrm{Mod})$ for the full subcategory *bounded-below A-modules of finite type*; namely those A-modules whose cohomology groups are degreewise finite-dimensional and vanish identically below dimension N for some $N \in \mathbb{Z}$.
- $Ho_{\mathbb{Q}}(\mathrm{Sp}_X)$ for the *rational homotopy category* of X-spectra; namely, the localisation of the homotopy category $Ho(\mathrm{Sp}_X)$ at rational homotopy equivalences.
- $\mathbb{Q}\mathrm{Coh}(X) \hookrightarrow Ho_{\mathbb{Q}}(\mathrm{Sp}_X)$ for the smallest full subcategory containing the trivial $\mathbb{S}$-bundle $\mathbb{S}_X$ and closed under homotopy pushouts, homotopy pullbacks and retracts. Objects of $\mathbb{Q}\mathrm{Coh}(X)$ are called *rationally coherent X-spectra*.
- $Ho_{\mathbb{Q}}(\mathrm{Sp}_X)_{\mathrm{ft,bbl}} \hookrightarrow Ho_{\mathbb{Q}}(\mathrm{Sp}_X)$ for the full subcategory of *bounded-below X-spectra of finite rational type*; namely those parametrised spectra $P \in \mathrm{Sp}_X$ all of whose homotopy fibre spectra have stable homotopy groups of degreewise finite rank, such that the rank is identically zero below dimension N for some $N \in \mathbb{Z}$.

We are now fully equipped to prove the main result of this section, which is a sort of "homotopical Sullivan–de Rham–Serre–Swan Theorem":

Theorem 2.7.34. *Let A be a cofibrant connected cDGA of finite type. Then the derived adjunction*

$$\begin{tikzcd} Ho(\mathrm{Sp}^{\mathbb{N}}_{\mathcal{S}(A)}) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & Ho(A\text{–}\mathrm{Mod})^{\mathrm{op}} \arrow[l, shift left] \end{tikzcd}$$

induces an equivalence of categories

$$\begin{tikzcd} \mathbb{Q}\mathrm{Coh}(\mathcal{S}(A)) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Perf}(A)^{\mathrm{op}}. \arrow[l, shift left] \end{tikzcd}$$

Proof. We argue using the Quillen adjunction of Lemma 2.7.25. By Example 2.7.28, the quasi-isomorphism

$$A \longrightarrow \mathcal{M}_A\big(\mathbb{S}_{\mathcal{S}(A)}\big) \cong \mathcal{O}(\mathcal{S}(A))$$

[2] this can be inferred from the results of §2.3.2, though we have not quite proven it. One way to see this is by using the following formulae for fibrewise smash products of parametrised spectra [MS06]: for a map of spaces $f\colon X \to Y$ we have

$$f_!(f^* A \wedge_X P) \cong A \wedge_Y f_! P \quad \text{and} \quad f^*(A \wedge_Y B) \cong f^* A \wedge_X f^* B$$

for X-spectra P and Y-spectra A and B at the level of homotopy categories. We conclude that fibrewise rationalisation is implemented by $X^* H\mathbb{Q} \wedge_X (-)$, and the relation $X_!(X^* H\mathbb{Q} \wedge_X (-)) \cong H\mathbb{Q} \wedge X_!(-)$ implies that $X_!\colon Ho(\mathrm{Sp}_X) \to Ho(\mathrm{Sp})$ sends fibrewise rational stable weak equivalences to rational stable homotopy equivalences.

exhibits a fibrant replacement of $\mathcal{M}_A(\mathbb{S}_{\mathcal{S}(A)})$ in $A-\mathrm{Mod}^{\mathrm{op}}$. The component of the derived unit at $\mathbb{S}_{\mathcal{S}(A)}$ is thus represented by the map of parametrised spectra

$$\mathbb{S}_{\mathcal{S}(A)} \longrightarrow \mathcal{P}_A(A). \tag{2.14}$$

We claim that this is a fibrewise stable rational homotopy equivalence. Indeed, fix an augmentation $\varrho\colon A \to \mathbb{Q}$ which becomes $p\colon * \to \mathcal{S}(A)$ under the Sullivan–de Rham functor $\mathcal{S}\colon \mathrm{DGAlg}^{\mathrm{op}} \to \mathrm{sSet}$. Since $\mathcal{S}(A)$ is connected [BG76, Proposition 8.12], it suffices to check that (2.14) is a rational stable weak equivalence on the homotopy fibres at p. The homotopy fibre spectrum of $\mathbb{S}_{\mathcal{S}(A)}$ at p is stably weak equivalent to $\mathbb{S}$ (as observed in Construction 1.2.36). To compute the stable homotopy groups of the homotopy fibre spectrum of $\mathcal{P}_A(A)$ at p, we employ the diagram of right Quillen functors

$$\begin{tikzcd} \mathrm{Sp}^{\mathbb{N}} & Ch^{\mathrm{op}} \arrow[l, "\mathcal{P}_{\mathbb{Q}}"'] \\ \mathrm{Sp}^{\mathbb{N}}_{\mathcal{S}(A)} \arrow[u, "p^*"] & A-\mathrm{Mod}^{\mathrm{op}} \arrow[l, "\mathcal{P}_A"'] \arrow[u, "\varrho_!"'] \end{tikzcd}$$

of Corollary 2.7.26. Since A is fibrant in $A-\mathrm{Mod}^{\mathrm{op}}$ the homotopy fibre spectrum of $\mathcal{P}_A(A)$ at p is $\mathcal{P}_{\mathbb{Q}}(\varrho_! A) = \mathcal{P}_{\mathbb{Q}}(\mathbb{Q})$, which is a fibrant Ω-spectrum. The n-th space of the spectrum $\mathcal{P}_{\mathbb{Q}}(\mathbb{Q})$ is

$$\mathcal{P}_{\mathbb{Q}}(\mathbb{Q})_n = \mathcal{S}(\mathrm{Sym}_{\mathbb{Q}}(\mathbb{Q}[-n])).$$

The indecomposables of $\mathrm{Sym}_{\mathbb{Q}}(\mathbb{Q}[-n])$ are $Q\,\mathrm{Sym}_{\mathbb{Q}}(\mathbb{Q}[-n]) = \mathbb{Q}[-n]$. By Remark 2.1.7, when $n \geq 1$ and $k \geq 2$, we therefore have group isomorphisms

$$\pi_k(\mathcal{P}_{\mathbb{Q}}(\mathbb{Q})_n) \cong \begin{cases} \mathbb{Q} & \text{if } n = k \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\pi_0^{\mathrm{st}}(\mathcal{P}_{\mathbb{Q}}(\mathbb{Q})) = \mathbb{Q}$, with all other stable homotopy groups equal to zero. We conclude that (2.14) is a fibrewise stable rational homotopy equivalence.

On the other hand, factoring (2.14) into a cofibration followed by an acyclic fibration:

$$\mathbb{S}_{\mathcal{S}(A)} \rightarrowtail \mathcal{P}'_A(A) \xrightarrow{\sim} \mathcal{P}_A(A)$$

produces a fibrewise rational stable weak equivalence $\mathbb{S}_{\mathcal{S}(A)} \to \mathcal{P}'_A(A)$. Applying $\mathcal{M}_A$ yields a quasi-isomorphism.

We have shown that the derived unit at $\mathbb{S}_{\mathcal{S}(A)}$ is a fibrewise rational stable weak equivalence, and that the derived counit at A is an isomorphism in the unbounded derived category of A. In a stable model category, the homotopy pushout squares coincide with homotopy pullback squares [Hov99, Remark 7.1.12], and homotopy pullback squares in $Ho(A-\mathrm{Mod})^{\mathrm{op}}$ coincide with homotopy pushout squares in $Ho(A-\mathrm{Mod})$. Since the derived functors are exact, we therefore have an induced adjunction

$$\begin{tikzcd} \mathrm{QCoh}(\mathcal{S}(A)) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Perf}(A)^{\mathrm{op}} \arrow[l, shift left] \end{tikzcd}$$

with respect to which the unit and counit are natural isomorphisms. This completes the proof. □

Remark 2.7.35. Let A be a cofibrant connected cDGA of finite type with augmentation $\varrho\colon A \to \mathbb{Q}$ and M a cofibrant A-module. The argument of the proof above shows that the fibrant $\Omega_{\mathcal{S}(A)}$-spectrum $\mathcal{P}_A(M)$ has fibrewise stable homotopy group

$$\mathrm{Hom}_{\mathbb{Q}}(H^n(\varrho_! M), \mathbb{Q})$$

in dimension n. In particular, if M is a minimal A-module, so that the underlying graded vector space of M is $A \otimes V$, then the fibrewise stable homotopy groups of $\mathcal{P}_A(M)$ are $V^\vee$.

2.7.4 The 1-Connected Case

In the case that $\mathcal{S}(A)$ is 1-connected we can say a fair bit more than Theorem 2.7.34. Firstly, vanishing of the fundamental group allows for a simple characterisation of fibrewise Eilenberg–Mac Lane spectra:

Lemma 2.7.36. *Let X be a 1-connected space and P be an X-spectrum such that the homotopy fibre spectrum of P at some (and hence any) point $x \in X$ has stable homotopy groups concentrated in a single dimension $\pi_k^{\mathrm{st}}(x^*P) \cong G$. Then $\Omega_X^k P$ is isomorphic to $X^*HG \cong HG \wedge_X \mathbb{S}_X$ in $Ho(\mathrm{Sp}_X)$.*

Proof. Since Ω_X and Σ_X have the effect of shifting fibrewise stable homotopy groups down and up by one respectively, it suffices to consider $k = 0$. The hypotheses then imply that $x^*P \cong HG$ in the stable homotopy category. Under the $Ho(\mathrm{Sp})$-enriched equivalence between $Ho(\mathrm{Sp}_X)$ and $Ho(\Omega X_+ - \mathrm{Mod})$ (Theorem 2.2.13) the parametrised X-spectrum P is equivalent to the data of an ΩX_+-action on HG:

$$\rho \colon \Omega X_+ \wedge HG \longrightarrow HG.$$

Since HG is coconnective and $\Omega X_+ \wedge HG$ is 0-connected, any $\rho \colon \Omega X_+ \wedge HG \to HG$ must be null-homotopic, so that the ΩX_+-action on $x^*P \cong HG$ is trivial. Observe that under the $Ho(\mathrm{Sp})$-enriched equivalence between $Ho(\mathrm{Sp}_X)$ and $Ho(\Omega X_+ - \mathrm{Mod})$, $\mathbb{S}_X$ corresponds to $\mathbb{S}$ with the trivial ΩX_+-action, so that the result now follows by smashing (fibrewise) with HG. □

We shall also need some structural results regarding fibrewise connective covers. This material is largely a straightforward adaptation of familiar properties of the stable homotopy category (compare [Sch12, Ch. II]). We use that the homotopy category of a stable model category is triangulated, together with the various tools that this fact affords us, without much further comment (see [Hov99, Ch. 6 & 7] and §A.3.4).

Definition 2.7.37. An X-spectrum P is $(k-1)$-*connected* (or k-*connective*) if the homotopy fibre spectrum x^*P has vanishing stable homotopy groups in dimensions $n < k$ for all $x \colon * \to X$.

Lemma 2.7.38. *Let X be connected and fix $x \in X$ and $k \in \mathbb{Z}$. Then the right-closed class $\langle x_! \mathbb{S}^k \rangle_+$ generated by $x_! \Sigma^k \mathbb{S}$ coincides with the class of k-connective X-spectra.*

Proof. The homotopy fibre spectrum of $x_! \Sigma^k \mathbb{S}$ at x is $\Sigma_+^k \Omega X$, which is k-connective. The derived functor $x^* \colon Ho(\mathrm{Sp}_X) \to Ho(\mathrm{Sp})$ is exact and preserves filtered homotopy colimits[3], from which we deduce that k-connective X-spectra are closed under

[3]That the derived functor x^* preserves filtered homotopy colimits can be seen from the fact that it has a derived right adjoint x_* [MS06], or by observing that it is naturally isomorphic to the composite

$$Ho(\mathrm{Sp}_X) \xrightarrow{\beta_X^+} Ho(\Omega X_+ - \mathrm{Mod}) \xrightarrow{\text{forget}} Ho(\mathrm{Sp})$$

of derived functors which preserve filtered homotopy colimits.

coproducts and extensions to the right since the same is true of k-connective spectra. In particular, the class $\langle x_! \mathbb{S}^k \rangle_+$ consists of k-connective X-spectra.

Since any right-closed class is closed under forming suspensions, $\langle x_! \mathbb{S}^k \rangle_+$ coincides with the right-closed class $\langle \mathcal{C}_{\geq k} \rangle_+$ generated by the set of compact objects $\mathcal{C}_{\geq k} = \{x_! \Sigma^p \mathbb{S}\}_{p \geq k}$. For any X-spectrum P, by Brown Representability (Lemma A.3.36) there is an object $P\langle k \rangle \in \langle x_! \mathbb{S}^k \rangle_+$ and a morphism $P\langle k \rangle \to P$ such that

$$Ho(\mathrm{Sp}_X)(x_! \mathbb{S}^p, P\langle k \rangle) \longrightarrow Ho(\mathrm{Sp}_X)(x_! \mathbb{S}^p, P)$$

is an isomorphism for all $p \geq k$. Since $x_! \mathbb{S}^p$ corepresents $P \mapsto \pi_p^{\mathrm{st}}(x^* P)$ by the derived $(x_! \dashv x^*)$-adjunction, when P is k-connective the map $P\langle k \rangle \to P$ is an isomorphism in $Ho(\mathrm{Sp}_X)$. In particular, $\langle x_! \mathbb{S}^k \rangle_+$ contains all k-connective X-spectra. □

Corollary 2.7.39. *For connected base space X and any $k \in \mathbb{Z}$, the inclusion of the full triangulated subcategory $Ho(\mathrm{Sp}_X)_{\geq k} \to Ho(\mathrm{Sp}_X)$ on k-connective X-spectra admits a right adjoint which sends an X-spectrum P to its k-connective cover $P\langle k \rangle$.*

Proof. Letting $P\langle k \rangle$ be the k-connective cover as in the proof of Lemma 2.7.38, we claim that for all k-connective $A \in Ho(\mathrm{Sp}_X)$ the map

$$Ho(\mathrm{Sp}_X)(A, P\langle k \rangle) \longrightarrow Ho(\mathrm{Sp}_X)(A, P)$$

is an isomorphism. For this, let Z be an X-spectrum whose homotopy fibre spectrum has vanishing stable homotopy groups in dimensions $n \geq k$. We then have

$$Ho(\mathrm{Sp}_X)(x_! \Sigma^k \mathbb{S}, Z) \cong \pi_k^{\mathrm{st}}(x^* Z) = 0.$$

Since $Ho(\mathrm{Sp}_X)(-, Z)$ sends homotopy cofibre sequences to exact sequences and sends coproducts to products it follows that $Ho(\mathrm{Sp}_X)(A, Z) = 0$ for all k-connective X-spectra A by the Lemma. Let $P_{\leq k-1}$ be the homotopy cofibre of $P\langle k \rangle \to P$, so that the homotopy fibre spectrum of $P_{\leq (k-1)}$ has vanishing homotopy groups in dimensions $n \geq k$. Applying $Ho(\mathrm{Sp}_X)(A, -)$ to the homotopy fibre sequence

$$P\langle k \rangle \longrightarrow P \longrightarrow P_{\leq (k-1)}$$

yields an exact sequence of homotopy mapping groups. Since

$$Ho(\mathrm{Sp}_X)(\Sigma_X A, P_{\leq (k-1)}) = 0 = Ho(\mathrm{Sp}_X)(A, P_{\leq (k-1)})$$

by k-connectivity of A and $\Sigma_X A$, the map $Ho(\mathrm{Sp}_X)(A, P\langle k \rangle) \to Ho(\mathrm{Sp}_X)(A, P)$ is an isomorphism. This isomorphism is natural in A and P by construction, so that we obtain the desired adjunction. □

Corollary 2.7.40. *For any map $f\colon X \to Y$ between connected spaces, the derived pushforward functor*

$$f_!\colon Ho(\mathrm{Sp}_X) \longrightarrow Ho(\mathrm{Sp}_Y)$$

sends k-connective X-spectra to k-connective Y-spectra.

Proof. Fixing $x\colon * \to X$, we have that $f_!(x_! \Sigma^k \mathbb{S}) = f(x)_! \Sigma^k \mathbb{S}$. Since $f_!$ is left exact and preserves filtered homotopy colimits, the right-closed class $\langle x_! \mathbb{S}^k \rangle_+$ is sent into $\langle f(x)_! \mathbb{S}^k \rangle_+$ by $f_!$. □

For the remainder of this section, we work with a fixed minimal 1-connected cDGA A of finite type, so that $\mathcal{S}(A)$ is 1-connected. To ease the notation, we write $X = \mathcal{S}(A)$ in what follows.

Remark 2.7.41. Since A is minimal, the connectivity assumption on $\mathcal{S}(A)$ is equivalent to the requirement $A^0 = \mathbb{Q}$ and $A^1 = 0$: the low-degree behaviour of A is highly constrained.

Suppose that P is a bounded-below X-spectrum of finite rational type, and let $N \in \mathbb{Z}$ be such that $\pi_k^{\mathrm{st}}(x^*P)$ has zero rank for all $k < N$. In particular, this implies that $P\langle N\rangle \to P$ is a rational homotopy equivalence of X-spectra. We show that the unit of the derived adjunction

$$\begin{tikzcd} Ho(\mathrm{Sp}_X) \arrow[r, shift left, "\mathcal{M}"] \arrow[r, phantom, "\perp"] & Ho(A\text{--}\mathrm{Mod})^{\mathrm{op}} \arrow[l, shift left, "\mathcal{P}"] \end{tikzcd} \tag{2.15}$$

at P is a rational homotopy equivalence. The argument works by induction over Postnikov stages.

(o) By Remark 2.7.33 and the above, we may replace P with $P\langle N\rangle$ and so assume that P is N-connective without loss of generality.

(i) The N-th Postnikov stage $P_{\leq N}$, obtained as the homotopy cofibre

$$P\langle N+1\rangle \longrightarrow P \longrightarrow P_{\leq N}, \tag{2.16}$$

has fibrewise stable homotopy groups concentrated in dimension N. By Lemma 2.7.36 we have an isomorphism $P_{\leq N} \cong H(\pi_N^{\mathrm{st}}(x^*P)) \wedge_X \Sigma_X^N \mathbb{S}_X$ in $Ho(\mathrm{Sp}_X)$. The latter X-spectrum is in turn rational homotopy equivalent to

$$\bigoplus_{r(N)} \Sigma_X^N \mathbb{S}_X,$$

where $r(N)$ is the rank of $\pi_N^{\mathrm{st}}(x^*P)$. By Theorem 2.7.34, we have that the component of the unit of the derived adjunction (2.15) at $P_{\leq N}$ is a rational homotopy equivalence. Observe that $P_{\leq N}$ is identified with the A-module $\bigoplus_{r(N)} A[-N]$.

(ii) Suppose that the unit of the derived adjunction (2.15) is a rational homotopy equivalence at $P_{\leq k}$ for all $N \leq k \leq n$. We have a homotopy fibre sequence

$$P[n+1] \longrightarrow P_{\leq(n+1)} \longrightarrow P_{\leq n} \tag{2.17}$$

such that the fibrewise stable homotopy groups of $P[n+1]$ are concentrated in dimension $(n+1)$. Homotopy fibre and cofibre sequences coincide in $Ho(\mathrm{Sp}_X)$ by stability, and by shifting the above cofibre sequence to the right we obtain the (co)fibre sequence

$$P_{\leq(n+1)} \longrightarrow P_{\leq n} \longrightarrow \Sigma_X P[n+1]. \tag{2.18}$$

By (ii), the derived unit at $\Sigma_X P[n+1]$ is a rational homotopy equivalence, and it is at $P_{\leq n}$ by hypothesis. Since the derived adjoint functors of (2.15) are exact, we deduce from the above (co)fibre sequence that the derived unit at $P_{\leq(n+1)}$ is also a rational homotopy equivalence of X-spectra. By induction, the unit of the derived adjunction is a rational homotopy equivalence at $P_{\leq k}$ for all $k \geq N$.

(iii) The homotopy limit over Postnikov stages gives $P \cong \mathrm{holim}_k P_{\leq k}$ in $Ho(\mathrm{Sp}_X)$. On the algebraic side, we show that $\mathcal{M}(P)$ is the homotopy colimit of the A-modules $\mathcal{M}(P_{\leq k})$ as follows. Shifting the cofibre sequence (2.16) to the right yields a cofibre sequence

$$P \longrightarrow P_{\leq k} \longrightarrow \Sigma_X P\langle k+1\rangle, \tag{2.19}$$

where $\Sigma_X P\langle k+1\rangle$ is $(k+2)$-connective. Corollary 2.7.40, Lemma 2.7.30 and the naturality square of Corollary 2.7.26 applied to the unit $\eta\colon \mathbb{Q} \to A$ together imply that $\mathcal{M}(\Sigma_X P\langle k+1\rangle) \cong \mathcal{M}(P\langle k+1\rangle)[-1]$ is a $(k+1)$-connected A-module. Applying $\mathcal{M}$ to the (co)fibre sequence (2.19) yields the (co)fibre sequence of A-modules

$$\mathcal{M}(P) \longleftarrow \mathcal{M}(P_{\leq k}) \longleftarrow \mathcal{M}(P\langle k+1\rangle)[-1].$$

Taking the homotopy colimit of this family of cofibre sequences over k yields an isomorphism $\mathcal{M}(P) \cong \operatorname{hocolim}_k \mathcal{M}(P_{\leq k})$ since $\operatorname{hocolim}_k \mathcal{M}(P\langle k+1\rangle)[-1]$ is acyclic.

Taking the homotopy limit over the family of rational homotopy equivalences $P_{\leq k} \to \mathcal{P}\mathcal{M}(P_{\leq k})$ yields the diagonal rational homotopy equivalence in the commutative diagram

$$\begin{tikzcd}
P \arrow[d] \arrow[dr, "\sim_{\text{r.h.e.}}"] & \\
\mathcal{P}\mathcal{M}(P) \arrow[r, "\sim"] & \mathcal{P}\big(\operatorname{hocolim}_k \mathcal{M}(P_{\leq k})\big) \cong \operatorname{holim}_k \mathcal{P}\mathcal{M}(P_{\leq k}).
\end{tikzcd}$$

The bottom arrow in this diagram is an isomorphism in $Ho(\mathrm{Sp}_X)$ since $\mathcal{M}(P) \cong \operatorname{hocolim}_k \mathcal{M}(P_{\leq k})$. We conclude that the unit of the derived adjunction (2.15) at P is a rational homotopy equivalence of X-spectra.

For our fixed minimal cDGA A, suppose that M is a bounded-below A-module of finite type. We show that the counit of the derived adjunction (2.15) at M is an isomorphism. As before, the argument works by induction over "Postnikov stages".

(o) By Remark 2.7.29, we may suppose that M is an N-connective minimal A-module for some $N \in \mathbb{Z}$. In particular, the underlying graded vector space of M is $A \otimes V$ where the graded vector space V is N-connective. Let $\{v_i\}_{i\in\mathcal{I}}$ be a basis for V per the definition of minimality, and let $\mathcal{I}(k) = \{i \in \mathcal{I} \mid |v_i| \leq k\}$ so that the $\mathcal{I}(k)$ form an exhaustive increasing filtration of $\mathcal{I}$. We write

$$M_{\leq k} := A \otimes V_{\leq k},$$

where $V_{\leq k} = \langle v_i \mid i \in I(k)\rangle$, regarded as a differential graded A-module with respect to the differential inherited from the minimal presentation of M. We call $M_{\leq k}$ the *k-th Postnikov stage* of M, noting that $M \cong \operatorname{hocolim}_k M(k)$ and that for each k there is a cofibre exact sequence

$$M_{\leq k} \longrightarrow M \longrightarrow M\langle k+1\rangle, \tag{2.20}$$

such that the underlying graded vector space of the A-module $M\langle k+1\rangle$ is $A \otimes \langle v_i \mid i \notin I(k)\rangle$. In particular, $M\langle k+1\rangle$ is $(k+1)$-connective.

(i) The N-th Postnikov stage $M_{\leq N}$ has underlying graded vector space $A \otimes W$ for W a vector space concentrated in dimension N. Since $M_{\leq N}$ is a minimal A-module, the low-degree behaviour of A (Remark 2.7.41) forces all elements of $W \hookrightarrow M(N)$ to be closed so that $M_{\leq N} \cong A \otimes W$ is a free A-module. (This is the algebraic analogue of Lemma 2.7.36.)

Feeding $M_{\leq N}$ through the derived adjunction (2.15), the X-spectrum $\mathcal{P}(M_{\leq N})$ has fibrewise stable homotopy groups concentrated in dimension N, where they are $W^\vee \cong W$ (Remark 2.7.35). By Lemma 2.7.36, $\mathcal{P}(M_{\leq N})$ is then rational homotopy equivalent to

$$\bigoplus_{\dim W} \Sigma^N_X \mathbb{S}_X,$$

in $Ho(\mathrm{Sp}_X)$. The left adjoint $\mathcal{M}$ sends rational homotopy equivalences of X-spectra to quasi-isomorphisms of A-modules and sends $\bigoplus_{\dim W} \Sigma^N_X \mathbb{S}_X$ to the A-module $\bigoplus_{\dim W} A[-N] \cong A \otimes W[-N]$. We conclude that the counit of the derived adjunction (2.15) is an isomorphism at $M_{\leq N}$.

(ii) Suppose by way of induction that the counit of the derived adjunction (2.15) is an isomorphism at $M_{\leq k}$ for all $N \leq k \leq n$. We have a cofibre exact sequence of A-modules

$$M_{\leq n} \longrightarrow M_{\leq (n+1)} \longrightarrow M[n+1] \tag{2.21}$$

in which $M[n+1]$ has underlying graded vector space $M[n+1] \cong A \otimes W$ for W a vector space concentrated in dimension $(n+1)$. By (ii) above, the counit of the derived adjunction (2.15) at $M[n+1]$ is an isomorphism in $Ho(A\mathrm{-Mod})^{\mathrm{op}}$. Since the functors $\mathcal{M}$ and $\mathcal{P}$ are exact and the components of the derived counit at $M_{\leq n}$ and $M[n+1]$ are isomorphisms, the component at $M_{\leq(n+1)}$ must be an isomorphism too, since $Ho(A\mathrm{-Mod})^{\mathrm{op}}$ is triangulated. By induction, the counit of the derived adjunction is an isomorphism at $M_{\leq n}$ for all $n \geq N$.

(iii) We have $M \cong \mathrm{hocolim}_k M_{\leq k}$ in $Ho(A\mathrm{-Mod})$, so that $\mathcal{P}(M) \cong \mathrm{holim}_k \mathcal{P}(M_{\leq k})$ in $Ho(\mathrm{Sp}_X)$. Applying $\mathcal{P}$ to the (co)fibre sequence (2.20) yields the (co)fibre sequence

$$\mathcal{P}(M\langle k+1\rangle) \longrightarrow \mathcal{P}(M) \longrightarrow \mathcal{P}(M_{\leq k}) \tag{2.22}$$

in $\mathrm{Ho}(\mathrm{Sp}_X)$. In item (o) above, we noted that the underlying graded vector space of $M\langle k+1\rangle$ is $A \otimes \langle v_i \mid i \notin I(k)\rangle$. If $\varrho\colon A \to \mathbb{Q}$ is the augmentation, the underlying graded vector space of $\varrho_! M\langle k+1\rangle$ is $\langle v_i \mid i \notin I(k)\rangle$. In particular, $\varrho_! M\langle k+1\rangle$ is $(k+1)$-connective which implies that $\mathcal{P}(M\langle k+1\rangle)$ is a $(k+1)$-connective X-spectrum (Remark 2.7.35). In turn, this guarantees that $\mathcal{MP}(M\langle k+1\rangle)$ is a $(k+1)$-connective A-module (by Corollary 2.7.40, Lemma 2.7.30 and Corollary 2.7.26 applied to the unit $\eta\colon \mathbb{Q} \to A$).

Applying $\mathcal{M}$ to the (co)fibre sequence (2.22) yields the (co)fibre sequence

$$\mathcal{MP}(M_{\leq k}) \longrightarrow \mathcal{MP}(M) \longrightarrow \mathcal{MP}(M\langle k+1\rangle)$$

Taking the homotopy colimit over k then implies $\mathrm{hocolim}_k \mathcal{MP}(M_{\leq k}) \cong \mathcal{MP}(M)$ since $\mathrm{hocolim}_k \mathcal{MP}(M\langle k+1\rangle)$ is acyclic.

Putting it all together, we obtain a commuting diagram in $Ho(A\mathrm{-Mod})$

$$\begin{tikzcd}
\mathrm{hocolim}_k M_{\leq k} \arrow[r, "\sim"] \arrow[d, "\sim"'] & M \arrow[d] \\
\mathrm{hocolim}_k \mathcal{MP}(M_{\leq k}) \arrow[r, "\sim"] & \mathcal{MP}(M)
\end{tikzcd}$$

with isomorphisms as marked (the left-hand vertical isomorphism by item (ii) above). We conclude that the counit of the derived adjunction (2.15) is an isomorphism at M.

This completes the proof of the

Theorem 2.7.42. *Let A be a minimal cDGA of finite type such that $\mathcal{S}(A)$ is 1-connected. Then the derived adjunction*

$$Ho(\mathrm{Sp}^{\mathbb{N}}_{\mathcal{S}(A)}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} Ho(A\text{--}\mathrm{Mod})^{\mathrm{op}}$$

induces an equivalence of categories

$$Ho_{\mathbb{Q}}(\mathrm{Sp}_{\mathcal{S}(A)})_{\mathrm{ft,bbl}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} Ho(A\text{--}\mathrm{Mod})^{\mathrm{op}}_{\mathrm{ft,bbl}}.$$

Remark 2.7.43. We can remove the minimality requirement on A in the Theorem, provided that A is a cofibrant connected cDGA of finite type for which $\mathcal{S}(A)$ is 1-connected. This is achieved simply by taking a minimal resolution of A, applying the Theorem, and then invoking naturality (Corollary 2.7.26).

Remark 2.7.44. We can remove the boundedness hypothesis when $A = \mathbb{Q}$ by taking homotopy colimits over connective covers and invoking the argument of Lemma 2.7.30.

Example 2.7.45. Let A be a minimal cDGA of finite type such that $X = \mathcal{S}(A)$ is 1-connected. Writing $\varrho\colon A \to \mathbb{Q}$ for the augmentation, we have a diagram of right Quillen functors

$$\begin{tikzcd} \mathrm{Sp}^{\mathbb{N}} & Ch^{\mathrm{op}} \arrow[l, "\mathcal{P}_{\mathbb{Q}}"'] \\ \mathrm{Sp}^{\mathbb{N}}_{\mathcal{S}(A)} \arrow[u, "p^*"] & A\text{--}\mathrm{Mod}^{\mathrm{op}} \arrow[l, "\mathcal{P}_A"'] \arrow[u, "\varrho_!"'] \end{tikzcd}$$

commuting up to natural isomorphism. Regarding $\mathbb{Q}$ as an A-module via ϱ, the bar resolution $B^\bullet(A, A, \mathbb{Q}) \to \mathbb{Q}$ is a fibrant resolution of $\mathbb{Q}$ in $A\text{--}\mathrm{Mod}^{\mathrm{op}}$. Applying $\varrho_!$ yields the bar resolution $B^\bullet(\mathbb{Q}, A, \mathbb{Q})$ computing the rational cohomology of ΩX by the Eilenberg–Moore Theorem (for this, 1-connectedness of X is crucial—see [McC01, Ch. 7]). We have

$$\pi_k^{\mathrm{st}}(\mathcal{P}_{\mathbb{Q}}(B^\bullet(\mathbb{Q}, A, \mathbb{Q})) \cong \mathrm{Hom}_{\mathbb{Q}}(H^k(B^\bullet(\mathbb{Q}, A, \mathbb{Q})), \mathbb{Q}) \cong H_k(\Omega X; \mathbb{Q})$$

by Remark 2.7.35, the finiteness hypothesis on A and the Universal Coefficient Theorem. In particular, $\mathbb{R}\mathcal{P}_A(\mathbb{Q})$ is a parametrised X-spectrum with homotopy fibre spectrum $H\mathbb{Q} \wedge \Omega X$.

Under the equivalence of Theorem 2.7.42, the A-module $\mathbb{Q}$ corresponds to the X-spectrum $x_!\mathbb{S}$, where $x\colon * \to X$ is any point in $X = \mathcal{S}(A)$.

Chapter 3

Vistas

In this final chapter, we combine the results of Chapters 1 and 2 to produce examples of twisted differential cohomology theories. Though specific examples such as twisted differential K-theory have been studied in the literature (see [BN14, §1.3] and the references given therein), our results in this chapter provide the first mechanism that systematically produces families of examples of twisted differential cohomology theories. The mechanism is analogous to Lie integration; passing from the infinitesimal/algebraic realm of higher Lie algebroids and their representations to the homotopy theory of smooth parametrised spectra. This technique has important ramifications in quantum field theory and M-theory, where it allows us to understand the work of [FSS17a; FSS17b] as the infinitesimal/algebraic version of a story whose true mathematical home is twisted differential cohomology.

Our main accomplishment in this section is to realise the stacky Lie integration procedure of [FSS12] as one half of an adjunction relating smooth ∞-stacks to algebra (Theorem 3.1.10). Stabilising this adjunction using the methods of §2.7 provides a mechanism by which we can present twisted differential cohomology theories in terms of algebraic data (Theorem 3.2.7 and Corollary 3.2.9). As a first basic example, we show that the module of sections of a vector bundle over a smooth manifold gives rise to a smooth parametrised spectrum whose corresponding twisted differential cohomology theory computes sections of pullback bundles (Example 3.2.10).

We conclude the chapter with a summary and with a discussion of two natural questions which arise from our work. The final section is less formal than the preceding ones: the reader is warned that inaccuracies may be lurking in those speculative waters.

3.1 ∞-Lie Theory . . .

In this section we summarise a higher Lie integration procedure which produces smooth ∞-stacks associated to connective L_∞-algebroids. The underlying idea is to interpret Sullivan's spatial realisation functor $\mathcal{S}\colon \mathrm{DGAlg}^{\mathrm{op}} \to \mathrm{sSet}$ as implementing a generalised Lie integration procedure when applied to Chevalley–Eilenberg cochain algebras. This idea is originally due to [Get09], following [Hin97], where $\mathcal{S}$ is used to integrate nilpotent L_∞-algebras to higher groupoids. An adaptation of this procedure due to Henriques [Hen08] lands in simplicial Banach manifolds, thereby producing smooth higher Lie groupoids. A further refinement introduced in [FSS12] produces smooth ∞-stacks also allows for a concomitant theory of higher secondary characteristic classes.

We begin this section by following the treatment of higher Lie integration given in [FSS12, §4]. We then realise this construction as one half of a Quillen adjunction between cDGAs over $\mathbb{R}$ and smooth ∞-stacks; this is a new perspective which sets the scene for exploring some of the homotopy-theoretic aspects of higher Lie theory.

Throughout, we adopt the dual perspective in which L_∞-algebroids are encoded by their Chevalley–Eilenberg cochain algebras:

Definition 3.1.1. Let R be a commutative algebra over $\mathbb{R}$ and let $\mathfrak{A}$ be a connective chain complex of R-modules which is degreewise finitely-generated projective. An L_∞*-algebroid* structure on $\mathfrak{A}$ is the datum of a differential d on the graded algebra $\mathrm{Sym}^\bullet_R(\mathfrak{A}^\vee[-1])$, where $\mathfrak{A}^\vee$ is the R-linear dual of $\mathfrak{A}$. The cDGA (over $\mathbb{R}$) given by

$$\mathrm{CE}(\mathfrak{A}) := \big(\mathrm{Sym}^\bullet_R(\mathfrak{A}^\vee[-1]), d\big)$$

is the *Chevalley–Eilenberg algebra* of the L_∞-algebroid $\mathfrak{A}$. A morphism of L_∞-algebroids $\mathfrak{A} \to \mathfrak{B}$ is defined in terms of Chevalley–Eilenberg algebras as a morphism of cDGAs $\mathrm{CE}(\mathfrak{B}) \to \mathrm{CE}(\mathfrak{A})$. We write $L_\infty\mathrm{-Algd} \hookrightarrow \mathrm{DGAlg}^{\mathrm{op}}_{\mathbb{R}}$ for the full subcategory of L_∞-algebroids.

Example 3.1.2. We briefly recall some of the standard examples of L_∞-algebroids, incarnated via their Chevalley–Eilenberg algebras.

- For $\mathfrak{g}$ an ordinary (finite-dimensional) Lie algebra over $\mathbb{R}$, $\mathrm{CE}(\mathfrak{g})$ is the cDGA whose underlying graded algebra is freely generated by $\mathfrak{g}^*$ in degree 1. The differential on $\mathrm{CE}(\mathfrak{g})$ is defined on generators by the dual of the Lie bracket $d = [-,-]^*$ and is uniquely extended as a graded derivation to all of $\mathrm{CE}(\mathfrak{g})$.

- For $n \in \mathbb{N}$, the L_∞-algebra $\mathbf{b}^{n-1}\mathbb{R}$ has $\mathrm{CE}(\mathbf{b}^{n-1}\mathbb{R})$ freely generated in degree n by a single generator with vanishing differential. Via the Sullivan–de Rham adjunction (with $\mathbb{R}$-coefficients), we observe that $\mathrm{CE}(\mathbf{b}^{n-1}\mathbb{R})$ defines a $K(n,\mathbb{R})$ after applying the spatial realisation functor $\mathcal{S}$.

- For M a smooth manifold, its *tangent Lie algebroid* has

$$\mathrm{CE}(TM) := \mathrm{Sym}^\bullet_{C^\infty(M)}(\mathfrak{X}^\vee_M[-1]) = \Omega^\bullet(M)$$

  equipped with the de Rham differential.

We consider a version of the path method of Lie integration which lands in smooth ∞-prestacks. Recall that a smooth ∞-prestack is a fibrant object for the projective model structure on simplicial presheaves $\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet})_{\mathrm{proj}}$ (Remark 1.3.17). Applied to an L_∞-algebroid $\mathfrak{A}$, this procedure produces a simplicial presheaf which sends the Cartesian space U to a Kan complex whose n-simplices are the smoothly U-parametrised families of smooth flat $\mathfrak{A}$-valued connections on the smooth n-simplex Δ^n. We view Δ^n as a submanifold (with corners) of $\mathbb{R}^{n+1}$ via the standard presentation. For the Lie integration procedure to come out correctly, we must impose some boundary conditions on smooth data defined on the smooth simplices.

Definition 3.1.3. For any point $x \in \Delta^n$, let Δ_x be the lowest dimensional sub-simplex of Δ^n to which x belongs, and let π_x be the orthogonal projection onto the affine subspace spanned by Δ_x. For $\epsilon > 0$, a differential form ω on Δ^n has ϵ*-sitting instants* if for all $x \in \partial\Delta^n$ there is a neighbourhood V_x of Δ_x containing the tubular neighbourhood of Δ_x of radius ϵ such that $\omega|_{V_x} = \pi_x^*(\omega|_{\Delta_x})$. Let $\Omega^\bullet_{\mathrm{si}}(\Delta^n) \subset \Omega^\bullet(\Delta^n)$ be the sub-DG-algebra of forms which have ϵ-sitting instants for some $\epsilon > 0$.

For a Cartesian space U and $\epsilon > 0$, a differential form ω on $U \times \Delta^n$ has ϵ*-sitting instants* if for all points $u \colon * \to U$ the pullback $(u \times \mathrm{id})^*\omega$ is a form with ϵ-sitting instants on Δ^n. We write $\Omega^\bullet_{\mathrm{si}}(U \times \Delta^n) \subset \Omega^\bullet(U \times \Delta^n)$ for the sub-DG-algebra of forms which have ϵ-sitting instants for some $\epsilon > 0$.

Finally, we write $\Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^n)$ for the cDGA of vertical differential forms ω with respect to the projection $\mathrm{pr}_1 \colon U \times \Delta^n \to U$ and which have ϵ-sitting instants for some $\epsilon > 0$.

Remark 3.1.4. The definition of $\Omega^\bullet_{\mathrm{si}}(\Delta^n)$ that we have given above is the same as that of [FSS12, Definition 4.2.1]. Our definition of $\Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^n)$, however, is different as we impose a uniform lower bound on the size of the sitting instant neighbourhoods as we vary in U. This distinction is crucial for the proof of Lemma 3.1.15 below.

Remark 3.1.5. It is instructive to think of forms in $\Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^n)$ as families of elements of $\Omega^\bullet_{\mathrm{si}}(\Delta^n)$ which are smoothly parametrised by $x \in U$.

Construction 3.1.6. For $\mathfrak{A}$ an L_∞-algebroid, its *exponential smooth ∞-prestack* $\exp_\Delta(\mathfrak{A})$ is the simplicial presheaf on CartSp defined as

$$\exp_\Delta(\mathfrak{A}) \colon \big(U, [n]\big) \longmapsto \mathrm{DGAlg}_{\mathbb{R}}\big(\mathrm{CE}(\mathfrak{A}), \Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^n)\big).$$

An n-simplex of $\exp_\Delta(\mathfrak{A})(U)$ can thus be thought of as a smooth U-parametrised family of flat $\mathfrak{A}$-valued forms on Δ^n; indeed, this is exactly the case when $\mathfrak{A} = \mathfrak{g}$ is an ordinary Lie algebra. The sitting instants condition guarantees that we can glue together such smooth families along faces of simplices. Observe that the assignment $\mathfrak{A} \mapsto \exp_\Delta(\mathfrak{A})$ is functorial in $\mathfrak{A}$.

Lemma 3.1.7. *For any L_∞-algebroid $\mathfrak{A}$, the simplicial presheaf $\exp_\Delta(\mathfrak{A})$ is a smooth ∞-prestack.*

Proof. This is a consequence of the fact that the smooth horn inclusion $h^n_k \colon \Lambda^n_k \hookrightarrow \Delta^n$ is a smooth deformation retract. Indeed, if $r_k \colon \Delta^n \to \Lambda^n_k$ is the standard smooth retraction map of the smooth n-simplex to its k-th horn, we have a commuting diagram of cDGAs:

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{si,vert}}(U \times \Lambda^n_k) \arrow[r, "(\mathrm{id}\times r_k)^*"] \arrow[dr, "\mathrm{id}"'] & \Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^n) \arrow[d, "(\mathrm{id}\times h^n_k)^*"] \\
 & \Omega^\bullet_{\mathrm{si,vert}}(U \times \Lambda^n_k)
\end{tikzcd}$$

for all $U \in \mathrm{CartSp}$. Note that the sitting instants condition is required for the pullbacks along $\mathrm{id} \times r_k$ to be well-defined. In particular, the Kan condition for $\exp_\Delta(\mathfrak{A})(U)_\bullet$ is satisfied since the map

$$\mathrm{DGAlg}_{\mathbb{R}}\big(\mathrm{CE}(\mathfrak{A}), \Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^n)\big) \xrightarrow{(\mathrm{id}\times h^n_k)^*} \mathrm{DGAlg}_{\mathbb{R}}\big(\mathrm{CE}(\mathfrak{A}), \Omega^\bullet_{\mathrm{si,vert}}(U \times \Lambda^n_k)\big)$$

is surjective. □

Remark 3.1.8. In order to obtain a genuine smooth ∞-stack from $\mathfrak{A}$ via this Lie integration procedure, in general it is necessary to apply the "∞-stackification functor". In terms of model categories, ∞-stackification is implemented by the left Bousfield localisation

$$\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet})_{\mathrm{proj}} \longrightarrow \mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet})_{\mathrm{proj,loc}};$$

namely, we take locally fibrant replacement of $\exp_\Delta(\mathfrak{A})$ (which by the above Lemma is already projectively fibrant). Although the ∞-stackification procedure is rather opaque, it does preserve finite homotopy limits, meaning that finite homotopy limits of smooth ∞-stacks can be presented already at the level of smooth ∞-prestacks. This is extremely useful for the analysis of homotopy fibres [FSS12, §6].

The attentive reader will have noticed that the proof of Lemma 3.1.7 does not depend on $\mathfrak{A}$ in any way: we can replace the Chevalley–Eilenberg algebra $\mathrm{CE}(\mathfrak{A})$ by any cDGA over $\mathbb{R}$ and the result still holds. In [FSS12], the restriction to the full subcategory $L_\infty-\mathrm{Algd}$ is due to the thematic focus of that paper on higher characteristic classes and ∞-Chern–Weil theory. We now pursue a different approach, which allows us to couch Construction 3.1.6 in terms of axiomatic homotopy theory:

Construction 3.1.9. Let $\mathcal{O}^\infty\colon \mathrm{CartSp}\times\Delta \to \mathrm{DGAlg}_{\mathbb{R}}^{\mathrm{op}}$ be the functor determined on objects by the assignment

$$\mathcal{O}^\infty\colon (U,[n]) \longmapsto \Omega^\bullet_{\mathrm{si,vert}}(U\times\Delta^n).$$

By the co-Yoneda Lemma, every simplicial presheaf on CartSp is a colimit over representables of the form $\underline{U}\otimes\Delta[n]$. We abuse notation by writing $\mathcal{O}^\infty$ for the left Kan extension of $\mathcal{O}^\infty$ along the Yoneda embedding $\mathcal{Y}\colon \mathrm{CartSp}\times\Delta \hookrightarrow \mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}},\mathrm{sSet})$:

$$\begin{tikzcd}
\mathrm{CartSp}\times\Delta \arrow[rr, "\mathcal{O}^\infty"] \arrow[dr, hook, "\mathcal{Y}"'] & {} \arrow[d, phantom, "\Downarrow"] & \mathrm{DGAlg}_{\mathbb{R}}^{\mathrm{op}} \\
& \mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}},\mathrm{sSet}). \arrow[ur, "\mathcal{O}^\infty"'] &
\end{tikzcd}$$

The general yoga of nerve and realisation functors implies that $\mathcal{O}^\infty$ (which is a realisation functor) has a right adjoint $\mathcal{B}$. In terms of the (co)end calculus, we have

$$\mathcal{O}^\infty\colon X\cong \int^{(U,[n])}(\underline{U}\otimes\Delta[n])\times X(U)_n \longmapsto \int_{(U,[n])}\prod_{X(U)_n}\Omega^\bullet_{\mathrm{si,vert}}(U\times\Delta^n),$$

where $X\cong\int^{(U,[n])}(\underline{U}\otimes\Delta[n])\times X(U)_n$ is expressed as a colimit of representables by the co-Yoneda Lemma. The right adjoint functor $\mathcal{B}$ sends the cDGA A over $\mathbb{R}$ to the simplicial presheaf

$$\mathcal{B}(A)\colon (U,[n]) \longmapsto \mathrm{DGAlg}_{\mathbb{R}}\big(A,\Omega^\bullet_{\mathrm{si,vert}}(U\times\Delta^n)\big).$$

In particular, $\mathcal{B}(\mathrm{CE}(\mathfrak{A})) = \exp_\Delta(\mathfrak{A})$ for any L_∞-algebroid $\mathfrak{A}$.

The category $\mathrm{DGAlg}_{\mathbb{R}}$ of connective cDGAs over $\mathbb{R}$ supports a projective model structure for which weak equivalences and fibrations are the quasi-isomorphisms and surjections respectively (compare Remark 2.1.6). With respect to this model structure, we have the following

Theorem 3.1.10. *The adjunction*

$$\begin{tikzcd}
\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}},\mathrm{sSet})_{\mathrm{proj,loc}} \arrow[r, shift left, "\mathcal{O}^\infty"] \arrow[r, phantom, "\perp"] & \mathrm{DGAlg}_{\mathbb{R}}^{\mathrm{op}} \arrow[l, shift left, "\mathcal{B}"]
\end{tikzcd}$$

is Quillen.

Proof. We establish this first at the level of the projective model structure on simplicial presheaves and then show that the adjunction descends to the local model structure. The projective model structure on $\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet})$ is cofibrantly generated, with generating cofibrations

$$\mathcal{I}_{\mathrm{smth}} := \{\underline{U} \otimes i_n \colon \underline{U} \otimes \partial\Delta[n] \to \underline{U} \otimes \Delta[n] \mid U \in \mathrm{CartSp},\, n \geq 0\}$$

and generating acyclic cofibrations

$$\mathcal{J}_{\mathrm{smth}} := \{\underline{U} \otimes h^n_k \colon \underline{U} \otimes \Lambda^n_k \to \underline{U} \otimes \Delta[n] \mid U \in \mathrm{CartSp},\, n \geq 0,\, 0 \leq k \leq n\},$$

where i_n and h^n_k are the usual boundary and horn inclusion maps [Hir03, Theorem 11.6.1]. Applying $\mathcal{O}^\infty$ to $\underline{U} \otimes i_n$ yields the restriction map

$$\Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^n) \longrightarrow \Omega^\bullet_{\mathrm{si,vert}}(U \times \partial\Delta^n),$$

which is surjective, as can be seen by extending $\omega \in \Omega^\bullet_{\mathrm{si,vert}}(U \times \partial\Delta[n])$ to the whole n-simplex using bump functions with sitting instants. Applying $\mathcal{O}^\infty$ to $\underline{U} \otimes h^n_k$ yields the restriction map

$$\Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^n) \longrightarrow \Omega^\bullet_{\mathrm{si,vert}}(U \times \Lambda^n_k). \tag{3.1}$$

By using bump functions with sitting instants, the morphisms (3.1) are all seen to be surjections. We wish to show that they are also quasi-isomorphisms, for which we shall use a hypercohomology argument[1]. Consider the sheaf $\mathcal{F}^p_{\Delta^n}$ on Δ^n defined by sheafification of the presheaf

$$V \longmapsto \Omega^p_{\mathrm{si,vert}}(U \times V),$$

and similarly for Λ^n_k. Sheafification is necessary here because of the uniform bound on the size of the sitting instant neighbourhoods. Nevertheless, we have

$$\Gamma(\mathcal{F}^p_{\Delta^n}) = \Omega^p_{\mathrm{si,vert}}(U \times \Delta^n) \quad \text{and} \quad \Gamma(\mathcal{F}^p_{\Lambda^n_k}) = \Omega^p_{\mathrm{si,vert}}(U \times \Lambda^n_k).$$

By using bump functions with sitting instants, we find that the sheaves $\mathcal{F}^p_{(-)}$ are fine for all $p \geq 0$. By Lemma 3.1.15, the complex of sheaves

$$0 \longrightarrow \underline{C}^\infty(U) \hookrightarrow \mathcal{F}^1_{\Delta^n} \longrightarrow \mathcal{F}^2_{\Delta^n} \longrightarrow \cdots$$

is an acyclic resolution of the constant sheaf with coefficients in $C^\infty(U)$. Computing hypercohomology, we thus have

$$\mathbb{H}^p(\Delta^n; \mathcal{F}^\bullet_{\Delta^n}) \cong \mathbb{H}^p(\Delta^n; \underline{C}^\infty(U)) = H^p(\Delta^n; \underline{C}^\infty(U)) = \begin{cases} C^\infty(U) & p = 0 \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, since the sheaves $\mathcal{F}^p_{\Delta^n}$ are acyclic, the hypercohomology spectral sequence for $\mathcal{F}^\bullet_{\Delta^n}$ degenerates at the second page, with

$$E_2^{p,0} = H^p\big(\Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^n)\big)$$

[1] this argument expands upon [FSS12, Remark 4.2.2].

and all other terms zero. Thus

$$H^p(\Omega^\bullet_{\mathrm{si,vert}}(U\times\Delta^n)) = \begin{cases} C^\infty(U) & p=0 \\ 0 & \text{otherwise.}\end{cases}$$

A similar argument applies to Λ^n_k, so we conclude that the maps (3.1) are indeed quasi-isomorphisms. This shows that $\mathcal{O}^\infty$ sends $\mathcal{I}_{\mathrm{smth}}$ and $\mathcal{J}_{\mathrm{smth}}$ to cofibrations and acyclic cofibrations in $\mathrm{DGAlg}_{\mathbb{R}}^{\mathrm{op}}$ respectively, hence $\mathcal{O}^\infty$ is a left Quillen functor for the projective model structure on simplicial presheaves.

For $\mathcal{O}^\infty$ to descend to the local projective model structure, we must show that for each $U \in \mathrm{CartSp}$ and good open cover $\mathcal{U} = \{U_i \to U\}_{i\in\mathcal{I}}$ of U, the Čech nerve $\check{C}(\mathcal{U}) \to U$ is sent to a weak equivalence by $\mathcal{O}^\infty$. For this, we recall that there is a weak equivalence $\check{C}(\mathcal{U}) \cong \mathrm{hocolim}_{\Delta^{\mathrm{op}}}\check{C}(\mathcal{U})_\bullet$, since regarded as a simplicial diagram in $\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}},\mathrm{sSet})_{\mathrm{proj}}$, the object $[n] \mapsto \check{C}(\mathcal{U})_n$ is Reedy cofibrant (see (1.15) and the reference immediately following it). Applying $\mathcal{O}^\infty$ we thus obtain a Reedy fibrant cosimplicial object

$$\mathcal{C}^\infty_{\mathcal{U}} \colon [n] \longmapsto \mathcal{O}^\infty(\check{C}(\mathcal{U})_n)$$

of $\mathrm{DGAlg}_{\mathbb{R}}$. It is sufficient to show that the map

$$C^\infty(U) \longrightarrow \operatorname*{holim}_{\Delta} \mathcal{C}^\infty_{\mathcal{U}} \tag{3.2}$$

is a weak equivalence, and for this we may argue at the level of underlying cochain complexes since the forgetful functor $Ho(\mathrm{DGAlg}_{\mathbb{R}}) \to Ho(Ch^+_{\mathbb{R}})$ is conservative. We showed above that for any $U \in \mathrm{CartSp}$ and $n \geq 0$, the canonical map

$$C^\infty(U) \longrightarrow \Omega^\bullet_{\mathrm{si,vert}}(U\times\Delta^n)$$

is a weak equivalence. Hence, we have a morphism of Reedy fibrant cosimplicial objects in $Ch^+_{\mathbb{R}}$,

$$\begin{tikzcd} \prod_{i\in\mathcal{I}} C^\infty(U_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_{i,j\in\mathcal{I}} C^\infty(U_{ij}) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & \cdots \\ \prod_{i\in\mathcal{I}} C^\infty(U_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j\in\mathcal{I}} \Omega^\bullet_{\mathrm{si,vert}}(U_{ij}\times\Delta^1) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots \end{tikzcd}$$

in which each vertical morphism is a weak equivalence. By Reedy fibrancy, we have that (the underlying chain complex of) the homotopy limit $\mathrm{holim}_\Delta \mathcal{C}^\infty_{\mathcal{U}}$ is quasi-isomorphic to the total complex of the double complex

$$\prod_{i\in\mathcal{I}} C^\infty(U_i) \xrightarrow{\ \partial\ } \prod_{i,j\in\mathcal{I}} C^\infty(U_{ij}) \xrightarrow{\ \partial\ } \cdots$$

where ∂ is the alternating sum of cosimplicial coface maps. But this is simply the unordered Čech complex for the sheaf $V \mapsto C^\infty(V)$ on U, which has cohomology $C^\infty(U)$ concentrated in degree zero by the standard argument on partitions of unity. We conclude that (3.2) is a quasi-isomorphism on underlying cochain complexes. □

Corollary 3.1.11. *If $A \in \mathrm{DGAlg}_{\mathbb{R}}$ is cofibrant, then $\mathcal{B}(A)$ is a smooth ∞-stack.*

Proof. By the Theorem, $\mathcal{B}$ sends fibrant objects to fibrant objects. □

Example 3.1.12. For $\mathfrak{g}$ an ordinary Lie algebra, the Chevalley–Eilenberg algebra $\mathrm{CE}(\mathfrak{g})$ is cofibrant in $\mathrm{DGAlg}_{\mathbb{R}}$ precisely if $\mathfrak{g}$ is nilpotent. Thus, if $\mathfrak{g}$ is a nilpotent Lie algebra then $\exp_\Delta(\mathfrak{g}) = \mathcal{B}(\mathrm{CE}(\mathfrak{g}))$ is already a smooth ∞-stack. This applies more generally for $\mathfrak{g}$ a nilpotent L_∞-algebra (in the sense of [Get09, Definition 4.2]).

Example 3.1.13. For M a (second-countable) smooth manifold, its ring of smooth functions $C^\infty(M)$ is cofibrant when regarded as a cDGA concentrated in degree zero (the left lifting property with respect to surjective quasi-isomorphisms is verified directly). The resulting smooth ∞-stack $\mathcal{B}(C^\infty(M))$ is isomorphic to the simplicial presheaf

$$\underline{M} \colon U \longmapsto \mathrm{Man}(U, M) \cong \mathbb{R}{-}\mathrm{Alg}(C^\infty(M), C^\infty(U)).$$

In the other direction, for any paracompact manifold M by picking a good open cover the associated Čech nerve $\check{C}(\mathcal{U}) \to \underline{M}$ is a cofibrant resolution of $\underline{M}$ for the local projective model structure. Applying $\mathcal{O}^\infty$, an argument as in Theorem 3.1.10 shows that the comparison map $C^\infty(M) \to \mathcal{O}^\infty(\check{C}(\mathcal{U}))$ is a quasi-isomorphism.

On the other hand, if X is a simplicial set then the simplicial presheaf $cX \colon U \mapsto X$ is cofibrant. A hypercohomology argument similar to that of Theorem 3.1.10 shows that $\mathcal{O}^\infty(cX)$ computes the real simplicial cohomology of X. Combining these two examples, we see that the adjunction $(\mathcal{O}^\infty \dashv \mathcal{B})$ of Theorem 3.1.10 captures a subtle mixture of both differential geometry and real (equivalently, rational) homotopy theory.

Remark 3.1.14. The reader may be perturbed by the apparent lack of smoothness on one side of the $(\mathcal{O}^\infty \dashv \mathcal{B})$-adjunction. There are two fundamental reasons that we can get away with working with plain cDGAs over $\mathbb{R}$, rather than more exotic algebras with smooth structure:

(1) One of the miracles of differential geometry[2] known as *Milnor's exercise* is that the functor

$$C^\infty(-) \colon \mathrm{Man} \longrightarrow \mathbb{R}{-}\mathrm{Alg}^{\mathrm{op}}$$

sending a manifold to its ring of smooth functions is fully-faithful on the full subcategory of second-countable manifolds [KMS93, Corollary 35.10].

(2) In unpublished work, Carchedi and Roytenberg construct a category DGAlg_∞ of *smooth cDGAs* [CR12, §6]. A smooth cDGA is a cDGA A over $\mathbb{R}$ such that the subalgebra $Z^0(A)$ of zero-cochains is a C^∞-ring: together with addition and multiplication, each smooth function $\mathbb{R}^n \to \mathbb{R}$ defines an n-ary operation on $Z^0(A)$ such that these operations are compatible with composition of smooth functions in the obvious way. The prototypical example of a C^∞-ring is the ring of smooth functions on a manifold though by no means are all C^∞-rings of this form; in general a smooth cDGA is thus an admixture of generalised smooth data as captured by the zero cochains and real homotopy-theoretic data as captured by information in non-zero degrees. In particular, each smooth cDGA is a cDGA over $\mathbb{R}$ and there is a forgetful functor $\mathrm{DGAlg}_\infty \to \mathrm{DGAlg}_{\mathbb{R}}$. This forgetful functor has a left adjoint called the C^∞-*completion*, and the adjunction

$$\begin{tikzcd} \mathrm{DGAlg}_{\mathbb{R}} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{DGAlg}_\infty \arrow[l, shift left] \end{tikzcd}$$

equips DGAlg_∞ with the structure of a cofibrantly generated model category. It is not hard to see that the $(\mathcal{O}^\infty \dashv \mathcal{B})$-adjunction factors as a composite of

[2] the author thanks Urs Schreiber for patiently driving home this point and also for the nice turn of phrase.

Quillen adjunctions

$$\begin{tikzcd} \mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet})_{\mathrm{proj,loc}} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{DGAlg}^{\mathrm{op}}_{\infty} \arrow[l, shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{DGAlg}^{\mathrm{op}}_{\mathbb{R}}. \arrow[l, shift left] \end{tikzcd}$$

In particular, this shows that in some sense the functor $\mathcal{B}$ gives us smoothness for free.

We conclude this section with a technical result which was used in the proof of Theorem 3.1.10:

Lemma 3.1.15. *For all $U \in \mathrm{CartSp}$ and $n \geq 0$, the complex of sheaves $\mathcal{F}^{\bullet}_{\Delta^n}$ on Δ^n satisfies the stalkwise Poincaré Lemma. In particular, each stalk has cohomology $C^\infty(U)$ concentrated in degree zero.*

Proof. It is sufficient to show that for any $x \in \Delta^n$, if $\omega \in \Omega^{\bullet}_{\mathrm{si,vert}}(U \times W)$ is closed in a neighbourhood W of x, then ω is exact in a neighbourhood $V \subset W$ of x.

In the case that x is an interior point of Δ^n, we can take V to be a sufficiently small open ball centred at x such that $V \cap \partial\Delta^n = \varnothing$. In this case, the sitting instants condition is vacuous, so by radially contracting V to x we find a primitive for ω by the standard Poincaré Lemma argument. We spell this out, to make sure that the smoothness and verticality along U are indeed preserved. Let $R\colon V \times [0,1] \to V$ be the radial contraction homotopy, so that $R(x + \mathbf{v}, t) := x + t\mathbf{v}$ for all $x + \mathbf{v} \in V$. We can view any $\alpha \in \Omega^{\bullet}_{\mathrm{si,vert}}(U \times V) = \Omega^{\bullet}_{\mathrm{vert}}(U \times V)$ as a function $\underline{\alpha}\colon U \to \Omega^{\bullet}(V)$ such that all the coefficient functions of $\underline{\alpha}$ depend smoothly on $u \in U$ and conversely; namely if

$$\underline{\alpha}(u) = \sum_{|I| \leq n} f^I(u, t_1, \cdots, t_n) dt^I,$$

then the functions f^I depend smoothly on $u \in U$ in addition to their smooth dependence on the coordinates t_i on $V \subset \Delta^n$. Writing d_v momentarily for the vertical differential on $\Omega^{\bullet}_{\mathrm{vert}}(U \times V)$, it is clear that $\underline{d_v \alpha} = d \circ \underline{\alpha}$. Now, if s is the coordinate along $[0,1]$ and $\theta \in \Omega^{\bullet}(V)$, set

$$I\theta := \int_0^1 ds\, \theta_s^{(B)},$$

where $R^*\theta = \theta_s^{(A)} + ds \wedge \theta_s^{(B)}$ is the unique decomposition such that $\theta_s^{(A)}$ has no ds-term. A standard argument shows that

$$\theta - x^*\theta = (I \circ d + d \circ I)\, \theta,$$

where $x\colon V \to V$ sends all points to x. Applying this to a closed $\omega \in \Omega^p_{\mathrm{vert}}(U \times V)$ for $p \geq 1$, we have

$$\underline{\omega}(u) = dI(\underline{\omega}(u)).$$

The assignment $u \mapsto I(\underline{\omega}(u))$ determines a vertical $(p-1)$-form τ on $U \times V$ such that $d_v \tau = \omega$.

Let us now turn to the case that $x \in \partial\Delta^n$, let $\epsilon > 0$ be such that the closed vertical differential $(p \geq 1)$-form $\omega \in \Omega^p_{\mathrm{si,vert}}(U \times W)$ has ϵ-sitting instants. Let k be the least integer such that $x \in \mathrm{sk}_k\Delta^n$ and let V be an open ball of radius $\delta < \epsilon$ centred at x such that V does not intersect the $(k-1)$-skeleton of Δ^n, nor does it intersect any k-face to which x does not belong. Then for all $u\colon * \to U$, we have

$$\big((u \times \mathrm{id})^*\omega\big)|_V = \pi_x^*\big((u \times \mathrm{id})^*\omega|_{(V \cap \mathrm{sk}_k\Delta^n)}\big), \tag{3.3}$$

with notation as in Definition 3.1.3. We can smoothly contract $V \cap \mathrm{sk}_k\Delta^n$ to x, so by the above argument we have a primitive for $\omega|_{(V\cap \mathrm{sk}_k\Delta^n)}$. By (3.3), pulling back this primitive along $\pi_x^*\colon U \times V \to U \times (V \cap \mathrm{sk}_k\Delta^n)$ yields the sought-after primitive for ω on V. $\square$

3.2 ...and Twisted Differential Cohomology

In this section, we outline a process by which twisted differential cohomology theories can be presented in terms of algebraic data. The fundamental idea is to leverage the same stabilisation procedure with which we obtained Theorem 2.7.34, but applied to the adjunction of Theorem 3.1.10 instead. The main result of this section is Theorem 3.2.7, which shows that cofibrant modules over cofibrant cDGAs produce twisted differential cohomology theories via higher Lie integration. We conclude with Example 3.2.10, which clarifies the relationship between f.g.p. modules, vector bundles and twisted differential cohomology.

Our discussion in this section is rather terse as many of the technical arguments are similar to those of §2.7. As a first step, observe that for any cDGA A over $\mathbb{R}$, there is an induced Quillen adjunction on categories of retractive objects

$$\Big(\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet})_{/\!/\mathcal{B}(A)}\Big)_{\mathrm{proj,loc}} \underset{\mathcal{B}_A}{\overset{\mathcal{O}_A^\infty}{\rightleftarrows}} \Big((\mathrm{DGAlg}_{\mathbb{R}})_{/\!/A}\Big)^{\mathrm{op}}$$

(where $\mathcal{C}_{/\!/X} := (\mathcal{C}_{/X})^{X/} \cong (\mathcal{C}^{X/})_{/X}$). The proof is similar Lemma 2.7.1. We wish to stabilise this adjunction following the logic of §2.7, so we must first establish that $\mathcal{O}^\infty$ is sufficiently well-behaved with respect to suspension functors.

Construction 3.2.1. Let A be a cDGA over $\mathbb{R}$ and let B be an augmented A-algebra. As $\mathrm{DGAlg}_{\mathbb{R}}$ is right proper, the pullback diagrams

$$\begin{tikzcd} \partial\Delta[1]\otimes^A B \arrow[r] \arrow[d] & B\oplus B \arrow[d] \\ A \arrow[r] & A\oplus A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Delta[1]\otimes^A B \arrow[r] \arrow[d] & \Omega^\bullet_{\mathrm{si}}(\Delta^1)\otimes B \arrow[d] \\ A \arrow[r] & \Omega^\bullet_{\mathrm{si}}(\Delta^1)\otimes A \end{tikzcd} \tag{3.4}$$

determine right Quillen endofunctors of $\mathrm{DGAlg}_{/A}^{A/}$ (left adjoints are guaranteed by the Adjoint Functor Theorem). Consequently, sending B to the pullback

$$\begin{tikzcd} \Sigma'_A B \arrow[r] \arrow[d] & \Delta[1]\otimes^A B \arrow[d] \\ A \arrow[r] & \partial\Delta[1]\otimes^A B \end{tikzcd}$$

determines a right Quillen endofunctor Σ'_A on $\mathrm{DGAlg}_{\mathbb{R}}$.

Lemma 3.2.2. *Let A be a cDGA over $\mathbb{R}$. Then the right Quillen endofunctor Σ'_A models suspension on $Ho((\mathrm{DGAlg}_{\mathbb{R}})_{/\!/A})^{\mathrm{op}}$.*

Proof. Observe that all objects of $(\mathrm{DGAlg}_{\mathbb{R}})_{/\!/A}$ are fibrant and that $\partial\Delta[1]\otimes^A B$ coincides with $B\prod_A B$. Moreover, $\Delta[1]\otimes^A B$ is a good path object for B: the diagrams of (3.4) are homotopy pullbacks by right properness, so since $\mathbb{R}\to\Omega^\bullet_{\mathrm{si}}(\Delta^1)$ is a weak equivalence, the map of augmented A-algebras $B \to \Delta[1] \otimes^A B$ is one too. The map $\Delta[1] \otimes^A B \to B \prod_A B$ is surjective, so is a fibration. $\square$

Lemma 3.2.3. *For any $A \in \mathrm{DGAlg}_{\mathbb{R}}$, there is a natural transformation of functors*

$$\xi\colon \mathcal{O}^\infty_A \Sigma_{\mathcal{B}(A)} \Longrightarrow \Sigma'_A \mathcal{O}^\infty_A.$$

Proof. We use the co-Yoneda Lemma to write the simplicial presheaves $\mathcal{B}(A)$ and $X \in \mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet})_{//\mathcal{B}(A)}$ as

$$\mathcal{B}(A) \cong \int^{(U,[n])} (\underline{U} \otimes \Delta[n]) \times \mathcal{B}(A)(U)_n \quad \text{and} \quad X \cong \int^{(U,[n])} (\underline{U} \otimes \Delta[n]) \times X(U)_n,$$

noting that each of the $X(U)_n$ retracts onto $\mathcal{B}(A)(U)_n$. In terms of these coend expressions, we find that

$$\Delta[1] \otimes X \cong \int^{(U,[n])} (\underline{U} \otimes (\Delta[n] \times \Delta[1])) \times X(U)_n,$$

for instance. Using the usual shuffle decomposition of $\Delta[n] \times \Delta[1]$ into $(n+1)$-simplices, we obtain maps

$$\Omega^\bullet_{\mathrm{si}}(\Delta^1) \otimes \Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^n) \longrightarrow \mathcal{O}^\infty\big(\underline{U} \otimes (\Delta[n] \times \Delta[1])\big) \tag{3.5}$$

which are natural in U and n. In order to guarantee naturality, we identify $\Delta[1]$ with $\{[0]\} \times \Delta[1] \hookrightarrow \Delta[n] \times \Delta[1]$ and $\Delta[n]$ with $\Delta[n] \times \{[0]\} \hookrightarrow \Delta[n] \times \Delta[1]$. Feeding these maps into the definitions of $\Sigma_{\mathcal{B}(A)}$ and Σ'_A gives rise to the sought-after natural transformation ξ. $\square$

Although $\mathcal{O}^\infty_A$ does not preserve suspension functors on the nose, Lemma 3.2.3 is sufficient to produce a Quillen adjunction

$$\mathrm{Sp}^{\mathbb{N}}\Big(\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet})_{//\mathcal{B}(A)}; \Sigma_{\mathcal{B}(A)}\Big) \underset{\mathcal{B}_A}{\overset{\overline{\mathcal{O}}^\infty_A}{\rightleftarrows}} \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A)^{\mathrm{op}}. \tag{3.6}$$

In this Quillen adjunction, the category of spectra on the left-hand side is given the projective model structure (for which fibrations and weak equivalences are the maps which are levelwise fibrations and weak equivalences of simplicial presheaves with respect to the local projective model structure) and $\mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A)$ is the category of sequential cospectra of augmented A-algebras with respect to the suspension endofunctor Σ'_A of Construction 3.2.1. The proof of Lemma 2.7.8 carries over to this setting, so that $\mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A)$ is equipped with the injective model structure. The right adjoint in (3.6) is defined by levelwise application of $\mathcal{B}_A$, using the dual natural transformation ξ to define the spectrum structure maps. The left adjoint is defined using (co)monadicity in a manner completely analogous to the construction of the functor aug^∞_A in Lemma 2.7.20.

Next, we pass from cospectra of augmented algebras to cospectra of connective modules using the augmentation ideal functor $\mathrm{aug}_A\colon (\mathrm{DGAlg}_{\mathbb{R}})_{//A} \to A{-}\mathrm{Mod}^+$. For this, we need the following

Lemma 3.2.4. *For any augmented A-algebra B, there is a natural weak equivalence*

$$\tau_B\colon (\mathrm{aug}_A B)[-1] \longrightarrow \mathrm{aug}_A(\Sigma'_A B).$$

Proof. Consider the map of cochain complexes $\Omega^\bullet_{\mathrm{si}}(\Delta^1) \to N^\bullet(\Delta[1])$ given by

$$\begin{tikzcd}
\Big[C^\infty_{\mathrm{si}}(\Delta^1) \arrow[r, "d"] \arrow[d, "f\mapsto(f(0),f(1))"'] & \Omega^1_{\mathrm{si}}(\Delta^1) \arrow[r] \arrow[d, "\omega\mapsto\int_{\Delta^1}\omega"] & 0 \arrow[r] & \cdots\Big] \\
\Big[\mathbb{R}\oplus\mathbb{R} \arrow[r, "(a,b)\mapsto b-a"] & \mathbb{R} \arrow[r] & 0 \arrow[r] & \cdots\Big],
\end{tikzcd}$$

noting that this is a weak equivalence. We have a map of cospans

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \mathbb{R}\oplus\mathbb{R} \arrow[d, equal] & \Omega^1_{\mathrm{si}}(\Delta^1) \arrow[l, two heads] \arrow[d] \\
0 \arrow[r] & \mathbb{R}\oplus\mathbb{R} & N^\bullet(\Delta[1]) \arrow[l, two heads]
\end{tikzcd}$$

which, by right properness of $Ch^+_{\mathbb{R}}$, implies a weak equivalence of the (homotopy) pullbacks $P \to \mathbb{R}[-1]$. Choosing a bump function ρ on Δ^1 such that $\rho \equiv 0$ in a neighbourhood of 0 and 1 with the property that $\int_{\Delta^1}\rho = 1$, the map

$$\begin{aligned}
\mathbb{R}[-1] &\longrightarrow \Omega^\bullet_{\mathrm{si}}(\Delta^1) \\
a &\longmapsto a\cdot\rho dt_1
\end{aligned}$$

factors through P. The resulting map $\mathbb{R}[-1] \to P$ is a section of $P \to \mathbb{R}[-1]$ and hence is a weak equivalence. Arguing as in Lemma 2.7.16, we find that $P \otimes \mathrm{aug}_A B \cong \mathrm{aug}_A(\Sigma'_A B)$, so that tensoring $\mathrm{aug}_A B$ with $\mathbb{R}[-1] \to P$ yields the desired natural weak equivalence by the Künneth Theorem. □

The proof of Lemma 2.7.20 now carries over to this setting more or less verbatim to prove

Lemma 3.2.5. *For any $A \in \mathrm{DGAlg}_{\mathbb{R}}$, there is a Quillen adjunction*

$$\begin{tikzcd}
\mathrm{CoSp}^{\mathbb{N}}_{\mathrm{mod}}(A) \arrow[r, shift left, "\mathrm{Sym}_A"] \arrow[r, phantom, "\perp"] & \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{alg}}(A) \arrow[l, shift left, "\mathrm{aug}^\infty_A"]
\end{tikzcd}$$

where the left adjoint is given by applying of $\mathrm{Sym}_A\colon A{-}\mathrm{Mod}^+ \to (\mathrm{DGAlg}_{\mathbb{R}})_{//A}$ *levelwise.*

The disassembly-assembly Quillen adjunction $(\mathcal{D} \dashv \mathcal{A})\colon A{-}\mathrm{Mod} \to \mathrm{CoSp}^{\mathbb{N}}_{\mathrm{mod}}(A)$ of Lemma 2.7.23 also carries over to the present setting, simply by replacing $\mathbb{Q}$ by $\mathbb{R}$. Composing $(\overline{\mathcal{O}}^\infty_A \dashv \mathcal{B}_A)$ with the opposite adjunctions $(\mathrm{aug}^\infty_A \dashv \mathrm{Sym}_A)$ and $(\mathcal{A} \dashv \mathcal{D})$, we obtain a composite Quillen adjunction

$$\begin{tikzcd}
\mathrm{Sp}^{\mathbb{N}}\Big(\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet})_{//\mathcal{B}(A)}; \Sigma_{\mathcal{B}(A)}\Big) \arrow[r, shift left, "\mathcal{M}^\infty_A"] \arrow[r, phantom, "\perp"] & A{-}\mathrm{Mod}^{\mathrm{op}}. \arrow[l, shift left, "\mathcal{E}_A"]
\end{tikzcd} \tag{3.7}$$

Remark 3.2.6. In order to bring the notation under control, from now on we write

$$R^\infty_{\mathcal{B}(A)} := \mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet})_{//\mathcal{B}(A)}$$

and

$$\mathrm{Sp}^{\infty}_{\mathcal{B}(A)} := \mathrm{Sp}^{\mathbb{N}}\Big(\mathrm{Fun}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet})_{//\mathcal{B}(A)}; \Sigma_{\mathcal{B}(A)}\Big).$$

The superscript "∞" is meant to remind the reader that we work in a context of generalised smooth objects.

We now show that the adjunctions of (3.7) descend to the stable model structure:

Theorem 3.2.7. *For any $A \in \mathrm{DGAlg}_{\mathbb{R}}$, there is a Quillen adjunction*

$$\begin{tikzcd} \mathrm{Sp}^{\infty}_{\mathcal{B}(A)} \arrow[r, shift left, "\mathcal{M}^{\infty}_A"] \arrow[r, phantom, "\perp"] & A\mathrm{-Mod}^{\mathrm{op}} \arrow[l, shift left, "\mathcal{E}_A"] \end{tikzcd}$$

of stable model categories.

Proof. Passage to the stable model structure on $\mathrm{Sp}^{\infty}_{\mathcal{B}(A)}$ is implemented by left Bousfield localisation with respect to the set of maps between cofibrant objects

$$\mathbf{S} = \big\{\zeta_n(C) \colon F_{n+1}(\Sigma_{\mathcal{B}(A)}C) \to F_n(C)\big\},$$

where $F_n \colon R^{\infty}_{\mathcal{B}(A)} \to \mathrm{Sp}^{\infty}_{\mathcal{B}(A)}$ is left adjoint to the functor which evaluates the n-th term of a sequential spectrum, and C ranges over the domains and codomains of the set of generating cofibrations

$$\mathcal{I}_{\mathcal{B}(A)} := \left\{ \begin{tikzcd} \mathcal{B}(A) \arrow[rr] \arrow[d] & & \mathcal{B}(A) \coprod (\underline{U} \otimes \Delta[n]) \arrow[d, "\mathrm{id}+\sigma_x"] \\ \mathcal{B}(A) \coprod (\underline{U} \otimes \partial\Delta[n]) \arrow[rru] \arrow[rr] & & \mathcal{B}(A) \end{tikzcd} \right\} \subset \mathrm{Mor}\big(R^{\infty}_{\mathcal{B}(A)}\big),$$

where σ_x represents an n-simplex $x \in \mathcal{B}(A)(U)_n$.

We show that $\mathcal{M}^{\infty}_A$ sends the set $\mathbf{S}$ to weak equivalences in $A\mathrm{-Mod}^{\mathrm{op}}$, arguing first on codomains of morphisms in $\mathcal{I}_{\mathcal{B}(A)}$. As in the proof of Lemma 2.7.25, we compute that $\mathcal{M}^{\infty}_A$ sends $F_n(\mathcal{B}(A) \coprod (\underline{U} \otimes \Delta[m]))$ to $\Omega^{\bullet}_{\mathrm{si,vert}}(U \times \Delta^m)[n]$, regarded as an A-module via the map

$$A \longrightarrow \mathcal{O}^{\infty}(\mathcal{B}(A)) \xrightarrow{\mathcal{O}^{\infty}(\sigma_x)} \Omega^{\infty}_{\mathrm{si,vert}}(U \times \Delta^m).$$

On the other hand, $\mathcal{M}^{\infty}_A$ sends $F_{n+1}(\Sigma_{\mathcal{B}(A)}(\mathcal{B}(A) \coprod (\underline{U} \otimes \Delta[m])))$ to $P_{U,m}[n+1]$, where $P_{U,m}$ is the pullback of the cospan of A-modules

$$0 \longrightarrow \Omega^{\bullet}_{\mathrm{si,vert}}(U \times \Delta^m) \oplus \Omega^{\bullet}_{\mathrm{si,vert}}(U \times \Delta^m) \longleftarrow \mathcal{O}^{\infty}\big(\underline{U} \otimes (\Delta[m] \times \Delta[1])\big).$$

Via the shuffle decomposition of $\Delta[m] \times \Delta[1]$ into non-degenerate $(m+1)$-simplices, we find that

$$\mathcal{O}^{\infty}\big(\underline{U} \otimes (\Delta[m] \times \Delta[1])\big) \cong \lim_{\substack{\Delta[k] \to \Delta[m] \times \Delta[1] \\ k \leq m+1}} \Omega^{\bullet}_{\mathrm{si,vert}}(U \times \Delta^k),$$

so that $\mathcal{O}^{\infty}(\underline{U} \otimes (\Delta[m] \times \Delta[1]))$ can be viewed as the global sections of a sheaf $\mathcal{F}_{U,m}$ on $\Delta^m \times \Delta^1$ whose local sections on $V \subset \Delta^m \times \Delta^1$ are the vertical differential forms ω on $U \times V$ such that ω has ϵ-sitting instants near all boundary faces of the sub-$(m+1)$-simplices in the shuffle decomposition for some $\epsilon > 0$. Employing the same hypercohomology argument as in Theorem 3.1.10, we find that the cohomology of $\mathcal{O}^\infty(\underline{U} \otimes (\Delta[m] \times \Delta[1]))$ is $C^\infty(U)$ concentrated in degree zero, and so the maps

$$\Omega^\bullet_{\mathrm{si}}(\Delta^1) \otimes \Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^m) \longrightarrow \mathcal{O}^\infty\big(\underline{U} \otimes (\Delta[m] \times \Delta[1])\big)$$

are weak equivalences. We thus have a morphism of cospans of A-modules

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & \Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^m) \oplus \Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^m) \arrow[d, equal] & \big(\Omega^\bullet_{\mathrm{si}}(\Delta^1) \otimes \Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^m)\big) \arrow[l, two heads] \arrow[d, "\wr"] \\ 0 \arrow[r] & \Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^m) \oplus \Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^m) & \mathcal{O}^\infty\big(\underline{U} \otimes (\Delta[m] \times \Delta[1])\big). \arrow[l, two heads] \end{tikzcd}$$

By right properness of $A-\mathrm{Mod}$, the pullbacks of each of these cospans is already a homotopy pullback. Combined with Lemma 3.2.4, we have that the map of A-modules

$$\Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^m)[-1] \longrightarrow P_{U,m}$$

is a weak equivalence. Writing $C = \mathcal{B}(A) \coprod(\underline{U} \otimes \Delta[n])$, we thus have a commuting diagram of weak equivalences.

$$\begin{tikzcd} \Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^m)[n] \arrow[r, "\cong"] \arrow[dr, "\mathcal{M}^\infty_A(\zeta_n(C))"'] & \big(\Omega^\bullet_{\mathrm{si,vert}}(U \times \Delta^m)[-1]\big)[n+1] \arrow[d] \\ & P_{U,m}[n+1]. \end{tikzcd}$$

A similar hypercohomology argument to the above shows that the natural map

$$\Omega^\bullet_{\mathrm{si}}(\Delta^1) \otimes \mathcal{O}^\infty(\underline{U} \otimes \partial\Delta[k]) \longrightarrow \mathcal{O}^\infty\big(\underline{U} \otimes (\partial\Delta[k] \times \Delta[1])\big)$$

is a weak equivalence for all k. Feeding this into the above argument shows that $\mathcal{M}^\infty_A$ sends each morphism in $\mathbf{S}$ to a weak equivalence.

By the universal property of the left Bousfield localisation, it now follows that the adjunction $(\mathcal{M}^\infty_A \dashv \mathcal{E}_A)$ is Quillen for the stable model structure on $\mathrm{Sp}^\infty_{\mathcal{B}(A)}$. □

Remark 3.2.8. In the case that $X\colon \mathrm{CartSp}^{\mathrm{op}} \to \mathrm{sSet}$ is a smooth ∞-stack (which is to say, a fibrant object for the local projective model structure), a fibrant object for the stable model structure on Sp^∞_X is an X-twisted differential cohomology theory (cf. Remark 1.3.21). Indeed, the *projectively* fibrant objects $P \in \mathrm{Sp}^\infty_X$ are precisely those for which each $P_n \in R^\infty_X$ is fibrant—since X is a smooth ∞-stack this implies that so too is each P_n. The condition of $\mathbf{S}$-locality on P is equivalent to the requirement that

$$P_n(U) \longrightarrow \Omega_X P_{n+1}(U)$$

is a weak equivalence for all Cartesian spaces U. Thus, if P is stably fibrant and X is a smooth ∞-stack then it satisfies the conditions of Lemma 1.3.19, hence is an X-twisted differential cohomology theory.

Corollary 3.2.9. *Let A be a cofibrant cDGA over $\mathbb{R}$ and let $M \in A-\mathrm{Mod}$ be a cofibrant A-module. Then $\mathcal{E}_A(M)$ is a $\mathcal{B}(A)$-twisted differential cohomology theory.*

Proof. If A is cofibrant $\mathcal{B}(A)$ is a smooth ∞-stack by Theorem 2.1.10. If $M \in A-\mathrm{Mod}$ is cofibrant then it is fibrant in the opposite model category, so that $\mathcal{E}_A(M)$ is stably fibrant in $\mathrm{Sp}^\infty_{\mathcal{B}(A)}$ by the Theorem. The observation of Remark 3.2.8 completes the proof. □

Example 3.2.10. In Example 3.1.13 we observed that $\mathcal{B}(C^\infty(M)) \cong \underline{M}$ for any second-countable manifold M. In this extended example, we discuss the behaviour of vector bundles under the functors of Theorem 3.2.7.

For a vector bundle $V \to M$ of finite rank, the global sections $\mathcal{V} := \Gamma_M(V)$ form a finitely-generated projective $C^\infty(M)$-module by the smooth Serre–Swan Theorem [Nes03, §11.33]. The same is true of the $C^\infty(M)$-linear dual $\mathcal{V}^\vee \cong \Gamma_M(V^*)$; in particular $\mathcal{V}$ and $\mathcal{V}^\vee$ are both cofibrant in $C^\infty(M){-}\mathrm{Mod}$. Corollary 3.2.9 now implies that $\mathcal{E}_{C^\infty(M)}(\mathcal{V})$ and $\mathcal{E}_{C^\infty(M)}(\mathcal{V}^\vee)$ are M-twisted differential cohomology theories. For N a second-countable smooth manifold and $f\colon N \to M$ a smooth map realised as a map of smooth ∞-stacks $\underline{N} \to \underline{M}$ in the natural way, the f-twisted differential $\mathcal{E}_{C^\infty(M)}(\mathcal{V}^\vee)$-cohomology groups of N are concentrated in degree zero, where they compute the sections $\Gamma_N(f^*V)$. This follows from Claim 3.2.11 below.

Recall the right Quillen functor

$$\overline{\Omega}^\infty_{(-)}\colon \mathrm{Fun}\big(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Sp}^{\mathbb{N}}_{\mathrm{sSet}}\big)_{\mathrm{proj,loc}} \longrightarrow \mathrm{Fun}\big(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}\big)_{\mathrm{proj,loc}}$$

from Remark 1.3.20. Applying this functor to $\mathcal{E}_{C^\infty(M)}(\mathcal{V}^\vee)$ we obtain the smooth ∞-stack

$$\overline{V}\colon U \longmapsto \mathrm{DGAlg}_{\mathbb{R}}\Big(\mathrm{Sym}_{C^\infty(M)}(\mathcal{V}^\vee), C^\infty(U)\Big).$$

Claim 3.2.11. $\overline{V}$ is isomorphic to $\underline{V}$, the smooth ∞-stack presented by the total space of the vector bundle $V \to M$.

Sketch of Proof. One way to see this is via the theory of C^∞-rings alluded to in Remark 3.1.14: there is a relative completion functor

$$\kappa^{C^\infty(M)}\colon \mathbb{R}{-}\mathrm{Alg}^{C^\infty(M)/} \longrightarrow \mathrm{C}^\infty{-}\mathrm{Alg}^{C^\infty(M)/}$$

which is left adjoint to the forgetful functor. Precomposing this functor with the free symmetric algebra functor $\mathrm{Sym}_{C^\infty(M)}$, uniqueness of adjoints implies that for all $n \geq 0$ the relative C^∞-ring

$$\kappa^{C^\infty(M)}\mathrm{Sym}_{C^\infty(M)}\big(C^\infty(M)^{\oplus n}\big) \cong C^\infty(M \times \mathbb{R}^n)$$

is the ring of smooth functions on $M \times \mathbb{R}^n$ which is a relative C^∞-ring via the map $C^\infty(M) \to C^\infty(M \times \mathbb{R}^n)$ computing pullback along the projection $M \times \mathbb{R}^n \to M$. Using this, we can show that for any finitely-generated projective $C^\infty(M)$-module $\mathcal{W}$ there is an isomorphism

$$\kappa^{C^\infty(M)}\mathrm{Sym}_{C^\infty(M)}(\mathcal{W}^\vee) \xrightarrow{\cong} C^\infty(W), \tag{3.8}$$

where $W \to M$ is a vector bundle with $\mathcal{W} \cong \Gamma_M(W)$ (by the Serre–Swan Theorem once more). The isomorphism of (3.8) is the adjunct of the map of $C^\infty(M)$-modules $\mathcal{W}^\vee \to C^\infty(W)$ which sends a global section of W^* to the smooth function on W which it defines via the dual pairing.

Turning now to $\overline{V}$, the map of $C^\infty(M)$-modules $\mathcal{V}^\vee \to C^\infty(V)$ gives rise to a diagram in $\mathbb{R}{-}\mathrm{Alg}$

$$\begin{tikzcd} & C^\infty(M) \arrow[dl] \arrow[dr] & \\ \mathrm{Sym}_{C^\infty(M)}(\mathcal{V}^\vee) \arrow[rr] & & C^\infty(V), \end{tikzcd}$$

which in turn gives rise to a diagram of smooth ∞-stacks

$$\begin{tikzcd} \underline{V} \arrow[rr] \arrow[dr] & & \overline{V} \arrow[dl] \\ & \underline{M}. & \end{tikzcd} \tag{3.9}$$

Evaluating at $U \in \mathrm{CartSp}$, (3.8) shows that $\underline{V}(U) \to \overline{V}(U)$ is an isomorphism in the fibre over $\rho\colon * \to \underline{M}(U)$. As this is true for all Cartesian spaces U and points in $\underline{M}(U)$ (corresponding to smooth maps $U \to M$), we conclude that the morphism $\underline{V} \to \overline{V}$ of (3.9) is an isomorphism of simplicial presheaves. □

3.3 Vistas

In this chapter, we have outlined how algebraic data can be used to present smooth ∞-stacks and twisted differential cohomology theories. The underlying principle is higher Lie theory: ignoring descent issues, a cDGA A over $\mathbb{R}$ encodes a sort of generalised L_∞-algebroid and integrates to a smooth ∞-stack $\mathcal{B}(A)$. An A-module is thus a sort of L_∞-representation, which we have seen integrates to a smooth parametrised spectrum encoding a higher vector bundle over $\mathcal{B}(A)$ (compare Example 3.2.10). If we summarise ordinary Lie theory by means of the schematic

$$\begin{array}{ccc} \boxed{\begin{array}{c}\mathfrak{g}\\ \text{Lie algebra}\end{array}} & \xrightarrow{\text{infinitesimal version of}} & \boxed{\begin{array}{c}\mathbf{B}G\\ \text{smooth classifying stack}\end{array}} \\ \\ \boxed{\begin{array}{c}M\\ \mathfrak{g}\text{-representation}\end{array}} & \xrightarrow{\text{infinitesimal version of}} & \boxed{\begin{array}{c}\mathbf{E}G\times_\rho M \longrightarrow \mathbf{B}G\\ \text{universal vector bundle}\end{array}} \end{array} \tag{3.10}$$

then the logic of higher Lie theory is summarised in the schematic

$$\begin{array}{ccc} \boxed{\begin{array}{c}A \text{ cDGA over } \mathbb{R}\\ (\text{generalised } L_\infty\text{-algebroid})\end{array}} & \xrightarrow[\text{(presented by } \mathcal{B})]{\text{infinitesimal version of}} & \boxed{\begin{array}{c}\mathcal{B}A\\ \text{smooth } \infty\text{-stack}\end{array}} \\ \\ \boxed{\begin{array}{c}M \text{ dg module over } A\\ (L_\infty\text{-representation})\end{array}} & \xrightarrow[\text{(presented by } \mathcal{E}_{\mathcal{B}(A)})]{\text{infinitesimal version of}} & \boxed{\begin{array}{c}\mathcal{E}_{\mathcal{B}(A)}(M)\\ \text{smooth parametrised spectrum}\end{array}} \end{array} \tag{3.11}$$

There are many subtleties involved with interpreting our work in this chapter as higher Lie theory; nevertheless, (3.11) provides a useful analogy deserving of further scrutiny due to its potential ramifications to the study of twisted differential cohomology and higher vector bundles.

We conclude with a discussion of two natural questions that arise from our work in this chapter:

Problem 3.3.1. In Example 3.1.13 we saw that the functor $\mathcal{O}^\infty$ captures a mixture of differential geometry and real homotopy theory. For G a Lie group, the derived functor $\mathbb{L}\mathcal{O}^\infty$ applied to the classifying stack $\mathbf{B}G = *//G$ computes the smooth group cohomology of G with $\mathbb{R}$ coefficients. More generally, for $\mathcal{G}$ a Lie n-groupoid ($n > 0$), we can realise $\mathcal{G}$ as an n-geometric ∞-stack encoding non-trivial diffeo-geometric as well as non-trivial homotopy-theoretic information. Via the $(\mathcal{O}^\infty \dashv \mathcal{B})$-adjunction, we find that the n-th cohomology group of $H^n(\mathbb{L}\mathcal{O}^\infty(\mathcal{G}))$ classifies $\mathbf{B}^{n-1}\mathbb{R}$-principal ∞-bundles over $\mathcal{G}$. A natural question is then

How do we effectively compute the cohomology algebra of $\mathbb{L}\mathcal{O}^\infty(\mathcal{G})$?

Some first examples to consider are the higher deloopings $\mathbf{B}^n U(1)$ for $n \geq 2$.

Problem 3.3.2. In §1.3 we likened (untwisted) differential cohomology to geometric bundles with connection, where the fracture squares (1.11) encode relations between curvature characteristic classes and the topological shape of the underlying smooth bundles. A similar interpretation is available in the unstable setting, where it is a consequence of the fact that $\mathbf{H}$ is a *cohesive* $(\infty, 1)$*-topos* (Remark 1.3.4 and [Sch17]). An example of this is given by the path method of Lie integration: for $\mathfrak{g}$ an ordinary Lie algebra and G the unique simply-connected Lie group integrating it, the smooth classifying stack $\mathbf{B}G$ can be recovered as a truncation of $\exp_\Delta(\mathfrak{g})$. The smooth ∞-stack $\exp_\Delta(\mathfrak{g})$ therefore encodes primary characteristic classes of G-bundles, but does not know about either connective structures or secondary characteristic classes. The main technical innovation of the article [FSS12] is the extension of the assignment $\mathfrak{g} \mapsto \exp_\Delta(\mathfrak{g})$ to encode such higher diffeo-geometric data. The end result is a sort of "∞-Chern–Weil theory" for higher principal bundles.

In the setting of Theorem 3.2.7, the implication is that the twisted differential cohomology theories obtained via the functors $\mathcal{E}_A$ only encode primary characteristic classes—our Lie integration procedure does not produce "full" twisted differential cohomology theories. This state of affairs is illustrated in Example 3.2.10, where we recover smooth vector bundles as opposed to, say, vector bundles equipped with connection. The natural question arising at this point is

Can we extend ∞-Chern–Weil theory to algebraically-presented smooth parametrised spectra?

Finding an affirmative answer to this question is a necessary key ingredient in lifting the rational-homotopy-theoretic derivation of D-brane charges in twisted K-theory [FSS17a; FSS17b] away from the rational approximation and toward full twisted differential cohomology.

Another potential application is to twisted differential string structures and their associated higher vector bundles. Modulo subtleties related to taking truncations, the string 2-Lie algebra $\mathfrak{string}(n)$ integrates to a smooth ∞-stack $\mathbf{B}\mathrm{String}(n)$ classifying $\mathrm{String}(n)$-∞-bundles [FSS12, §4.2.3]. Representations of $\mathfrak{string}(n)$ integrate to smooth spectra parametrised by $\mathbf{B}\mathrm{String}(n)$, interpreted as higher vector bundles associated to the given $\mathfrak{string}(n)$-representations (compare Example 3.2.10). Feeding this through our speculative ∞-Chern–Weil theory produces higher $\mathrm{String}(n)$-vector bundles with connection. For M an n-manifold with string structure encoded by a differential cocycle $\tau\colon M \to \mathbf{B}\mathrm{String}(n)$, pulling back along τ yields a $\mathrm{String}(n)$-vector bundle with connection on M. Optimistically, taking surface holonomies of the pullback higher connections on M would then produce Segal elliptic objects [Seg07] along with their differential analogues, similarly to the sketch of [AR12, §§6.4–6.5]. Such an approach could help shed light on the spectrum tmf of topological modular forms, the string orientation of tmf and their differential refinements.

Appendix A

A Model-Categorical Toolbox

"Perhaps the purpose of categorical algebra is to show that which is formal is formally formal."

- J. P. MAY

In this appendix, we summarise many of the model-category-theoretic tools that are used throughout the thesis. We do not claim any originality for these results. Our primary references for the underlying theory of model categories are [Hov99; Hir03].

A.1 Frequently Used Results

At various stages throughout this thesis, we appeal to certain structural results or principles in our arguments. In this section, we record the most commonly occuring and important of these.

A.1.1 The Adjoint Functor Theorem

The Adjoint Functor Theorem is a useful tool for determining when a given functor has a left or right adjoint. Throughout the thesis, we always refer to the Adjoint Functor Theorem for locally presentable categories (see [AR94]), namely

Theorem A.1.1. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a fuctor between locally presentable categories. Then*

- *F has a right adjoint precisely if it preserves all small colimits;*
- *F has a left adjoint precisely if it is an accessible functor functor that preserves all small limits.*

Proof. Let $\mathcal{C}$ be locally λ-presentable for some regular cardinal λ and write S_λ for the full subcategory of λ-small objects of $\mathcal{C}$. By [AR94, Theorem 1.46], $\mathcal{C}$ is equivalent to the full subcategory of functors $S_\lambda^{\mathrm{op}} \to \mathrm{Set}$ which preserve λ-small limits. Since F is cocontinuous, the functor $h_{\mathcal{D}}\colon \mathcal{D} \to \mathrm{Fun}(S_\lambda^{\mathrm{op}}, \mathrm{Set})$ which sends

$$d \longmapsto \mathcal{C}\big(F(-), d\big)$$

is such that $h_{\mathcal{D}}(d)$ preserves λ-small limits for all $d \in \mathcal{D}$. Hence $h_{\mathcal{D}}$ factors via $\mathcal{C}$ and defines a right adjoint to F. The second statement is [AR94, Theorem 1.66]. $\square$

A.1.2 Cofibrantly Generated Model Categories

Most of the model categories which we consider are *cofibrantly generated*, meaning that there are sets of cofibrations and acyclic cofibrations which generate the classes of (acyclic) cofibrations under pushouts, transfinite compositions and retracts. Standard references on cofibrantly generated model categories are [Hov99, §2.1.3] and [Hir03, Ch. 11].

Quillen Bifunctors. We frequently make use of cofibrant generation to check that an adjunction of two variables is a Quillen bifunctor [Hov99, §§4.1–4.2]. Recall that for a bifunctor $\otimes\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, where $\mathcal{E}$ has finite colimits, the *pushout-product* of maps $f\colon x \to y$ in $\mathcal{C}$ and $g\colon c \to d$ in $\mathcal{D}$ is the morphism

$$f \,\square\, g\colon (y \otimes c) \coprod_{(x \otimes c)} (x \otimes d) \longrightarrow y \otimes d$$

in $\mathcal{E}$. When $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are model categories, such a bifunctor $\otimes$ is a *Quillen bifunctor* if $f \,\square\, g$ is a cofibration whenever f and g are, and is moreover an acyclic cofibration when either f or g is acyclic.

Lemma A.1.2. *Suppose that $\otimes\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is an adjunction of two variables between model categories. Moreover, suppose that $\mathcal{C}$ and $\mathcal{D}$ are cofibrantly generated, with generating cofibrations $\mathcal{I}$ and $\mathcal{I}'$ and generating trivial cofibrations $\mathcal{J}$ and $\mathcal{J}'$ respectively. Then $\otimes$ is a Quillen bifunctor if and only if $\mathcal{I} \,\square\, \mathcal{I}'$ consists of cofibrations and both $\mathcal{I} \,\square\, \mathcal{J}'$ and $\mathcal{J} \,\square\, \mathcal{I}'$ consist of trivial cofibrations.*

Proof. This is [Hov99, Corollary 4.2.5]. The main idea is to use the adjoints of $\otimes$ (which are specified as part of the data of an adjunction of two variables) together with the lifting properties that characterise cofibrations and acyclic cofibrations. □

Transfer Theorems. Another extremely important property of cofibrantly generated model categories is that they allow model structures to be transferred along adjunctions:

Theorem A.1.3 (Right Transfer Theorem). *Let $\mathcal{C}$ be a cofibrantly generated model category with generating cofibrations $\mathcal{I}$ and generating acyclic cofibrations $\mathcal{J}$. Suppose that $\mathcal{D}$ has all small limits and colimits and that we have an adjunction*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D}. \arrow[l, shift left, "U"] \end{tikzcd}$$

Let $F\mathcal{I} := \{F(i) \mid i \in \mathcal{I}\}$ and $F\mathcal{J} := \{F(j) \mid j \in \mathcal{J}\}$, if

(i) *both $F\mathcal{I}$ and $F\mathcal{J}$ admit the small object argument; and*

(ii) *U sends transfinite compositions of pushouts along morphisms in $F\mathcal{J}$ to weak equivalences in $\mathcal{C}$,*

then $\mathcal{D}$ is a cofibrantly generated model category with generating cofibrations $F\mathcal{I}$ and generating acyclic cofibrations $F\mathcal{J}$ such that the functor U creates weak equivalences and fibrations in $\mathcal{D}$.

Proof. This is a result of Kan, reproduced as [Hir03, Theorem 11.3.2]. □

Recall that a model category $\mathcal{M}$ is *combinatorial* if it is cofibrantly generated and the underlying category is locally presentable. For combinatorial model categories, we have the following

Theorem A.1.4 (Left Transfer Theorem). *Let $\mathcal{M}$ be a combinatorial model category and $\mathcal{C}$ be locally presentable. Suppose we have an adjunction*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "V"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left, "K"] \end{tikzcd}$$

and call a morphism $f\colon x \to y$ a cofibration *or* weak equivalence *precisely if $V(f)$ is a cofibration or weak equivalence in $\mathcal{M}$ respectively. If $\mathcal{C}$ admits*

(i) *cofibrant replacements $\varepsilon_x\colon \mathcal{Q}x \xrightarrow{\sim} x$ for all objects x such that for each morphism $f\colon x \to y$ there exists a morphism $\mathcal{Q}f\colon \mathcal{Q}x \to \mathcal{Q}y$ such that the diagram*

$$\begin{tikzcd} \mathcal{Q}x \arrow[r, "\mathcal{Q}f"] \arrow[d, "\varepsilon_x"'] & \mathcal{Q}y \arrow[d, "\varepsilon_y"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

commutes; and

(ii) *good cylinder objects*

$$\mathcal{Q}x \coprod \mathcal{Q}x \rightarrowtail \mathrm{Cyl}(\mathcal{Q}x) \xrightarrow{\sim} \mathcal{Q}x$$

for all x,

then $\mathcal{C}$ has a combinatorial model structure with cofibrations and weak equivalences created by the left adjoint V.

Proof. This specific transfer theorem appears as [HS16, Theorem A.5], where it is cited as a special case of the general results of [Bay+15]. □

A Useful Criterion for Weak Equivalences. For a morphism $f\colon x \to y$ in any model category $\mathcal{M}$, the following are equivalent

- f is a weak equivalence;
- for all cofibrant objects $c \in \mathcal{M}$, the induced map of homotopy function complexes $\mathrm{map}_{\mathcal{M}}(c, f)\colon \mathrm{map}_{\mathcal{M}}(c, x) \to \mathrm{map}_{\mathcal{M}}(c, y)$ is a weak equivalence; and
- for all fibrant objects $w \in \mathcal{M}$, the induced map of homotopy function complexes $\mathrm{map}_{\mathcal{M}}(f, w)\colon \mathrm{map}_{\mathcal{M}}(x, w) \to \mathrm{map}_{\mathcal{M}}(y, w)$ is a weak equivalence

[Hir03, Theorem 17.7.7]. In the case that $\mathcal{M}$ is a left proper cofibrantly generated model category, we can characterise weak equivalences in terms of a small set of cofirbant objects:

Lemma A.1.5. *Let $\mathcal{M}$ be a left proper cofibrantly generated model category with generating cofibrations $\mathcal{I}$. Then a morphism $f\colon x \to y$ in $\mathcal{M}$ is a weak equivalence precisely if $\mathrm{map}_{\mathcal{M}}(c, x) \to \mathrm{map}_{\mathcal{M}}(c, y)$ is a weak equivalence for all domains and codomains c of maps in $\mathcal{I}$.*

Proof. This is proven as [Hov01, Proposition 3.2] and as [Dug01b, Proposition A.5]. Both proofs require the results of [Hir03, §10] in an essential way. □

A.1.3 A Useful Criterion for Quillen Equivalence

We record a well-known, useful criterion for checking when a given Quillen adjunction is a Quillen equivalence:

Lemma A.1.6. *Let*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left=2, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left=2, "R"] \end{tikzcd}$$

be a Quillen adjunction in which R creates weak equivalences. If the unit $x \to RLx$ is a weak equivalence for all cofibrant $x \in \mathcal{C}$, then $(L \dashv R)$ is a Quillen equivalence.

Proof. We must show that for any cofibrant $x \in \mathcal{C}$ and fibrant $d \in \mathcal{D}$, a morphism $f\colon Lx \to d$ is a weak equivalence in $\mathcal{D}$ precisely if its adjunct $f^\vee\colon x \to Rd$ is a weak equivalence in $\mathcal{C}$. Recall that the adjunct $f^\vee$ is obtained as the composite

$$x \longrightarrow RLx \xrightarrow{Rf} Rd.$$

Thus, if f is a weak equivalence so too is $f^\vee$ by our hypotheses on the adjunction $(L \dashv R)$. Conversely, if $f^\vee$ is a weak equivalence then so too is Rf by our hypotheses and the 2-out-of-3 property. But R reflects weak equivalences, so that f is a weak equivalence in $\mathcal{D}$. □

A variant of this result requires slightly weaker hypotheses:

Lemma A.1.7. *Let*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left=2, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left=2, "R"] \end{tikzcd}$$

be a Quillen adjunction in which R reflects weak equivalences between fibrant objects. Let $\mathcal{R}$ be a fibrant replacement functor for $\mathcal{D}$, and suppose that for all cofibrant $x \in \mathcal{C}$ the derived unit

$$x \longrightarrow RLx \longrightarrow R(\mathcal{R}(LX))$$

is a weak equivalence. Then $(L \dashv R)$ is a Quillen equivalence.

Proof. The first published account of this result seems to be [HSS00, Lemma 4.1.7]. Suppose $x \in \mathcal{C}$ and $d \in \mathcal{D}$ are cofibrant and fibrant respectively and that we have a map $f\colon Lx \to d$ in $\mathcal{D}$. Consider the commuting diagram in $\mathcal{C}$

$$\begin{tikzcd} x \arrow[r] \arrow[rr, bend left, "f^\vee"] & RLx \arrow[r, "Rf"] \arrow[d] & Rd \arrow[d] \\ & R(\mathcal{R}(Lx)) \arrow[r, "R(\mathcal{R}(f))"] & R(\mathcal{R}(d)). \end{tikzcd}$$

Since $d \to \mathcal{R}(d)$ is a weak equivalence between fibrant objects, Ken Brown's Lemma implies that is $Rd \to R(\mathcal{R}(d))$ too. Thus, the above diagram and the 2-out-of-3 property imply that $f^\vee$ is a weak equivalence if and only if $R(\mathcal{R}(Lx)) \to R(\mathcal{R}(d))$ is. But this latter map is a weak equivalence if and only if $\mathcal{R}(Lx) \to \mathcal{R}(d)$ is a weak equivalence, by hypothesis. Finally, $\mathcal{R}(f)\colon \mathcal{R}(Lx) \to \mathcal{R}(d)$ is a weak equivalence precisely if f is, by the 2-out-of-3 property, so that assertion is proven. □

A.2 Left Bousfield Localisation

Here we briefly review the theory of left Bousfield localisations of model categories, following [Hir03; Bar10].

Definition A.2.1. Let $\mathcal{M}$ be a model category and S a set of maps in $\mathcal{M}$.

(1) A *left Bousfield localisation of* $\mathcal{M}$ *with respect to* S is a model category $L_S\mathcal{M}$ equipped with a left Quillen functor $\mathcal{M} \to L_S\mathcal{M}$ that is initial among left Quillen functors $F\colon \mathcal{M} \to \mathcal{N}$ such that $\mathbb{L}F(s)$ is an isomorphism in $Ho(\mathcal{N})$ for all $s \in S$.

(2) An object $x \in \mathcal{M}$ is *S-local* if for any $f\colon a \to b$ in S, the induced map of homotopy function complexes

$$\mathrm{map}_{\mathcal{M}}(f, x)\colon \mathrm{map}_{\mathcal{M}}(b, x) \longrightarrow \mathrm{map}_{\mathcal{M}}(a, x)$$

is a weak equivalence.

(3) A morphism $\phi\colon x \to y$ is a *S-local equivalence* if for any S-local object z, the map of homotopy function complexes

$$\mathrm{map}_{\mathcal{M}}(\phi, z)\colon \mathrm{map}_{\mathcal{M}}(y, z) \longrightarrow \mathrm{map}_{\mathcal{M}}(x, z)$$

is a weak equivalence.

Note that $L_S\mathcal{M}$ is unique up to unique Quillen equivalence if it exists.

For $\mathcal{M}$ a left proper combinatorial model category, Smith's Existence Theorem (see [Bar10, Theorem 4.7]) guarantees that left Bousfield localisations exist:

Theorem A.2.2. *Let $\mathcal{M}$ be a left proper combinatorial model category and S a set of morphisms in $\mathcal{M}$. Then the left Bousfield localisation $L_S\mathcal{M}$ exists and satisfies the following conditions:*

(i) *$L_S\mathcal{M}$ is left proper and combinatorial;*

(ii) *the underlying categories of $L_S\mathcal{M}$ and $\mathcal{M}$ coincide;*

(iii) *the cofibrations of $L_S\mathcal{M}$ coincide with those of $\mathcal{M}$;*

(iv) *the fibrant objects of $L_S\mathcal{M}$ are precisely the S-local fibrant objects of $\mathcal{M}$; and*

(v) *the weak equivalences of $L_S\mathcal{M}$ are precisely the S-local equivalences.*

Lemma A.2.3. *Let $x, y \in L_S\mathcal{M}$ be fibrant and consider a morphism $f\colon x \to y$. The following are equivalent:*

(1) *$f\colon x \to y$ is a weak equivalence in $\mathcal{M}$; and*

(2) *$f\colon x \to y$ is a weak equivalence in $L_S\mathcal{M}$.*

Proof. By [Hir03, Theorem 17.7.7], the weak equivalences in $\mathcal{M}$ are S-local equivalences so that $(1) \Rightarrow (2)$.

Conversely, suppose that $f\colon x \to y$ is an S-local equivalence. Let $Ho_S(\mathcal{M})$ be the full subcategory of $Ho(\mathcal{M})$ on the S-local objects. Then since f is an S-local equivalence the map $Ho_S(\mathcal{M})(f, z)$ is an isomorphism for all S-local z. By the Yoneda Lemma, this means that f is an isomorphism in $Ho_S(\mathcal{M})$ and hence also in $Ho(\mathcal{M})$. By [Hov99, Theorem 1.2.10 (iv)], f is a weak equivalence in $\mathcal{M}$. $\square$

A.3 Stabilisation Machines for Model Categories

In this section, we recall two general constructions for the passage from unstable to stable homotopy theory due to Hovey [Hov01]. Throughout this section, $\mathcal{M}$ denotes a *left proper combinatorial model category* and $\Sigma\colon \mathcal{M} \to \mathcal{M}$ is a left Quillen endofunctor of $\mathcal{M}$, with right adjoint Ω. The notation $(\Sigma \dashv \Omega)$ is suggestive of suspension and loop space endofunctors, such as are considered in Chapters 1 and 2, but the stabilisation machinery we discuss is more generally applicable.

A.3.1 Sequential Stabilisation

The first stabilisation machine that we consider is a generalisation of the work of Bousfield–Friedlander [BF78]:

Definition A.3.1. A *sequential spectrum* in $\mathcal{M}$ is a sequence $\{x_n\}_{n\in\mathbb{N}}$ of objects of $\mathcal{M}$ equipped with a collection of structure maps $\{\sigma_n\colon \Sigma x_n \to x_{n+1}\}_{n\in\mathbb{N}}$. We shall commonly abbreviate these data simply to x, in which case a sequence of objects and structure maps are understood.

A *morphism of sequential spectra* $f\colon x \to y$ is a sequence of morphisms $f_n\colon x_n \to y_n$ in $\mathcal{M}$ which commute with the structure maps, so that there are commuting diagrams

$$\begin{tikzcd} \Sigma x_n \arrow[r, "\Sigma f_n"] \arrow[d, "\sigma_n^x"'] & \Sigma y_n \arrow[d, "\sigma_n^y"] \\ x_{n+1} \arrow[r, "f_{n+1}"] & y_{n+1} \end{tikzcd}$$

for each n. With these definitions, we obtain the *category of sequential spectra in $\mathcal{M}$*, denoted by $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma)$ or simply by $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M})$ when the left Quillen endofunctor Σ is understood.

Lemma A.3.2. *The category $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma)$ is locally presentable. In particular, $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma)$ is complete and cocomplete. Moreover, the forgetful functor $U\colon \mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma) \to \mathrm{Fun}(\mathbb{N},\mathcal{M})$ which sends a sequential spectrum to its underlying sequence creates limits and colimits.*

Proof. We show that there is an accessible monad T^{sp} on the category of sequences $\mathrm{Fun}(\mathbb{N},\mathcal{M})$ together with a canonical isomorphism of categories between $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma)$ and $T^{\mathrm{sp}}-\mathrm{Alg}$[1]. Since T^{sp} is an accessible monad and $\mathrm{Fun}(\mathbb{N},\mathcal{M})$ is locally presentable, $T^{\mathrm{sp}}-\mathrm{Alg}$ is locally presentable [Bor94, Theorem 5.5.9].

Let T^{sp} be the endofunctor on $\mathrm{Fun}(\mathbb{N},\mathcal{M})$ that sends $x = \{x_n\}_{n\in\mathbb{N}}$ to the sequence whose n-th term is

$$T^{\mathrm{sp}}(x)_n := \coprod_{i=0}^{n} \Sigma^{n-i} x_i.$$

The monad multiplication $\mu\colon T^{\mathrm{sp}}T^{\mathrm{sp}} \to T^{\mathrm{sp}}$ is defined on components by the folding maps

$$T^{\mathrm{sp}}T^{\mathrm{sp}}(x)_n = \coprod_{i=0}^{n} \left(\Sigma^{n-i} x_i\right)^{\amalg(n+1-i)} \longrightarrow \coprod_{i=0}^{n} \Sigma^{n-i} x_i,$$

and the monadic unit $\eta\colon \mathrm{id} \to T^{\mathrm{sp}}$ is induced by the coprojections $x_n \to \coprod_{i=0}^{n} \Sigma^{n-i} x_i$. The triple $(T^{\mathrm{sp}},\mu,\eta)$ is easily seen to satisfy the monad axioms, and moreover accessible since T^{sp} preserves all colimits of sequences.

[1]the author is convinced that T^{sp} deserves the name "spectral triple".

There is a canonical isomorphism of categories $T^{\mathrm{sp}}-\mathrm{Alg} \cong \mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma)$. Indeed, suppose that x is a T^{sp}-algebra, so that the action $\rho\colon T^{\mathrm{sp}}(x) \to x$ has n-th component

$$\rho_n\colon \coprod_{i=0}^{n} \Sigma^{n-i} x_i \longrightarrow x_n.$$

Let $\sigma_{n-1}\colon \Sigma x_{n-1} \to x_n$ be the restriction of ρ_n to the Σx_{n-1}-summand. The the condition that ρ is a T^{sp}-action is equivalent to the requirement that ρ_n is determined on the summand $\Sigma^{n-i}x_i$ as the composite

$$\Sigma^{n-i} x_i \xrightarrow{\Sigma^{i-1}\sigma_{n-i}} \cdots \xrightarrow{\Sigma\sigma_{n-2}} \Sigma x_{n-1} \xrightarrow{\sigma_{n-1}} x_n$$

for $i < n$ and as the identity $x_n \to x_n$ on the remaining summand. From this description, it is clear that T^{sp}-algebras correspond to sequential spectra in a canonical way. This identification lifts naturally to morphisms and defines the sought-after isomorphism of categories.

Finally, the forgetful functor U creates limits by monadicity. for a small diagram $D\colon \mathcal{I} \to \mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma)$ it is easy to see that the sequence

$$n \longmapsto \operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Ev}_n D(i)$$

equipped with structure maps

$$\Sigma\left(\operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Ev}_n D(i)\right) \cong \operatorname*{colim}_{i\in\mathcal{I}} \Sigma\left(\mathrm{Ev}_n D(i)\right) \xrightarrow{\operatorname*{colim}_{i\in\mathcal{I}} \sigma_n^{D(i)}} \operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Ev}_{n+1} D(i)$$

is a colimit of the diagram D. □

Construction A.3.3. For each $n \in \mathbb{N}$, the assignment $x \mapsto x_n$ taking a sequential spectrum to its n-term determines the *evaluation functor* $\mathrm{Ev}_n\colon \mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma) \to \mathcal{M}$. The evaluation functor has a left adjoint $F_n^{\mathcal{M}}\colon \mathcal{M} \to \mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma)$ which sends $y \in \mathcal{M}$ to the spectrum with m-th term

$$F_n^{\mathcal{M}}(y)_m := \begin{cases} \Sigma^{m-n} y & \text{if } m \geq n \\ 0 & \text{otherwise} \end{cases}$$

equipped with the obvious structure maps.

The evaluation functor also has a right adjoint $C_n^{\mathcal{M}}$, which sends $z \in \mathcal{M}$ to the spectrum with m-th term

$$C_n^{\mathcal{M}}(z)_m := \begin{cases} \Omega^{n-m} z & \text{if } m \leq n \\ 0 & \text{otherwise,} \end{cases}$$

with structure maps $\sigma_k\colon \Sigma C_n^{\mathcal{M}}(z)_k \to C_n^{\mathcal{M}}(z)_{k+1}$ given by the $(\Sigma \dashv \Omega)$-adjunct of the identity map $\Omega^{n+1-k}z \to \Omega^{n+1-k}z$ for $k < n$, and by the zero maps otherwise.

Let $\iota_n\colon \mathcal{M} \to \mathrm{Fun}(\mathbb{N},\mathcal{M})$ be the functor that sends $x \in \mathcal{M}$ to the sequence whose n-th term is x, with all other terms given by the zero object 0. Evaluating a sequence at its n-th term provides a two-sided adjoint $\mathrm{ev}_n\colon \mathrm{Fun}(\mathbb{N},\mathcal{M}) \to \mathcal{M}$ to ι_n. Using the characterisation of sequential spectra as algebras over the monad T^{sp} given in the proof of Lemma A.3.2, we have for each n a composite adjunction

$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left, "\iota_n"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(\mathbb{N},\mathcal{M}) \arrow[l, shift left, "\mathrm{ev}_n"] \arrow[r, shift left, "T^{\mathrm{sp}}"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma) \arrow[l, shift left, "U"] \end{tikzcd}$$

in which $T^{\mathrm{sp}} \circ \iota_n = F_n^{\mathcal{M}}$ and $\mathrm{ev}_n \circ U = \mathrm{Ev}_n$ are the functors of Construction A.3.3 above.

Definition A.3.4. Let $\mathcal{M}$ and $\mathcal{N}$ be a left proper combinatorial model categories, and let Σ and Σ' be left Quillen endofunctors on $\mathcal{M}$ and $\mathcal{N}$ respectively. A *lax map of pre-stable pairs* $(\mathcal{M},\Sigma) \to (\mathcal{N},\Sigma')$ is a Quillen adjunction $(L \dashv R)\colon \mathcal{M} \to \mathcal{N}$ together with a natural transformation $\tau\colon L\Sigma \Rightarrow \Sigma' L$ such that τ_x is a weak equivalence for all $x \in \mathcal{M}$. A lax map of pre-stable pairs is a *strong map of pre-stable pairs* if τ is a natural isomorphism.

Lemma A.3.5. *A lax map of pre-stable pairs $(\mathcal{M},\Sigma) \to (\mathcal{N},\Sigma')$ with underlying Quillen adjuntion $(L \dashv R)\colon \mathcal{M} \to \mathcal{N}$ prolongs to an adjunction*

$$\begin{tikzcd} \mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma) \arrow[r, shift left, "L^{\infty}"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\mathbb{N}}(\mathcal{N};\Sigma') \arrow[l, shift left, "R"] \end{tikzcd}$$

on categories of sequential spectra. The prolonged right adjoint is given by levelwise application of R and there are natural isomorphisms $L^{\infty} \circ F_n^{\mathcal{M}} \cong F_n^{\mathcal{N}} \circ L$ for each n. If $(\mathcal{M},\Sigma) \to (\mathcal{N},\Sigma')$ is a strong map of pre-stable pairs, then L^{∞} is also given by levelwise application of L.

Proof. Let $\tau^{\vee}\colon R\Omega' \Rightarrow \Omega R$ be the dual natural transformation to τ. For y a sequential spectrum in $\mathcal{N}$, we define $Ry \in \mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma)$ by $R(y)_n := R(y_n)$ equipped with structure maps adjunct to

$$R(y_n) \xrightarrow{R(\sigma_n^{\vee})} R(\Omega' y_{n+1}) \xrightarrow{\tau^{\vee}_{y_{n+1}}} \Omega R(y_{n+1}),$$

where $\sigma_n^{\vee}\colon y_n \to \Omega' y_{n+1}$ is the adjunct of the structure map $\sigma_n\colon \Sigma' y_n \to y_{n+1}$. The functor $R\colon \mathrm{Sp}^{\mathbb{N}}(\mathcal{N};\Sigma') \to \mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma)$ so obtained is a limit-preserving accessible functor between locally presentable categories, so has a left adjoint L^{∞} by the Adjoint Functor Theorem. Observe that $L^{\infty} \circ F_n^{\mathcal{M}}$ and $F_n^{\mathcal{N}} \circ L$ are both left adjoints of $\mathrm{Ev}_n \circ R = R \circ \mathrm{Ev}_n$, so are naturally isomorphic.

In the case that we have a strong map of pre-stable pairs, for $x \in \mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma)$ we define a spectrum Lx in $\mathcal{N}$ by $(Lx)_n := L(x_n)$ with structure maps

$$\Sigma' L x_n \xrightarrow{\tau_{x_n}} L\Sigma x_n \xrightarrow{L\sigma_n} L x_{n+1},$$

where $\sigma_n\colon \Sigma x_n \to x_{n+1}$ are the structure maps of x. Levelwise application of the $(L \dashv R)$-unit and counit provide a unit and counit for the prolonged adjunction. In particular, L^{∞} is given by levelwise application of L in this case. ☐

Corollary A.3.6. *The adjunction $(\Sigma \dashv \Omega)$ on $\mathcal{M}$ prolongs to an adjunction $(\Sigma \dashv \Omega)$ on $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M},\Sigma)$ by levelwise application.*

The functor category $\mathrm{Fun}(\mathbb{N},\mathcal{M})$ inherits a model structure from $\mathcal{M}$: a map of sequences $f\colon x \to y$ is a weak equivalence, fibration or cofibration precisely if each map $f_n\colon x_n \to y_n$ is such in $\mathcal{M}$. Suppose that we have chosen generating sets $\mathcal{I}_{\mathcal{M}}$

of cofibrations and $\mathcal{J}_{\mathcal{M}}$ of acyclic cofibrations for $\mathcal{M}$. The category of sequences $\mathrm{Fun}(\mathbb{N}, \mathcal{M})$ is then cofibrantly generated, with generating sets

$$\mathcal{I}^{\mathrm{seq}}_{\mathcal{M}} := \{\imath_n(i) \mid i \in \mathcal{I}_{\mathcal{M}} \text{ and } n \in \mathbb{N}\} \quad \text{and} \quad \mathcal{J}^{\mathrm{seq}}_{\mathcal{M}} := \{\imath_n(j) \mid j \in \mathcal{J}_{\mathcal{M}} \text{ and } n \in \mathbb{N}\}$$

of cofibrations and acyclic cofibrations respectively.

Lemma A.3.7. *The adjunction*

$$\begin{tikzcd} \mathrm{Fun}(\mathbb{N}, \mathcal{M}) \arrow[r, shift left, "T^{\mathrm{sp}}"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\mathbb{N}}(\mathcal{M}; \Sigma) \arrow[l, shift left, "U"] \end{tikzcd}$$

equips $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}, \Sigma)$ *with a left proper combinatorial model structure. This is the* projective model structure *on sequential spectra, which we denote* $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}, \Sigma)_{\mathrm{proj}}$.

Proof. We first check that the adjunction $(T^{\mathrm{sp}} \dashv U)$ satisfies the conditions of the Right Transfer Theorem A.1.3. Indeed, the sets $\mathcal{I}^{\mathbb{N}}_{\mathcal{M}} := T^{\mathrm{sp}}(\mathcal{I}^{\mathrm{seq}}_{\mathcal{M}})$ and $\mathcal{J}^{\mathbb{N}}_{\mathcal{M}} := T^{\mathrm{sp}}(\mathcal{J}^{\mathrm{seq}}_{\mathcal{M}})$ automatically admit the small object argument since $\mathrm{Sp}^{\Sigma}(\mathcal{M}; \Sigma)$ is locally presentable by Lemma A.3.2. To deduce the existence of the projective model structure, it is now sufficient to show that U carries transfinite compositions of pushouts of morphisms in $\mathcal{J}^{\mathbb{N}}_{\mathcal{M}}$ to weak equivalences. Observe that the set $U(\mathcal{J}^{\mathbb{N}}_{\mathcal{M}})$ consists of acyclic cofibrations, since acyclic cofibrations in $\mathcal{M}$ are preserved by coproducts and by the left Quillen functor Σ. Since U creates colimits, we see that the hypotheses of the Right Transfer Theorem are satisfied.

The projective model structure on $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}, \Sigma)$ is combinatorial by fiat, since it is a cofibrant model structure on a locally presentable category. Note that $U(\mathcal{I}^{\mathbb{N}}_{\mathcal{M}})$ consists of cofibrations by the above argument, from which it immediately follows that the projective model structure inherits left properness from $\mathcal{M}$. □

Corollary A.3.8. *For each n, the $(F^{\mathcal{M}}_n \dashv \mathrm{Ev}_n)$-adjunction is Quillen for the projective model structure.*

We force the prolonged adjunction $(\Sigma \dashv \Omega)$ of Corollary A.3.6 to be a Quillen equivalence on sequential spectra by passing to a left Bousfield localisation. It is in order to guarantee that the left Bousfield localisation exists that we require $\mathcal{M}$ to be left proper and combinatorial.

Definition A.3.9. Let $\mathcal{I}_{\mathcal{M}}$ be a chosen set of generating cofibrations for $\mathcal{M}$. The corresponding *stabilising set* of the pair $(\mathcal{M}, \Sigma)$ is

$$\mathbf{S}(\mathcal{I}_{\mathcal{M}}) := \left\{\zeta^{\mathcal{M}}_n(\mathcal{Q}c) \colon F^{\mathcal{M}}_{n+1}(\Sigma\mathcal{Q}c) \to F^{\mathcal{M}}_n(\mathcal{Q}c)\right\},$$

where $\zeta^{M}_n(x)$ is the $(F^{\mathcal{M}}_{n+1} \dashv \mathrm{Ev}_{n+1})$-adjunct of the identity map on $\Sigma x = \mathrm{Ev}_{n+1} F^{\mathcal{M}}_n(x)$, $\mathcal{Q}$ is a chosen cofibrant replacement functor on $\mathcal{M}$, $n \in \mathbb{N}$, and c ranges over the domains and codomains of morphisms in $\mathcal{I}_{\mathcal{M}}$. Note that each morphism in $\mathcal{S}(\mathcal{I}_{\mathcal{M}})$ is a projective cofibration.

Since the projective model structure on $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}; \Sigma)$ is left proper and combinatorial, the left Bousfield localisation at $\mathbf{S}(\mathcal{I}_{\mathcal{M}})$ exists and is itself left proper and combinatorial (Theorem A.2.2). We call the left Bousfield localisation

$$L_{\mathbf{S}(\mathcal{I}_{\mathcal{M}})} \mathrm{Sp}^{\mathbb{N}}(\mathcal{M}; \Sigma)_{\mathrm{proj}}$$

the *stable model structure* on sequential spectra in $\mathcal{M}$.

Lemma A.3.10. *The stable model structure on $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma)$ has the following properties:*

(1) *the fibrant objects are prescisely the Ω-spectra; namely those sequential spectra x such that $x_n \in \mathcal{M}$ is fibrant for all n and such that the maps $\sigma_n^{\vee}\colon x_n \to \Omega x_{n+1}$ are weak equivalences in $\mathcal{M}$; and*

(2) *for each cofibrant object a of $\mathcal{M}$ and $n \in \mathbb{N}$, the map $\zeta_n^{\mathcal{M}}(a)$ is a weak equivalence.*

In particular, the stable model structure is independent of the choice of stabilising set.

Proof. A spectrum x is fibrant for the stable model structure precisely if it is projectively fibrant and $\mathbf{S}(\mathcal{I}_{\mathcal{M}})$-local. Being projectively fibrant simply means that each of the x_n is fibrant in $\mathcal{M}$. The condition of $\mathcal{S}(\mathcal{I}_{\mathcal{M}})$-locality amounts to the requirement that $\mathrm{map}(s,x)$ is a weak equivalence of homotopy function complexes for all $s \in \mathcal{S}(\mathcal{I}_{\mathcal{M}})$. That is, we require that for each n and domain or codomain c of a morphism in $\mathcal{I}$, the map

$$\mathrm{map}(F_n^{\mathcal{M}}(\mathcal{Q}c),x) \cong \mathrm{map}(\mathcal{Q}c,x_n) \longrightarrow \mathrm{map}(F_{n+1}^{\mathcal{M}}(\Sigma\mathcal{Q}c),x) \cong \mathrm{map}(\mathcal{Q}c,\Omega x_{n+1})$$

is a weak equivalence. By Lemma A.1.5, this is equivalent to requiring that each of the adjoint structure maps $\sigma_n^{\vee}\colon x_n \to \Omega x_{n+1}$ is a weak equivalence, so that x is an Ω-spectrum.

For any cofibrant $a \in \mathcal{M}$, the map $\zeta_n^{\mathcal{M}}(a)$ is a weak equivalence in the stable model structure if and only if $\mathrm{map}(\zeta_n^{\mathcal{M}}(a),x)$ is a weak equivalence for all stably fibrant x [Hir03, Theorem 17.7.7]. By adjointness, the map $\mathrm{map}(\zeta_n^{\mathcal{M}}(a),x)$ is identified with $\mathrm{map}(a,x_n) \to \mathrm{map}(a,\Omega x_{n+1})$. Since $x_n \to \Omega x_{n+1}$ is a weak equivalence of fibrant objects in $\mathcal{M}$, we have that $\mathrm{map}(\zeta_n^{\mathcal{M}}(a),x)$ is a weak equivalence by [Hir03, Theorem 17.7.7] once more.

Let $\mathcal{I}_{\mathcal{M}}$ and $\mathcal{I}'_{\mathcal{M}}$ be generating sets of cofibrations of $\mathcal{M}$, giving rise to stabilising sets $\mathbf{S}(\mathcal{I}_{\mathcal{M}})$ and $\mathbf{S}(\mathcal{I}'_{\mathcal{M}})$ respectively. Write $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma)$ and $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma)'$ for the left Bousfield localisations of the projective model structure at $\mathbf{S}(\mathcal{I}_{\mathcal{M}})$ and $\mathbf{S}(\mathcal{I}'_{\mathcal{M}})$ respectively. By (2) and the universal property of left Bousfield localisation, the identity determines left Quillen functors

$$\mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma) \longrightarrow \mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma)' \quad \text{and} \quad \mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma)' \longrightarrow \mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma),$$

so that the model structures $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma)$ and $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma)'$ coincide. $\square$

Remark A.3.11. For each n, the $(F_n^{\mathcal{M}} \dashv \mathrm{Ev}_n)$-adjunction is Quillen for the stable model structure.

Lemma A.3.12. *The prolonged adjunction*

$$(\Sigma \dashv \Omega)\colon \mathrm{Sp}^{\mathbb{N}}(\mathcal{M},\Sigma) \longrightarrow \mathrm{Sp}^{\mathbb{N}}(\mathcal{M},\Sigma)$$

is a Quillen equivalence for the stable model structure.

Proof. The prolonged adjunction of Corollary A.3.6 is Quillen for the projective model structures since $\Omega\colon \mathcal{M} \to \mathcal{M}$ preserves levelwise fibrations and acyclic fibrations. To see that $(\Sigma \dashv \Omega)$ descends to the stable model structure, it is sufficient to observe that $\Sigma\colon \mathrm{Sp}^{\mathbb{N}}(\mathcal{M},\Sigma) \to \mathrm{Sp}^{\mathbb{N}}(\mathcal{M},\Sigma)$ carries morphisms in a given stabilising set $\mathbf{S}(\mathcal{I}_{\mathcal{M}})$ to acyclic cofibrations; indeed, each $s \in \mathbf{S}(\mathcal{I}_{\mathcal{M}})$ is a levelwise cofibration, so is sent to a cofibration by Σ. Since $\Sigma F_n^{\mathcal{M}} = F_n^{\mathcal{M}}\Sigma$ for all n, Lemma A.3.10 implies that $\Sigma(s)$ is a stable weak equivalence for all $s \in \mathbf{S}(\mathcal{I}_{\mathcal{M}})$.

We now show that $(\Sigma \dashv \Omega)$ is a Quillen equivalence for the stable model structure. Consider the auxiliary shift endofunctors s_- and s_+ on $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M},\Sigma)$ defined as follows. For x a sequential spectrum, $s_-(x)$ is the sequential spectrum with n-th term $s_-(x) = x_{n+1}$ and structure maps inherited from x. Dually, $s_+(x)$ has n-th term $s_+(x)_n = x_{n-1}$ for $n > 0$ and $s_+(x)_0 = 0$, with the obvious structure maps. It is clear that we have an adjunction $(s_+ \dashv s_-)$, which is Quillen for the stable model structures by essentially the same argument as above. Moreover, a direct calculation shows that $s_+\Sigma \cong \Sigma s_+$, so that $s_-\Omega \cong \Omega s_-$ by adjointness.

For a sequential spectrum x the adjoint structure maps $x_n \to \Omega x_{n+1}$ define a map $x \to s_-\Omega x$. When x is stably fibrant, the characterisation of Lemma A.3.10 shows that this is a levelwise weak equivalence and hence a stable weak equivalence. Passing to the stable homotopy category (which is to say, $\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}(\mathcal{M},\Sigma)_{\mathrm{stab}})$), we have that the derived functor $\mathbb{R}(s_-\Omega)$ is naturally isomorphic to the identity. Since $\mathbb{R}(s_-\Omega) \cong \mathbb{R}(s_-) \circ \mathbb{R}(\Omega) \cong \mathbb{R}(\Omega) \circ \mathbb{R}(s_-)$, we have that $\mathbb{R}(s_-)$ and $\mathbb{R}(\Omega)$ are inverse equivalences of the stable homotopy category. Moreover, we can choose unit and counit maps so that $(\mathbb{R}(s_-) \dashv \mathbb{R}(\Omega))$ is an adjoint equivalence. This shows that the functors $\mathbb{R}(s_-) \cong \mathbb{L}(\Sigma)$ and $\mathbb{R}(\Omega)$ define inverse equivalences of the stable homotopy category, proving the assertion. □

The sequential stabilisation machine is idempotent in the following sense:

Lemma A.3.13. *Suppose that $(\Sigma \dashv \Omega)$ is already a Quillen equivalence on $\mathcal{M}$. Then the adjunction*

$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left, "F_0^{\mathcal{M}}"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\mathbb{N}}(\mathcal{M},\Sigma)_{\mathrm{stab}} \arrow[l, shift left, "\mathrm{Ev}_0"] \end{tikzcd}$$

is a Quillen equivalence.

Proof. This is proven as [Hov01, Theorem 5.1]. □

Importantly for our arguments in Chapter 1, the sequential stabilisation machine is functorial in maps of pre-stable pairs:

Lemma A.3.14. *For any lax map of pre-stable pairs $(\mathcal{M},\Sigma) \to (\mathcal{N},\Sigma')$, the prolonged adjunction*

$$\begin{tikzcd} \mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma) \arrow[r, shift left, "L^\infty"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\mathbb{N}}(\mathcal{N};\Sigma') \arrow[l, shift left, "R"] \end{tikzcd}$$

is Quillen for the stable model structures. If the underlying adjunction $(L \dashv R)\colon \mathcal{M} \to \mathcal{N}$ is a Quillen equivalence and either

- *the domains of generating cofibrations in $\mathcal{M}$ can be chosen to be cofibrant; or*
- *$\tau\colon L\Sigma \Rightarrow \Sigma' L$ is a natural weak equivalence,*

then the prolonged adjunction $(L^\infty \dashv R)$ is a Quillen equivalence.

Proof. This is proven as [Hov01, Proposition 5.5 & Theorem 5.7]. □

We are particuarly interested in the case that $\mathcal{M}$ is a sSet_*-model category and the functor Σ is given by tensoring with the pointed simplicial set $S^1 := \Delta[1]/\partial\Delta[1]$. In the enriched setting, we have the following

Theorem A.3.15. *Let $\mathcal{C}$ be a cofibrantly generated symmetric monoidal model category, and suppose that we can take the domains of the generating cofibrations of $\mathcal{C}$ to be cofibrant. Suppose that $\mathcal{M}$ is a left proper combinatorial $\mathcal{C}$-model category and that Σ is a $\mathcal{C}$-Quillen endofunctor of $\mathcal{M}$. Then the stable model structure on $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma)$ is a $\mathcal{C}$-model structure, the prolonged adjunction $(\Sigma \dashv \Omega)$ is a $\mathcal{C}$-Quillen autoequivalence of $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma)$, and the functors $F_n^{\mathcal{M}}\colon \mathcal{M} \to \mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma)$ are $\mathcal{C}$-Quillen functors.*

Proof. This is proven as [Hov01, Theorem 6.3]. □

Corollary A.3.16. *Let $\mathcal{C}$ be a cofibrantly generated symmetric monoidal model category as in Theorem A.3.15. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are left proper combinatorial $\mathcal{C}$-model categories, and that Σ and Σ' are $\mathcal{C}$-Quillen endofunctors of $\mathcal{M}$ and $\mathcal{N}$ respectively. Let $(\mathcal{M},\Sigma) \to (\mathcal{N},\Sigma')$ be a strong map of pre-stable pairs such that the underlying adjunction $(L \dashv R)$ is a $\mathcal{C}$-Quillen adjunction. Then the prolonged adjunction on stable model categories*

$$\begin{tikzcd} \mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma) \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\mathbb{N}}(\mathcal{N};\Sigma') \arrow[l, shift left, "R"] \end{tikzcd}$$

is a $\mathcal{C}$-Quillen adjunction.

Proof. Inspecting the proof of [Hov01, Theorem 6.3], it is immediate that the prolonged functor $L\colon \mathrm{Sp}^{\mathbb{N}}(\mathcal{M};\Sigma) \to \mathrm{Sp}^{\mathbb{N}}(\mathcal{N};\Sigma')$ preserves $\mathcal{C}$-tensors, which is sufficient to deduce the result. □

A.3.2 The Symmetric Stabilisation Machine

One drawback of the sequential stabilisation machine considered in the previous section is that it generally does not preserve monoidal structures. The failure of the smash product on pointed spaces to extend to a monoidal structure on sequential spectra was the original motivation for sequential spectra in [HSS00]. In this section, we recall the symmetric stabilisation machine, which is one solution to the problem of extending monoidal structures to stable homotopy theory.

Throughout this section, $\mathcal{C}$ is a symmetric monoidal model category and $K \in \mathcal{C}$ is a fixed cofibrant object. We fix a $\mathcal{C}$-model category $\mathcal{M}$, and we are concerned with stabilising with respect to the left Quillen endofunctor $\Sigma := K \otimes (-)$ on $\mathcal{M}$. Note that $K \otimes (-)$ is left Quillen by cofibrancy of K and the pushout-product axiom for the $\mathcal{C}$-tensoring on $\mathcal{M}$. Furthermore, $\mathcal{C}$ and $\mathcal{M}$ are assumed to be left proper and combinatorial, so that left Bousfield localisations exist.

Definition A.3.17. Let $\mathbf{\Sigma}$ be the category with objects $n \in \mathbb{N}$, morphisms

$$\mathbf{\Sigma}(n,m) := \begin{cases} \Sigma_n & \text{if } n = m \\ \varnothing & \text{otherwise} \end{cases}$$

and composition defined by left multiplication. The assignment $(n,m) \mapsto n+m$ determines a symmetric monoidal structure on $\mathbf{\Sigma}$.

A *symmetric sequence* in $\mathcal{M}$ is a functor $\mathbf{\Sigma} \to \mathcal{M}$. The category of symmetric sequences in $\mathcal{M}$ is $\mathrm{Fun}(\mathbf{\Sigma},\mathcal{M})$.

The category of symmetric sequences in $\mathcal{C}$ is a closed symmetric monoidal category with respect to the Day convolution product. For symmetric sequences A and

B in $\mathcal{C}$, their Day convolution is the symmetric sequence

$$A \otimes_{\mathrm{Day}} B \colon n \longmapsto \int^{p,q \in \mathbf{\Sigma}} \mathbf{\Sigma}(p+q, n) \otimes A(p) \otimes B(q)$$

in $\mathcal{C}$. Similarly, $\mathrm{Fun}(\mathbf{\Sigma}, \mathcal{M})$ is tensored and cotensored over $\mathrm{Fun}(\mathbf{\Sigma}, \mathcal{C})$. The symmetric monoidal functor $\mathbf{\Sigma} \to \mathcal{C}$ which sends

$$n \longmapsto \underbrace{K \otimes \cdots \otimes K}_{n \text{ times}}$$

gives rise to a commutative monoid $\mathrm{Sym}(K)$ in $\mathrm{Fun}(\mathbf{\Sigma}, \mathcal{C})$.

Definition A.3.18. The category of *symmetric spectra* $\mathrm{Sp}^{\Sigma}(\mathcal{M}; K)$ is the category of (left) $\mathrm{Sym}(K)$-modules in $\mathrm{Fun}(\mathbf{\Sigma}, \mathcal{M})$. Spelling this out, a *symmetric K-spectrum in $\mathcal{M}$* is the data of

- a sequence $\{x_n\}_{n \in \mathbb{N}}$ of objects of $\mathcal{M}$ such that x_n has an action of the permutation group Σ_n; and
- for each $n \in \mathbb{N}$ a Σ_n-equivariant map $\sigma_n^x \colon K \otimes x_n \to x_{n+1}$

such that for all $p, q \geq 0$, the composite

$$K^{\otimes p} \otimes x_q \xrightarrow{K^{\otimes(p-1)} \otimes \sigma_q} K^{\otimes(p-1)} \otimes x_{q+1} \xrightarrow{K^{\otimes(p-2)} \otimes \sigma_{q+1}} \cdots \xrightarrow{\sigma_{q+p-1}} x_{p+q}$$

is $(\Sigma_p \times \Sigma_q)$-equivariant. A map of symmetric spectra $x \to y$ is a collection of Σ_n-equivariant maps $x_n \to y_n$ compatible with the structure maps.

Construction A.3.19. For each $n \in \mathbb{N}$ there is a free-forgetful adjunction

$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left, "\mathrm{F}^{\mathcal{M}}"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\Sigma}(\mathcal{M}; K). \arrow[l, shift left, "\mathrm{Ev}_n"] \end{tikzcd}$$

As for sequential spectra, the right adjoint $\mathrm{Ev}_n \colon x \mapsto x_n$ extracts the n-th term of a symmetric spectrum (forgetting the Σ_n-action). Because of the symmetric group actions, the free functors $\mathrm{F}_n^{\mathcal{M}}$ are more subtle than their sequential counterparts $F_n^{\mathcal{M}}$ and arise in the following way. For fixed $n \in \mathbb{N}$ the functor $\mathrm{Fun}(\mathbf{\Sigma}, \mathcal{M}) \to \mathcal{M}$ which sends a symmetric sequence to its n-th term has left adjoint

$$x \longmapsto \mathrm{s}_n(x)_m := \begin{cases} \Sigma_n \otimes x & \text{if } m = n \\ \varnothing & \text{otherwise,} \end{cases}$$

with $\varnothing \in \mathcal{M}$ the initial object. The symmetric spectrum $\mathrm{F}_n^{\mathcal{M}}(x)$ is the free $\mathrm{Sym}(K)$-module on the symmetric sequence $\mathrm{s}_n(x)$. Explicitly, the m-th term is

$$\mathrm{F}_n^{\mathcal{M}}(x)_m := \begin{cases} \Sigma_m \otimes_{\Sigma_{m-n}} \left(K^{\otimes(m-n)} \otimes x\right) & \text{for } m \geq n \\ \varnothing & \text{otherwise} \end{cases}$$

with structure maps either the identity or the initial map depending on m and n.

Construction A.3.20. The category $\mathrm{Sp}^{\Sigma}(\mathcal{C};K)$ is closed symmetric monoidal with respect to the *smash product* of symmetric spectra, which is defined via the formula

$$x \wedge y = x \otimes_{\mathrm{Sym}(K)} y := \mathrm{colim}\Big(x \otimes_{\mathrm{Day}} \mathrm{Sym}(K) \otimes_{\mathrm{Day}} y \overset{\rho_x}{\underset{\rho_y}{\rightrightarrows}} x \otimes_{\mathrm{Day}} y \Big) \tag{A.1}$$

where ρ_x (resp. ρ_y) is the $\mathrm{Sym}(K)$-action on x (resp. y). For any $n, m \in \mathbb{N}$ and $c, d \in \mathcal{C}$, we have a natural isomorphism

$$\mathrm{F}_n^{\mathcal{M}}(c) \wedge \mathrm{F}_m^{\mathcal{M}}(d) \cong \mathrm{F}_{n+m}^{\mathcal{M}}(c \otimes d). \tag{A.2}$$

In particular, $\mathrm{F}_0^{\mathcal{F}}$ is strong symmetric monoidal. In a similar fashion, $\mathrm{Sp}^{\Sigma}(\mathcal{M};K)$ is enriched, tensored and cotensored over $\mathrm{Sp}^{\Sigma}(\mathcal{C};K)$. The $\mathrm{Sp}^{\Sigma}(\mathcal{C};K)$-tensoring, for instance, is defined using the formula (A.1) by replacing the monoidal structure $\otimes_{\mathrm{Day}}$ on $\mathrm{Fun}(\mathbf{\Sigma}, \mathcal{C})$ by the $\mathrm{Fun}(\mathbf{\Sigma}, \mathcal{C})$-tensoring on $\mathrm{Fun}(\mathbf{\Sigma}, \mathcal{M})$ where appropriate.

Note that the categories of symmetric sequences $\mathrm{Fun}(\mathbf{\Sigma}, \mathcal{C})$ and $\mathrm{Fun}(\mathbf{\Sigma}, \mathcal{M})$ are locally presentable, and in both cases the monad $\mathrm{Sym}(K) \otimes (-)$ preserves colimits, so is accessible. By [Bor94, Theorem 5.5.9], it follows that $\mathrm{Sp}^{\Sigma}(\mathcal{C};K)$ and $\mathrm{Sp}^{\Sigma}(\mathcal{M};K)$ are locally presentable. Moreover, since the monad $\mathrm{Sym}(K) \otimes (-)$ preserves colimits, both limits and colimits in $\mathrm{Fun}(\mathbf{\Sigma}, \mathcal{M})$ are created by the forgetful functor to symmetric sequences (likewise for $\mathcal{C}$).

Lemma A.3.21. *The category* $\mathrm{Sp}^{\Sigma}(\mathcal{M};K)$ *has a* projective model structure, *with respect to which a map of symmetric spectra is a weak equivalence or fibration precisely if the underlying map of symmetric sequences is such. The projective model structure on* $\mathrm{Sp}^{\Sigma}(\mathcal{M};K)$ *is left proper and combinatorial. Moreover,* $\mathrm{Sp}^{\Sigma}(\mathcal{M};K)_{\mathrm{proj}}$ *is a* $\mathrm{Sp}^{\Sigma}(\mathcal{C};K)_{\mathrm{proj}}$*-model category.*

Proof. This is [Hov01, Theorems 8.2 & 8.3]. Alternatively, the existence of the model structure is obtained similarly to Lemma A.3.7 using the Right Transfer Theorem A.1.3 applied to the free-forgetful adjunction

$$\begin{tikzcd} \mathrm{Fun}(\mathbf{\Sigma}, \mathcal{M}) \arrow[r, shift left, "\mathrm{Sym}(K)\otimes(-)"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\Sigma}(\mathcal{M};K), \arrow[l, shift left] \end{tikzcd}$$

where $\mathrm{Fun}(\mathbf{\Sigma}, \mathcal{M})$ is equipped with the level model structure. Left properness follows from the facts that projective cofibrations are level cofibrations and that $\mathcal{M}$ is left proper. The pushout-product axiom for the $\mathrm{Sp}^{\Sigma}(\mathcal{C};K)$-tensoring on $\mathrm{Sp}^{\Sigma}(\mathcal{M};K)$ is checked directly using cofibrant generation (Lemma A.1.2) and (A.2). □

Corollary A.3.22. *For each* n, $(\mathrm{F}_n^{\mathcal{M}} \dashv \mathrm{Ev}_n)$ *is a* $\mathcal{C}$*-Quillen adjunction for the projective model structure on* $\mathrm{Sp}^{\Sigma}(\mathcal{M};K)$.

As for sequential spectra, in the symmetric setting we implement stabilisation by taking a left Bousfield localisation of the projective model structure.

Definition A.3.23. Let $\mathcal{I}_{\mathcal{M}}$ be a set of generating cofibrations for $\mathcal{M}$. The corresponding *stabilising set* is

$$\mathbf{S}(\mathcal{I}_{\mathcal{M}}) := \left\{ \zeta_n^{\mathcal{M}}(\mathcal{Q}c) \colon \mathrm{F}_{n+1}^{\mathcal{M}}(K \otimes \mathcal{Q}c) \longrightarrow \mathrm{F}_n^{\mathcal{M}}(\mathcal{Q}c) \right\},$$

where $\zeta_n^{\mathcal{M}}(x)$ is the adjunct of the map

$$K \otimes x \longrightarrow \Sigma_{n+1} \otimes \big(K \otimes x\big) = \mathrm{Ev}_{n+1}\mathrm{F}_n^{\mathcal{M}}(x)$$

determined by the Set-tensoring with unit map $* \to \Sigma_{n+1}$, $n \in \mathbb{N}$, $\mathcal{Q}$ is a fixed cofibrant replacement functor for $\mathcal{M}$, $n \in \mathbb{N}$, and c ranges over the domains and codomains of morphisms in $\mathcal{I}_{\mathcal{M}}$.

We define the *stable model structure* on symmetric spectra in $\mathcal{M}$ as the left Bousfield localisation

$$\mathrm{Sp}^{\Sigma}(\mathcal{M};K)_{\mathrm{stab}} := L_{\mathbf{S}(\mathcal{I}_{\mathcal{M}})}\mathrm{Sp}^{\Sigma}(\mathcal{M};K)_{\mathrm{proj}},$$

noting that this exists since the projective model structure is left proper and combinatorial.

Lemma A.3.24. *The stable model structure on* $\mathrm{Sp}^{\Sigma}(\mathcal{M};\Sigma)$ *has the following properties:*

(1) *the fibrant objects are prescisely the* Ω*-spectra; namely those symmetric spectra* x *such that* $x_n \in \mathcal{M}$ *is fibrant for all* n *and such that the maps* $\sigma_n^{\vee}\colon x_n \to K \pitchfork x_{n+1}$ *are weak equivalences in* $\mathcal{M}$*; and*

(2) *for each cofibrant object* a *of* $\mathcal{M}$ *and* $n \in \mathbb{N}$*, the map* $\xi_n^{\mathcal{M}}(a)$ *is a weak equivalence.*

In particular, the stable model structure is independent of the choice of stabilising set.

Proof. The proof is completely analogous to that of Lemma A.3.10. □

Lemma A.3.25. *Suppose that the domains of the generating cofibrations of* $\mathcal{C}$ *and* $\mathcal{M}$ *are cofibrant. Then* $\mathrm{Sp}^{\Sigma}(\mathcal{C};K)_{\mathrm{stab}}$ *is a symmetric monoidal model category with respect to the smash product, and* $\mathrm{Sp}^{\Sigma}(\mathcal{M};K)_{\mathrm{stab}}$ *is a* $\mathrm{Sp}^{\Sigma}(\mathcal{C};K)_{\mathrm{stab}}$*-model structure.*

Proof. This is [Hov01, Theorem 8.11], and is proven from Lemma A.3.21 by using (A.2) to verify that smashing cofibrant objects in $\mathrm{Sp}^{\Sigma}(\mathcal{C};K)$ with maps in the stabilising set $\mathbf{S}(\mathcal{I}_{\mathcal{M}})$ yields stable weak equivalences. □

The stable model structure on $\mathrm{Sp}^{\Sigma}(\mathcal{M};K)$ does indeed invert K up to homotopy:

Lemma A.3.26. *The* $\mathcal{C}$*-tensoring* $x \mapsto K \otimes x$ *on* $\mathrm{Sp}^{\Sigma}(\mathcal{M};K)_{\mathrm{stab}}$ *is a left Quillen equivalence.*

Proof. This is the symmetric spectrum analogue of Lemma A.3.12, and is proven in essentially the same way by constructing auxiliary shift functors. □

Similarly to Lemma A.3.13, the symmetric stabilisation machine is idempotent:

Lemma A.3.27. *Suppose that* $K \otimes (-)$ *is a left Quillen autoequivalence of* $\mathcal{M}$*. Then* $\mathrm{F}_0^{\mathcal{M}}\colon \mathcal{M} \to \mathrm{Sp}^{\Sigma}(\mathcal{M};K)$ *is a Quillen equivalence.*

Proof. This is [Hov01, Theorem 9.1]. □

We conclude this section with a discussion of the functoriality properties of the symmetric stabilisation machine:

Theorem A.3.28. *Let* $\mathcal{C}$ *be a left proper combinatorial model category, and let* $\mathcal{M}$*,* $\mathcal{N}$ *be left proper combinatorial* $\mathcal{C}$*-model categories. Suppose moreover that the domains of the generating cofibrations of* $\mathcal{C}$*,* $\mathcal{M}$ *and* $\mathcal{N}$ *can be taken to be cofibrant. Then any left* $\mathcal{C}$*-Quillen functor* $L\colon \mathcal{M} \to \mathcal{N}$ *extends naturally to a left* $\mathrm{Sp}^{\Sigma}(\mathcal{C};K)_{\mathrm{stab}}$*-Quillen functor*

$$L\colon \mathrm{Sp}^{\Sigma}(\mathcal{M};K)_{\mathrm{stab}} \longrightarrow \mathrm{Sp}^{\Sigma}(\mathcal{N};K)_{\mathrm{stab}}.$$

Additionally, if $L\colon \mathcal{M} \to \mathcal{N}$ *is a left Quillen equivalence then so too is its prolongation.*

Proof. This is proven as [Hov01, Theorem 9.3]. □

Remark A.3.29. The prolonged Quillen adjunction of Theorem A.3.28 is given by levelwise application of the adjunction $(L \dashv R)\colon \mathcal{M} \to \mathcal{N}$. In particular, if $L\colon \mathcal{C} \to \mathcal{D}$ is a strong monoidal left Quillen functor then the prolonged left Quillen functor

$$L\colon \mathrm{Sp}^{\Sigma}(\mathcal{C}; K) \longrightarrow \mathrm{Sp}^{\Sigma}(\mathcal{D}; L(K))$$

is strong symmetric monoidal with respect to the smash product.

A.3.3 Quillen Invariance for $\mathrm{Sp}^{\Sigma}(ch_+)$

In this section, we record a technical result required in the proof of Theorem 2.4.6. We first recall the following

Definition A.3.30. Let $\mathcal{M}$ be a monoidal model category such that the forgetful functors create model structures of modules over any monoid. Then *Quillen invariance holds for* $\mathcal{M}$ if for any weak equivalence of monoids $\psi\colon A \to B$ in $\mathcal{M}$, the extension and restriction of scalars adjunction

$$\begin{tikzcd} A-\mathrm{Mod} \arrow[r, shift left=1ex, "\psi_!"] \arrow[r, phantom, "\perp"] & B-\mathrm{Mod} \arrow[l, shift left=1ex, "\psi^*"] \end{tikzcd}$$

is a Quillen equivalence.

Lemma A.3.31. *Quillen invariance holds for the stable model structure on* $\mathrm{Sp}^{\Sigma}(ch_+)$.

Proof. Inspecting [Shi07, §3], we find that the proof of [Shi07, Corollary 3.4] applies, replacing $\mathbb{Z}$ by $\mathbb{Q}$. □

Remark A.3.32. Following [HSS00, §5], the proof of [Shi07, Corollary 3.4] works by constructing an auxiliary *injective* stable model structure, and showing that the injective stable model structure is a Quillen module over the (projective) stable model structure.

A.3.4 Brown Representability

Let $\mathcal{M}$ be a *stable* model category, namely a model category for which the suspension endofunctor

$$\Sigma\colon Ho(\mathcal{M}) \longrightarrow Ho(\mathcal{M})$$

is an autoequivalence. Then $Ho(\mathcal{M})$ is a triangulated category, with distinguished triangles defined as the homotopy cofibre sequences [Hov99, Ch. 6 & 7].

In this section, we recall a version of Brown's Representability Theorem, following the presentation of [Sch12, Ch. II.5]. Let $\mathcal{T}$ be a triangulated category with infinite sums, and recall that an object $K \in \mathcal{T}$ is *compact* if the map

$$\bigoplus_{i\in\mathcal{I}} \mathcal{T}(K, X_i) \longrightarrow \mathcal{T}\Big(K, \bigoplus_{i\in\mathcal{I}} X_i\Big)$$

is an isomorphism for all indexing sets $\mathcal{I}$. Recall that a *homotopy colimit* of a sequence of morphisms

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots$$

in $\mathcal{T}$ is an object $\operatorname{hocolim}_n X_n$ equipped with morphisms $q_n\colon X_n \to \operatorname{hocolim}_n X_n$ such that $q_{n+1} \circ f_n = q_n$ for all n, and such that there exists a distinguished triangle

$$\bigoplus_n X_n \xrightarrow{1-f} \bigoplus_n X_n \xrightarrow{\oplus q_n} \operatorname{hocolim}_n X_n \longrightarrow \bigoplus_n \Sigma X_n,$$

where $1-f$ is defined on the i-th summand as the difference between the maps

$$X_i \to \bigoplus_n X_n \quad \text{and} \quad X_i \xrightarrow{f_i} X_{i+1} \to \bigoplus_n X_n.$$

In the case that $\mathcal{T} = Ho(\mathcal{M})$ is the homotopy category of a stable model category, the homotopy colimit as just defined is isomorphic to the homotopy colimit defined using model-theoretic means.

Definition A.3.33. For $\mathcal{T}$ a triangulated category, a functor $E\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ is *cohomological* if it takes sums in $\mathcal{T}$ to products of abelian groups, and if for every distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A,$$

the sequence

$$E(\Sigma A) \longrightarrow E(C) \longrightarrow E(B) \longrightarrow E(A)$$

is exact.

Remark A.3.34. Any representable functor $\mathcal{T}(-,X)\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ is cohomological, since $\mathcal{T}(-,X)$ converts distinguished triangles into exact sequences.

Definition A.3.35. Let $\mathcal{T}$ be a triangulated category with infinite sums and $\mathcal{G}$ a set of compact objects of $\mathcal{T}$. The *right-closed class* generated by $\mathcal{G}$ is the smallest class $\langle \mathcal{G} \rangle_+$ of objects of $\mathcal{T}$ which

- contains $\mathcal{G}$;
- is closed under sums; and
- is closed under extensions to the right, meaning that if

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

  is a distinguished triangle for which $A, B \in \langle \mathcal{G} \rangle_+$, then $C \in \langle \mathcal{G} \rangle_+$.

Lemma A.3.36 (Brown Representability)**.** *Let $\mathcal{T}$ be a triangulated category with infinite sums and let $\mathcal{G}$ be a set of compact objects. For every cohomological functor $E\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ there exists an object $X \in \langle \mathcal{G} \rangle_+$ and an element $u \in E(X)$ such that for every $G \in \mathcal{G}$, the evaluation map*

$$f \longmapsto E(f)(u)$$

determines an isomorphism $\mathcal{T}(G,X) \to E(G)$.

Proof. Set

$$X_0 := \bigoplus_{G \in \mathcal{G}} \bigoplus_{x \in E(G)} G,$$

then we have an isomorphism

$$E(X_0) \cong \prod_{G \in \mathcal{G}} \prod_{x \in E(G)} E(G),$$

so that in particular there is an element $u_0 \in E(X_0)$ that restricts to $x \in E(G)$ on the corresponding summand. The object $X_0 \in \langle \mathcal{G} \rangle_+$, and evaluation at u_0 induces an epimorphism $\mathcal{T}(G, X_0) \to E(G)$ for all $G \in \mathcal{G}$. We kill the kernel of this map by an inductive argument as follows.

Suppose we have defined $X_k \in \langle \mathcal{G} \rangle_+$ for $0 \leq k \leq n-1$ together with maps $i_k \colon X_k \to X_{k+1}$ and elements $u_k \in E(X_k)$ such that $E(i_k)(u_{k+1}) = u_k$. Write $I_n(G)$ for the kernel of the map $\mathcal{T}(G, X_n) \to E(G)$ given by evaluating at u_n, and set

$$K_n := \bigoplus_{G \in \mathcal{G}} \bigoplus_{x \in I_n(G)} G,$$

which is furnished with a morphism $\tau \colon K_n \to X_n$ for which $\tau^* u_n$ equals x when restricted to the summand indexed by $x \in I_n(G)$. Completing $\tau \colon K_n \to X_n$ to a distinguished triangle

$$K_n \xrightarrow{\tau} X_n \xrightarrow{i_n} X_{n+1} \longrightarrow \Sigma K_n,$$

since E is cohomological we obtain an exact sequence

$$E(\Sigma K_n) \longrightarrow E(X_{n+1}) \xrightarrow{E(i_n)} E(X_n) \xrightarrow{E(\tau)} E(K_n).$$

But $E(\tau)$ takes u_n to zero by definition of K_n, so lifts to an element $u_{n+1} \in E(X_{n+1})$ by exactness. Observe that $X_{n+1} \in \langle \mathcal{G} \rangle_+$.

Let X be a homotopy colimit of the X_n, so that $X \in \langle \mathcal{G} \rangle_+$, and the exact sequence

$$0 \longrightarrow \lim^1_n E(X_n) \longrightarrow E(X) \longrightarrow \lim_n E(X_n) \longrightarrow 0$$

implies that there is some $u \in E(X)$ restricting to $u_n \in E(X_n)$ for each n. For any $G \in \langle \mathcal{G} \rangle_+$, the composite

$$\mathcal{T}(G, X_0) \longrightarrow \mathcal{T}(G, X) \xrightarrow{\mathrm{ev}_u} E(G)$$

is evaluation at u_0, which is surjective by construction. In particular, the evaluation map $\mathcal{T}(G, X) \to E(G)$ is surjective. For injectivity, suppose that $\alpha \colon G \to X$ is sent to zero by $E(-)(u)$. Since G is compact, the functor $\mathcal{T}(G, -)$ induces an isomorphism

$$\mathcal{T}(G, X) \cong \mathrm{colim}_n\, \mathcal{T}(G, X_n),$$

so that α factors through one of the finite stages via $\alpha' \colon G \to X_n$, say. But then we have $\alpha' \in I_n(G)$, so that α' indexes one of the summands of K_n and hence $\alpha' = \tau \circ \alpha''$ for some $\alpha'' \colon G \to K_n$. Since the composite $K_n \to X_n \to X_{n+1}$ is zero, it follows that

$$\alpha = q_n \circ \alpha' = q_n \circ \tau \circ \alpha'' = q_{n+1} \circ i_n \circ \tau \circ \alpha'' = 0.$$

Thus, the evaluation map $\mathcal{T}(G, X) \to E(G)$ is an isomorphism for all $G \in \langle \mathcal{G} \rangle_+$. $\square$

Remark A.3.37. Taking $\mathcal{T}$ to be the stable homotopy category, the more familiar form of the Brown Representability Theorem is obtained by applying the Lemma to the right-closed class generated by suspensions and desuspensions of the sphere spectrum and then using that $\mathbb{S}$ is a weak compact generator.

A.4 The Grothendieck Construction for Model Categories

This section is devoted to a brief treatment of the Grothendieck construction for model categories (Theorem A.4.6), which is a key tool for studying families of model categories parametrised by a model category. The Grothendieck construction for model categories was first established in [Roi94b]; our treatment closely follows that of [HP15]. We first fix some notation:

Definition A.4.1. Let **Cat** be the 2-category of categories, functors and natural transformations. Write **Adj** for the $(2,1)$-category of adjunctions, where

- objects are categories $\mathcal{C}$;
- morphisms are adjunctions $(L \dashv R)\colon \mathcal{C} \to \mathcal{D}$; and
- 2-morphisms are *pseudonatural transformations of adjunctions*

  $$(\eta, \xi)\colon (L \dashv R) \Longrightarrow (L' \dashv R'),$$

  which is to say, pairs of natural isomoprhisms $\eta\colon L \Rightarrow L'$ and $\xi\colon R' \Rightarrow R$ such that the diagram

  $$\begin{tikzcd} \mathcal{C}(x, R'y) \arrow[r, "\cong"] \arrow[d, "(\xi_y)_*"'] & \mathcal{D}(L'x, y) \arrow[d, "\eta_x^*"] \\ \mathcal{C}(x, Ry) \arrow[r, "\cong"] & \mathcal{D}(Lx, y) \end{tikzcd}$$

  commutes for all $x \in \mathcal{C}$ and $y \in \mathcal{D}$.

We write **Mod** for the $(2,1)$-category which has

- objects the model categories;
- morphisms $\mathcal{M} \to \mathcal{N}$ are Quillen adjunctions $(L \dashv R)\colon \mathcal{M} \to \mathcal{N}$; and
- 2-morphisms are pseudonatural transformations of adjunctions.

Definition A.4.2. Let $F\colon \mathcal{C} \to \mathbf{Cat}$ be a pseudofunctor. The *Grothendieck construction of F* is the category $\int_{\mathcal{C}} F$ with

- objects the pairs (x, a) for $x \in \mathcal{C}$ and $a \in F(c)$; and
- morphisms $(x, a) \to (y, b)$ given by pairs (f, ϕ), with $f\colon x \to y$ a morphism in $\mathcal{C}$ and $\phi\colon F(f)(a) \to b$ a morphism in $F(y)$.

The Grothendieck construction is equipped with a functor $\int_{\mathcal{C}} F \to \mathcal{C}$ determined on objects by $(x, a) \mapsto x$.

Remark A.4.3. The functor $\int_{\mathcal{C}} F \to \mathcal{C}$ is a co-Cartesian fibration, and the Grothendieck construction establishes an equivalence of 2-categories

$$\int\colon \mathbf{PsFun}(\mathcal{C}, \mathbf{Cat}) \longrightarrow \mathbf{coCart}(\mathcal{C})$$

between the 2-categories of pseudofunctors $\mathcal{C} \to \mathbf{Cat}$ of co-Cartesian fibrations over $\mathcal{C}$. Dually, for a pseudofunctor $G\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ we obtain a Cartesian fibration[2]

[2]beware that in this case, morphisms in the Grothendieck construction $(x, a) \to (y, b)$ are defined as pairs (f, ψ) with $f\colon x \to y$ a morphism in $\mathcal{C}$ and $\psi\colon a \to F(f)(b)$ a morphism in $F(x)$.

$\int_{\mathcal{C}^{\mathrm{op}}} G \to \mathcal{C}$ together with an equivalence of 2-categories

$$\int : \mathbf{PsFun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}) \longrightarrow \mathbf{Cart}(\mathcal{C}).$$

In the case that we have a pseudofunctor $F\colon \mathcal{C} \to \mathbf{Adj}$, the Grothendieck construction $\int_{\mathcal{C}} F \to \mathcal{C}$ is a bi-Cartesian fibration, meaning that it is both Cartesian and co-Cartesian. The Grothendieck construction determines an equivalence of $(2,1)$-categories

$$\int : \mathbf{PsFun}(\mathcal{C}, \mathbf{Adj}) \longrightarrow \mathbf{biCart}(\mathcal{C}).$$

For details, we refer the reader to [HP15, §§2.1–2.2].

Suppose that $\mathcal{M}$ let $F\colon \mathcal{M} \to \mathbf{Mod}$ be a pseudofunctor. Thus, to each $x \in \mathcal{M}$ we assign a model category $F(x)$ and to each morphism $f\colon x \to y$ in $\mathcal{M}$ we assign a Quillen adjunction

$$\begin{tikzcd} (f_! \dashv f^*)\ F(x) \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] & F(y). \arrow[l, shift left, "f^*"] \end{tikzcd}$$

For a morphism $(f,\phi)\colon (x,a) \to (y,b)$ in the Grothendieck construction $\int_{\mathcal{M}} F$, let $\phi^{\vee}\colon a \to f^*b$ be the $(f_! \dashv f^*)$-adjunct of $\phi\colon f_!a \to b$. A straightforward argument shows that $\int_{\mathcal{M}} F$ has all small limits and colimits (see also [HP15, §2.4]).

We would like to equip $\int_{\mathcal{M}} F$ with a model structure combining the model structures of $\mathcal{M}$ and of the $F(x)$. A natural requirement is that the pseudofunctor F sends weak equivalences in $\mathcal{M}$ to Quillen equivalences:

Definition A.4.4. The pseudofunctor $F\colon \mathcal{M} \to \mathbf{Mod}$ is *relative* if for every weak equivalence $f\colon x \to y$ in $\mathcal{M}$, the adjunction $(f_! \dashv f^*)$ is a Quillen equivalence.

In this setting, the bi-Cartesian fibration $\int_{\mathcal{M}} F \to \mathcal{M}$ has both a left and a right adjoint. The left adjoint is determined on objects by $x \mapsto (x, \varnothing_{F(x)})$, with $\varnothing_{F(x)} \in F(x)$ the initial object, and the right adjoint is determined similarly via terminal objects. Another natural requirement of the model structure on $\int_{\mathcal{M}} F$ is that the bi-Cartesian fibration $\int_{\mathcal{M}} F \to \mathcal{M}$ is a left and right Quillen functor. In order for this to be the case, F must be *proper*:

Definition A.4.5. The pseudofunctor $F\colon \mathcal{M} \to \mathbf{Mod}$ is

- *left proper* if whenever $f\colon x \to y$ is an acyclic cofibration in $\mathcal{M}$, the left Quillen functor $f_!\colon F(x) \to F(y)$ preserves weak equivalences; and
- *right proper* if whenever $f\colon x \to y$ is an acyclic fibration in $\mathcal{M}$, the right Quillen functor $f^*\colon F(y) \to F(x)$ preserves weak equivalences.

The pseudofunctor F is *proper* if it is both right and left proper.

Theorem A.4.6. *Let $\mathcal{M}$ be a model category and let $F\colon \mathcal{M} \to$* **Mod** *be a proper relative pseudofunctor. Then the Grothendieck construction $\int_{\mathcal{M}} F$ has a model structure with respect to which a morphism $(f,\phi)\colon (x,a) \to (y,b)$ in $\int_{\mathcal{M}} F$ is*

- *a* weak equivalence *if $f\colon x \to y$ is a weak equivalence and the composite*

$$f_!(a^{\mathrm{cof}}) \longrightarrow f_!(a) \longrightarrow b$$

is a weak equivalence in $F(y)$, where $a^{\mathrm{cof}} \to a$ is any cofibrant replacement of a in $F(x)$;

- *a* cofibration *if $f\colon x \to y$ is a cofibration and $\phi\colon f_! a \to b$ is a cofibration in $F(y)$; and*
- *a* fibration *if $f\colon x \to y$ is a fibration and $\phi^\vee\colon a \to f^* b$ is a fibration in $F(x)$.*

The model strucutre on $\int_{\mathcal{M}} F$ so obtained is called the integral model structure. *With respect to the integral model structure, the bi-Cartesian fibration $\int_{\mathcal{M}} F \to \mathcal{M}$ is a left and right Quillen functor.*

Proof. This is proven as [HP15, Theorem 3.0.12]. Note that $\int_{\mathcal{M}} F \to \mathcal{M}$ is left and right Quillen as it preserves weak equivalences, cofibrations and fibrations. □

Remark A.4.7. For $F\colon \mathcal{M} \to \mathbf{Mod}$ a proper relative pseudofunctor, we can define a functor of $(\infty,1)$-categories

$$F^\circ\colon \mathcal{M}^\circ \longrightarrow \mathcal{C}at$$

as follows. Recall that the *relative categories* studied in [BK12] present $(\infty,1)$-categories. From F, we build a relative functor

$$F^{\mathrm{cof}}\colon \mathcal{M}^{\mathrm{cof}} \longrightarrow \mathrm{RelCat}$$

by restricting to the cofibrant objects $\mathcal{M}^{\mathrm{cof}}$ and setting $F^{\mathrm{cof}}(x) := (F(x))^{\mathrm{cof}}$; in either case, the category in question is relative with respect to the wide subcategory of weak equivalences. Since F is relative, we can show that the relative functor F^{cof} sends weak equivalences in $\mathcal{M}^{\mathrm{cof}}$ to Dwyer–Kan equivalences of relative categories. Composing with the homotopy coherent nerve $\mathcal{N}\colon \mathrm{RelCat} \to \mathrm{sSet}^+$, where sSet^+ is the category of marked simplicial sets, we have

$$\mathcal{N}F^{\mathrm{cof}}\colon \mathcal{M}^{\mathrm{cof}} \longrightarrow \mathrm{sSet}^+.$$

Applying the homotopy coherent nerve to this relative functor yields a morphism of marked simplicial sets

$$F^\circ\colon \mathcal{M}^\circ \longrightarrow \mathcal{C}at := (\mathrm{sSet}^+)^\circ$$

(we are glossing over size issues here). Since sSet^+ presents the homotopy theory of $(\infty,1)$-categories [Lur09], this is a functor of $(\infty,1)$-categories. Applying the Grothendieck construction for $(\infty,1)$-categories [Lur09, Ch. 3] yields a co-Cartesian fibration of $(\infty,1)$-categories

$$\int_{\mathcal{M}^\circ} F^\circ \longrightarrow \mathcal{M}^\circ.$$

On the other hand, the bi-Cartesian fibration $\int_{\mathcal{M}} F \to \mathcal{M}$ gives rise to a co-Cartesian fibration of $(\infty,1)$-categories

$$\left(\int_{\mathcal{M}} F\right)^\circ \longrightarrow \mathcal{M}^\circ.$$

According to [HP15, Proposition 3.1.2], there is a natural equivalence of $(\infty,1)$-categories over $\mathcal{M}^\circ$

$$\begin{tikzcd}
(\int_{\mathcal{M}} F)^\circ \arrow[rr, "\cong"] \arrow[dr] & & \int_{\mathcal{M}^\circ} F^\circ \arrow[dl] \\
& \mathcal{M}^\circ, &
\end{tikzcd}$$

so that the Grothendieck construction for model categories models the Grothendieck constrution for $(\infty, 1)$-categories. The proof of this result depends on [Hin16] in an essential way.

Finally, suppose we have proper relative pseudofunctors $F\colon \mathcal{M} \to \mathbf{Mod}$ and $G\colon \mathcal{M} \to \mathbf{Mod}$, and a *pseudonatural Quillen transformation* $\Phi\colon F \Rightarrow G$. That is to say, for each $x \in \mathcal{M}$ there is a Quillen adjunction

$$\begin{tikzcd} F(x) \arrow[r, shift left, "\Phi_!^x"] \arrow[r, phantom, "\perp"] & G(x) \arrow[l, shift left, "\Phi_x^*"] \end{tikzcd}$$

such that for each $f\colon x \to y$ in $\mathcal{M}$, the diagram of left Quillen functors

$$\begin{tikzcd} F(x) \arrow[r, "f_!"] \arrow[d, "\Phi_!^x"'] & F(y) \arrow[d, "\Phi_!^y"] \\ G(x) \arrow[r, "f_!"] & G(y) \end{tikzcd}$$

commutes up to natural isomorphism. The data of the adjunctions $(\Phi_!^x \dashv \Phi_x^*)$ are subject to some natural coherence conditions which we do not spell out. A pseudo natural Quillen transformation is a pseudonatural Quillen equivalence if the adjunctions $(\Phi_!^x \dashv \Phi_x^*)$ are Quillen equivalences for all cofibrant $x \in \mathcal{M}$.

Theorem A.4.8. *Let $F, G\colon \mathcal{M} \to \mathbf{Mod}$ be proper relative pseudofunctors and $\Phi\colon F \Rightarrow G$ a pseudonatural Quillen transformation. Then there is a Quillen adjunction*

$$\begin{tikzcd} \int_{\mathcal{M}} F \arrow[r, shift left, "\Phi_!"] \arrow[r, phantom, "\perp"] & \int_{\mathcal{M}} G \arrow[l, shift left, "\Phi^*"] \end{tikzcd}$$

commuting with the projection functors to $\mathcal{M}$. If Φ is a pseudonatural Quillen equivalence, then $(\Phi_! \dashv \Phi^*)$ is a Quillen equivalence.

Proof. This is a special case of [HP15, Theorem 4.1.3], though our terminology here is different. The left adjoint $\Phi_!$ sends $(x, a) \mapsto (x, \Phi_!^x(a))$ whereas the right adjoint Φ^* sends $(y, \beta) \mapsto (y, \Phi_y^*(\beta))$. The adjunction $(\Phi_! \dashv \Phi^*)$ is straightforward to verify directly. With respect to the integral model structures, it is easy to see that $\Phi_!$ preserves cofibrations and acyclic cofibrations (the latter uses left properness of F and G).

Now suppose that Φ is a pseudonatural Quillen equivalence. Let $(x, a) \in \int_{\mathcal{M}} F$ and $(y, \beta) \in \int_{\mathcal{M}} G$ be cofibrant and fibrant objects respectively. Suppose we have a morphism $(f, \psi)\colon (x, a) \to \Phi^*(y, \beta) = (y, \Phi_y^*(\beta))$, so that we have $f\colon x \to y$ in $\mathcal{M}$ and $\psi\colon f_! a \to \Phi_y^*(\beta)$ in $F(y)$. Since $a \in F(x)$ is cofibrant, the morphism (f, ψ) is a weak equivalence in $\int_{\mathcal{M}} F$ precisely if f and ψ are weak equivalences in $\mathcal{M}$ and $F(y)$ respectively. Since f is a weak equivalence, the adjunction $(f_! \dashv f^*)\colon F(x) \to F(y)$ is a Quillen equivalence, so that the adjunct

$$a \longrightarrow f^* \Phi_y^*(\beta) \cong \Phi_x^* f^*(\beta)$$

is a weak equivalence in $F(x)$. Since $x \in \mathcal{M}$ is cofibrant, by hypothesis the adjunction $(\Phi_!^x \dashv \Phi_x^*)\colon F(x) \to G(x)$ is a Quillen equivalence, so that the adjunct

$$\Phi_!^x(a) \longrightarrow f^*(\beta)$$

is a weak equivalence in $G(x)$. Finally, $(f_! \dashv f^*)\colon G(x) \to G(y)$ is Quillen equivalence so that

$$\psi^\vee\colon f_!\Phi_!^x(a) \longrightarrow \beta$$

is a weak equivalence in $G(y)$. Thus, if $(f,\psi)\colon (x,a) \to \Phi^*(y,\beta)$ is a weak equivalence then its adjunct $(f,\psi^\vee)\colon \Phi_!(x,a) \to (y,\beta)$ is a weak equivalence. The converse is proven in exactly the same way, so that $(\Phi_! \dashv \Phi^*)$ is a Quillen equivalence. $\square$

Appendix B

Simplicial Group Actions

In this appendix, we recall some aspects of the homotopy theory of simplicial groups and their actions. A standard reference for much of this material is [GJ09, Ch. 5].

B.1 Simplicial G-Spaces

We begin by recalling the model structure on the category of simplicial G-spaces:

Theorem B.1.1. *For G a simplicial group, the category $G-\mathrm{sSet}$ of G-spaces has a proper combinatorial simplicial model structure for which a morphism $f\colon X \to Y$ is*

- *a weak equivalence if and only if f is a weak equivalence of simplicial sets;*
- *a fibration if and only if f is a Kan fibration; and*
- *a cofibration if and only if f has the left lifting property against acyclic fibrations. In particular, f is a monomorphism and $Y_n \setminus f(X_n)$ is a free G_n-set for eacn $n \in \mathbb{N}$.*

Proof. The free-forgetful adjunction

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, shift left, "G\times(-)"] \arrow[r, phantom, "\perp"] & G-\mathrm{sSet} \arrow[l, shift left, "U"] \end{tikzcd}$$

exhibits $G-\mathrm{sSet}$ as the category of algebras over the accessible monad $G \times (-)$ on sSet. We conclude that $G-\mathrm{sSet}$ is locally presentable. The forgetful functor U has a right adjoint and so preserves colimits. Applying the Right Transfer Theorem A.1.3, we obtain a cofibrantly generated model structure on $G-\mathrm{sSet}$ with weak equivalences and fibrations created by U. Note that the conditions of the Right Transfer Theorem are met since the monad $G \times (-)$ on sSet preserves acyclic cofibrations by the pushout-product axiom.

By the Transfer Theorem, the cofibrations in $G-\mathrm{sSet}$ are precisely the retracts of transfinite compositions of pushouts of morphisms in $\mathcal{I}^G_{\mathrm{Kan}} := \{G \times i \mid i \in \mathrm{I}_{\mathrm{Kan}}\}$. Suppose that $f\colon X \to Y$ is a cofibration: since colimits in $G-\mathrm{sSet}$ are preserved by U, it follows that f is a monomorphism. Moreover, each of the morphisms in $\mathcal{I}^G_{\mathrm{Kan}}$ is a monomorphism of the form specified in the statement of the Theorem, and this levelwise freeness property is closed under retracts, so it is sufficient to suppose that f is a transfinite composition of morphisms in $\mathcal{I}^G_{\mathrm{Kan}}$. This transfinite composition can be rearranged so as to be the colimit of a sequence

$$X = Z(0) \longrightarrow Z(1) \longrightarrow \cdots \longrightarrow Z(n) \longrightarrow Z(n+1) \longrightarrow \cdots$$

indexed by $n \in \mathbb{N}$, where $Z(n+1)$ is obtained from $Z(n)$ by a pushout diagram of the form

$$\begin{tikzcd} \coprod_{\alpha(n)} G \times \partial\Delta[n] \arrow[r] \arrow[d] & Z(n) \arrow[d, "f_n"] \\ \coprod_{\alpha(n)} G \times \Delta[n] \arrow[r] & Z(n+1). \end{tikzcd}$$

We have that $Z(n+1)_k \setminus f_n(Z(n))_k = \coprod_{\alpha(n)} G_k \times (\Delta[n]_k \setminus \partial\Delta[n]_k)$ is a free G_k-set for all k and n. Taking the colimit over n obtains the desired characterisation of cofibrations. In particular, U preserves cofibrations.

Properness is inherited from the Kan model structure, using that U preserves cofibrations and creates fibrations, weak equivalences, limits and colimits.

Finally, each $K \in$ sSet can be equipped with the trivial G-action. The sSet-tensoring of K with $X \in G-$sSet is then given by $K \otimes X = K \times X$ equipped with the diagonal G-action. This bifunctor preserves colimits in both variables, so to conclude it is sufficient to verify the pushout-product axiom. For $i_m \in \mathcal{I}_{\mathrm{Kan}}$ and $G \times i_n \in \mathcal{I}^G_{\mathrm{Kan}}$, it is easy to see that $i_m \square (G \times i_n) \cong G \times (i_m \boxtimes i_n)$, where $\boxtimes$ denotes the pushout-product for the cartesian product in sSet. Since $G \times (-)$ is left Quillen, we conclude that $\mathcal{I}_{\mathrm{Kan}} \square \mathcal{I}^G_{\mathrm{Kan}}$ consists of cofibrations. By cofibrant generation, we conclude that the sSet-tensoring is a Quillen bifunctor; the acyclicity clause of the pushout-product axiom is easy to verify directly. This completes the proof. □

Lemma B.1.2. *A G-space X is cofibrant if and only if each X_n is a free G_n-space.*

Proof. One direction is immediate from the Theorem. Let us then suppose that X is a G-space for which each X_n is a free G_n-space. We define a filtration of X by setting X_n to be the pullback

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{sk}_n(X/G) \arrow[r] & X/G. \end{tikzcd}$$

We have the standard pushout diagram relating the $(n-1)$- and n-skeleta

$$\begin{tikzcd} \coprod_{\alpha(n)} \partial\Delta[n] \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}(X/G) \arrow[d] \\ \coprod_{\alpha(n)} \Delta[n] \arrow[r] & \mathrm{sk}_n(X/G), \end{tikzcd}$$

where $\alpha(n)$ indexes the non-degenerate n-simplices of X/G. Each term in this diagram has a natural map to X/G, so by pulling back along $X \to X/G$ and invoking universality of colimits in sSet we obtain a pushout diagram

$$\begin{tikzcd} \coprod_{\alpha(n)} G \times \partial\Delta[n] \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ \coprod_{\alpha(n)} G \times \Delta[n] \arrow[r] & X_n. \end{tikzcd}$$

In this last step, we have used the fact that for $x\colon \Delta[n] \to X/G$ an n-simplex of the quotient, a choice of lift of x to X induces an isomorphism $x^*(X \to X/G) \cong G \times \Delta[n]$ since X_n is a free G_n-set. Our argument shows that for X a levelwise free G-set we have a presentation $X = \mathrm{colim}_n X_n$ as a $\mathcal{I}^G_{\mathrm{Kan}}$-cell complex, so that X is cofibrant. □

Lemma B.1.3. *Suppose that $f\colon Y \to X$ is a morphism of G-spaces where X is cofibrant. If the induced map on the quotients $Y/G \to X/G$ is an isomorphism then f is an isomorphism.*

Proof. This is easy to see directly, using the characterisation of cofibrant G-spaces of Lemma B.1.2. □

We now recall the adjunction

$$\begin{tikzcd} \mathrm{sSet}_0 \arrow[r, shift left, "\mathbb{G}"] \arrow[r, phantom, "\perp"] & \mathrm{sGrp}, \arrow[l, shift left, "\overline{W}"] \end{tikzcd}$$

originally due to Kan [Kan58], which provides simplicial models $\mathbb{G}$ and $\overline{W}$ for the loopspace and classifying space functors respectively. Along the way, we recall various classical results related to these adjoint functors.

Construction B.1.4. For a simplicial group G, the reduced simplicial set $\overline{W}G$ has n-simplices

$$\overline{W}G_n := G_{n-1} \times \cdots \times G_0$$

for $n > 0$. Face and degeneracy maps are defined by the formulae

$$d_i(g_{n-1}, \ldots, g_0) := \begin{cases} (g_{n-2}, \ldots, g_0) & i = 0 \\ (d_{i-1}g_{n-1}, \ldots, d_1 g_{n-i}, d_0 g_{n-i-1} \cdot g_{n-i-2}, g_{n-i-3}, \ldots, g_0) & i > 0 \end{cases}$$

$$s_i(g_{n-2}, \ldots, g_0) := \begin{cases} (e, g_{n-2}, \ldots, g_0) & i = 0 \\ (s_{i-1}g_{n-2}, \ldots, s_0 g_{n-i-1}, e, g_{n-i-2} \ldots, g_0) & i > 0, \end{cases}$$

where e always denotes the group unit. This assignment extends to a functor in the obvious way.

Projection to the first factor gives rise to a *twisting function* $\tau_G\colon \overline{W}G_n \to G_{n-1}$. This allows us to construct the *twisted Cartesian product* $WG := G \times_\tau \overline{W}G$ whose underlying simplicial set is $WG_n := G_n \times \overline{W}G_n$, with face and degeneracy maps

$$d_i(g_n, (g_{n-1}, \ldots, g_0)) := \begin{cases} (d_0(g_n) \cdot g_{n-1}, (g_{n-1}, \ldots, g_0)) & i = 0 \\ (d_i(g_n), d_i(g_{n-1}, \ldots, g_0)) & i > 0 \end{cases}$$
$$s_i(g_{n-1}, (g_{n-2}, \ldots, g_0)) := (s_i(g_{n-1}), s_i(g_{n-2}, \ldots, g_0)),$$

noting that $g_{n-1} = \tau_G(g_{n-1}, \ldots, g_0)$ in the formula for d_0. There is a left action of G on WG by left multiplication on the first factor.

Lemma B.1.5. *WG is a contractible, cofibrant G-space. The quotient map $WG \to WG/G = \overline{W}G$ is a fibration.*

Proof. Cofibrancy is immediate from the construction and Lemma B.1.2. Moreover, the assignment

$$(g_n, (g_{n-1}, \ldots, g_0)) \longmapsto (e, (g_n, g_{n-1}, \ldots, g_0))$$

defines an extra degeneracy on WG, from which we conclude that WG is contractible (compare [GJ09, §III.5]). To complete the proof, recall that the underlying simplicial set of a simplicial group is a Kan complex by Moore's Theorem ([GJ09, Lemma I.3.4]). A straightforward argument using this and Lemma B.1.3 shows that the quotient map $X \to X/G$ is a fibration for any cofibrant G-space X. □

Corollary B.1.6. *The homotopy fibre of $WG \to \overline{W}G$ is G.*

Construction B.1.7. For a reduced simplicial set X, we define the simplicial group $\mathbb{G}X$ levelwise as the $\mathbb{G}X_n := F(X_{n+1} \setminus s_0(X_n))$, where F is the free group functor and $s_0 \colon X_n \to X_{n+1}$ is the zeroth degeneracy map. For $\theta \colon [m] \to [n]$ a morphism in Δ, we take $\tilde{\theta} \colon [m+1] \to [n+1]$ to be given by

$$\tilde{\theta}(i) := \begin{cases} 0 & \text{if } i = 0 \\ \theta(i-1)+1 & \text{if } i \geq 1 \end{cases}$$

and $c\theta = \sigma_0\tilde{\theta} \colon [m+1] \to [n]$, with $\sigma_0 \colon [n+1] \to [n]$ the zeroth co-degeneracy. We then define the corresponding simplicial structure map $\theta^* \colon \mathbb{G}X_n \to \mathbb{G}X_m$ on generators $\langle x \rangle$ for $x \in X_{n+1} \setminus s_0(X_n)$ by $\theta^*(\langle x \rangle) := \langle \tilde{\theta}^* x \rangle \cdot \langle c\theta^*(d_0 x) \rangle^{-1}$. A direct check verifies that we obtain in this manner a functor $\mathbb{G}X \colon \Delta^{\mathrm{op}} \to \mathrm{Grp}$, and hence a simplicial group. The assignment $X \mapsto \mathbb{G}X$ extends to a functor $\mathrm{sSet} \to \mathrm{sGrp}$ in the obvious fashion. There is a natural transformation $\eta \colon \mathrm{id}_{\mathrm{sSet}} \Rightarrow \overline{W}\mathbb{G}$ whose component at X is determined on n-simplices as

$$x \longmapsto \big(\langle x \rangle, \langle d_0 x \rangle, \ldots, \langle d_0^{n-1} x \rangle\big),$$

where we take all factors to be the identity if $x \in s_0(X_{n-1})$.

Theorem B.1.8. *The natural transformation η is the unit of a Quillen adjunction*

$$\begin{tikzcd} \mathrm{sSet}_0 \arrow[r, shift left, "\mathbb{G}"] \arrow[r, phantom, "\perp"] & \mathrm{sGrp}. \arrow[l, shift left, "\overline{W}"] \end{tikzcd}$$

A morphism $f \colon \mathbb{G}X \to G$ is a weak equivalence precisely if its adjunct $f^\vee \colon X \to \overline{W}G$ is. In particular, $(\mathbb{G} \dashv \overline{W})$ is a Quillen equivalence.

Proof. This is proven as [GJ09, Proposition III.6.3]. □

Construction B.1.9. Let $\mathbb{P}X$ denote the pullback

$$\begin{tikzcd} \mathbb{P}X \arrow[r] \arrow[d, "\pi_X"'] & W\mathbb{G}X \arrow[d] \\ X \arrow[r, "\eta_X"] & \overline{W}\mathbb{G}X, \end{tikzcd}$$

so that the induced projection $\mathbb{P}X \to X$ is a fibration with homotopy fibre $\mathbb{G}X$. By right properness of sSet, the map $\mathbb{P}X \to W\mathbb{G}X$ is a weak equivalence so that $\mathbb{P}X$ is weakly contractible. This justifies the notation: $\mathbb{P}X \to X$ is a particular simplicial model for the path fibration.

The space $\mathbb{P}X$ inherits a $\mathbb{G}X$-action from $W\mathbb{G}X$ and the defining pullback diagram becomes a pullback diagram of $\mathbb{G}X$-spaces, where the spaces in the bottom row are equipped with the trivial action. It is immediate that $\mathbb{P}X$ is a levelwise free $\mathbb{G}X$-space, so is cofibrant by Lemma B.1.2, and moreover that $\mathbb{P}X/\mathbb{G}X = X$.

B.2 Products of G-Spaces

In this section, we show that the model category of G-spaces of Theorem B.1.1 is a symmetric monoidal model category with respect to the product of G-spaces. This is surely well-known, but the author was not able to find a reference.

Lemma B.2.1. *The product of free G-spaces is a free G-space.*

Proof. For free G-spaces $G \times K$ and $G \times L$, their product in $G-\mathrm{sSet}$ is $G \times G \times K \times L$ equipped with the diagonal G-action. But $G^2 \cong G \times \Delta_G$ as G-spaces, where $\Delta_G = (G \times G)/G$ is the quotient by the diagonal action. Hence $(G \times K) \times (G \times L) \cong G \times (\Delta_G \times K \times L)$ is a free G-space. □

Lemma B.2.2. *For G a simplicial group, $G-\mathrm{sSet}$ is a symmetric monoidal model category with respect to the categorical product.*

Proof. By the usual argument, the categorical product is a symmetric monoidal structure with unit the G-space $*$ equipped with the trivial G-action. The resulting bifunctor $G-\mathrm{sSet} \times G-\mathrm{sSet} \to G-\mathrm{sSet}$ preserves colimits in both arguments, so by the Adjoint Functor Theorem is part of an adjunction of two variables.

It remains to check that the categorical product is compatible with the model structure. To verify the unit axiom, we use the cofibrant replacement $WG \to *$. For any G-space X, the induced map $WG \times X \to * \times X \cong X$ is then a weak equivalence.

We now turn to verification of the pushout-product axiom. Fixing m and n, via the usual description of the non-degenerate simplices of $\Delta[m] \times \Delta[n]$ in terms of (m,n)-shuffles, we write

$$(\partial\Delta[n] \times \Delta[m]) \coprod_{(\partial\Delta[n] \times \partial\Delta[m])} (\Delta[n] \times \partial\Delta[m]) \longrightarrow \Delta[m] \times \Delta[n]$$

as a finite sequence of attachments of simplices $\Delta[k]$ along their boundaries $\partial\Delta[k]$. The pushout-product of the generating cofibrations $(G \times i_n) \,\square\, (G \times i_m)$ is the map of G-spaces

$$(G^2 \times \partial\Delta[m] \times \Delta[n]) \coprod_{(G^2 \times \partial\Delta[m] \times \partial\Delta[n])} (G^2 \times \Delta[m] \times \partial\Delta[n]) \longrightarrow G^2 \times \Delta[m] \times \Delta[n],$$

which via the shuffle decomposition can be decomposed as a finite sequence of pushouts along maps of the form

$$G \times \Delta_G \times \partial\Delta[k] \longrightarrow G \times \Delta_G \times \Delta[k],$$

using the isomorphism of G-spaces $G \times G \cong G \times \Delta_G$ of Lemma B.2.1. Since such maps are cofibrations of G-spaces, we find that $\mathcal{I}^G_{\mathrm{Kan}} \,\square\, \mathcal{I}^G_{\mathrm{Kan}}$ consists of cofibrations. Hence, by cofibrant generation, the pushout-product of any two cofibrations is a cofibration. In particular, if X is a cofibrant G-space, the endofunctor $X \times (-)$ on $G-\mathrm{sSet}$ preserves cofibrations.

Finally, take $i = G \times i_n \in \mathcal{I}^G_{\mathrm{Kan}}$ and $j = G \times h^m_k \in \mathcal{J}^G_{\mathrm{Kan}}$ and consider the diagram defining the pushout-product:

$$\begin{tikzcd}
(G \times \partial\Delta[n]) \times (G \times \Lambda^m_k) \arrow[r] \arrow[d, "\alpha"'] & (G \times \Delta[n]) \times (G \times \Lambda^m_k) \arrow[d, "\beta"'] \arrow[dr, "\gamma"] & \\
(G \times \partial\Delta[n]) \times (G \times \Delta[k]) \arrow[r] & P \arrow[r, "i \square j"] & (G \times \Delta[n]) \times (G \times \Delta[m]),
\end{tikzcd}$$

where P denotes the pushout. The morphisms marked α and γ are cofibrations by the above and are acyclic by inspection. Acyclic cofibrations are stable under pushouts so that β is an acyclic cofibration too. By the 2-out-of-3 property, it follows that $i \,\square\, j$ is an acyclic cofibration. The pushout-product axiom now follows from symmetry and once more invoking cofibrant generation. □

Bibliography

[AR94] J. Adámek and J. Rosický. *Locally presentable and accessible categories*. Vol. 189. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1994, pp. xiv+316. ISBN: 0-521-42261-2. URL: https://doi.org/10.1017/CBO9780511600579.

[And+14] M. Ando, A. J. Blumberg, D. Gepner, Hopkins, M. J., and C. Rezk. "An ∞-categorical approach to R-line bundles, R-module Thom spectra, and twisted R-homology". In: *J. Topol.* 7.3 (2014), pp. 869–893. ISSN: 1753-8416. URL: https://doi.org/10.1112/jtopol/jtt035.

[AR12] C. Ausoni and J. Rognes. "Rational algebraic K-theory of topological K-theory". In: *Geom. Topol.* 16.4 (2012), pp. 2037–2065. ISSN: 1465-3060. URL: https://doi.org/10.2140/gt.2012.16.2037.

[Bar10] C. Barwick. "On left and right model categories and left and right Bousfield localizations". In: *Homology Homotopy Appl.* 12.2 (2010), pp. 245–320. ISSN: 1532-0073. URL: http://projecteuclid.org/euclid.hha/1296223884.

[BK12] C. Barwick and D. M. Kan. "Relative categories: another model for the homotopy theory of homotopy theories". In: *Indag. Math. (N.S.)* 23.1-2 (2012), pp. 42–68. ISSN: 0019-3577. URL: https://doi.org/10.1016/j.indag.2011.10.002.

[Bay+15] M. Bayeh, K. Hess, V. Karpova, M. Kedziorek, E. Riehl, and B. Shipley. "Left-induced model structures and diagram categories". In: *Women in topology: collaborations in homotopy theory*. Vol. 641. Contemp. Math. Amer. Math. Soc., Providence, RI, 2015, pp. 49–81. URL: https://doi.org/10.1090/conm/641/12859.

[Bor94] F. Borceux. *Handbook of categorical algebra. 2*. Vol. 51. Encyclopedia of Mathematics and its Applications. Categories and structures. Cambridge University Press, Cambridge, 1994, pp. xviii+443. ISBN: 0-521-44179-X.

[Bor48] K. Borsuk. "On the imbedding of systems of compacta in simplicial complexes". In: *Fund. Math.* 35 (1948), pp. 217–234. ISSN: 0016-2736. URL: https://doi.org/10.4064/fm-35-1-217-234.

[BF78] A. K. Bousfield and E. M. Friedlander. "Homotopy theory of Γ-spaces, spectra, and bisimplicial sets". In: *Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II*. Vol. 658. Lecture Notes in Math. Springer, Berlin, 1978, pp. 80–130.

[BG76] A. K. Bousfield and V. K. A. M. Gugenheim. "On PL de Rham theory and rational homotopy type". In: *Mem. Amer. Math. Soc.* 8.179 (1976), pp. ix+94. ISSN: 0065-9266. URL: https://doi.org/10.1090/memo/0179.

[BN14] U. Bunke and T. Nikolaus. "Twisted differential cohomology". arXiv:1406.3231. 2014.

[BNV16] U. Bunke, T. Nikolaus, and M. Völkl. "Differential cohomology theories as sheaves of spectra". In: *J. Homotopy Relat. Struct.* 11.1 (2016), pp. 1–66. ISSN: 2193-8407. URL: https://doi.org/10.1007/s40062-014-0092-5.

[BS12] U. Bunke and T. Schick. "Differential K-theory: a survey". In: *Global differential geometry*. Vol. 17. Springer Proc. Math. Springer, Heidelberg, 2012, pp. 303–357. URL: https://doi.org/10.1007/978-3-642-22842-1_11.

[CR12] D. Carchedi and D. Roytenberg. "Homological algebra for superalgebras of differentiable functions". arXiv:1212.3745. 2012.

[CS85] J. Cheeger and J. Simons. "Differential characters and geometric invariants". In: *Geometry and topology (College Park, Md., 1983/84)*. Vol. 1167. Lecture Notes in Math. Springer, Berlin, 1985, pp. 50–80. URL: https://doi.org/10.1007/BFb0075216.

[CR14] M. Ching and E. Riehl. "Coalgebraic models for combinatorial model categories". In: *Homology Homotopy Appl.* 16.2 (2014), pp. 171–184. ISSN: 1532-0073. URL: https://doi.org/10.4310/HHA.2014.v16.n2.a9.

[Dug01a] D. Dugger. "Combinatorial model categories have presentations". In: *Adv. Math.* 164.1 (2001), pp. 177–201. ISSN: 0001-8708. URL: https://doi.org/10.1006/aima.2001.2015.

[Dug01b] D. Dugger. "Replacing model categories with simplicial ones". In: *Trans. Amer. Math. Soc.* 353.12 (2001), pp. 5003–5027. ISSN: 0002-9947. URL: https://doi.org/10.1090/S0002-9947-01-02661-7.

[Dug01c] D. Dugger. "Universal homotopy theories". In: *Adv. Math.* 164.1 (2001), pp. 144–176. ISSN: 0001-8708. URL: https://doi.org/10.1006/aima.2001.2014.

[DHI04] D. Dugger, S. Hollander, and D. Isaksen. "Hypercovers and simplicial presheaves". In: *Math. Proc. Cambridge Philos. Soc.* 136.1 (2004), pp. 9–51. ISSN: 0305-0041. URL: https://doi.org/10.1017/S0305004103007175.

[FSS17a] D. Fiorenza, H. Sati, and U. Schreiber. "Rational sphere valued supercocycles in M-theory and type IIA string theory". In: *J. Geom. Phys.* 114 (2017), pp. 91–108. ISSN: 0393-0440. URL: https://doi.org/10.1016/j.geomphys.2016.11.024.

[FSS17b] D. Fiorenza, H. Sati, and U. Schreiber. "T-Duality from super Lie n-algebra cocycles for super p-branes". Manuscript at https://ncatlab.org/schreiber/show/T-Duality+from+super+Lie+n-algebra+cocycles+for+super+p-branes (second version). 2017.

[FSS12] D. Fiorenza, U. Schreiber, and J. Stasheff. "Čech cocycles for differential characteristic classes: an ∞-Lie theoretic construction". In: *Adv. Theor. Math. Phys.* 16.1 (2012), pp. 149–250. ISSN: 1095-0761. URL: http://projecteuclid.org/euclid.atmp/1358950853.

[Get09] E. Getzler. "Lie theory for nilpotent L_∞-algebras". In: *Ann. of Math. (2)* 170.1 (2009), pp. 271–301. ISSN: 0003-486X. URL: https://doi.org/10.4007/annals.2009.170.271.

[GJ09] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory*. Modern Birkhäuser Classics. Reprint of the 1999 edition. Birkhäuser Verlag, Basel, 2009, pp. xvi+510. ISBN: 978-3-0346-0188-7. URL: https://doi.org/10.1007/978-3-0346-0189-4.

[GS17] D. Grady and H. Sati. "Twisted differential generalized cohomology theories and their Atiyah-Hirzebruch spectral sequence". arXiv:1711.06650. 2017.

[HP15] Y. Harpaz and M. Prasma. "The Grothendieck construction for model categories". In: *Adv. Math.* 281 (2015), pp. 1306–1363. ISSN: 0001-8708. URL: https://doi.org/10.1016/j.aim.2015.03.031.

[Hen08] A. Henriques. "Integrating L_∞-algebras". In: *Compos. Math.* 144.4 (2008), pp. 1017–1045. ISSN: 0010-437X. URL: https://doi.org/10.1112/S0010437X07003405.

[HS16] K. Hess and B. Shipley. "Waldhausen K-theory of spaces via comodules". In: *Adv. Math.* 290 (2016), pp. 1079–1137. ISSN: 0001-8708. URL: https://doi.org/10.1016/j.aim.2015.12.019.

[Hin97] V. Hinich. "Descent of Deligne groupoids". In: *Internat. Math. Res. Notices* 5 (1997), pp. 223–239. ISSN: 1073-7928. URL: https://doi.org/10.1155/S1073792897000160.

[Hin16] V. Hinich. "Dwyer-Kan localization revisited". In: *Homology Homotopy Appl.* 18.1 (2016), pp. 27–48. ISSN: 1532-0073. URL: https://doi.org/10.4310/HHA.2016.v18.n1.a3.

[Hir03] P. S. Hirschhorn. *Model categories and their localizations*. Vol. 99. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003, pp. xvi+457. ISBN: 0-8218-3279-4.

[Hir05] P. S. Hirschhorn. *Overcategories and undercategories of model categories*. Note available online at http://www-math.mit.edu/~psh/undercat.pdf. 2005.

[HS05] M. J. Hopkins and I. M. Singer. "Quadratic functions in geometry, topology, and M-theory". In: *J. Differential Geom.* 70.3 (2005), pp. 329–452. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1143642908.

[Hov99] M. Hovey. *Model categories*. Vol. 63. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1999, pp. xii+209. ISBN: 0-8218-1359-5.

[Hov01] M. Hovey. "Spectra and symmetric spectra in general model categories". In: *J. Pure Appl. Algebra* 165.1 (2001), pp. 63–127. ISSN: 0022-4049. URL: https://doi.org/10.1016/S0022-4049(00)00172-9.

[HSS00] M. Hovey, B. Shipley, and J. Smith. "Symmetric spectra". In: *J. Amer. Math. Soc.* 13.1 (2000), pp. 149–208. ISSN: 0894-0347. URL: https://doi.org/10.1090/S0894-0347-99-00320-3.

[Kan58] D. M. Kan. "On homotopy theory and c.s.s. groups". In: *Ann. of Math. (2)* 68 (1958), pp. 38–53. ISSN: 0003-486X. URL: https://doi.org/10.2307/1970042.

[KMS93] I. Kolář, P. W. Michor, and J. Slovák. *Natural operations in differential geometry*. Springer-Verlag, Berlin, 1993, pp. vi+434. ISBN: 3-540-56235-4. URL: https://doi.org/10.1007/978-3-662-02950-3.

[LS69] R. G. Larson and M. E. Sweedler. "An associative orthogonal bilinear form for Hopf algebras". In: *Amer. J. Math.* 91 (1969), pp. 75–94. ISSN: 0002-9327. URL: https://doi.org/10.2307/2373270.

[Lur09] J. Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. URL: https://doi.org/10.1515/9781400830558.

[Lur11] J. Lurie. "Derived Algebraic Geometry VII: Spectral Schemes". Manuscript at http://www.math.harvard.edu/~lurie/ (version of 5.11.2011). 2011.

[Lur17] J. Lurie. "Higher algebra". Manuscript at http://www.math.harvard.edu/~lurie/ (version of 18.09.2017). 2017.

[May67] J. P. May. *Simplicial objects in algebraic topology*. Van Nostrand Mathematical Studies, No. 11. D. Van Nostrand Co., Princeton, N.J.-Toronto, Ont.-London, 1967, pp. vi+161.

[MS06] J. P. May and J. Sigurdsson. *Parametrized homotopy theory*. Vol. 132. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2006, pp. x+441. ISBN: 978-0-8218-3922-5; 0-8218-3922-5. URL: https://doi.org/10.1090/surv/132.

[McC01] J. McCleary. *A user's guide to spectral sequences*. Second. Vol. 58. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2001, pp. xvi+561. ISBN: 0-521-56759-9.

[Nes03] J. Nestruev. *Smooth manifolds and observables*. Vol. 220. Graduate Texts in Mathematics. Springer-Verlag, New York, 2003, pp. xiv+222. ISBN: 0-387-95543-7.

[Pos11] L. Positselski. "Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence". In: *Mem. Amer. Math. Soc.* 212.996 (2011), pp. vi+133. ISSN: 0065-9266. URL: https://doi.org/10.1090/S0065-9266-2010-00631-8.

[Qui67] D. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967, iv+156 pp. (not consecutively paged).

[Qui69] D. Quillen. "Rational homotopy theory". In: *Ann. of Math. (2)* 90 (1969), pp. 205–295. ISSN: 0003-486X. URL: https://doi.org/10.2307/1970725.

[Roi94a] A. Roig. "Formalizability of dg modules and morphisms of cdg algebras". In: *Illinois J. Math.* 38.3 (1994), pp. 434–451. ISSN: 0019-2082. URL: http://projecteuclid.org/euclid.ijm/1255986724.

[Roi94b] A. Roig. "Model category structures in bifibred categories". In: *Journal of Pure and Applied Algebra* 95.2 (1994), pp. 203–223. ISSN: 0022-4049. URL: https://doi.org/10.1016/0022-4049(94)90074-4.

[RSA00] A. Roig and M. Saralegi-Aranguren. "Minimal models for non-free circle actions". In: *Illinois J. Math.* 44.4 (2000), pp. 784–820. ISSN: 0019-2082. URL: http://projecteuclid.org/euclid.ijm/1255984692.

[Sch17] U. Schreiber. "Differential cohomology in a cohesive ∞-topos". Manuscript at https://ncatlab.org/schreiber/show/differential+cohomology+in+a+cohesive+topos (second version). 2017.

[Sch12] S. Schwede. "Symmetric spectra". Manuscript available at http://www.math.uni-bonn.de/people/schwede/SymSpec-v3.pdf (version of 12.04.2012). 2012.

[SS00] S. Schwede and B. Shipley. "Algebras and modules in monoidal model categories". In: *Proc. London Math. Soc. (3)* 80.2 (2000), pp. 491–511. ISSN: 0024-6115. URL: https://doi.org/10.1112/S002461150001220X.

[SS03a] S. Schwede and B. Shipley. "Equivalences of monoidal model categories". In: *Algebr. Geom. Topol.* 3 (2003), pp. 287–334. ISSN: 1472-2747. URL: https://doi.org/10.2140/agt.2003.3.287.

[SS03b] S. Schwede and B. Shipley. "Stable model categories are categories of modules". In: *Topology* 42.1 (2003), pp. 103–153. ISSN: 0040-9383. URL: https://doi.org/10.1016/S0040-9383(02)00006-X.

[Seg07] Graeme Segal. "What is an elliptic object?" In: *Elliptic cohomology*. Vol. 342. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2007, pp. 306–317. URL: https://doi.org/10.1017/CBO9780511721489.016.

[Shi04] B. Shipley. “A convenient model category for commutative ring spectra”. In: *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*. Vol. 346. Contemp. Math. Amer. Math. Soc., Providence, RI, 2004, pp. 473–483. URL: https://doi.org/10.1090/conm/346/06300.

[Shi07] B. Shipley. “$H\mathbb{Z}$-algebra spectra are differential graded algebras”. In: *Amer. J. Math.* 129.2 (2007), pp. 351–379. ISSN: 0002-9327. URL: https://doi.org/10.1353/ajm.2007.0014.

[Spa95] E. H. Spanier. *Algebraic topology*. Corrected reprint of the 1966 original. Springer-Verlag, New York, 1995, pp. xvi+528. ISBN: 0-387-94426-5.

[Wei94] C. A. Weibel. *An introduction to homological algebra*. Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994, pp. xiv+450. ISBN: 0-521-43500-5; 0-521-55987-1. URL: https://doi.org/10.1017/CBO9781139644136.

[Zee57] E. C. Zeeman. “A proof of the comparison theorem for spectral sequences”. In: *Proc. Cambridge Philos. Soc.* 53 (1957), pp. 57–62.