Towards verified hardware-aware Topological Quantum Programming

Urs Schreiber @ CQTS CENTER FOR QUANTUM & TOPOLOGICAL SYSTEMS

presenting at:

CQTS and TII Workshop

NYU Abu Dhabi, 24 Feb 2023

slides and references at: ncatlab.org/schreiber/show/TQP#TalkForTII

Real quantum computation

$\underbrace{\text{scalable, universal, reliable}}$

Real quantum computation

scalable, universal, reliable

Real quantum computation will require:

scalable, universal, reliable

Real quantum computation will require

scalable, universal, reliable

Real quantum computation will require

stabilization

$\underbrace{\text{scalable, universal, reliable}}$

Real quantum computation will require

stabilization

compilation

$\underbrace{\text{Real}}_{}$

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Real quantum computation will require

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$\overbrace{\text{Real}}^{\text{scalable, universal, reliable}}$ quantum computation will require

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Real quantum computation will require

stabilization ↔ *topological* gates

compilation

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scalable, universal, reliable

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> “*small* [NISQ] *machines are unlikely to uncover truly macroscopic quantum phenomena, which have no classical analogs.*
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J. Sau: *Roadmap for Scalable Topological Quantum Computers*
Physics **10** (2017) 68
`[physics.aps.org/articles/v10/68]`

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MIT Technology Review (March 2022)
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> *“The qubit systems we have today are a tremendous scientific achievement, but they take us no closer to having a quantum computer that can solve a problem that anybody cares about.* [...] *What is missing is the breakthrough* [...] *bypassing quantum error correction by using far-more-stable qubits, in an approach called topological quantum computing.”*

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“Why did discovering quantum teleportation take 60 years?

Bob Coecke: *Kindergarten QM* `[arXiv:quant-ph/0510032]`
Quantum Picturalism `[arXiv:0908.1787]`

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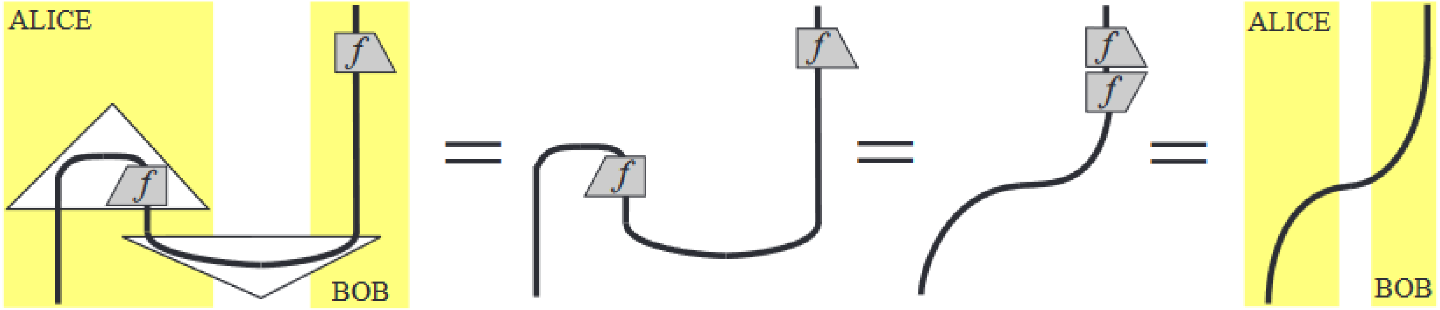
> *“Why did discovering quantum teleportation take 60 years?*
> *We claim that this is due to a ‘bad quantum formalism’*
> *I claim that a good formalism exists:* [linear circuit logic]



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topological gate set is highly constrained,

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⇒ topological quantum compilation intricate

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to declare the **data type** of all data

scalable, universal, reliable

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to declare the **data type** of all data
formally specifying the admissible
data construction and behaviour

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to declare the **data type** of all data
formally specifying the admissible
data construction and behaviour
(aka: “formal methods”)

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> *“We argue that quantum programs demand machine-checkable proofs of correctness. We justify this on the basis of*

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> **Thesis Statement**:
>
> Quantum programming is not only amenable to formal verification: it demands it.”

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all the more for topological quantum computing:
due to exotic gates in complex & unituitive circuits

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but existing quantum circuit verification languages
such as `QWIRE` or `Quipper`
lack support for topological gates

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Real quantum programming is

scalable, universal, reliable

Real quantum computation will require

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Real quantum programming is *homotopy*

scalable, universal, reliable

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Real quantum programming is *linear homotopy*

scalable, universal, reliable

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Real quantum programming is *linear homotopy typed* programming.

Foundations Project
@ CQTS

scalable, universal, reliable

Part I

Verifying realistic topological quantum gates

Real quantum programming is *linear homotopy typed* programming.

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Part I

Verifying realistic topological quantum gates

Part II

Verifying their compilation into quantum circuits

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Part I
Verifying realistic topological quantum gates

Part II
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Towards verifying realistic topological quantum gates

Taken at traditional face value,
formally specifying & certifying
realistic topological quantum gates

seems a formidable task and
grossly inefficient even if possible.

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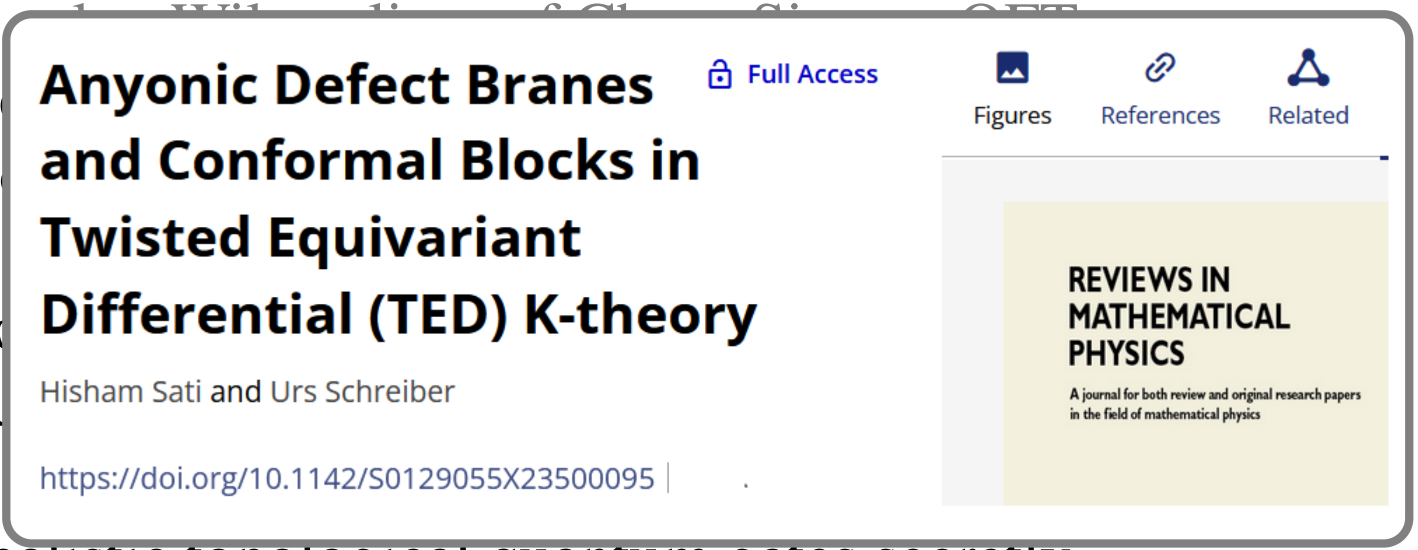

Anyonic Defect Branes and Conformal Blocks in Twisted Equivariant Differential (TED) K-theory

Hisham Sati and Urs Schreiber

https://doi.org/10.1142/S0129055X23500095

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A journal for both review and original research papers in the field of mathematical physics

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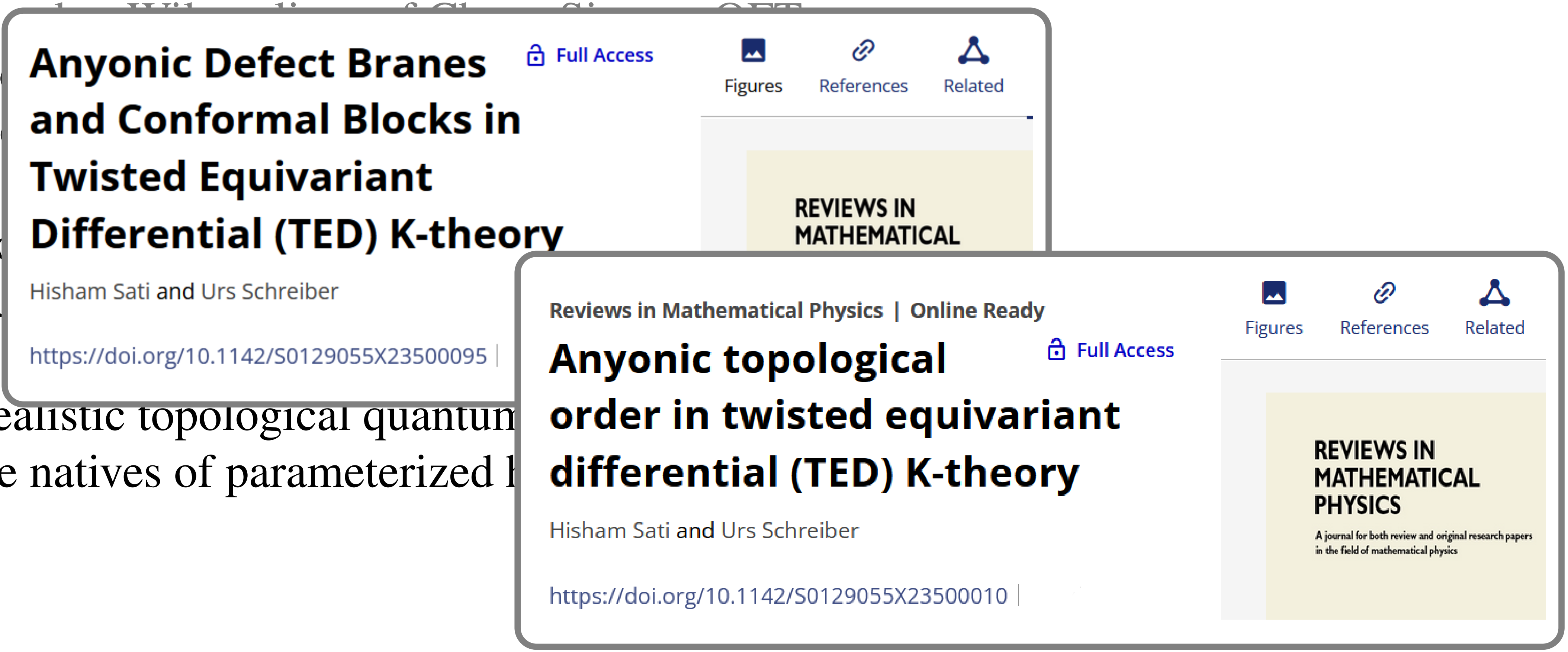

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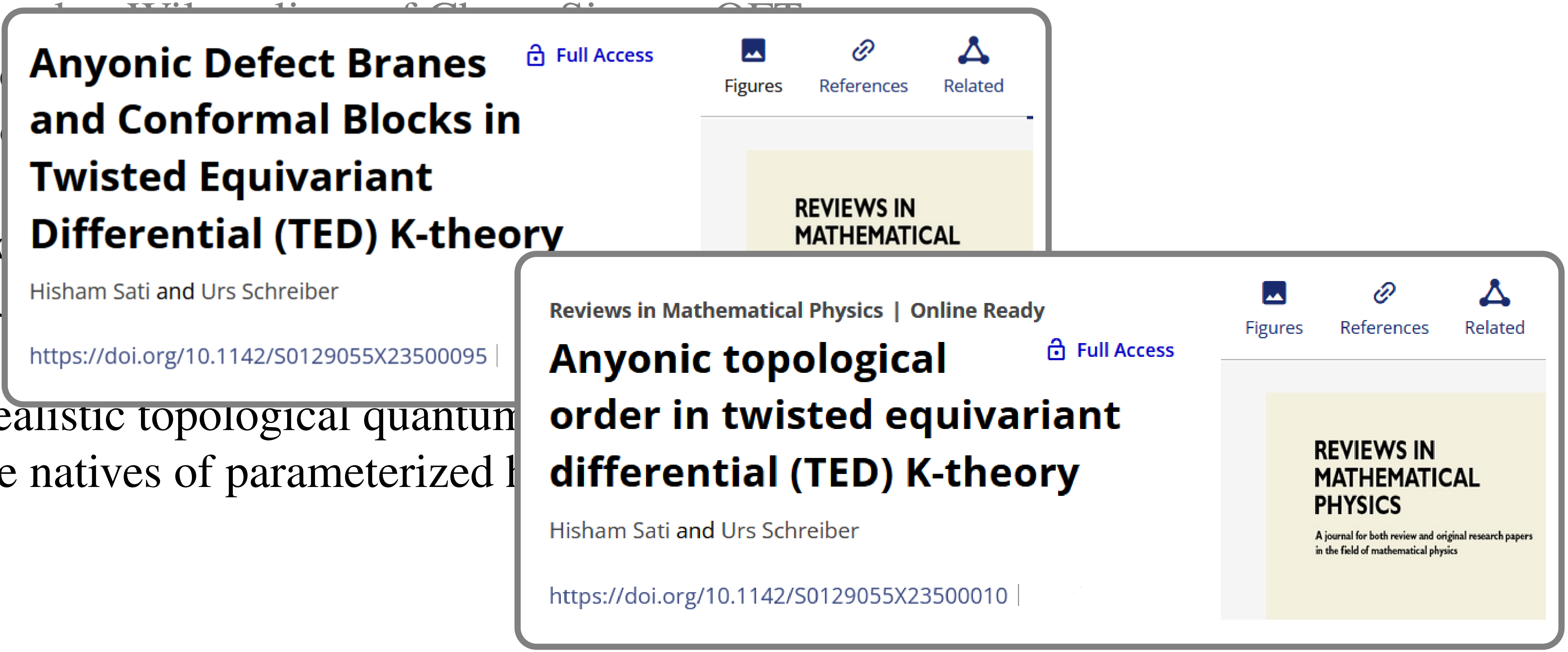

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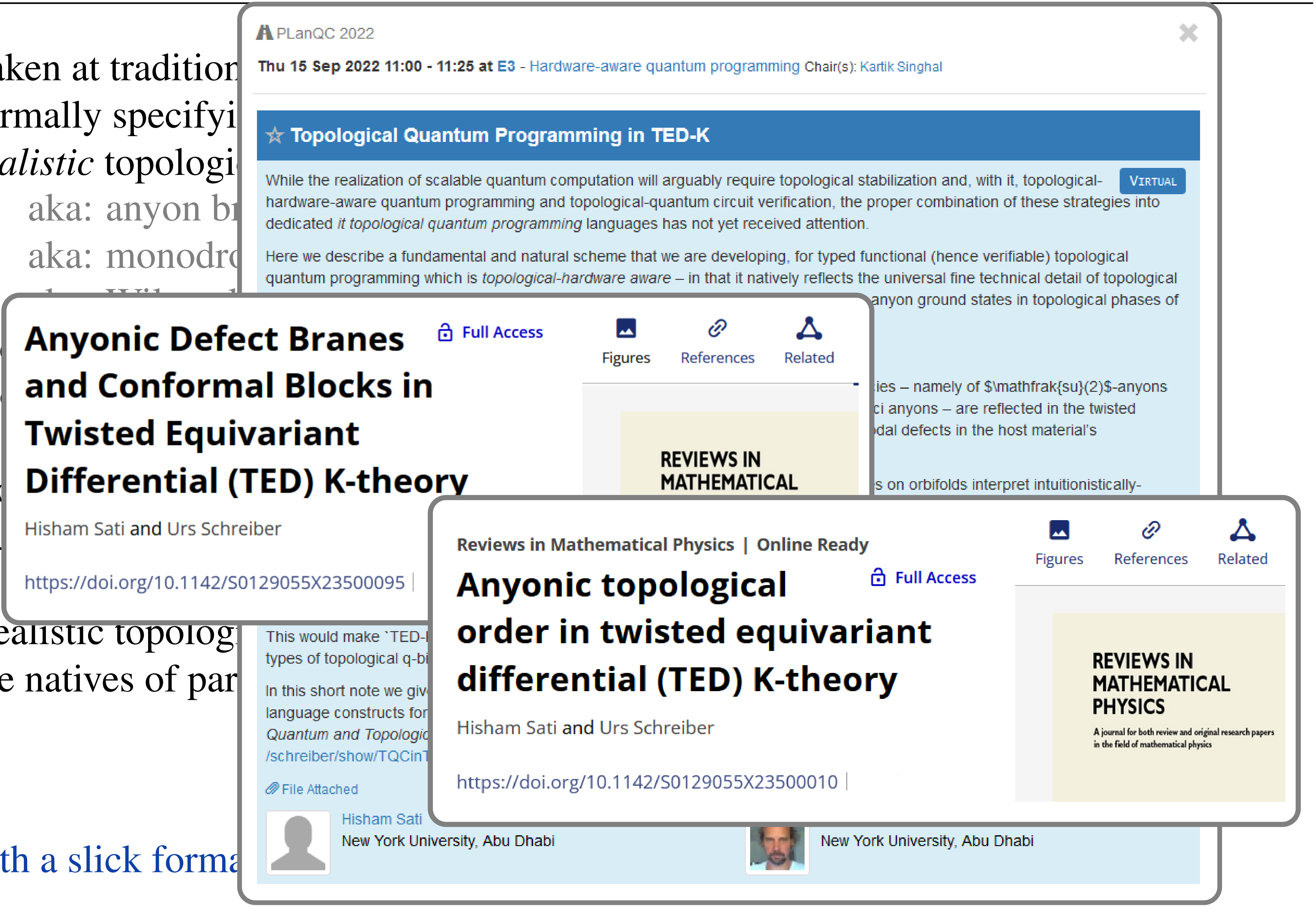

PLanQC 2022

Thu 15 Sep 2022 11:00 - 11:25 at E3 - Hardware-aware quantum programming Chair(s): Kartik Singhal

☆ Topological Quantum Programming in TED-K

VIRTUAL

While the realization of scalable quantum computation will arguably require topological stabilization and, with it, topological-hardware-aware quantum programming and topological-quantum circuit verification, the proper combination of these strategies into dedicated *it topological quantum programming* languages has not yet received attention.

Here we describe a fundamental and natural scheme that we are developing, for typed functional (hence verifiable) topological quantum programming which is *topological-hardware aware* – in that it natively reflects the universal fine technical detail of topological

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File Attached

Hisham Sati

New York University, Abu Dhabi

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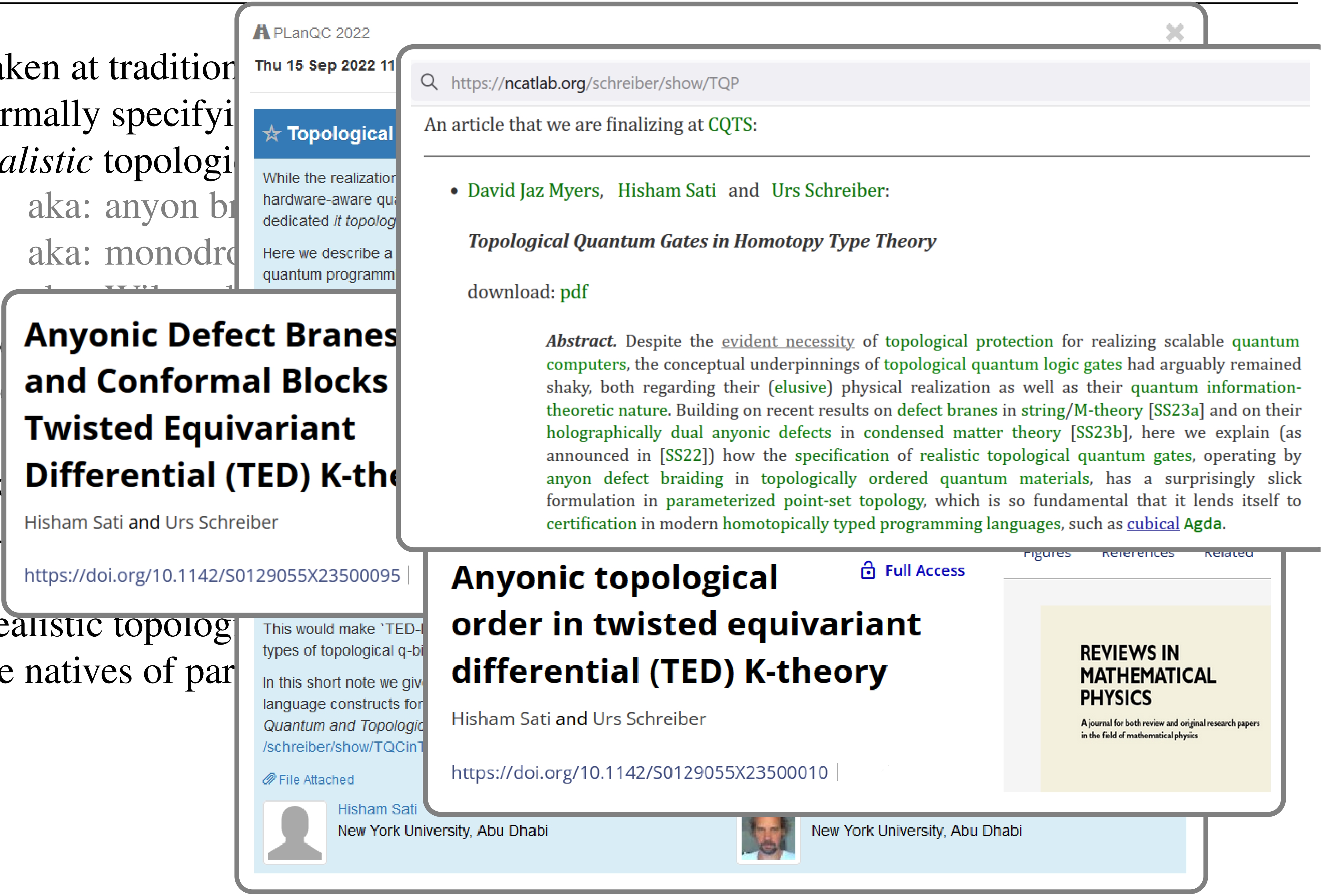

PLanQC 2022

Thu 15 Sep 2022 11

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https://ncatlab.org/schreiber/show/TQP

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- David Jaz Myers, Hisham Sati and Urs Schreiber:

  Topological Quantum Gates in Homotopy Type Theory

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  Abstract. Despite the evident necessity of topological protection for realizing scalable quantum computers, the conceptual underpinnings of topological quantum logic gates had arguably remained shaky, both regarding their (elusive) physical realization as well as their quantum information-theoretic nature. Building on recent results on defect branes in string/M-theory [SS23a] and on their holographically dual anyonic defects in condensed matter theory [SS23b], here we explain (as announced in [SS22]) how the specification of realistic topological quantum gates, operating by anyon defect braiding in topologically ordered quantum materials, has a surprisingly slick formulation in parameterized point-set topology, which is so fundamental that it lends itself to certification in modern homotopically typed programming languages, such as cubical Agda.

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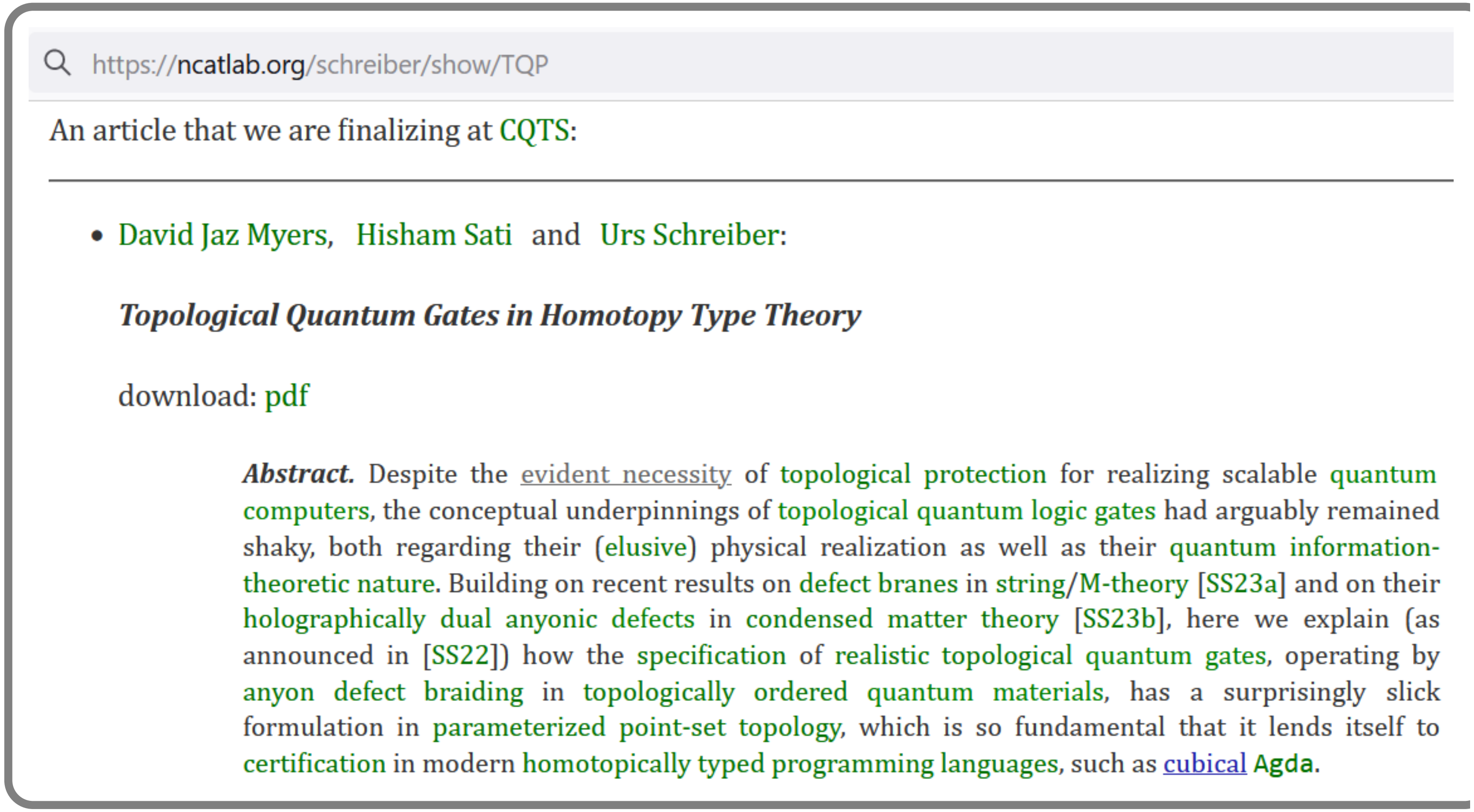

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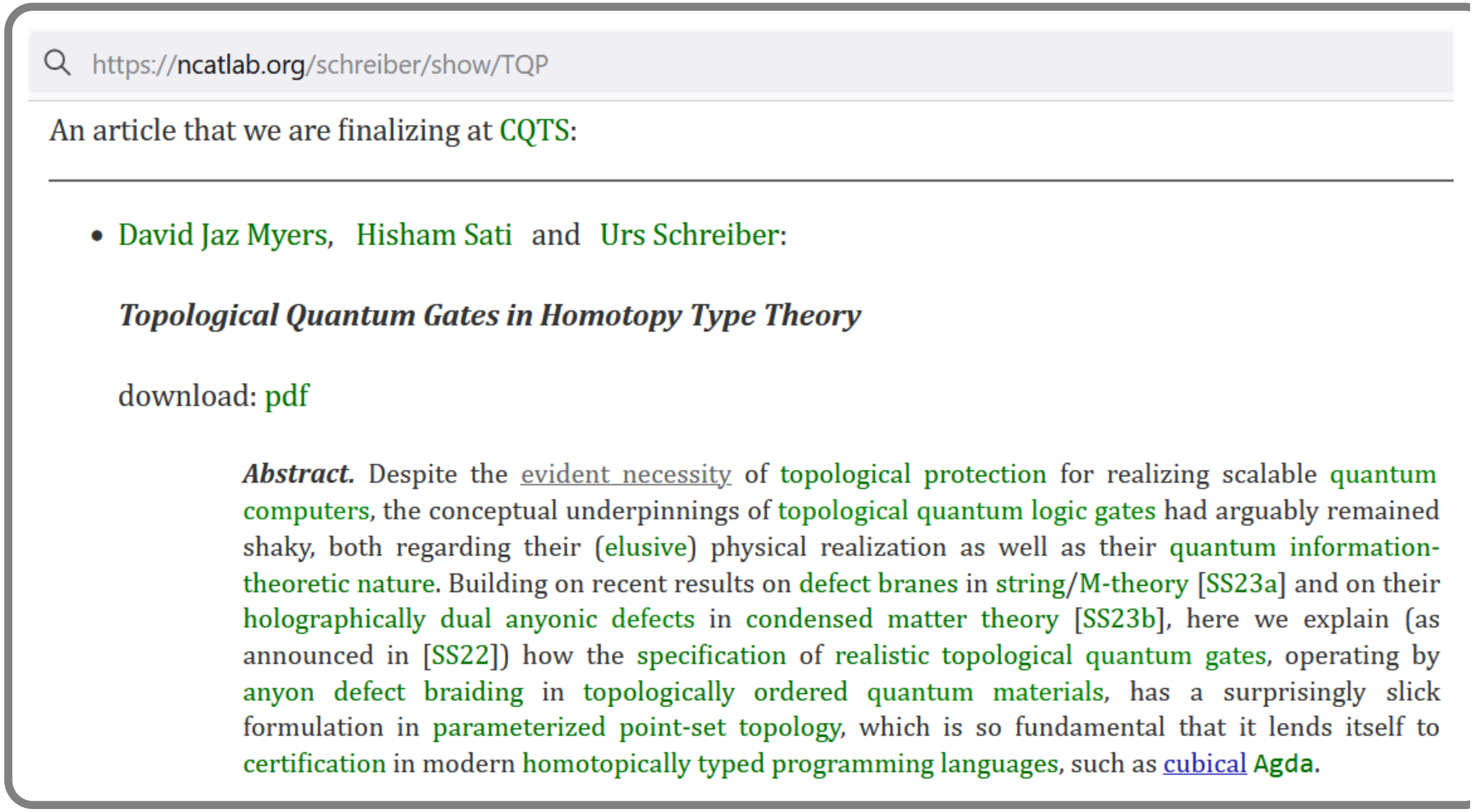

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Now to say all this in more detail ⟶

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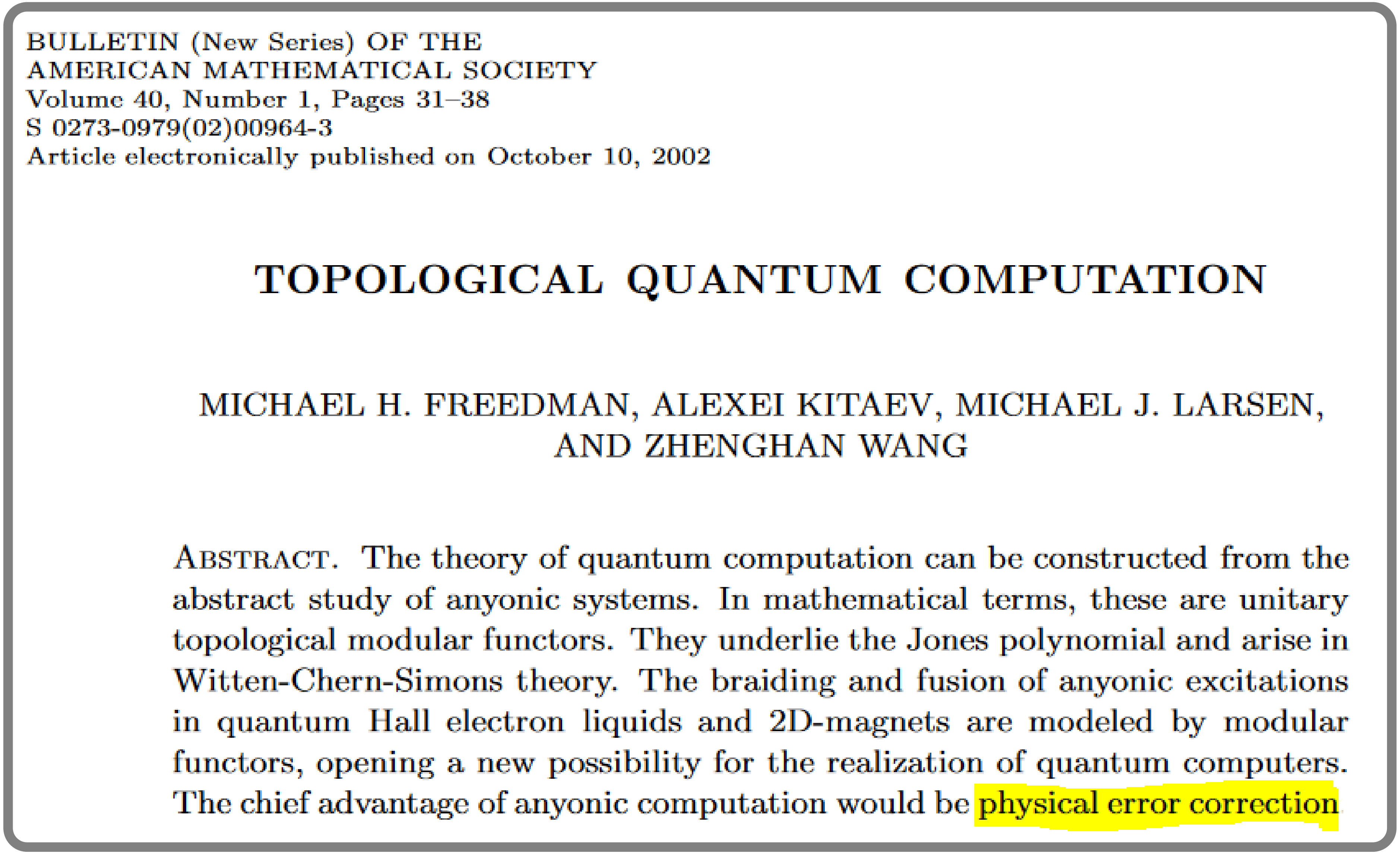

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 40, Number 1, Pages 31–38
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Article electronically published on October 10, 2002

TOPOLOGICAL QUANTUM COMPUTATION

MICHAEL H. FREEDMAN, ALEXEI KITAEV, MICHAEL J. LARSEN,
AND ZHENGHAN WANG

Abstract. The theory of quantum computation can be constructed from the abstract study of anyonic systems. In mathematical terms, these are unitary topological modular functors. They underlie the Jones polynomial and arise in Witten-Chern-Simons theory. The braiding and fusion of anyonic excitations in quantum Hall electron liquids and 2D-magnets are modeled by modular functors, opening a new possibility for the realization of quantum computers. The chief advantage of anyonic computation would be physical error correction

There are good arguments that
if Quantum Computation is to be a practical reality
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Das Sarma, MIT Tech Rev (2022):

“*The quantum-bit systems we have today are a tremendous scientific achievement,*

but they take us no closer to having a quantum computer that can solve a problem that anybody cares about.

What is missing is the breakthrough bypassing quantum error correction by using far-more-stable quantum-bits, in an approach called topological quantum computing.”

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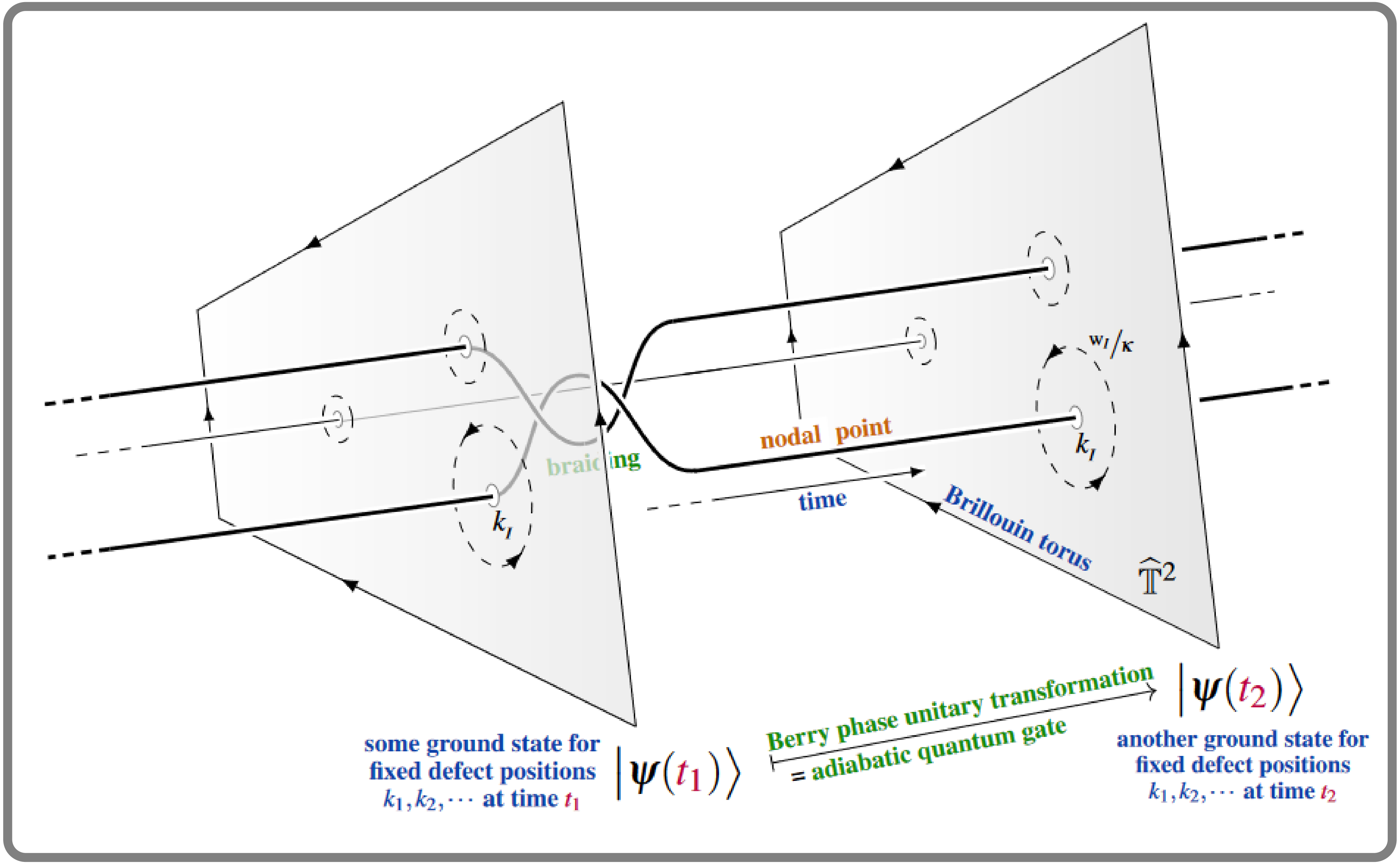

some ground state for fixed defect positions $k_1, k_2, \cdots$ at time t_1 $\left|\psi(t_1)\right\rangle$

Berry phase unitary transformation = adiabatic quantum gate

$\left|\psi(t_2)\right\rangle$ another ground state for fixed defect positions $k_1, k_2, \cdots$ at time t_2

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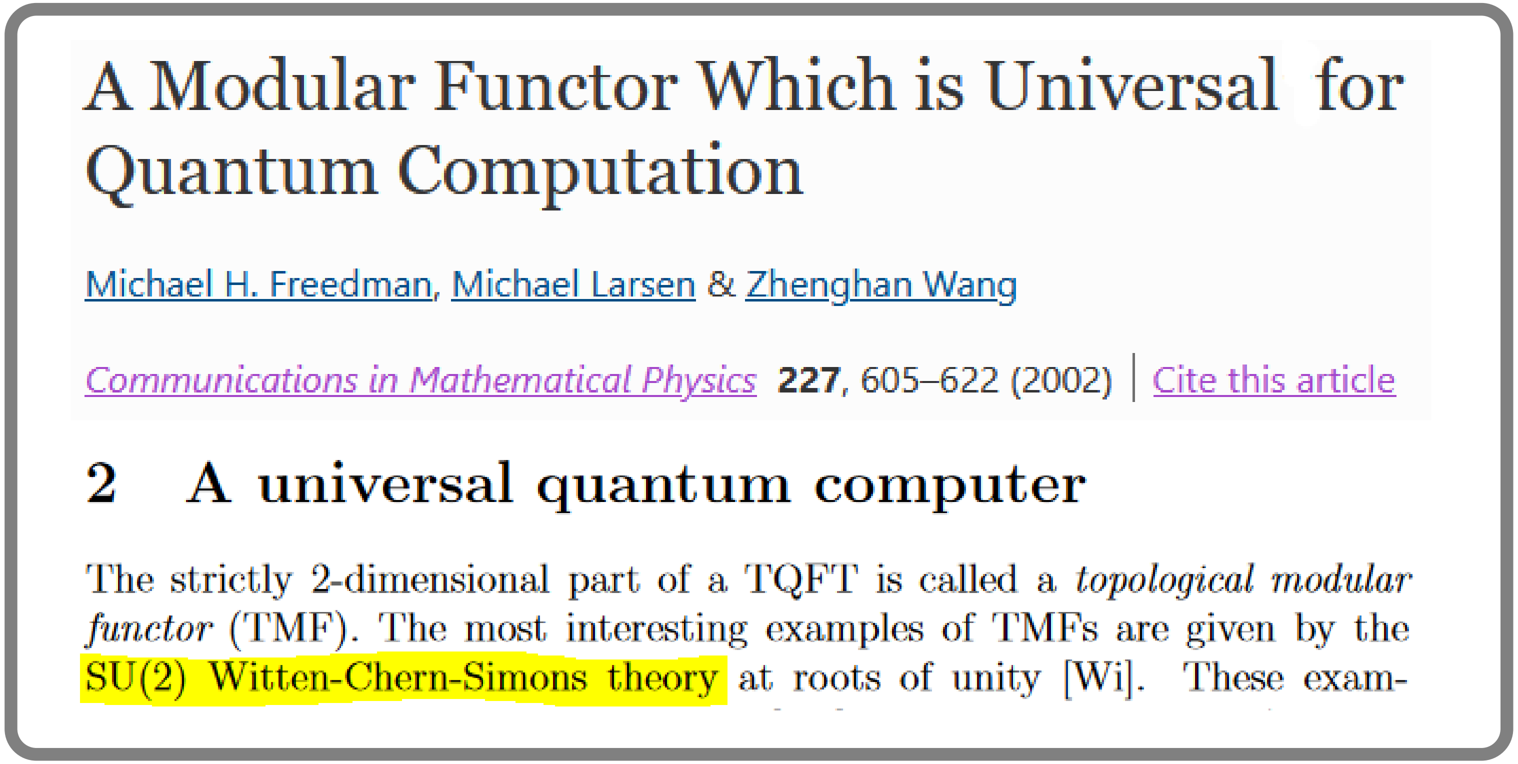

A Modular Functor Which is Universal for Quantum Computation

Michael H. Freedman, Michael Larsen & Zhenghan Wang

Communications in Mathematical Physics **227**, 605–622 (2002) | Cite this article

2 A universal quantum computer

The strictly 2-dimensional part of a TQFT is called a *topological modular functor* (TMF). The most interesting examples of TMFs are given by the SU(2) Witten-Chern-Simons theory at roots of unity [Wi]. These exam-

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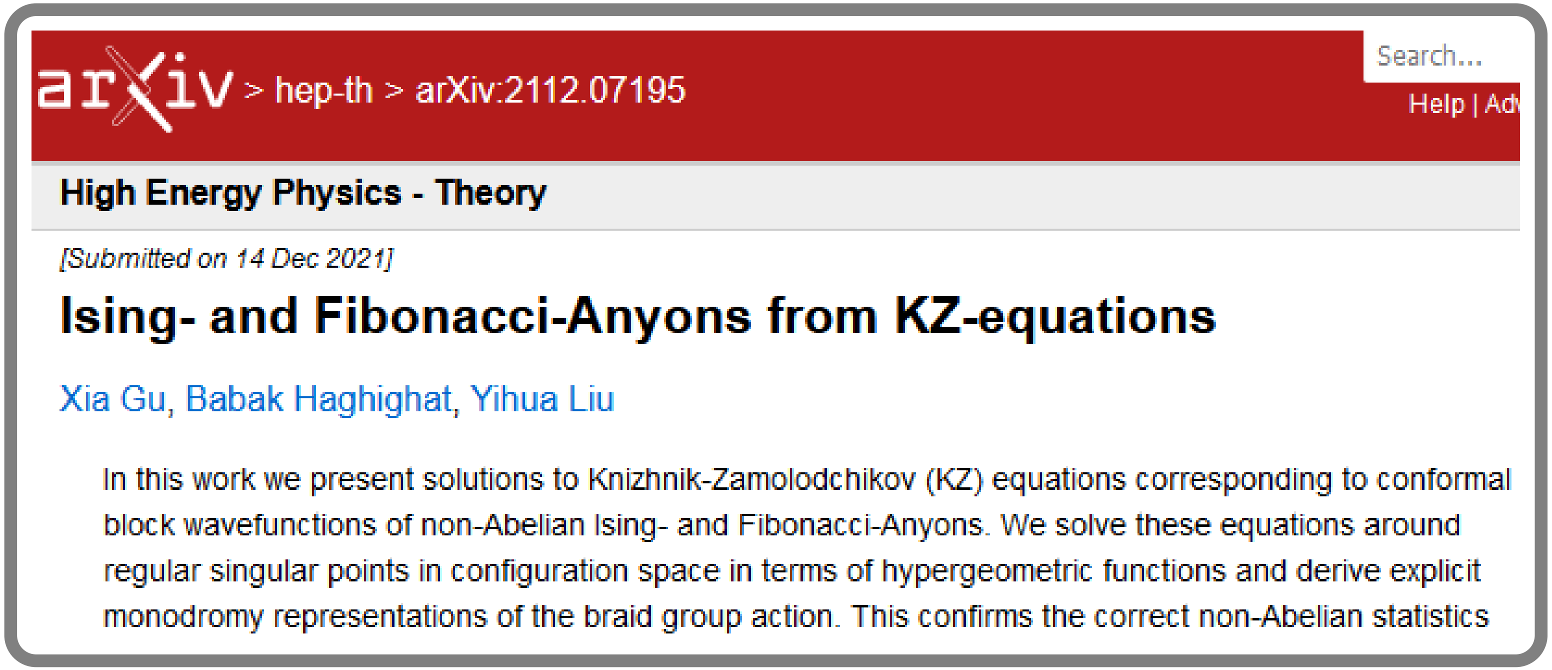

High Energy Physics - Theory

[Submitted on 14 Dec 2021]

Ising- and Fibonacci-Anyons from KZ-equations

Xia Gu, Babak Haghighat, Yihua Liu

In this work we present solutions to Knizhnik-Zamolodchikov (KZ) equations corresponding to conformal block wavefunctions of non-Abelian Ising- and Fibonacci-Anyons. We solve these equations around regular singular points in configuration space in terms of hypergeometric functions and derive explicit monodromy representations of the braid group action. This confirms the correct non-Abelian statistics

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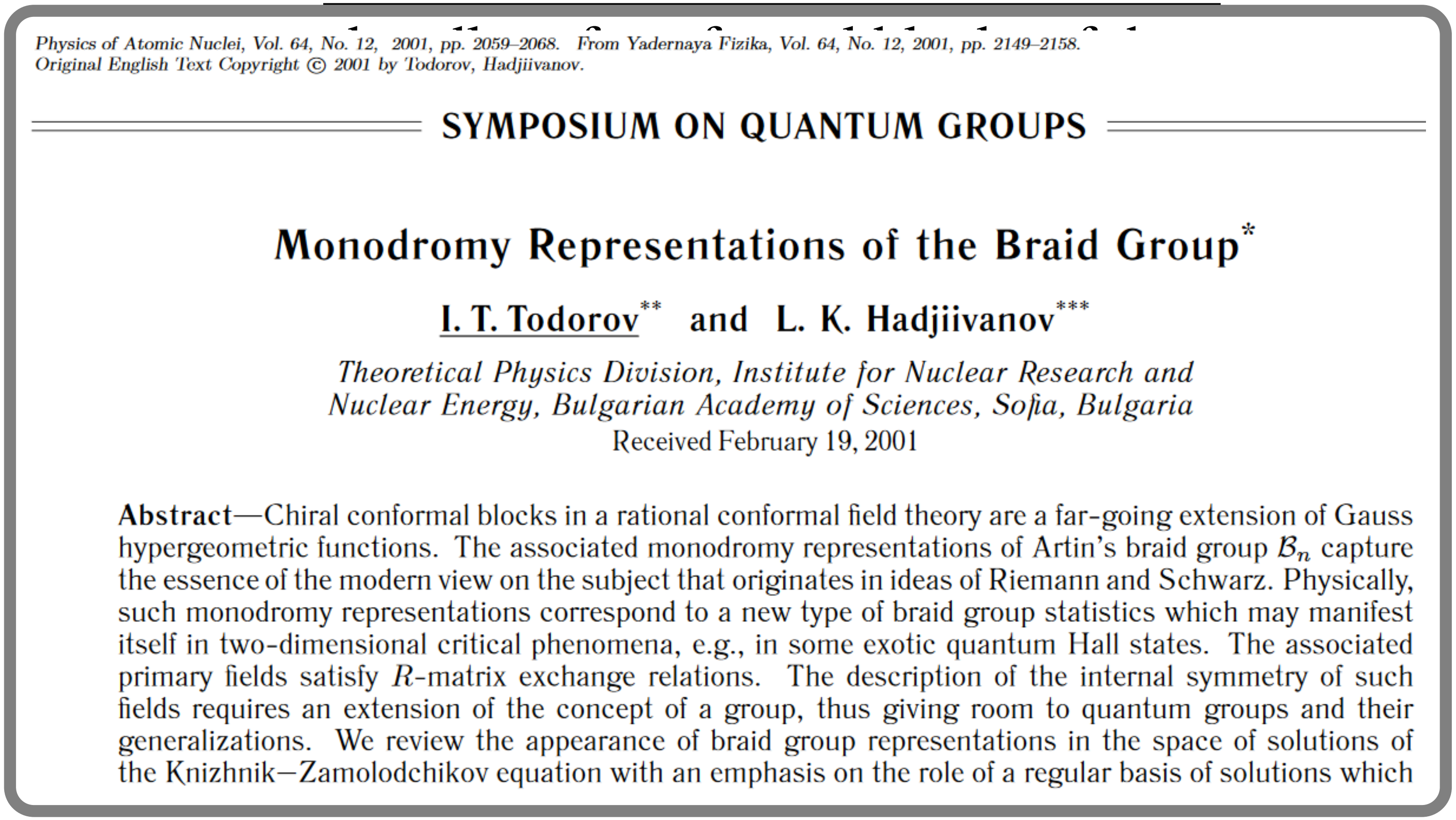

Physics of Atomic Nuclei, Vol. 64, No. 12, 2001, pp. 2059–2068. From Yadernaya Fizika, Vol. 64, No. 12, 2001, pp. 2149–2158.
Original English Text Copyright © 2001 by Todorov, Hadjiivanov.

SYMPOSIUM ON QUANTUM GROUPS

Monodromy Representations of the Braid Group*

I. T. Todorov** and **L. K. Hadjiivanov*****

Theoretical Physics Division, Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Sofia, Bulgaria

Received February 19, 2001

Abstract—Chiral conformal blocks in a rational conformal field theory are a far-going extension of Gauss hypergeometric functions. The associated monodromy representations of Artin's braid group $\mathcal{B}_n$ capture the essence of the modern view on the subject that originates in ideas of Riemann and Schwarz. Physically, such monodromy representations correspond to a new type of braid group statistics which may manifest itself in two-dimensional critical phenomena, e.g., in some exotic quantum Hall states. The associated primary fields satisfy R-matrix exchange relations. The description of the internal symmetry of such fields requires an extension of the concept of a group, thus giving room to quantum groups and their generalizations. We review the appearance of braid group representations in the space of solutions of the Knizhnik–Zamolodchikov equation with an emphasis on the role of a regular basis of solutions which

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Efficient programming of topological quantum computers

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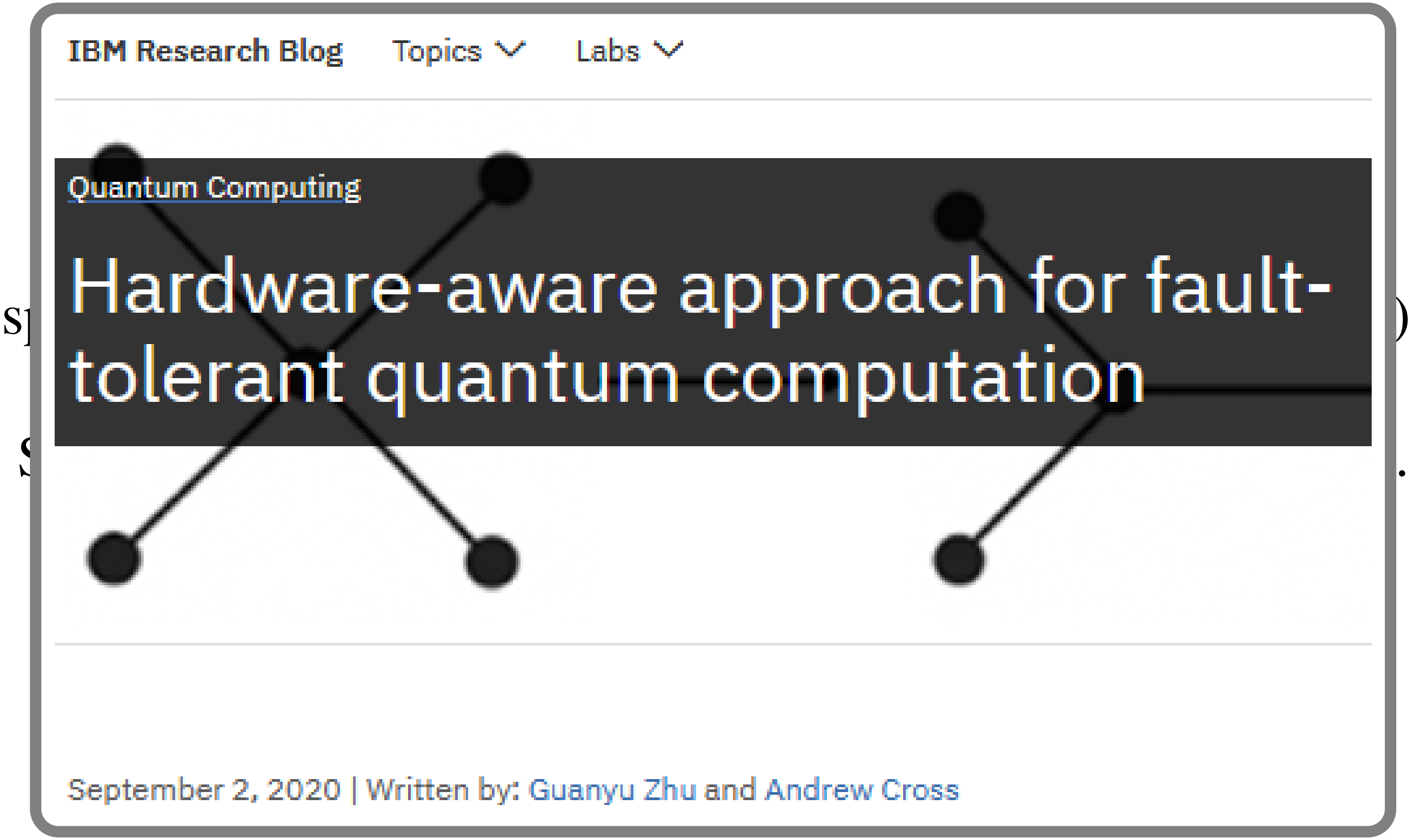

Quantum Computing

Hardware-aware approach for fault-tolerant quantum computation

September 2, 2020 | Written by: Guanyu Zhu and Andrew Cross

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Programming languages suited for describing *bundles* are *dependently typed*

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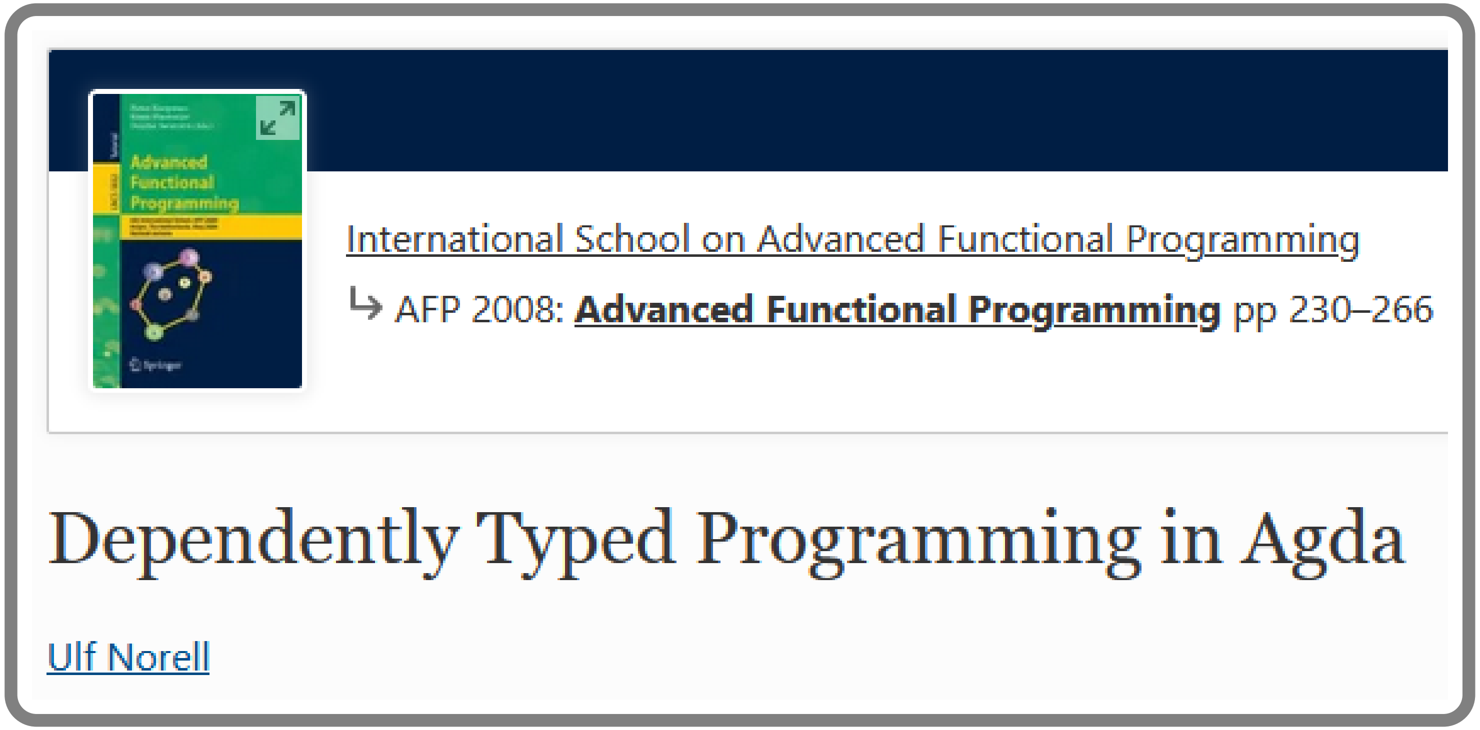

International School on Advanced Functional Programming

↳ AFP 2008: **Advanced Functional Programming** pp 230–266

Dependently Typed Programming in Agda

Ulf Norell

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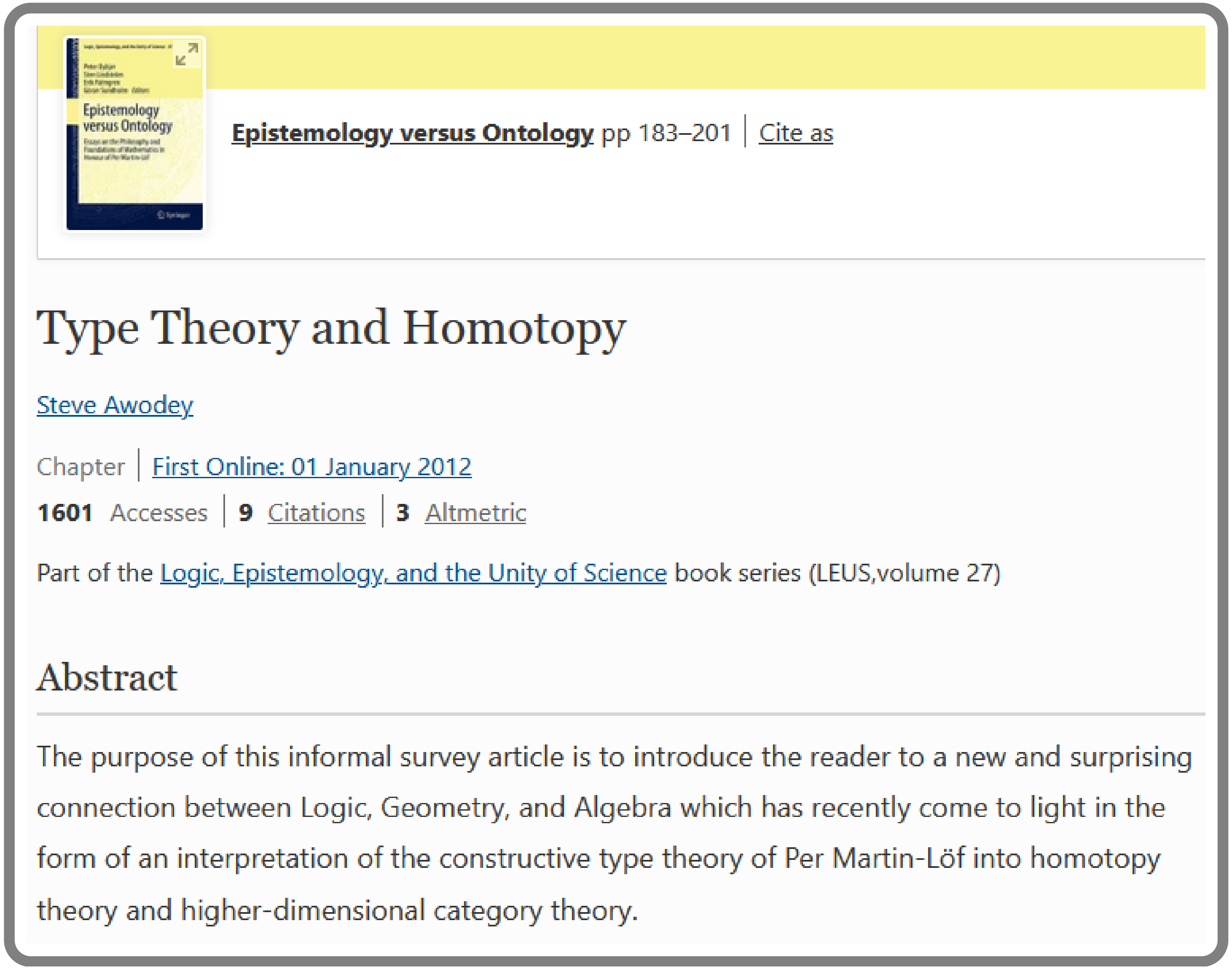

Epistemology versus Ontology pp 183–201 | Cite as

Type Theory and Homotopy

Steve Awodey

Chapter | First Online: 01 January 2012

1601 Accesses | **9** Citations | **3** Altmetric

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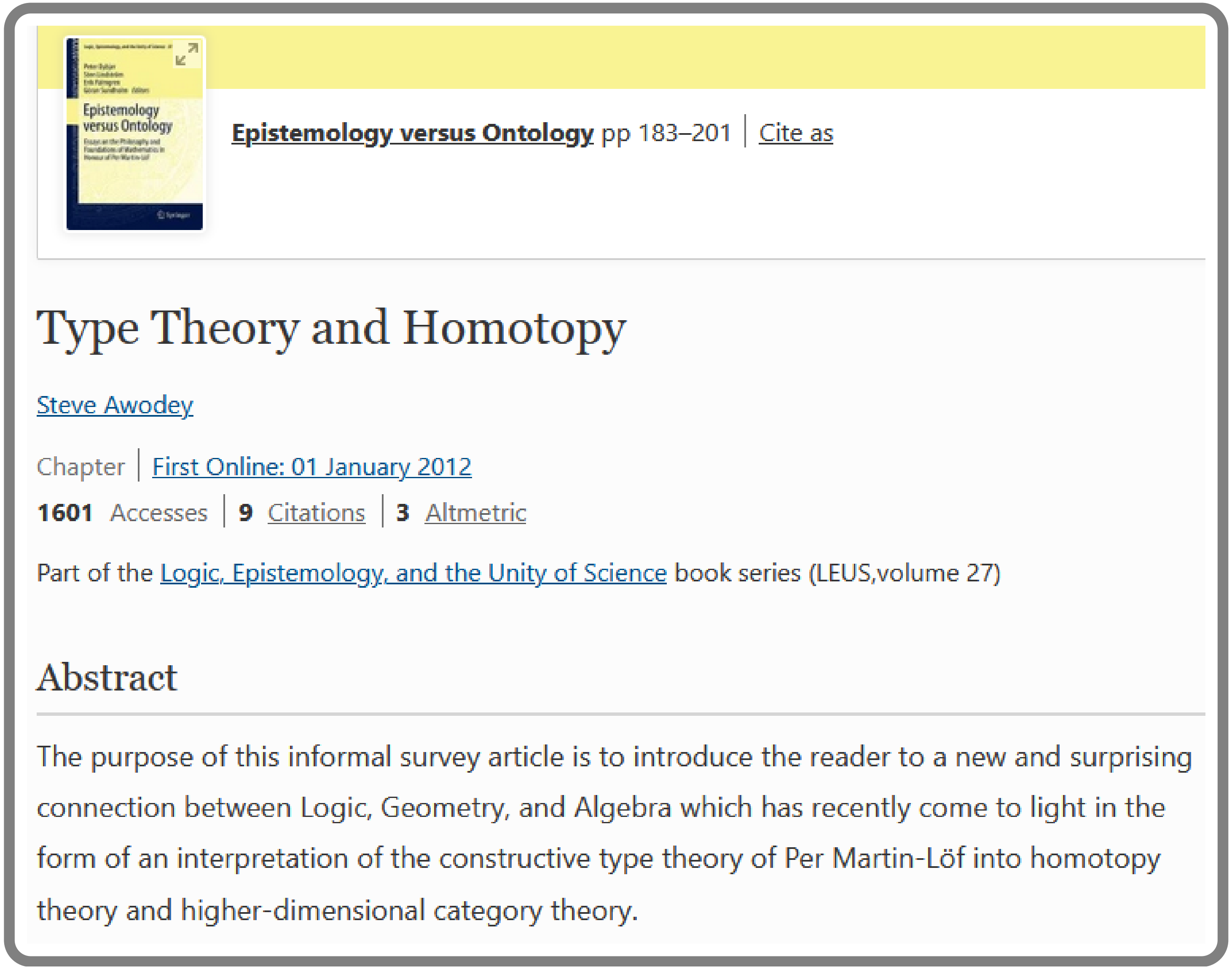
In HoTT, data types come with *paths* between their terms



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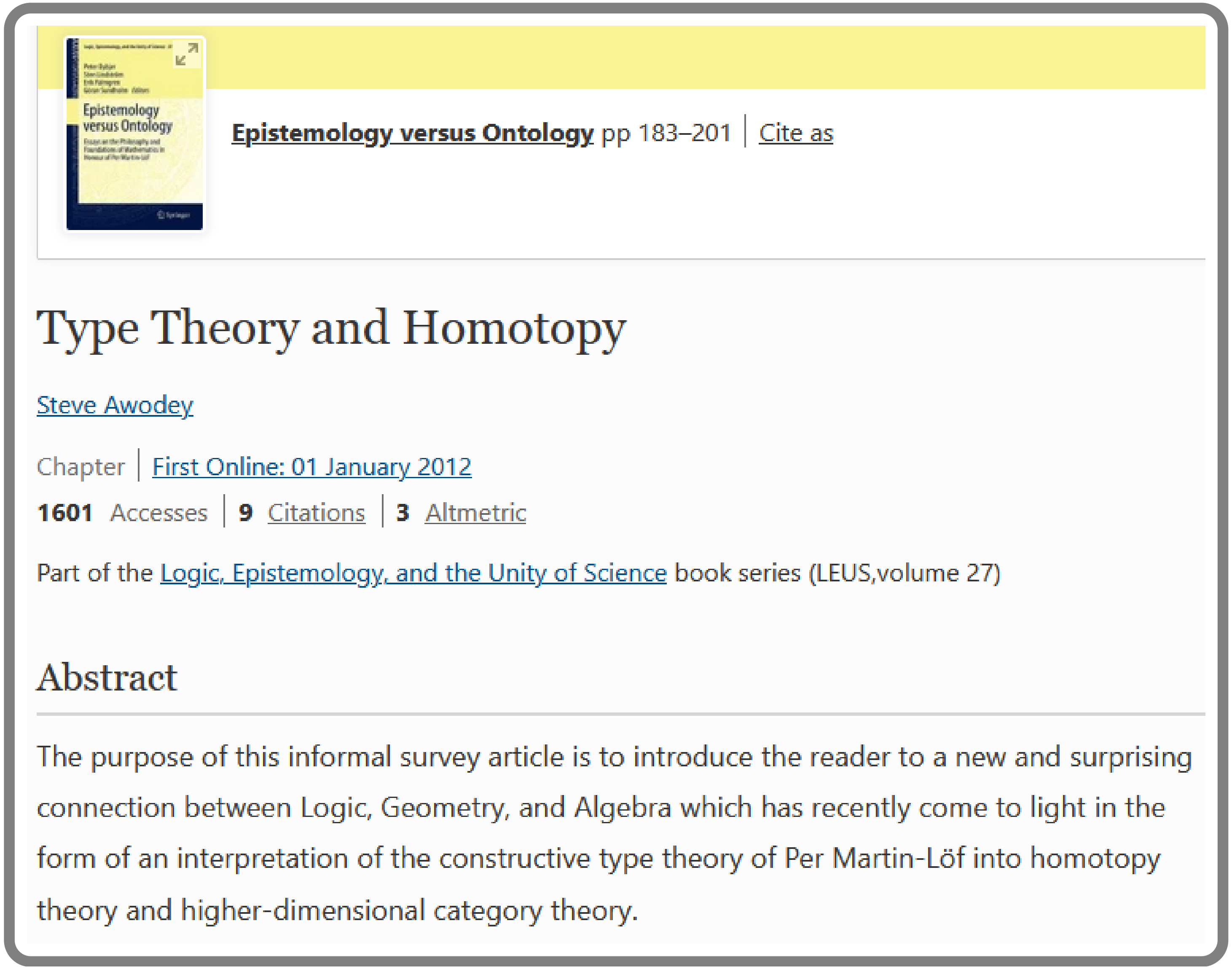

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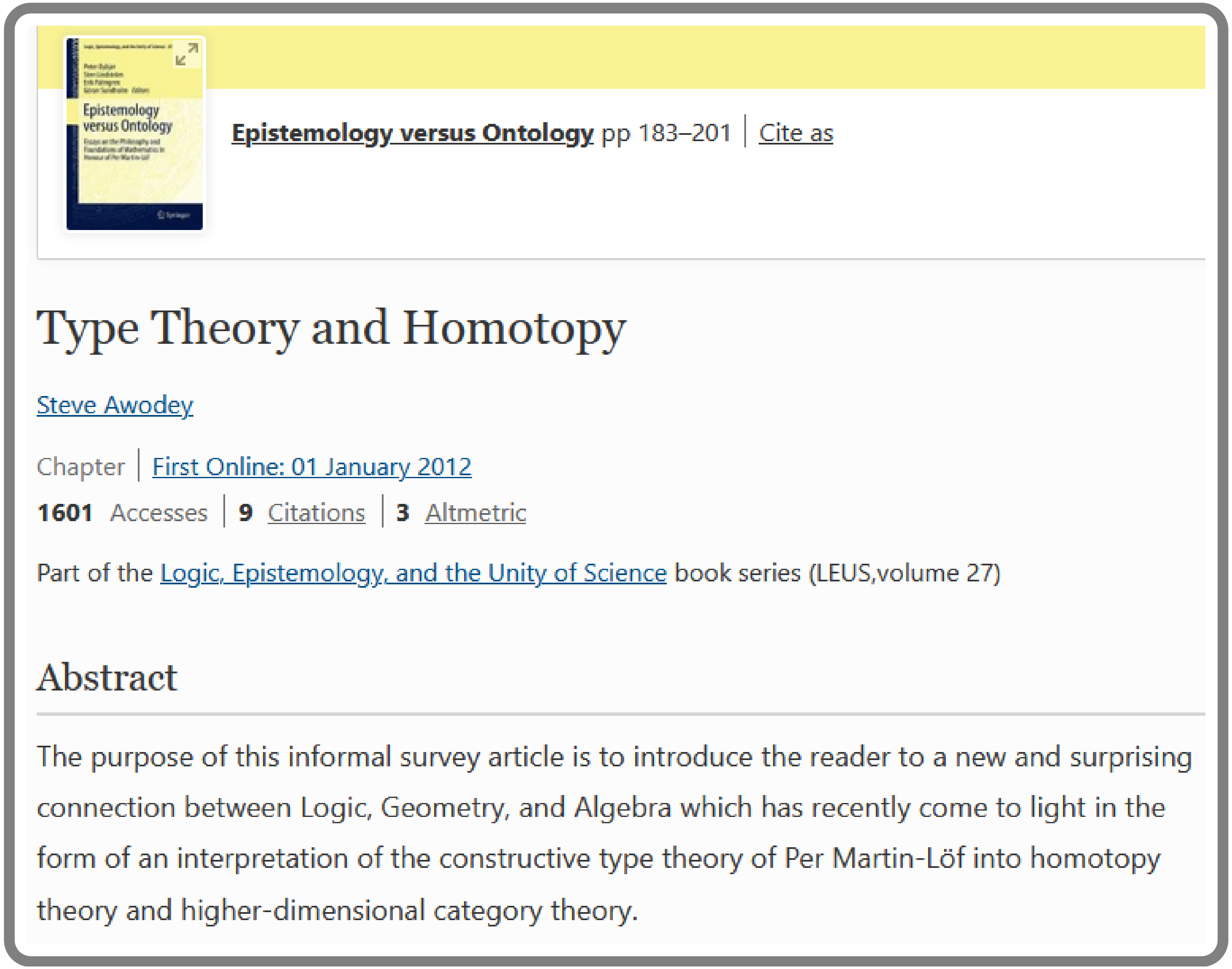

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OXFORD LOGIC GUIDES • 36

Twenty-Five Years of Constructive Type Theory

GIOVANNI SAMBIN
and
JAN SMITH

CHAPTER

6 The groupoid interpretation of type theory

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Martin Hofmann, Thomas Streicher

https://doi.org/10.1093/oso/9780198501275.003.0008 Pages 83–112

Published: October 1998

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Homotopy theoretic models of identity types

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An X-dependent type family $\boxed{x \in X \;\vdash\; P(x) \in \mathrm{Types}}$ inherits *transport* (monodromy!) along base paths:

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E.g.: if G is a finitely presented group, then we get a type $\mathbf{B}G$ with essentially unique $* \in \mathbf{B}G$ s.t. $\mathrm{Paths}_{\mathbf{B}G}(*,*) \simeq G$.
For $G = \mathrm{Br}(n)$ an Artin braid group this is the homotopy type of configurations of points: $\mathbf{B}\mathrm{Br}(n) \simeq \int \mathrm{Conf}_n(\mathbb{C})$.

An X-dependent type family $\boxed{x \in X \;\vdash\; P(x) \in \mathrm{Types}}$
inherits *transport* (monodromy!) along base paths:

$P : X \to \mathrm{Types}$

$$\begin{tikzcd}
 & P(y) \arrow[dr, "\mathrm{tr}(\gamma_2)"] & \\
P(x) \arrow[ur, "\mathrm{tr}(\gamma_1)"] \arrow[rr, "\mathrm{tr}(\gamma_3)"] & & P(z)
\end{tikzcd}$$

$X : \mathrm{Types}$

$$\begin{tikzcd}
 & y \arrow[dr, squiggly, "\gamma_2"] & \\
x \arrow[ur, squiggly, "\gamma_1"] \arrow[rr, squiggly, "\gamma_3"] & & z
\end{tikzcd}$$

In HoTT, data types come with paths between their terms

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An X-dependent type family $\boxed{x \in X \;\;\vdash\;\; P(x) \,\in\, \mathrm{Types}}$

and compatible *path lifting*:

$P : \; X \longrightarrow \mathrm{Types}$

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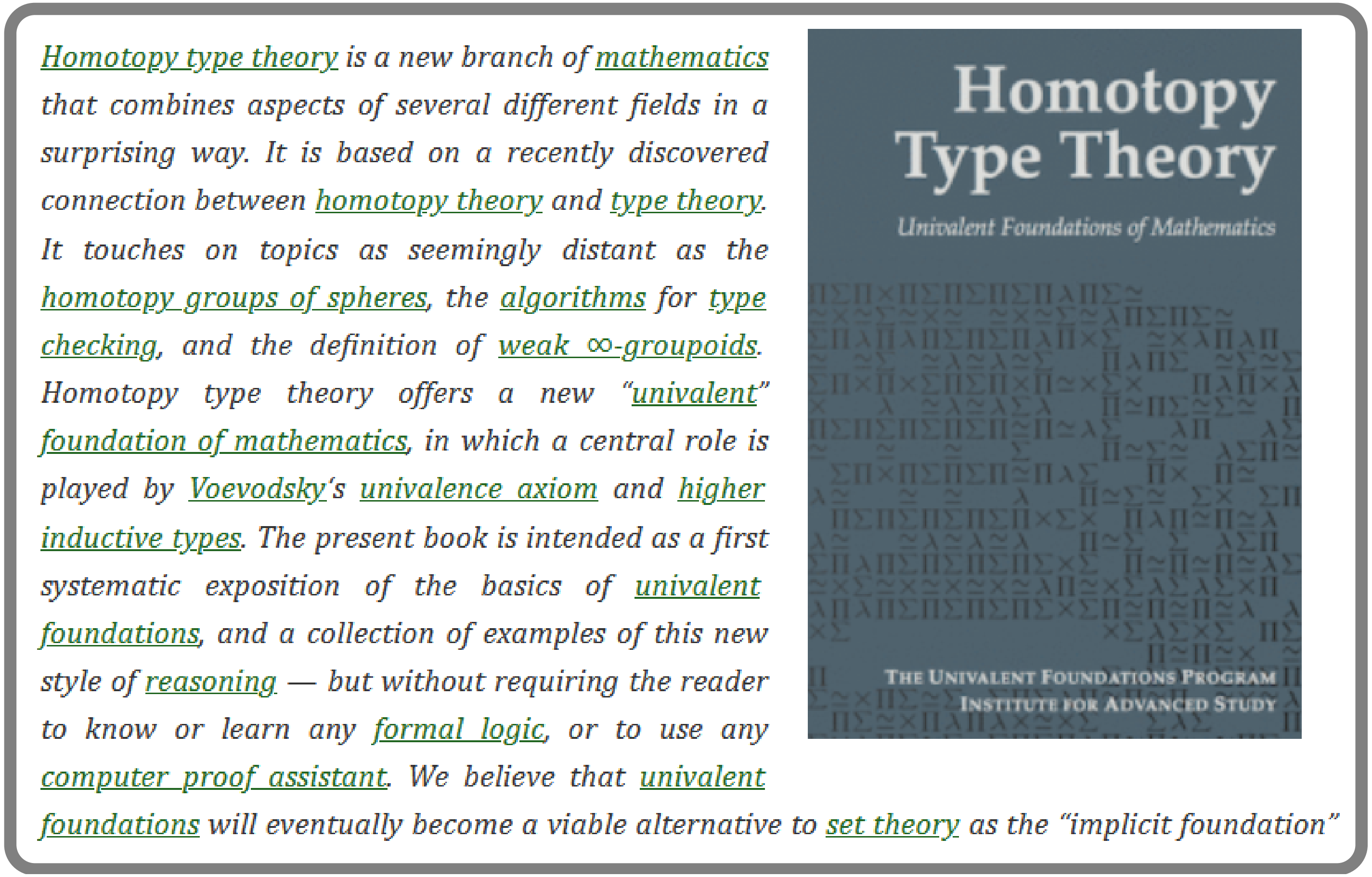

Homotopy type theory is a new branch of mathematics that combines aspects of several different fields in a surprising way. It is based on a recently discovered connection between homotopy theory and type theory. It touches on topics as seemingly distant as the homotopy groups of spheres, the algorithms for type checking, and the definition of weak ∞-groupoids. Homotopy type theory offers a new "univalent" foundation of mathematics, in which a central role is played by Voevodsky's univalence axiom and higher inductive types. The present book is intended as a first systematic exposition of the basics of univalent foundations, and a collection of examples of this new style of reasoning — but without requiring the reader to know or learn any formal logic, or to use any computer proof assistant. We believe that univalent foundations will eventually become a viable alternative to set theory as the "implicit foundation"

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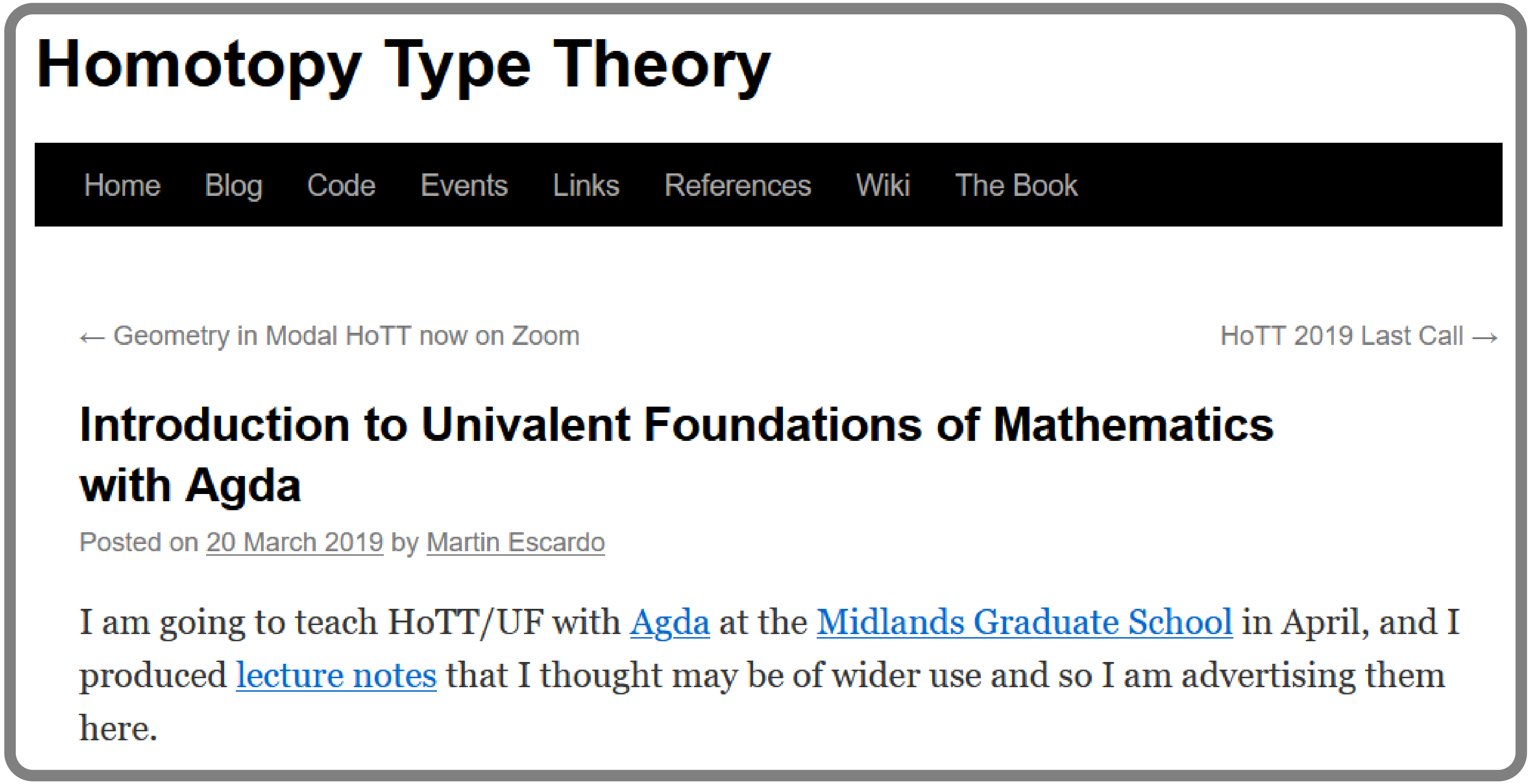

← Geometry in Modal HoTT now on Zoom

HoTT 2019 Last Call →

Introduction to Univalent Foundations of Mathematics with Agda

Posted on 20 March 2019 by Martin Escardo

I am going to teach HoTT/UF with Agda at the Midlands Graduate School in April, and I produced lecture notes that I thought may be of wider use and so I am advertising them here.

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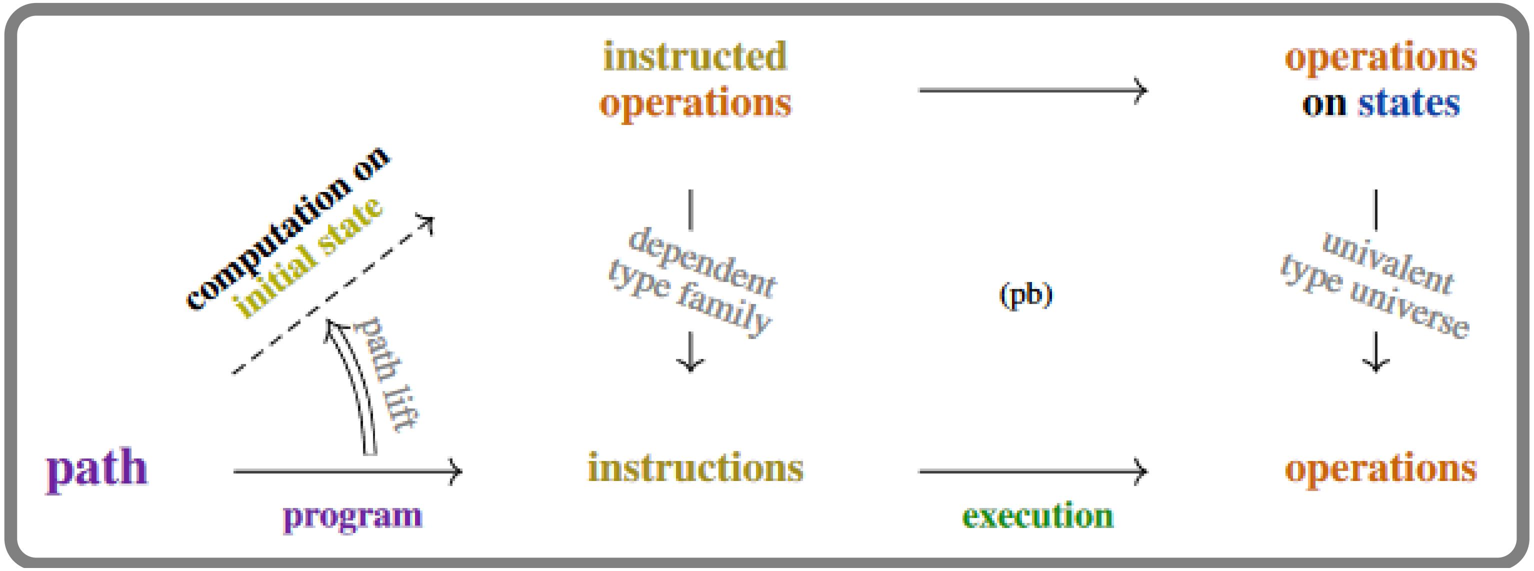

$$\begin{tikzcd}
 & \text{instructed operations} \arrow[r] \arrow[d, "\text{dependent type family}"] \arrow[dr, phantom, "\text{(pb)}"] & \text{operations on states} \arrow[d, "\text{univalent type universe}"] \\
\text{path} \arrow[r, "\text{program}"', ""{name=P}] \arrow[ur, dashed, "\text{computation on initial state}", ""{name=C, swap}] & \text{instructions} \arrow[r, "\text{execution}"'] & \text{operations}
\arrow[Rightarrow, from=P, to=C, "\text{path lift}"']
\end{tikzcd}$$

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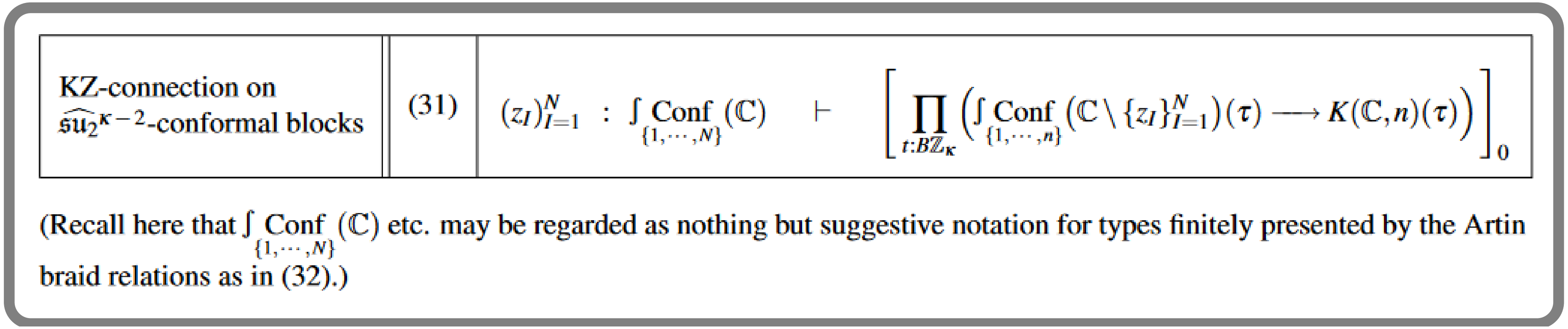

KZ-connection on $\widehat{\mathfrak{su}_2}^{\kappa-2}$-conformal blocks	(31)	$(z_I)_{I=1}^N \;:\; \int \underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C}) \quad \vdash \quad \left[\prod_{t:B\mathbb{Z}_\kappa} \left(\int \underset{\{1,\cdots,n\}}{\mathrm{Conf}}(\mathbb{C} \setminus \{z_I\}_{I=1}^N)(\tau) \longrightarrow K(\mathbb{C},n)(\tau) \right) \right]_0$

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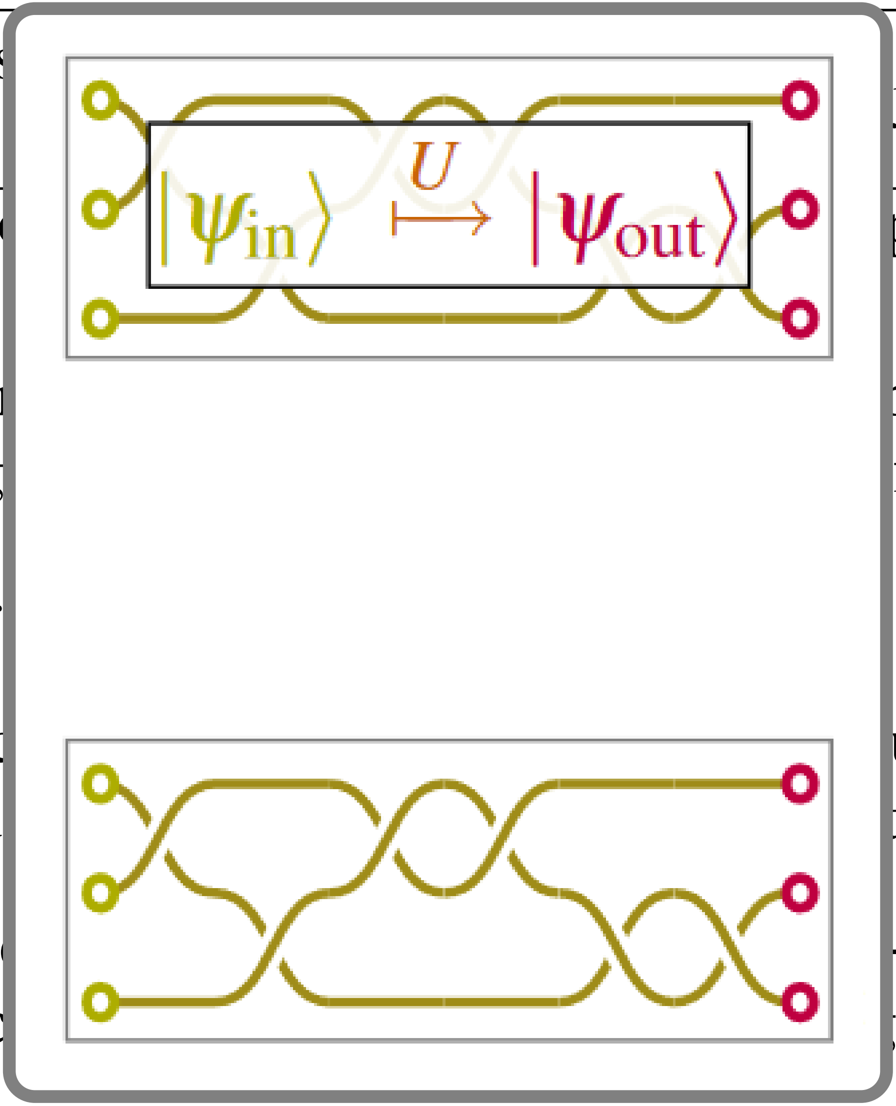

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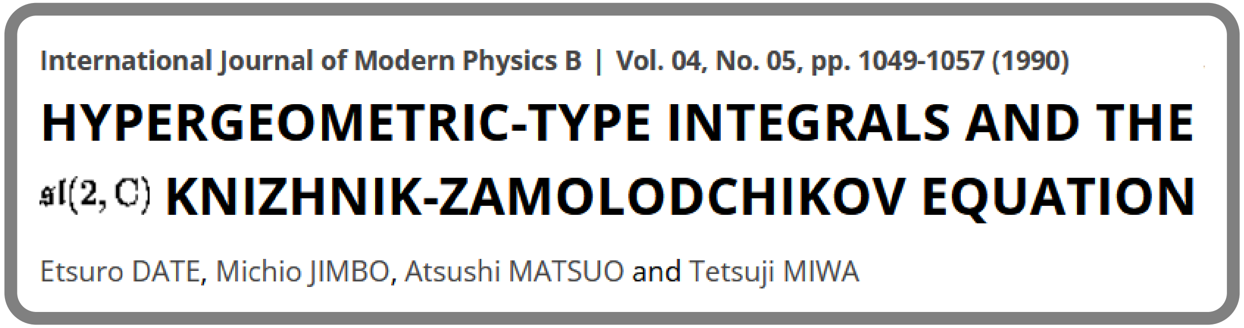

International Journal of Modern Physics B | Vol. 04, No. 05, pp. 1049-1057 (1990)

HYPERGEOMETRIC-TYPE INTEGRALS AND THE $\mathfrak{sl}(2, \mathbb{C})$ KNIZHNIK-ZAMOLODCHIKOV EQUATION

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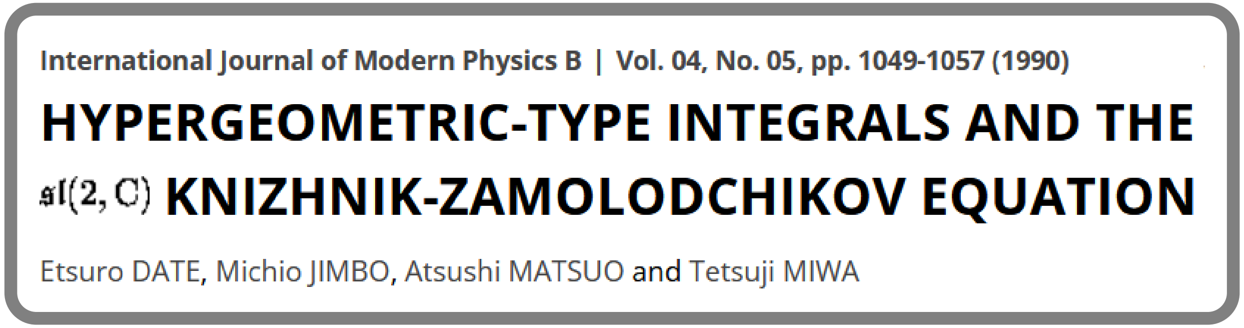

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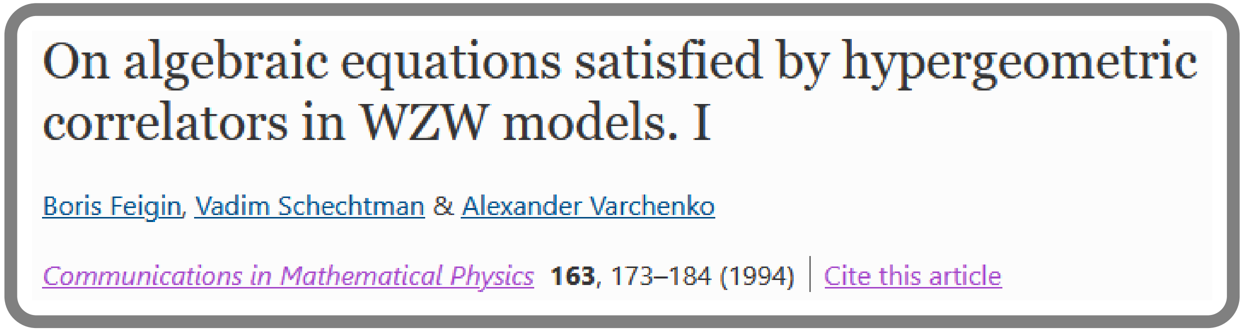

On algebraic equations satisfied by hypergeometric correlators in WZW models. I

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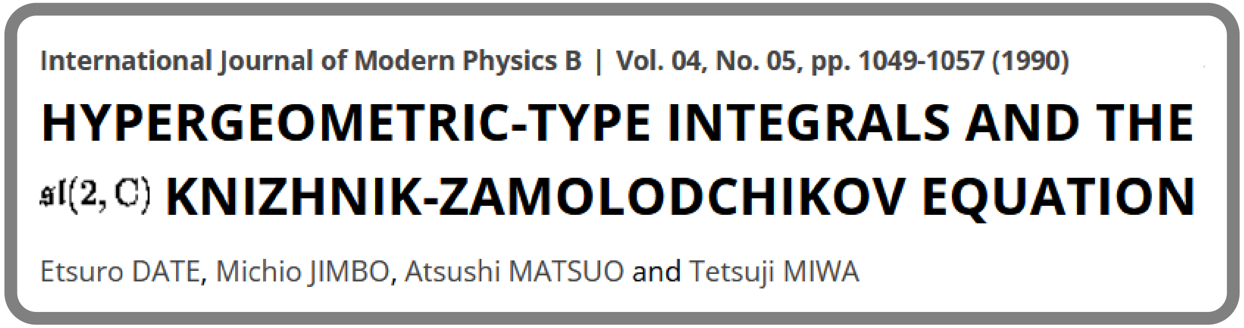

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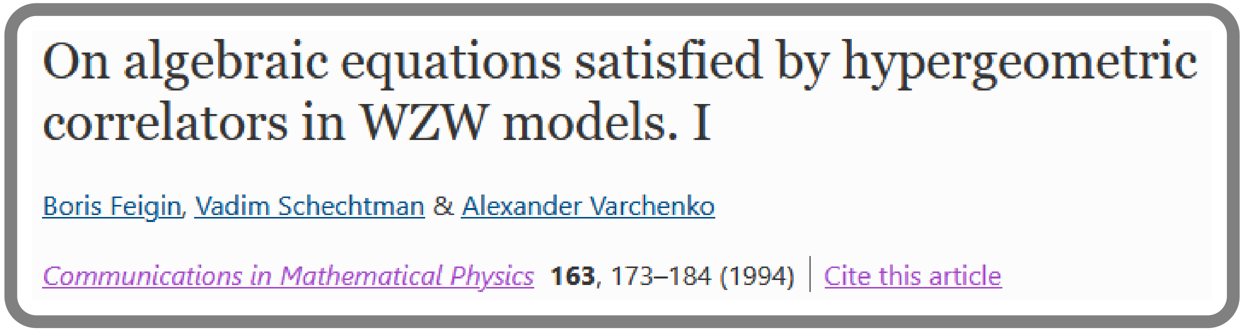

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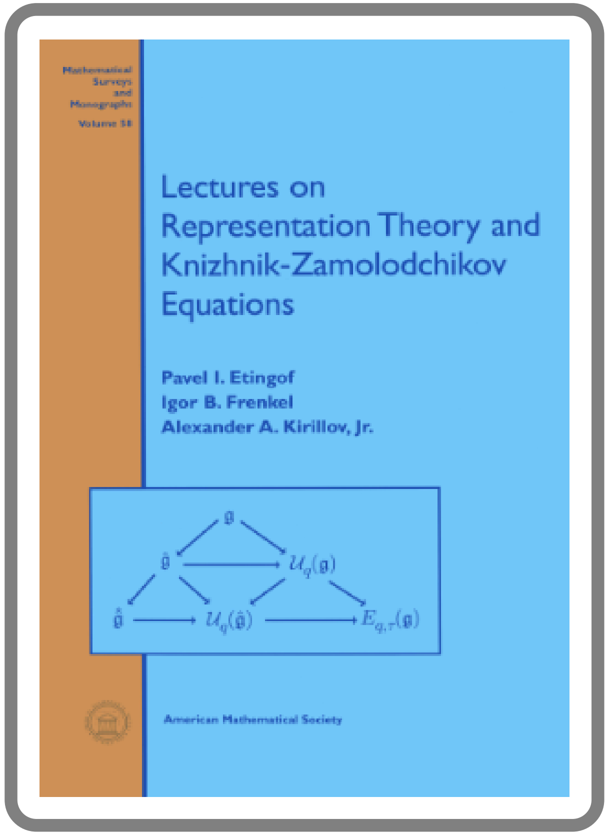

Mathematical
Surveys
and
Monographs
Volume 58

Lectures on
Representation Theory and
Knizhnik-Zamolodchikov
Equations

Pavel I. Etingof
Igor B. Frenkel
Alexander A. Kirillov, Jr.

$$\begin{tikzcd}
 & & \mathfrak{g} \arrow[dl] \arrow[drr] & & \\
 & \hat{\mathfrak{g}} \arrow[rrr] \arrow[dl] \arrow[dr] & & & \mathcal{U}_q(\mathfrak{g}) \arrow[dll] \arrow[dr] \\
\hat{\hat{\mathfrak{g}}} \arrow[rr] & & \mathcal{U}_q(\hat{\mathfrak{g}}) \arrow[rrr] & & & E_{q,\tau}(\mathfrak{g})
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American Mathematical Society

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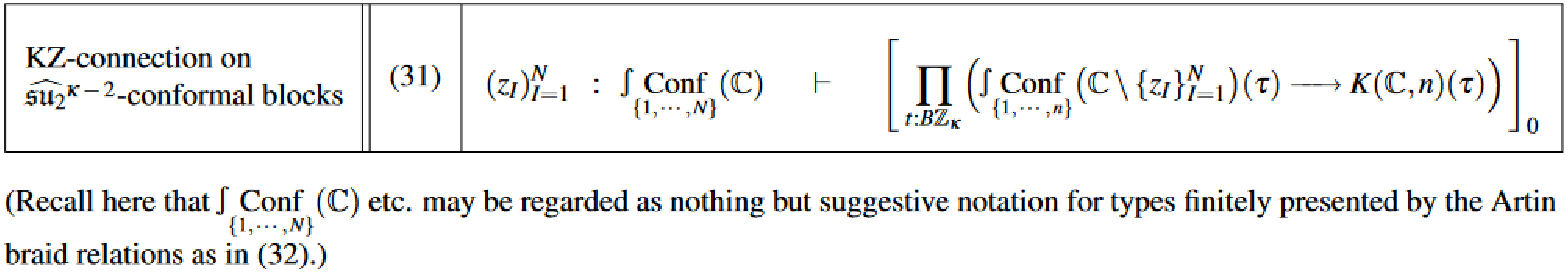

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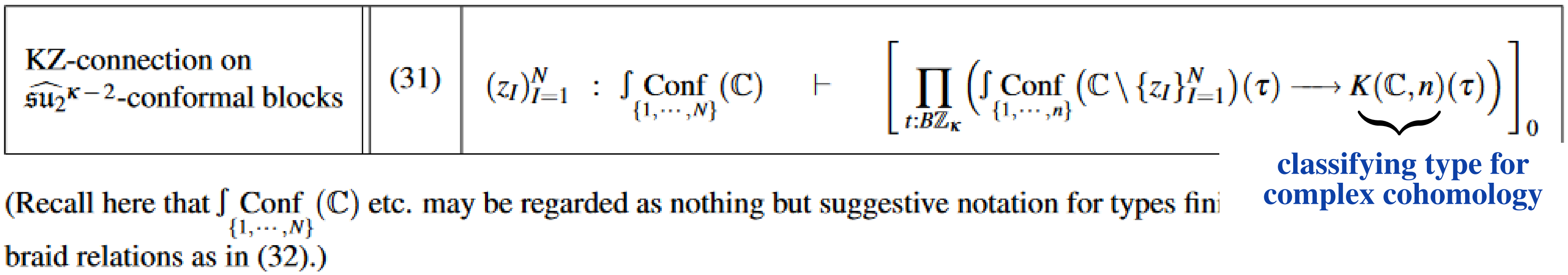

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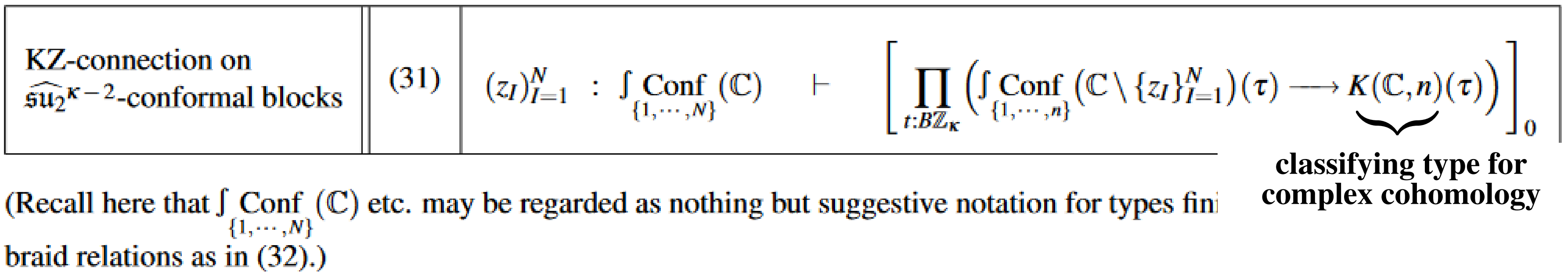

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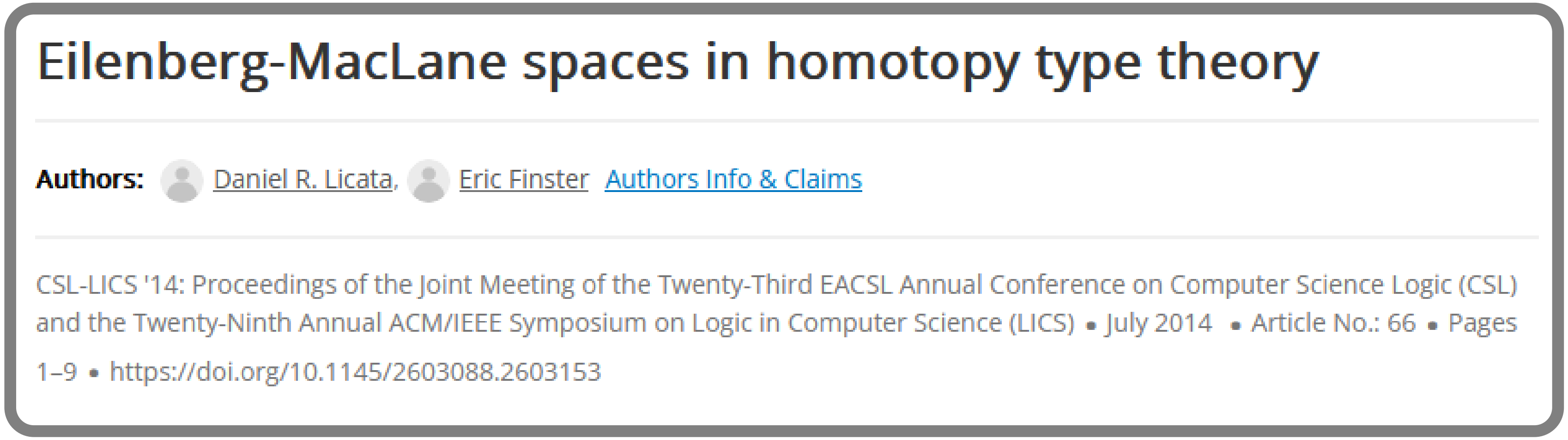

Eilenberg-MacLane spaces in homotopy type theory

Authors: Daniel R. Licata, Eric Finster Authors Info & Claims

CSL-LICS '14: Proceedings of the Joint Meeting of the Twenty-Third EACSL Annual Conference on Computer Science Logic (CSL) and the Twenty-Ninth Annual ACM/IEEE Symposium on Logic in Computer Science (LICS) • July 2014 • Article No.: 66 • Pages 1–9 • https://doi.org/10.1145/2603088.2603153

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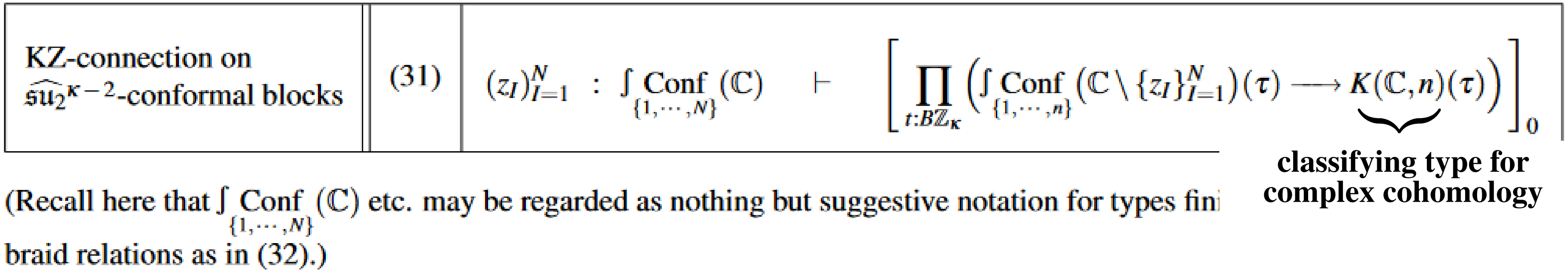

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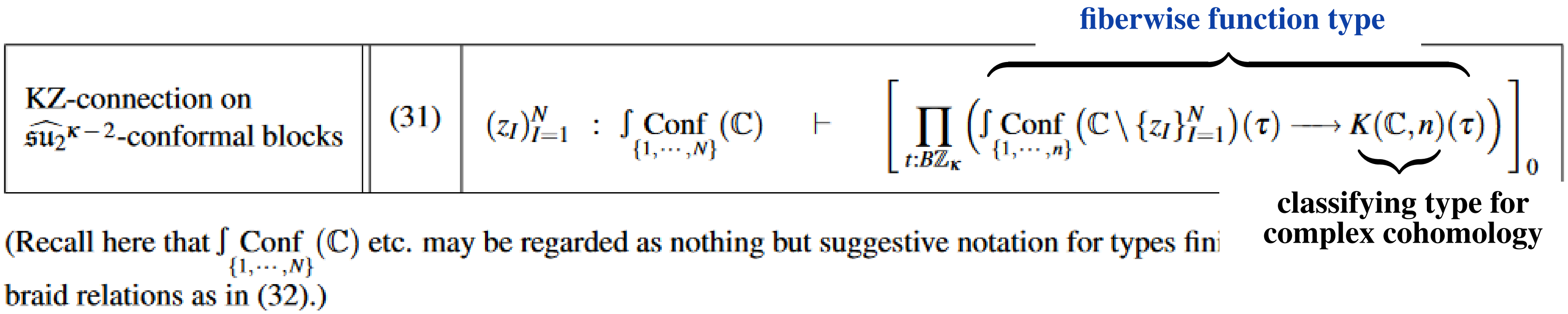

KZ-connection on $\widehat{\mathfrak{su}_2}^{\kappa-2}$-conformal blocks	(31)	$(z_I)_{I=1}^N \;:\; \int \underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C}) \quad \vdash \quad \left[\prod_{t:B\mathbb{Z}_\kappa} \overbrace{\left(\int \underset{\{1,\cdots,n\}}{\mathrm{Conf}}(\mathbb{C}\setminus\{z_I\}_{I=1}^N)(\tau) \longrightarrow K\underbrace{(\mathbb{C},n)}_{\text{classifying type for complex cohomology}}(\tau)\right)}^{\text{fiberwise function type}}\right]_0$

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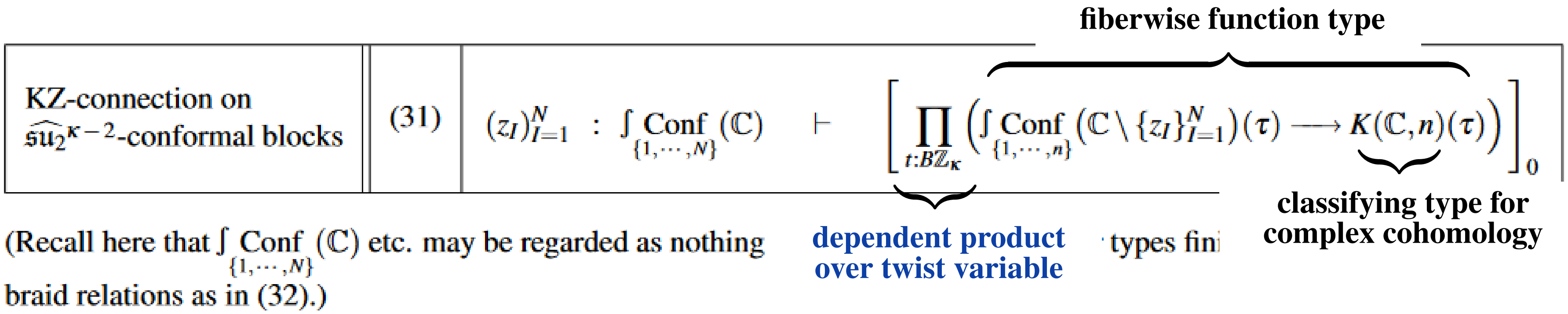

KZ-connection on $\widehat{\mathfrak{su}_2}^{\kappa-2}$-conformal blocks	(31)	$(z_I)_{I=1}^N \;:\; \int \mathrm{Conf}_{\{1,\cdots,N\}}(\mathbb{C}) \;\vdash\; \Big[\underbrace{\prod_{t:B\mathbb{Z}_\kappa}}_{\text{dependent product over twist variable}} \Big(\overbrace{\int \mathrm{Conf}_{\{1,\cdots,n\}}\big(\mathbb{C}\setminus\{z_I\}_{I=1}^N\big)(\tau) \longrightarrow K\underbrace{(\mathbb{C},n)}_{\text{classifying type for complex cohomology}}(\tau)}^{\text{fiberwise function type}} \Big) \Big]_0$

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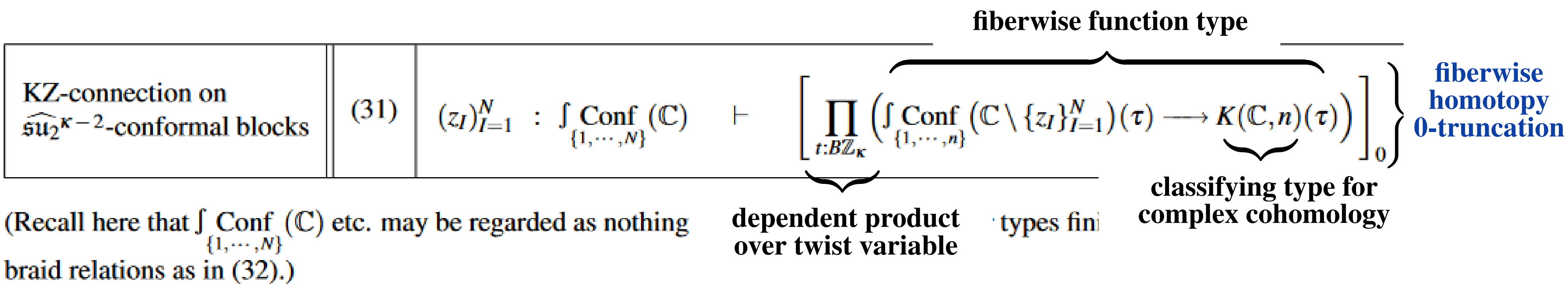

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fiberwise homotopy 0-truncation

(Recall here that $\int \mathrm{Conf}_{\{1,\cdots,N\}}(\mathbb{C})$ etc. may be regarded as nothing types fini
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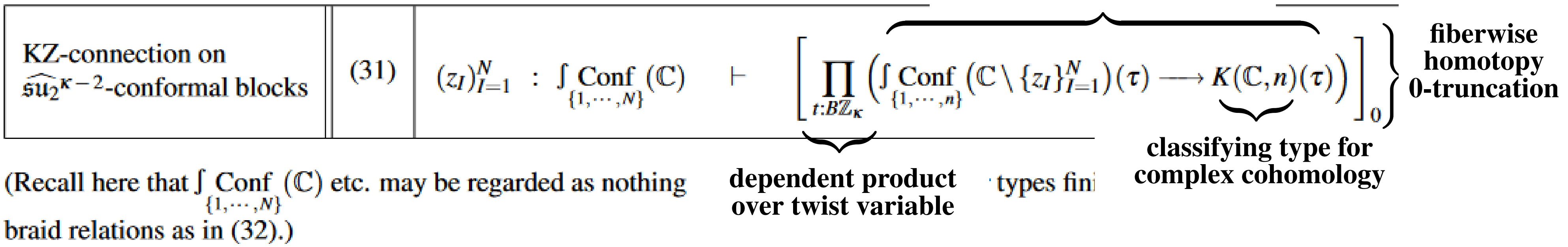

KZ-connection on $\widehat{\mathfrak{su}_2}^{\kappa-2}$-conformal blocks	(31)	$(z_I)_{I=1}^N \;:\; \int \underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C}) \;\vdash\; \left.\left[\underbrace{\prod_{t:B\mathbb{Z}_\kappa}}_{\text{dependent product over twist variable}} \overbrace{\left(\int \underset{\{1,\cdots,n\}}{\mathrm{Conf}}\left(\mathbb{C}\setminus\{z_I\}_{I=1}^N\right)(\tau) \longrightarrow \underbrace{K(\mathbb{C},n)}_{\text{classifying type for complex cohomology}}(\tau)\right)}^{\text{fiberwise function type}}\right]_0\right\}\text{fiberwise homotopy 0-truncation}$

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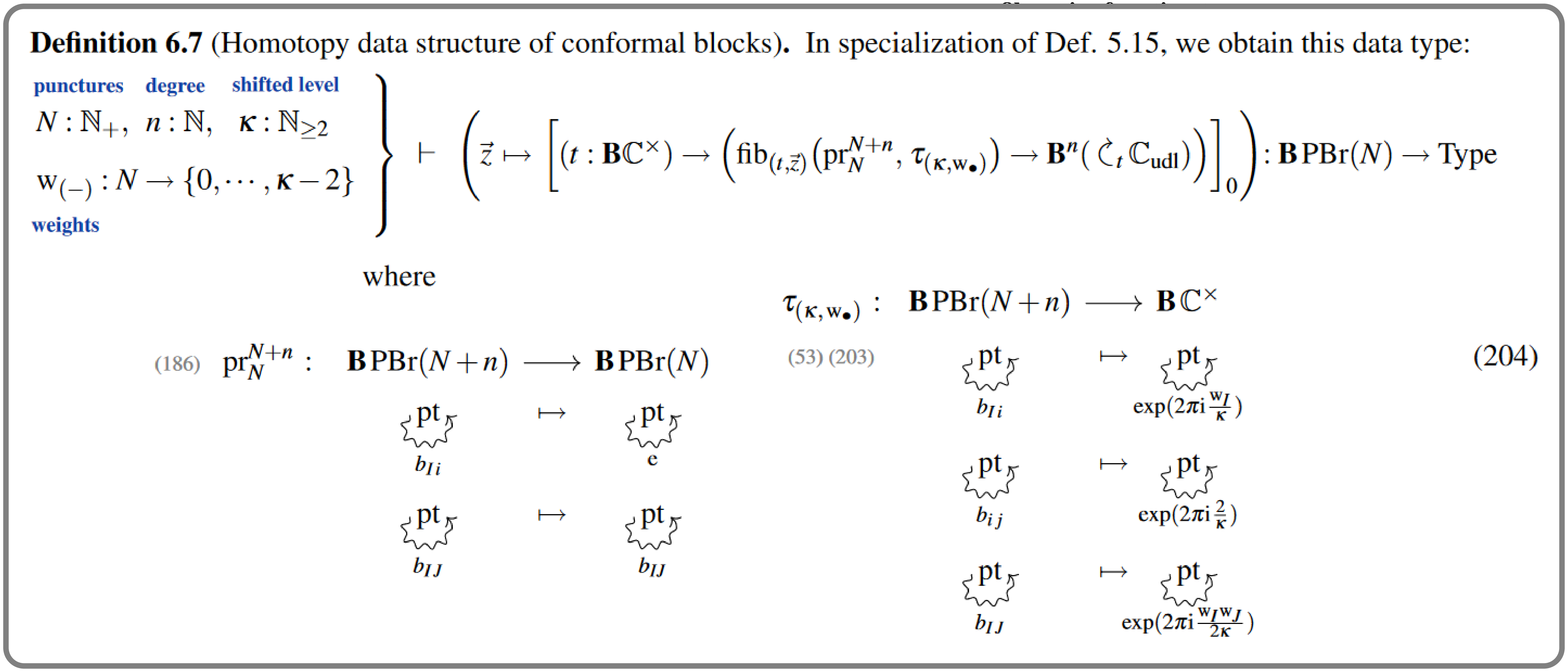

Definition 6.7 (Homotopy data structure of conformal blocks)**.** In specialization of Def. 5.15, we obtain this data type:

$$\left.\begin{array}{l}\overset{\text{punctures}}{N : \mathbb{N}_+}, \;\; \overset{\text{degree}}{n : \mathbb{N}}, \;\; \overset{\text{shifted level}}{\kappa : \mathbb{N}_{\geq 2}} \\ \underset{\text{weights}}{\mathrm{w}_{(-)} : N \to \{0, \cdots, \kappa - 2\}}\end{array}\right\} \vdash \left(\vec{z} \mapsto \Big[(t : \mathbf{B}\mathbb{C}^\times) \to \Big(\mathrm{fib}_{(t,\vec{z})}\big(\mathrm{pr}_N^{N+n}, \tau_{(\kappa,\mathrm{w}_\bullet)}\big) \to \mathbf{B}^n\big(\circlearrowleft_t \mathbb{C}_{\mathrm{udl}}\big)\Big)\Big]_0\right) : \mathbf{B}\,\mathrm{PBr}(N) \to \mathrm{Type}$$

where

$$\begin{array}{lllll} & & & \tau_{(\kappa,\mathrm{w}_\bullet)} : & \mathbf{B}\,\mathrm{PBr}(N+n) \longrightarrow \mathbf{B}\mathbb{C}^\times \\ \text{(186)} \;\; \mathrm{pr}_N^{N+n} : & \mathbf{B}\,\mathrm{PBr}(N+n) \longrightarrow \mathbf{B}\,\mathrm{PBr}(N) & & \text{(53) (203)} & \underset{b_{Ii}}{\mathrm{pt}\circlearrowright} \;\mapsto\; \underset{\exp(2\pi\mathrm{i}\frac{\mathrm{w}_I}{\kappa})}{\mathrm{pt}\circlearrowright} \\ & \underset{b_{Ii}}{\mathrm{pt}\circlearrowright} \;\mapsto\; \underset{\mathrm{e}}{\mathrm{pt}\circlearrowright} & & & \underset{b_{ij}}{\mathrm{pt}\circlearrowright} \;\mapsto\; \underset{\exp(2\pi\mathrm{i}\frac{2}{\kappa})}{\mathrm{pt}\circlearrowright} \\ & \underset{b_{IJ}}{\mathrm{pt}\circlearrowright} \;\mapsto\; \underset{b_{IJ}}{\mathrm{pt}\circlearrowright} & & & \underset{b_{IJ}}{\mathrm{pt}\circlearrowright} \;\mapsto\; \underset{\exp(2\pi\mathrm{i}\frac{\mathrm{w}_I \mathrm{w}_J}{2\kappa})}{\mathrm{pt}\circlearrowright} \end{array} \qquad (204)$$

Myers, Sati & Schreiber: *Topological Quantum Gates in Homotopy Type Theory*

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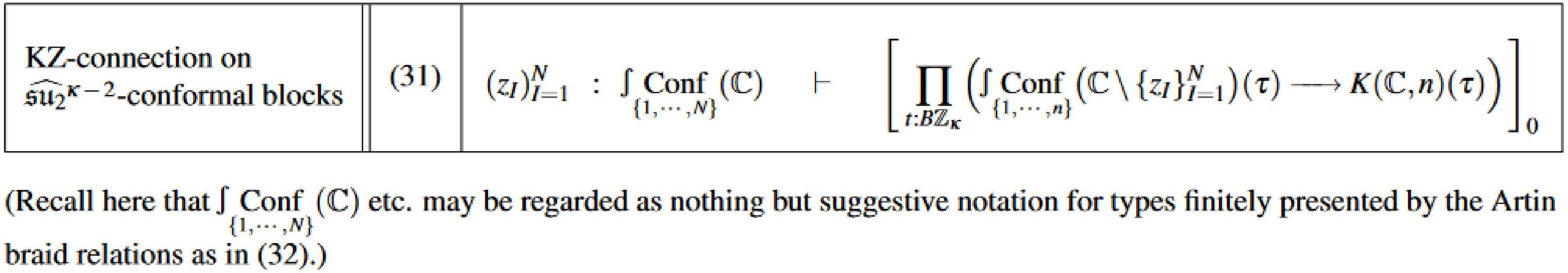

KZ-connection on $\widehat{\mathfrak{su}_2}^{\kappa-2}$-conformal blocks	(31)	$(z_I)_{I=1}^N \;:\; \int \underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C}) \quad\vdash\quad \Big[\prod_{t:B\mathbb{Z}_\kappa}\Big(\int \underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{z_I\}_{I=1}^N\big)(\tau) \longrightarrow K(\mathbb{C},n)(\tau)\Big)\Big]_0$

(Recall here that $\int \underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C})$ etc. may be regarded as nothing but suggestive notation for types finitely presented by the Artin braid relations as in (32).)

This $\left\langle \begin{array}{c}\text{works because}\\ \text{and uses that}\end{array} \right\rangle$ HoTT has categorical semantics
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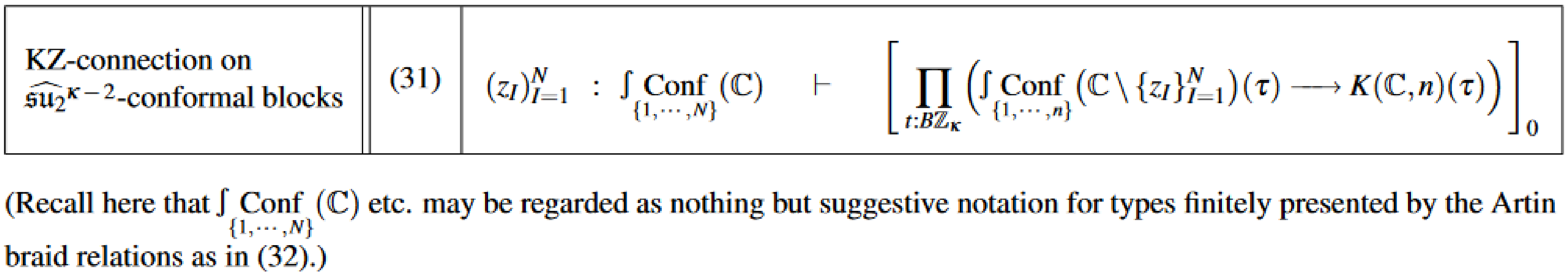

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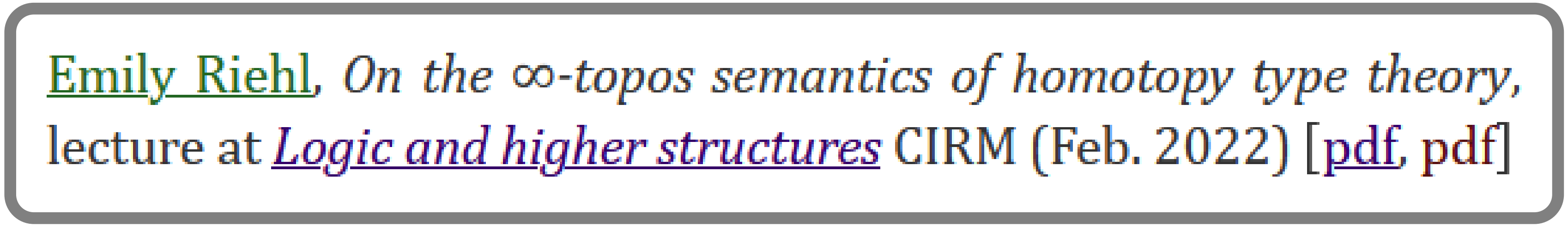

Emily Riehl, *On the ∞-topos semantics of homotopy type theory*, lecture at *Logic and higher structures* CIRM (Feb. 2022) [pdf, pdf]

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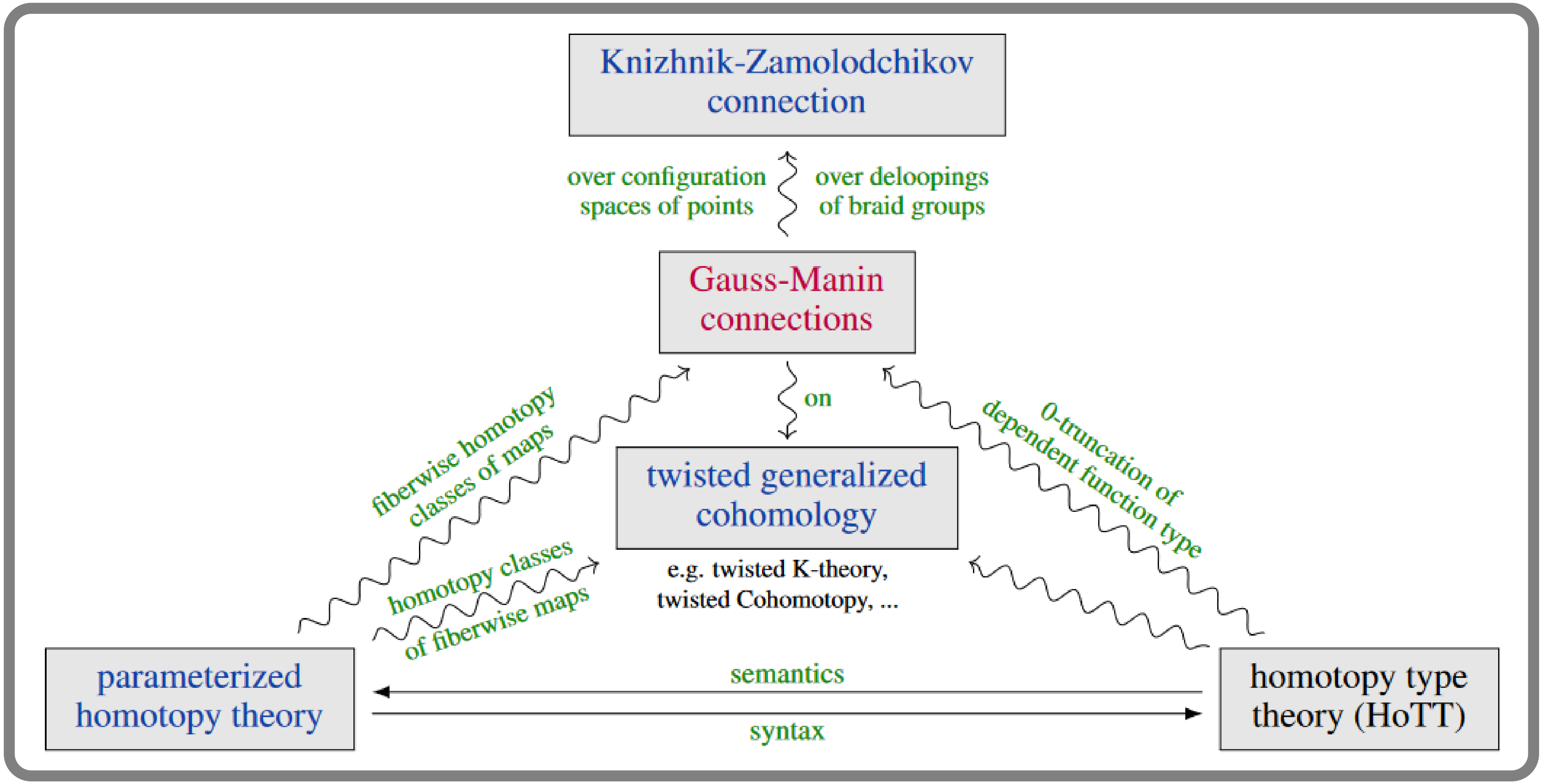

Knizhnik-Zamolodchikov connection

over configuration spaces of points

over deloopings of braid groups

Gauss-Manin connections

on

twisted generalized cohomology

e.g. twisted K-theory,
twisted Cohomotopy, ...

fiberwise homotopy classes of maps

homotopy classes of fiberwise maps

0-truncation of dependent function type

parameterized homotopy theory

semantics

syntax

homotopy type theory (HoTT)

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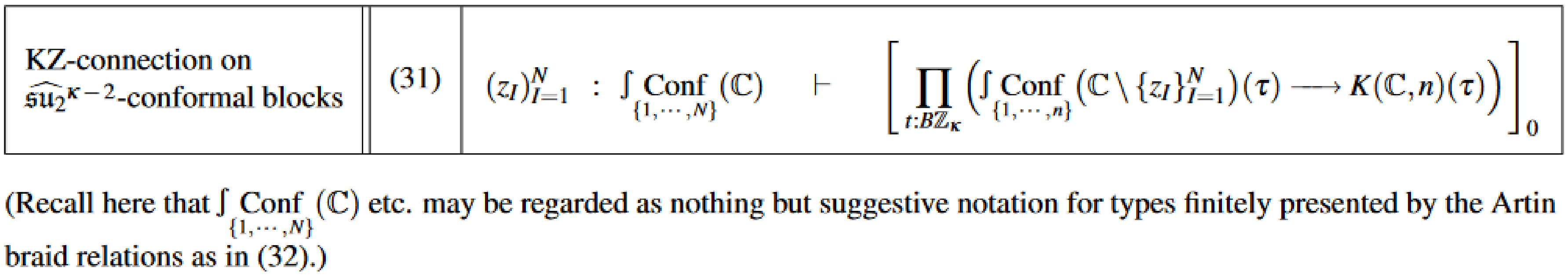

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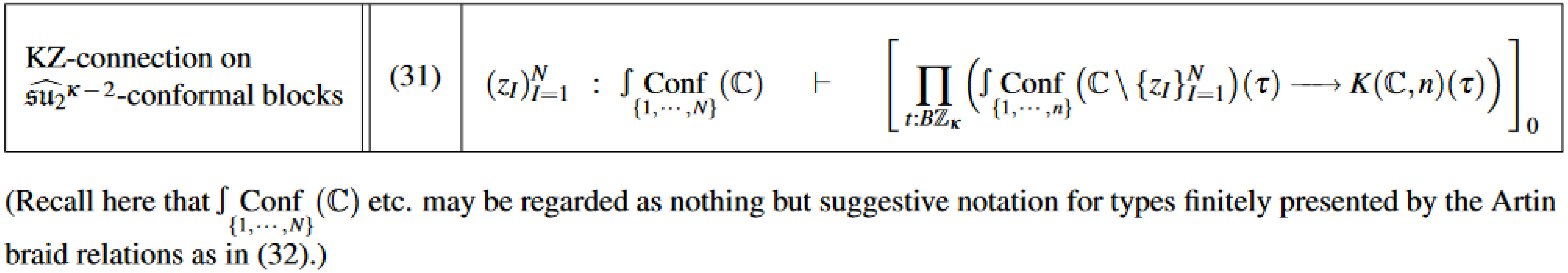

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Claim: Its transport operation is the monodromy braid representation

$$P : \; X \longrightarrow \mathrm{Types}$$

$$\begin{tikzcd}
& P(y) \arrow[dr, "\mathrm{tr}(\gamma_2)"] & \\
P(x) \arrow[ur, "\mathrm{tr}(\gamma_1)"] \arrow[rr, "\mathrm{tr}(\gamma_3)"] & & P(z)
\end{tikzcd}$$

$$X : \; \mathrm{Types}$$

$$\begin{tikzcd}
& y \arrow[dr, squiggly, "\gamma_2"] & \\
x \arrow[ur, squiggly, "\gamma_1"] \arrow[rr, squiggly, "\gamma_3"] & & z
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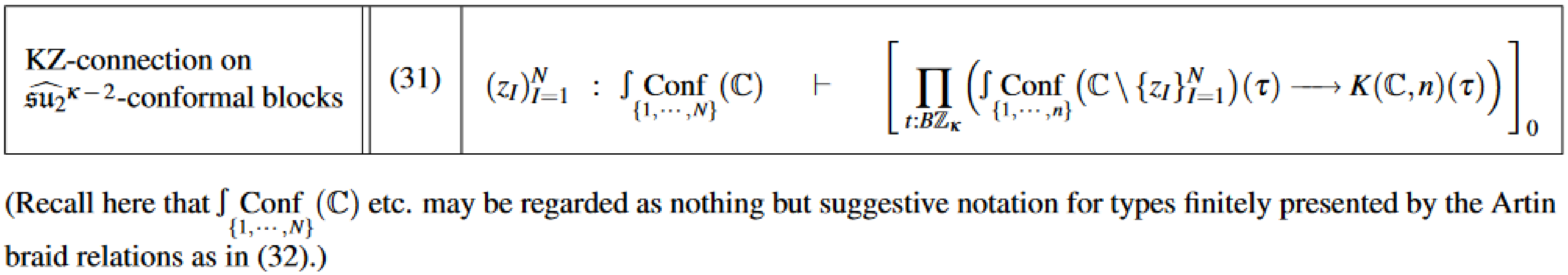

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Claim: Its transport operation is the monodromy braid representation
and its path lifting is execution of $\mathfrak{su}(2)$-anyon braid gates.

$P : X \longrightarrow \mathrm{Types}$

$$\begin{tikzcd}
 & p_y \arrow[rd, rightsquigarrow, "\widehat{\gamma_2}"] & \\
p_x \arrow[ru, rightsquigarrow, "\widehat{\gamma_1}"] \arrow[rr, rightsquigarrow, "\widehat{\gamma_3}"] & & p_z
\end{tikzcd}$$

$$\begin{tikzcd}
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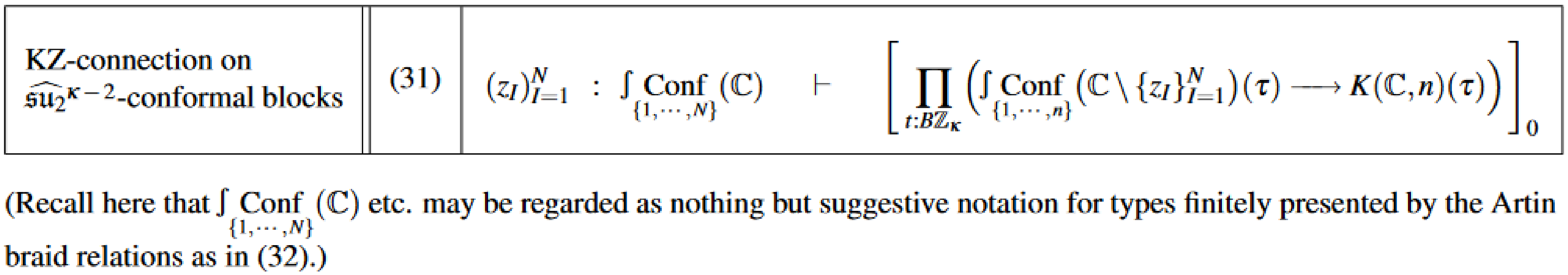

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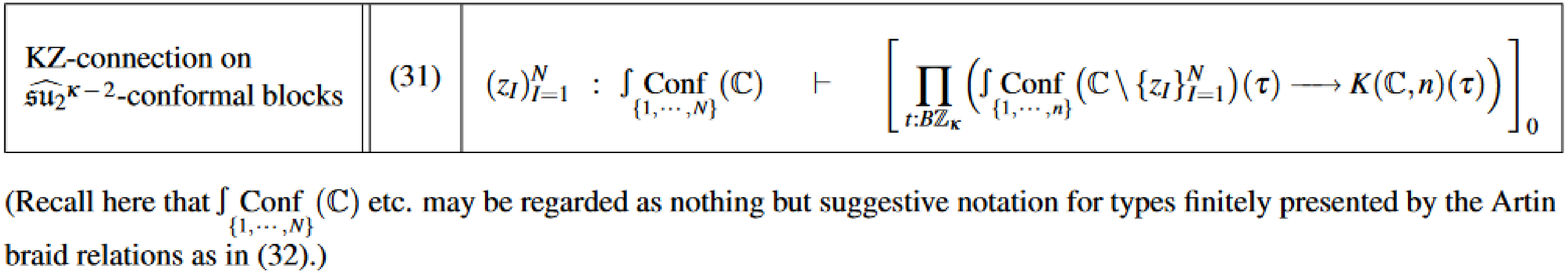

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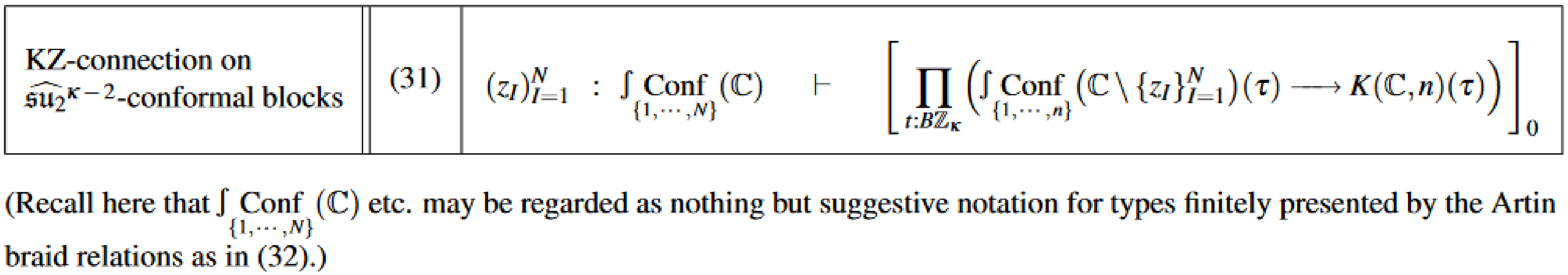

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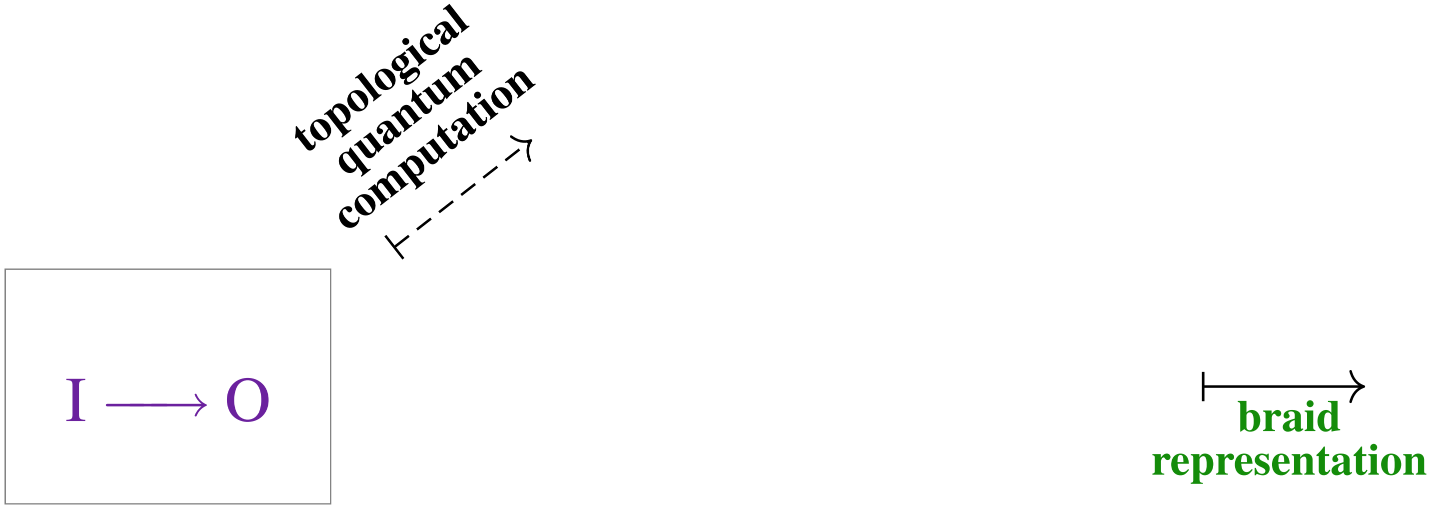
To compute is

the case of
Topological Quantum Computation
[Sati & Schreiber, PlanQC **2022** 33 (2022)]

To compute is to **execute**

the case of
Topological Quantum Computation
[Sati & Schreiber, PlanQC **2022** 33 (2022)]



$I \longrightarrow O$

topological quantum computation

braid representation

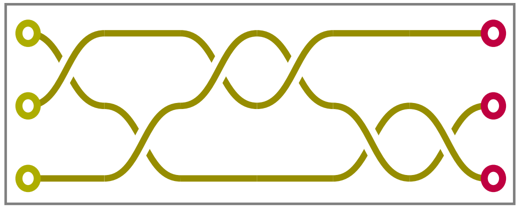
To compute is to **execute**
sequences of **instructions**

the case of
Topological Quantum Computation
[Sati & Schreiber, PlanQC **2022** 33 (2022)]

topological quantum computation

$I \longrightarrow O$

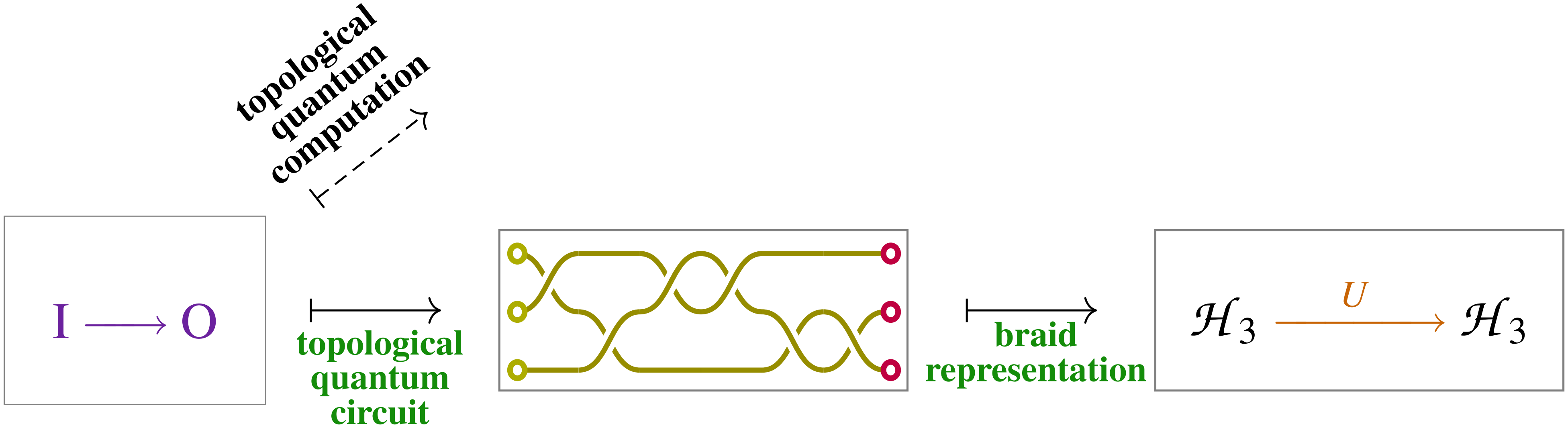
topological quantum circuit



braid representation

To compute is to **execute** sequences of **instructions** as composable **operations**

the case of
Topological Quantum Computation
[Sati & Schreiber, PlanQC **2022** 33 (2022)]



$\mathrm{I} \longrightarrow \mathrm{O}$

topological quantum computation

topological quantum circuit

braid representation

$\mathcal{H}_3 \xrightarrow{U} \mathcal{H}_3$

To compute is to **execute**
sequences of **instructions**
as composable **operations**
on a chosen **state space**,

the case of
Topological Quantum Computation
[Sati & Schreiber, PlanQC **2022** 33 (2022)]

$$\begin{tikzcd} \mathcal{H}_3 \arrow[r, "U"] & \mathcal{H}_3 \\ |\psi_{\mathrm{in}}\rangle \arrow[r, mapsto] & |\psi_{\mathrm{out}}\rangle \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[d, mapsto] \\ {} \end{tikzcd}$$

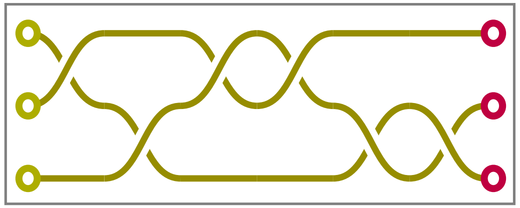
topological quantum computation

$$\begin{tikzcd} {} \arrow[r, mapsto, dashed] & {} \end{tikzcd}$$

$$\mathrm{I} \longrightarrow \mathrm{O}$$

$$\begin{tikzcd} {} \arrow[r, mapsto] & {} \end{tikzcd}$$

topological quantum circuit



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braid representation

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To compute is to **execute**
sequences of **instructions**
as composable **operations**
on a chosen **state space**,
turning a given **initial state**

the case of
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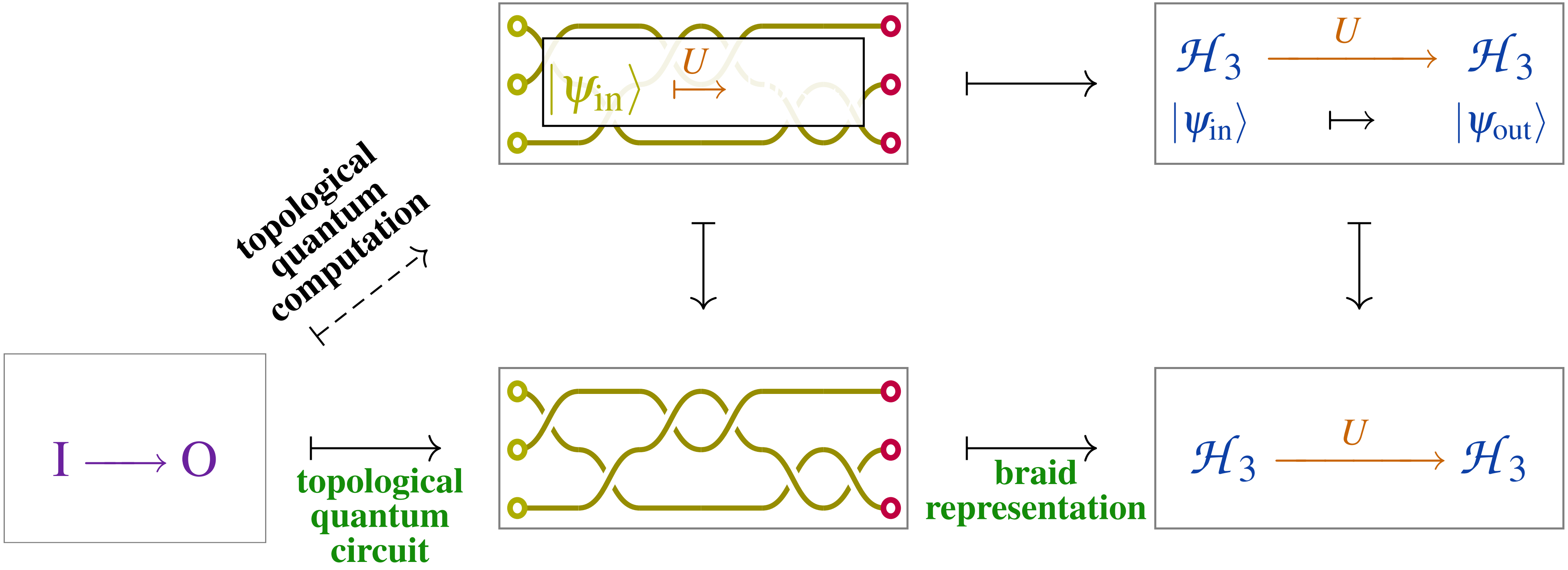
[Sati & Schreiber, PlanQC **2022** 33 (2022)]



To compute is to **execute**
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as composable **operations**
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turning a given **initial state**
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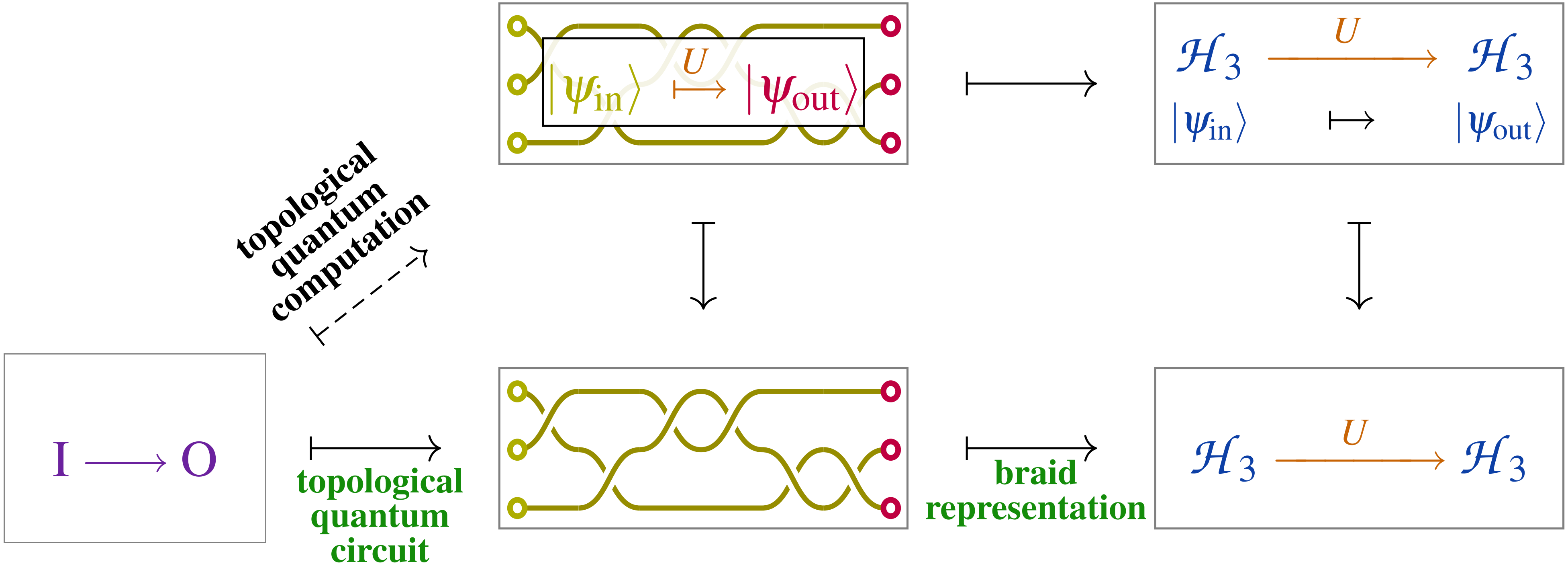
the case of
Topological Quantum Computation
[Sati & Schreiber, PlanQC **2022** 33 (2022)]



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the case of
Topological Quantum Computation
[Sati & Schreiber, PlanQC **2022** 33 (2022)]

$$\begin{tikzcd}
 & \textbf{bundle of conformal blocks} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \textbf{quantum states in Hilbert spaces} \arrow[d] \\
\textbf{path} \arrow[ur, dashed, "\textbf{topological quantum computation}"] \arrow[r, "\textbf{topological quantum program}"'] & \textbf{configuration space of distinct points} \arrow[r, mapsto, "\textbf{braid representation}"'] & \textbf{unitary operators}
\end{tikzcd}$$

To compute is to **execute**
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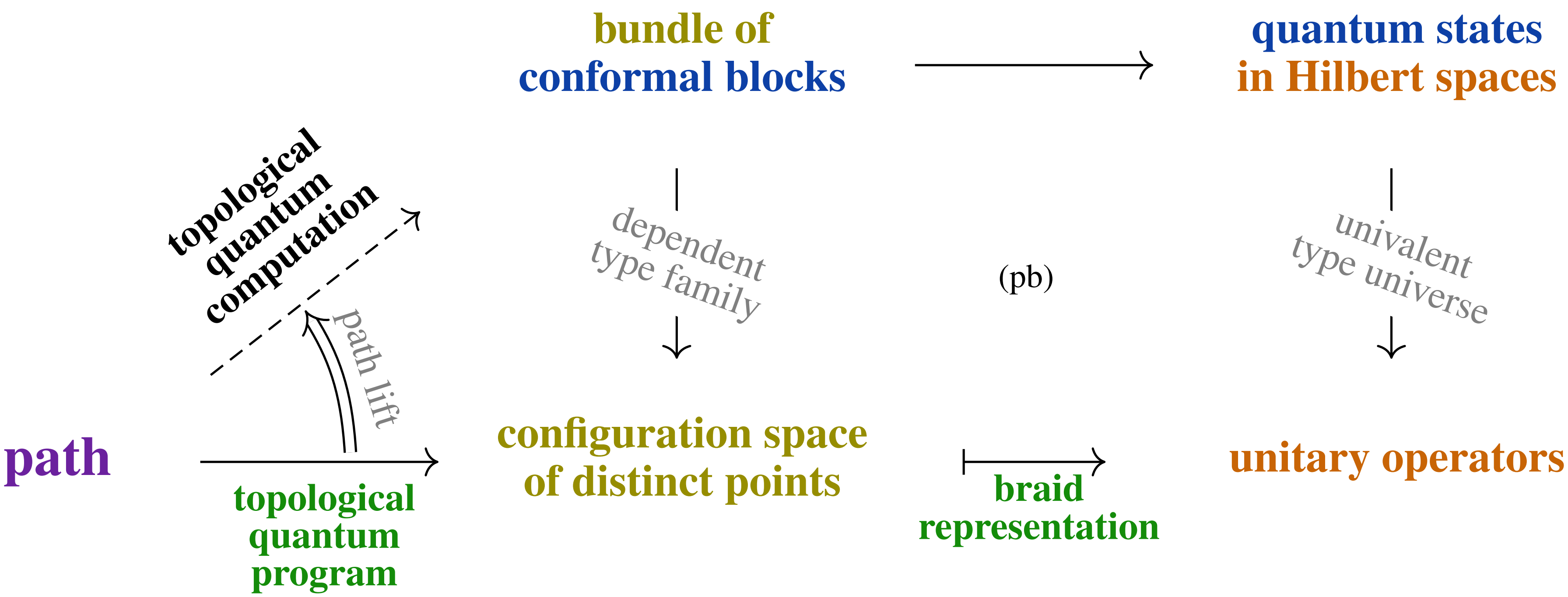
the case of
Topological Quantum Computation
[Sati & Schreiber, PlanQC **2022** 33 (2022)]

Claim: This has natural construction in HoTT languages:



bundle of
conformal blocks

quantum states
in Hilbert spaces

dependent
type family

(pb)

univalent
type universe

topological
quantum
computation

path lift

path

topological
quantum
program

configuration space
of distinct points

braid
representation

unitary operators

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$$\begin{tikzcd}[column sep=small]
\overset{\text{even-periodic de Rham cohomology}}{\underset{\substack{d \in \mathbb{Z} \\ 1 \leq r \leq \kappa}}{\bigoplus} H^{n+2d}\big(\Omega^\bullet_{\mathrm{dR}}(\mathrm{X};\mathbb{C}), \overset{\substack{\text{twisted by tensor powers of} \\ \text{the flat complex line bundle}}}{\mathrm{d} + r\cdot\omega_1\wedge}\big)} \arrow[rr] \arrow[dr, "\substack{\text{secondary} \\ \text{Chern character}}"'] & & \overset{\substack{BC^{+}_{\kappa}\text{-twisted} \\ C_\kappa\text{-equivariant} \\ \text{differential K-theory}}}{\underset{\substack{\text{smooth manifold inside} \\ C_\kappa\text{-orbi-singularity}}}{\mathrm{KU}^{n-1+[\omega_1]}_{\mathrm{diff}}\big(\mathrm{X}\times * /\!\!/ C_\kappa\big)}} \arrow[rr] \arrow[dr, "\text{forget}"'] & & \overset{\text{even-periodic twisted de Rham cohomology in opposite degree}}{\underset{\substack{d \in \mathbb{Z} \\ 1 \leq r \leq \kappa}}{\bigoplus} H^{n-1+2d}\big(\Omega^\bullet_{\mathrm{dR}}(\mathrm{X};\mathbb{C}), \mathrm{d} + r\cdot\omega_1\wedge\big)}. \\
 & \underset{\text{flat TED-K-theory}}{\mathrm{KU}^{n-1+[\omega_1]}_{\flat}\big(\mathrm{X}\times * /\!\!/ C_\kappa\big)} \arrow[ur, "\text{include}"] \arrow[rr] & & \underset{\text{underlying twisted equivariant K-theory}}{\mathrm{KU}^{n-1+[\omega_1]}\big(\mathrm{X}\times * /\!\!/ C_\kappa\big)} \arrow[ur, "\text{Chern character}"'] &
\end{tikzcd}$$

arXiv > hep-th > arXiv:2203.11838

High Energy Physics - Theory

[Submitted on 22 Mar 2022]

Anyonic Defect Branes and Conformal Blocks in Twisted Equivariant Differential (TED) K-theory

Hisham Sati, Urs Schreiber

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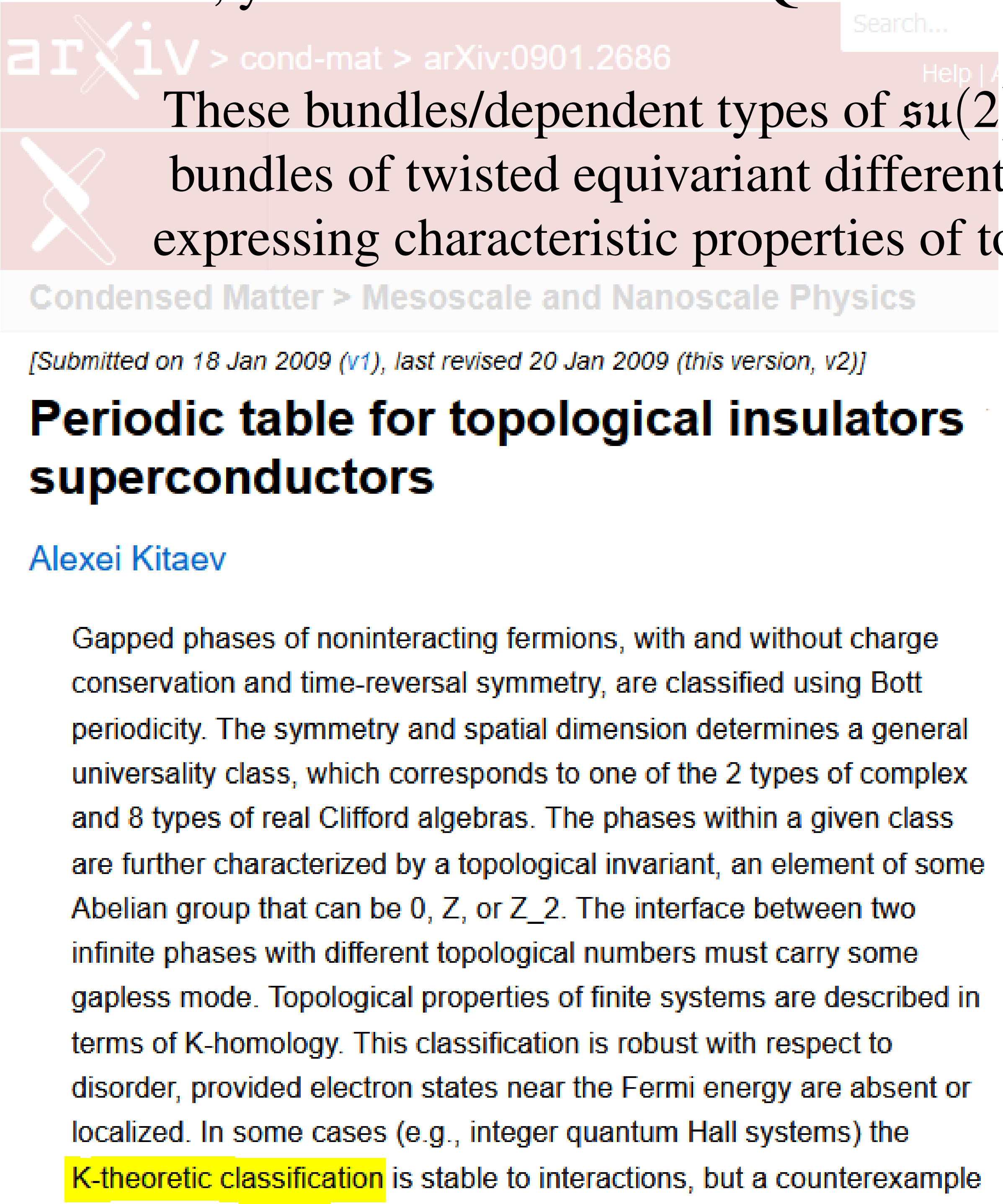

Condensed Matter > Mesoscale and Nanoscale Physics

[Submitted on 18 Jan 2009 (v1), last revised 20 Jan 2009 (this version, v2)]

Periodic table for topological insulators superconductors

Alexei Kitaev

Gapped phases of noninteracting fermions, with and without charge conservation and time-reversal symmetry, are classified using Bott periodicity. The symmetry and spatial dimension determines a general universality class, which corresponds to one of the 2 types of complex and 8 types of real Clifford algebras. The phases within a given class are further characterized by a topological invariant, an element of some Abelian group that can be 0, Z, or Z_2. The interface between two infinite phases with different topological numbers must carry some gapless mode. Topological properties of finite systems are described in terms of K-homology. This classification is robust with respect to disorder, provided electron states near the Fermi energy are absent or localized. In some cases (e.g., integer quantum Hall systems) the K-theoretic classification is stable to interactions, but a counterexample

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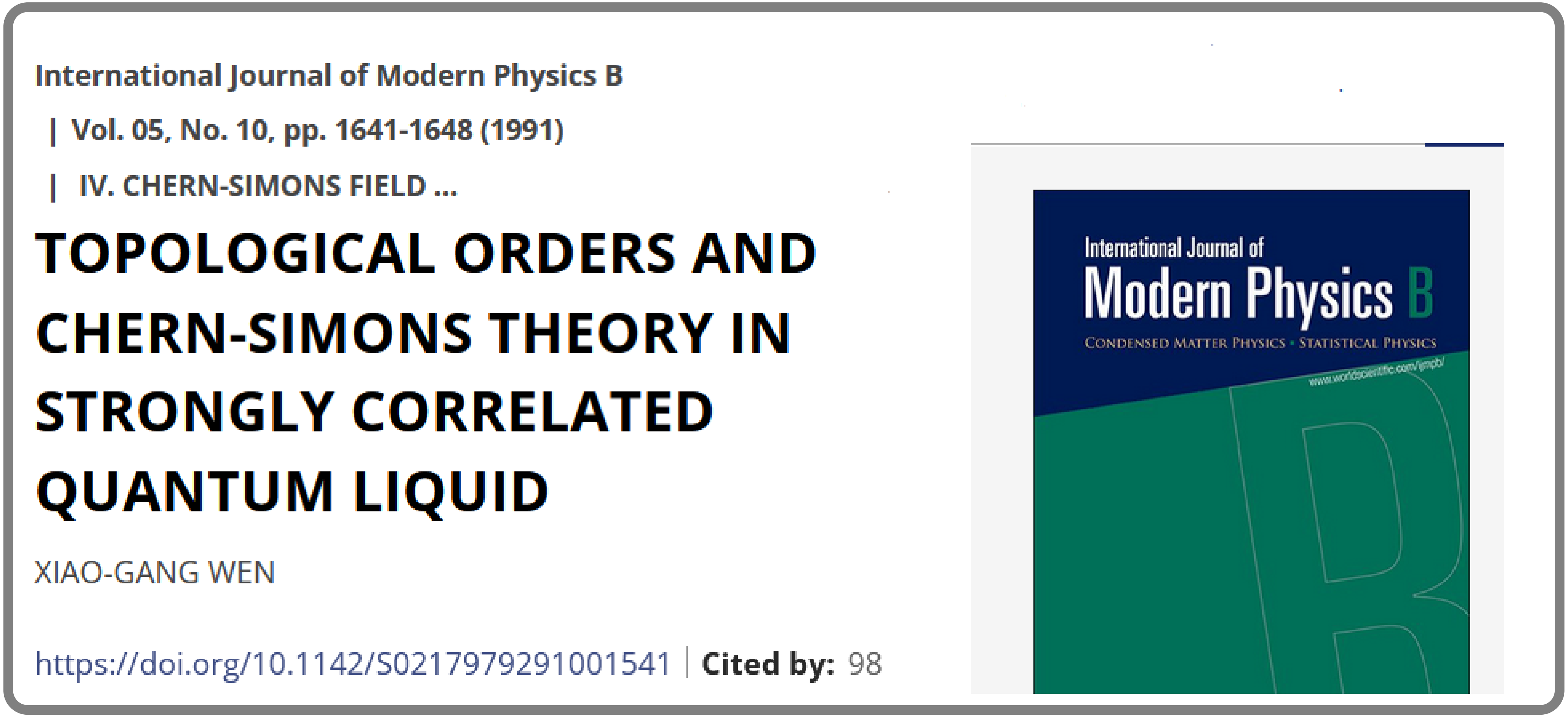

International Journal of Modern Physics B

| Vol. 05, No. 10, pp. 1641-1648 (1991)

| IV. CHERN-SIMONS FIELD ...

TOPOLOGICAL ORDERS AND CHERN-SIMONS THEORY IN STRONGLY CORRELATED QUANTUM LIQUID

XIAO-GANG WEN

https://doi.org/10.1142/S0217979291001541 | **Cited by:** 98

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arXiv > hep-th > arXiv:2206.13563

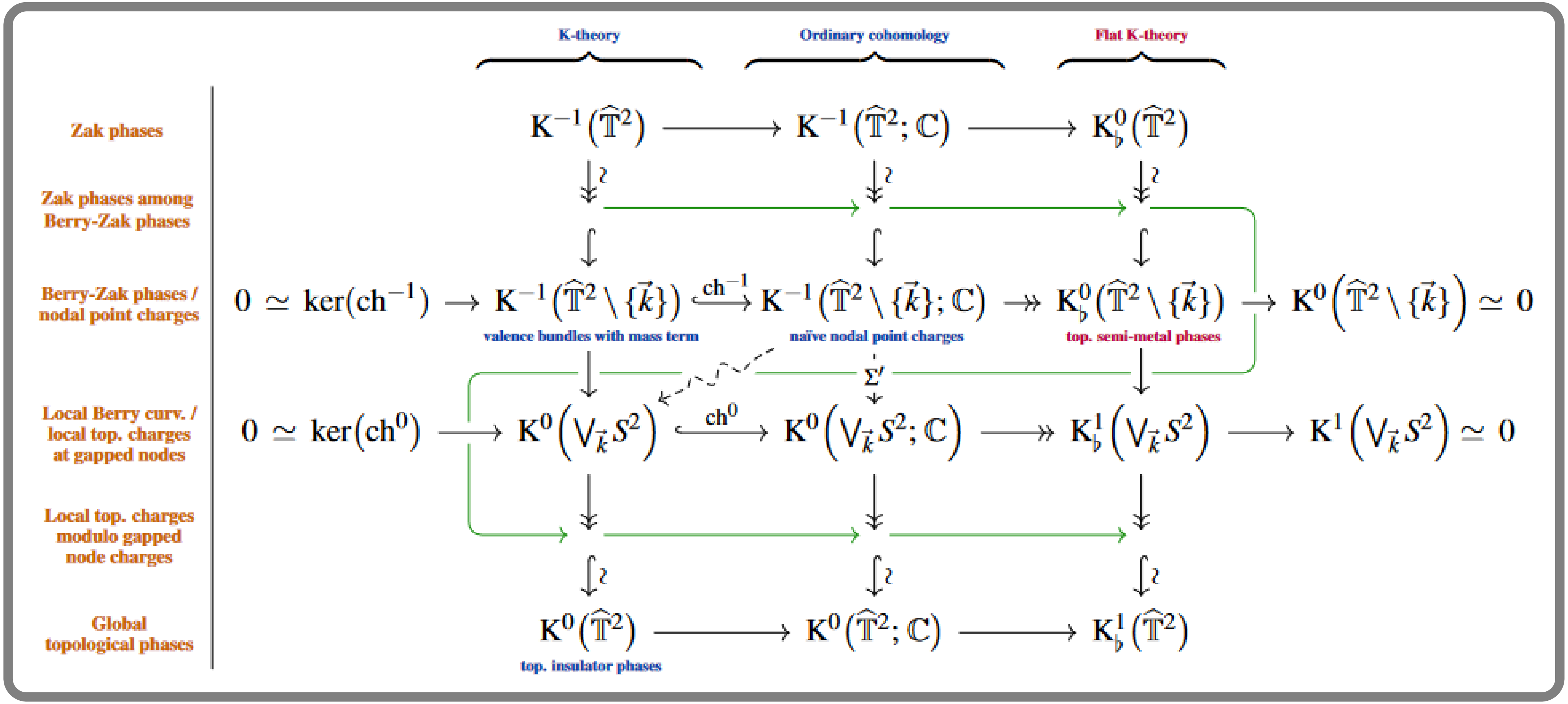
High Energy Physics - Theory

[Submitted on 27 Jun 2022]

Anyonic Topological Order in Twisted Equivariant Differential (TED) K-Theory

Hisham Sati, Urs Schreiber

expressing characteristic properties of topological phases of matter hosting anyonic defects in their topologically ordered ground states.



K-theory

Ordinary cohomology

Flat K-theory

Zak phases

Zak phases among Berry-Zak phases

Berry-Zak phases / nodal point charges

Local Berry curv. / local top. charges at gapped nodes

Local top. charges modulo gapped node charges

Global topological phases

$$\begin{tikzcd}
 & K^{-1}\big(\widehat{\mathbb{T}}^2\big) \arrow[r] \arrow[d, two heads, "\wr"] & K^{-1}\big(\widehat{\mathbb{T}}^2;\mathbb{C}\big) \arrow[r] \arrow[d, two heads, "\wr"] & K^0_\flat\big(\widehat{\mathbb{T}}^2\big) \arrow[d, two heads, "\wr"] & \\
 & {} \arrow[r] \arrow[d, hook] & {} \arrow[r] \arrow[d, hook] & {} \arrow[d, hook] \arrow[dddll] & \\
0 \simeq \ker(\mathrm{ch}^{-1}) \arrow[r] & K^{-1}\big(\widehat{\mathbb{T}}^2 \setminus \{\vec{k}\}\big) \arrow[r, hook, "\mathrm{ch}^{-1}"] \arrow[d] & K^{-1}\big(\widehat{\mathbb{T}}^2 \setminus \{\vec{k}\};\mathbb{C}\big) \arrow[r, two heads] \arrow[d, dashed, "\Sigma'"] \arrow[dl, dashed, squiggly] & K^0_\flat\big(\widehat{\mathbb{T}}^2 \setminus \{\vec{k}\}\big) \arrow[r] \arrow[d] & K^0\Big(\widehat{\mathbb{T}}^2 \setminus \{\vec{k}\}\Big) \simeq 0 \\
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valence bundles with mass term

naïve nodal point charges

top. semi-metal phases

top. insulator phases

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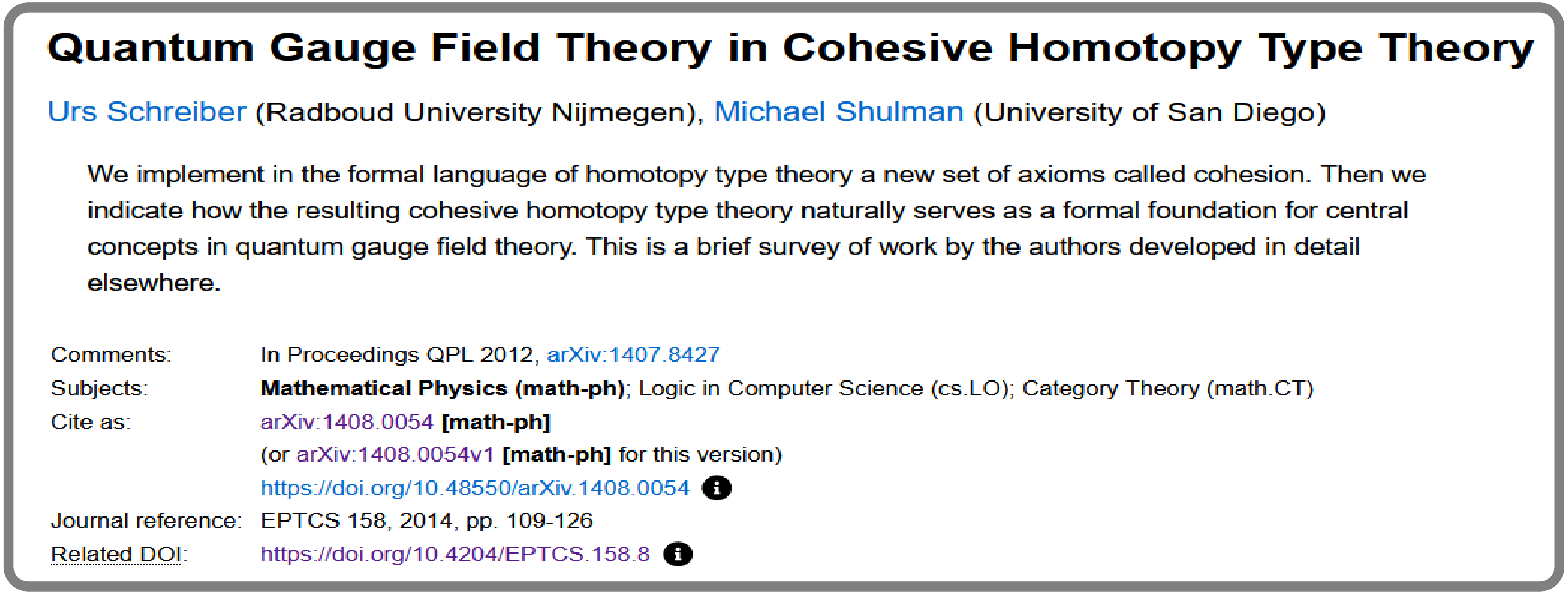

Quantum Gauge Field Theory in Cohesive Homotopy Type Theory

Urs Schreiber (Radboud University Nijmegen), Michael Shulman (University of San Diego)

We implement in the formal language of homotopy type theory a new set of axioms called cohesion. Then we indicate how the resulting cohesive homotopy type theory naturally serves as a formal foundation for central concepts in quantum gauge field theory. This is a brief survey of work by the authors developed in detail elsewhere.

Comments:	In Proceedings QPL 2012, arXiv:1407.8427
Subjects:	**Mathematical Physics (math-ph)**; Logic in Computer Science (cs.LO); Category Theory (math.CT)
Cite as:	arXiv:1408.0054 **[math-ph]** (or arXiv:1408.0054v1 **[math-ph]** for this version) https://doi.org/10.48550/arXiv.1408.0054 ℹ
Journal reference:	EPTCS 158, 2014, pp. 109-126
Related DOI:	https://doi.org/10.4204/EPTCS.158.8 ℹ

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In fa able:

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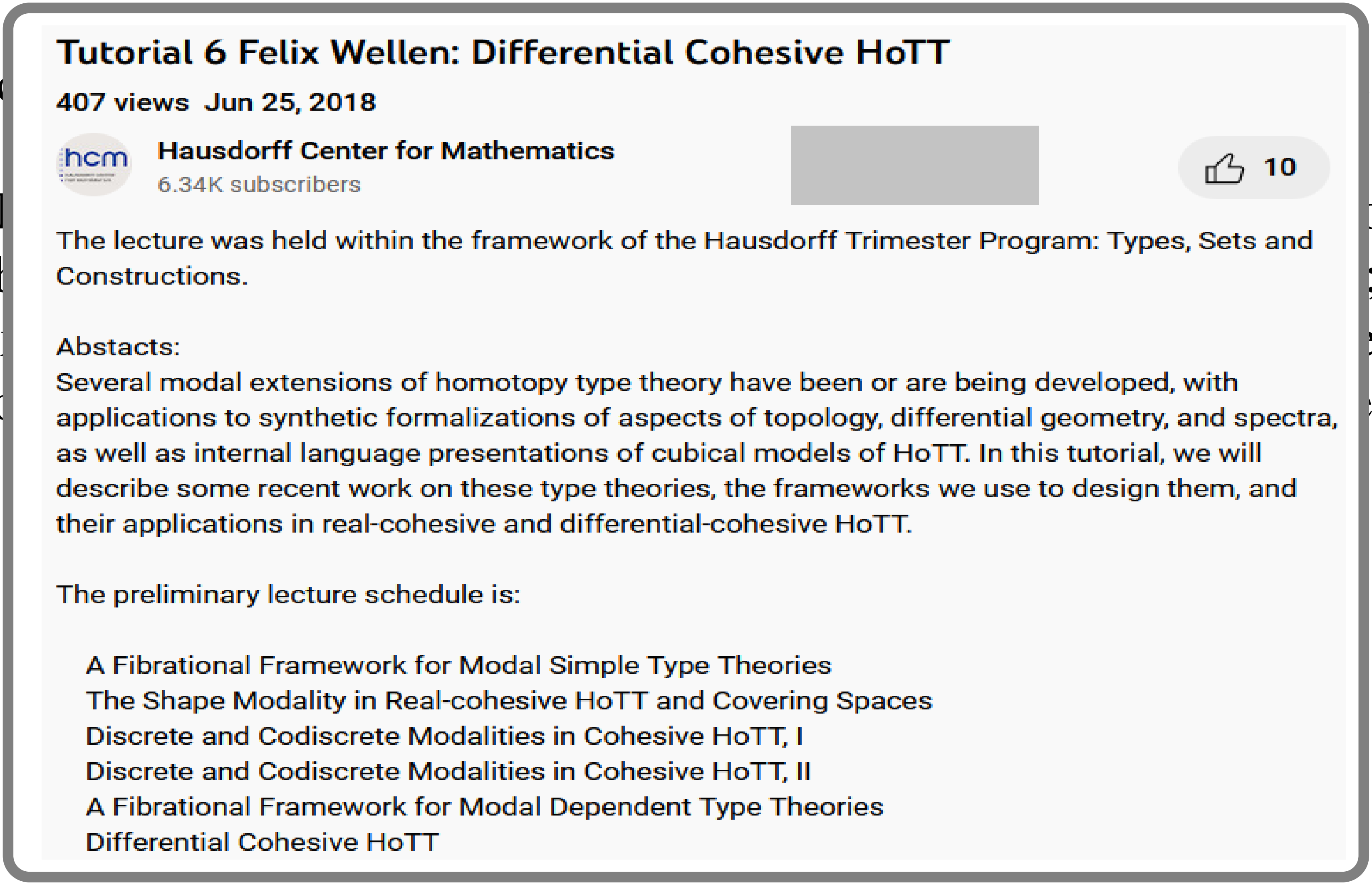

Tutorial 6 Felix Wellen: Differential Cohesive HoTT

407 views Jun 25, 2018

Hausdorff Center for Mathematics
6.34K subscribers

The lecture was held within the framework of the Hausdorff Trimester Program: Types, Sets and Constructions.

Abstacts:
Several modal extensions of homotopy type theory have been or are being developed, with applications to synthetic formalizations of aspects of topology, differential geometry, and spectra, as well as internal language presentations of cubical models of HoTT. In this tutorial, we will describe some recent work on these type theories, the frameworks we use to design them, and their applications in real-cohesive and differential-cohesive HoTT.

The preliminary lecture schedule is:

- A Fibrational Framework for Modal Simple Type Theories
- The Shape Modality in Real-cohesive HoTT and Covering Spaces
- Discrete and Codiscrete Modalities in Cohesive HoTT, I
- Discrete and Codiscrete Modalities in Cohesive HoTT, II
- A Fibrational Framework for Modal Dependent Type Theories
- Differential Cohesive HoTT

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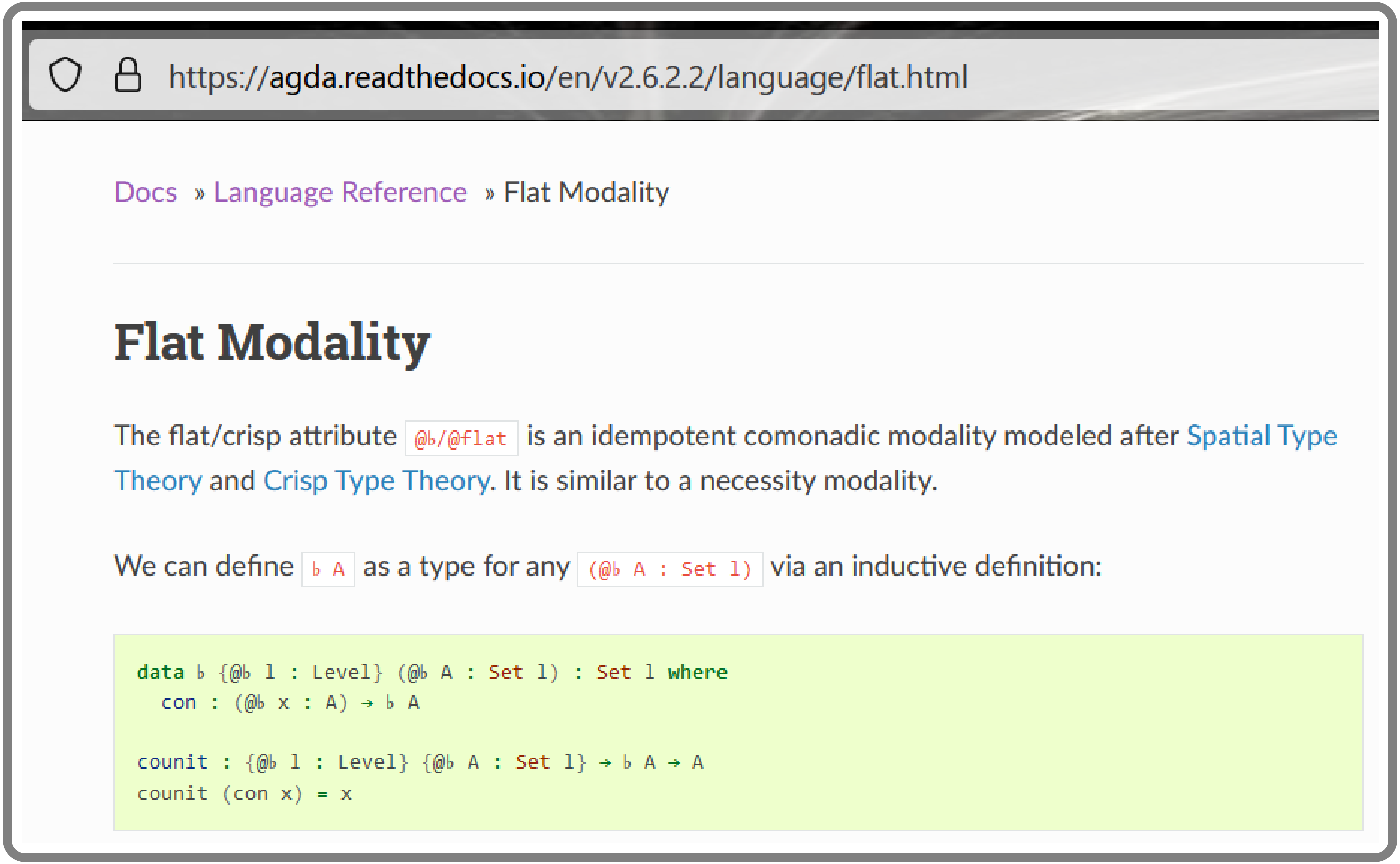
Parts of Cohesive HoTT have already been implemented in `Agda`.



https://agda.readthedocs.io/en/v2.6.2.2/language/flat.html

Docs » Language Reference » Flat Modality

Flat Modality

The flat/crisp attribute `@♭/@flat` is an idempotent comonadic modality modeled after Spatial Type Theory and Crisp Type Theory. It is similar to a necessity modality.

We can define `♭ A` as a type for any `(@♭ A : Set l)` via an inductive definition:

```
data ♭ {@♭ l : Level} (@♭ A : Set l) : Set l where
  con : (@♭ x : A) → ♭ A

counit : {@♭ l : Level} {@♭ A : Set l} → ♭ A → A
counit (con x) = x
```

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Further development at CQTS.

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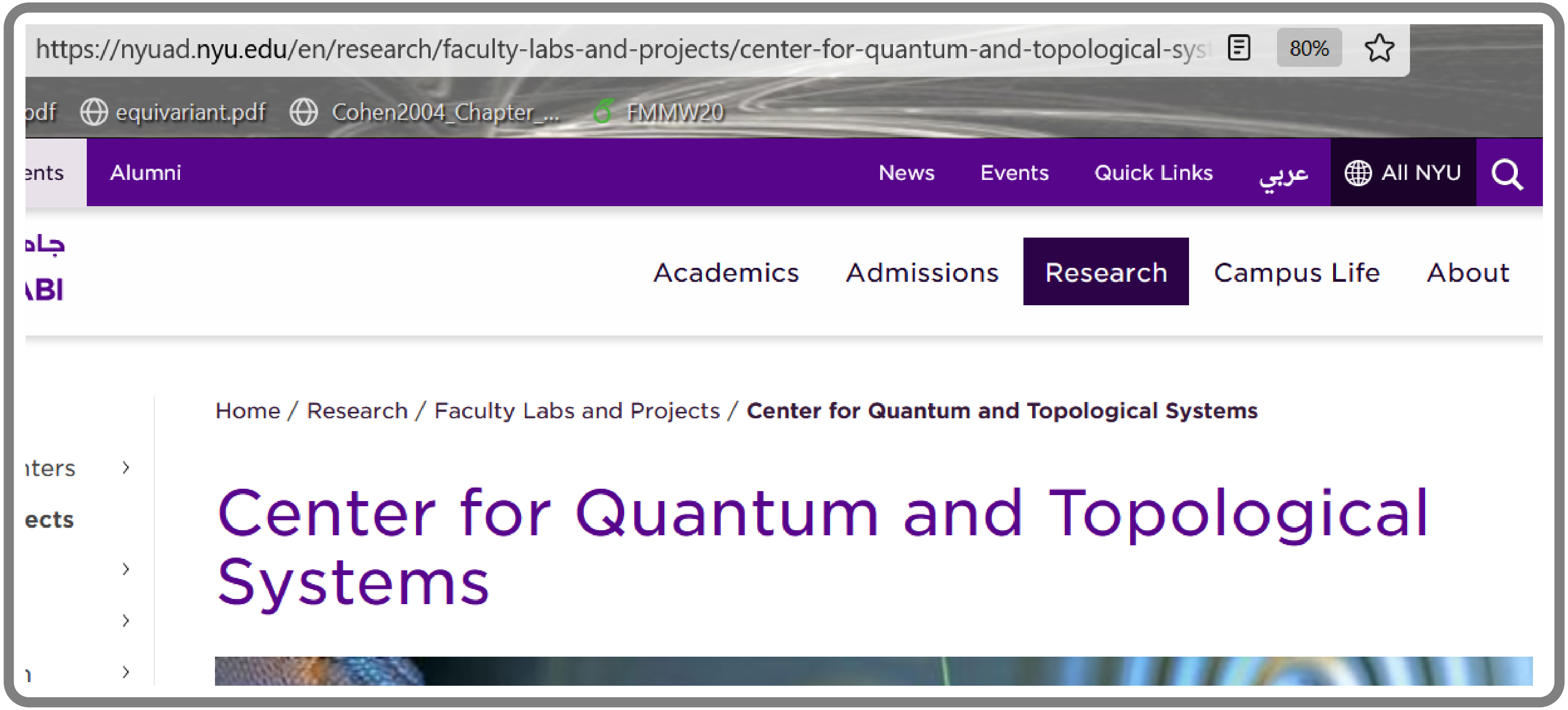


Home / Research / Faculty Labs and Projects / **Center for Quantum and Topological Systems**

Center for Quantum and Topological Systems

Further development at CQTS.

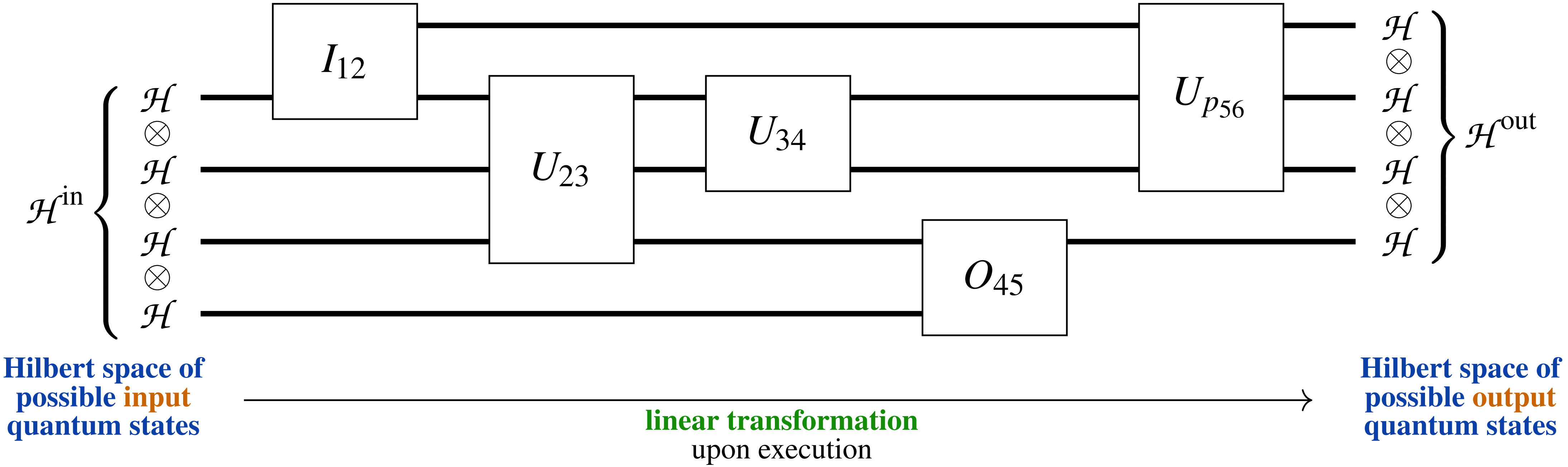
Part I
Verifying realistic topological quantum gates

Part II
Verifying their compilation into quantum circuits

The Problem

Pure quantum circuits are easy...

Linear operator composed & tensored from given *quantum logic gates*



Hilbert space of possible input quantum states

linear transformation
upon execution

Hilbert space of possible output quantum states

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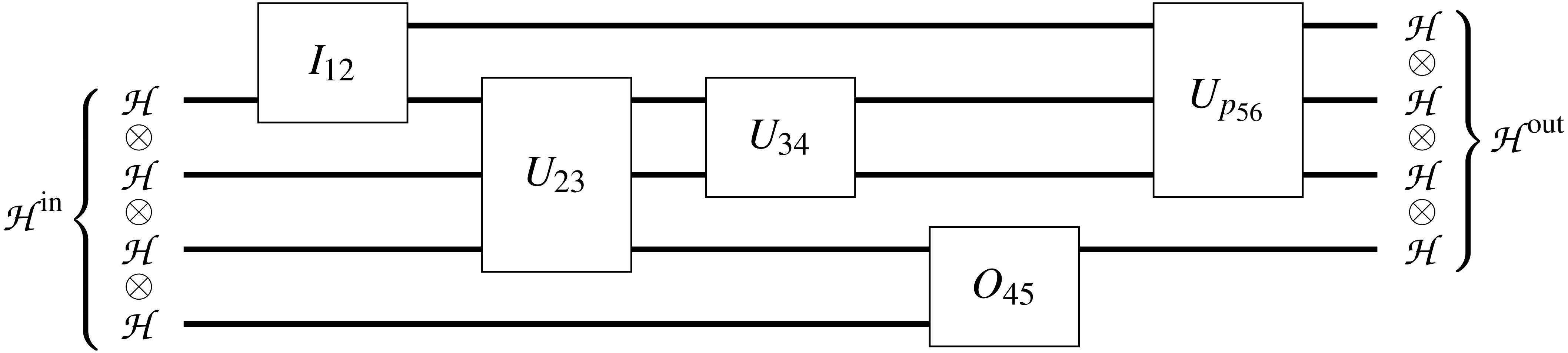


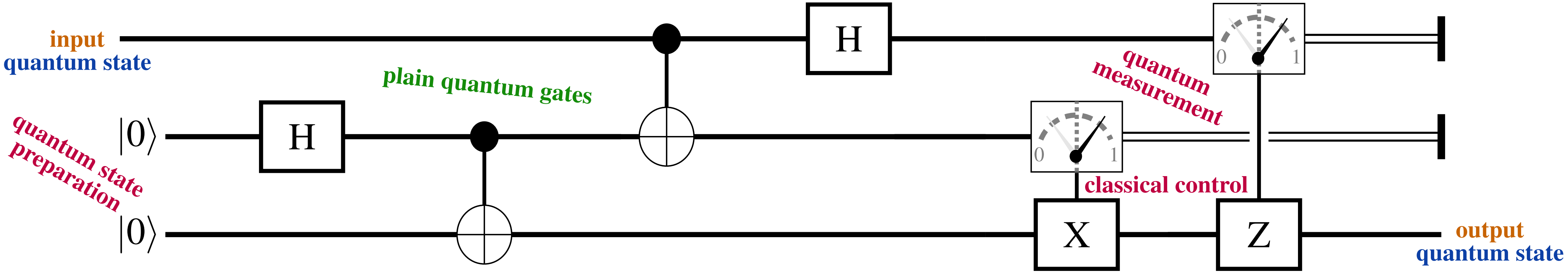
Hilbert space of possible input quantum states

linear transformation
upon execution

Hilbert space of possible output quantum states

but real quantum circuits have classical control & effects

(Example: QBit Teleportation protocol)



input quantum state

plain quantum gates

quantum state preparation

quantum measurement

classical control

output quantum state

full reality is a loop: $\text{Classical} \underset{\text{prepare}}{\overset{\text{measure}}{\leftrightarrows}} \text{Quantum}$

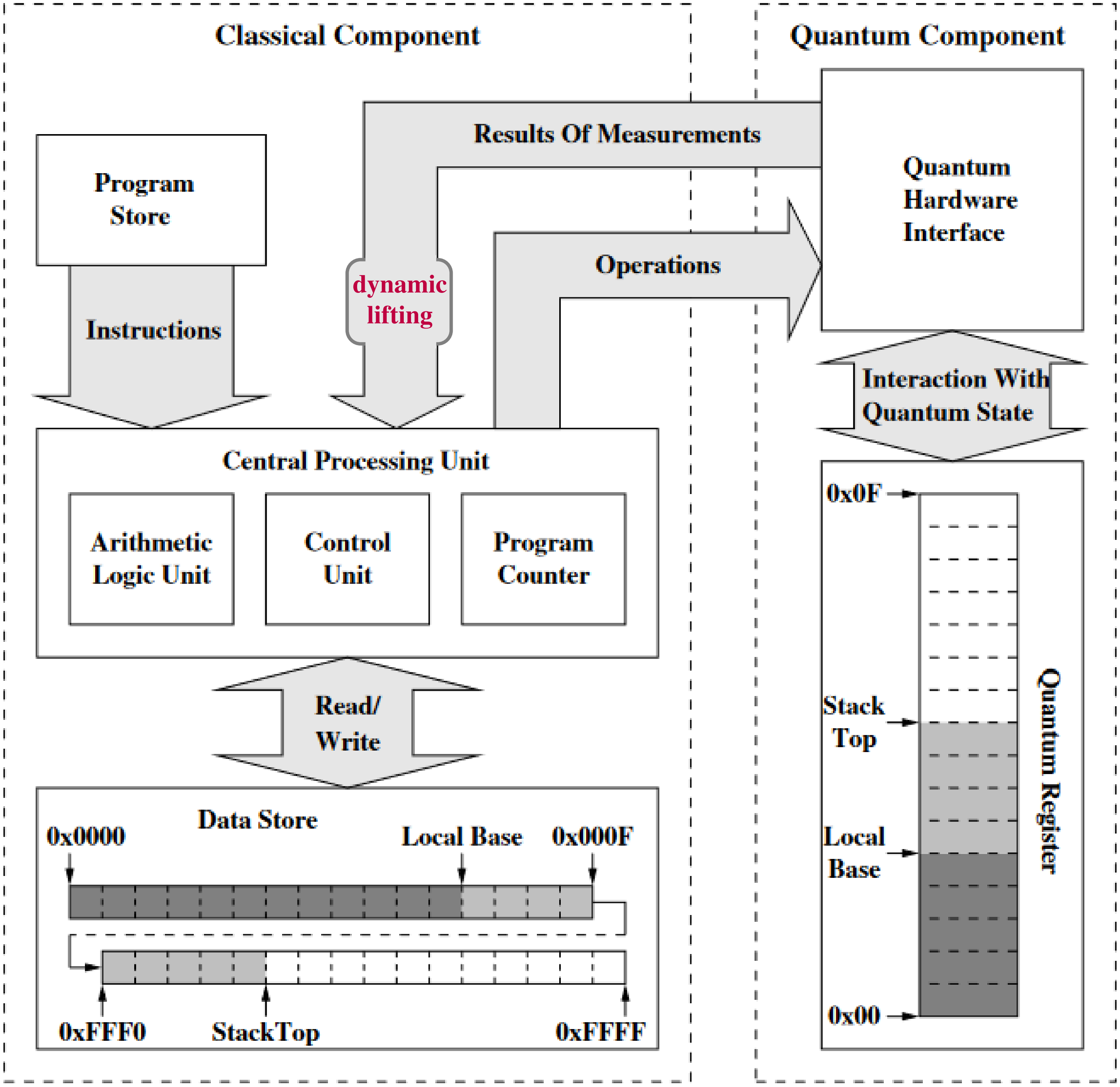


diagram adapted from Nagarajan et al. (2007)

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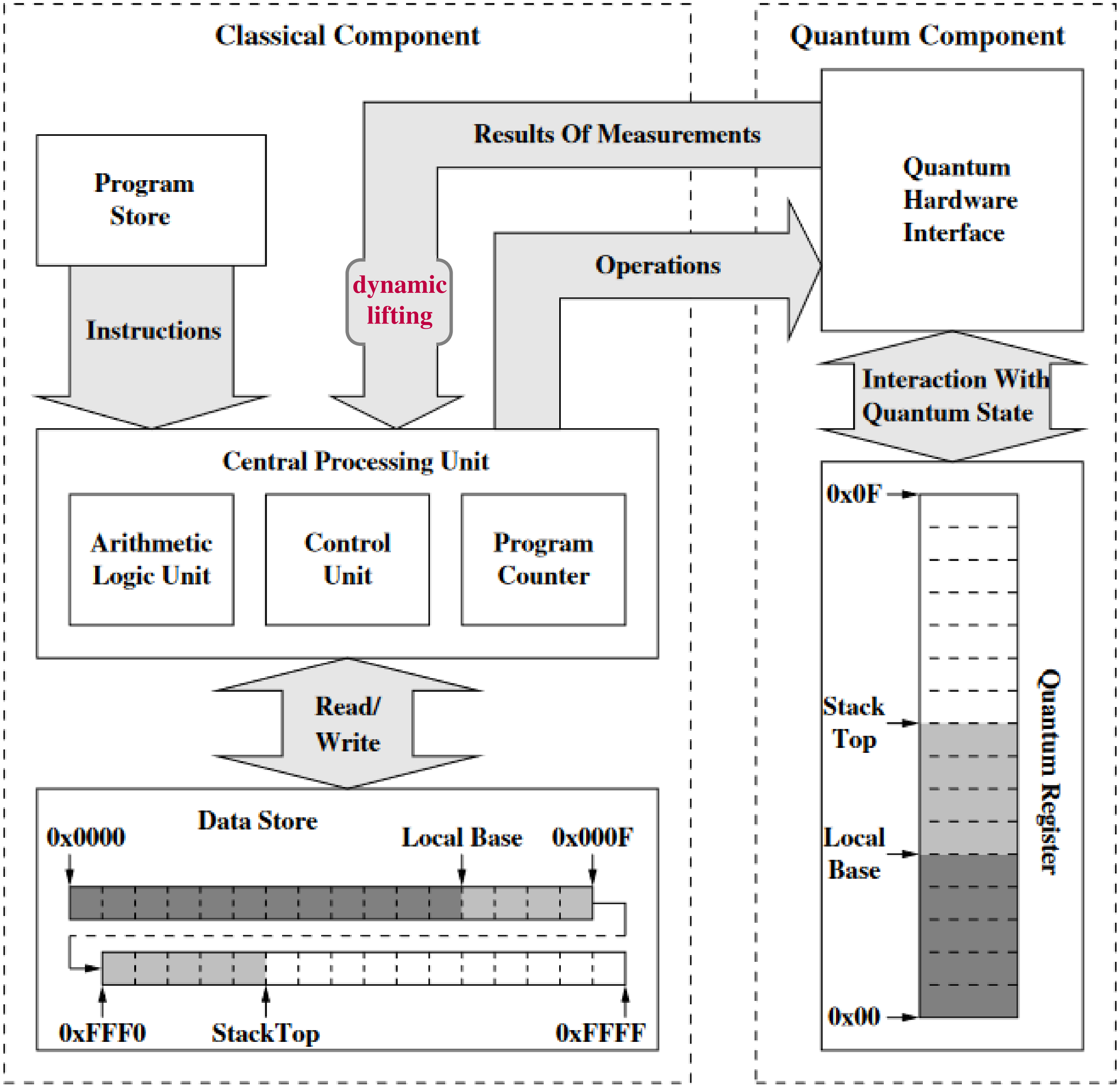

full loop needed e.g. for quantum error correction

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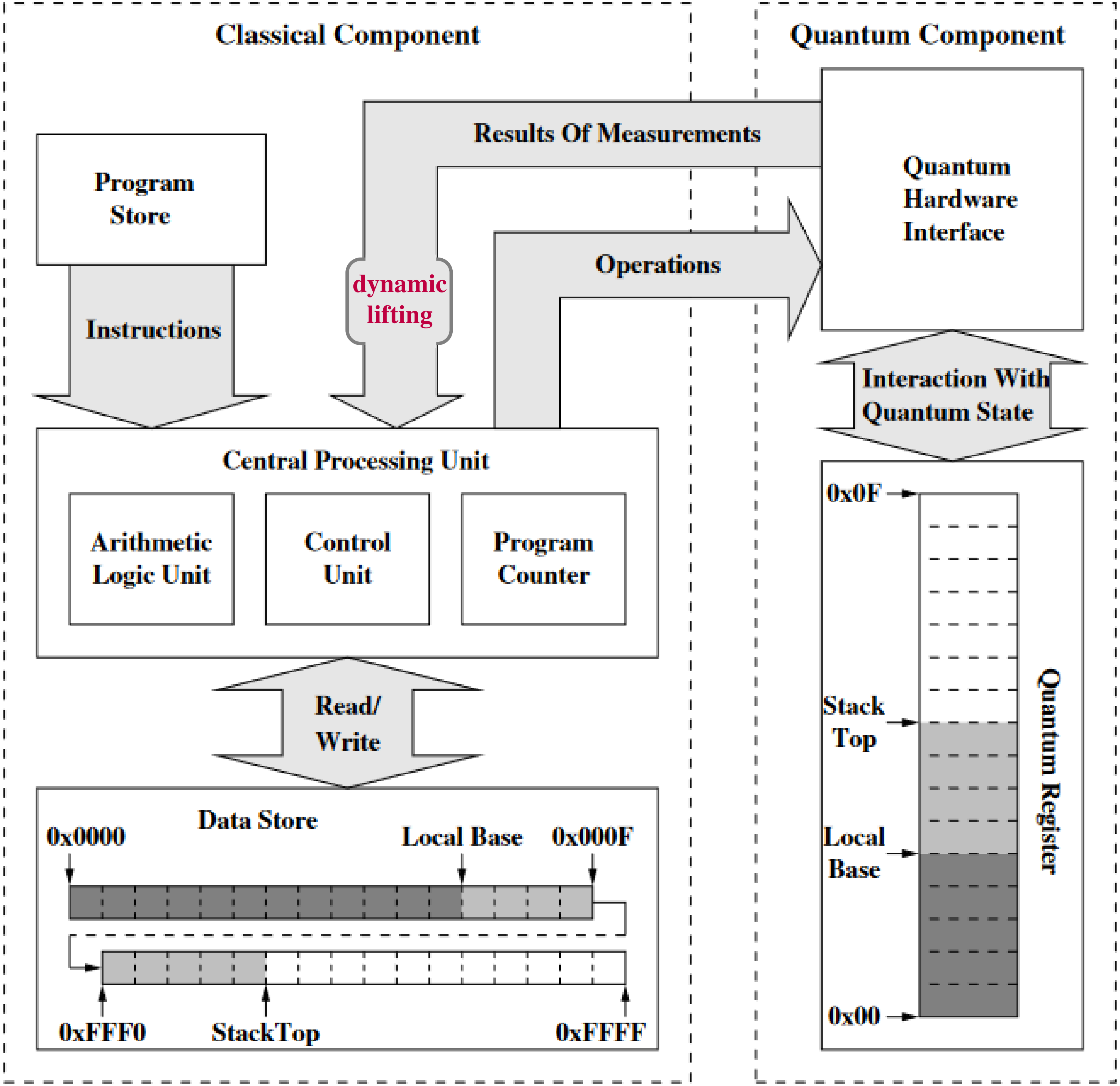

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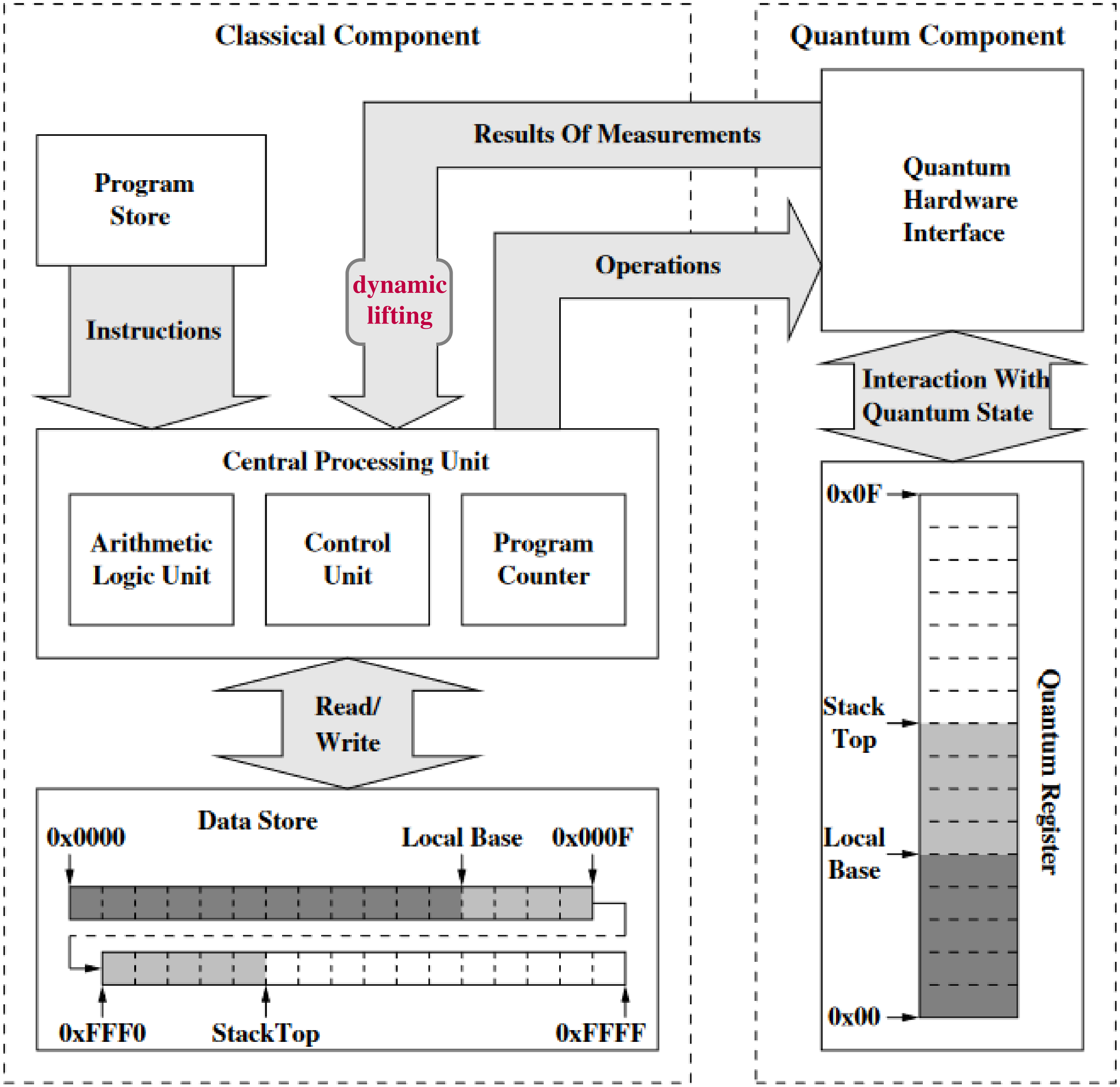

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existing models for dynamic lifting are ad hoc & unverified

Existing quantum typed circuit languages

are embedded inside *classical* type theories:

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Classical Type Theory (e.g. `Haskell`, `Coq`, ...)

for universal classical computation

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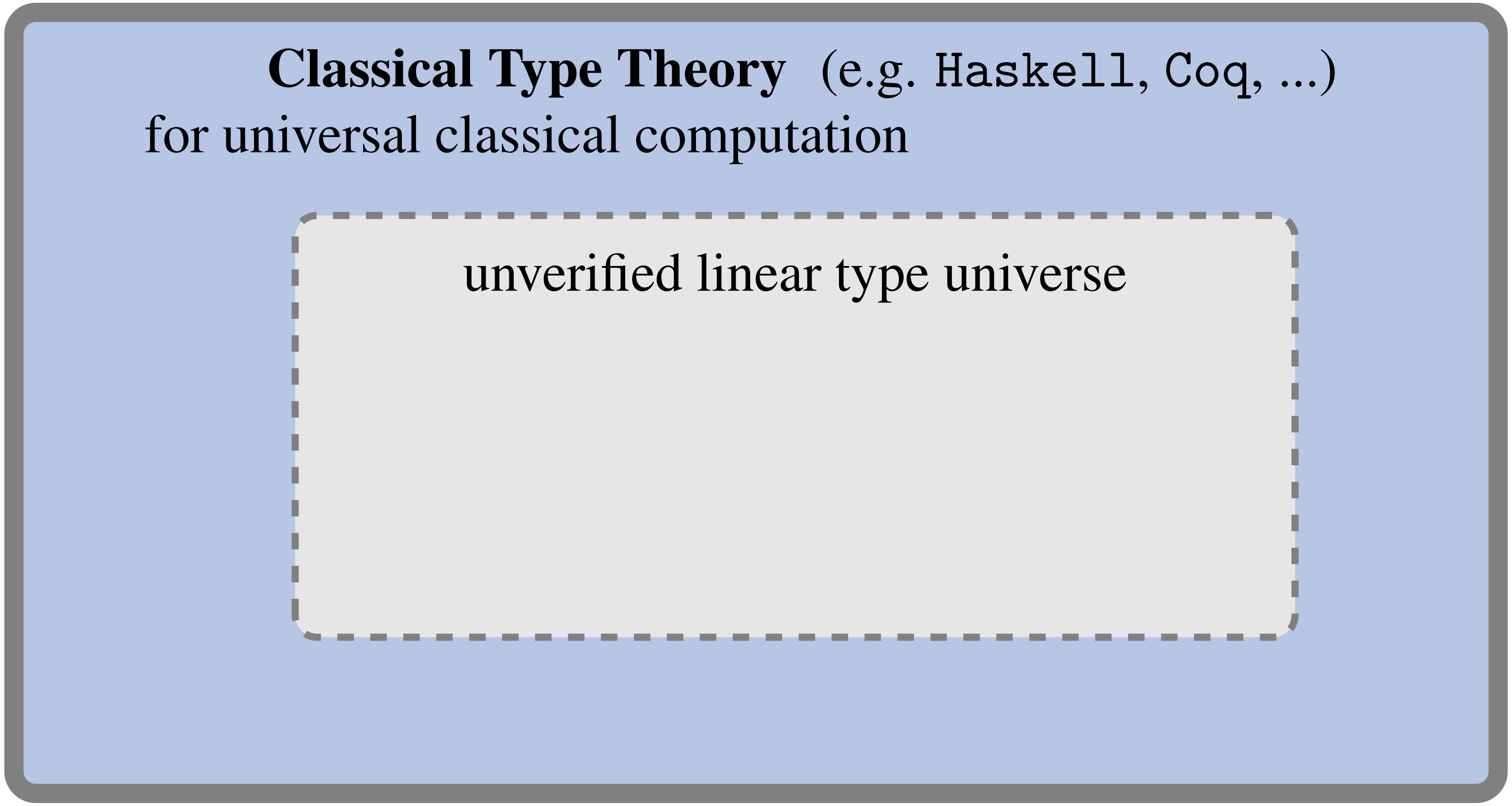

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unverified linear type universe

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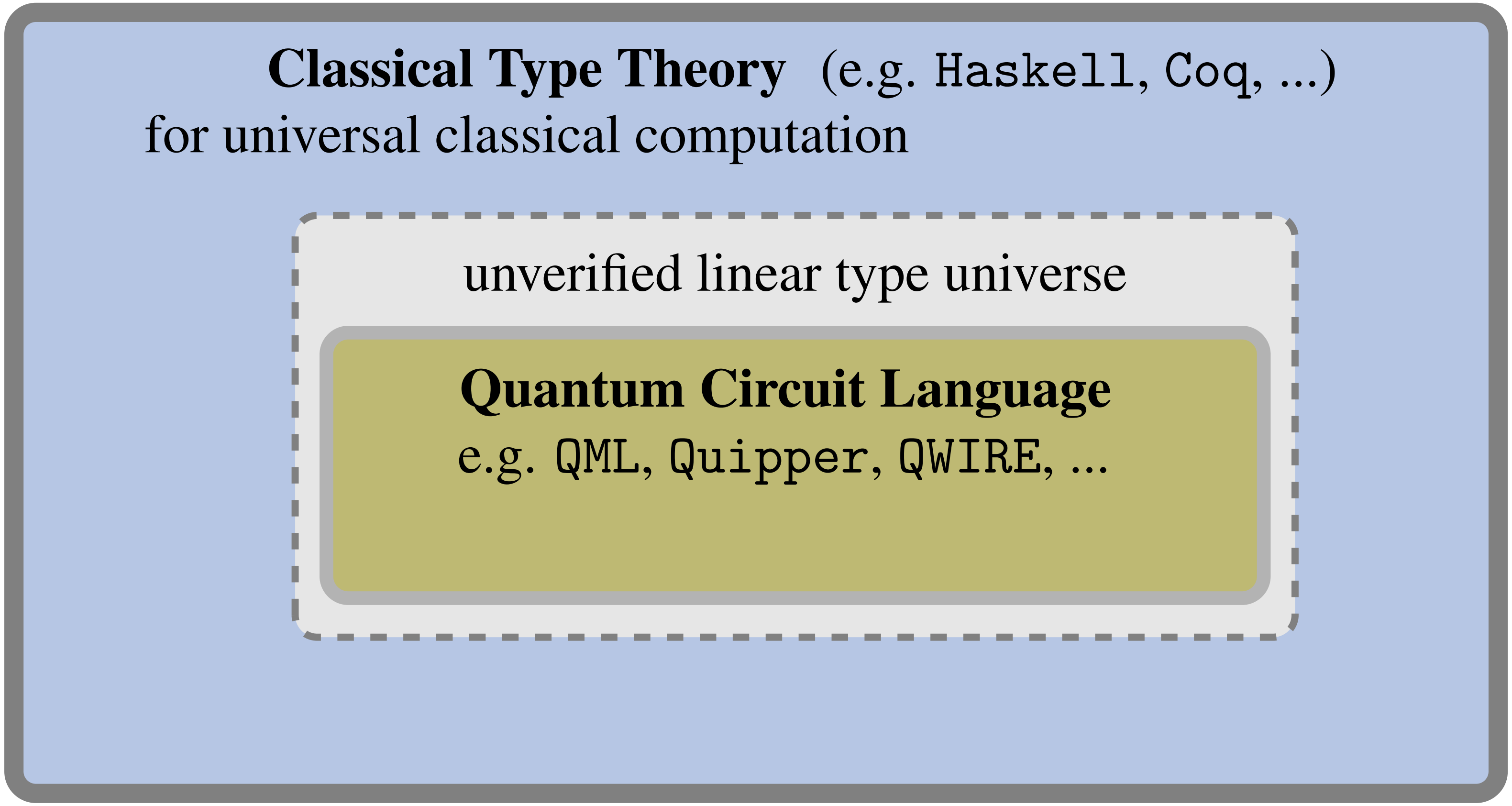

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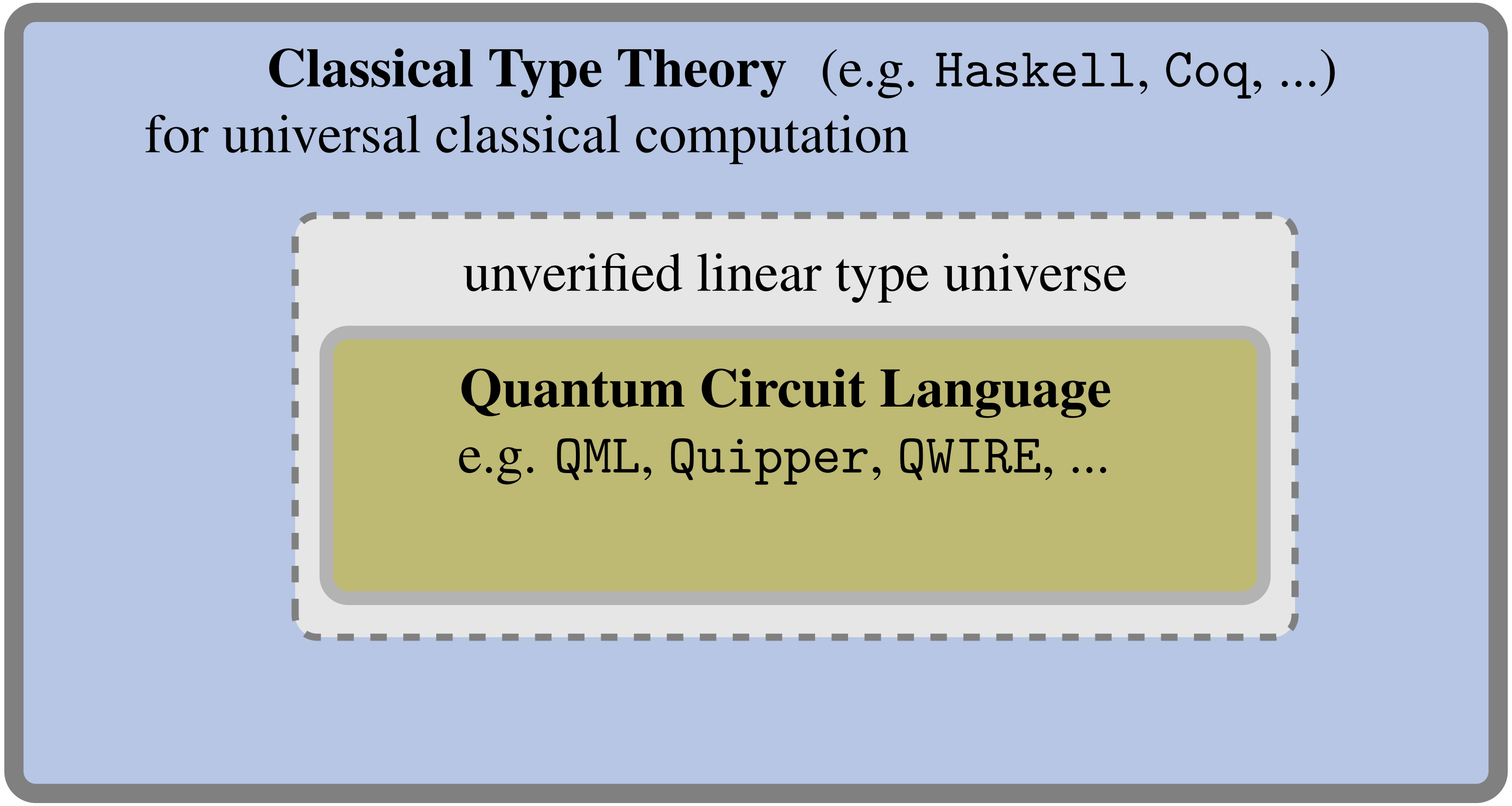

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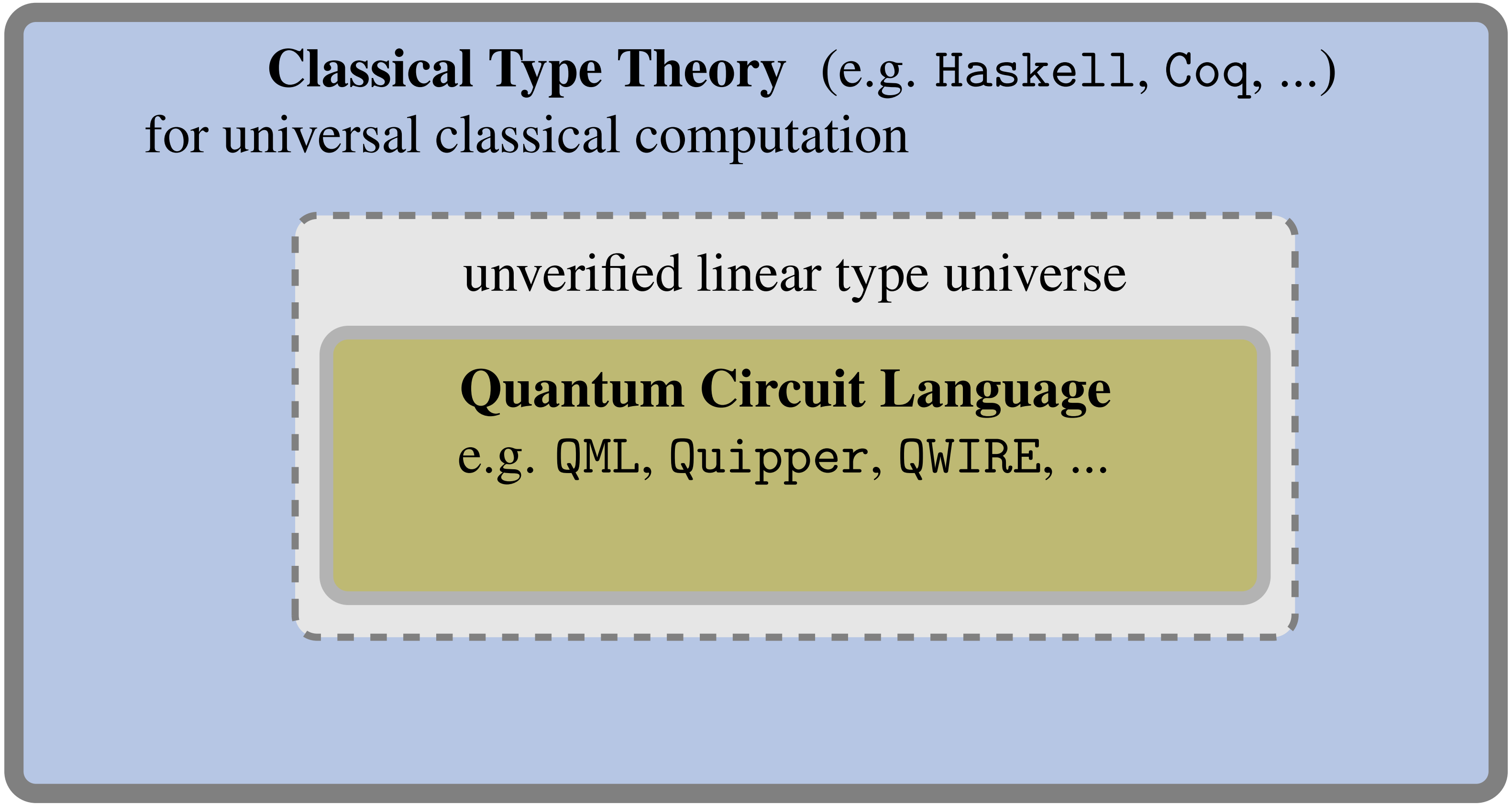

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Until now...

Our Solution

Dependent Linear Homotopy Type Theory (`LHoTT`)

Theorem [M. Riley (2022), `doi:10.14418/wes01.3.139`]:

Dependent Linear Homotopy Type Theory (`LHoTT`)

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$\exists$ *universal* quantum+classical specification language

conservative over classical *Homotopy Type Theory* (`HoTT`)

Dependent Linear Homotopy Type Theory (`LHoTT`)

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(i.e. Grothendieck's six operations *à la* Wirthmüller — more on all this below)

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`LHoTT` is like a quantum microscope for Classical Data Types

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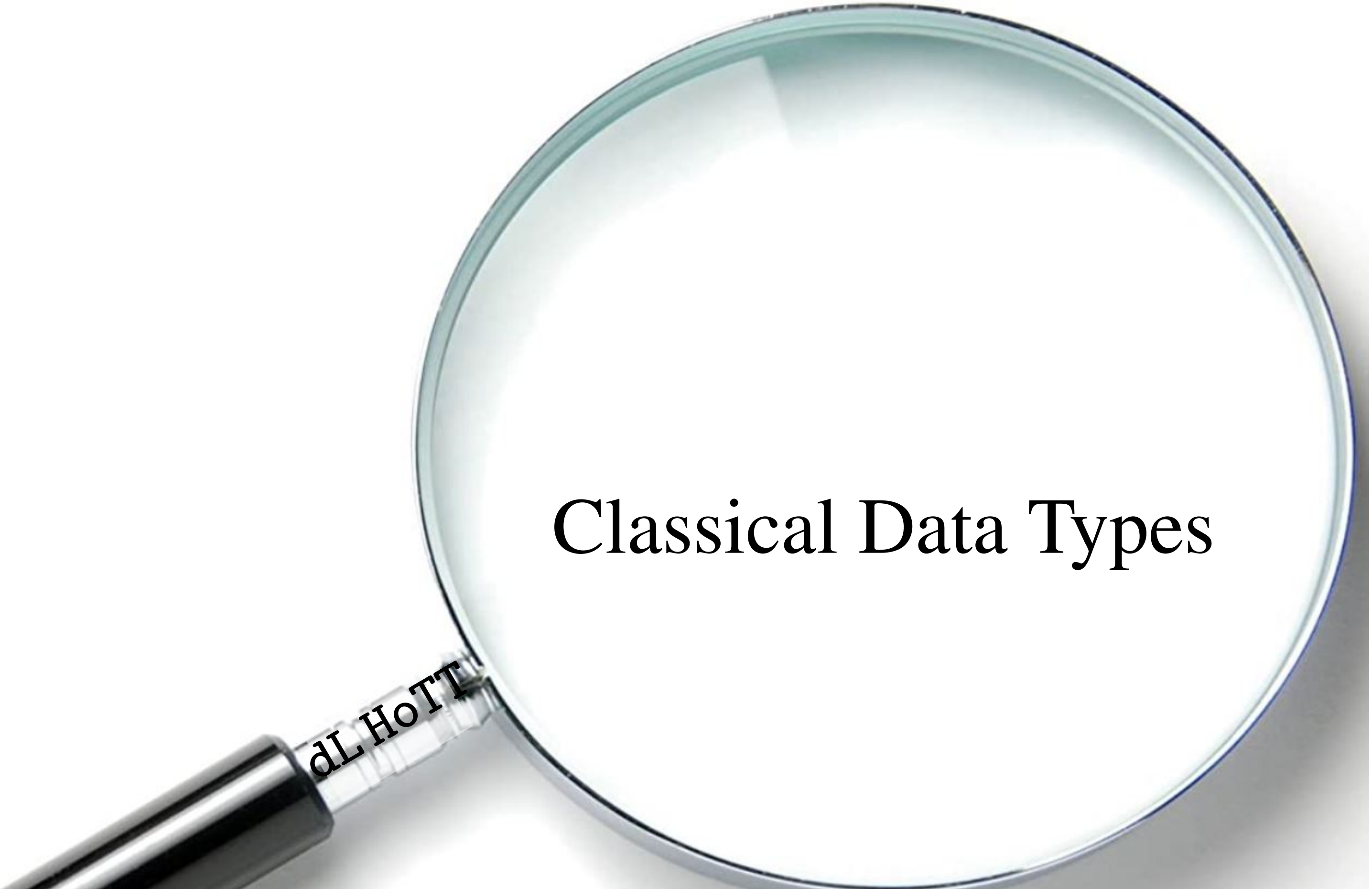

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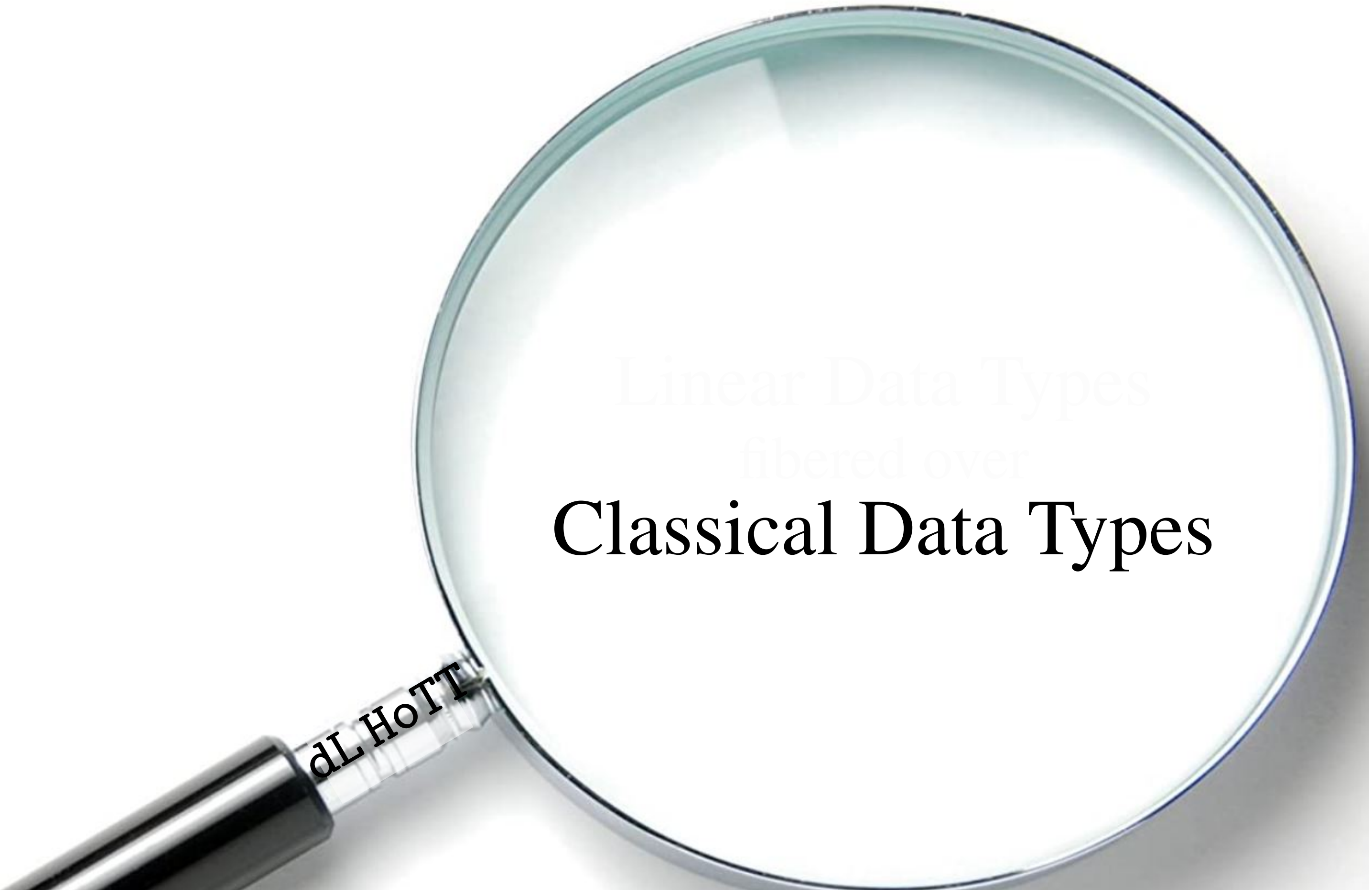

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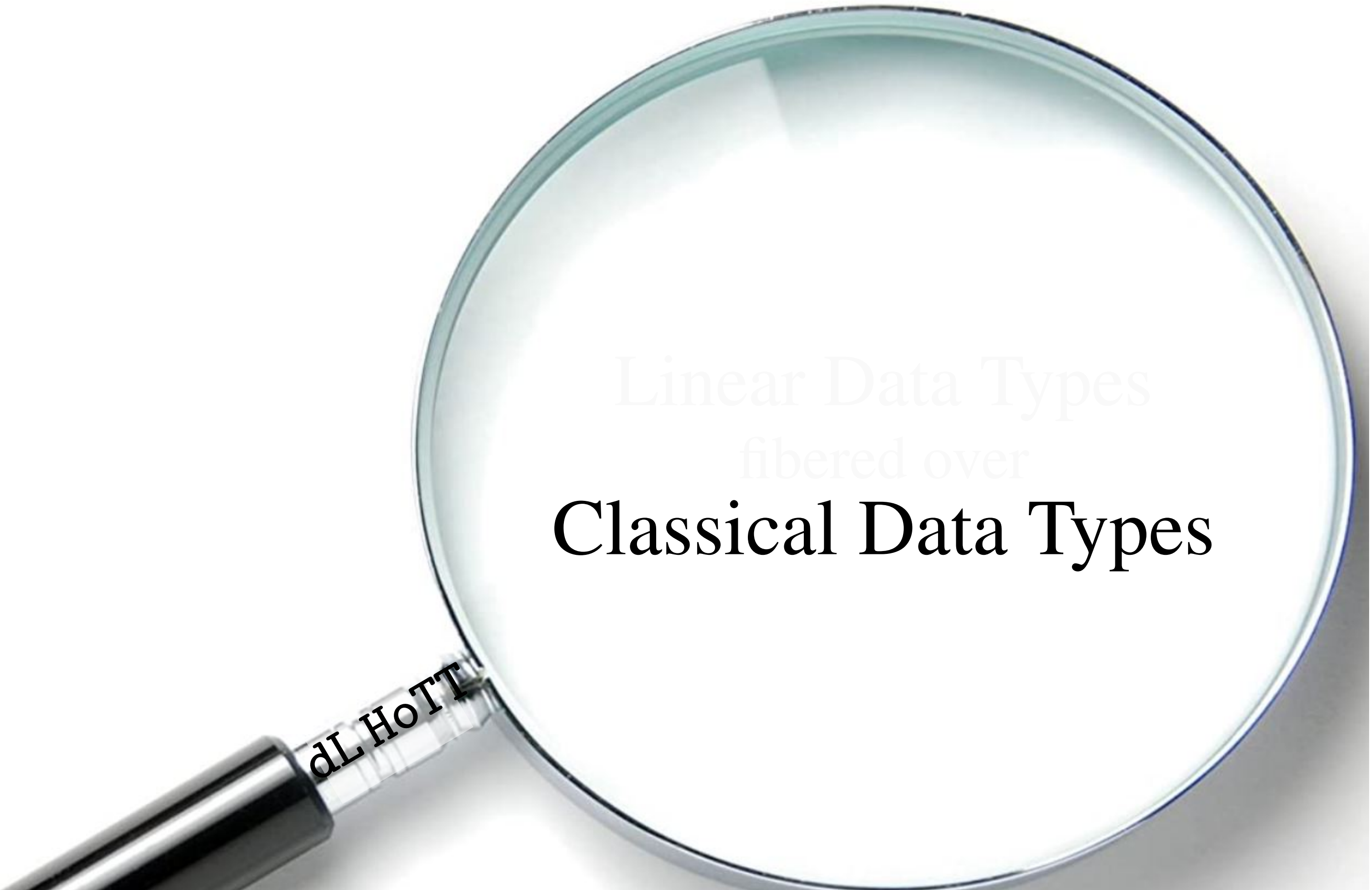

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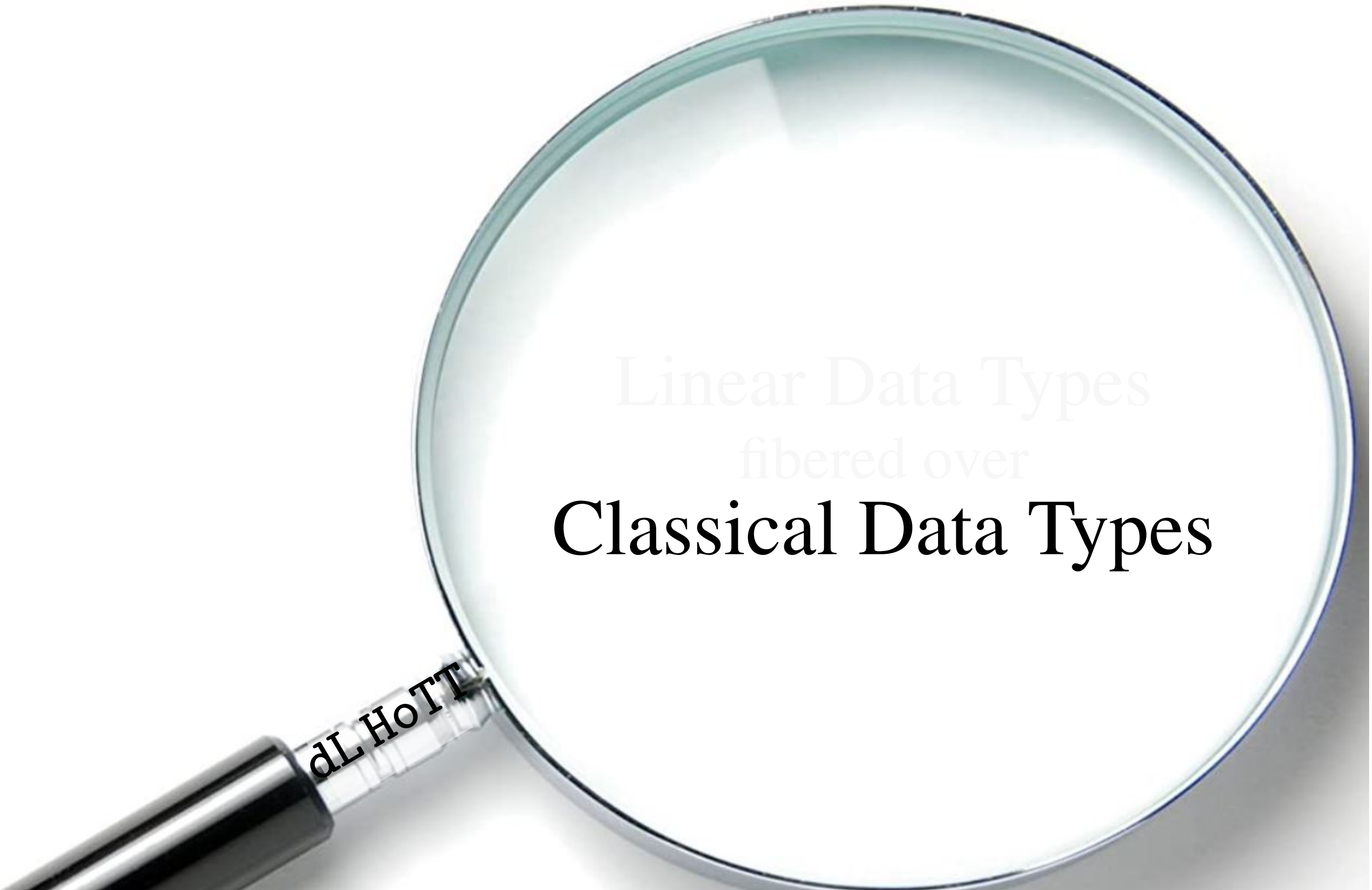

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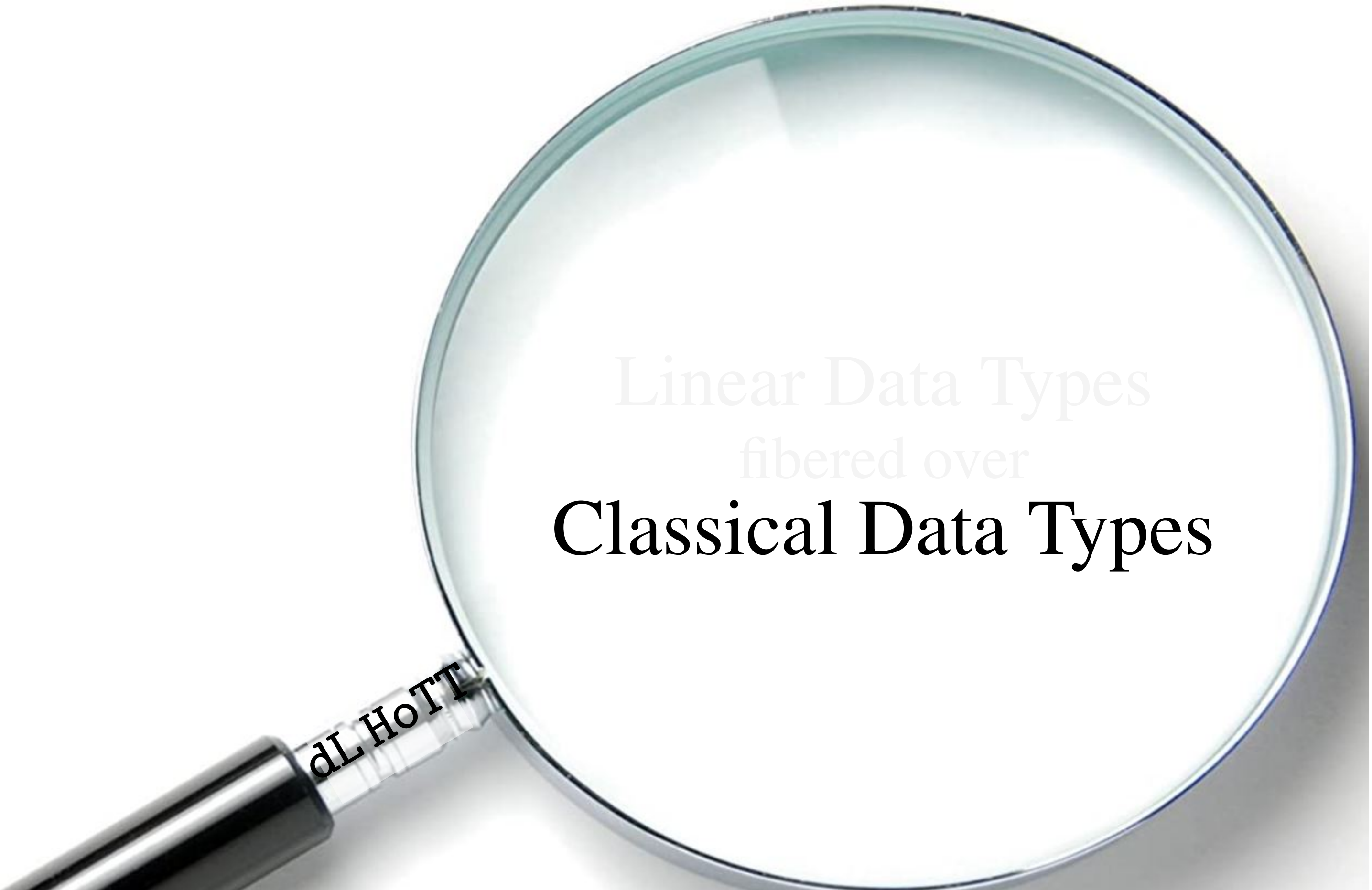

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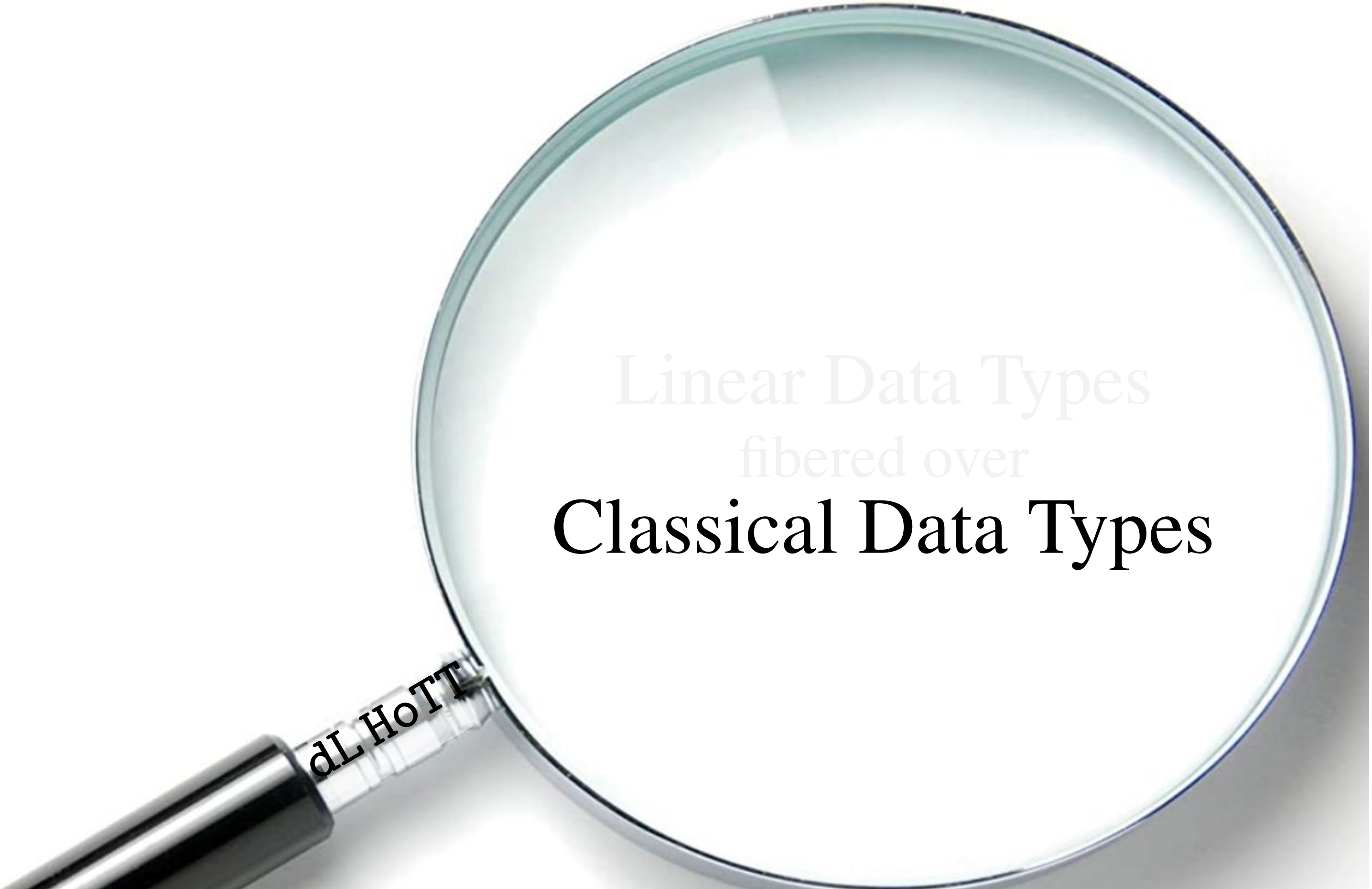

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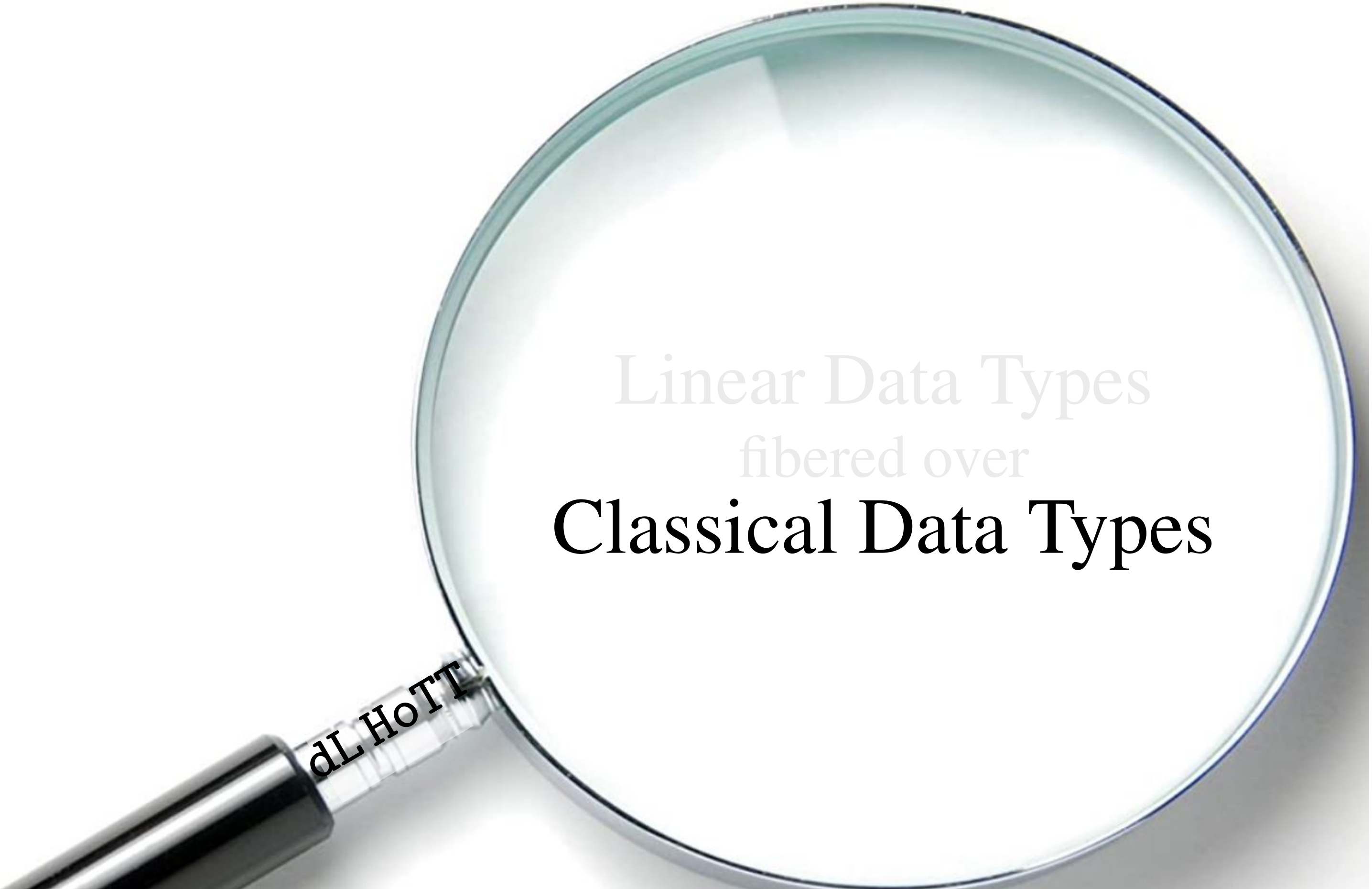

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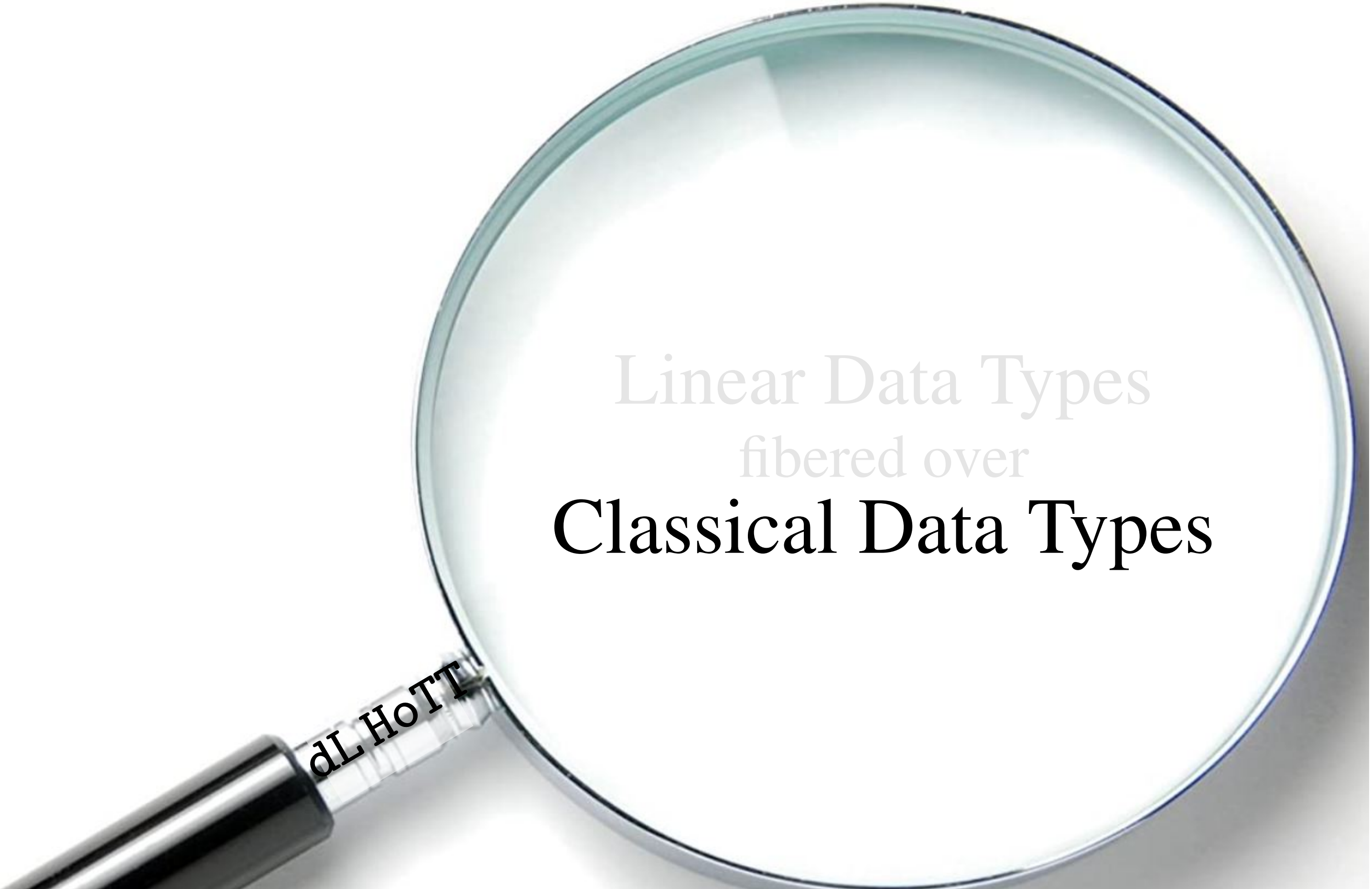

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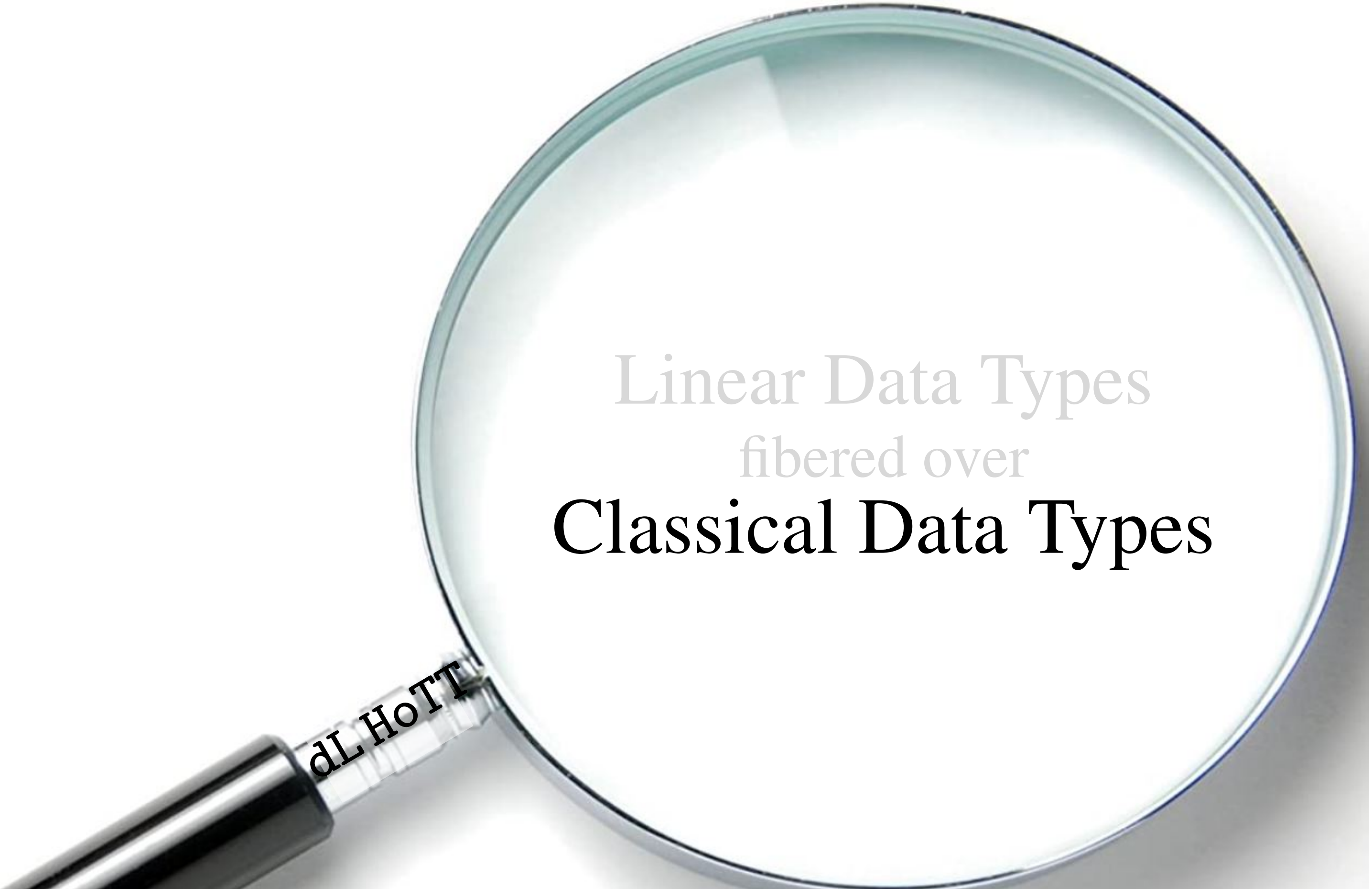

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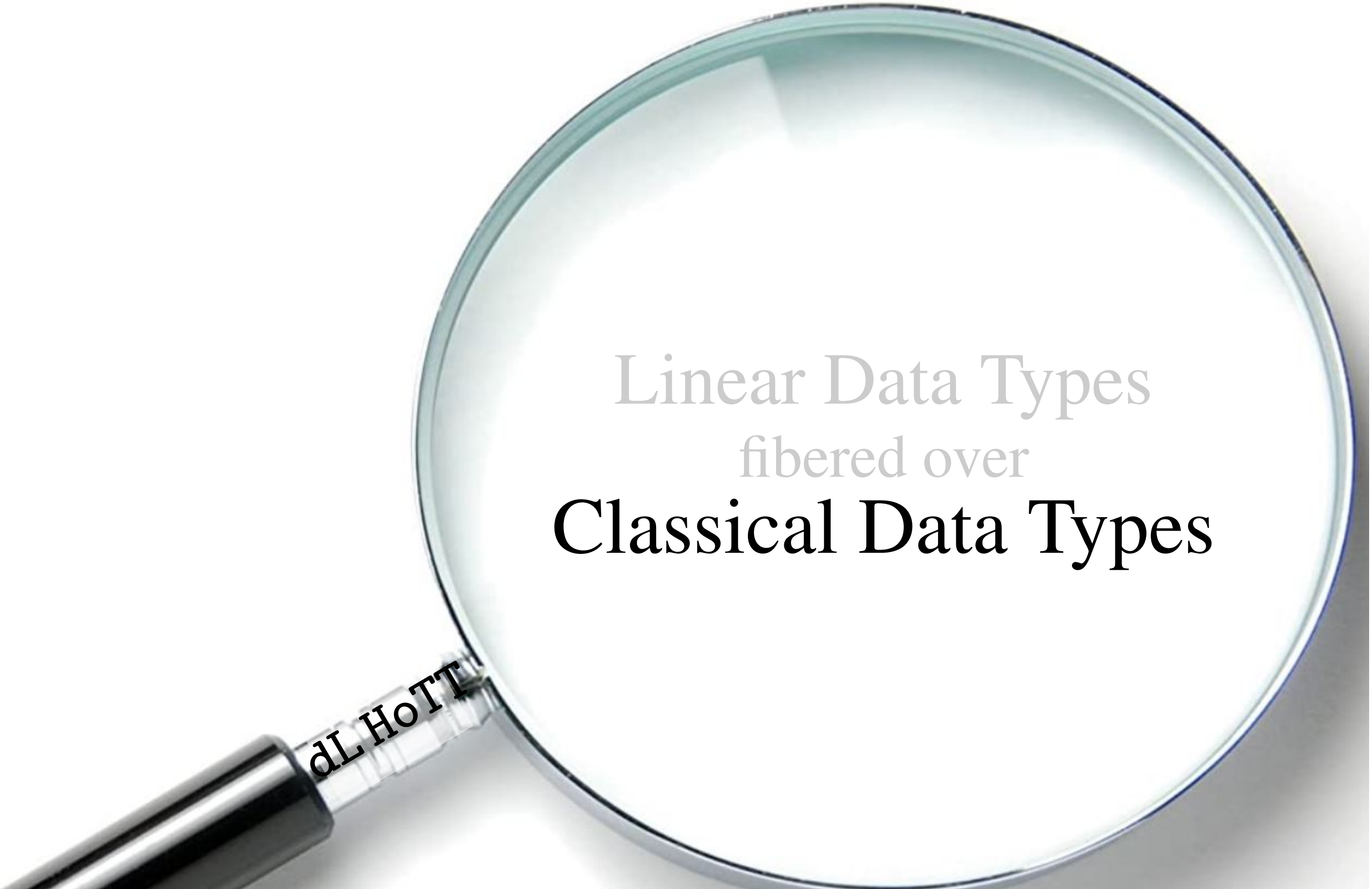

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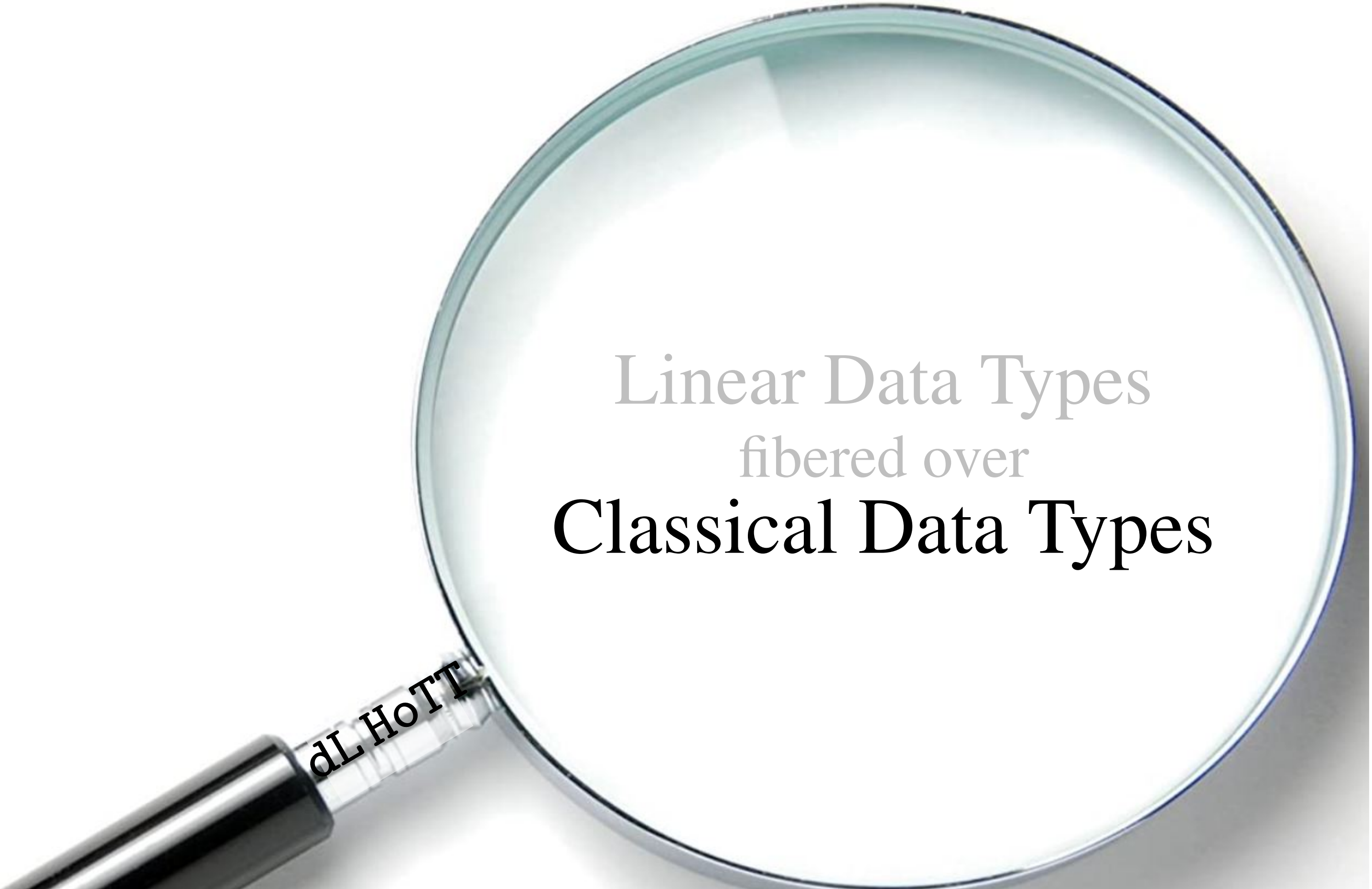

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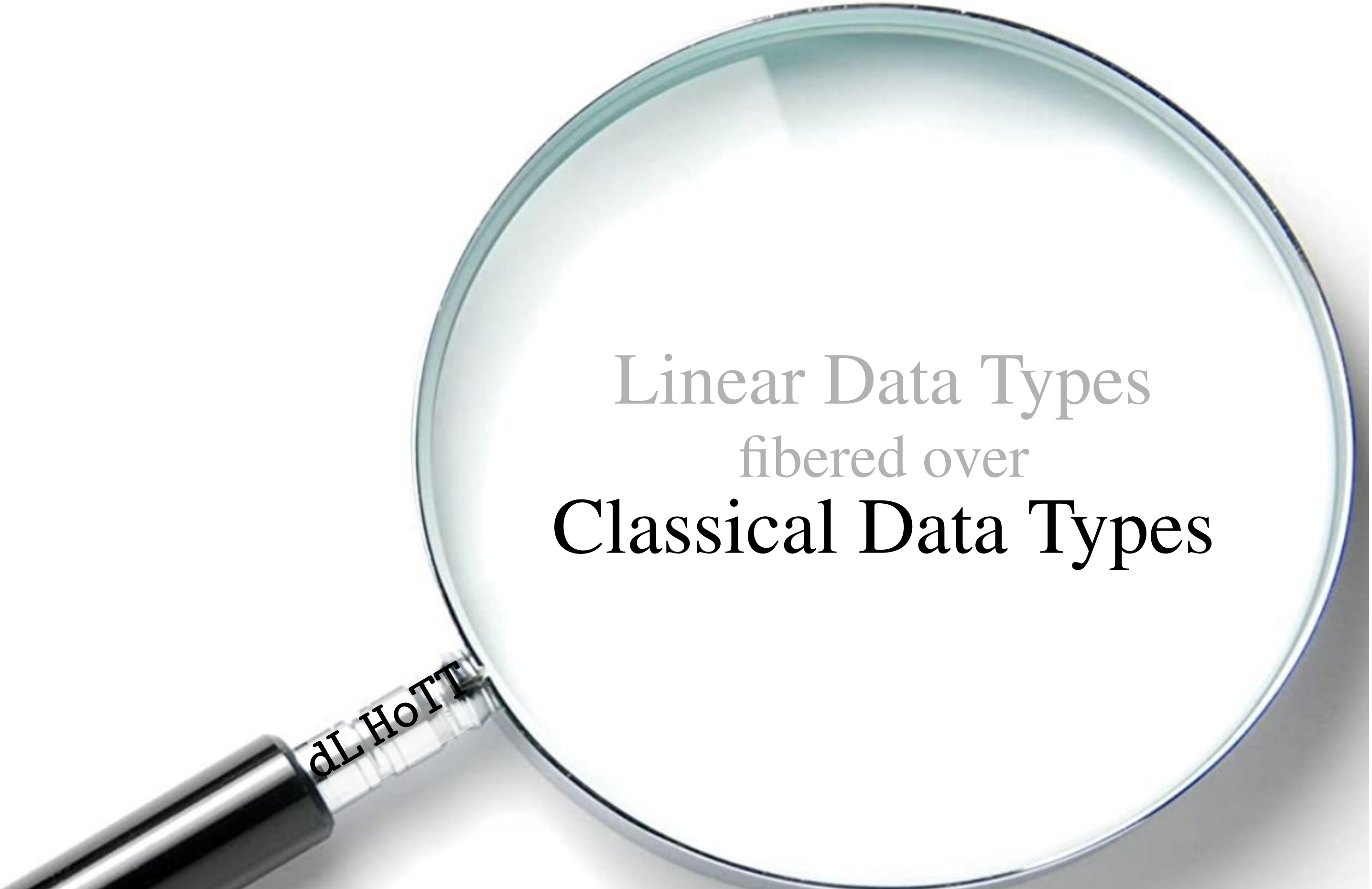

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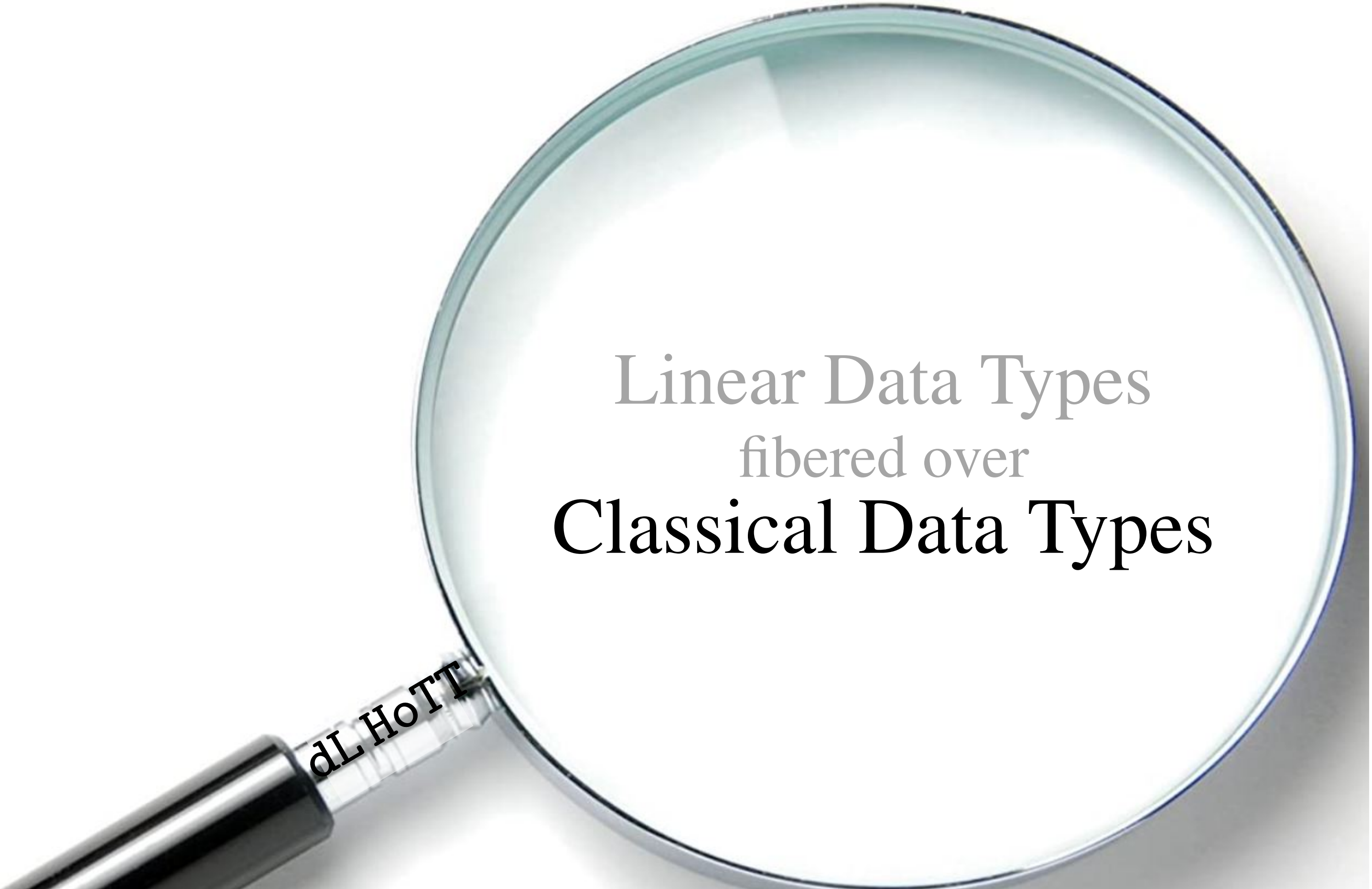

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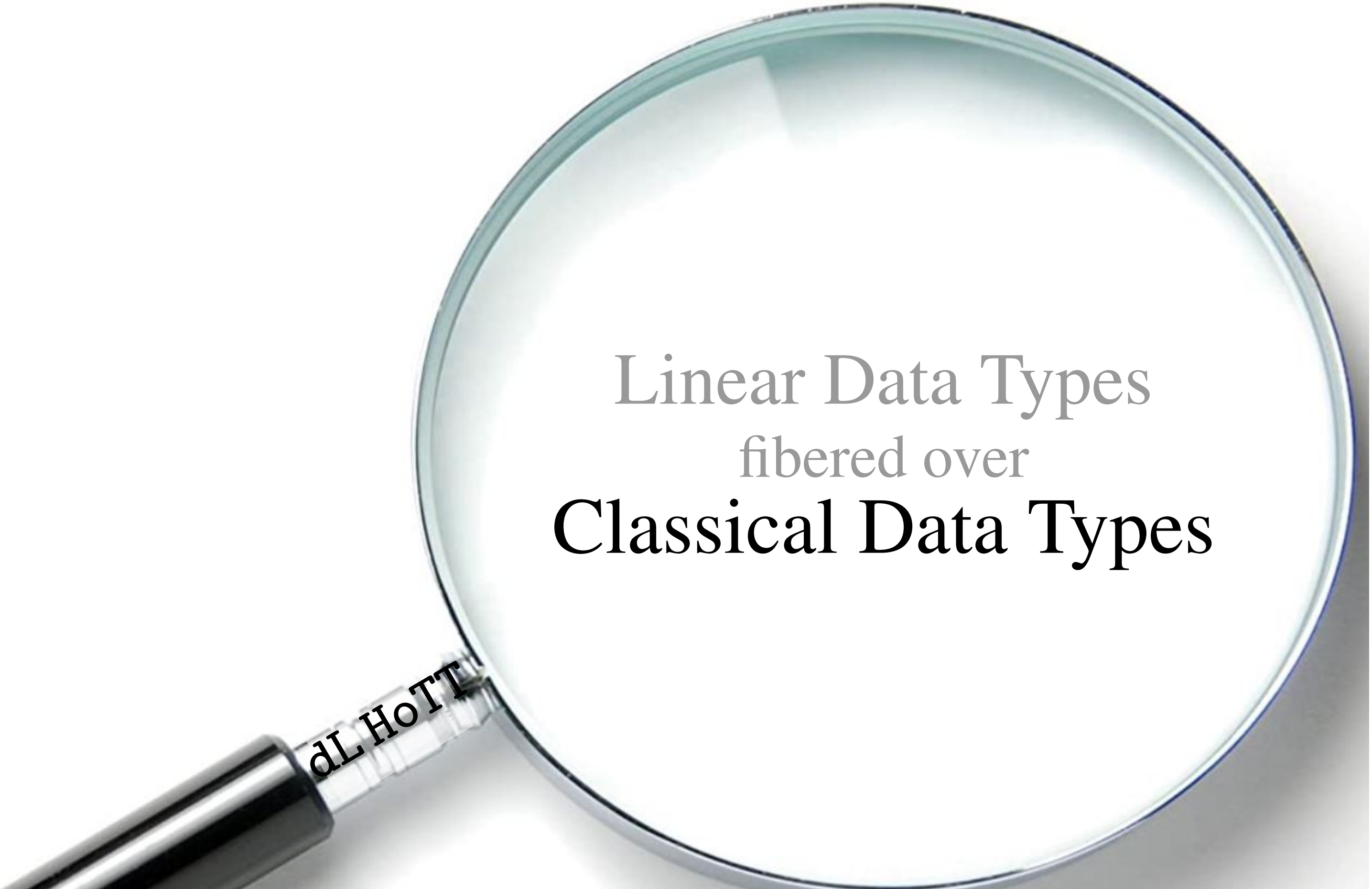

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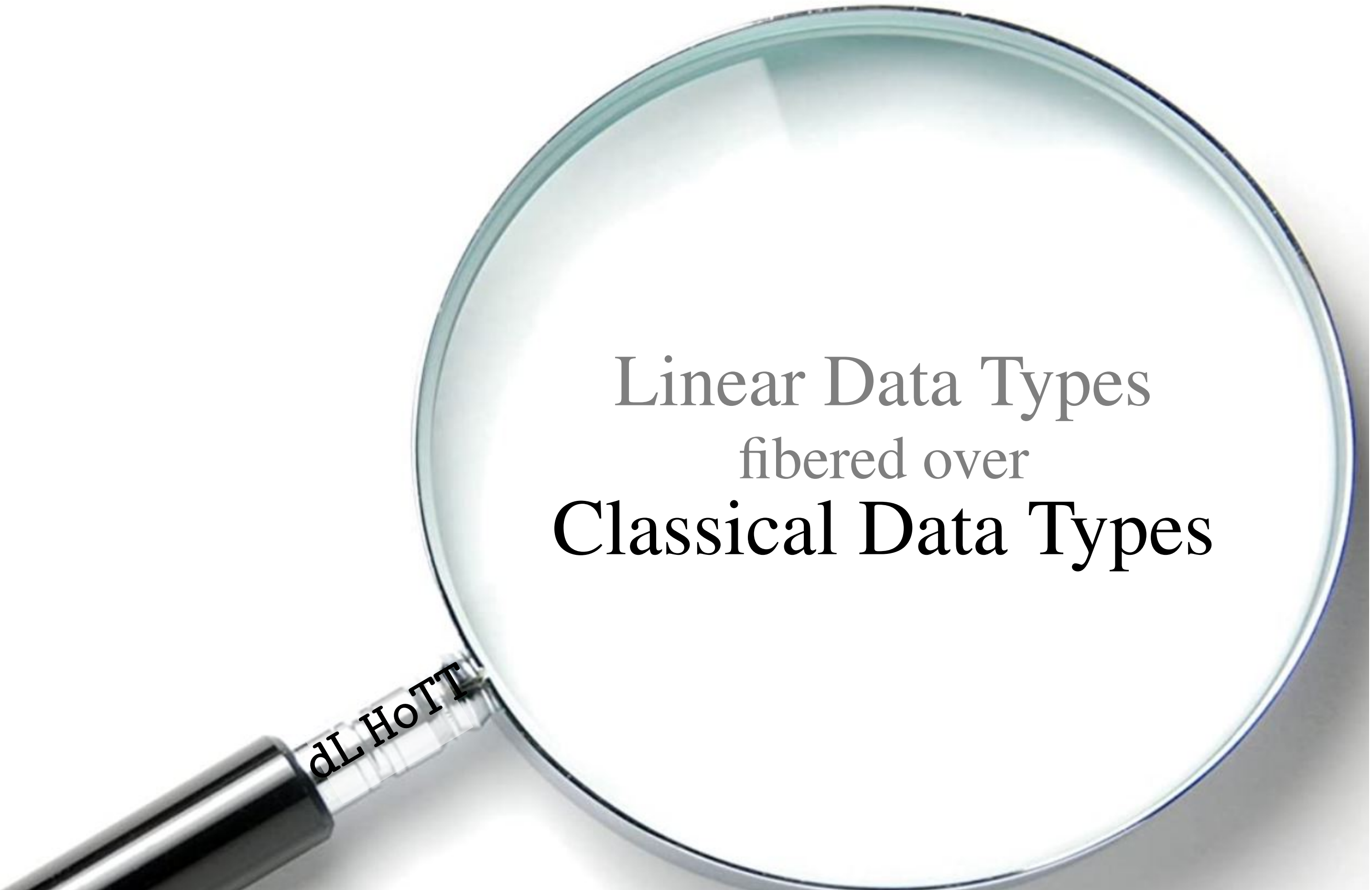

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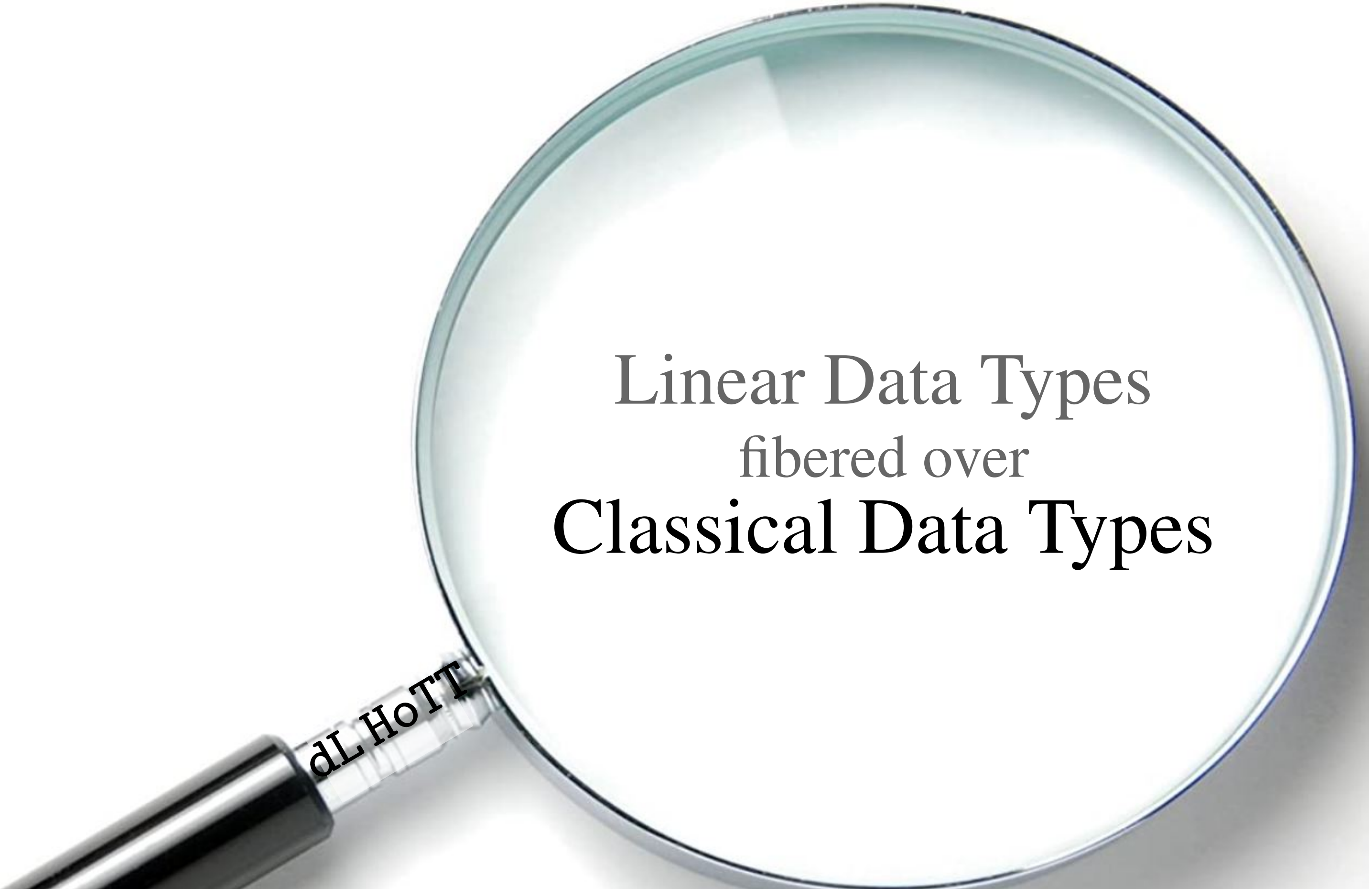

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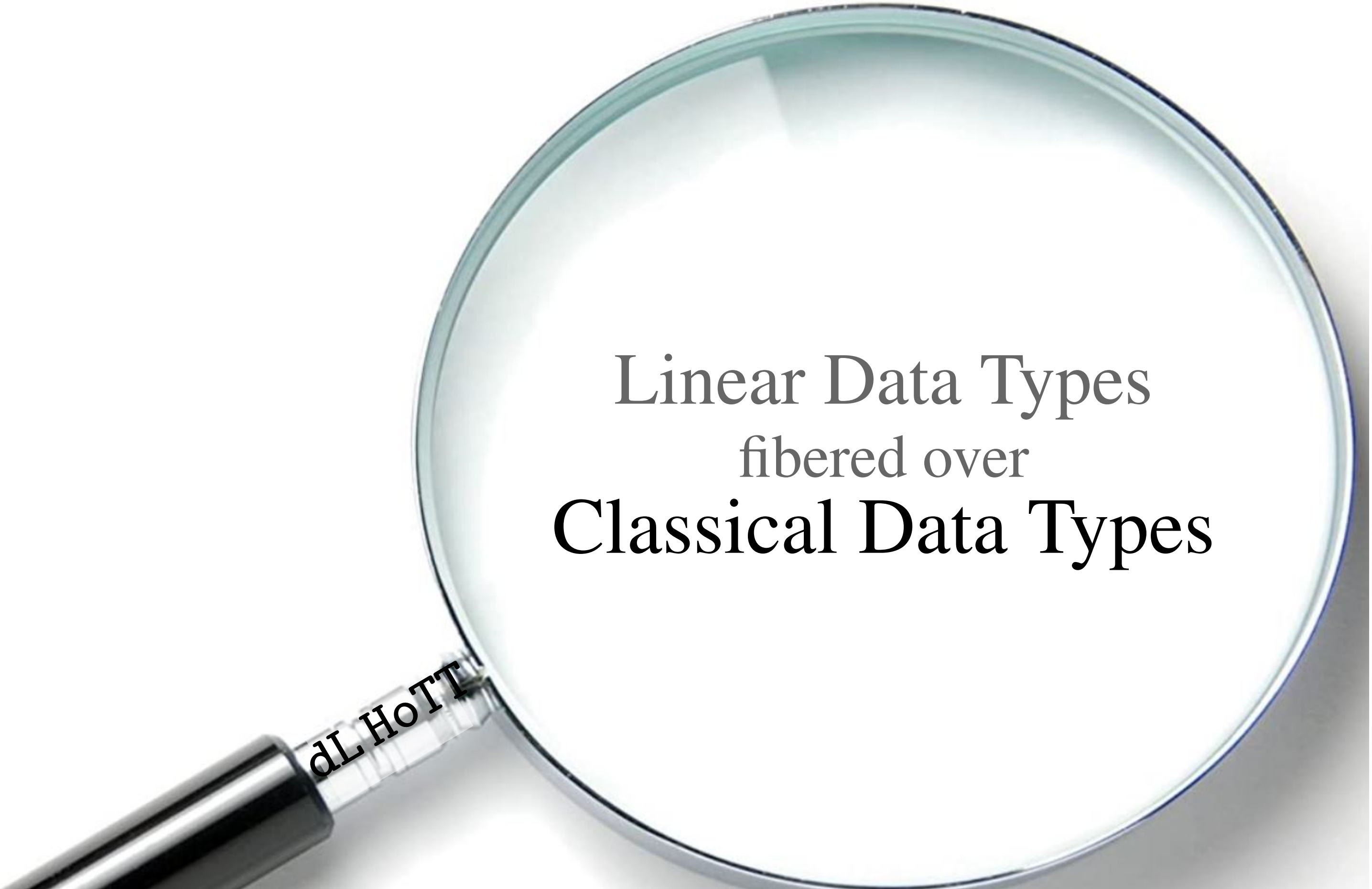

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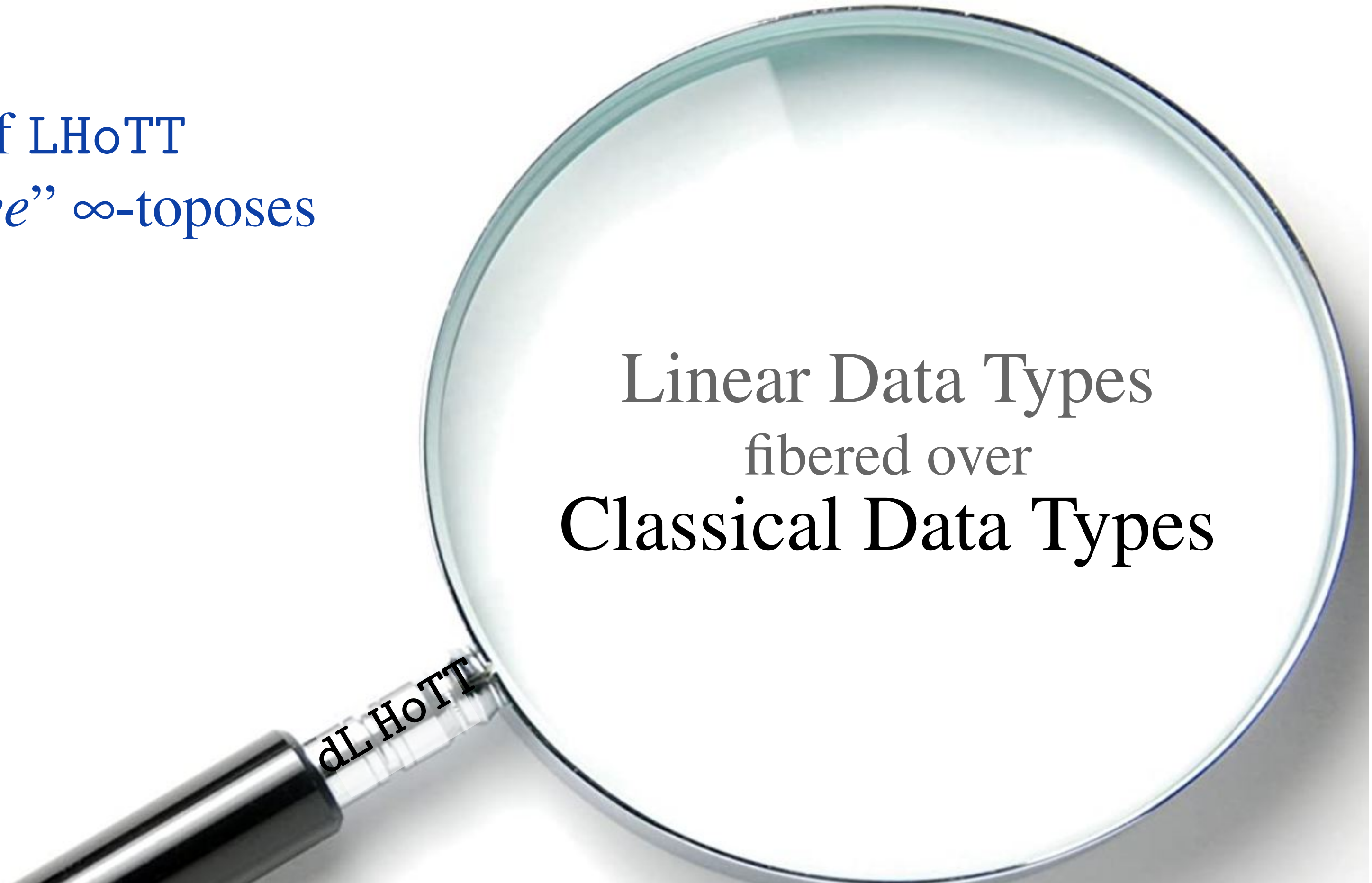

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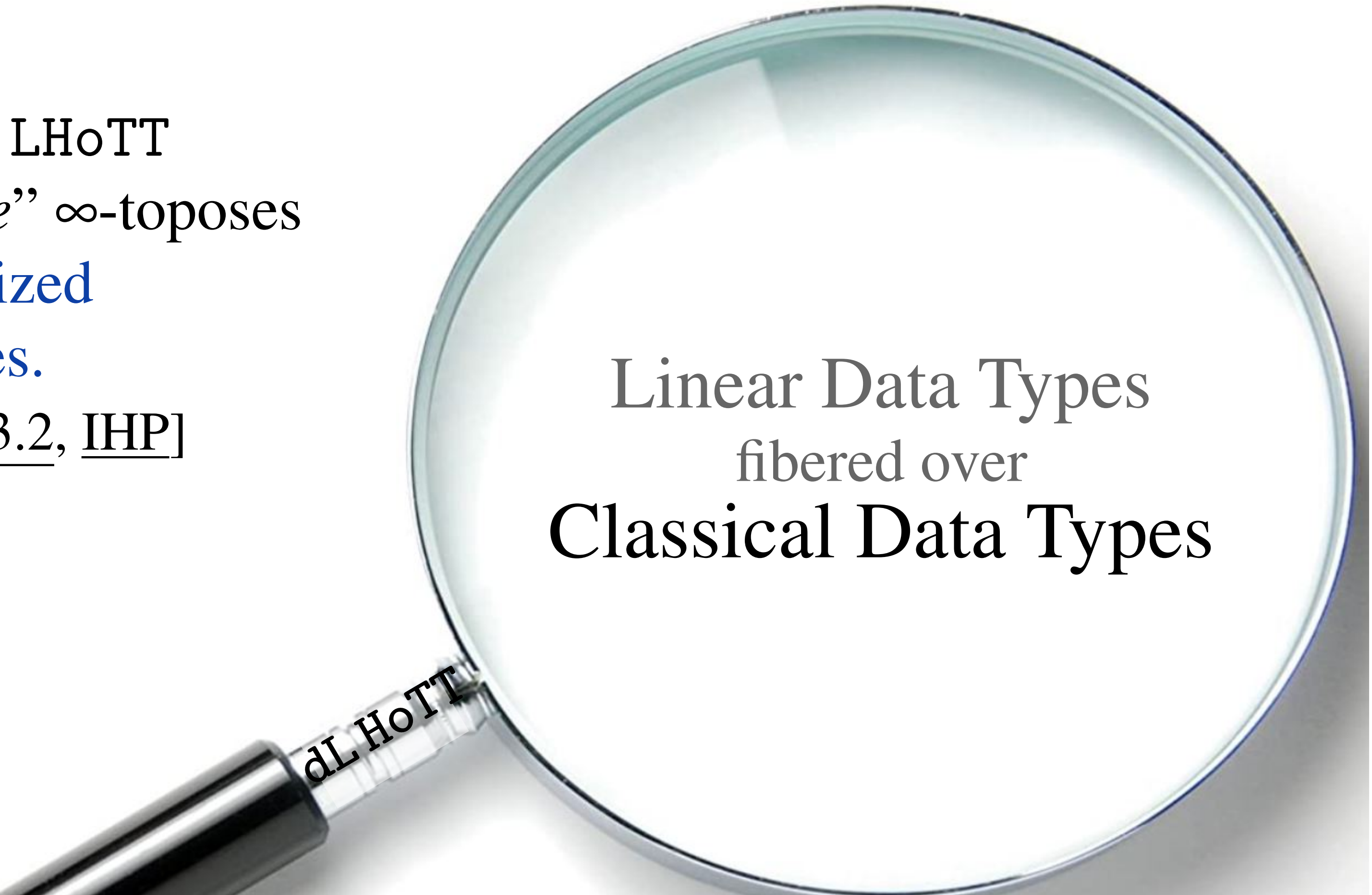

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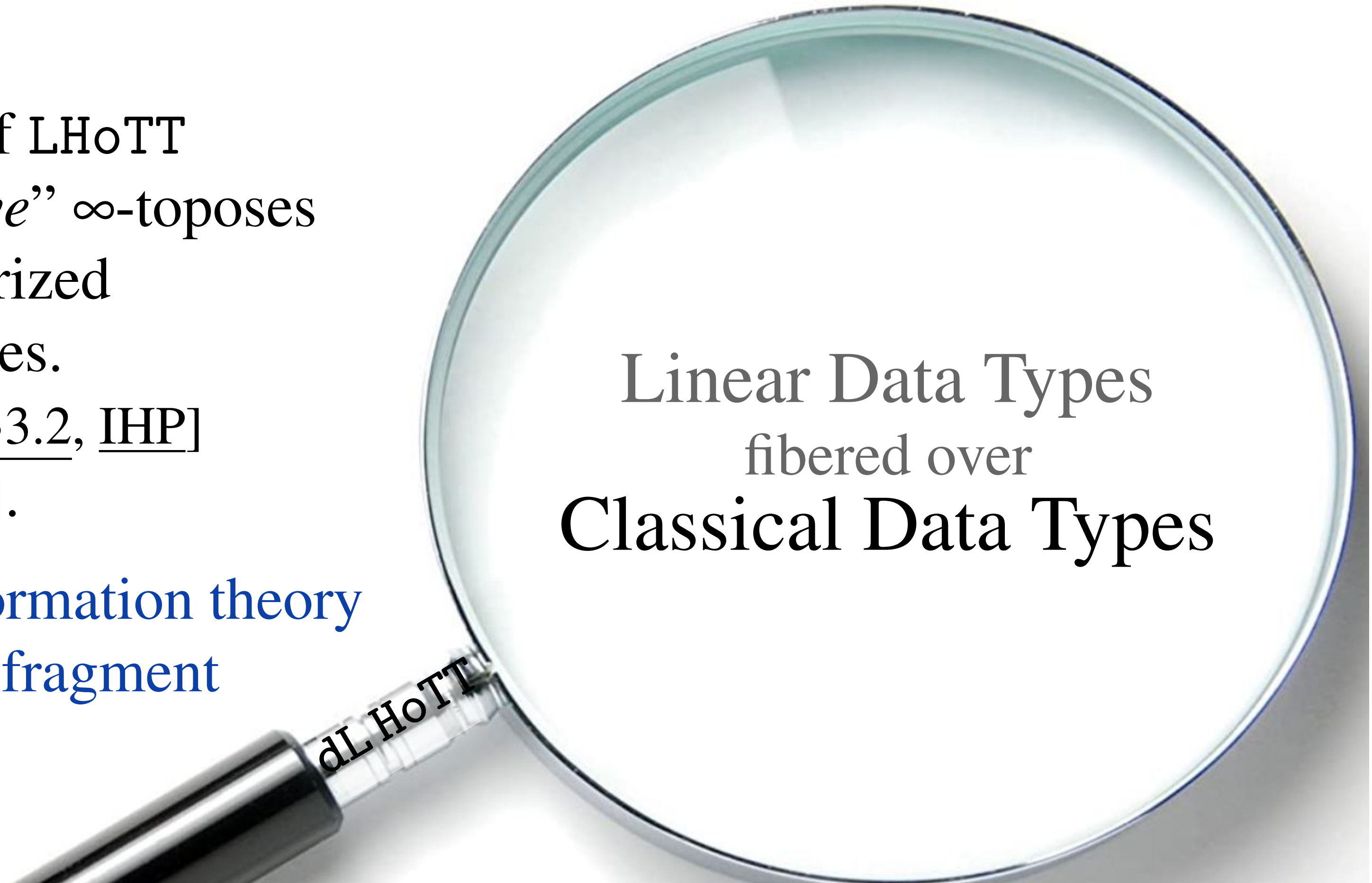

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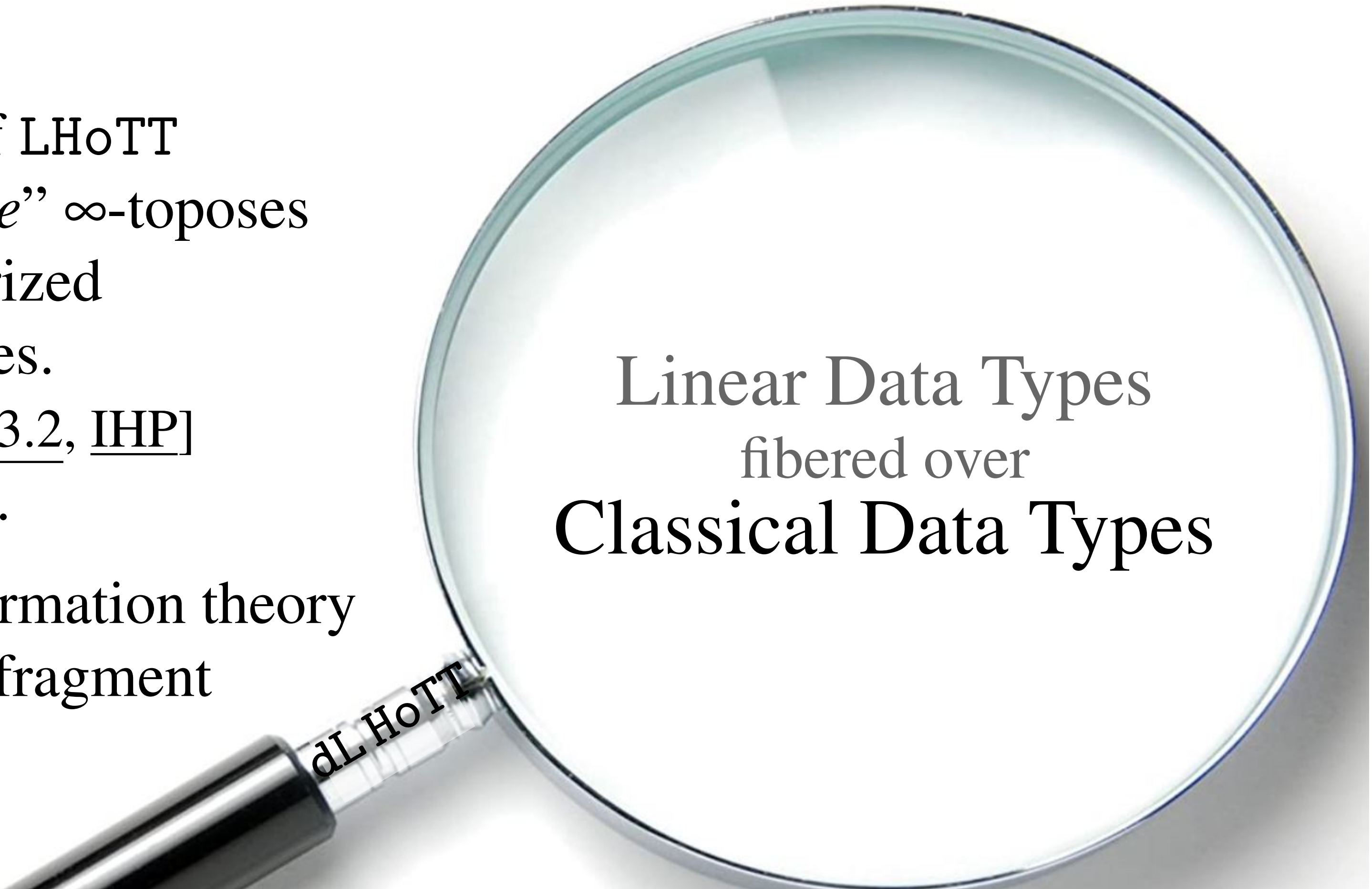
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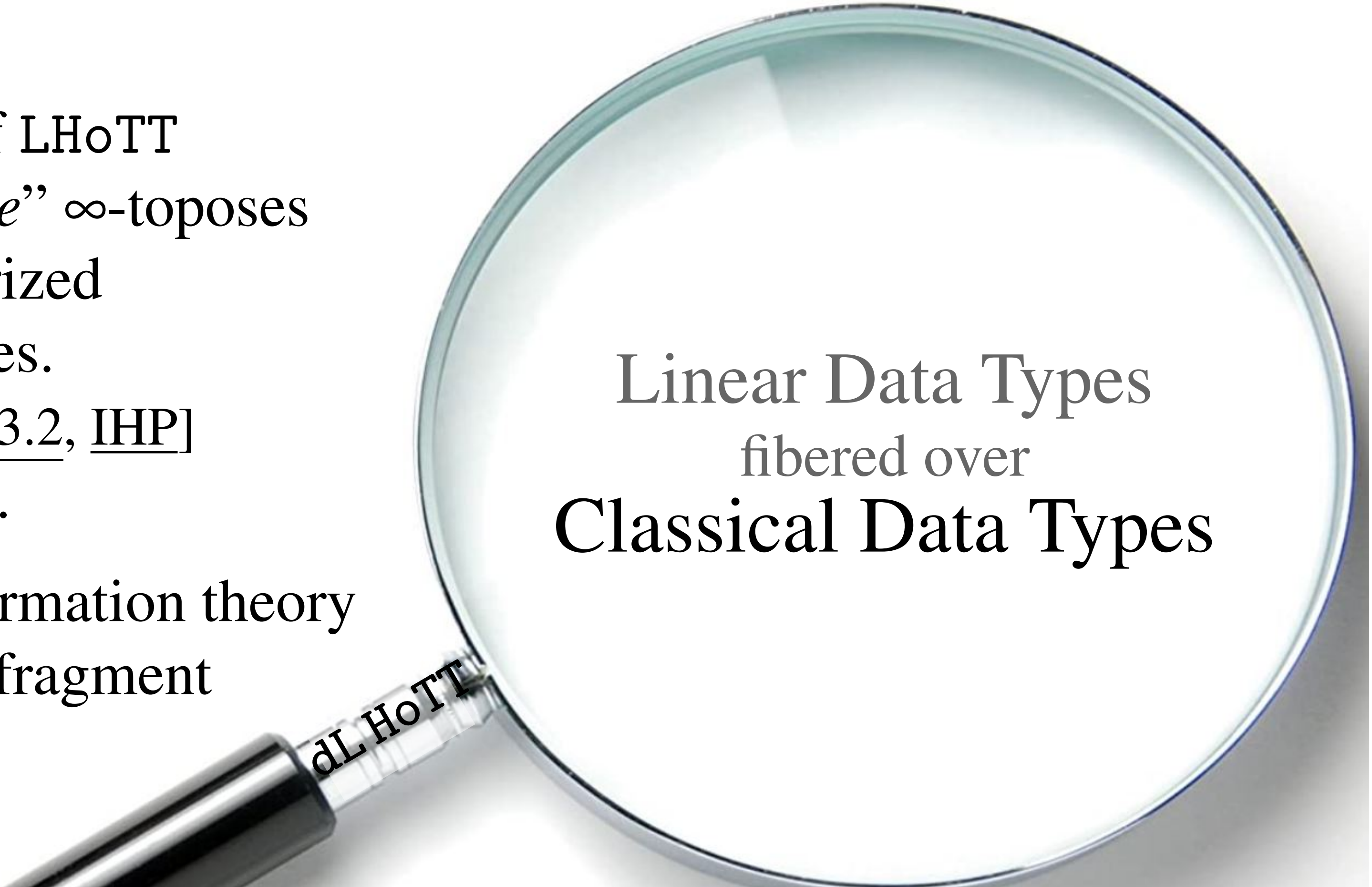

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Dependent Linear Homotopy Type Theory (`LHoTT`)
for universal algorithmic quantum computation

QS – Quantum Systems language @ CQTS

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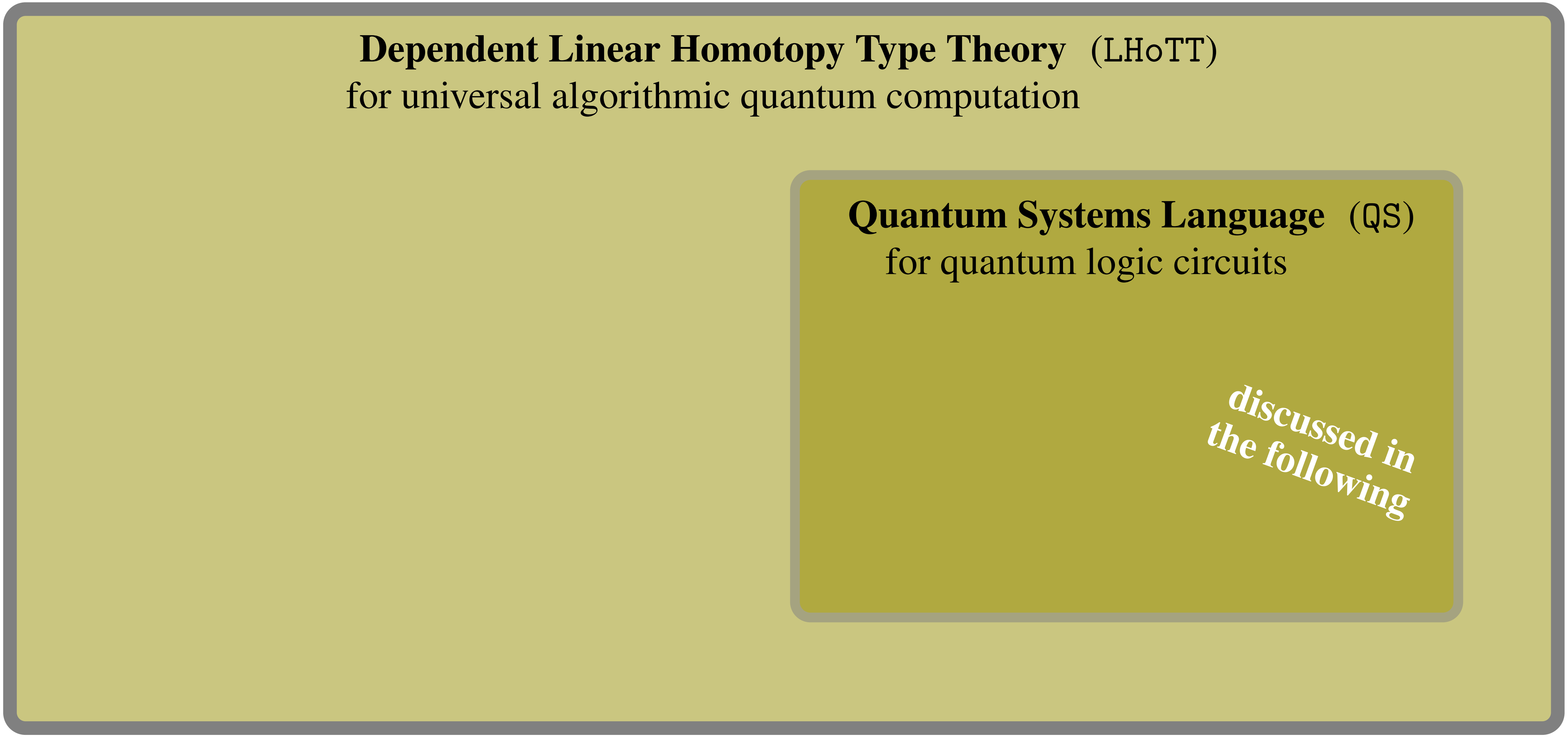

Dependent Linear Homotopy Type Theory (LHoTT)
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Quantum Systems Language (QS)
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discussed in the following

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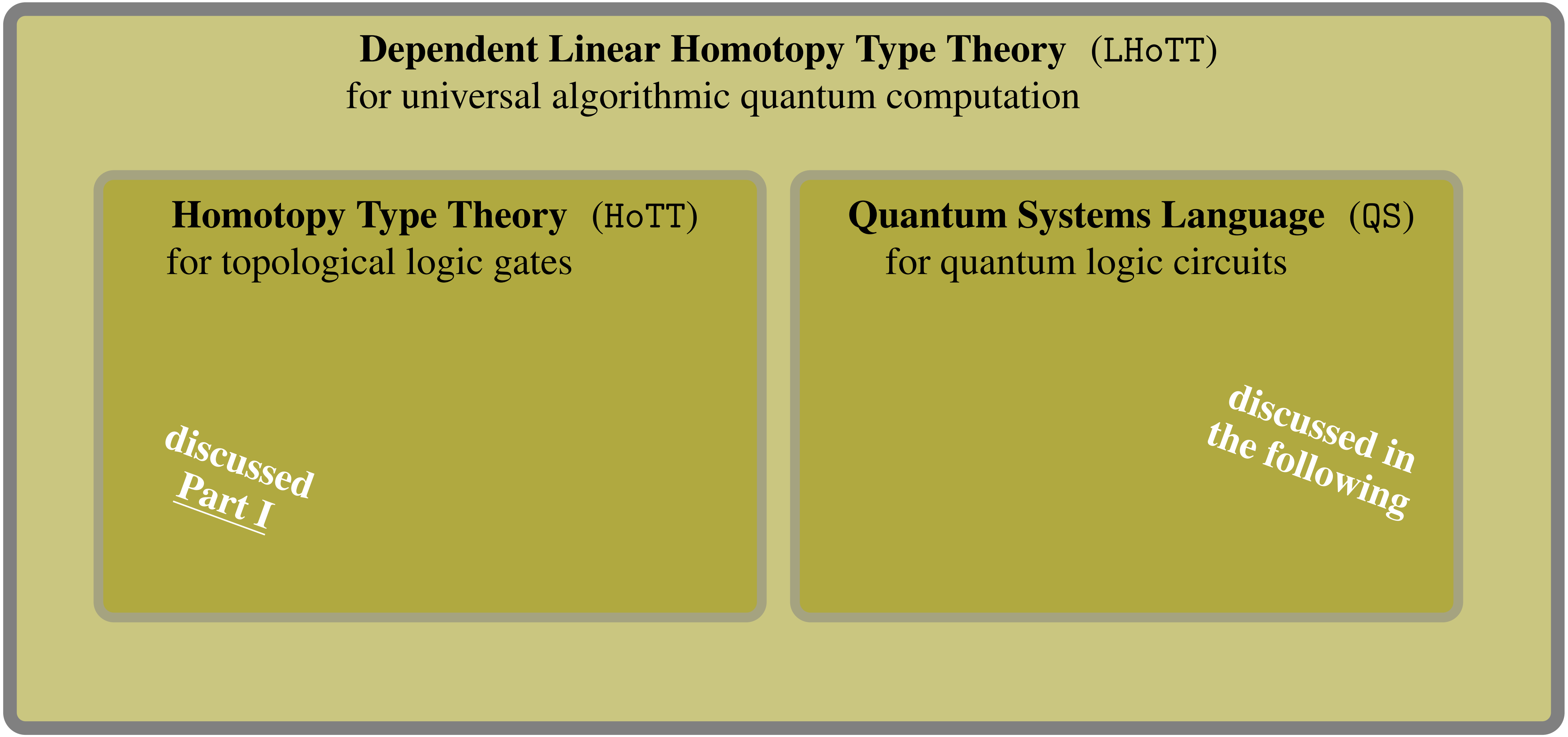

Dependent Linear Homotopy Type Theory (LHoTT)
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Homotopy Type Theory (HoTT)
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discussed Part I

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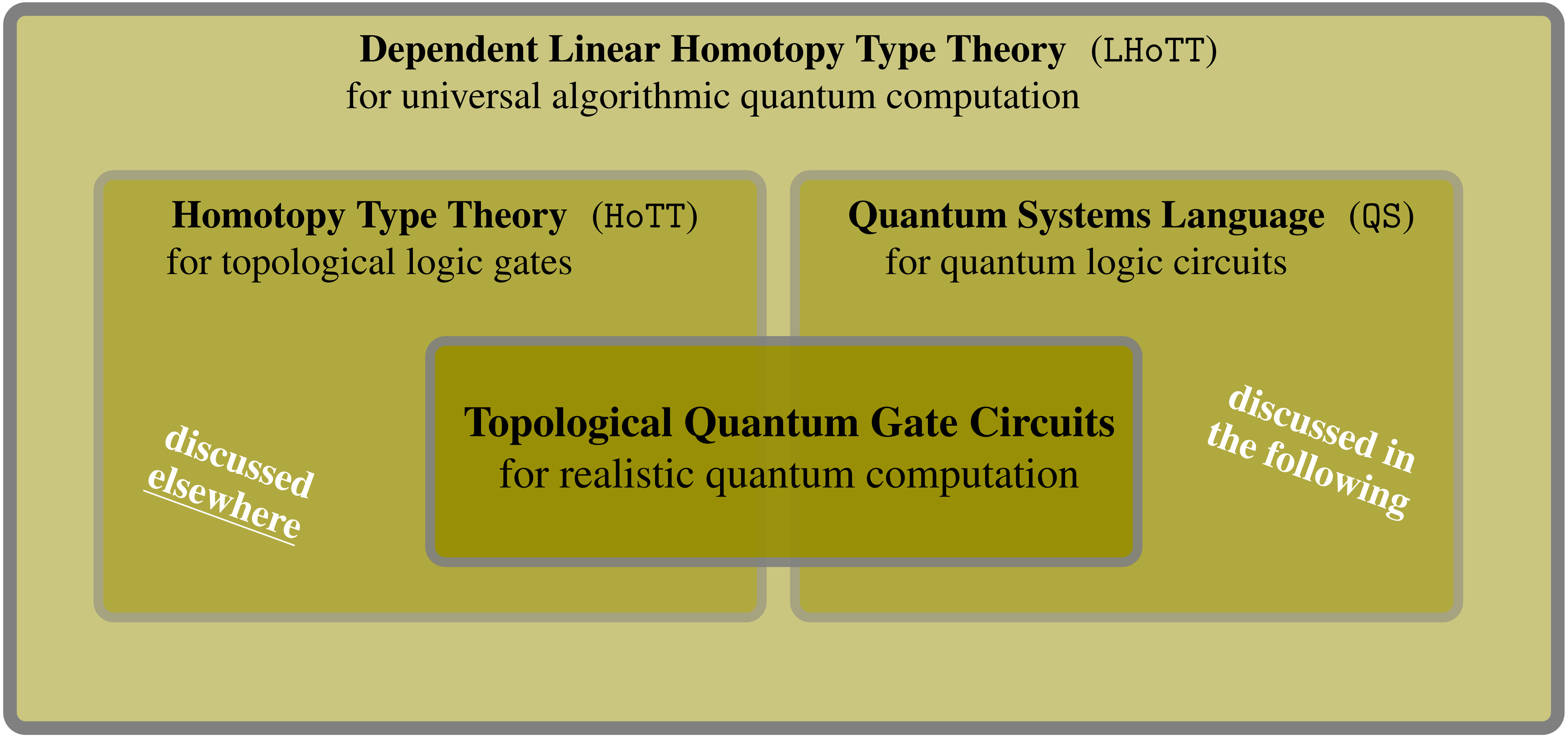

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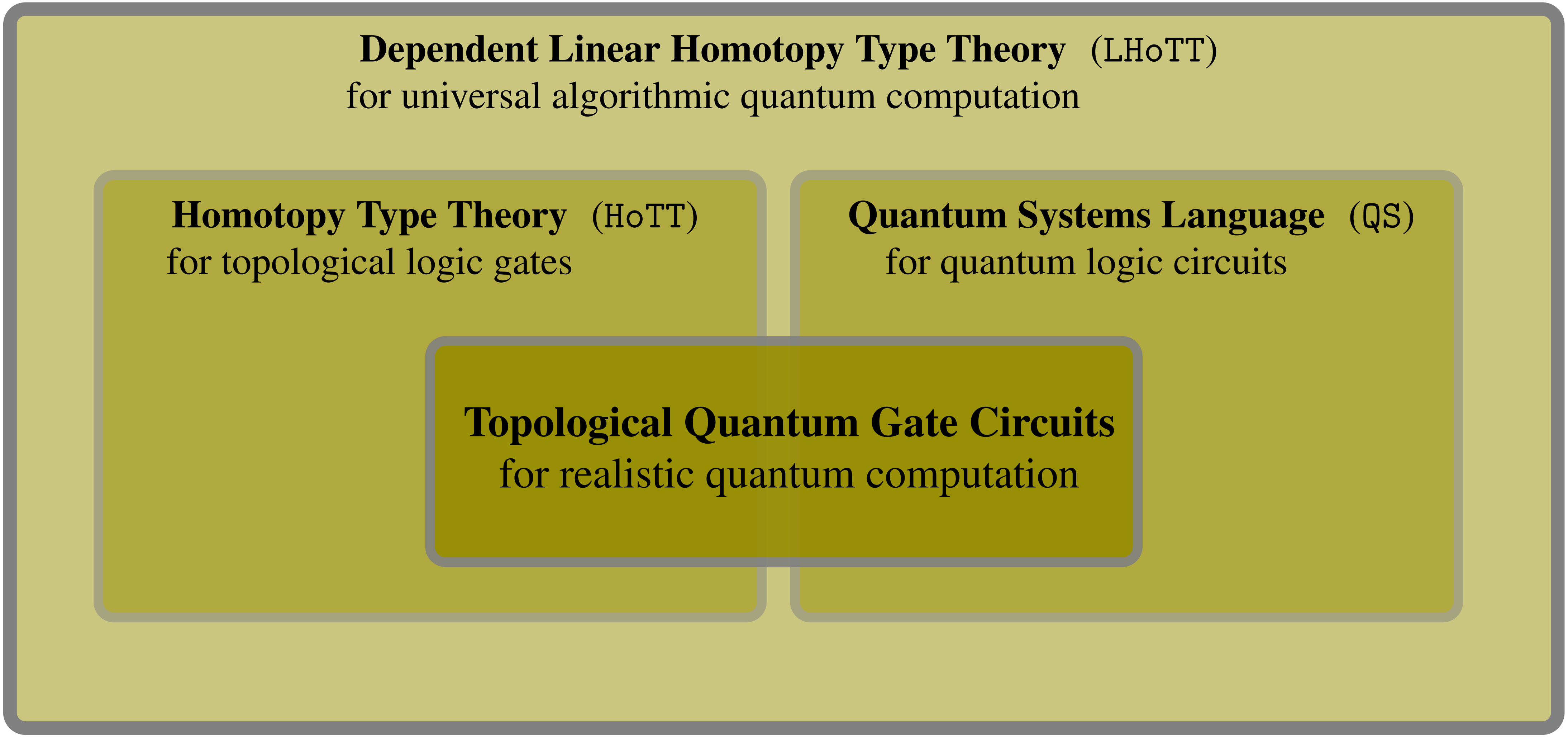

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Homotopy Type Theory (HoTT)
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Topological Quantum Gate Circuits
for realistic quantum computation

ambient LHoTT	verifies	classically dependent quantum linear types
ambient HoTT	provides	specification of topological quantum gates
ambient dTT	provides	full verified classical control

Quantum Data Types

Linear/Quantum Data Types

<table>
<tr><td>Characteristic Property</td><td></td><td></td><td></td></tr>
<tr><td>Symbol</td><td></td><td></td><td></td></tr>
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Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:		
Symbol			
Formula (for B : FinType)			
AlgTop Jargon			
Linear Logic			
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Linear/Quantum Data Types

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Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	
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Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
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Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
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AlgTop Jargon	biproduct, stability, ambidexterity	Frobenius reciprocity	mapping spectrum
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Physics Meaning	superselection sectors / quantum parallelism	compound quantum systems / quantum entanglement	QRAM systems

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Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd}
B \arrow[r, "p_B"] & * \\
\mathrm{BType}_B & \mathrm{BType}
\end{tikzcd}$$

classical type system dependent on context

classical type system

Linear/Quantum Data Types

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Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

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\end{tikzcd}$$

classical context extension

classical type system dependent on context

$$\begin{tikzcd}
\mathrm{BType}_B & \mathrm{BType} \arrow[l, "*_B \times"']
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classical type system

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Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd} B \arrow[r, "p_B"] & * \end{tikzcd}$$

co-product

$$\begin{tikzcd} \mathrm{BType}_B \arrow[r, bend left, "\coprod_{b:B}"] \arrow[r, phantom, "\perp"] & \mathrm{BType} \arrow[l, bend left, "*_B \times"] \end{tikzcd}$$

classical type system dependent on context

classical type system

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Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd}
B \arrow[r, "p_B"] & *
\end{tikzcd}$$

classical type system dependent on context

$$\begin{tikzcd}[column sep=huge]
\mathrm{BType}_B \arrow[r, bend left=50, "\overset{\text{co-product}}{\coprod_{b:B}}"] \arrow[r, bend right=50, "\underset{\text{product}}{\prod_{b:B}}"'] \arrow[r, phantom, shift left=3ex, "\perp"] \arrow[r, phantom, shift right=3ex, "\perp"] & \mathrm{BType} \arrow[l, "*_B \times"']
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classical type system

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Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd}[column sep=huge]
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co-product

classical type system dependent on context

$$\begin{tikzcd}[column sep=huge, row sep=large]
\mathrm{BType}_B \arrow[r, shift left=6, "\coprod_{b:B}"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, shift right=6, "\prod_{b:B}"'] & \mathrm{BType} \arrow[l, "*_B \times"']
\end{tikzcd}$$

classical type system

product

classical base change / classical quantification

Linear/Quantum Data Types

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Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd}
B \arrow[r, "p_B"] & \ast
\end{tikzcd}$$

classical type system dependent on context

co-product

$$\begin{tikzcd}[column sep=huge]
\mathrm{BType}_B \arrow[r, shift left=8, "\coprod_{b:B}"] \arrow[r, shift right=8, "\prod_{b:B}"'] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] & \mathrm{BType} \arrow[l, "\ast_B \times"']
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product

classical type system

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Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd}
B \arrow[rr, "p_B"] && *
\end{tikzcd}$$

classical type system dependent on context

$$\begin{tikzcd}
\mathrm{BType}_B \arrow[rr, shift left=8, "\overset{\text{co-product}}{\coprod_{b:B}}"] \arrow[rr, phantom, shift left=4, "\perp"] \arrow[rr, phantom, shift right=4, "\perp"] \arrow[rr, shift right=8, "\underset{\text{product}}{\prod_{b:B}}"'] && \mathrm{BType} \arrow[ll, "{*_B \times}"']
\end{tikzcd}$$

classical type system

classical base change / classical quantification

linear type system in classical context

$$\Big(\mathrm{LType}_B, \overset{\text{tensor}}{\otimes_B}\Big) \qquad \Big(\mathrm{LType}, \overset{\text{tensor}}{\otimes}\Big)$$

linear type system

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Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd} B \arrow[r, "p_B"] & * \end{tikzcd}$$

classical type system dependent on context

$$\begin{tikzcd}[column sep=huge]
\mathrm{BType}_B \arrow[r, bend left=40, "\overset{\text{co-product}}{\coprod_{b:B}}"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, bend right=40, "\underset{\text{product}}{\prod_{b:B}}"'] & \mathrm{BType} \arrow[l, "*_B \times"']
\end{tikzcd}$$

classical type system

linear type system in classical context

$$\begin{tikzcd}[column sep=huge]
\Big(\mathrm{LType}_B, \overset{\text{tensor}}{\otimes_B}\Big) & \Big(\mathrm{LType}, \overset{\text{tensor}}{\otimes}\Big) \arrow[l, "\overset{\text{linear context extension}}{\mathbb{1}_B \otimes}"']
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linear type system

classical base change / classical quantification

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Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

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classical type system dependent on context

$$\begin{tikzcd}[column sep=huge, row sep=large]
\mathrm{BType}_B \arrow[r, shift left=8, "\overset{\text{co-product}}{\coprod_{b:B}}"] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] \arrow[r, shift right=8, "\underset{\text{product}}{\prod_{b:B}}"'] & \mathrm{BType} \arrow[l, "*_B \times"{description}]
\end{tikzcd}$$

classical type system

classical base change / classical quantification

linear type system in classical context

$$\begin{tikzcd}[column sep=huge, row sep=large]
\Big(\mathrm{LType}_B, \overset{\text{tensor}}{\otimes_B}\Big) \arrow[r, shift left=8, "\overset{\text{direct sum}}{\bigoplus_{b:B}}"] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] \arrow[r, shift right=8, "\bigoplus_{b:B}"'] & \Big(\mathrm{LType}, \overset{\text{tensor}}{\otimes}\Big) \arrow[l, "\mathbb{1}_B \otimes"{description}]
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Formula (for B : FinType)	$\overset{\textbf{cart. product}}{\prod_B \mathcal{H}_b} \simeq \underset{\textbf{direct sum}}{\bigoplus_B \mathcal{H}_b} \simeq \overset{\textbf{co-product}}{\coprod_B \mathcal{H}_b}$	$\mathcal{V} \otimes \Big(\bigoplus_{b:B} \mathcal{H}_b \Big) \simeq \bigoplus_{b:B} (\mathcal{V} \otimes \mathcal{H}_b)$	$(\mathcal{V} \otimes \mathcal{H}) \multimap \mathcal{K}$ $\simeq \mathcal{V} \multimap (\mathcal{H} \multimap \mathcal{K})$

Dependent linear Type Formers

finite classical context (variables, parameters, ...) **reference context**

$$\begin{tikzcd}[column sep=huge] B \arrow[r, "p_B"] & * \end{tikzcd}$$

classical type system dependent on context

$$\begin{tikzcd}[column sep=huge]
\mathrm{BType}_B
\arrow[r, shift left=14pt, "\overset{\text{co-product}}{\coprod_{b:B}}"]
\arrow[r, phantom, shift left=7pt, "\perp"]
\arrow[r, phantom, shift right=7pt, "\perp"]
\arrow[r, shift right=14pt, "\underset{\text{product}}{\prod_{b:B}}"']
& \mathrm{BType} \arrow[l, "*_B \times" description]
\end{tikzcd}$$

classical type system

classical base change / classical quantification

linear type system in classical context

$$\begin{tikzcd}[column sep=huge]
\big(\mathrm{LType}_B, \overset{\text{tensor}}{\otimes_B}\big)
\arrow[r, shift left=14pt, "\overset{\text{direct sum}}{\bigoplus_{b:B}}"]
\arrow[r, phantom, shift left=7pt, "\perp"]
\arrow[r, phantom, shift right=7pt, "\perp"]
\arrow[r, shift right=14pt, "\bigoplus_{b:B}"']
& \big(\mathrm{LType}, \overset{\text{tensor}}{\otimes}\big) \arrow[l, "\mathbb{1}_B \otimes" description]
\end{tikzcd}$$

linear type system

quantum base change / Motivic Yoga

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
Symbol	$\bigoplus$ direct sum	$\otimes$ tensor product	$\multimap$ linear function type
Formula (for B : FinType)	$\overset{\textbf{cart. product}}{\prod_B \mathcal{H}_b} \simeq \underset{\textbf{direct sum}}{\bigoplus_B \mathcal{H}_b} \simeq \overset{\textbf{co-product}}{\coprod_B \mathcal{H}_b}$	$\mathcal{V} \otimes \Big(\bigoplus_{b:B} \mathcal{H}_b \Big) \simeq \bigoplus_{b:B} (\mathcal{V} \otimes \mathcal{H}_b)$	$(\mathcal{V} \otimes \mathcal{H}) \multimap \mathcal{K}$ $\simeq \mathcal{V} \multimap (\mathcal{H} \multimap \mathcal{K})$

Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd} B \arrow[rr, "p_B"] & & * \end{tikzcd}$$

classical type system dependent on context

$$\begin{tikzcd}[column sep=huge] \mathrm{BType}_B \arrow[r, shift left=8, "\overset{\text{co-product}}{\coprod_{b:B}}"] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] \arrow[r, shift right=8, "\underset{\text{product}}{\prod_{b:B}}"'] & \mathrm{BType} \arrow[l, "*_B \times" description] \end{tikzcd}$$

classical type system

classical base change / classical quantification

linear type system in classical context

$$\begin{tikzcd}[column sep=huge] \big(\mathrm{LType}_B, \overset{\text{tensor}}{\otimes_B}\big) \arrow[r, shift left=8, "\overset{\text{direct sum}}{\bigoplus_{b:B}}"] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] \arrow[r, shift right=8, "\bigoplus_{b:B}"'] & \big(\mathrm{LType}, \overset{\text{tensor}}{\otimes}\big) \arrow[l, "\mathbb{1}_B \otimes" description] \end{tikzcd}$$

linear type system

quantum base change / Motivic Yoga

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
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Formula (for B : FinType)	$\overset{\text{cart. product}}{\prod_B \mathcal{H}_b} \simeq \underset{\text{direct sum}}{\bigoplus_B \mathcal{H}_b} \simeq \overset{\text{co-product}}{\coprod_B \mathcal{H}_b}$	$\mathcal{V} \otimes \Big(\bigoplus_{b:B} \mathcal{H}_b \Big) \simeq \bigoplus_{b:B} (\mathcal{V} \otimes \mathcal{H}_b)$	$(\mathcal{V} \otimes \mathcal{H}) \multimap \mathcal{K}$ $\simeq \mathcal{V} \multimap (\mathcal{H} \multimap \mathcal{K})$

Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd}[column sep=huge]
B \arrow[r, "p_B"] & *
\end{tikzcd}$$

all available in LHoTT

classical type system dependent on context

$$\begin{tikzcd}[column sep=huge]
\mathrm{BType}_B \arrow[r, shift left=6, "\overset{\text{co-product}}{\coprod_{b:B}}" description] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, shift right=6, "\underset{\text{product}}{\prod_{b:B}}" description] \arrow[r, phantom, shift right=3, "\perp"] & \mathrm{BType} \arrow[l, "{*_B}\times" description]
\end{tikzcd}$$

classical type system

classical base change / classical quantification

linear type system in classical context

$$\begin{tikzcd}[column sep=huge]
\Big(\mathrm{LType}_B, \overset{\text{tensor}}{\otimes_B}\Big) \arrow[r, shift left=6, "\overset{\text{direct sum}}{\bigoplus_{b:B}}" description] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, shift right=6, "\bigoplus_{b:B}" description] \arrow[r, phantom, shift right=3, "\perp"] & \Big(\mathrm{LType}, \overset{\text{tensor}}{\otimes}\Big) \arrow[l, "\mathbb{1}_B \otimes" description]
\end{tikzcd}$$

linear type system

quantum base change / Motivic Yoga

Quantum Effects

Recall: **Monadic computational effects.**

A *monad* $\mathscr{E}(-)$ on a data type system encodes *computational effects*:

effectful program

$$D_1 \xrightarrow{\mathrm{prog}_{12}} \mathscr{E}(D_2)$$

output data of nominal type D_2
causing effects of type $\mathscr{E}(-)$

Recall: **Monadic computational effects.**

A *monad* $\mathscr{E}(-)$ on a data type system encodes *computational effects*:

first program

$$D_1 \xrightarrow{\mathrm{prog}_{12}} \mathscr{E}(D_2)$$

output data of nominal type D_2
causing effects of type $\mathscr{E}(-)$

second program

$$D_2 \xrightarrow{\mathrm{prog}_{23}} \mathscr{E}(D_3)$$

input data of type D_2
causing effects of type $\mathscr{E}(-)$

Recall: **Monadic computational effects.**

A *monad* $\mathscr{E}(-)$ on a data type system encodes *computational effects*:

first program

second program

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}"] & \mathscr{E}(D_2) & D_2 \arrow[r, "\mathrm{prog}_{23}"] & \mathscr{E}(D_3) \arrow[d, mapsto, bend left] \\
D_1 \arrow[r, "\mathrm{prog}_{12}"] & \mathscr{E}(D_2) & \mathscr{E}(D_2) \arrow[r, "\mathrm{bind}^{\mathscr{E}}\mathrm{prog}_{23}"] & \mathscr{E}(D_3)
\end{tikzcd}$$

output data of nominal type D_2
causing effects of type $\mathscr{E}(-)$

input data of type D_2
causing effects of type $\mathscr{E}(-)$

bind previous effects
into second program

carry any previous $\mathscr{E}(-)$-effects along

Recall: **Monadic computational effects.**

A *monad* $\mathscr{E}(-)$ on a data type system encodes *computational effects*:

first program

$$D_1 \xrightarrow{\;\mathrm{prog}_{12}\;} \mathscr{E}(D_2)$$

output data of nominal type D_2
causing effects of type $\mathscr{E}(-)$

second program

$$D_2 \xrightarrow{\;\mathrm{prog}_{23}\;} \mathscr{E}(D_3)$$

input data of type D_2
causing effects of type $\mathscr{E}(-)$

$\mapsto$ **bind previous effects**
into second program

$$D_1 \xrightarrow{\;\mathrm{prog}_{12}\;} \mathscr{E}(D_2) \qquad \mathscr{E}(D_2) \xrightarrow{\;\mathrm{bind}^{\mathscr{E}}\,\mathrm{prog}_{23}\;} \mathscr{E}(D_3)$$

carry any previous $\mathscr{E}(-)$-effects along

$\mapsto$ **compose**

$\mathscr{E}$-composite program

$$D_1 \xrightarrow{\qquad\;\mathrm{bind}^{\mathscr{E}}\,\mathrm{prog}_{23} \circ \mathrm{prog}_{12}\;\qquad} \mathscr{E}(D_3)$$

causing cumulative $\mathscr{E}(-)$-effects

Recall: **Monadic effect handlers.**

$$D_1 \xrightarrow{\text{prog}_{12}} D_2$$ **data type to absorb $\mathscr{E}$-effects**

in-effectful program

Recall: **Monadic effect handlers.**

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}", "\textbf{in-effectful program}"'] & D_2 \arrow[d, mapsto, bend left, "\begin{array}{c}\textbf{incorporate handling}\\ \textbf{of } \mathscr{E}(-)\textbf{-effects}\end{array}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\,\mathrm{prog}_{12}", "\begin{array}{c}\textbf{in-effectful program}\\ \textbf{handling effects of type } \mathscr{E}(-)\end{array}"'] & D_2
\end{tikzcd}$$

Recall: **Monadic effect handlers.**

$$\begin{tikzcd}[column sep=huge, row sep=large]
D_1 \arrow[r, "\mathrm{prog}_{12}", "\textbf{in-effectful program}"'] & D_2 \arrow[d, mapsto, bend left, "\begin{array}{c}\textbf{incorporate handling}\\ \textbf{of } \mathscr{E}(-)\textbf{-effects}\end{array}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}", "\begin{array}{c}\textbf{in-effectful program}\\ \textbf{handling effects of type } \mathscr{E}(-)\end{array}"'] & D_2
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
D_1 \arrow[r, "\mathrm{ret}^{\mathscr{E}}_{D_1}", "\begin{array}{c}\textbf{produce}\\ \textbf{trivial effect}\end{array}"'] \arrow[rr, bend right, "\begin{array}{c}\mathrm{prog}_{12}\\ \textbf{no effect}\end{array}"'] & \mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}", "\begin{array}{c}\textbf{handle effects}\\ \textbf{running program}\end{array}"'] & D_2
\end{tikzcd}$$

consistency conditions

Recall: **Monadic effect handlers.**

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}", "\text{in-effectful program}"'] & D_2 \arrow[d, mapsto, bend left, "\text{incorporate handling of } \mathscr{E}(-)\text{-effects}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\,\mathrm{prog}_{12}", "\text{in-effectful program handling effects of type } \mathscr{E}(-)"'] & D_2
\end{tikzcd}$$

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{ret}^{\mathscr{E}}_{D_1}", "\text{produce trivial effect}"'] \arrow[rr, bend right, "\mathrm{prog}_{12}\ \text{no effect}"'] & \mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\,\mathrm{prog}_{12}", "\text{handle effects running program}"'] & D_2
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{E}(D_0) \arrow[r, "\mathrm{bind}^{\mathscr{E}}\,\mathrm{prog}_{01}", "\text{carry effects along}"'] \arrow[rr, bend right, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\left(D_0 \xrightarrow{\mathrm{prog}_{01}} \mathscr{E}(D_1) \xrightarrow{\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}} D_2\right)"'] & \mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\,\mathrm{prog}_{12}", "\text{handle cumulative effects}"'] & D_2
\end{tikzcd}$$

handle effects... **consecutively**

consistency conditions

Recall: **Data type system of Monadic effect handlers.**

$$\begin{tikzcd}[column sep=huge, row sep=huge]
D_1 \arrow[r, "\mathrm{prog}_{12}", "\textbf{in-effectful program}"'] & D_2 \arrow[d, mapsto, bend left=60, "\textbf{incorporate handling of } \mathscr{E}(-)\textbf{-effects}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\,\mathrm{prog}_{12}", "{\textbf{in-effectful program handling effects of type } \mathscr{E}(-)}"'] & D_2
\end{tikzcd}$$

“$\mathscr{E}$-modal data type”

Monadicity:

$\mathscr{E}$**-modales in** Type
(“EM-category”)

$\mathrm{Type}^{\mathscr{E}}$

Recall: **Data type system of Monadic effect handlers.**

$$\begin{tikzcd} D_1 \arrow[r, "\mathrm{prog}_{12}", "\text{in-effectful program}"'] & D_2 \arrow[d, mapsto, bend left=60, "\text{incorporate handling of } \mathscr{E}(-)\text{-effects}"] \\ \mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}", "\text{in-effectful program handling effects of type } \mathscr{E}(-)"'] & D_2 \end{tikzcd}$$

“$\mathscr{E}$-modal data type”

Monadicity:

$$\begin{tikzcd} \mathrm{Type} \arrow[loop right, "\mathscr{E}"] \end{tikzcd}$$

monad

$\mathscr{E}$-modales in Type (“EM-category”) $\mathrm{Type}^{\mathscr{E}}$

Recall: **Data type system of Monadic effect handlers.**

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}"] & D_2 \arrow[d, mapsto, bend left] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\,\mathrm{prog}_{12}"] & D_2
\end{tikzcd}$$

in-effectful program

incorporate handling of $\mathscr{E}(-)$-effects

“$\mathscr{E}$-modal data type”

in-effectful program handling effects of type $\mathscr{E}(-)$

Monadicity:

$$\begin{tikzcd}
& \mathrm{Type} \arrow[loop right, "\mathscr{E}"] \arrow[dl, shift right=2, "F^{\mathscr{E}}"'] \\
\mathrm{Type}^{\mathscr{E}} \arrow[ur, shift right=2, "U^{\mathscr{E}}"'] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}$$

induced monad

monadic adjunction

$\mathscr{E}$**-modales in** Type

(“EM-category”)

Recall: **Data type system of Monadic effect handlers.**

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}"] & D_2 \arrow[d, mapsto, bend left, "\text{incorporate handling of } \mathscr{E}(-)\text{-effects}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}"] & D_2
\end{tikzcd}$$

in-effectful program

“$\mathscr{E}$-modal data type”

in-effectful program
handling effects of type $\mathscr{E}(-)$

Monadicity:

$$\begin{tikzcd}
\mathrm{Type}' \arrow[d, "K^{UF}"] \arrow[rr, shift right, "U"'] \arrow[rr, phantom, "\perp"] & & \mathrm{Type} \arrow[ll, shift right, "F"'] \arrow[loop right, "\mathscr{E}"] \arrow[dll, shift right, "F^{\mathscr{E}}"'] \\
\mathrm{Type}^{\mathscr{E}} \arrow[urr, shift right, "U^{\mathscr{E}}"'] \arrow[urr, phantom, "\perp"] & &
\end{tikzcd}$$

any adjunction
on type system

induced monad

terminal comparison functor

monadic adjunction

$\mathscr{E}$-modales in Type
(“EM-category”)

Recall: **Data type system of Monadic effect handlers.**

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}", "\text{in-effectful program}"'] & D_2 \arrow[d, mapsto, bend left, "\text{incorporate handling of } \mathscr{E}(-)\text{-effects}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\,\mathrm{prog}_{12}", "\text{in-effectful program handling effects of type } \mathscr{E}(-)"'] & D_2 \;\; \text{“}\mathscr{E}\text{-modal data type”}
\end{tikzcd}$$

Monadicity:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathrm{Type}_{\mathscr{E}} \arrow[d, hook, "K_{UF}"] \arrow[dr, shift right, "U_{\mathscr{E}}"'] & & \\
\mathrm{Type}' \arrow[d, "K^{UF}"] \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{Type} \arrow[l, shift right, "F"'] \arrow[ul, two heads, shift right, "F_{\mathscr{E}}"'] \arrow[ul, phantom, "\perp"] \arrow[dl, shift right, "F^{\mathscr{E}}"'] \arrow[dl, phantom, "\perp"] \arrow[loop right, "\mathscr{E}"] \\
\mathrm{Type}^{\mathscr{E}} \arrow[ur, shift right, "U^{\mathscr{E}}"'] & &
\end{tikzcd}$$

free $\mathscr{E}$-modales in Type
(“Kleisli category”)

initial comparison functor

any adjunction
on type system

terminal comparison functor

$\mathscr{E}$-modales in Type
(“EM-category”)

monadic adjunction

induced monad

Recall: **Data type system of Monadic effect handlers.**

$$\begin{tikzcd} D_1 \arrow[r, "\mathrm{prog}_{12}"] & D_2 & {} \arrow[d, mapsto, bend left, "\text{incorporate handling of } \mathscr{E}(-)\text{-effects}"] \\ \mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}"] & D_2 & {} \end{tikzcd}$$

in-effectful program

incorporate handling of $\mathscr{E}(-)$-effects

“$\mathscr{E}$-modal data type”

in-effectful program
handling effects of type $\mathscr{E}(-)$

Monadicity:

$$\begin{tikzcd} \mathrm{Type}_{\mathscr{E}} \arrow[d, hook, "K_{UF}"] \arrow[dr, shift left, "U_{\mathscr{E}}"'] & \\ \mathrm{Type}' \arrow[d, "K^{UF}"] \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{Type} \arrow[l, shift right, "F"'] \arrow[ul, two heads, shift left, "F_{\mathscr{E}}"'] \arrow[dl, shift right, "F^{\mathscr{E}}"] \arrow[loop right, "\mathscr{E}"] \\ \mathrm{Type}^{\mathscr{E}} \arrow[ur, shift right, "U^{\mathscr{E}}"'] & \end{tikzcd}$$

free $\mathscr{E}$-modales in Type
(“Kleisli category”)

initial comparison functor

any adjunction
on type system

terminal comparison functor

$\mathscr{E}$-modales in Type
(“EM-category”)

induced monad

monadic adjunction

Now just to work this out
for the **effects induced by**
dependent data type formers
in LHoTT →

Given $B : \mathrm{BType}$ of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent data type formers** constitute modalities of **actual and potential B-measurements**:

classical data types

possibly **randomly**

$$\begin{tikzcd}[column sep=huge]
\mathrm{Type}_B \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] \arrow[r, shift left=8, "\coprod_B"] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] \arrow[r, shift right=8, "\prod_B"'] & \mathrm{Type} \arrow[l, "\times B" description] \arrow[loop above, "\star_B"'] \arrow[loop below, "\bigcirc_B"]
\end{tikzcd}$$

actual data $\perp$ $\perp$ **potential data**

necessarily **indefinitely**

Given $B : \mathrm{BType}$ of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent data type formers** constitute modalities of **actual and potential B-measurements**:

classical
data types

possibly **randomly**

$$\begin{tikzcd}
\mathrm{Type}_B \arrow[loop above, "\lozenge_B"] \arrow[loop below, "\square_B"'] \arrow[r, shift left=3, "\coprod_B"] \arrow[r, shift right=3, "\prod_B"'] & \mathrm{Type} \arrow[l, "\times B" description] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"']
\end{tikzcd}$$

actual data $\perp$ $\quad$ $\perp$ **potential data**

necessarily **indefinitely**

---

necessarily $P_\bullet$

$$\square_B\, P_\bullet$$

$$b : B \;\vdash\; \prod_{b' : B} P_{b'}$$

Given B : BType of possible measurement outcomes ("possible worlds")
the monadic effects of B-dependent data type formers constitute modalities of **actual and potential B-measurements**:

classical data types

possibly **randomly**

$$\begin{tikzcd}
\mathrm{Type}_B \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] \arrow[r, shift left=4, "\coprod_B"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, shift right=4, "\prod_B"'] & \mathrm{Type} \arrow[l, "\times B"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"']
\end{tikzcd}$$

actual data $\perp$ $\perp$ **potential data**

necessarily **indefinitely**

---

necessarily $P_\bullet$ **entails** **actually $P_\bullet$**

$$\Box_B P_\bullet \xrightarrow{\;\epsilon^{\Box_B}_{P_\bullet}\;} P_\bullet$$

$$b : B \;\vdash\; \prod_{b':B} P_{b'} \xrightarrow{(p_{b'})_{b':B} \,\mapsto\, p_b} P_b$$

Given B : BType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent data type formers** constitute modalities of **actual and potential B-measurements**:

classical data types

possibly **randomly**

$$\begin{tikzcd}
\mathrm{Type}_B \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] \arrow[r, shift left=4, "\coprod_B"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, shift right=4, "\prod_B"'] & \mathrm{Type} \arrow[l, "\times B" description] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"']
\end{tikzcd}$$

actual data $\perp$ $\perp$ **potential data**

necessarily **indefinitely**

---

necessarily $P_\bullet$ **entails** **actually $P_\bullet$** **entails** **possibly $P_\bullet$**

$$\Box_B P_\bullet \xrightarrow{\;\epsilon^{\Box_B}_{P_\bullet}\;} P_\bullet \xrightarrow{\;\eta^{\Diamond_B}_{P_\bullet}\;} \Diamond_B P_\bullet$$

$$b : B \;\vdash\; \prod_{b':B} P_{b'} \xrightarrow{\;(p_{b'})_{b':B} \;\mapsto\; p_b\;} P_b \xrightarrow{\;p_b \;\mapsto\; (p_b)_b\;} \coprod_{b':B} P_{b'}$$

Given B : BType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent data type formers** constitute modalities of **actual and potential B-measurements**:

classical data types

possibly **randomly**

actual data $\perp$ **potential data**

necessarily **indefinitely**

$$\begin{tikzcd}
\mathrm{Type}_B \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"] \arrow[r, shift left=6, "\coprod_B"] \arrow[r, shift right=6, "\prod_B"'] & \mathrm{Type} \arrow[l, "\times B"' description] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"]
\end{tikzcd}$$

$$\begin{tikzcd}
\coprod_B \arrow[d, phantom, "\perp"] \\
\times B \arrow[d, phantom, "\perp"] \\
\prod_B
\end{tikzcd}$$

---

necessarily $P_\bullet$ **entails** **actually $P_\bullet$** **entails** **possibly $P_\bullet$**

$$\begin{tikzcd}
\Box_B P_\bullet \arrow[r, "\epsilon^{\Box_B}_{P_\bullet}" description] & P_\bullet \arrow[r, "\eta^{\Diamond_B}_{P_\bullet}" description] & \Diamond_B P_\bullet
\end{tikzcd}$$

$$b : B \ \vdash \ \begin{tikzcd}
\prod\limits_{b':B} P_{b'} \arrow[r, "(p_{b'})_{b':B} \,\mapsto\, p_b"] & P_b \arrow[r, "p_b \,\mapsto\, (p_b)_b"] & \coprod\limits_{b':B} P_{b'}
\end{tikzcd}$$

randomly P

$$\star_B P$$

$$\coprod_{b:B} P$$

Given $B :$ BType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent data type formers** constitute modalities of **actual and potential B-measurements**:

classical data types

possibly **randomly**

actual data **potential data**

$$\begin{tikzcd}
\perp & \mathrm{Type}_B \arrow[loop above, "\lozenge_B"] \arrow[loop below, "\Box_B"'] \arrow[rr, shift left=4, "\coprod_B"] \arrow[rr, shift left=2, phantom, "\perp"] \arrow[rr, shift right=2, phantom, "\perp"] \arrow[rr, shift right=4, "\prod_B"'] & & \mathrm{Type} \arrow[ll, "\times B" description] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"'] & \perp
\end{tikzcd}$$

necessarily **indefinitely**

necessarily $P_\bullet$ **entails** **actually $P_\bullet$** **entails** **possibly $P_\bullet$**

$$\Box_B P_\bullet \xrightarrow{\epsilon^{\Box_B}_{P_\bullet}} P_\bullet \xrightarrow{\eta^{\lozenge_B}_{P_\bullet}} \lozenge_B P_\bullet$$

$$b : B \;\vdash\; \prod_{b':B} P_{b'} \xrightarrow{(p_{b'})_{b':B} \;\mapsto\; p_b} P_b \xrightarrow{p_b \;\mapsto\; (p_b)_b} \coprod_{b':B} P_{b'}$$

randomly P **entails** **potentially P**

$$\star_B P \xrightarrow{\epsilon^{\star_B}_{P}} P$$

$$\coprod_{b:B} P \xrightarrow{(p)_b \;\mapsto\; p} P$$

Given B : BType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent data type formers** constitute modalities of **actual and potential B-measurements**:

classical data types

possibly **randomly**

actual data $\perp$ **potential data**

necessarily **indefinitely**

$$\begin{tikzcd}
\mathrm{Type}_B \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"] \arrow[r, shift left=4, "\coprod_B"] \arrow[r, shift right=4, "\prod_B"'] & \mathrm{Type} \arrow[l, "\times B" description] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"]
\end{tikzcd}$$

$$\coprod_B \dashv \times B \dashv \prod_B \qquad \Box_B \perp \Diamond_B \qquad \star_B \perp \bigcirc_B$$

necessarily $P_\bullet$ **entails** **actually $P_\bullet$** **entails** **possibly $P_\bullet$**

$$\Box_B P_\bullet \xrightarrow{\;\varepsilon^{\Box_B}_{P_\bullet}\;} P_\bullet \xrightarrow{\;\eta^{\Diamond_B}_{P_\bullet}\;} \Diamond_B P_\bullet$$

$$b : B \;\vdash\; \prod_{b':B} P_{b'} \xrightarrow{(p_{b'})_{b':B} \,\mapsto\, p_b} P_b \xrightarrow{p_b \,\mapsto\, (p_b)_b} \coprod_{b':B} P_{b'}$$

randomly P **entails** **potentially P** **entails** **indefinitely P**

$$\star_B P \xrightarrow{\;\varepsilon^{\star_B}_{P}\;} P \xrightarrow{\;\eta^{\bigcirc_B}_{P}\;} \bigcirc_B P$$

$$\coprod_{b:B} P \xrightarrow{(p)_b \,\mapsto\, p} P \xrightarrow{p \,\mapsto\, (p)_{b:B}} \prod_{b:B} P$$

Given B : BType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements**.

quantum data types

$$\begin{tikzcd}
& \text{possibly} &&&& \text{randomly} \\
&& (\mathrm{LType}_B)_{\Diamond_B} \arrow[r, hook, "\sim"] & (\mathrm{LType}_B)^{\Diamond_B} \arrow[dr, phantom, "\simeq"] && \\
\text{actual data } \perp & \mathrm{LType}_B \simeq \mathrm{LType}^{\bigcirc_B} \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] \arrow[ur, two heads, "\oplus_B"] \arrow[dr, two heads, "\oplus_B"'] & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook] \arrow[u, phantom, "\perp"] \arrow[d, phantom, "\perp"] && \mathrm{LType} \arrow[ll, two heads, "\mathbb{1}_B \otimes"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"'] & \perp \text{ potential data} \\
&& (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B} \arrow[ur, phantom, "\simeq"] && \\
& \text{necessarily} &&&& \text{indefinitely}
\end{tikzcd}$$

Given $B : \mathrm{BType}$ of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements**.

quantum data types

possibly **randomly**

actual data $\perp$ $\perp$ **potential data**

$$\begin{tikzcd}
& (\mathrm{LType}_B)_{\Diamond_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\Diamond_B} \\
\mathrm{LType}_B \simeq \mathrm{LType}^{\bigcirc_B} \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"] \arrow[ur, two heads, "\oplus_B"] \arrow[dr, two heads, "\oplus_B"'] & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[d, phantom, "\perp"] & \mathrm{LType} \arrow[l, two heads, "\mathbb{1}_B \otimes"'] \arrow[u, phantom, "\simeq"] \arrow[d, phantom, "\simeq"] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"] \\
& (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B}
\end{tikzcd}$$

necessarily **indefinitely**

necessarily $\mathcal{H}_\bullet$

$\Box_B \, \mathcal{H}_\bullet$

Given... **obtain...**

$b : B \;\vdash\; \mathcal{H}$

measurement result

where $\mathcal{H} \;:=\; \bigoplus_{b' : B} \mathcal{H}_{b'}$

Given B : BType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements**.

quantum
data types

possibly **randomly**

$$\begin{tikzcd}
{} & {} & (\mathrm{LType}_B)_{\Diamond_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\Diamond_B} & {} \\
\mathrm{LType}_B \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"] \arrow[r, phantom, "\simeq"] \arrow[urr, two heads, "\oplus_B"] \arrow[drr, two heads, "\oplus_B"'] & \mathrm{LType}^{\bigcirc_B} & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[d, phantom, "\perp"] & {} & \mathrm{LType} \arrow[ll, two heads, "\mathbb{1}_B\otimes"'] \arrow[ul, dash, "\wr"'] \arrow[dl, dash, "\wr"] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"] \\
{} & {} & (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B} & {}
\end{tikzcd}$$

actual data $\perp$ $\perp$ **potential data**

necessarily **indefinitely**

necessarily $\mathcal{H}_\bullet$ **entails** **actually $\mathcal{H}_\bullet$**

$$\begin{tikzcd}
\Box_B \mathcal{H}_\bullet \arrow[rr, "\varepsilon^{\Box_B}_{\mathcal{H}_\bullet}"] & & \mathcal{H}_\bullet
\end{tikzcd}$$

Given... **obtain...**

$$b : B \;\vdash\; \mathcal{H} \xrightarrow[\text{measurement collapse}]{\sum_{b'} |\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle} \mathcal{H}_b \qquad \text{where } \mathcal{H} := \bigoplus_{b':B} \mathcal{H}_{b'}$$

measurement result

Given $B : \mathrm{BType}$ of possible measurement outcomes ("possible worlds") **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements**.

quantum
data types

possibly

actual data $\perp$

necessarily

randomly

$\perp$ **potential data**

indefinitely

$$\begin{tikzcd}
& (\mathrm{LType}_B)_{\lozenge_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\lozenge_B} \arrow[dr, dash, "\sim"] & \\
\mathrm{LType}_B \simeq \mathrm{LType}^{\bigcirc_B} \arrow[loop above, "\lozenge_B"] \arrow[loop below, "\square_B"] \arrow[ur, two heads, "\oplus_B"] \arrow[dr, two heads, "\oplus_B"'] & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[d, phantom, "\perp"] & & \mathrm{LType} \arrow[ll, two heads, "\mathbb{1}_B \otimes"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"] \\
& (\mathrm{LType}_B)_{\square_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\square_B} \arrow[ur, dash, "\sim"'] &
\end{tikzcd}$$

necessarily $\mathcal{H}_\bullet$ **entails** **actually $\mathcal{H}_\bullet$** **entails** **possibly $\mathcal{H}_\bullet$**

$$\begin{tikzcd}
\square_B \mathcal{H}_\bullet \arrow[r, "\epsilon^{\square_B}_{\mathcal{H}_\bullet}"] & \mathcal{H}_\bullet \arrow[r, "\eta^{\lozenge_B}_{\mathcal{H}_\bullet}"] & \lozenge_B \mathcal{H}_\bullet
\end{tikzcd}$$

Given... **obtain...**

$b : B \vdash$

measurement result

$$\begin{tikzcd}
\mathcal{H} \arrow[r, two heads, "\sum_{b'} |\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle", "\text{measurement collapse}"'] \arrow[rr, bend right, "\text{linear projector onto sub-Hilbert space } \mathcal{H}_b"'] & \mathcal{H}_b \arrow[r, hook, "|\psi_b\rangle \mapsto \oplus_{b'} \begin{cases} |\psi_b\rangle & \text{if } b'=b \\ 0 & \text{else} \end{cases}", "\text{state preparation}"'] & \mathcal{H},
\end{tikzcd}$$

where $\mathcal{H} := \bigoplus\limits_{b':B} \mathcal{H}_{b'}$

Given $B : \mathrm{BType}$ of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements**.

quantum data types

possibly **randomly**

actual data $\perp$ $\quad$ $\perp$ **potential data**

$$\begin{tikzcd}
 & & (\mathrm{LType}_B)_{\Diamond_B} \arrow[r, hook, "\sim"] & (\mathrm{LType}_B)^{\Diamond_B} & \\
\mathrm{LType}_B \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] \arrow[urr, two heads, "\oplus_B"] \arrow[drr, two heads, "\oplus_B"'] \arrow[r, phantom, "\simeq"] & \mathrm{LType}^{\bigcirc_B} & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[u, phantom, "\perp"] \arrow[d, phantom, "\perp"] & & \mathrm{LType} \arrow[ll, two heads, "\mathbb{1}_B \otimes"'] \arrow[ul, phantom, "\simeq"] \arrow[dl, phantom, "\simeq"] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"'] \\
 & & (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B} &
\end{tikzcd}$$

necessarily **indefinitely**

---

principle of quantum compulsion:

necessarily $\mathcal{H}_\bullet$ **entails** **actually $\mathcal{H}_\bullet$** **entails** **possibly $\mathcal{H}_\bullet$** **is** **necessarily $\mathcal{H}_\bullet$**

$$\begin{tikzcd}
\Box_B \mathcal{H}_\bullet \arrow[r, "\varepsilon^{\Box_B}_{\mathcal{H}_\bullet}"] & \mathcal{H}_\bullet \arrow[r, "\eta^{\Diamond_B}_{\mathcal{H}_\bullet}"] & \Diamond_B \mathcal{H}_\bullet \arrow[r, phantom, "\simeq"'] & \Box_B \mathcal{H}_\bullet
\end{tikzcd}$$

ambidexterity

Given... **obtain...**

$$\begin{tikzcd}
b : B \vdash & \mathcal{H} \arrow[r, two heads, "\Sigma_{b'} |\psi_{b'}\rangle \,\mapsto\, |\psi_b\rangle", "\text{measurement collapse}"'] \arrow[rr, bend right, "\text{linear projector onto sub-Hilbert space } \mathcal{H}_b"'] & \mathcal{H}_b \arrow[r, hook, "|\psi_b\rangle \mapsto \oplus_{b'} \begin{cases} |\psi_b\rangle & \text{if } b'=b \\ 0 & \text{else} \end{cases}", "\text{state preparation}"'] & \mathcal{H},
\end{tikzcd}
\qquad \text{where } \mathcal{H} := \bigoplus_{b' : B} \mathcal{H}_{b'}$$

measurement result

Given $B : \mathrm{BType}$ of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements**.

quantum data types

possibly **randomly**

actual data $\perp$ $\quad$ $\perp$ **potential data**

$$\begin{tikzcd}
 & (\mathrm{LType}_B)_{\Diamond_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\Diamond_B} \arrow[dr, phantom, "\simeq"] & \\
\mathrm{LType}_B \simeq \mathrm{LType}^{\bigcirc_B} \arrow[ur, two heads, "\oplus_B"] \arrow[dr, two heads, "\oplus_B"'] \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[d, phantom, "\perp"] & & \mathrm{LType} \arrow[ll, two heads, "\mathbb{1}_B \otimes"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"'] \\
 & (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B} \arrow[ur, phantom, "\simeq"] &
\end{tikzcd}$$

necessarily **indefinitely**

---

principle of quantum compulsion:

necessarily $\mathcal{H}_\bullet$ **entails** **actually $\mathcal{H}_\bullet$** **entails** **possibly $\mathcal{H}_\bullet$** **is** **necessarily $\mathcal{H}_\bullet$**

$$\Box_B \mathcal{H}_\bullet \xrightarrow{\epsilon^{\Box_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet \xrightarrow{\eta^{\Diamond_B}_{\mathcal{H}_\bullet}} \Diamond_B \mathcal{H}_\bullet \underset{\text{ambidexterity}}{\simeq} \Box_B \mathcal{H}_\bullet$$

Given... **obtain...**

$$\underset{\text{measurement result}}{b : B} \;\vdash\; \begin{tikzcd}
\mathcal{H} \arrow[r, two heads, "\sum_{b'} |\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle", "\text{measurement collapse}"'] \arrow[rr, bend right, "\text{linear projector onto sub-Hilbert space } \mathcal{H}_b"'] & \mathcal{H}_b \arrow[r, hook, "|\psi_b\rangle \mapsto \oplus_{b'} \begin{cases} |\psi_b\rangle & \text{if } b'=b \\ 0 & \text{else} \end{cases}", "\text{state preparation}"'] & \mathcal{H},
\end{tikzcd} \quad \text{where } \mathcal{H} := \bigoplus_{b':B} \mathcal{H}_{b'}$$

---

randomly $\mathcal{H}$

$$\star_B \mathcal{H}$$

$$\bigoplus_{b:B} \mathcal{H}$$

Given B : BType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements**.

quantum data types

possibly **randomly**

actual data $\perp$ $\perp$ **potential data**

$$\begin{tikzcd}
& (\mathrm{LType}_B)_{\lozenge_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\lozenge_B} \arrow[dr, phantom, "\simeq"] & \\
\mathrm{LType}_B \simeq \mathrm{LType}^{\bigcirc_B} \arrow[ur, two heads, "\bigoplus_B"] \arrow[dr, two heads, "\bigoplus_B"'] \arrow[loop above, "\lozenge_B"] \arrow[loop below, "\Box_B"] & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[d, phantom, "\perp"] & & \mathrm{LType} \arrow[ll, two heads, "\mathbb{1}_B \otimes"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"] \\
& (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B} \arrow[ur, phantom, "\simeq"] &
\end{tikzcd}$$

necessarily **indefinitely**

---

principle of quantum compulsion:

necessarily $\mathcal{H}_\bullet$ **entails** **actually $\mathcal{H}_\bullet$** **entails** **possibly $\mathcal{H}_\bullet$** **is** **necessarily $\mathcal{H}_\bullet$**

$$\Box_B \mathcal{H}_\bullet \xrightarrow{\;\epsilon^{\Box_B}_{\mathcal{H}_\bullet}\;} \mathcal{H}_\bullet \xrightarrow{\;\eta^{\lozenge_B}_{\mathcal{H}_\bullet}\;} \lozenge_B \mathcal{H}_\bullet \;\simeq\; \Box_B \mathcal{H}_\bullet$$

ambidexterity

Given... **obtain...**

$$b : B \;\vdash\; \begin{tikzcd}
\mathcal{H} \arrow[r, two heads, "\sum_{b'} |\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle", "\text{measurement collapse}"'] \arrow[rr, bend right, "\text{linear projector onto sub-Hilbert space } \mathcal{H}_b"'] & \mathcal{H}_b \arrow[r, hook, "|\psi_b\rangle \mapsto \oplus_{b'} \left\{ \begin{array}{ll} |\psi_b\rangle & \text{if } b'{=}b \\ 0 & \text{else} \end{array} \right.", "\text{state preparation}"'] & \mathcal{H},
\end{tikzcd} \quad \text{where } \mathcal{H} := \bigoplus_{b' : B} \mathcal{H}_{b'}$$

measurement result

measurement collapse **state preparation**

linear projector onto sub-Hilbert space $\mathcal{H}_b$

---

randomly $\mathcal{H}$ **entails** **potentially $\mathcal{H}$**

$$\star_B \mathcal{H} \xrightarrow{\;\epsilon^{\star_B}_{\mathcal{H}}\;} \mathcal{H}$$

$$\bigoplus_{b:B} \mathcal{H} \xrightarrow[\text{quantum superposition}]{\;\oplus_b |\psi_b\rangle \,\mapsto\, \sum_b |\psi_b\rangle\;} \mathcal{H}$$

quantum superposition

Given B : BType of possible measurement outcomes (“possible worlds”)
the monadic effects of B-dependent linear data type formers constitute modalities of **actual and potential quantum B-measurements.**

quantum
data types

$$\begin{tikzcd}
 & & (\mathrm{LType}_B)_{\Diamond_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\Diamond_B} \arrow[rd, phantom, "\sim"] & \\
\text{actual data } \perp & \mathrm{LType}_B \simeq \mathrm{LType}^{\bigcirc_B} \arrow[loop above, "\text{possibly } \Diamond_B"] \arrow[loop below, "\text{necessarily } \Box_B"'] \arrow[ru, two heads, "\oplus_B"] \arrow[rd, two heads, "\oplus_B"'] & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[d, phantom, "\perp"] & & \mathrm{LType} \arrow[ll, two heads, "\mathbb{1}_B\otimes"'] \arrow[loop above, "\text{randomly } \star_B"] \arrow[loop below, "\text{indefinitely } \bigcirc_B"'] \quad \perp \text{ potential data} \\
 & & (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B} \arrow[ru, phantom, "\sim"] &
\end{tikzcd}$$

---

principle of quantum compulsion:

necessarily $\mathcal{H}_\bullet$ **entails** **actually $\mathcal{H}_\bullet$** **entails** **possibly $\mathcal{H}_\bullet$** **is** **necessarily $\mathcal{H}_\bullet$**

$$\begin{tikzcd}
\Box_B \mathcal{H}_\bullet \arrow[r, "\epsilon^{\Box_B}_{\mathcal{H}_\bullet}"] & \mathcal{H}_\bullet \arrow[r, "\eta^{\Diamond_B}_{\mathcal{H}_\bullet}"] & \Diamond_B \mathcal{H}_\bullet \arrow[r, phantom, "\simeq"] & \Box_B \mathcal{H}_\bullet
\end{tikzcd}$$

ambidexterity

Given... **obtain...**

$$\begin{tikzcd}
b : B \ \vdash & \mathcal{H} \arrow[r, two heads, "\sum_{b'} |\psi_{b'}\rangle \ \mapsto \ |\psi_b\rangle", "\text{measurement collapse}"'] \arrow[rr, bend right, "\text{linear projector onto sub-Hilbert space } \mathcal{H}_b"'] & \mathcal{H}_b \arrow[r, hook, "|\psi_b\rangle \mapsto \oplus_{b'} \begin{cases} |\psi_b\rangle & \text{if } b'=b \\ 0 & \text{else} \end{cases}", "\text{state preparation}"'] & \mathcal{H},
\end{tikzcd}$$

measurement result

$$\text{where } \mathcal{H} := \bigoplus_{b':B} \mathcal{H}_{b'}$$

- -

randomly $\mathcal{H}$ **entails** **potentially $\mathcal{H}$** **entails** **indefinitely $\mathcal{H}$**

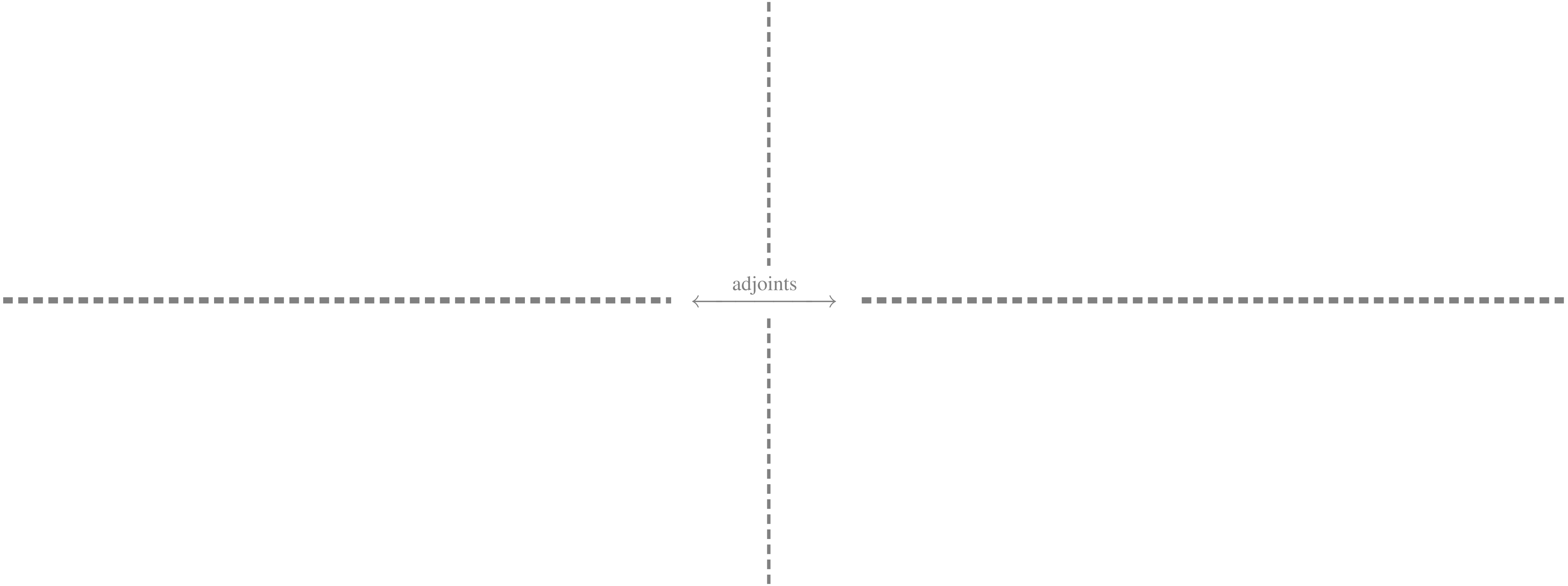
$$\begin{tikzcd}
\star_B \mathcal{H} \arrow[r, "\epsilon^{\star_B}_{\mathcal{H}}"] & \mathcal{H} \arrow[r, "\eta^{\bigcirc_B}_{\mathcal{H}}"] & \bigcirc_B \mathcal{H}
\end{tikzcd}$$

$$\begin{tikzcd}
\bigoplus_{b:B} \mathcal{H} \arrow[r, "\oplus_b |\psi_b\rangle \mapsto \sum_b |\psi_b\rangle", "\text{quantum superposition}"'] & \mathcal{H} \arrow[r, "|\psi\rangle \mapsto \oplus_b |\psi\rangle_b", "\text{quantum parallelization}"'] & \bigoplus_{b:B} \mathcal{H}
\end{tikzcd}$$

The pure effects of these modalities of dependent linear data type formation

are remarkable in their sheer quantum information-theoretic content.

To repeat:



The pure effects of these modalities of dependent linear data type formation

are remarkable in their sheer quantum information-theoretic content.

To repeat:

$$\overbrace{(p_B)^*(p_B)_*}^{\Box_B}\mathcal{H}_\bullet \xrightarrow[\textbf{necessity counit}]{\varepsilon^{\Box_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet$$

$$b : B \vdash \bigoplus_{b':B}\mathcal{H}_{b'} \xrightarrow[\text{quantum }\textbf{measurement}]{\oplus_{b'}|\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle} \mathcal{H}_b$$

“ the necessary becomes actual ”

adjoints $\longleftrightarrow$

The pure effects of these modalities of dependent linear data type formation

are remarkable in their sheer quantum information-theoretic content.

To repeat:

$$\overbrace{(p_B)^*(p_B)_*}^{\Box_B}\mathcal{H}_\bullet \xrightarrow[\textbf{necessity counit}]{\varepsilon^{\Box_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet$$

$$b : B \vdash \bigoplus_{b':B}\mathcal{H}_{b'} \xrightarrow[\text{quantum }\textbf{measurement}]{\oplus_{b'}|\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle} \mathcal{H}_b$$

“ the necessary becomes actual ”

adjoints

↔

$$\mathcal{H}_\bullet \xrightarrow[\textbf{possibility unit}]{\eta^{\Diamond_B}_{\mathcal{H}_\bullet}} \overbrace{(p_B)^*(p_B)_!}^{\Diamond_B}\mathcal{H}_\bullet$$

$$b : B \vdash \mathcal{H}_b \xrightarrow[\text{quantum }\textbf{state preparation}]{|\psi_b\rangle \;\mapsto\; \oplus_{b'}\begin{cases}|\psi_b\rangle & \text{if } b'=b \\ 0 & \text{else}\end{cases}} \bigoplus_{b':B}\mathcal{H}_{b'}$$

“ the actual is possible ”

The pure effects of these modalities of dependent linear data type formation

are remarkable in their sheer quantum information-theoretic content.

To repeat:

$$\overbrace{(p_B)^*(p_B)_*}^{\Box_B}\mathcal{H}_\bullet \xrightarrow[\textbf{necessity counit}]{\epsilon^{\Box_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet$$

$$b : B \vdash \bigoplus_{b':B}\mathcal{H}_{b'} \xrightarrow[\text{quantum } \textbf{measurement}]{\oplus_{b'}|\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle} \mathcal{H}_b$$

“ the necessary becomes actual ”

$$\mathcal{H}_\bullet \xrightarrow[\textbf{possibility unit}]{\eta^{\Diamond_B}_{\mathcal{H}_\bullet}} \overbrace{(p_B)^*(p_B)_!}^{\Diamond_B}\mathcal{H}_\bullet$$

$$b : B \vdash \mathcal{H}_b \xrightarrow[\text{quantum } \textbf{state preparation}]{|\psi_b\rangle \;\mapsto\; \oplus_{b'}\left\{\begin{array}{ll}|\psi_b\rangle & \text{if } b'=b \\ 0 & \text{else}\end{array}\right.} \bigoplus_{b':B}\mathcal{H}_{b'}$$

“ the actual is possible ”

$$\xleftrightarrow{\text{adjoints}}$$

“ the random becomes potential ”

$$\overbrace{(p_B)_!(p_B)^*}^{\star_B}\mathcal{H} \xhookrightarrow[\textbf{randomness counit}]{\epsilon^{\star_B}_{\mathcal{H}}} \mathcal{H}$$

$$\bigoplus_{b:B}\mathcal{H} \xrightarrow[\text{quantum } \textbf{superposition}]{\oplus_b|\psi_b\rangle \;\mapsto\; \Sigma_b|\psi_b\rangle} \mathcal{H}$$

The pure effects of these modalities of dependent linear data type formation

are remarkable in their sheer quantum information-theoretic content.

To repeat:

$$\overbrace{(p_B)^*(p_B)_*}^{\Box_B}\mathcal{H}_\bullet \xrightarrow[\textbf{necessity counit}]{\epsilon^{\Box_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet$$

$$b : B \vdash \bigoplus_{b':B}\mathcal{H}_{b'} \xrightarrow[\text{quantum }\textbf{measurement}]{\oplus_{b'}|\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle} \mathcal{H}_b$$

“ the necessary becomes actual ”

$$\mathcal{H}_\bullet \xrightarrow[\textbf{possibility unit}]{\eta^{\Diamond_B}_{\mathcal{H}_\bullet}} \overbrace{(p_B)^*(p_B)_!}^{\Diamond_B}\mathcal{H}_\bullet$$

$$b : B \vdash \mathcal{H}_b \xrightarrow[\text{quantum }\textbf{state preparation}]{|\psi_b\rangle \;\mapsto\; \oplus_{b'}\begin{cases}|\psi_b\rangle & \text{if } b'{=}b\\ 0 & \text{else}\end{cases}} \bigoplus_{b':B}\mathcal{H}_{b'}$$

“ the actual is possible ”

$\longleftrightarrow$ adjoints

“ the random becomes potential ”

$$\overbrace{(p_B)_!(p_B)^*}^{\star_B}\mathcal{H} \xhookrightarrow[\textbf{randomness counit}]{\epsilon^{\star_B}_{\mathcal{H}}} \mathcal{H}$$

$$\bigoplus_{b:B}\mathcal{H} \xrightarrow[\text{quantum }\textbf{superposition}]{\oplus_b|\psi_b\rangle \;\mapsto\; \Sigma_b|\psi_b\rangle} \mathcal{H}$$

“ the potential is inderminate ”

$$\mathcal{H} \xrightarrow[\textbf{indeterminacy unit}]{\eta^{\bigcirc_B}_{\mathcal{H}}} \overbrace{(p_B)_*(p_B)^*}^{\bigcirc_B}\mathcal{H}$$

$$\mathcal{H} \xrightarrow[\text{quantum }\textbf{parallelism}]{|\psi\rangle \;\mapsto\; \oplus_b|\psi\rangle_b} \bigoplus_{b:B}\mathcal{H}$$

Q-bits are the free linear indeterminacy-effect handlers over $\mathrm{Bit} = \{0, 1\}$

Coherent q-bits:

$$\begin{tikzcd} \mathrm{QBit} \arrow[d, equal, "{:}"'] & : & \mathrm{LType} \arrow[r, hook, "\mathbb{1}_{\mathrm{Bit}}\otimes"] & \mathrm{LType}_{\mathrm{Bit}} \arrow[r, "\oplus_{\mathrm{Bit}}", "\sim"'] & \mathrm{LType}^{\bigcirc_B} \\ \bigcirc_{\mathrm{Bit}}\mathbb{1} \end{tikzcd}$$

De-cohered (measured) q-bits:

Quantum gate with q-bit output:

Q-bits are the free linear indeterminacy-effect handlers over $\mathrm{Bit} = \{0, 1\}$

Coherent q-bits:

$$\begin{tikzcd}
\mathrm{QBit} \arrow[d, equal, ":"] & : \; \mathrm{LType} \arrow[r, hook, "\mathbb{1}_{\mathrm{Bit}}\otimes"] & \mathrm{LType}_{\mathrm{Bit}} \arrow[r, "\bigoplus_{\mathrm{Bit}}", "\sim"'] & \mathrm{LType}^{\bigcirc_B} \\
\bigcirc_{\mathrm{Bit}}\mathbb{1} \; = \; \bigoplus_{\{0,1\}}\mathbb{C} \; = \; \mathbb{C}\cdot|0\rangle \oplus \mathbb{C}\cdot|1\rangle & & &
\end{tikzcd}$$

Quantum gate with q-bit output:

De-cohered (measured) q-bits:

Q-bits are the free linear indeterminacy-effect handlers over $\mathrm{Bit} = \{0,1\}$

Coherent q-bits:

———

$$\begin{array}{l} \mathrm{QBit} \;:\; \mathrm{LType} \xhookrightarrow{\mathbb{1}_{\mathrm{Bit}}\otimes} \mathrm{LType}_{\mathrm{Bit}} \underset{\sim}{\xrightarrow{\oplus_{\mathrm{Bit}}}} \mathrm{LType}^{\bigcirc_B} \\ \quad\overset{\cdot\cdot}{\|} \\ \bigcirc_{\mathrm{Bit}}\mathbb{1} \;=\; \bigoplus_{\{0,1\}}\mathbb{C} \;=\; \mathbb{C}\cdot|0\rangle \oplus \mathbb{C}\cdot|1\rangle \end{array}$$

Quantum gate with q-bit output:

De-cohered (measured) q-bits:

Q-bits are the free linear indeterminacy-effect handlers over $\mathrm{Bit} = \{0,1\}$

Coherent q-bits:

$$\mathrm{QBit} \;:\; \mathrm{LType} \xhookrightarrow{\mathbb{1}_{\mathrm{Bit}}\otimes} \mathrm{LType}_{\mathrm{Bit}} \xrightarrow[\sim]{\bigoplus_{\mathrm{Bit}}} \mathrm{LType}^{\bigcirc_B}$$

$$\overset{\cdot\cdot}{\Vert}$$

$$\bigcirc_{\mathrm{Bit}}\mathbb{1} \;=\; \bigoplus_{\{0,1\}}\mathbb{C} \;=\; \mathbb{C}\cdot|0\rangle \oplus \mathbb{C}\cdot|1\rangle$$

$$\mathrm{QBit}$$

$$\otimes$$

$$\mathcal{H}$$

$$\Vert$$

$$\bigcirc_{\mathrm{Bit}}\mathcal{H} \;=\; \bigoplus_{\{0,1\}}\mathcal{H} \;=\; \mathcal{H}\otimes|0\rangle \;\oplus\; \mathcal{H}\otimes|0\rangle$$

De-cohered (measured) q-bits:

Quantum gate with q-bit output:

Q-bits are the free linear indeterminacy-effect handlers over $\mathrm{Bit} = \{0, 1\}$

Coherent q-bits:

$$\begin{tikzcd}
\mathrm{QBit} \, : \, \mathrm{LType} \arrow[r, hook, "\mathbb{1}_{\mathrm{Bit}}\otimes"] & \mathrm{LType}_{\mathrm{Bit}} \arrow[r, "\bigoplus_{\mathrm{Bit}}", "\sim"'] & \mathrm{LType}^{\bigcirc_B}
\end{tikzcd}$$

$$\overset{..}{\Vert}$$

$$\bigcirc_{\mathrm{Bit}} \mathbb{1} \;=\; \bigoplus_{\{0,1\}} \mathbb{C} \;=\; \mathbb{C} \cdot |0\rangle \oplus \mathbb{C} \cdot |1\rangle$$

$$\mathrm{QBit}$$

$$\otimes$$

$$\mathcal{H}$$

$$\Vert$$

$$\bigcirc_{\mathrm{Bit}} \mathcal{H} \;=\; \bigoplus_{\{0,1\}} \mathcal{H} \;=\; \mathcal{H} \otimes |0\rangle \;\oplus\; \mathcal{H} \otimes |0\rangle$$

De-cohered (measured) q-bits:

$$\begin{tikzcd}
\mathbb{1}_{\mathrm{Bit}} \, : \, \mathrm{LType}_{\mathrm{Bit}} \arrow[r, "\bigoplus_{\mathrm{Bit}}", "\sim"'] & \mathrm{LType}^{\bigcirc_{\mathrm{Bit}}}
\end{tikzcd}$$

$$b : \mathrm{Bit} \quad \vdash \quad \mathbb{C} \cdot |b\rangle \, : \, \mathrm{LType}$$

Quantum gate with q-bit output:

Q-bits are the free linear indeterminacy-effect handlers over $\mathrm{Bit} = \{0,1\}$

Coherent q-bits:

$$\mathrm{QBit} \;:\; \mathrm{LType} \xhookrightarrow{\mathbb{1}_{\mathrm{Bit}}{}^{\otimes}} \mathrm{LType}_{\mathrm{Bit}} \underset{\sim}{\xrightarrow{\bigoplus_{\mathrm{Bit}}}} \mathrm{LType}^{\bigcirc_B}$$

$$\overset{\cdot\cdot}{\|}$$

$$\bigcirc_{\mathrm{Bit}} \mathbb{1} \;=\; \bigoplus_{\{0,1\}} \mathbb{C} \;=\; \mathbb{C}\cdot|0\rangle \oplus \mathbb{C}\cdot|1\rangle$$

$$\mathrm{QBit}$$

$$\otimes$$

$$\mathcal{H}$$

$$\|$$

$$\bigcirc_{\mathrm{Bit}} \mathcal{H} \;=\; \bigoplus_{\{0,1\}} \mathcal{H} \;=\; \mathcal{H}\otimes|0\rangle \;\oplus\; \mathcal{H}\otimes|0\rangle$$

De-cohered (measured) q-bits:

$$\mathbb{1}_{\mathrm{Bit}} \;:\; \mathrm{LType}_{\mathrm{Bit}} \underset{\sim}{\xrightarrow{\bigoplus_{\mathrm{Bit}}}} \mathrm{LType}^{\bigcirc_{\mathrm{Bit}}}$$

$$b : \mathrm{Bit} \quad \vdash \quad \mathbb{C}\cdot|b\rangle \;:\; \mathrm{LType}$$

$$\mathbb{1}_{\mathrm{Bit}}$$

$$\otimes \qquad b : \mathrm{Bit} \quad \vdash \quad \mathcal{H}\otimes|b\rangle \;:\; \mathrm{LType}$$

$$\mathcal{H}$$

Quantum gate with q-bit output:

Q-bits are the free linear indeterminacy-effect handlers over $\mathrm{Bit} = \{0,1\}$

Coherent q-bits:

$$\mathrm{QBit} : \mathrm{LType} \xhookrightarrow{\mathbb{1}_{\mathrm{Bit}}\otimes} \mathrm{LType}_{\mathrm{Bit}} \xrightarrow[\sim]{\oplus_{\mathrm{Bit}}} \mathrm{LType}^{\bigcirc_B}$$

$$\overset{\cdot\cdot}{\shortparallel}$$

$$\bigcirc_{\mathrm{Bit}}\mathbb{1} = \bigoplus_{\{0,1\}}\mathbb{C} = \mathbb{C}\cdot|0\rangle \oplus \mathbb{C}\cdot|1\rangle$$

$$\mathrm{QBit}$$

$$\otimes$$

$$\mathcal{H}$$

$$\shortparallel$$

$$\bigcirc_{\mathrm{Bit}}\mathcal{H} = \bigoplus_{\{0,1\}}\mathcal{H} = \mathcal{H}\otimes|0\rangle \oplus \mathcal{H}\otimes|0\rangle$$

De-cohered (measured) q-bits:

$$\mathbb{1}_{\mathrm{Bit}} : \mathrm{LType}_{\mathrm{Bit}} \xrightarrow[\sim]{\oplus_{\mathrm{Bit}}} \mathrm{LType}^{\bigcirc_{\mathrm{Bit}}}$$

$$b : \mathrm{Bit} \quad \vdash \quad \mathbb{C}\cdot|b\rangle : \mathrm{LType}$$

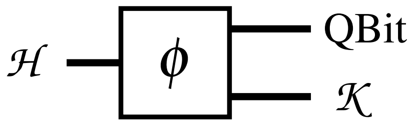
$$\mathbb{1}_{\mathrm{Bit}}$$

$$\otimes \qquad b : \mathrm{Bit} \quad \vdash \quad \mathcal{H}\otimes|b\rangle : \mathrm{LType}$$

$$\mathcal{H}$$

Quantum gate with q-bit output:

A quantum gate which
may handle $\bigcirc_{\mathrm{Bit}}$-effects
is one with a QBit-output:



$$\mathcal{H} \xrightarrow{\phi} \mathrm{QBit}\otimes\mathcal{K} \simeq \bigcirc_{\mathrm{Bit}}\mathcal{K}$$

Q-bits are the free linear indeterminacy-effect handlers over $\mathrm{Bit} = \{0,1\}$

Coherent q-bits:

$$\mathrm{QBit} \;:\; \mathrm{LType} \xhookrightarrow{\mathbb{1}_{\mathrm{Bit}}\otimes} \mathrm{LType}_{\mathrm{Bit}} \xrightarrow[\sim]{\bigoplus_{\mathrm{Bit}}} \mathrm{LType}^{\bigcirc_B}$$

$$\overset{..}{||}$$

$$\bigcirc_{\mathrm{Bit}}\mathbb{1} \;=\; \bigoplus_{\{0,1\}}\mathbb{C} \;=\; \mathbb{C}\cdot|0\rangle \oplus \mathbb{C}\cdot|1\rangle$$

$$\mathrm{QBit}$$
$$\otimes$$
$$\mathcal{H}$$
$$||$$
$$\bigcirc_{\mathrm{Bit}}\mathcal{H} \;=\; \bigoplus_{\{0,1\}}\mathcal{H} \;=\; \mathcal{H}\otimes|0\rangle \;\oplus\; \mathcal{H}\otimes|0\rangle$$

De-cohered (measured) q-bits:

$$\mathbb{1}_{\mathrm{Bit}} \;:\; \mathrm{LType}_{\mathrm{Bit}} \xrightarrow[\sim]{\bigoplus_{\mathrm{Bit}}} \mathrm{LType}^{\bigcirc_{\mathrm{Bit}}}$$

$$b : \mathrm{Bit} \quad\vdash\quad \mathbb{C}\cdot|b\rangle \;:\; \mathrm{LType}$$

$$\mathbb{1}_{\mathrm{Bit}}$$
$$\otimes \qquad b : \mathrm{Bit} \quad\vdash\quad \mathcal{H}\otimes|b\rangle \;:\; \mathrm{LType}$$
$$\mathcal{H}$$

Quantum gate with q-bit output:

A quantum gate which
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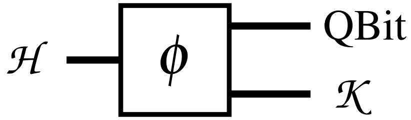
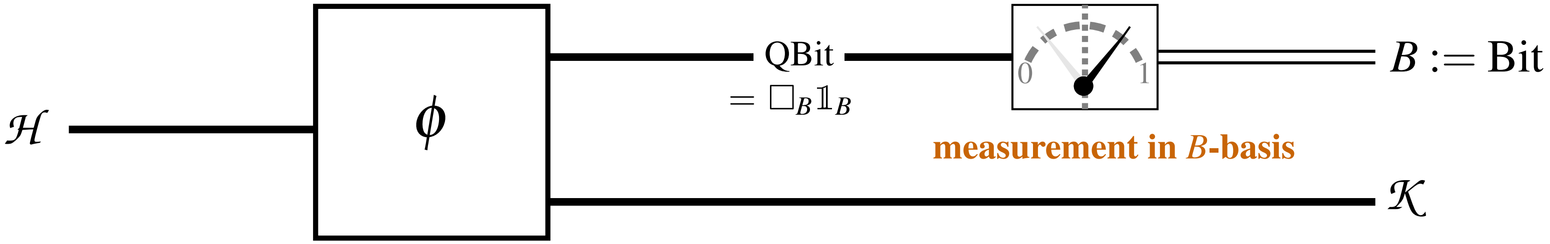


$$\mathcal{H} \xrightarrow{\phi} \mathrm{QBit}\otimes\mathcal{K} \simeq \bigcirc_{\mathrm{Bit}}\mathcal{K}$$

Quantum measurement is Linear indefiniteness-effect handling.

quantum circuit



quantum gate

$$\mathcal{H} \xrightarrow{\phi} \mathrm{QBit} \otimes \mathcal{K} \simeq \bigcirc_B \mathcal{K}$$

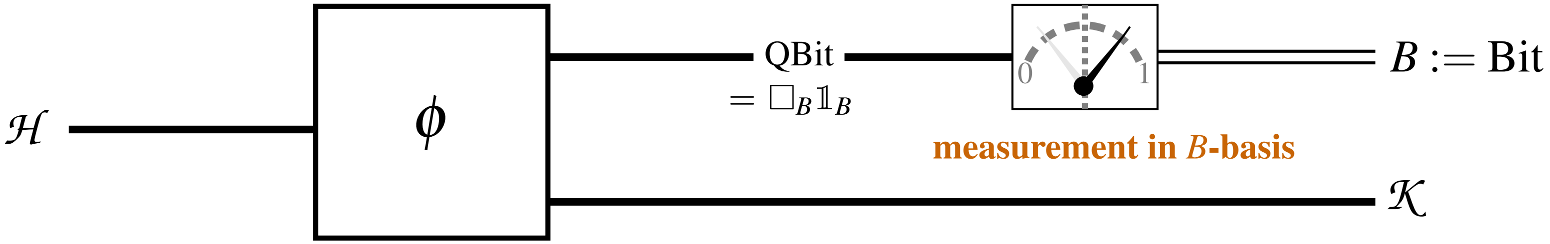
$\bigcirc_B$-effect handling

Quantum measurement is Linear indefiniteness-effect handling.

quantum circuit

$\bigcirc_B$-modal linear types

$\mathrm{LType}_{\bigcirc_B}$



measurement in B-basis

quantum gate

$$\mathcal{H} \xrightarrow{\phi} \mathrm{QBit} \otimes \mathcal{K} \simeq \bigcirc_B \mathcal{K}$$

$\bigcirc_B$-effect handling

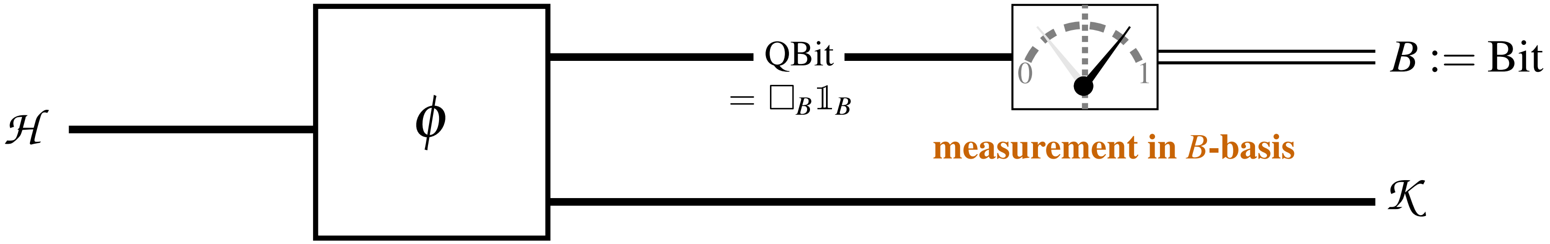
Quantum measurement is Linear indefiniteness-effect handling.

quantum circuit

$$\begin{tikzcd}
\text{quantum circuit} \arrow[d, squiggly, "\text{formalization}"] \\
\mathrm{LType}_{\bigcirc_B} \arrow[d, hook, "K_{(p_B)_*(p_B)^*}"', "\text{comparison functor}"] \\
\mathrm{LType}_B
\end{tikzcd}$$

$\bigcirc_B$-modal linear types

B-dependent linear types



measurement in B-basis

subjection to $\bigcirc_B$-effects

quantum gate

$$\begin{tikzcd}
\bigcirc_B \mathcal{H} \arrow[loop above, "\bigcirc_B"] \arrow[r, "\bigcirc_B\left(\mathcal{H} \xrightarrow{\phi} \mathrm{QBit} \otimes \mathcal{K} \simeq \bigcirc_B \mathcal{K}\right)"] & \bigcirc_B \bigcirc_B \mathcal{K} \arrow[loop above, "\bigcirc_B"] \arrow[r, "\mathrm{hndl}^{\bigcirc_B}_{\bigcirc_B \mathcal{K}}", "\bigcirc_B\text{-effect handling}"'] & \bigcirc_B \mathcal{K} \arrow[loop above, "\bigcirc_B"]
\end{tikzcd}$$

Quantum measurement is Linear indefiniteness-effect handling.

quantum circuit

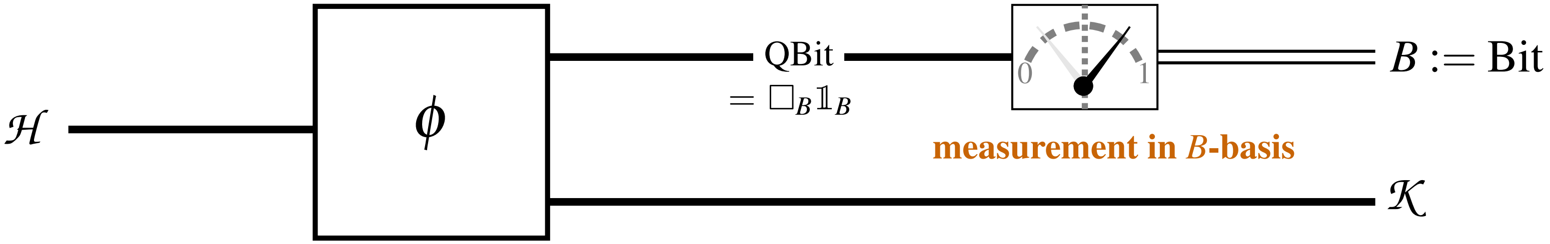
$$\begin{tikzcd} \text{quantum circuit} \arrow[d, squiggly, "\text{formalization}"] \\ \mathrm{LType}_{\bigcirc_B} \arrow[d, "K_{(p_B)^*(p_B)_*}"', "\text{comparison functor}"] \\ \mathrm{LType}_B \end{tikzcd}$$

$\bigcirc_B$-modal linear types

$\mathrm{LType}_{\bigcirc_B}$

LType_B

B-dependent linear types



subjection to $\bigcirc_B$-effects

quantum gate

$$\begin{tikzcd} \bigcirc_B \mathcal{H} \arrow[loop above, "\bigcirc_B"] \arrow[rr, "\bigcirc_B\left(\mathcal{H}\xrightarrow{\phi}\mathrm{QBit}\otimes\mathcal{K}\simeq\bigcirc_B\mathcal{K}\right)"] && \bigcirc_B\bigcirc_B\mathcal{K} \arrow[loop above, "\bigcirc_B"] \arrow[rr, "\mathrm{hndl}^{\bigcirc_B}_{\bigcirc_B\mathcal{K}}", "\bigcirc_B\text{-effect handling}"'] && \bigcirc_B\mathcal{K} \arrow[loop above, "\bigcirc_B"] \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{1}_B \otimes \mathcal{H} \arrow[rr, "\mathbb{1}_B\otimes\phi"] && \square_B \mathbb{1}_B \otimes \mathcal{K} \arrow[rr, "\epsilon^{\square_B}_{\mathbb{1}_B}\otimes\mathcal{K}", "\square_B\text{-counit}"'] && \mathbb{1}_B \otimes \mathcal{K} \end{tikzcd}$$

$$\begin{tikzcd} b : B \vdash \mathcal{H} \arrow[rr, "\phi"] && \mathrm{QBit}\otimes\mathcal{K} \arrow[rr, "\text{quantum state collapse}"] && \mathcal{K} \end{tikzcd}$$

measurement result

$$\begin{array}{c} |0\rangle\otimes|\psi_0\rangle \\ +|1\rangle\otimes|\psi_1\rangle \end{array} \mapsto |\psi_b\rangle$$

Quantum measurement is Linear indefiniteness-effect handling.

quantum circuit

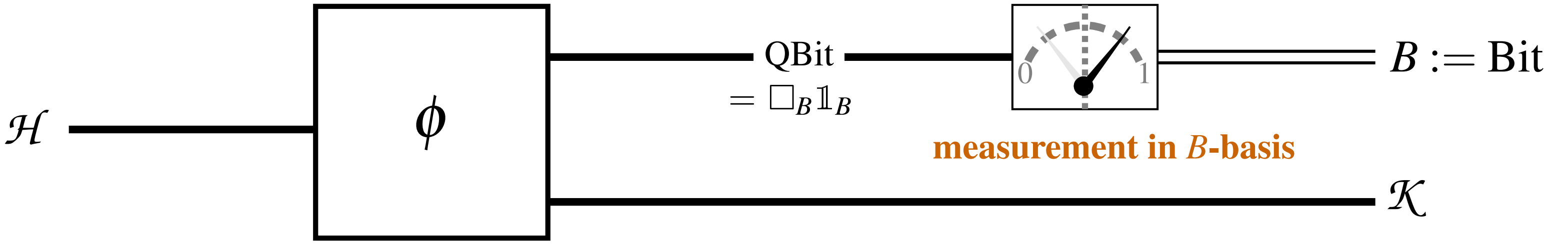
$$\begin{tikzcd}
\text{quantum circuit} \arrow[d, squiggly, "\text{formalization}"] \\
\mathrm{LType}_{\bigcirc_B} \arrow[d, "\text{comparison functor}", "K_{(p_B)^*(p_B)_*}"'] \\
\mathrm{LType}_B
\end{tikzcd}$$

$\bigcirc_B$-modal linear types

$\mathrm{LType}_{\bigcirc_B}$

LType_B

***B*-dependent linear types**



measurement in *B*-basis

$$\begin{tikzcd}
\bigcirc_B \mathcal{H} \arrow[loop above, "\bigcirc_B"] \arrow[r, "\bigcirc_B\left(\mathcal{H} \xrightarrow{\phi} \mathrm{QBit} \otimes \mathcal{K} \simeq \bigcirc_B \mathcal{K}\right)"] & \bigcirc_B \bigcirc_B \mathcal{K} \arrow[loop above, "\bigcirc_B"] \arrow[r, "\mathrm{hndl}^{\bigcirc_B}_{\bigcirc_B \mathcal{K}}", "\bigcirc_B\text{-effect handling}"'] & \bigcirc_B \mathcal{K} \arrow[loop above, "\bigcirc_B"]
\end{tikzcd}$$

subjection to $\bigcirc_B$-effects

quantum gate

$$\begin{tikzcd}
\mathbb{1}_B \otimes \mathcal{H} \arrow[r, "\mathbb{1}_B \otimes \phi"] & \Box_B \mathbb{1}_B \otimes \mathcal{K} \arrow[r, "\epsilon^{\Box_B}_{\mathbb{1}_B} \otimes \mathcal{K}", "\Box_B\text{-counit}"'] & \mathbb{1}_B \otimes \mathcal{K}
\end{tikzcd}$$

$$\begin{tikzcd}
b : B \vdash \quad \mathcal{H} \arrow[r, "\phi"] & \mathrm{QBit} \otimes \mathcal{K} \arrow[r, "\text{quantum state collapse}"] & \mathcal{K}
\end{tikzcd}$$

measurement result

$$\begin{array}{c} |0\rangle \otimes |\psi_0\rangle \\ + |1\rangle \otimes |\psi_1\rangle \end{array} \mapsto |\psi_b\rangle$$

full linearly-typed detail of
quantum measurement logic
is emergent effect in LHoTT

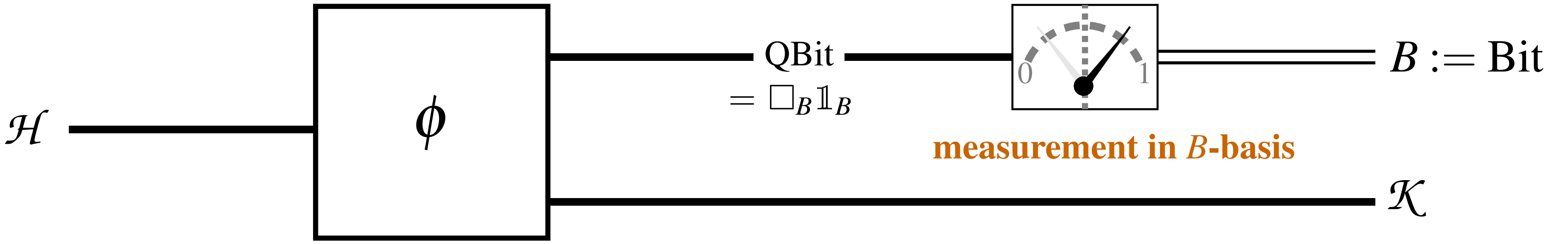
Quantum measurement is Linear indefiniteness-effect handling.

quantum circuit

$$\begin{tikzcd}
\text{quantum circuit} \arrow[d, rightsquigarrow, "\text{formalization}"] \\
\text{LType}_{\bigcirc_B} \arrow[d, "K_{(p_B)^*(p_B)_*}"', "\text{comparison functor}"] \\
\text{LType}_B
\end{tikzcd}$$

$\bigcirc_B$-modal linear types

B-dependent linear types



measurement in B-basis

$$\begin{tikzcd}
\bigcirc_B \mathcal{H} \arrow[loop above, "\bigcirc_B"] \arrow[rr, "{\bigcirc_B\left(\mathcal{H} \xrightarrow{\phi} \text{QBit} \otimes \mathcal{K} \simeq \bigcirc_B \mathcal{K}\right)}"] & & \bigcirc_B \bigcirc_B \mathcal{K} \arrow[loop above, "\bigcirc_B"] \arrow[rr, "\text{hndl}^{\bigcirc_B}_{\bigcirc_B \mathcal{K}}", "\bigcirc_B\text{-effect handling}"'] & & \bigcirc_B \mathcal{K} \arrow[loop above, "\bigcirc_B"]
\end{tikzcd}$$

subjection to $\bigcirc_B$-effects

quantum gate

$$\begin{tikzcd}
\mathbb{1}_B \otimes \mathcal{H} \arrow[rr, "\mathbb{1}_B \otimes \phi"] & & \Box_B \mathbb{1}_B \otimes \mathcal{K} \arrow[rr, "\epsilon^{\Box_B}_{\mathbb{1}_B} \otimes \mathcal{K}", "\Box_B\text{-counit}"'] & & \mathbb{1}_B \otimes \mathcal{K}
\end{tikzcd}$$

$$\begin{tikzcd}
b : B \vdash \mathcal{H} \arrow[rr, "\phi"] & & \text{QBit} \otimes \mathcal{K} \arrow[rr, "\text{quantum state collapse}"] & & \mathcal{K}
\end{tikzcd}$$

measurement result

$$\begin{array}{l} |0\rangle \otimes |\psi_0\rangle \\ + |1\rangle \otimes |\psi_1\rangle \end{array} \mapsto |\psi_b\rangle$$

full linearly-typed detail of
quantum measurement logic
is emergent effect in LHoTT

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$$\bigcirc_B$$

$$\pitchfork$$

$$\mathrm{Monad}\big(\mathrm{LType}\big)$$

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$$\begin{array}{c}
\bigcirc_B \\
\rotatebox[origin=c]{-90}{$=:$} \\
B\text{-Reader} \\
\\
\\
\rotatebox[origin=c]{90}{$\in$} \\
\mathrm{Monad}\big(\mathrm{LType}\big)
\end{array}$$

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$$B : \mathrm{FinType} \;\; \vdash \qquad \begin{tikzcd} \bigcirc_B \arrow[d, phantom, "\overset{!}{=}" rotate=90] \\ B\text{-Reader} \arrow[d, phantom, "\wr\!|"] \\ \mathbb{1}^B\text{-Writer} \arrow[d, phantom, "\Cap"] \\ \mathrm{Monad}\big(\mathrm{LType}\big) \end{tikzcd}$$

$$\mathbb{1}^B\text{-Writer}(D) := \mathbb{1}^B \oplus D$$

$$\mathrm{bind}^{\mathbb{1}^B\mathrm{Writer}}\Big(D_1 \xrightarrow{\mathrm{prog}} \mathbb{1}^B \oplus D_2\Big) :=$$

$$\mathbb{1}^B \oplus D_1 \xrightarrow{\mathbb{1}^B \oplus \mathrm{prog}} \mathbb{1}^B \oplus \mathbb{1}^B \oplus D_2 \xrightarrow{\mu \oplus \mathrm{id}_{D_2}} \mathbb{1}^B \oplus D_2$$

Where $\mathbb{1}^B = \bigoplus\limits_{b:B} \mathbb{C} \cdot P_b \in \mathrm{CMon}\big(\mathrm{LType}\big)$ is algebra of B-projection operators :

unit

$$\mathbb{1} \xrightarrow[\; 1 \mapsto \sum\limits_{b:B} P_b \;]{\eta} \mathbb{1}^B$$

product

$$\mathbb{1}^B \otimes \mathbb{1}^B \xrightarrow[\; P_b \otimes P_{b'} \mapsto \left\{ \begin{array}{l} P_b \text{ if } b=b' \\ 0 \text{ else} \end{array} \right. \;]{\mu} \mathbb{1}^B$$

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$$\begin{tikzcd}
 & & \bigcirc_B \arrow[d, phantom, "\shortparallel"] \\
 & & B\text{-Reader} \arrow[d, phantom, "\wr"] \\
 & & \mathbb{1}^B\text{-Writer} \\
B : \mathrm{FinType} \;\; \vdash & \mathbb{1}^B\text{-DualWriter} \arrow[ur, phantom, "\simeq"] & \\
 & & \mathrm{Monad}(\mathrm{LType}) \arrow[ul, phantom, "\subset"] \\
 & \mathrm{FrobMonad}(\mathrm{LType}) \arrow[ur] &
\end{tikzcd}$$

$$\mathbb{1}^B\text{-Writer}(D) := \mathbb{1}^B \oplus D$$

$$\mathrm{bind}^{\mathbb{1}^B\mathrm{Writer}}\Big(D_1 \xrightarrow{\mathrm{prog}} \mathbb{1}^B \oplus D_2\Big) := \mathbb{1}^B \oplus D_1 \xrightarrow{\mathbb{1}^B \oplus \mathrm{prog}} \mathbb{1}^B \oplus \mathbb{1}^B \oplus D_2 \xrightarrow{\mu \oplus \mathrm{id}_{D_2}} \mathbb{1}^B \oplus D_2$$

Where $\mathbb{1}^B = \bigoplus_{b:B} \mathbb{C} \cdot P_b \in \mathrm{CMon}(\mathrm{LType})$ is Frobenius algebra of B-projection operators :

$$\mathbb{1} \xrightarrow[1 \mapsto \sum\limits_{b:B} P_b]{\substack{\textbf{unit} \\ \eta}} \mathbb{1}^B \xrightarrow[P_b \mapsto P_b \otimes P_b]{\substack{\textbf{co-product} \\ \delta}} \mathbb{1}^B \otimes \mathbb{1}^B \xrightarrow[P_b \otimes P_{b'} \mapsto \left\{\begin{array}{l} P_b \text{ if } b=b' \\ 0 \text{ else} \end{array}\right.]{\substack{\textbf{product} \\ \mu}} \mathbb{1}^B \xrightarrow[P_b \mapsto 1]{\substack{\textbf{co-unit} \\ \varepsilon}} \mathbb{1}$$

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$$B : \mathrm{FinType} \;\; \vdash$$

$$\begin{tikzcd}
 & & \bigcirc_B \arrow[d, phantom, "\shortparallel"] \\
 & & B\text{-Reader} \arrow[d, phantom, "\wr"] \\
\mathbb{1}^B\text{-CoWriter} & & \mathbb{1}^B\text{-Writer} \\
 & \mathbb{1}^B\text{-DualWriter} \arrow[ul, phantom, "\simeq"] \arrow[ur, phantom, "\simeq"] \arrow[d, phantom, "\in"] & \\
\mathrm{CoMonad}\big(\mathrm{LType}\big) & & \mathrm{Monad}\big(\mathrm{LType}\big) \\
 & \mathrm{FrobMonad}\big(\mathrm{LType}\big) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$$\mathbb{1}^B\text{-Writer}(D) :\equiv \mathbb{1}^B \oplus D$$

$$\mathrm{bind}^{\mathbb{1}^B\mathrm{Writer}}\Big(D_1 \xrightarrow{\mathrm{prog}} \mathbb{1}^B \oplus D_2\Big) :\equiv \mathbb{1}^B \oplus D_1 \xrightarrow{\mathbb{1}^B \oplus \mathrm{prog}} \mathbb{1}^B \oplus \mathbb{1}^B \oplus D_2 \xrightarrow{\mu \oplus \mathrm{id}_{D_2}} \mathbb{1}^B \oplus D_2$$

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$$\mathbb{1} \xrightarrow[1 \mapsto \sum_{b:B} P_b]{\overset{\textbf{unit}}{\eta}} \mathbb{1}^B \xrightarrow[P_b \mapsto P_b \otimes P_b]{\overset{\textbf{co-product}}{\delta}} \mathbb{1}^B \otimes \mathbb{1}^B \xrightarrow[P_b \otimes P_{b'} \mapsto \begin{cases} P_b & \text{if } b=b' \\ 0 & \text{else} \end{cases}]{\overset{\textbf{product}}{\mu}} \mathbb{1}^B \xrightarrow[P_b \mapsto 1]{\overset{\textbf{co-unit}}{\varepsilon}} \mathbb{1}$$

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$$\begin{tikzcd}[column sep=small]
 & & & \bigcirc_B \\
 & B\text{-CoReader} \arrow[d, phantom, "\wr|" description] & & B\text{-Reader} \arrow[u, phantom, "\overset{!}{=}" description] \arrow[d, phantom, "\wr|" description] \\
 & \mathbb{1}^B\text{-CoWriter} & & \mathbb{1}^B\text{-Writer} \\
B : \mathrm{FinType} \;\vdash & & \mathbb{1}^B\text{-DualWriter} \arrow[ul, phantom, "\simeq" sloped] \arrow[ur, phantom, "\simeq" sloped] & \\
 & \mathrm{CoMonad}\big(\mathrm{LType}\big) & \arrow[u, phantom, "\in" sloped] & \mathrm{Monad}\big(\mathrm{LType}\big) \\
 & & \mathrm{FrobMonad}\big(\mathrm{LType}\big) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$$\mathbb{1}^B\text{-Writer}(D) :\equiv \mathbb{1}^B \oplus D$$

$$\mathrm{bind}^{\mathbb{1}^B\mathrm{Writer}}\Big(D_1 \xrightarrow{\mathrm{prog}} \mathbb{1}^B \oplus D_2\Big) :\equiv \mathbb{1}^B \oplus D_1 \xrightarrow{\mathbb{1}^B \oplus \mathrm{prog}} \mathbb{1}^B \oplus \mathbb{1}^B \oplus D_2 \xrightarrow{\mu \oplus \mathrm{id}_{D_2}} \mathbb{1}^B \oplus D_2$$

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$$\mathbb{1} \xrightarrow[1 \,\mapsto\, \sum_{b:B} P_b]{\overset{\textbf{unit}}{\eta}} \mathbb{1}^B \xrightarrow[P_b \,\mapsto\, P_b \otimes P_b]{\overset{\textbf{co-product}}{\delta}} \mathbb{1}^B \otimes \mathbb{1}^B \xrightarrow[P_b \otimes P_{b'} \,\mapsto\, \begin{cases} P_b \text{ if } b = b' \\ 0 \text{ else} \end{cases}]{\overset{\textbf{product}}{\mu}} \mathbb{1}^B \xrightarrow[P_b \,\mapsto\, 1]{\overset{\textbf{co-unit}}{\varepsilon}} \mathbb{1}$$

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(see nLab:quantum+reader+monad)

$$\begin{tikzcd}
& \star_B \arrow[d, phantom, "\rotatebox{-90}{$=:$}"] & & \bigcirc_B \arrow[d, phantom, "\rotatebox{-90}{$=:$}"] \\
& B\text{-CoReader} \arrow[d, phantom, "\wr"] & & B\text{-Reader} \arrow[d, phantom, "\wr"] \\
& \mathbb{1}^B\text{-CoWriter} \arrow[dr, phantom, "\simeq"] & & \mathbb{1}^B\text{-Writer} \arrow[dl, phantom, "\simeq"] \\
B : \text{FinType} \;\; \vdash & & \mathbb{1}^B\text{-DualWriter} \arrow[dd, phantom, "\rotatebox{90}{$\in$}"] & \\
& \text{CoMonad}(\text{LType}) & & \text{Monad}(\text{LType}) \\
& & \text{FrobMonad}(\text{LType}) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$\mathbb{1}^B\text{-Writer}(D) :\equiv \mathbb{1}^B \oplus D$

$\text{bind}^{\mathbb{1}^B\text{Writer}}\Big(D_1 \xrightarrow{\text{prog}} \mathbb{1}^B \oplus D_2\Big) :\equiv$

$\mathbb{1}^B \oplus D_1 \xrightarrow{\mathbb{1}^B \oplus \text{prog}} \mathbb{1}^B \oplus \mathbb{1}^B \oplus D_2 \xrightarrow{\mu \oplus \text{id}_{D_2}} \mathbb{1}^B \oplus D_2$

Where $\mathbb{1}^B = \bigoplus_{b:B} \mathbb{C} \cdot P_b \;\in\; \text{CMon}(\text{LType})$ is Frobenius algebra of B-projection operators :

$$\begin{tikzcd}[column sep=huge]
\mathbb{1} \arrow[r, "\overset{\textbf{unit}}{\eta}", "1 \mapsto \sum\limits_{b:B} P_b"'] & \mathbb{1}^B \arrow[r, "\overset{\textbf{co-product}}{\delta}", "P_b \mapsto P_b \otimes P_b"'] & \mathbb{1}^B \otimes \mathbb{1}^B \arrow[r, "\overset{\textbf{product}}{\mu}", "P_b \otimes P_{b'} \mapsto \begin{cases} P_b \text{ if } b=b' \\ 0 \text{ else} \end{cases}"'] & \mathbb{1}^B \arrow[r, "\overset{\textbf{co-unit}}{\varepsilon}", "P_b \mapsto 1"'] & \mathbb{1}
\end{tikzcd}$$

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$$\begin{tikzcd}
 & \star_B \arrow[rr, equal] \arrow[dr, equal] & & \bigcirc_B \arrow[dl, equal] & \\
B : \mathrm{FinType} \;\; \vdash & & \mathbb{1}^B\text{-DualWriter} & & \\
 & & \pitchfork & & \\
 & \mathrm{CoMonad}\big(\mathrm{LType}\big) & & \mathrm{Monad}\big(\mathrm{LType}\big) & \\
 & & \mathrm{FrobMonad}\big(\mathrm{LType}\big) \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

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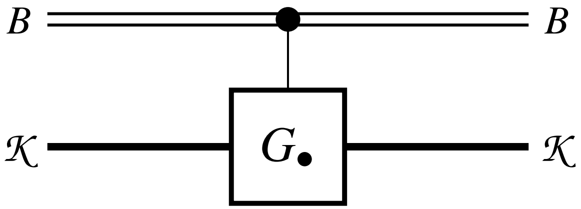
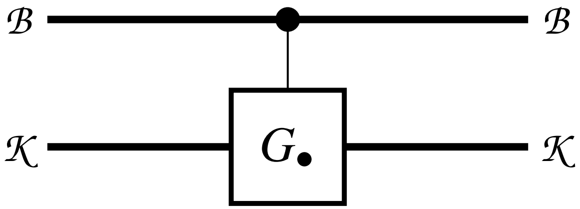
Exmp: Deferred measurement principle – Proven by monadic effect logic.

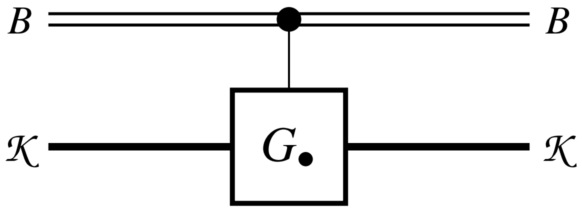
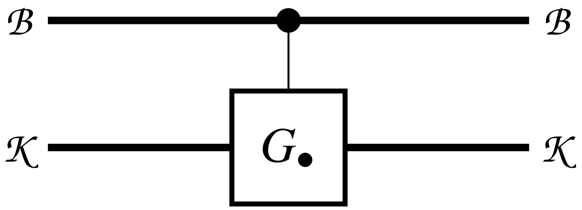
Deferred Measurement Principle

classically controlled gate	**quantumly controlled gate**
 $\mathcal{B}_\bullet \boxtimes \mathcal{K} \xrightarrow{G_\bullet} \mathcal{B}_\bullet \boxtimes \mathcal{K}$ $b : B \;\vdash\; \mathcal{K} \xrightarrow{G_b} \mathcal{K}$	 $\square_B \mathcal{B}_\bullet \boxtimes \mathcal{K} \xrightarrow{\square_B G_\bullet} \square_B \mathcal{B}_\bullet \boxtimes \mathcal{K}$ $b : B \;\vdash\; \bigoplus_{b' B} \mathcal{K} \xrightarrow{\bigoplus_{b' : B} G_{b'}} \bigoplus_{b' B} \mathcal{K}$

Exmp: Deferred measurement principle – Proven by monadic effect logic.

$$\Box_B \mathcal{H}_\bullet \xrightarrow{F} \Box_B \mathcal{H}_\bullet \xrightarrow{\varepsilon^{\Box_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet \xrightarrow{G_\bullet} \mathcal{H}_\bullet \;\mapsto\; \Box_B \mathcal{H}_\bullet \xrightarrow{\mathrm{diag}_B(G_\bullet)\circ F} \Box_B \mathcal{H}_\bullet \;\mapsto\; \Box_B \mathcal{H}_\bullet \xrightarrow{\mathrm{diag}_B(G_\bullet)\circ F} \Box_B \mathcal{H}_\bullet \xrightarrow{\varepsilon^{\Box_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet$$

measurement-controlled quantum gate — quantum-controlled quantum gate... — ...followed by measurement

classically controlled gate	**quantumly controlled gate**
 $\mathcal{B}_\bullet \boxtimes \mathcal{K} \xrightarrow{G_\bullet} \mathcal{B}_\bullet \boxtimes \mathcal{K}$ $b : B \;\vdash\; \mathcal{K} \xrightarrow{G_b} \mathcal{K}$	 $\Box_B \mathcal{B}_\bullet \boxtimes \mathcal{K} \xrightarrow{\Box_B G_\bullet} \Box_B \mathcal{B}_\bullet \boxtimes \mathcal{K}$ $b : B \;\vdash\; \bigoplus_{b' B} \mathcal{K} \xrightarrow{\bigoplus_{b' : B} G_{b'}} \bigoplus_{b' B} \mathcal{K}$

Exmp: Deferred measurement principle – Proven by monadic effect logic.

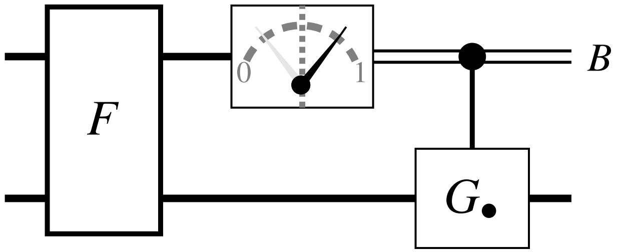
$$\begin{tikzcd} \mathrm{Kl}(\Box_B) \arrow[r, "\sim", "\delta^B \circ \Box_B(-)"'] \arrow[rr, bend left, "\mathrm{id}"] & \mathrm{LType}_{B\Box_B} \arrow[r, "\sim", "\varepsilon^{\Box_B} \circ (-)"'] & \mathrm{Kl}(\Box_B) \end{tikzcd}$$

Kleisli equivalence

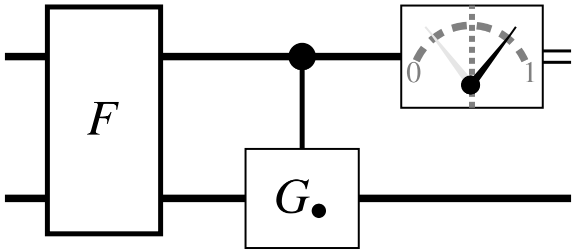
$\Box_B$-Kleisli morphisms | **$\Box_B$-coalgebra homomorphisms** | **$\Box_B$-Kleisli morphisms**

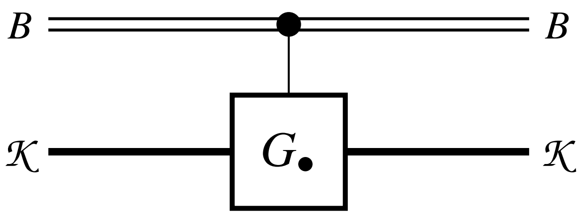
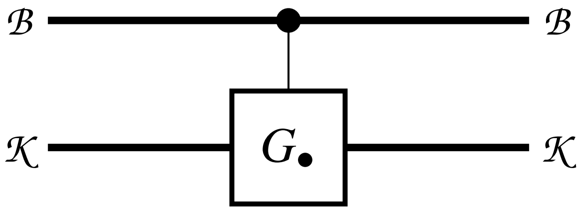
$$\Box_B \mathcal{H}_\bullet \xrightarrow{F} \Box_B \mathcal{H}_\bullet \xrightarrow{\varepsilon^{\Box_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet \xrightarrow{G_\bullet} \mathcal{H}_\bullet \quad \mapsto \quad \Box_B \mathcal{H}_\bullet \xrightarrow{\mathrm{diag}_B(G_\bullet)\circ F} \Box_B \mathcal{H}_\bullet \quad \mapsto \quad \Box_B \mathcal{H}_\bullet \xrightarrow{\mathrm{diag}_B(G_\bullet)\circ F} \Box_B \mathcal{H}_\bullet \xrightarrow{\varepsilon^{\Box_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet$$

measurement-controlled quantum gate | **quantum-controlled quantum gate...** | **...followed by measurement**



Deferred Measurement Principle ⟷



classically controlled gate	**quantumly controlled gate**
	
$\mathcal{B}_\bullet \boxtimes \mathcal{K} \xrightarrow{G_\bullet} \mathcal{B}_\bullet \boxtimes \mathcal{K}$	$\Box_B \mathcal{B}_\bullet \boxtimes \mathcal{K} \xrightarrow{\Box_B G_\bullet} \Box_B \mathcal{B}_\bullet \boxtimes \mathcal{K}$
$b : B \ \vdash \quad \mathcal{K} \xrightarrow{G_b} \mathcal{K}$	$b : B \ \vdash \quad \bigoplus_{b' B} \mathcal{K} \xrightarrow{\bigoplus_{b' : B} G_{b'}} \bigoplus_{b' B} \mathcal{K}$

The Quantum modality.

Also the *exponential modality* traditionally postulated in linear logic is an emergent effect in `LHoTT`,

linear randomization

aka: stabilization/motivization

$$\begin{tikzcd}
\mathrm{Type} \arrow[r, shift left=2, "\Sigma^\infty_+ \,:\, B \;\mapsto\; \star_B \mathbb{1}"] \arrow[r, phantom, "\perp"] & \mathrm{LType} \arrow[l, shift left=2, "\Omega^\infty"] \arrow[loop right, "!"]
\end{tikzcd}$$

exponential modality

classical types

linear types

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Also the *exponential modality* traditionally postulated in linear logic is an emergent effect in `LHoTT`, as is the crucial *Quantum Modality*, not considered before:

linear randomization

aka: stabilization/motivization

quantum modality

$$\begin{tikzcd}
\mathrm{Type} \arrow[loop left, "Q"] \arrow[r, shift left=2, "\Sigma^\infty_+ \,:\, B \,\mapsto\, \star_B \mathbb{1}"] \arrow[r, phantom, "\bot"] & \mathrm{LType} \arrow[l, shift left=2, "\Omega^\infty"] \arrow[loop right, "!"]
\end{tikzcd}$$

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linear types

The Q-monad plays a crucial role in the full formulation of the `QS`-language.

It is the secret actor behind QBit $=$ Q(Bit)...

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exponential modality

classical types

linear types

The Q-monad plays a crucial role in the full formulation of the `QS`-language.
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Quantum Circuits

Quantum effects are compatible with tensor product.

Linear Randomness and Indefiniteness are “very strong” effects, in that:

$$\bigcirc_B\big(D \otimes D'\big) \;\simeq\; \big(\bigcirc_B D\big) \otimes D'\,, \qquad \star_B\big(D \otimes D'\big) \;\simeq\; \big(\star_B D\big) \otimes D'$$

There is a whole system of them:

$$\bigcirc_B \bigcirc_{B'} \;\simeq\; \bigcirc_{B \times B'}\,, \qquad \text{NB: } \bigcirc_B \bigcirc'_B \;\simeq\; \bigcirc_B \mathbb{1} \otimes \bigcirc'_B$$

which under dynamic lifting (monadicity comparison functor)
gives the external tensor product of dependent linear types:

free $\bigcirc_B$-effect handlers in linear data types

comparison functor

B-dependent linear data types

$$\begin{tikzcd} & \bigcirc_{B\times B'} \arrow[r, bend left, "\bigcirc_{B\times B'}"] & \bigcirc_{B\times B'}\mathcal{H} \arrow[d, mapsto] \\ \mathrm{LType}_{\bigcirc_B} \arrow[d, "K_{\oplus_B \mathbb{1}_B \otimes}"] & & \\ \mathrm{LType}_B & \big(\Box_B\, 1_B\big) \boxtimes \big(\Box_{B'}\, 1_{B'}\big) \otimes \mathcal{H} & \end{tikzcd}$$

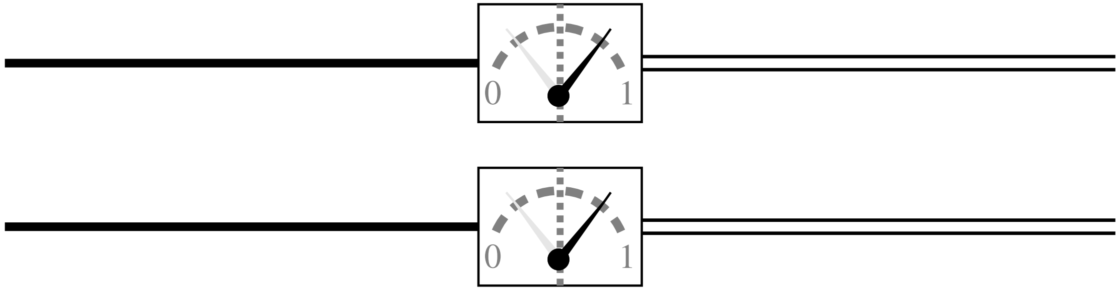
external tensor product

Quantum circuits with classical control & effects

are the *effectful* string diagrams in the linear type system

E.g.

The dependent linear type of a measurement on a pair of qbits:



type of a pair of coherent qbits

pair of measurements

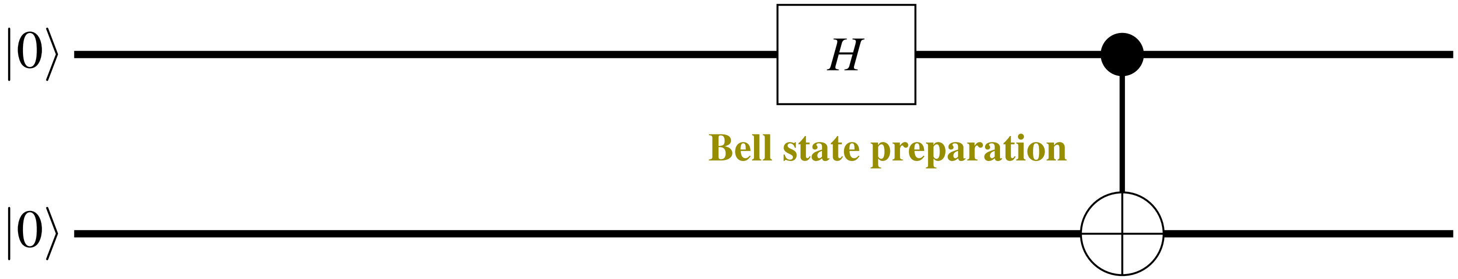
type of collapsed qbits dependent on measured bits b, b'

$$\Box_{\mathrm{Bit}^2}\big(\mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet\big) \xrightarrow{\;\varepsilon_{\mathrm{Bit}^2}(\mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet)\;} \mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet$$

measured bits

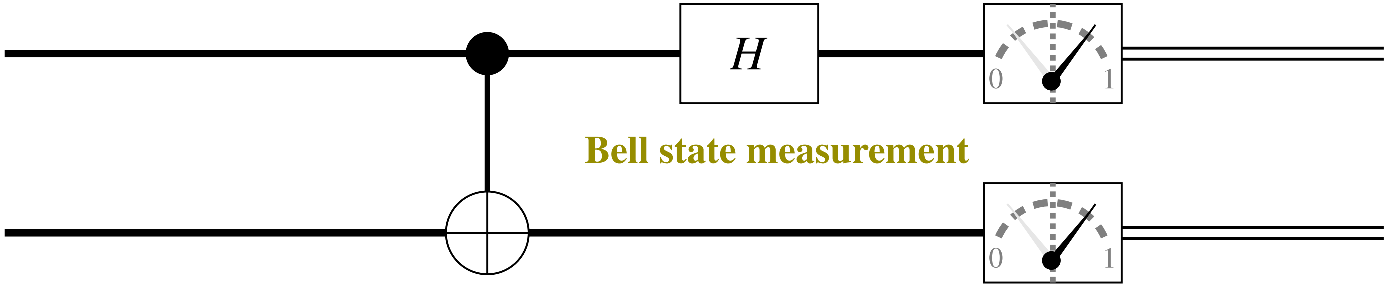
$$(b, b') : \mathrm{Bit}^2 \;\;\vdash\;\; \Box_{\mathrm{Bit}^2}\big(\mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet\big)_{(b,b')} \simeq \mathbb{C}^2 \otimes \mathbb{C}^2 \xrightarrow[\text{collapse of the quantum state}]{\;\Sigma_{d,d'}\, q_{dd'}\; |d\rangle \otimes |d'\rangle \;\mapsto\; q_{bb'}\; |b\rangle \otimes |b'\rangle\;} \mathbb{C}\,.$$

Example: Bell states of q-bits are typed as follows (regarded in $\mathrm{LType}_{\mathrm{Bit}\times\mathrm{Bit}}$):



$$\mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet \longrightarrow \big(\lozenge_{\mathrm{Bit}}\mathrm{QBit}_\bullet\big) \boxtimes \big(\lozenge_{\mathrm{Bit}}\mathrm{QBit}_\bullet\big) \simeq \square_{\mathrm{Bit}^2}\big(\mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet\big) \longrightarrow \square_{\mathrm{Bit}^2}\big(\mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet\big)$$

$$b, b' : \mathrm{Bit} \vdash \mathbb{C} \xrightarrow{\;1 \;\mapsto\; |0\rangle\otimes|0\rangle \;\mapsto\; \frac{1}{\sqrt{2}}\big(|0\rangle+|1\rangle\big)\otimes|0\rangle \;\mapsto\; \frac{1}{\sqrt{2}}\big(|0\rangle\otimes|0\rangle+|1\rangle\otimes|1\rangle\big)\;} \mathbb{C}^2 \otimes \mathbb{C}^2$$

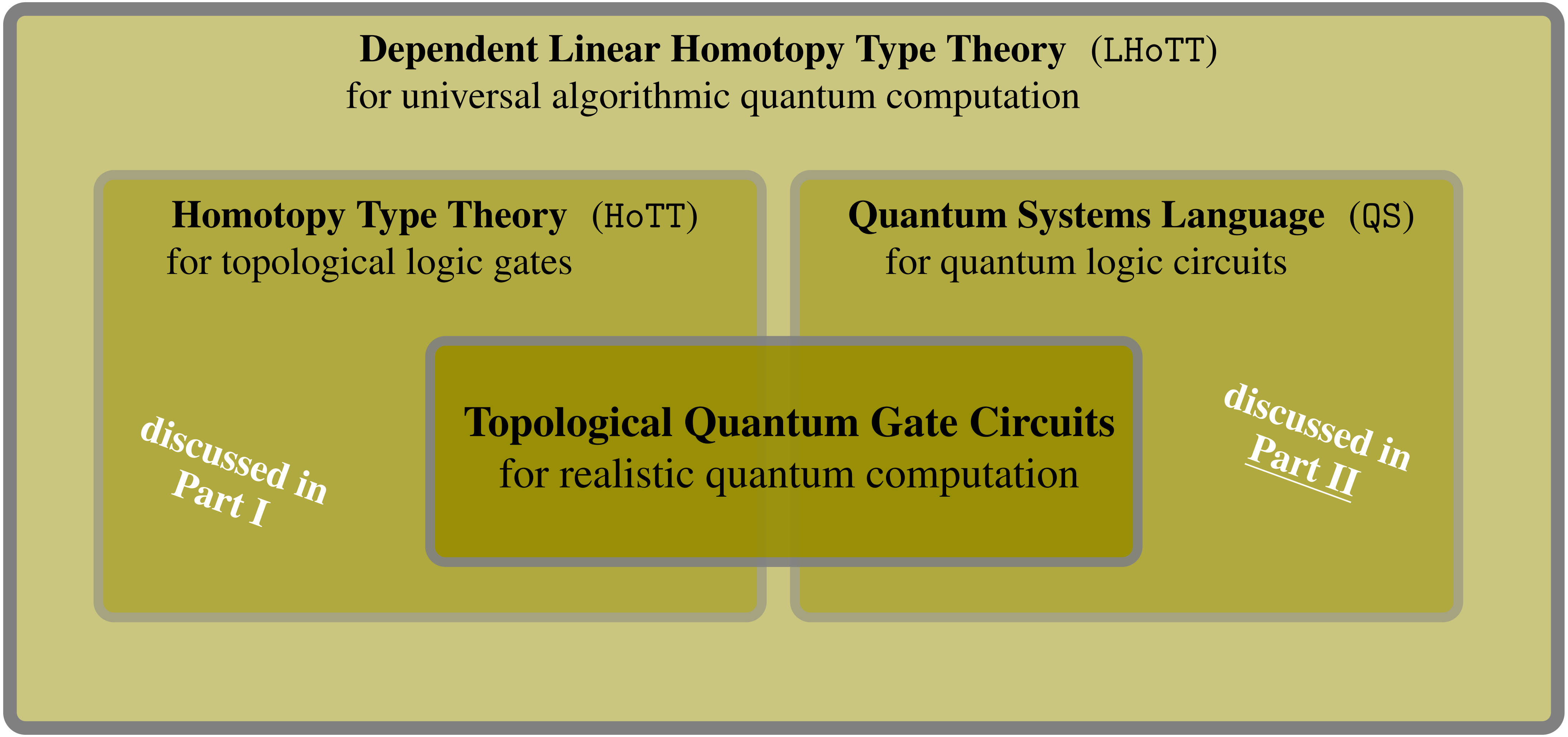


$$\square_{\mathrm{Bit}^2}\big(\mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet\big) \longrightarrow \mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet$$

$$b_1, b_2 : \mathrm{Bit} \vdash \mathbb{C}^2 \otimes \mathbb{C}^2 \xrightarrow{\;\Sigma_{b'_1 b'_2}\, q_{b'_1 b'_2} \cdot |b'_1\rangle\otimes|b'_2\rangle \;\mapsto\; \big(q_{0,b_2} + (-1)^{b_1} \cdot q_{1,(1-b_2)}\big)\cdot|b_1\rangle\otimes|b_2\rangle\;} \mathbb{C}$$

`QS` – Quantum Systems language @ CQTS

⇝ full-blown Quantum Systems language emerges embedded in `LHoTT`



Dependent Linear Homotopy Type Theory (`LHoTT`)
for universal algorithmic quantum computation

Homotopy Type Theory (`HoTT`)
for topological logic gates

Quantum Systems Language (`QS`)
for quantum logic circuits

Topological Quantum Gate Circuits
for realistic quantum computation