Utrecht University
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Master's thesis

Geometric quantization of symplectic and Poisson manifolds

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Abstract

The first part of this thesis provides an introduction to recent development in geometric quantization of symplectic and Poisson manifolds, including modern refinements involving Lie groupoid theory and index theory/K-theory. We start by giving a detailed treatment of traditional geometric quantization of symplectic manifolds, where we cover both the quantization scheme via polarization and via push-forward in K-theory. A different approach is needed for more general Poisson manifolds, which we treat by the geometric quantization of Poisson manifolds via the geometric quantization of their associated symplectic groupoids, due to Weinstein, Xu, Hawkins, et al. In the second part of the thesis we show that this geometric quantization via symplectic groupoids can naturally be understood as an instance of higher geometric quantization in higher geometry, namely as the boundary theory of the 2d Poisson sigma-model. This thesis closes with an outlook on the implications of this change of perspective.

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Contents

CHAPTER 1

Introduction

1. Motivation

The idea of quantization has evolved through time. At the beginning of the twentieth century a revolutionary change in our understanding of microscopic phenomena took place with the idea that at this scale certain physical quantities assume only discrete values. This discreteness was later understood within the Hilbert space formalism of quantum mechanics, where certain self-adjoint operators can have a discrete spectrum. As the understanding of this physics developed, it proved to be more about non-commutativity than discreteness. In 1925 Heisenberg recognized that the quantum mechanical observables should form a non-commutative algebra, which led to the Heisenberg uncertainty relation as a physical basis for quantum mechanics. A general formalism incorporating this idea was given in 1930 by Dirac [**9**]. His formalism was simple and beautiful but did not satisfy the requirement of mathematical rigor. A mathematical rigorous formalism of quantum mechanics, which still stands today, is due to von Neumann. He created a fully-fledged theory of Hilbert spaces and self-adjoint operators for this pupose. His theory is nowadays slightly generalized to allow also other C^*-algebars than $\mathcal{B}(\mathcal{H})$, the algebra of all bounded operators on a Hilbert space $\mathcal{H}$. We will see that these C^*-algebras that do not come from observables acting on a certain Hilbert space $\mathcal{H}$ correspond to certain Poisson manifolds that are not symplectic in the underlying classical theory.

There are two mathematical formulations of *quantization*, on the one hand *deformation quantization* and on the other hand *geometric quantization*. Deformation quantization, corresponds to the Heisenberg picture and focuses on the algebra of observables of the classical system. The commutative algebra of classical observables can be deformed to a non-commutative algebra of quantum observables, with a parameter $\hbar$ in such a way that the commutator is to leading order in $\hbar$ by the Poisson bracket on the observables

$$[f,g]_- = -i\hbar\{f,g\} + \mathcal{O}^2(\hbar). \tag{1.1}$$

Traditionally, deformation quantization refers to *formal* deformation quantization, in the sense that it produces a formal power series expansion in the formal parameter $\hbar$ of the product in the deformed algebra of observables, as is suggested by the notation in Eq. 1.1. In formal deformation quantization it is in general not possible to insert a specific value of $\hbar$ since it is just a formal power series. This is a drawback from the physical perspective where $\hbar$ is Planck's constant. So formal deformation quantization alone does not really describe what it was originally intended to. However deformation quantization is a systematic formal procedure and can be applied to any Poisson manifold, this can be done in a way that, up to natural automorphism, does not depend on any auxiliary choice (such as the choices need in geometric quantization)[**38**]. Deformation quantization can be interpreted as describing the infinitesimal aspects of a more concrete structure, as that produced by geometric quantization. Any result concerning deformation quantization should then have implication for geometric quantization. In principle, geometric quantization can be viewed in terms of formal expansion in deformation quantization[**40**].

Geometric quantization, corresponds to the Schrödinger picture and focuses on the space of states of the classical system. It gives a concrete procedure for constructing a C^*-algebra for each allowed value of $\hbar$. In the limit as $\hbar \to 0$, each of the these algebras can be linearly identified with the ordinary algebra of continuous functions. It is this approximate sense that the elements of the algebra can be thought of as being fixed while the product changes and satisfy Eq. 1.1. (See [**40**]). This programme was developed by Kostant and Souriau, with the aim to find a way of formulating the relationship between classical and quantum mechanics in a concrete way. Given a so called prequantizable symplectic manifold M, regarded as a classical phase space, the first step is constructing a prequantum line bundle over M, the second step is then to choose a polarization, which is splitting of the abstract phase space into "coordinates' and "momenta". After this quantization is carried out via tensoring the prequantum line bundle with the half-form bundle over M and choosing only those section that are polarized via our chosen polarization. This gives us a Hilbert space $\mathcal{H}$ that depends only on the "coordinates". To certain functions of the Poisson algebra $C^\infty(M)$ we can associate an operator acting on this Hilbert space $\mathcal{H}$. There are several drawbacks to this approach, first of all the symplectic manifold should be prequantizable, secondly the polarization may not exist, and if it exist it may not be unique. It is also not clear when quantization carried out with two different polarizations give equivelant results. Thirdly, the choice of a half-form bundle is equivalently to a choice of a metaplectic correction, which has similar existence and uniqueness conditions.

These ad hoc choices make geometric quantization less systematic than one would hope. This state of affairs has become widely known in the mathematical community and is expressed in the famous saying about quantization due to Nelson:

"First quantization is a mystery, but second quantization is a functor" (E. Nelson)

The second quantization is a construction to get from quantum description of a single-particle system to a non-interacting many-particle system, using Fock spaces. This second quantization is functorial and the deep problem suggested by this quote is the possible functoriality of first quantization, that is the quantization of Poisson manifolds. What is missing is a deeper mathematical understanding of what quantization is naturally supposed to be. In [**38**], Gukov and Witten hinted that the geometric quantization of a symplectic manifold can be formulated in terms of the quantization of a 2d quantum field theory, called the A-model, for which the symplectic manifold is a boundary. Gukov and Witten noted that

"The goal is to get closer to a systematic theory of quantization" (Gukov-Witten)

In the end of this thesis we will consider a similar situation, when we realize a Poisson manifold as a boundary of the 2d quantum field theory called, called the non-perturbative Poisson sigma-model.

2. Overview

In this thesis we will give a detailed review of geometric quantization, including the modern cohomological formulation of that. We will start by reviewing geometric quantization of symplectic manifolds due to Kostant and Souriau. This traditional quantization scheme via polarization and metaplectic corrections has the disadvantage that it is not very natural from a mathematical point of view. A more modern and natural approach is the formulation of geometric quantization in terms of Spin^c-structures, due to Bott. This notion of a Spin^c-structure is closely related to the notion of a metaplectic correction. The choice of such a structure together with a connection define an elliptic operator, which is called the Spin^c-Dirac operator. The Spin^c-quantization is then the index of the corresponding Spin^c-Dirac operator. This Spin^c-quantization is independent of the choice of Spin^c-Dirac operator. The choices of Spin^c-structures are in a way much less choices than the choices of metaplectic structures and polarizations, and the space of all possible choices of Spin^c-structures is very well understood. More importantly Spin^c-quantization is completely determined by the cohomology class of the symplectic form and hence the Spin^c-structure plays a purely auxiliary role in it. A drawback is that we must assume that we work on compact manifold, which is needed in order for the index of the Spin^c-Dirac operator to be well-defined. This definition of Spin^c-quantization is equivalent to geometric quantization via push-forward in complex K-theory of the prequantum line bundle to the point.

These constructions work only for symplectic manifolds and not for Poisson manifolds in general. As mentioned before, formal deformation quantization is a systematic formal procedure that can be applied to Poisson manifolds. Weinstein had already proposed that a more proper strict C^*-algebraic deformation quantization should proceed via geometric quantization of the symplectic groupoid that Lie integrates the Poisson Lie algebroid associated to the Poisson manifold. This program was finally brought close to completion by Hawkins.

> "A C^*-algebra $\mathcal{A}$ quantizes a Poisson manifold M if the Poisson algebra of functions on M approximates $\mathcal{A}$" (E. Hawkins)

He showed that an integrable Poisson manifold may be quantized by the polarized convolution algebra of the corresponding symplectic groupoid, twisted by its multiplicative prequantum bundle. In the case that the Poisson manifold is symplectic, we recover the standard C^*-algebra of geometric quantization of symplectic manifolds, namely the C^*-algebra of compact operators.

The geometric quantization of symplectic groupoid can be reinterpreted in terms of higher symplectic geometry and hence is a good test case of higher geometric quantization. The symplectic groupoid may be identified with the moduli stack of the 2d Chern-Simons theory, whose perturbative part is the Poisson sigma-model. In particular, the geometric quantization of Poisson manifolds can be seen as the boundary theory of the Poisson sigma-model. Not a long time ago a similar perspective has already been conceived by Gukov and Witten[**38**]. They pointed out that geometric quantization is more fundamentally understood as being a boundary theory of the quantization of a 2d quantum field theory. This may be a blueprint for the analogous situation in one dimension higher, where the 2d WZW theory via the holographic principle arises as the boundary of the 3d Chern-Simons theory. With this perspective in mind, the higher geometric quantization of a 2d theory yields a 2-vector space of quantum 2-states. Under the identification of 2-vector spaces with categories of modules over an associative algebra, the 2-basis of this space of quantum 2-states identifies, up to Morita equivalence, with an algebra. In this case, the algebra we get from Hawkins' solution to strict C^*-deformation quantization does only have meaning up to Morita equivalence. But for the case that the Poisson manifold is symplectic, the C^*-algebra of compact operators is Morita equivalent to the ground field, which reflects the fact that the symplectic groupoid is Morita equivalent to the point. The problem is that Hawkins' strict C^*-deformation quantization is not Morita faithful, in the sense that it distinguish Morita equivalent groupoids and Morita equivalent algebras.

3. Outlook

This thesis closes with the further implication of this change of perspective, which will be explored in more detail in the companion thesis[**67**] by Joost Nuiten, called "Cohomological quantization of local prequantum boundary field theory". The problem that Hawkins' strict C^*-deformation quantization is not Morita faithful, can be solved by quantizing the whole morphism $i : M \to \mathrm{SymplGpd}$, call it the *atlas*, from the Poisson manifold (M, π) into its symplectic groupoid SymplGpd. We can correct Hawkins' convolution quantization under Morita equivalence, not by assigning just the twisted convolution algebra $C^*_{\nabla_0}(\mathrm{SymplGpd})$ to the multiplicative prequantum line bundle, but assigning to the atlas $i : M \to \mathrm{SymplGpd}$ a Hilbert bimodule of C^*-algebras. The multiplicative prequantum line bundle over SymplGpd which twist the convolution algebra can be seen as a bundle gerbe $\nabla_0 : \mathrm{SymplGpd} \to \mathbf{B}^2U(1)$. In the language of higher smooth stacks the prequantum field theory is given by the diagram

$$\begin{tikzcd}
 & M \arrow[dl] \arrow[dr, "i"] & \\
* \arrow[dr] \arrow[rr, phantom, "\Leftarrow \xi"] & & \mathrm{SymplGpd} \arrow[dl, "\nabla^0"] \\
 & \mathbf{B}^2U(1) &
\end{tikzcd}$$

This correspondence can be applied to the (2,1)-functor $C^*(-)$ to give a cospan of Hilbert bimodules

$$\mathbb{C} \xrightarrow{\xi} C^*_{i^*\nabla^0}(M) \xleftarrow{i^*} C^*_{\nabla^0}(\mathrm{SymplGpd})$$

In the geometric quantization scheme for Poisson manifolds we had the dependence on choices of polarization, this corresponds to choices of K-orientation in K-theory of the atlas. In this case we have a canonical dual map $(i^*)^\vee$ and with composition of the Thom isomorphim Th we get the pull-push quantization in KK-theory, that is

$$\mathbb{C} \xrightarrow{\xi} C^*_{i^*\nabla_0}(M) \xrightarrow{i_!} C^*_{\nabla_0}(\mathrm{SymplGpd})^\vee \in KK(\mathbb{C}, C^*_{\nabla_0}(\mathrm{SymplGpd})^\vee) = K_0(C^*_{\nabla_0}(\mathrm{SymplGpd})^\vee)$$

where $i_! = (i^*)^\vee \circ Th$. In the case of symplectic manifolds we should get precisely $i_!(\mathbb{L}) = \mathrm{index}(D_{\mathbb{L}}) \in KK(\mathbb{C}, \mathbb{C}) \in \mathbb{Z}$, where $\mathbb{L}$ is here ξ. That KK-theory is a natural codomain for quantization of Poisson manifolds has long been amplified by Landsman, but Landsman focused mainly on the issue of symplectic reduction, while here the point is to define the quantization of Poisson manifolds. This pull-push quantization thought of as a 2-geometric quantization of symplectic groupoids is the endpoint of the list of quantization procedures that I will present in this thesis.

4. Outline

This thesis runs as follows. In chapter 2 we give a review of standard geometric quantization of symplectic manifolds. In section 1 we gives an introduction to geometric quantization and in the next three sections we review the two major quantization schemes. We treat in section 2 the prequantization of symplectic manifolds, this gives the construction of a prequantum line bundle, which is the first step in constructing the quantum Hilbert space. In section 3 we treat the traditional quantization scheme via polarization and metaplectic corrections and in section 4 we treat the second quantization scheme via Spin^c-structures. This chapter motivates the approach for geometric quantizing a Poisson manifold.

In chapter 3 we review the geometric quantization of a Poisson manifold due to Hawkins, that is via the geometric quantization of their associated symplectic groupoids. This approach is very similar to the approach of geometric quantizing a symplectic manifold via polarization and metaplectic corrections. In section 1 we treat the integration of a Poisson manifold to a symplectic groupoid. In section 2 we treat the prequantization of this symplectic groupoid which gives the multiplicative prequantum line bundle. These bundles give rise to a twisted convolution algebra which is shown in section 3. In section 4 we define a polarization of the symplectic groupoid, which in turn, as is described in section 5, give rise to a polarized twisted convolution algebra. In section 6 we treat a Bohr-Sommerfeld condition, which we encountered also in chapter 1. In section 7 we discuss how this polarized twisted convolution algebra can be completed to a C^*-algebra. Finally, in section 8 we show that this approach due to Hawkins reproduce the geometric quantization of symplectic manifolds and give rise to the Moyal quantization of Poisson vector spaces.

In chapter 4 we discuss how the results of chapter 2 and 3 can be reinterpreted in terms of higher geometric quantization. To place us in the right setting we treat in section 1 3d Chern-Simons theory as a motivating example. In section 2 we give a brief outline of the basic constructions and facts about higher geometry and show how the prequantization steps of chapter 2 and 3 can be interpreted in higher prequantum geometry. In section 3 we show how this can interpreted in higher symplectic geometry. We will show how the symplectic groupoid give rise to a degree 3-cocycle in the simplicial de Rham cohomology, how the non-degeneracy of this cocycle is encoded in the associated symplectic Lie algebroid and how this gives rise to a Poisson σ-model. We will see also that this Poisson σ-model can be Lie integrated to a 2d Poisson-Chern-Simons theory which has the Poisson manifold as its boundary theory. In section 4 we show that the higher geometric quantization of a 2d field theory yields a 2-vector space of quantum 2-states, which has as 2-basis the algebra that we found in chapter 3. These algebras make only sense up to Morita equivalence which reflects the fact that in higher geometry Lie groupoids make only sense up to Morita equivalence. This shows that the geometric quantization of Poisson manifolds as described in chapter 3 is not Morita faithful.

In chapter 5, we discuss the implications of this change of perspective and hint at a solution to the problem that the results of chapter 3 are not Morita faithful. We conclude with an outlook for further research.

CHAPTER 2

Geometric Quantization of Symplectic Manifolds

In this chapter we give a review of standard geometric quantization of symplectic manifolds. We start by explaining what geometric quantization is all about and what its difficulties are. There are two major quantization schemes which geometrically quantize symplectic manifolds and which deal with these difficulties each in their own manner. The first step in both quantization schemes is the construction of some prequantum line bundle over the symplectic manifold, which one need in order to define the actual quantum Hilbert space. In the first quantization scheme we review the traditional quantization scheme via polarization and metaplectic corrections and in the second quantization scheme we review the more modern approach via Spin^c-structures.

1. What is geometric quantization all about?

Traditionally, "quantization" refer to the process which associates to a classical system its corresponding quantum system. The classical system is described by the commutative algebra of functions on the phase space of the system. Quantization associates to each classical system a Hilbert space $\mathcal{H}$ of quantum states and defines a map $\mathcal{Q}$ from a subset of this commutative algebra to the space of operators on $\mathcal{H}$. Usually, the phase space is described by a symplectic manifold (M, ω) and the commutative algebra is the Poisson algebra $C^\infty(M)$. For a certain sub-algebra $S \subset C^\infty(M)$ the quantization is the assignment

$$\mathcal{Q} : S \to Op(\mathcal{H})$$

mapping smooth functions $f : M \to \mathbb{R}$ to operators $\mathcal{Q}(f) : \mathcal{H} \to \mathcal{H}$. One would like the following *Dirac axioms* to hold:

- **Q1:** $\mathbb{R}$-linearity: $\mathcal{Q}(rf + g) = r\,\mathcal{Q}(f) + \mathcal{Q}(g)\ \forall r \in \mathbb{R}, f, g \in S$
- **Q2:** Normalization: $\mathcal{Q}(1) = \mathbf{1}$, where 1 is the constant function and $\mathbf{1}$ the identity operator on $\mathcal{H}$.
- **Q3:** Hermiticity: $\mathcal{Q}(f)^* = \mathcal{Q}(f)$
- **Q4:** Diracs quantum condition: $[\mathcal{Q}(f), \mathcal{Q}(g)] = -i\hbar\,\mathcal{Q}(\{f, g\})$
- **Q5:** Irreducibility condition: If $\{f_1, ..., f_k\}$ is a complete set of observables, then $\{\mathcal{Q}(f_1), ..., \mathcal{Q}(f_k)\}$ is a complete set of operators.

A set of observables $\{f_1, ..., f_k\}$ is defined to be complete if the only observables which Poisson commute with every element of $\{f_1, ..., f_k\}$ are the constant functions. The set of operators is called complete if it acts irreducibly on $\mathcal{H}$. By Schur's lemma, this means in complete analogy that the only set of operators that commute with all of them are multiples of the identity (See [**2, 9**] for more detail).

There are many known examples of representations of Poisson subalgebras which do not comply with the irreducibility condition, for instance the so called prequantizations obtained through the geometric quantization scheme. It is generally acknowledged that these examples are manifestations of a general obstruction to quantization: it is impossible to quantize the entire Poisson algebra $C^\infty(M)$ while satisfying simultaneously the Dirac quantum condition on the whole algebra and the irreducibility condition. This was first proved for $\mathbb{R}^{2n}$ by the no-go theorem of Gröenewald and van Hove [**36, 44, 45**] and later similar no-go results were proven for other symplectic manifolds [**35**].

Example 1.0.1. Let $Q = \mathbb{R}^n$, $M = T^*Q$ with the standard symplectic form $\omega = \sum dq_j \wedge dp^j$, where $\{q^j\}$ are the coordinates of the configuration space Q and $\{p_j\}$ the corresponding momentum coordinates in the fibers. Note that these coordinate functions q^k and p_l form a complete set of observables. Using the corresponding Poisson bracket

$$\{f, g\} = \sum_{j=1}^{n} \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p^j} - \frac{\partial f}{\partial p^j} \frac{\partial g}{\partial q_j}$$

we require according to (Q4) the canonical commutation relations:

$$[\mathcal{Q}(q^k), \mathcal{Q}(q^l)] = [\mathcal{Q}(p_k), \mathcal{Q}(p_l)] = 0$$
$$[\mathcal{Q}(q^k), \mathcal{Q}(p_l)] = i\hbar\delta_l^k$$

This means that the operators form the *Heisenberg algebra* and, by Schur's lemma, (Q5) implies that we need to find an irreducible representation of this algebra. By the Stone-Von Neumann Theorem, any such representation (that exponentiates to a representation of the Heissenberg group) is unitarily equivalent to $L^2(Q) = L^2(\mathbb{R}^n)$ with

$$\mathcal{Q}(q^k)\psi(x) = x^k\psi(x)$$
$$\mathcal{Q}(p_l)\psi(x) = -i\hbar\frac{\partial \psi}{\partial x^l}(x)$$

This gives us the famous Schrödinger representation.The fact that the wave function $\psi(x)$ of the Hilbert space $L^2(Q)$ depends only on the configuration space is a consequence of representation theory.

The kinetic energy $p^2 = \sum p_j p_j$ can be represented by

$$\mathcal{Q}(p^2) = \hbar^2 \sum \frac{\partial^2}{\partial x^j \partial x^j} = \hbar^2 \Delta$$

By imposing (Q3) and (Q4) one finds

$$\mathcal{Q}(p_k q^l) = \frac{1}{2}\left(\mathcal{Q}(p_k)\,\mathcal{Q}(q^l) + \mathcal{Q}(q^l)\,\mathcal{Q}(p_k)\right)$$

Which is known in quantum physics as *operator ordening* of $\mathcal{Q}(p_k q^l) \sim \mathcal{Q}(p_k)\,\mathcal{Q}(q^l)$. Note that in general we have $\mathcal{Q}(fg) \neq \mathcal{Q}(f)\,\mathcal{Q}(g)$. It turns out that the quadratic observables form a closed Lie algebra under Poisson brackets, the *symplectic Lie algebra* $\mathfrak{sp}(n)$. When we quantize a symplectic vector space, we always obtain a representation of the symplectic Lie algebra. We have now quantized all the linear and quadratic observables in a consistent way. Unfortunately, we are unable to quantize cubic observables, hence even in the simplest case, no full quantization is possible[**3**].

Therefore the quantization procedure requires the selection of a preferred sub-algebra of $C^\infty(M)$. There are no specific rules that tells us which complete set of observables to choose, nor is it ruled out that different choices of complete sets will lead to different quantum theories with different physical results. It is here, where extraneous information and requirements enter the construction of a quantum theory. Certain symmetries or geometric properties of the classical system make one complete set more 'preferred' than another. Geometric quantization is a procedure that addresses the question of existence and classification of quantizations satisfying (Q1) through (Q5) in general.

2. Prequantization of a symplectic manifold

Geometric quantization is a quantization scheme which construct the Hilbert space $\mathcal{H}$ and the map $\mathcal{Q}$ from a symplectic manifold in a geometric way. This Hilbert space $\mathcal{H}$ will be defined as a certain subspace of the space of sections of a complex line bundle $\mathbb{L}$ over the symplectic manifold M. The construction of this line bundle is the first step towards geometric quantization, and is called *prequantization*.

2.1. Line bundles and the first Chern characteristic class. We start by recalling a number of standard definitions and results concerning line bundles and their first Chern characteristic classes.

DEFINITION 2.1.1. Let M be a manifold. By a *complex line bundle* $\mathbb{L}$ over M, we mean a vector bundle $\pi : \mathbb{L} \to M$ with $\mathbb{C}$, the complex numbers, as fibers.

Thus $\mathbb{L}$ is a manifold and the projection map π is smooth, and if for any $p \in M$ one puts $\mathbb{L}_p = \pi^{-1}(p)$, then $\mathbb{L}_p$ is a one dimensional vector space over $\mathbb{C}$. Moreover there exists a open covering $\mathcal{U} = \{U_i | i \in I\}$ of M and nowhere vanishing smooth sections $s_i : U_i \to \mathbb{L}$ on U_i, $i \in I$, such that the map $\eta_i : \mathbb{C} \times U_i \to \pi^{-1}(U_i)$ given by $\eta_i((z, q)) = zs_i(q)$, is a diffeomorphism.

The set of pairs $\{(U_i, s_i)|i \in I\}$ is called a *local system* for $\mathbb{L}$. Given such local system, the corresponding set of transition functions are the elements $c_{ij} \in C^\infty(U_i \cap U_j)$, $i, j \in I$ defined by $c_{ij}s_i = s_j$ on $U_i \cap U_j$. This gives the relations

$$c_{ij} = c_{ji}^{-1} \quad \text{and} \quad c_{ij}c_{jk} = c_{jk} \ \text{ on } \ U_i \cap U_j \cap U_k$$

We denote by $\Gamma(L)$ the space of all smooth sections $s : M \to \mathbb{L}$, which form under pointwise multiplication a $C^\infty(M)$-module.

Two complex line bundles $\mathbb{L}^1$ and $\mathbb{L}^2$ over M are called *equivalent* if there exists a diffeomorphism $\tau : \mathbb{L}^1 \to \mathbb{L}^2$ such that for any $p \in M$, τ induces a linear isomorphism $\mathbb{L}^1_p \to \mathbb{L}^2_p$. This gives an equivalence relation on the set of all line bundle over M and let $\mathcal{L} = \mathcal{L}(M)$ be the set of these equivalence classes.

Let M be a smooth manifold and let $\mathcal{U} = \{U_i | i \in I\}$ be a open contractible cover of M, that is each of the open sets U_i, $U_i \cap U_j$, $U_i \cap U_j \cap U_k$, ... is either empty or can be smoothly contracted to a point.

A *k-simplex* is any $k+1$-tuple $(i_0, i_1, ..., i_k) \in I^{k+1}$ such that $U_{i_0} \cap U_{i_1} \cap ... \cap U_{i_k} \neq \emptyset$. Let A be an abelian group. A *k-cochain* relative to $\mathcal{U}$ is any totally skew map

$$a : (i_0, i_1, ..., i_k) \mapsto a(i_0, i_1, ..., i_k) \in A$$

from the set of k-simplices into A. The set of all k-cochains form a abelian group $C^k(\mathcal{U}, A)$ under addition of functions, and one obtains a group homomorphism $\delta : C^k(\mathcal{U}, A) \to C^{k+1}(\mathcal{U}, A)$ defined by

$$\delta a(i_0, i_1, ..., i_{k+1}) = \sum_{i=0}^{k+1} (-1)^j a(i_0, i_1, ..., \hat{i}_j, ..., i_{k+1})$$

This δ is called the *coboundary operator*. Note that δ^2 is trivial.

A cochain $a \in C^k(\mathcal{U}; A)$ such that $\delta a = 0$ is called a *k-cocycle*. If, in addition, $a = \delta b$ for some $b \in C^{k-1}(\mathcal{U}; A)$ then a is called a *k-coboundary*. The set of k-cocycles is denoted by $Z^k(\mathcal{U}; A)$. The k-coboundaries form a subgroup of $Z^k(\mathcal{U}; A)$ and the quotient

$$H^k(\mathcal{U}; A) = \frac{Z^k(\mathcal{U}; A)}{\delta(C^{k-1}(\mathcal{U}; A))}$$

is called the *k^{th}-cohomology group* and defines the *Čech cohomology* of $\mathcal{U}$, denoted by $\check{H}(\mathcal{U}; A)$. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then there is a homomorphism $H^k(\mathcal{U}; A) \to H^k(\mathcal{V}; A)$. Taking the inductive limit over all coverings under refinement, gives us the cohomology group $H^k(M; A)$. For any particular covering $\mathcal{U}$ one has the natural homomorphism $H^k(\mathcal{U}; A) \to H^k(M; A)$, which is happens to be an isomorphism if $\mathcal{U}$ is contractible [**33**]. Thus we can identify $H^k(\mathcal{U}; A)$ with $H^k(M; A)$, when $\mathcal{U}$ is contractible.

Now suppose that $\pi : \mathbb{L} \to M$ is a complex line bundle and that $\{U_i, s_i\}$ is a local system for $\mathbb{L}$, with every U_i contractible. The transition functions $c_{ij} : U_i \cap U_j \to \mathbb{C}^\times$ form a Čech 1-cocycle, and thus $\mathbb{L}$ determines an equivalence class $[c] \in H^1(\mathcal{U}; \mathbb{C}^\times)$. Furthermore if $\pi_1 : \mathbb{L}_1 \to M$ and $\pi_2 : \mathbb{L}_2 \to M$ are two equivalent complex line bundles, with corresponding transition functions c_{ij} and b_{ij} respectively. These two cocycles will be related by functions $g_i : U_i \to \mathbb{C}^\times$, such that $c_{ij} = g_i b_{ij} g_j^{-1}$. This means that c_{ij} and b_{ij} differ by the 0-coboundary, which shows that equivalent complex line bundles define the same equivalence class in $H^1(M; \mathbb{C}^\times)$. Conversely, it can easily be shown that every one-cocycle in $[c]$, will lead to equivalent complex line bundles[**48**]. The fact that the group $H^1(M; \mathbb{C}^\times)$ is isomorphic to $H^2(M; 2\pi\mathbb{Z})$, follows from the long exact sequences in cohomology, coming from the short exact sequence of group homomorphisms

$$0 \to 2\pi\mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times \to 0$$

where the map $2\pi\mathbb{Z} \to \mathbb{C}$ is inclusion and the map $\mathbb{C} \to \mathbb{C}^*$ is given by $z \mapsto e^{iz}$. With induced long exact sequence

$$... \to H^1(M; \mathbb{C}) \to H^1(M; \mathbb{C}^\times) \xrightarrow{\epsilon} H^2(M; 2\pi\mathbb{Z}) \to H^2(M; \mathbb{C}) \to ... \to$$

implying $H^1(M; \mathbb{C}^\times) \cong H^2(M; 2\pi\mathbb{Z})$, since $H^1(M; \mathbb{C}) = H^2(M; \mathbb{C}) = 0$ ($\mathbb{C}$ is contractible). If $\mathbb{L}$ is complex line bundle over M and $\beta(\mathbb{L})$ is the corresponding element of $H^1(M; \mathbb{C}^\times)$, then the equivalence class $c_1(\mathbb{L}) = \frac{1}{2\pi}\epsilon\beta(\mathbb{L}) \in H^2(M; \mathbb{Z})$ is called the *first Chern characteristic class* of $\mathbb{L}$[**48**].

Consider a complex line bundle $\pi : \mathbb{L} \to M$. For any $k \in \mathbb{N}$, let

$$\Omega^k(M, \mathbb{L}) = \Gamma(M, \mathbb{L} \otimes \wedge^k(T^*M))$$

be the space of k-forms with values in $\mathbb{L}$.

Definition 2.1.2. A *Hermitian metric* on $\mathbb{L}$ is an Hermitian inner product $\langle \cdot, \cdot \rangle$ on each fiber with the property that, for any $s, t \in \Gamma(\mathbb{L})$, the function $\langle s, t \rangle : M \to \mathbb{C}$ defined by $m \mapsto \langle s(m), t(m) \rangle$ is smooth.

Definition 2.1.3. A *connection* of $\mathbb{L}$ is a linear map $\nabla : \Omega^0(M, \mathbb{L}) \to \Omega^1(M, \mathbb{L})$ satisfying

$$\nabla(fs) = df \otimes s + f\nabla s$$

for any section $s \in \Gamma(M, \mathbb{L})$ and function $f \in C^\infty(M)$.

These two structures are required to be compatible if we have

$$d\langle s, t \rangle = \langle \nabla s, t \rangle + \langle s, \nabla t \rangle, \quad \forall s, t \in \Gamma(M, \mathbb{L})$$

Definition 2.1.4. A complex line bundle with connection ∇ and a compatible Hermitian metric is called a *Hermitian line bundle with connection* and is denoted by $(\mathbb{L}, \langle \cdot, \cdot, \rangle, \nabla)$.

Example 2.1.5. Let $\mathbb{L}$ be the trivial bundle $M \times \mathbb{C}$, so that $\Omega^k(M, \mathbb{L}) = \Omega^k(M)$. Then for any $\beta \in \Omega^1(M)$, we can define a connection ∇ by

$$\nabla f = df + f\beta, \quad f \in C^\infty(M)$$

Conversely, all the connections of the trivial bundle have this form, since we can set $\beta = \nabla 1$ and compute $\nabla(f\ 1) = df + f\nabla 1$.

Given a connection ∇ of a complex line bundle $\mathbb{L}$ on M and a vector field X of M, then the covariant derivative with respect to X is defined by

$$\nabla_X : \Gamma(M, \mathbb{L}) \to \Gamma(M, \mathbb{L}) \quad \nabla_X s = \nabla s(X)$$

If (U, s) is a pair of a local system and $\beta = \frac{\nabla s}{s}$, then $\nabla_X(fs) = Xf \otimes s + \beta(X) \otimes s$. Furthermore there exist an $\omega \in \Omega^2(M)$ such that for any vector fields X, Y we have

$$\omega(X, Y) = \nabla_x \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

which is called the *curvature* of ∇.

Let M be a smooth manifold with an open contractible cover $\mathcal{U}$, then it can be shown that the Čech cohomology $\check{H}(M;\mathbb{R})$ is precisely isomorphic to the de Rham cohomology $H_{dR}(M;\mathbb{R})$[**83**]. Consider in addition an Hermitian line bundle with connection $(\mathbb{L}, \langle\cdot,\cdot,\rangle, \nabla)$ on M, then the curvature of this connection ∇ defines a 2-form which de Rham class is an element of $H^2_{dR}(M;\mathbb{R})$. This de Rham class of the curvature of the Hermitian line bundle $\mathbb{L}$ is closely related to the first Chern charactersitic class of $\mathbb{L}$. The first Chern class $c_1(\mathbb{L})$ maps to $\frac{1}{2\pi}[\omega]$ under the natural homomorphism

$$i : H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{R})$$

Consequently $\frac{1}{2\pi}[\omega]$ has to be integral[**83**]. For the converse we have the following result due to Weil (see [**48**]).

THEOREM 2.1.6. *(Weil's theorem) Let σ be a closed 2-form on smooth manifold M such that its de Rham cohomology class lies in the image of $i : H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{R})$. Then there exists an Hermitian line bundle with connection $(\mathbb{L}, \langle\cdot,\cdot,\rangle, \nabla)$ on M such that its curvature equals $2\pi\sigma$.*

The $\ker i$ is exactly the torsion subgroup of $H^2(M;\mathbb{Z})$, that is the Hermitian line bundle with connection $(\mathbb{L}, \langle\cdot,\cdot,\rangle, \nabla)$ is determined by $[\sigma]$ uniquely up to torsion element of $H^2(M;\mathbb{Z})$.

2.2. Prequantum line bundle. This theorem of Weil gives us a prequantization condition for a symplectic manifold (M,ω). A symplectic manifold (M,ω) is called *prequantizable* if $\frac{1}{2\pi}[\omega]$ is integral. Which gives the following definition:

DEFINITION 2.2.1. A *prequantization* of (M,ω) is a Hermitian line bundle with connection $(\mathbb{L}, \langle,\rangle, \nabla)$ such that the curvature of the connection ∇ is ω. We call $\mathbb{L}$ a *prequantum line bundle* on (M,ω).

This formulation is due to Kostant, an equivalent formulation is made by Souriau by using principal $U(1)$-bundles associated to $\mathbb{L}$. Recall that a principal $U(1)$-bundle consists of a fiber bundle $\pi : P \to M$ with fiber $U(1)$ and for any $x \in M$, an action of $U(1)$ on P_x which is free and transitive. Furthermore P need to be locally trivializable, that is for any $x \in M$ there exists a neighborhood U and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times U(1)$ such that $\phi(p) = (\pi(p), \tau(p))$, where $\tau : \pi^{-1}(U) \to U(1)$ satisfy $\tau(\theta \cdot p) = \theta + \tau(p)$ for $\theta \in U(1)$. For any $\theta \in U(1)$ we denote the action by $L_\theta : P \to P$, then the vector field ∂_θ of P is given by

$$\partial_\theta|_p = \frac{d}{dt}|_{t=0} L_t(p)$$

for $p \in P$. A connection on a principal $U(1)$-bundle $\pi : P \to M$ is a 1-form $\Theta \in \Omega^1(P,\mathbb{R})$ which is $U(1)$-invariant, i.e. $L_\theta^*\Theta = \Theta$ for any $\theta \in U(1)$, and satisfies $\iota_{\partial_\theta}\Theta = 1$. These two conditions on Θ imply that there exist a unique 2-form $\omega \in \Omega^2(M;\mathbb{R})$ such that $\pi^*\omega + d\Theta = 0$, which is called the *curvature* of Θ. Given an Hermitian line bundle $(\mathbb{L}, \langle,\rangle)$, the associated principal $U(1)$-bundle is given by $P = \{v \in \mathbb{L} | \langle v, v\rangle = 1\}$. Conversely, to a principal $U(1)$-bundle P we can associate the Hermitian line bundle $\mathbb{L} = P \times_{U(1)} \mathbb{C}$. The prequantization $(\mathbb{L}, \langle,\rangle, \nabla)$ uniquely determines the prequantization (P,Θ) and vice verse, via the equation $\frac{\nabla s}{s} = \sqrt{-1}s^*\Theta$ for every $s \in \Gamma(P) \subset \Gamma(\mathbb{L})$. Hence a prequantization of (M,ω) is a principal $U(1)$-bundle $\pi : P \to M$ equipped with a connection θ on P such that the curvature of this connection is ω.

EXAMPLE 2.2.2. Let ω be an exact 2-form on M and let β be a 1-form on M such that $d\beta = -\omega$. Then (M,ω) can always be prequantized by the trivial $U(1)$-bundle $P = M \times U(1)$ and connection $\Theta = d\theta + \pi^*\beta$, where $\theta : P \to U(1)$ is the the angle coordinate on $U(1)$.

A map of Hermitian line bunles $\tau : \mathbb{L}' \to \mathbb{L}$, is a diffeomorphism which, (i) commutes with the projection $\pi \circ \tau = \pi'$, (ii) it restricts to a linear isomorphism $\tau_m : \mathbb{L}'_m \to \mathbb{L}_m$ for each $m \in M$ and (iii) the functions $H' : \mathbb{L}' \to \mathbb{R}$, $H : \mathbb{L} \to \mathbb{R}$ defined by $H'(s) = \langle s, s\rangle'$, $H(s) = \langle s, s\rangle$ satisfy $H \circ \tau = H'$. If $\mathbb{L}' = \mathbb{L}$, then we call τ a *gauge transformation/equivalence*. If in addition ∇ is a connection on $\mathbb{L}$, then we require that $\tau^*(\nabla) = \nabla$ in order for τ to be an gauge transformation of $(\mathbb{L}, \langle,\rangle, \nabla)$. Analogously, a map of principal $U(1)$-bundles $\phi : P' \to P$, is a smooth map of manifolds which commutes with the $U(1)$-action. If $P' = P$, then we call ϕ a *gauge transformation/equivalence*. If in addition Θ is a connection on P, then we require that $\phi^*(\Theta) = \Theta$ in order for ϕ to be an gauge transformation of (P, Θ). Two prequantization $(\mathbb{L}, \langle, \rangle, \nabla)$ are gauge equivalent if and only if the corresponding prequantizations (P, Θ) are gauge equivalent. Since a gauge equivalence is an isomorphism of bundles that respects additional structures, we will be interested in prequantum line bundles up to gauge equivalence. Due to Kostant we have the following result:

PROPOSITION 2.2.3. [**48**] *Let (M, ω) be a symplectic manifold, such that the cohomology class $\frac{1}{2\pi}[\omega]$ is integral. Let $(\mathbb{L}, \langle, \rangle, \nabla)$ be a prequantum line bundle. Then for every Hermitian structure $\langle, \rangle'$ on $\mathbb{L}$ there is a prequantization $(\mathbb{L}, \langle, \rangle', \nabla')$ for (M, ω). If M is simply connected, the prequantization structure on $\mathbb{L}$ is unique up to gauge equivalence. More generally the prequantization structures on $\mathbb{L}$ are classified by $H^1(M; \mathbb{R})/H^1(M; \mathbb{Z})$, up to gauge equivalence.*

A consequence is that if M is simply connected ,then all connections Θ with $d\Theta = -\pi^*\omega$ are equivalent by gauge transformations.

REMARK 2.2.4. If the classical system has a symmetry group G, then it is natural to require that the quantum system obtained also posses this symmetry. A Hamiltonian G-action on a symplectic manifold (M, ω) with associated moment map Φ, gives rise to a linear group action on the associated Hilbert space $\mathcal{H}$. The irreducibility condition can be reformulated by saying that the quantization of a transitive Hamiltonian G-action is an irreducible G-representation. This leads to the famous *Orbit method*, developed by Kirillov and Kostant [**47, 48**], which gives a quantization procedure for constructing an irreducible unitary representation of the group G starting from a G-orbit in the coadjoint representation. For more details, see [**32**].

3. Geometric quantization by polarization

The space of smooth sections of $\mathbb{L}$ is too large for geometric quantization. As we saw in example 1.0.1, the Hilbert space $L^2(Q)$ depends only on the configuration space Q, and not on the state space T^*Q. This Hilbert space space can be thought of as functions on T^*Q which are independent of the fiber variables. To cut down the variable dependency of the functions by half, is known as *polarization.* A choice of polarization is what in physics is called a choice of "canonical coordinates" and "canonical momenta", where the canonical momenta are the coordinates along a leaf and the canonical coordinates are the coordinates on the leaf space. The advantage of restricting to a smaller class of sections, namely the "polarized sections", is that the resulting prequantization may satisfy the "irreducibility axiom" of section 1. The space of smooth sections of $\mathbb{L}$ is just too big for this axiom to hold. In the language of physics these polarized sections are called "wave functions" and the polarization condition says precisely that these wave functions are functions on the phase space which depend only on canonical coordinates and not on canonical momenta.

3.1. Polarization.

Recall that a *complex distribution* is a complex subbundle $\mathcal{F} \subset T_{\mathbb{C}}M = TM \otimes \mathbb{C}$. If (M, ω) is symplectic, $\mathcal{F}$ is called *Lagrangian* if for every $p \in M$ the subspace $\mathcal{F}_p \subset T_pM \otimes \mathbb{C}$ is Lagrangian, i.e. $\dim_{\mathbb{C}} \mathcal{F}_p = \frac{1}{2} \dim M$ and the complex-valued two-form induced from ω vanishes on $\mathcal{F}_p$. A distribution $\mathcal{F}$ is said to be *involutive* if for any two vector fields u and v of $\mathcal{F}$, the Lie bracket $[u, v]$ is also a vector field of $\mathcal{F}$. A distribution $\mathcal{F}$ is said to be *integrable* if for all $p \in M$ there is a submanifold $S \subset M$ such that $p \in S$ and for all $q \in S$ we have $T_qS = \mathcal{F}_q$. By Frobenius theorem a distribution $\mathcal{F}$ is involutive if and only if it is integrable.

DEFINITION 3.1.1. A *polarization* of a symplectic manifold (M, ω) is a complex involutive Lagrangian distribution $\mathcal{F} \subset T_{\mathbb{C}}M$.

Definition 3.1.2. A *real polarization* is an involutive Lagrangian real distribution $\mathcal{F} \subset TM$. A polarization $\mathcal{F}$ is called *totally complex* (or *Kähler*) if $\mathcal{F} \cap \bar{\mathcal{F}} = \{0\}$.

If a polarization satisfies $\bar{\mathcal{F}} = \mathcal{F}$, then it is the complexification of a real polarization and hence we regard a real polarization as a special case of complex polarization.

Example 3.1.3. A real polarization $\mathcal{F}$ is an integrable subbundle of TM. Frobenius theorem states that it defines a foliation of M, whose leaves are Lagrangian submanifolds of M. Conversely, every Lagrangian foliation defines a real polarization.

A real polarization may not exist in general. For example, on a two-dimensional surface a real polarization corresponds to a nowhere vanishing vector field, thus if the two-sphere S^2 has a real polarization then it has a nowhere vanishing vector field, which contradicts the well-known Hairy ball theorem. This is why we consider involutive Lagrangian distributions of the complexified tangent bundle.

Example 3.1.4. For (M, J) an almost complex manifold. The almost complex structure $J : TM \to TM$ extends to a $\mathbb{C}$-linear bundle isomorphism on $T_{\mathbb{C}}M$ and has eigenvalues $\pm i$. The $\pm i$ eigenspaces of J are denoted by $T_{1,0}M$ and $T_{0,1}M$ and are spanned by vectors of the form $X \mp iJX$. These bundles are complex conjugate of each other and satisfy $T_{1,0}M \cap T_{0,1}M = \{0\}$, hence $T_{\mathbb{C}}M = T_{1,0}M \oplus T_{0,1}M$. The distribution $T_{0,1}M$ is Lagrangian if and only if the symplectic form ω is of type (1,1), that is $\omega \in \Omega^{1,1}(M)$. The fact that ω is a (1,1)-form implies that $\langle u, v\rangle := \omega(u, Jv)$ is a symmetric metric. If we require in addition that this symmetric metric is positive definite, then we say that the almost complex structure J is *compatible* with ω. By the Newlander-Nirenberg theorem, the distribution $T_{0,1}M$ is integrable if and only if J comes from a complex structure on M. In this case we call (M, J, ω) a *Kähler manifold*, which is a symplectic manifold with an integrable almost complex structure which is compatible with the symplectic form. For Kähler manifolds we have the natural polarization $T_{0,1}M$, which is called the *Kähler polarization*. Conversely, every Kähler polarization $\mathcal{F}$ give rise to an almost complex structure J on M, with the property that at a point $x \in M$, the vector $X + iJX$, for all $X \in T_xM$, span the space $\mathcal{F}_x$. By Newlander-Nirenberg theorem this J comes from a complex structure on M, and if in addition this J is compatible with ω, then it defines a Kähler manifold (M, J, ω) (See [**90**]).

In general for a polarization $\mathcal{F}$, the real part $\mathcal{F} \cap TM$ is not necessarily of constant rank. By requiring that it is of constant rank ensures us that it is an involutive distibution, i.e. a foliation. But the subbundle $\mathcal{F} + \bar{\mathcal{F}} \subset T_{\mathbb{C}}M$ is still not necessary involutive. Imposing the following conditions on $\mathcal{F}$ ensures us that the polarization is well behaved.

Definition 3.1.5. For a polarization $\mathcal{F} \subset T_{\mathbb{C}}M$, the distributions $D, E \subset TM$ are defined by $D_{\mathbb{C}} := \mathcal{F} \cap \bar{\mathcal{F}}$ and $E_{\mathbb{C}} := \mathcal{F} + \bar{\mathcal{F}}$. The polarization $\mathcal{F}$ is *strongly admissible* if there exist manifolds and surjective submersions

$$M \overset{p}{\twoheadrightarrow} M/D \overset{q}{\twoheadrightarrow} M/E$$

such that D and E are the kernel foliations $D = \ker Tp$ and $E = \ker T(q \circ p)$.

Example 3.1.6. A polarization satisfying $\mathcal{F} = \bar{\mathcal{F}}$ is the complexification of a real polarization. For such a polarization $D = E$, so that $\mathcal{F} = D_{\mathbb{C}}$ is strongly admissible if the space of leaves of the underlying real polarization D are smooth manifolds. For a Kähler polarization we have $\mathcal{F} \cap \bar{\mathcal{F}} = \{0\}$ so $D = \{0\}$ and hence $E = TM$, since $D_p^{\perp} = E_p$ for all $p \in M$, so that any Kähler polarization is strongly admissible.

In geometric quantization, one starts with a symplectic manifold (M, ω), together with a prequantization $(\mathbb{L}, \langle,\rangle, \nabla)$ and a polarization $\mathcal{F}$. Further we assume that $\mathcal{F}$ is strongly admissible. A section $s \in \Gamma(\mathbb{L})$ is *covariantly constant along* $\mathcal{F}$ if $\nabla_X s = 0$ for all $X \in \Gamma(\mathcal{F})$. We denote the space of sections of $\mathbb{L}$ covariantly constant along a polarization $\mathcal{F}$ by $\Gamma_{\mathcal{F}}(\mathbb{L})$, which is also called the space of *polarized sections.*

EXAMPLE 3.1.7. (Vertical polarization) Let Q be a manifold, and $M = T^*Q$ be its cotangent bundle, with projection map $\pi_Q : T^*Q \to Q$. The tautological one-form τ can locally be defined as $\tau = \sum p_i dq_i$, with the canonical symplectic form $\omega = -d\tau = \sum dq_i \wedge dp_i$. Let $\mathcal{F} \subset T_{\mathbb{C}}M$ be the subbundle

$$\mathcal{F} := \ker T_{\mathbb{C}}\pi_Q \cong T_{\mathbb{C}}Q \hookrightarrow T_{\mathbb{C}}M$$

Then $\mathcal{F}$ is a polarization of (M, ω), called the *vertical polarization.* In fact it is a real polarization, which is strongly admissible. Now M can be quantized by the trivial Hermitian line bundle $\mathbb{L} = M \times \mathbb{C}$, with Hermitian metric $(\alpha, \beta) = \alpha^*\beta$ for $\alpha, \beta \in \Gamma(\mathbb{L})$ and connection $\nabla = d + i\tau$,where $\nabla_X = \iota_X \nabla$. The curvature of this connection is $\omega = -d\tau$. The sections of $\mathbb{L}$ are functions $s : T^*Q \to \mathbb{C}$ and for each vertical vector field $X \in \Gamma(\mathcal{F})$, we have that $\nabla_X s = \iota_X ds = \mathcal{L}_X\, s = 0$. Hence the polarized sections are functions on T^*Q which are constant on the fibers. i.e. the pullback of functions on Q to T^*Q.

EXAMPLE 3.1.8. (Kähler polarization) Let (M, J, ω) be a Kähler manifold. We saw in example 3.1.4 that we have a natural Kähler polarization $T_{0,1}M$. If we have in addition a prequantum line bundle $(\mathbb{L}, \langle,\rangle, \nabla)$ for M, then $\mathbb{L}$ becomes a holomorphic line bundle and its polarized sections are exactly the holomorphic sections. To see this, first note that ω is of type (1,1) and closed, we have by the local exactness of the Dolbeault complex (see [**90**]) that there exists a (1,0)-form τ such that $\omega = \bar{\partial}\tau$ on some neighborhood around $p \in M$. This τ satisfies $\partial\tau = 0$, since $\bar{\partial}(\partial\tau) = -\partial\omega = 0$ and again by local exactness there exists a form α such that $\partial\tau = \bar{\partial}\alpha$ on some neighborhood around p. Since $\partial\tau$ is of type (2,0) and $\bar{\partial}\alpha$ is of type (1,1), we must have that $\partial\tau = 0$. This means that $d\tau = \partial\tau + \bar{\partial}\tau = \omega$. Locally we can trivialize the prequantum bundle $\mathbb{L}$, by passing to a smaller open U around p, such that the connection has the form $\nabla = d - i\tau$, where $\omega = d\tau$. By taking a constant section s of $\mathbb{L}$, i.e. $\mathcal{L}_X\, s = ds(X) = 0$ for $X \in \Gamma(TM)$, we have that

$$\frac{\nabla s}{s} = -i\tau \tag{3.1}$$

This section s is unique, since an arbitrary section of $\mathbb{L}$ is of the form fs, where $f \in C^\infty(M)$, and for this section condition (3.1) implies that $df = 0$, and hence f is constant. By assuming without loss of generality that U is a simply connected, proposition 2.2.3 implies that for every prequantum line bundle $(\mathbb{L}, \langle,\rangle, \nabla)$ on M we have a unique trivializing section s of $\mathbb{L}$ such that condition (3.1) holds. Since τ is op type (1,0) we have that

$$\nabla_{\frac{\partial}{\partial \bar{z}_i}} s = -i\tau(\frac{\partial}{\partial \bar{z}_i})s = 0$$

on U, with local coordinates z_i where $i = 1, ..., n$. Hence this s is a local holomorphic section of $\mathbb{L}$ over U and this gives a well-defined holomorphic structure on $\mathbb{L}$, which makes $\mathbb{L}$ a holomorphic line bundle. Furthermore the polarized sections of $\mathbb{L}$ over U is the product of s with a holomorphic function, hence the polarized sections are exactly the holomorphic sections.

By polarization we restricted to a smaller class of sections, namely the polarized sections. To obtain a Hilbert space we might be tempted to define this Hilbert space as the densely spanned space of polarized sections. The pitfall is that these polarized sections are covariantly constant along the leaves of D and if these leaves are noncompact, then these polarized sections are not square integrable with respect to the volume form ω^n. A remedy to this problem is to integrate the polarized sections not over M but over the space M/D of leaves of the polarization. However there is no natural volume form on M/D. You can tackle this problem by working with "half-densities" or with "half-forms".

3.2. Half-form bundle. We will tackle this problem by using the approach via "half-forms". Normally, if we want to integrate a volume form over a smooth manifold, we know that the "change of variables" formula for integration involves absolute values of Jacobians, hence integration of n-forms on M require the choice of orientation. The use of densities will circumvent this need, however in more complex systems the use of n-forms seems more suitable.

Let M be a manifold of dimension n. We can think of an *n-form* ν on M as a function which at $x \in M$ assigns to each basis $\{e_1, ..., e_n\}$ of T_xM a number that satisfies

$$\nu(eg) = \det(g) \cdot \nu(e)$$

for all $g \in \mathrm{GL}(n, \mathbb{R})$ and where $e = e_1 \wedge ... \wedge e_n$. Similarly, an *$\alpha$-density* is defined as an object ν which changes according to

$$\nu(eg) = |\det(g)|^{\alpha} \cdot \nu(e)$$

We would like to define a *half-form* as an object which changes according to

$$\nu(eg) = \det(g)^{1/2} \cdot \nu(e)$$

The problem is that $\det(g)^{1/2}$ is not a well-defined function on $\mathrm{GL}(n, \mathbb{R})$ and to remedy this we need to pass to a double covering of $\mathrm{GL}(n, \mathbb{R})$ and a corresponding double covering of the bundle of bases of TM. This double covering of the frame bundle is called a metalinear frame bundle. But before we construct this bundle let us first start with the definition of the double covering of the general linear group.

The general linear group $\mathrm{GL}(n, \mathbb{R})$ has two components, namely the matrices of positive and negative determinants. We expect that the double covering we are looking for should have four component and that $\det(g)^{1/2}$ should take values in the half lines $\mathbb{R}^+$, $i\mathbb{R}^+$, $-\mathbb{R}^+$ and $-i\mathbb{R}^+$. It will be convenient to regard $\mathrm{GL}(n, \mathbb{R})$ as a subgroup of real matrices lying in $\mathrm{GL}(n, \mathbb{C})$.

Consider the group isomorphism

$$\pi : (\mathbb{C} \times \mathrm{SL}(n, \mathbb{C}))/\mathbb{Z} \xrightarrow{\cong} \mathrm{GL}(n, \mathbb{C})$$

given by $(z, A) \mapsto e^z A$, where the action of $k \in \mathbb{Z}$ on (z, A) is given by $(z + \frac{2\pi i k}{n}, e^{-\frac{2\pi i k}{n}} A)$. We have the map $det : \mathrm{GL}(n, \mathbb{C}) \to \mathbb{C}^*$ and this pulls back to the map $\det \circ \pi$ on $\mathbb{C} \times \mathrm{SL}(n, \mathbb{C})$, where $\det \circ \pi(z, A) = e^{nz}$ and this has a well-defined holomorphic square root

$$\chi(z, A) := \sqrt{\det \circ \pi}(z, A) = e^{\frac{nz}{2}}$$

which is defined on the group $ML(n, \mathbb{C}) := (\mathbb{Z} \times \mathrm{SL}(n, \mathbb{C}))/2\mathbb{Z}$, which we call the *complex metalinear group*. It is a double cover of $\mathrm{GL}(n, \mathbb{C})$, with as double covering map $r : \mathrm{ML}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ the projection of $\mathrm{ML}(n, \mathbb{C})$ to $\mathrm{GL}(n, \mathbb{C})$, so that

$$\chi^2(z, A) = \det \circ r(z, A)$$

If we regard $\mathrm{GL}(n, \mathbb{R}) \hookrightarrow \mathrm{GL}(n, \mathbb{C})$ as the real matrices, then we obtain the *real metalinear group* $\mathrm{ML}(n, \mathbb{R})$ as a double cover of $\mathrm{GL}(n, \mathbb{R})$. Indeed it follows that χ can take on values in the four half lines $\mathbb{R}^+$, $i\mathbb{R}^+$, $-\mathbb{R}^+$ and $-i\mathbb{R}^+$ and thus the group $ML(n, \mathbb{R})$ has four components[**37**].

Definition 3.2.1. Let $p : P \to M$ be a vector bundle of finite rank n. Its *frame bundle* is the bundle $F(P) \to M$ over the same base, whose fiber over $x \in M$ is the set of all ordered bases of $P_x = p^{-1}(x)$.

The frame bundle has a natural action of $\mathrm{GL}(n, \mathbb{K})$, where $\mathbb{K}$ denotes the ground field, given by an ordered change of basis which is free and transitive, i.e. the frame bundle is a principal $\mathrm{GL}(n, \mathbb{K})$-bundle.

Definition 3.2.2. A *complex metalinear frame bundle* $\tilde{F}(P) \to M$ of a rank n complex vector bundle $P \to M$ is a principal $\mathrm{ML}(n,\mathbb{C})$-bundle together with a covering map $\rho : \tilde{F}(P) \to F(P)$ which makes the diagram

$$\begin{tikzcd} \tilde{F}(P) \times \mathrm{ML}(n,\mathbb{C}) \arrow[r] \arrow[d, "(\rho,r)"] & \tilde{F}(P) \arrow[d, "\rho"'] \\ F(P) \times \mathrm{GL}(n,\mathbb{C}) \arrow[r] & F(P) \end{tikzcd}$$

commutes. The horizontal arrows are the natural group actions. We call this complex metalinear frame bundle $\tilde{F}(P) \to M$ a *complex metalinear structure* on P. Analogously, we can define the a real metalinear frame bundle and a real metalinear structure on real vector bundles.

There is no guarantee that $\tilde{F}(P)$ exists and when it exists it will in general not be unique. The existence condition is that the obstruction class in $H^2(M,\mathbb{Z}_2)$ associated with P should vanish, and in this case the various possible choices for $\tilde{F}(P)$ are parametrized by the cohomology group $H^1(M,\mathbb{Z}_2)$ (See [**83**] for more details).

Given a metalinear structure $\tilde{F}(P)$, we can now define a *half-form* to be a smooth map $\nu : \tilde{F}(P) \to \mathbb{C}$ which satisfies

$$\nu(bg) = \chi(g)\nu(b) \quad b \in \tilde{F}(P), \quad g \in \mathrm{ML}(n,\mathbb{C})$$

Let us denote the space of half-forms on P by $\Omega^{1/2}(P)$, which we call the *half-form bundle* of P, let us denote the space of n-forms on P by $\Omega^n(P)$, and let us denote the space of α-densities on P by $|\Omega|^\alpha(P)$. We have a bilinear pairing

$$\Omega^{1/2}(P) \times \Omega^{1/2}(P) \to \Omega^n(P) \quad (\nu_1,\nu_2) \mapsto \nu_1\nu_2$$

A sesquilinear pairing

$$\Omega^{1/2}(P) \times \Omega^{1/2}(P) \to \Omega^n(P) \quad (\nu_1,\nu_2) \mapsto \nu_1\bar{\nu_2}$$

We have also the space of conjugate half-forms $\bar{\Omega}^{1/2}(P)$ where each element satisfy $\nu(bg) = \bar{\chi}(g)\nu(b)$ and the space of negative half-forms $\Omega^{-1/2}(P)$ which satisfy $\nu(bg) = \chi(g)^{-1}\nu(b)$. A metalinear structure on P induces a metalinear structure on its dual bundle P^*, such that $\Omega^{-1/2}(P) \cong \Omega^{1/2}(P^*)$ (see [**37**]).

If M is a smooth manifold of dimension n, and TM carries a metalinear structure then we say that M is a *metalinear manifold*. We denote the complex line bundle $\Omega^n(TM)$ by $\Omega^n(M)$ and call it the *canonical bundle*, and we denote the complex line bundle $\Omega^{1/2}(TM)$ by $\Omega^{1/2}(M)$ and call it the *half-form bundle* on M. In this case we have a natural way to differentiate sections of $\Omega^{1/2}(M)$ along vector fields on M. If $X \in \Gamma(TM)$ is a vector field on M, then we can push-forward tangent vectors and hence frames along X. This means that X lifts naturally to a vector field on $F(TM)$ and hence to a vector field $\tilde{X}$ on $\tilde{F}(TM)$, since the double covering $\tilde{F}(TM) \to F(TM)$ is a local diffeomorphism. On the other hand, a section of the half-form bundle $\Omega^{1/2}(M)$ can be interpreted as a function v on $\tilde{F}(TM)$. Hence we can define the derivative $\mathcal{L}_X$ of $v \in \Gamma(\Omega^{1/2}(M))$ along X by

$$\mathcal{L}_X\, v = \tilde{X}(v)$$

as functions on $\tilde{F}(TM)$.

We will be interested in the choice of a metalinear structure on each polarization $\mathcal{F}$, where (M,ω) is a symplectic manifold. The choice of a so called metaplectic structure on (M,ω) allows us to put a metalinear structure on each Lagrangian subspace of TM and in turn induce a metalinear structure on the polarization $\mathcal{F}$. This is done by considering instead of the frame bundle of TM, the symplectic frame bundle $\mathrm{Bp}(M)$ consisting of all symplectic frames of TM. A symplectic frame at each $x \in M$ is an ordered basis $\{e_1, ..., e_n, f_1, ..., f_n\}$ such that $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$ and $\omega(e_i, f_j) = \delta_{ij}$ for all $i, j \leq n$. The collection of symplectic frames at $x \in M$ is equivalent to the symplectic group $\mathrm{Sp}(n, \mathbb{R})$, i.e. the group of real $2n \times 2n$-matrices of the block form

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A, B, C, D $n \times n$-matrices and satisfying $S^T \sigma S = \sigma$, where $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Notice that $\mathrm{GL}(n, \mathbb{R})$ is naturally embedded in $\mathrm{Sp}(n, \mathbb{R})$ via the map sending $g \in \mathrm{GL}(n, \mathbb{R})$ to

$$\begin{pmatrix} g & 0 \\ 0 & (g^{-1})^T \end{pmatrix}$$

in $\mathrm{Sp}(n, \mathbb{R})$. As is explained in [**37**] $\mathrm{Sp}(n, \mathbb{R})$ is diffeomorphic to the product of the unitary group $U(n)$ and an Euclidean space. Therefore the fundamental group of $\mathrm{Sp}(n, \mathbb{R})$ is $\mathbb{Z}$ so that $\mathrm{Sp}(n, \mathbb{R})$ has a unique double covering, which we denote by $\mathrm{Mp}(n, \mathbb{R})$ and we call the *metaplectic group*. Note that the symplectic frame bundle $\mathrm{Bp}(M)$ has a canonical right action of $\mathrm{Sp}(n, \mathbb{R})$ and similar to the construction of the metalinear frame bundle we can now construct the metaplectic frame bundle.

Definition 3.2.3. A *metaplectic frame bundle* $\mathrm{Mp}(M) \to M$ is a principal $\mathrm{Mp}(n, \mathbb{R})$-bundle together with a covering map $\tilde{\rho} : \mathrm{Mp}(M) \to \mathrm{Bp}(M)$ which makes the diagram

$$\begin{tikzcd} \mathrm{Mp}(M) \times \mathrm{Mp}(n, \mathbb{R}) \arrow[r] \arrow[d, "(\tilde{\rho}, \tilde{r})"] & \mathrm{Mp}(M) \arrow[d, "\tilde{\rho}"'] \\ \mathrm{Bp}(M) \times \mathrm{Sp}(n, \mathbb{R}) \arrow[r] & \mathrm{Bp}(M) \end{tikzcd}$$

commutes. The horizontal arrows are the natural group actions. We call this metaplectic frame bundle $\mathrm{Mp}(M) \to M$ a *metaplectic structure* on M. The choice of a metaplectic structure on a symplectic manifold (M, ω) is also called a *metaplectic correction*.

Of course, such a lifting may not exist and if it exist it may not be unique. A symplectic manifold (M, ω) together with a metaplectic structure determines a metalinear structure for each Lagrangian subbundle of TM and hence on each polarization of M (See [**37**] for more details).

Now if we consider symplectic manifold (M, ω) together with a metaplectic structure and a real polarization $\mathcal{F} \subset TM$ that is strongly admissible, then we have a surjective submersion $p : M \to M/D$, where M/D is the space of integral surfaces. Now if we pull-back $T^*(M/D)$ to M along p, we get the annihilator bundle of $\mathcal{F}$, which we denote by $\mathcal{F}^\perp$. That is

$$\mathcal{F}^\perp := \{\xi \in T^*M : \ \forall X \in \mathcal{F}, \ \ \langle X, \xi \rangle = 0\}$$

At each $x \in M$ we have that $\mathcal{F}_x$ is the annihilator space of $\mathcal{F}_x^\perp$ and since $\mathcal{F}_x$ is Lagrangian, it is also the annihilator space of its image under the isomorphism of TM with T^*M given by the symplectic form ω, hence this gives an isomorphism $F(\mathcal{F})_x$ with $F(T^*(M/D))_{p(x)}$. The metaplectic structure on M induces a metalinear structure on $\mathcal{F}_x$, which in turn induce a metalinear structure on $T^*(M/D)_{p(x)}$ The metalinear structure on $T^*(M/D)_{p(x)}$ is obtained by covering the frame bundle $F(T^*(M/D))_{p(x)}$ by the bundle $\tilde{F}(\mathcal{F})_x$, via the isomorphism of $\mathcal{F}_x$ with $T^*(M/D)_{p(x)}$. This is independent of the choice of a point in the fiber $p^{-1}(y)$, $y \in M/D$, and does give a bundle covering of $F(T^*(M/D))$. A smooth section v of $\tilde{F}(\mathcal{F})$ is constant along a section X of $\mathcal{F}$ if it satisfies $\mathcal{L}_X v = 0$. These sections gives a section of $\tilde{F}(T^*(M/D))$ and in this way we get a metalinear structure on M/D. Hence we can identify the half-forms $\Omega^{1/2}(M/D)$ with those sections of $\Omega^{-1/2}(\mathcal{F})$ that are constant along $\mathcal{F}$ (See [**37**] for more details). Since $\mathcal{F}$ is in particular an involutive distribution, we have a *Bott connection* on $\mathcal{F}^\perp$, that is the flat $\mathcal{F}$-connection which equals the Lie derivative, i.e. $\nabla^B_X v := \mathcal{L}_X\, v$ for any $X \in \Gamma(\mathcal{F})$ and $v \in \Gamma(\mathcal{F}^\perp)$. This Bott connection extends to any bundle constructed from $\mathcal{F}^\perp$. By duality the half-form $\Omega^{1/2}(\mathcal{F}^\perp)$ is equal to $\Omega^{-1/2}(\mathcal{F})$ and hence the induced Bott connection on $\Omega^{1/2}(\mathcal{F}^\perp)$ induces a connection on $\Omega^{-1/2}(\mathcal{F})$, which correspond to our previous defined Lie derivative along $\mathcal{F}$. Hence we can identify the half-forms of $\Omega^{1/2}(M/D)$ with those sections of $\Omega^{1/2}(\mathcal{F}^\perp)$ that are $\mathcal{F}$-constant by the Bott connection.

Combining the connection ∇ of a prequantum line bundle $\mathbb{L}$ and the Bott connection gives a flat $\mathcal{F}$-connection $\tilde{\nabla}$ on $\mathbb{L} \otimes \Omega^{1/2}(\mathcal{F}^\perp)$. Define a section of $\mathbb{L} \otimes \Omega^{1/2}(\mathcal{F}^\perp)$ to be *polarized* if it is $\mathcal{F}$-constant by this connection, i.e. a sections $s \otimes v \in \Gamma(\mathbb{L} \otimes \Omega^{1/2}(\mathcal{F}^\perp))$ needs to satisfy $\nabla_X s = 0$ and $\mathcal{L}_X\, v = 0$ for all $X \in \Gamma(\mathcal{F})$. The inner product of two polarized sections gives a volume form, which can be integrated over M/D. This gives the pre-Hilbert space, consisting of all square integrable polarized sections of $\mathbb{L} \otimes \Omega^{1/2}(\mathcal{F}^\perp)$. The completion of this space gives us the quantum Hilbert space $L^2_{\mathcal{F}}(M, \mathbb{L})$. [**83**].

Now if we consider a polarization $\mathcal{F} \subset T_{\mathbb{C}}M$ that is not real, then the inner product is valued in $\bar{\Omega}^{1/2}(\mathcal{F}^\perp) \otimes \Omega^{1/2}(\mathcal{F}^\perp)$, rather than $\Omega^n(D^\perp_{\mathbb{C}})$. Instead one has the natural isomorphism[**39**]

$$\Omega^n(\bar{\mathcal{F}}^\perp) \otimes \Omega^n(\mathcal{F}^\perp) \cong \Omega^n(E^\perp_{\mathbb{C}}) \otimes \Omega^n(D^\perp_{\mathbb{C}})$$

and the exterior product with $\frac{\omega^k}{k!}$, with $2k := \mathrm{rk}E - \mathrm{rk}D$, defines a canonical isomorphism $\Omega^{n/2}(D^\perp) \cong \Omega^{n/2}(E^\perp)$. With these isomorphisms we can correct the inner product and define $L^2_{\mathcal{F}}(M, \mathbb{L})$ for every polarization $\mathcal{F}$.

3.3. Bohr-Sommerfeld variety. There are some complications involved if the leaves of the polarization are not simply connected, which lead to the notion of a Bohr-Sommerfeld variety. Let Λ be an integral manifold of D. Then the flat $\mathcal{F}$-connection $\tilde{\nabla}$ on $\mathbb{L} \otimes \Omega^{1/2}(\mathcal{F}^\perp)$ induces a flat $\mathcal{F}$-connection on the restriction of $\mathbb{L} \otimes \Omega^{1/2}(\mathcal{F}^\perp)$ to Λ. The holonomy group G_Λ of this connection is a subgroup of $\mathbb{C}^\times$. Now if Λ is not simply connected, then it is possible to have non-trivial holonomy along the non-contractible loops in Λ. Now since a section $s \otimes v|_\Lambda$ is a covariantly constant section, it does not change under parallel transport, and in particular, cannot pick up a phase from the non-trivial holonomy group G_Λ. Therefore either $s \otimes v|_\Lambda$ is the zero section or the holonomy group G_Λ is trivial, i.e. $G_\Lambda = \{1\}$. Now we call the union of all integral manifolds of D such that $G_\Lambda = \{1\}$ the *Bohr-Sommer variety* $M_{B\text{-}S}$. Hence in practice we restrict M to the Bohr-Sommerfeld variety $M_{B\text{-}S}$. In the case that all Λ are simply connected we have that $M = M_{B\text{-}S}$[**3**].

Another difficulty is that it seems that our constructed quantum Hilbert space could be zero, depending on the topology of the integral surfaces of $\mathcal{F}$. A solution may be given using "cohomological wave functions", where the local polarized section of $\Omega^{1/2}(\mathcal{F}^\perp)$ form a sheaf (see [**93**]). The global polarized sections corresponds to the zeroth sheaf cohomology space, but the higher sheaf cohomology spaces may not be zero. This hints to a relation between the quantum Hilbert space and higher cohomology spaces. In Spin^c-quantization, which we treat in the next section, we will consider the quantum Hilbert space as the index of a Spin^c-Dirac operator acting on a certain chain complex. For Spin^c-quantization you need instead of the choice of a metaplectic structure, the choice of a Spin^c-structure. The notion of a metaplectic structure is closely related to the notion of Spin^c-structure, the advantage of Spin^c-structures is that the space of all choice is very well understood. Furthermore, this may also allow us to describe the quantization of a symplectic manifold with a Hamiltonian G-action as a G-representation on a virtual vector space. In this setting Spin^c-quantization can also be described by a pushforward map in equivariant K-theory, as was first observed by [**61**].

4. Geometric quantization by Spin^c-structures

In the previous section we constructed the Hilbert space of quantum states as a certain subspace of the space of polarized sections of $\mathbb{L} \otimes \Omega^{1/2}(\mathcal{F}^{\perp})$. There is a second quantization scheme, that seems to be more natural and general from a mathematical point of view. In this case we need to choose, instead of a metaplectic structure, a *Spin*^{c} structure. This define the Hilbert space of quantum states as the index of a corresponding Dirac operator twisted by $\mathbb{L}$. Let us start with the case of Kähler polarization, since this fits completely within the framework of complex geometry.

4.1. Quantization of the Dolbeault-Dirac operator. Let (M, ω) be a compact symplectic manifold and let $(\mathbb{L}, \langle,\rangle, \nabla)$ be a prequantization of (M, ω). Let J be an almost complex structure on TM that is compatible with ω, i.e. the symmetric bilinear form $g := \omega(-, J-)$ is a Riemannian metric on M. Notice that every symplectic manifold admits an almost complex structure J that is compatible with the symplectic form[**32**]. This defines the Hermitian metric $h := g + i\omega$ on M and give rise to an Hermitian inner product $h^{0,1}$ on the vector bundle $T^{0,1}M := T^*_{0,1}M$. Remember from example 3.1.4, that the almost complex structure J give rise to a splitting $T_{\mathbb{C}}M = T_{1,0}M \oplus T_{0,1}M$. This Hermitian inner product is constructed using the composition of the induced complex linear isomorphism $\tilde{h} : T_xM \to T^*_xM$ and $i : T^*_xM \to T^{0,1}M$, from where we can construct a unitary frame locally on $T^{0,1}M$. This in turn gives us a Hermitian inner product on $T^{0,q}M := \wedge^q T^{0,1}M$. Similarly we have an Hermitian inner product on $T^{p,0}M$. The choice of a metaplectic structure on M gives us also a Hermitian inner product on $\Omega^{1/2}(T^{1,0}M)$. Since $\mathbb{L}$ is by construction equipped with a Hermitian inner product, one has a Hermitian inner product $h_{\tilde{\mathbb{L}} \otimes T^{0,q}M}$ on $\tilde{\mathbb{L}} \otimes T^{0,q}M$, where $\tilde{\mathbb{L}} = \mathbb{L} \otimes \Omega^{1/2}(T^{1,0}M)$. These Hermitian inner products and the volume form ω^n give an Hermitian L^2-inner product of two sections u and v of $\tilde{\mathbb{L}} \otimes T^{0,q}M$, which is defined as

$$(u, v) = \int_M h_{\tilde{\mathbb{L}} \otimes T^{0,q}M}(u, v)\omega^n$$

The connection ∇ on $\mathbb{L}$ and the Bott connection on $\Omega^{1/2}(T_{1,0}M)$ gives the flat $T_{0,1}M$-connection $\tilde{\nabla}$ on $\tilde{\mathbb{L}}$ and defines a differential operator

$$\tilde{\nabla} : \Omega^k(M; \tilde{\mathbb{L}}) \to \Omega^{k+1}(M; \tilde{\mathbb{L}})$$

such that for all $s \in \Gamma(M; \tilde{\mathbb{L}})$ and $\alpha \in \Omega^k(M)$, $\tilde{\nabla}(s \otimes \alpha) = \nabla s \otimes \alpha + s \otimes d\alpha$. Consider the projection

$$\pi^{0,*} : \Omega^*_{\mathbb{C}}(M; \tilde{\mathbb{L}}) \to \Omega^{0,*}(M; \tilde{\mathbb{L}})$$

according to the decomposition $\Omega^k_{\mathbb{C}}(M; \tilde{\mathbb{L}}) = \bigoplus_{p+q=k} \Omega^{p,q}(M; \tilde{\mathbb{L}})$. Define the differential operator

$$\overline{\partial} : \Omega^{0,q}(M; \tilde{\mathbb{L}}) \to \Omega^{0,q+1}(M; \tilde{\mathbb{L}})$$

by

$$\overline{\partial} := \pi^{0,*} \circ \tilde{\nabla}$$

The formal adjoint $\overline{\partial}^*$ of $\overline{\partial}$ is the differential operator

$$\overline{\partial}^* : \Omega^{0,q}(M; \tilde{\mathbb{L}}) \to \Omega^{0,q-1}(M; \tilde{\mathbb{L}})$$

defined by the above Hermitian L^2-inner product as

$$(\overline{\partial}(u), v) = (u, \overline{\partial}^*(v)), \quad u \in \Omega^{0,q}(M; \tilde{\mathbb{L}}),\ v \in \Omega^{0,q+1}(M; \tilde{\mathbb{L}})$$

Definition 4.1.1. The *Dolbeault-Dirac operator* is the elliptic differential operator

$$\overline{\partial} + \overline{\partial}^* : \Omega^{0,*}(M; \tilde{\mathbb{L}}) \to \Omega^{0,*}(M; \tilde{\mathbb{L}})$$

which maps even degree forms to odd degree, and vice versa.

The *Dolbeault-quantization* of (M,ω) is defined as the virtual vector space[**10**]

$$\mathcal{Q}(M) = \sum (-1)^k H^{0,k}(M;\tilde{\mathbb{L}})$$

which is by Hodge theory, the index of the Dolbeault-Dirac operator

$$\overline{\partial} + \overline{\partial}^* : \Omega^{0,even}(M;\tilde{\mathbb{L}}) \to \Omega^{0,odd}(M;\tilde{\mathbb{L}})$$

. In other words,

$$\mathcal{Q}(M) = \ker(\overline{\partial} + \overline{\partial}^*) - \operatorname{coker}(\overline{\partial} + \overline{\partial}^*)$$

This index is well-defined, because this operator is elliptic and M is compact. Note that we have $\operatorname{coker}((\overline{\partial} + \overline{\partial}^*)|_{\Omega^{0,even}(M;\tilde{\mathbb{L}})}) \cong \ker((\overline{\partial} + \overline{\partial}^*)|_{\Omega^{0,odd}(M;\tilde{\mathbb{L}})})$ hence

$$\mathcal{Q}(M) = \ker((\overline{\partial} + \overline{\partial}^*)|_{\Omega^{0,even}(M;\tilde{\mathbb{L}})}) - \ker((\overline{\partial} + \overline{\partial}^*)|_{\Omega^{0,odd}(M;\tilde{\mathbb{L}})})$$

REMARK 4.1.2. When M is compact the Dolbeault quantization is independent of the choice of J (See [**32**]).

This definition of quantization is a slight generalization of our first quantization scheme. To see this we consider a compact Kähler manifold (M,J,ω) and we fix a prequantization $(\mathbb{L},\langle,\rangle,\nabla)$. This Kähler manifold has the natural Kähler polarization given by $T_{0,1}M$. In this case $\mathbb{L}$ is a holomorphic line bundle and the space of polarized sections are the holomorphic sections. The choice of a metaplectic structure on M and the Kähler polarization induce the half-form bundle $\Omega^{1/2}(T^{1,0}M)$, since $T_{0,1}M^{\perp} = T^{1,0}M$. Now if the curvature of $\tilde{\mathbb{L}}$ is positive, then the higher cohomology spaces $H^{0,k}(M;\tilde{\mathbb{L}})$ vanishes for $k>0$, and the defined Hilbert space becomes $\mathcal{Q}(M) = H^{0,0}(M;\tilde{\mathbb{L}})$ (See [**90**]). But the zeroth cohomology space $H^{0,0}(M;\tilde{\mathbb{L}})$ is the space of global holomorphic section of $\tilde{\mathbb{L}}$, which we defined to be the Hilbert space of quantum states. But Dolbeault quantization refines this quantization even for the case where $H^{0,0}(M;\tilde{\mathbb{L}})$ is trivial, which happens for example if the curvature of the bundle $\mathbb{L}\otimes\Omega^{-1/2}(T^{1,0}M)$ is negative. This follows also from Kodaira vanishing theorem[**90**], which states that for a holomorphic line bundle L on M with negative curvature, we have

$$H^{0,0}(M; L\otimes\Omega^n(M)) = 0$$

If $L := \mathbb{L}\otimes\Omega^{-1/2}(T^{1,0}M)$ and has a negative curvature, then there are no non-zero holomorphic section of $\tilde{\mathbb{L}}$.

4.2. ***Spin^c*-quantization.** The prequantum line bundle and the complex structure of the Kähler manifold together with a metaplectic structure give, as we will see, a Spin^c-structure on M. This structure together with a connection define an elliptic operator, the Spin^c-Dirac operator, which in Spin^c-quantization will play the role of the Dolbeault-Dirac opertor in Dolbeault quantization. We begin by introducing the notion of Spin^c-structures on manifolds.

DEFINITION 4.2.1. The *Clifford algebra* $Cl(V,q)$ associated to a real vector space V with quadratic form q can be defined as

$$Cl(V,q) = T(V)/I(V,q)$$

where $T(V)$ is the tensor algebra of V and $I(V,q)$ is the ideal in $T(V)$ generated by elements $v\otimes v - q(v)$ for $v\in V$. Note the tensor algebra $T(V)$ is $\mathbb{Z}$-graded, and since the ideal $I(V,q)$ is generated by a quadratic elements, the quotient $Cl(V,q)$ has a $\mathbb{Z}_2$-grading, i.e. $Cl(V,q) = Cl_{even}(V,q)\oplus Cl_{odd}(V,q)$.

In case that $V=\mathbb{R}^n$ and $q(x_1,...,x_n) = -x_1^2 - ... - x_n^2$, we will denote the Clifford algebra as $Cl(n)$. $\mathbb{R}^n$ is itself a linear subspace of $Cl(n)$, $\mathbb{R}^n\subset Cl(n)$. For every $x\in\mathbb{R}^n$, the equality $x\otimes x = -\|x\|^2$ holds in $Cl(n)$ and hence the inverse element is given by $x^{-1} = -x/\|x\|^2$.

DEFINITION 4.2.2. $\mathrm{Pin}(n) \subset Cl(n)$ is the group which is multiplicatively generated by all vectors $x \in S^{n-1}$. Therefore, the elements of $\mathrm{Pin}(n)$ are the products $x_1 \otimes ... \otimes x_m$ with $x_i \in \mathbb{R}^n$, $\|x_i\| = 1$. The *spin group*, $\mathrm{Spin}(n)$, is defined as $\mathrm{Spin}(n) = \mathrm{Pin}(n) \cap Cl_{even}(n)$. The elements of of $\mathrm{Spin}(n)$ are the products $x_1 \otimes ... \otimes x_{2m}$ with $x_i \in \mathbb{R}^n$, $\|x_i\| = 1$.

The group $\mathrm{Spin}(n)$ is for $n > 2$ the simply connected double cover of $\mathrm{SO}(n)$ (See [**25**]). Hence $\mathbb{Z}_2$ is embedded into $\mathrm{Spin}(n)$ as the kernel of the covering map $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n)$, which gives exactly a spin group extension of the special orthogonal group of dimension n. Furthermore $\mathbb{Z}_2$ is embedded into $U(1)$ as the subgroup $\{\pm 1\}$.

DEFINITION 4.2.3. The group $\mathrm{Spin}^c(n) = (\mathrm{Spin}(n) \times U(1))/\{\pm 1\} = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} U(1)$. The elements are thus classes $[s, z]$ of pairs $(s, z) \in \mathrm{Spin}(n) \times U(1)$ under the equivalence relation $(s, z) \sim (-s, -z)$.

The projection onto its two factors gives rise to the maps

$$\pi : \mathrm{Spin}^c(n) \to \mathrm{SO}(n), \quad [s, z] \mapsto \lambda(s)$$

and

$$\det : \mathrm{Spin}^c(n) \to U(1), \quad [s.z] \mapsto z^2$$

which give rise to short exact sequences

$$1 \to U(1) \to \mathrm{Spin}^c(n) \xrightarrow{\pi} \mathrm{SO}(n) \to 1$$

and

$$1 \to \mathrm{Spin}(n) \to \mathrm{Spin}^c(n) \xrightarrow{\det} U(1) \to 1$$

DEFINITION 4.2.4. A Spin^c*-structure* on an oriented n-dimensional real vector bundle $E \to M$ is a principal $\mathrm{Spin}^c(n)$-bundle

$$P \to M$$

together with $\mathrm{Spin}^c(n)$-equivariant bundle map

$$p : P \to F(E)$$

Where $F(E)$ denote the frame bundle of E, i.e. the principal $\mathrm{GL}(n)$-bundle whose fiber over $p \in M$ is the set of bases of the vector space E_p, and the group $\mathrm{Spin}^c(n)$ acts on $F(E)$ through the composition $\mathrm{Spin}^c(n) \xrightarrow{\pi} \mathrm{SO}(n) \hookrightarrow \mathrm{GL}(n)$. We denote a Spin^c-structure on E by (P, p). A Spin^c-structure on a manifold M is a Spin^c-structure on its tangent bundle $E = TM$. A manifold equipped with a Spin^c-structure is called a Spin^c*-manifold*.

Since the $\mathrm{Spin}^c(n)$ acts on $\mathbb{R}^n$ via the homomorphim π, we have that the map p determines an isomorphism of vector bundles,

$$P \times_\pi \mathbb{R}^n \cong E,$$

and vice versa, such an isomorphism determines an equivariant map $p : P \to F(E)$. The Spin^c-structure on a vector bundle $E \to M$ induces a metric and orientation on E, obtained from the standard metric and orientation on $\mathbb{R}^n$, via the above isomorphism. If E was already equipped with these structures, then the isomorphism is supposed to be an isometric isomorphism of oriented vector bundles.

A Spin^c-structure (P, p) on a manifold M, together with the choice of a connection on P, give rise to a Spin^c-Dirac operator D acting on the space of sections of a certain vector bundle, which as we will see next is the spinor bundle. We define the quantization to be the index of this operator.

Now assume that the underlying manifold M has even dimension $n \in \mathbb{N}$. We denote the *canonical representation* of $Cl(n)$ by $c : Cl(n) \to End(\Delta_n)$ (see [**25**]). The vector space of complex n-spinors Δ_n is naturally isomorphic to $\mathbb{C}^{2^{n/2}}$. The restriction to $\mathrm{Spin}(n)$ of this representation decomposes into two irreducible subrepresentation $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$ of equal dimension. For $x \in \mathbb{R}^n \subset Cl(n)$ we have the so-called Clifford multiplication with spinors

$$x\Delta_n^+ := c(x)\Delta_n^+ \subset \Delta_n^-$$
$$x\Delta_n^- := c(x)\Delta_n^- \subset \Delta_n^+$$

The representation c of $\mathrm{Spin}(n)$ extends to the group $\mathrm{Spin}^c(n)$ via the formula $[s,z]\cdot\delta = z(s\cdot\delta)$ for $s \in \mathrm{Spin}(n)$, $z \in U(1)$ and $\delta \in \Delta_n$. The Spin^c-Dirac operator acts on sections of the *spinor bundle* associated to the Spin^c-structure on M.

Definition 4.2.5. Let (P,p) be a Spin^c-structure on an n-dimensional manifold M, with n even. The *spinor bundle* on M associated to this Spin^c-structure is the vector bundle

$$\mathcal{S} := P \times_{\mathrm{Spin}^c(n)} \Delta_n$$

The isomorphism $\Delta_n \cong \mathbb{C}^{2^{n/2}}$ induces a Hermitian metric on $\mathcal{S}$ and the decomposition $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$ induces a natural decomposition of the spinor bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$.

Since p gives a vector bundle isomorphism

$$TM \cong P \times_{\mathrm{Spin}^c(n)} \mathbb{R}^n$$

we can define an action of TM on $\mathcal{S}$, which we call the *Clifford action* and we denote by

$$c_{TM} : TM \otimes \mathcal{S} \to \mathcal{S}\,.$$

Let $[p,x] \in P \times_{\mathrm{Spin}^c(n)} \mathbb{R}^n \cong TM$ and $\delta \in \Delta_n$, the Clifford action is defined by

$$c_{TM}([p,x])[p,\delta] := [p, x\cdot\delta]$$

By definition of Clifford multiplication we see that the Clifford action interchanges the sub-bundles $\mathcal{S}^+$ and $\mathcal{S}^-$. This induces an action of vector fields on sections of the spinor bundle, which we also denote by c_{TM}.

Choose a connection on P, then this induces a connection on the spinor bundle $\mathcal{S}$ which we denote by

$$\nabla : \Gamma(\mathcal{S}) \to \Gamma(T^*M \otimes \mathcal{S})$$

The metric allows us to identify $T^*M = TM$. Then we can compose the maps c_{TM} and ∇ to get the Spin^c-Dirac operator.

Definition 4.2.6. The Spin^c-*Dirac operator* D on M, associated to the Spin^c-structure (P,p) and the connection on P, is defined by

$$D = c_{TM} \circ \nabla : \Gamma(S) \to \Gamma(T^*M \otimes S) = \Gamma(TM \otimes S) \to \Gamma(S)$$

Where ∇ is the induced connection on the spinor bundle $\mathcal{S}$. With respect to local orthonormal frame $\{e_1, ..., e_n\}$ of TM,

$$D = \sum_{i=1}^{n} c_{TM}(e_i)\nabla_{e_i}$$

This operator is a first order differential operator that maps sections of S^+ to sections of S^- and vice versa.

Its principal symbol $\sigma(D)(\xi) : S \to S$ for every $\xi \in T^*M/\{0\}$ is given by the Clifford action

$$\sigma(D)(\xi)\psi = c_{TM}(\xi^*)\psi$$

Where $\psi \in \mathcal{S}$ and $\xi^* \in TM/\{0\}$ is the tangent vector associated to ξ via the metric on M. The square of this principle symbol is given by scalar multiplication by $-\|\xi\|^2$, so that $\sigma(D)$ is invertible, and hence the Spin^c-Dirac operator is elliptic. For elliptic operators on compact manifolds we can calculate the index which is essential in order to define the Spin^c-quantization. But in order to define a Spin^c-prequantization we need to consider the following line bundle.

DEFINITION 4.2.7. The *determinant line bundle* associated with the Spin^c-structure (P,p) on M, is the line bundle

$$\mathbb{L}_{\det} = P \times_{\det} \mathbb{C}$$

over M, associated through the homomorphism $\det : \mathrm{Spin}^c(n) \to U(1)$.

In order to define Spin^c-quantization we first take a look at our previous notion of quantization. Previously for (M, J, ω) a compact Kähler manifold we had to choose beside the compatible complex structure J, a metaplectic structure in order to define the square root $\Omega^{1/2}(T^{1,0}M)$ of the bundle $\Omega(T^{1,0}M) := \Omega^n(T^{1,0}M)$. Together with a prequantum line bundle $\mathbb{L}$ we can define the line bundle $\tilde{\mathbb{L}} = \mathbb{L} \otimes \Omega^{1/2}(T_{1,0}M)$. Moreover these structures together induce naturally a Spin^c-structure on M.

PROPOSITION 4.2.8. [**68**] *Let (M, J, ω) a Kähler manifold, then the existence of a line bundle $\tilde{\mathbb{L}}$ such that $\tilde{\mathbb{L}}^2 \otimes \Omega^{-1}(T^{1,0}M)$ is a prequantum line bundle over $(M, 2\omega)$ is equivalent to the existence of a* Spin^c*-structure (P, p) such that its determinant line bundle $\mathbb{L}_{\det}$ is a prequantum line bundle over $(M, 2\omega)$.*

Note that $\tilde{\mathbb{L}}^2 \otimes \Omega^{-1}(T^{1,0}M) = \mathbb{L}^2$ which has a curvature of 2ω and hence we have that it is a prequantum line bundle over $(M, 2\omega)$. From the proposition 4.2.8 we can indeed conclude that these structures together give a Spin^c-structure which has the determinant line bundle $\mathbb{L}_{\det}$ as prequantum line bundle over $(M, 2\omega)$. The converse is not automatically true, since for the existence of the half-form bundle $\Omega^{1/2}(T^{1,0}M)$ we need a metaplectic structure. Actually the existence of a Spin-structure is enough.

PROPOSITION 4.2.9. [**42**] *A* Spin*-structure on a Kähler manifold (M, J, ω) exists if and only if there exists a holomorphic square root of the bundle $\Omega^n(T^{1,0}M)$.*

Since we have the inclusion $i : \mathrm{Spin}(n) \to \mathrm{Spin}(n)^c$, the converse is only true if the Spin^c-structure comes from a Spin-structure.

PROPOSITION 4.2.10. [**25**] *Let (M, J, ω) be a Kähler manifold with a fixed* Spin*-structure. Then the spinor bundle is isomorphic to $S = \Omega^{0,*}(M, \tilde{\mathbb{L}})$, with $\tilde{\mathbb{L}} := \mathbb{L} \otimes \Omega^{1/2}(T^{1,0}M)$. Furthermore the Dolbeault-Dirac operator and the* Spin^c*-Dirac operator have the same principal symbol and hence the same index.*

In light of proposition 4.2.8 we can now give the definition of Spin^c-quantization.

DEFINITION 4.2.11. A Spin^c*-prequantization* of (M, ω) is a Spin^c-structure (P, p) together with a connection ∇ on P such that the induced connection on its determinant line bundle has a curvature that is half of $[\omega]$. The Spin^c*-quantization* of (M, ω) is then the index of the corresponding Spin^c-Dirac operator

$$D : \Gamma(\mathcal{S}^+) \to \Gamma(\mathcal{S}^-)$$

on the spinor bundle $\mathcal{S}$. That is

$$\mathcal{Q}(M) := \ker D - \operatorname{coker} D$$

Remember the definition of the Spin^c-Dirac operator depends on the choice of a connection on the principlal Spin$^c(n)$-bundle $P \to M$. Since the space of such connections is connected, the choice of different connections give rise to homotopic Spin^c-Dirac operators. These operators have the same index, by the Fredholm homotopy invariance of the index[**32**]. Hence the index associated to a Spin^c-structure is well defined and does not depend on the choice of connection. Furthermore, the Spin^c-quantization is completely determined by the cohomology class $[\omega]$ and hence the Spin^c-structure plays a purely auxiliary role in it.

We see that in the case of a Kähler manifold (M, J, ω) with a fixed Spin-structure, the Dolbeault-quantization equals the Spin^c-quantization. The assumption that we have a Spin-structure is only needed for the comparison with the Dolbeault-quantization of Kähler manifolds, this is not necessary for Spin^c-quantization. This definition of Spin^c-quantization does not assume any choice of polarization, nor any choice of complex structure. On the other hand, every choice of almost complex structure, and in particular every Kähler polarization, does induce a Spin^c-structure[**32**].

The above construction can be generalized to the case where we have a Hamiltonian G-action on M and the index of the Spin^c-Dirac operator, viewed as a virtual vector space, is a space on which G acts, and this virtual representation of G is the Spin^c-quantization of the Spin^c G-manifold. We will refer the reader to [**32**] for more details about this. Furthermore these results can in turn be interpreted as a push-forward in G-equivariant K-theory of the prequantum line bundle to the point, which was observed by [**61**] and further worked out in [**67**].

CHAPTER 3

Geometric Quantization of Poisson manifolds

In this chapter we give a review of geometric quantization of Poisson manifolds. Since a symplectic manifold is a special type of Poisson manifold this generalizes the geometric quantization scheme for symplectic manifolds. In particular we review the geometric quantization of Poisson manifolds via the geometric quantization of their associated symplectic groupoids, due to Weinstein, Xu [**89**] and Hawkins [**39**]. This is most rigorously done by Hawkins [**39**], where he considers instead of the formal deformation quantization approach, the more concrete approach via *strict C^*-deformation quantization*, which was initiated by Rieffel[**71**]. His main objective was to systematically construct a single natural C^*-algebra from a Poisson manifold.

In section 2.1 we said that we associate to a symplectic manifold (M, ω) a Hilbert space $\mathcal{H}$ and to a certain subalgebra S of its associated Poisson algebra $C^\infty(M)$ an algebra of operators acting on $\mathcal{H}$. We know that for standard geometric quantization this algebra is the C^*-algebra of compact operators. We will see that a general Poisson manifold can be geometrically quantized to a certain C^*-algebra and this construction will give us the right C^*-algebra of compact operators for the symplectic case.

In Hawkins' approach to geometric quantization of Poisson manifolds, elements of geometric quantization are lifted from manifolds to Lie groupoids. This is done by integrating the associated Poisson Lie algebroid of the Poisson manifold to a symplectic groupoid, which is only possible if the Poisson manifold is integrable. The passage of the Poisson manifold to its associated Poisson Lie algebroid doubles the dimension, since we need to pass to the cotangent bundle of the Poisson manifold. This doubling has to be undone by introducing a symplectic groupoid polarization. In the case of symplectic manifolds, this symplectic groupoid polarization is precisely related to the polarization that we encountered in the geometric quantization of symplectic manifolds. In the geometric quantization procedure of symplectic manifolds, we introduced also a prequantization step by considering a prequantum line bundle over the symplectic manifold. This line bundle can be lifted to the symplectic groupoid in order to get a multiplicative line bundle over the symplectic groupoid, which is a line bundle that carries a twist. Such a twisted line bundle will define a prequantization for the symplectic groupoid. Together with the polarization we can construct a twisted polarized convolution algebra of the symplectic groupoid. In summary we have the following procedure:

(i) Find an integrating symplectic groupoid Σ of the Poisson manifold (M, π).
(ii) Construct a prequantization of Σ with data $(\mathbb{L}, \nabla, \sigma)$, where $\mathbb{L}$ is the multiplicative prequantum line bundle with connection ∇ and σ is a cocycle twist.
(iii) Choose a symplectic groupoid polarization $\mathcal{P}$ of Σ.
(iv) Construct a half-density bundle $\Omega^{1/2}_{\mathcal{P}}$.
(v) Construct the C^*-algebra of $\Sigma/\mathcal{P}$ as a twisted polarized convolution algebra.

This construction entails some existence and uniqueness issues and hence there exist a similar prequantization criteria as was the case in geometric quantization of symplectic manifolds. In order for a Poisson manifold to be prequantizable in the symplectic groupoid sense, it needs to be integrable and it needs to satisfy some integrality condition. In the next sections we will review this procedure in more detail.

1. Integration of a Poisson manifold

The first step towards geometric quantization of Poisson manifolds is its integration. In appendix A we show that we can integrate every Poisson manifold M by integrating their associated Poisson Lie algebroid $\mathcal{B}$ and by theorem A.1.0.22 this integrated Lie algebroid can be taken to be a s-simply connected symplectic groupoid $\Sigma(M)$. We know that a symplectic groupoid integrating a Poisson manifold M is unique modulo covering and isomorphism. The Poisson structure can be thought of as an infinitesimal structure that is integrated by the groupoid.

Take for example the case where M is a symplectic manifold. Example A.1.0.23 shows us that the s-simply connected integration of M is isomorphic to the fundamental groupoid $\Pi_1(M)$. The fact that the fundamental groupoid $\Pi_1(M)$ is the connected cover of of the pair groupoid $Pair(M)$, implies that $Pair(M)$ also integrates M. But the pair groupoid $Pair(M)$ is not isomorphic to $\Pi_1(M)$ unless M is simply connected.

Hence the first obstruction to quantization is integrability of the Poisson manifold and the non-uniqueness of integration is an ambiguity in the quantization process. For a clear account on the integrability of Poisson manifolds, see [**16, 17**]. Besides this obstruction and ambiguity, the next step will be to prequantize this integrated symplectic groupoid.

2. Prequantization of a symplectic groupoid

In this section we recall the relevant results of the prequantization of symplectic groupoids according to Weinstein and Xu [**89**], Crainic [**15**], Crainic and Zhu [**19**] and Hawkins [**39**]. Remember that a prequantization of a symplectic manifold was defined as a Hermitian line bundle with a curvature equal to the symplectic form. Rather than regarding the symplectic groupoid Σ as a symplectic manifold, the prequantization of Σ as a symplectic groupoid involves a little more structure. We will view the prequantization of a symplectic groupoid again in the Souriau picture, that is in terms of circle bundles and equivalently in the Konstant picture, that is in terms of line bundles, which is more suited for the construction of the C^*-algebra.

2.1. Multiplicative prequantum line bundle. Before we define the extra structure that is needed for the prequantization of a symplectic groupoid, we start with an example, where we consider the pair groupoid. That is let M be a symplectic manifold and consider the integrated symplectic groupoid $Pair(M)$. For geometric quantization of a symplectic manifold we first need to prequantize M by constructing a prequantum line bundle $\mathbb{L} \to M$, which is a Hermitian line bundle with curvature equal to the symplectic form of M. We know that a Hermitian line bundle is equivalent to a (principal) circle bundle. The following example shows how this circle bundle over M gives a natural circle bundle over $Pair(M)$ but with some additional structure.

EXAMPLE 2.1.1. Consider a circle bundle $\pi : P \to M$ (remember every circle bundle is a principal S^1-bundle) and let $Gauge(P)$ be the induced gauge groupoid over M, as is explained in example A.0.1.7. Then $(s,t) : Gauge(P) \to M \times M$ is induced by the surjective submersion $(\pi, \pi) : P \times P \to M \times M$, and hence it is itself a surjective submersion. This gives us a surjective groupoid morphism between the groupoids $Gauge(P)$ and $Pair(M)$, with an identity morphism on the base manifolds. Moreover the trivial bundle of Lie groups with fiber S^1, that is $S^1 \times M$, is a groupoid. The source and target maps are the projection maps and the multiplication is defined by $(z_1, x)(z_2, x) = (z_1 z_2, x)$. This groupoid can be embedded in $Gauge(P)$, that is $\iota : S^1 \times M \to P \times_{S^1} P$ is an embedding, such that $\operatorname{im} \iota = \ker(s,t)$. In order for ι to be a groupoid morphism with an identity morphism on the base manifolds, it needs to satisfy the condition $\iota((z,t(p)))p = p(\iota((z,s(p))))$ for all $z \in S^1$ and $p \in P \times_{S^1} P$. Altogether this gives us diagrammatically:

$$\begin{tikzcd} S^1 \times M \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & P \times_{S^1} P \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & M \times M \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, "\sim"] & M \arrow[r, "\sim"] & M \end{tikzcd}$$

Which is precisely a central extension of the groupoid $Pair(M)$ by the Abelian group S^1. The action of S^1 on $P \times_{S^1} P$ is given by $z \cdot p = \iota((z,t(p)))p \in P \times_{S^1} P$ where $p \in P \times_{S^1} P$. Since $P \times_{S^1} P$ is an extension of the Lie groupoid $M \times M$ by S^1, S^1 is mapped diffeomorphically onto each orbit by the action. Hence this makes $P \times_{S^1} P$ a circle bundle over $M \times M$.

This construction gives us a circle bundle over $Pair(M)$, but with the extra structure of a groupoid, which is what we need in order to define the convolution product. This construction can be formalized in the following definition.

Definition 2.1.2. A S^1-*central extension* or a *twist* of a Lie groupoid $\mathcal{G}$ is a sequence of Lie groupoids and smooth mappings

$$S^1 \times M \overset{\iota}{\hookrightarrow} R \overset{\pi}{\twoheadrightarrow} \mathcal{G}$$

where ι and π are injective immersion and surjective submersion groupoid morphisms over the diffeomorphism $M \overset{\sim}{\to} R_0$ and its inverse respectively, satisfying conditions

(i) $\mathrm{im}(\iota) = \ker(\pi)$,
(ii) $\iota((z,t(r)))r = r(\iota((z,s(r))))$ for all $z \in S^1$ and $r \in R$.

This last condition makes the extension central. We denote the action by $z \cdot r := \iota((z,t(r)))r$ for any $z \in S^1$ and $r \in R$.

Lemma 2.1.3. [**85**] *The central extension R of a Lie groupoid $\mathcal{G}$ by S^1 is a circle bundle over $\mathcal{G}$.*

Since $R|_M$ is trivial, one can choose a locally smooth section τ of π such that $\tau|_M$ coincides with the identifying map $M \overset{\sim}{\to} R_0$ and that is smooth in a neighborhood of M. A simple calculation shows that this map τ commutes with the source and target maps of R and $\mathcal{G}$. From this it follows that for $g,h \in \mathcal{G}_2$ the product $\tau(g)\tau(h)\tau(gh)^{-1}$ is well-defined in R. Since we have

$$\begin{aligned} \pi(\tau(g)\tau(h)\tau(gh)^{-1}) &= (\pi \circ \tau(g))(\pi \circ \tau(h))(\pi \circ \tau(gh)^{-1}) \\ &= gh(\pi \circ \tau(gh))^{-1} \\ &= gh(gh)^{-1} \\ &= s(g) \in M \end{aligned}$$

and since R is a central extension of $\mathcal{G}$, it follows that

$$\tau(g)\tau(h)\tau(gh)^{-1} \in \iota(\{s(g)\} \times S^1) \subset R$$

This determines an element $\sigma(g,h) \in S^1$. By direct computation it is shown that $\sigma : \mathcal{G}_2 \to S^1$ is a 2-cocycle on the groupoid $\mathcal{G}$ with coefficients in S^1 and the cohomology class $[\sigma] \in H^2_{es}(\mathcal{G};S^1)$ does not depend on the choice of the section $\tau : \mathcal{G} \to R$. We call two S^1-central extensions (R_1,ι_1,π_1) and (R_2,ι_2,π_2) of $\mathcal{G}$ equivalent, if there exists an isomorphism $\phi : R_1 \to R_2$ such that $\iota_2 = \phi \circ \iota_1$ and $\pi_1 = \pi_2 \circ \phi$. Under this equivalence, one gets the following result on the classification on central extensions of a groupoid $\mathcal{G}$ which is s-connected

Theorem 2.1.4. [**85**] *If $\mathcal{G}$ is s-connected, then the isomophism classes of S^1-central extensions of $\mathcal{G}$ are mapped isomorphically to the cohomology group $H^2_{es}(\mathcal{G};S^1)$.*

We denote this cohomology group by $Tw(\mathcal{G}) := H^2_{es}(\mathcal{G}; S^1)$ and for an extension R with 2-cocycle σ we denote $R := \mathcal{G}^\sigma$.

Given a S^1-central extension $\mathcal{G}^\sigma$, we can apply the Lie algebroid functor A to the extension $S^1 \times M \overset{\iota}{\hookrightarrow} \mathcal{G}^\sigma \overset{\pi}{\twoheadrightarrow} \mathcal{G}$ to get a Lie algebroid extension (See [**58**])

$$0 \to \mathbb{R} \times M \overset{\iota}{\hookrightarrow} A(\mathcal{G}^\sigma) \overset{\pi}{\twoheadrightarrow} A(\mathcal{G}) \to 0$$

Here $\mathbb{R} \times M$ is the trivial line bundle over M, viewed as a bundle of abelian Lie algebras, hence an algebroid with zero anchor map. It is central, since it satisfies $[\iota(r), \alpha] = 0$ for all $r \in \Gamma(\mathbb{R} \times M)$ and $\alpha \in \Gamma(A(\mathcal{G}^\sigma))$. This central extension can be split, since any short exact sequence of vector bundles splits, hence we can identify $A(\mathcal{G}^\sigma) \cong A(\mathcal{G}) \oplus (\mathbb{R} \times M)$ as vector bundles. The Lie bracket on $A(\mathcal{G}^\sigma)$ with this identification is

$$[(\alpha, f), (\beta, g)]_{A(\mathcal{G}^\sigma)} = ([\alpha, \beta]_{A(\mathcal{G})}, \mathcal{L}_{\rho(\alpha)}(g) - \mathcal{L}_{\rho(\beta)}(f) + c(\alpha, \beta)),$$

with the anchor $(\alpha, f) \mapsto \rho(\alpha)$. This term c is a Lie algebroid 2-cocycle. This defines a characteristic class map $\Psi : Tw(\mathcal{G}) \to H^2_{Lie}(A(\mathcal{G}))$, $[\sigma] \mapsto [c]$ to Lie algebroid cohomology (See [**39, 15, 85**] for more details).

THEOREM 2.1.5. [**39**] *Let $\mathcal{G}$ be any s-simply connected Lie groupoid, then $\Psi : Tw(\mathcal{G}) \to H^2_{Lie}(A(\mathcal{G}))$ is injective.*

REMARK 2.1.6. We stress here that the cohomology group of a central extension depend on the actual Lie groupoid, up to isomorphism of S^1-central extensions, and not on their associated Morita equivalence class, and hence the differentiable stack which they represent. For instance the Morita equivalence class of the pair groupoid is that of the trivial groupoid, that is the point, as is explained in example A.2.0.31. Therefore the only Morita equivalent S^1-central extensions of the pair groupoid, should be the trivial one. But by the above discussion, there are non-trivial S^1-central extensions of the pair groupoid. As is explained in example A.2.0.32, we will see indeed that $Gauge(P)$ of example 2.1.1 is Morita equivalent to the trivial S^1-central extension of the point. This will be important in the higher geometric picture of geometric quantization, but for the moment one can neglect this.

A S^1-central extension is the Souriau picture of a twist of a Lie groupoid, which is needed in order to construct the convolution product of the C^*-algebra. But there is an equivalent Konstant picture of the twist in terms of Hermitian line bundles. Let Σ be a symplectic groupoid. For Σ^σ a S^1-central extension of Σ, we have the corresponding circle bundle Σ^σ over Σ. Let $\mathbb{L} = \Sigma^\sigma \times_{S^1} \mathbb{C}$ be the associated Hermitian line bundle over Σ. The 2-cocycle σ that characterizes the extension, can also be seen as the multiplication on $\mathbb{L}$, that is the structure for multiplying fibers of $\mathbb{L}$ over different points in Σ. Hence we can think of σ as a section of $\partial^*\mathbb{L}^* := pr_1^*\mathbb{L}^* \otimes m^*\mathbb{L} \otimes pr_2^*\mathbb{L}^*$ over Σ_2. This means that for any composable morphism $(g, h) \in \Sigma_2$ we have a bilinear map $\sigma(g, h) : \mathbb{L}_g \times \mathbb{L}_h \to \mathbb{L}_{gh}$. It is furthermore associative, since the multiplicative coboundary of σ equals 1, remember σ is a cocycle. In order to construct a C^*-algebra we also need a norm and an involution. The norm on the Hermitian line bundle $\mathbb{L}$ is given by the Hermitian inner product, which makes this Hermitian line bundle a Banach bundle. Since a Hermitian line bundle is equivalent to the circle bundle, the cocycle σ must have a norm 1 everywhere. Futhermore, the involution on $\mathbb{L}$ is defined by complex conjugation. This extra structure of a multiplication, norm, and involution on a line bundle over a groupoid can also be summarized into the notion of a Fell bundle. Hence this Hermitian line bundle $\mathbb{L}$ with a norm 1 cocycle σ is a Fell bundle. Fell bundles were first defined by Yamagami in [**94**] where he called them C^*-algebras over groupoids, (see [**49**]).

Now we can turn to prequantization of symplectic groupoids. As said before a prequantization of the symplectic groupoid Σ gives in particular a prequantization of Σ as a symplectic manifold, hence the Hermitian line bundle $\mathbb{L}$ should give a prequantum line bundle over Σ. These structures of a connection and a twist on a Hermitian line bundle should satisfy a compatibility condition in order for the convolution to be compatible with polarization, which we explain in section 4. For now we only need to know that the compatibility condition means that the cocycle σ should be a covariantly constant section of $\partial^*\mathbb{L}^*$ over Σ_2.

DEFINITION 2.1.7. [**39**] A *prequantization of a symplectic groupoid* Σ is a Hermitian line bundle $(\mathbb{L}, \langle,\rangle)$ over Σ equipped with Hermitian connection ∇ and a section $\sigma \in \Gamma(\Sigma_2, \partial^*\mathbb{L}^*)$ such that

(i) The curvature of ∇ equals the symplectic form,
(ii) σ is a cocycle and has norm 1 at every point,
(iii) σ is covariantly constant

We call $\mathbb{L}$ a *multiplicative prequantum line bundle* on Σ.

REMARK 2.1.8. The definition of a multiplicative prequantum bundle can also be defined using a *bundle gerbe with connective structure*, whose base space is a Lie groupoid instead of just a manifold. A bundle gerbe over a manifold is equivalently a central extension of the Čech groupoid of the manifold. But here we are centrally extending Lie groupoids that are in general not Čech groupoid and not Morita equivalent to a manifold. We will come back to this in section 2.4.

The base manifold of a symplectic groupoid is always a Poisson manifold. But for a Poisson manifold (M, π), the Lie algebroid cohomology of T^*M is canonically isomorphic to the Poisson cohomology of (M, π), that is $H^*_{Lie}(T^*M) = H^*_\pi(M)$ (See [**89**]). Together with theorem 2.1.5, the problem of prequantization of a symplectic groupoid, comes down to finding an element of $Tw(\Sigma)$ in the preimage of $\Psi^{-1}[\pi]$. Crainic and Zhu [**19**] improved on this by proving a prequantization condition for Poisson manifolds.

First we note that a Poisson manifold M can be split into a collection of symplectic leaves. This splitting arises as from the foliation of M into leaves where the Poisson bivector has a constant rank. Each leaf of the foliation is itself a symplectic manifold (see [**88**]). Now suppose that $\phi : S^2 \to M$ is a smooth map whose image lies in a single symplectic leaf of a Poisson manifold M. Denote the symplectic form on this symplectic leaf by ω_{Leaf}. The monodromy of ϕ is the first order variation of the integral $\int_{S^2} \phi^*\omega_{Leaf}$. The map ϕ has trivial monodromy if this integral is unchanged to first order if ϕ is perturbed (See [**39**]).

DEFINITION 2.1.9. [**18**] The *periods* of a Poisson manifold M are the integrals

$$\int_{S^2} \phi^*\omega_{Leaf}$$

for such smooth maps $\phi : S^2 \to M$ with trivial monodromy.

THEOREM 2.1.10. [**15, 19**] *For an integrable Poisson manifold M, the symplectic groupoid $\Sigma(M)$ is prequantizable if and only if all the periods of M are integer multiples of 2π. If so, the prequantization of $\Sigma(M)$ is unique.*

Hence the prequantization condition of an integrable Poisson manifolds M is equivalent to what we call the *integrality condition*, that is all the periods of M need to be integer multiples of 2π. This integrality condition holds only for s-simply connected symplectic Lie groupoids. For the case that the symplectic groupoid is not s-simply connected, the prequantization may not be unique. This non-uniqueness is described as follows.

THEOREM 2.1.11. [**39**] *Let Σ be a prequantizable symplectic groupoid over M. Any prequantization of Σ determines a bijection from $H^1(\Sigma, M; S^1)$ to the set of isomorphism classes of prequantizations of Σ.*

For further (less general) prequantization conditions, we refer the reader to the results of Weinstein and Xu [**89**] and Crainic [**15**].

3. Twisted convolution algebra

Now that we have defined the multiplicative prequantum line bundle over a symplectic groupoid, we can twist the convolution algebra by its cocycle. There are two standard ways of constructing a twisted convolution algebra of a Lie groupoid, using either a Haar system or half-densities. In this section we use the half-density approach because it is closer to the Hilbert space construction in geometric quantization of symplectic manifolds. But first we need to define a convolution algebra on a Lie groupoid $\mathcal{G}$.

3.1. The convolution algebra of a Lie groupoid. Let $\mathcal{G}$ be a Lie groupoid. Remember that $T^t_{\mathbb{C}}\,\mathcal{G} := \ker Tt \otimes \mathbb{C}$ and $T^s_{\mathbb{C}}\,\mathcal{G} := \ker Ts \otimes \mathbb{C}$. We will define the half-densities on the bundle $\wedge^{max}(T^{t*}_{\mathbb{C}}\,\mathcal{G} \oplus T^{s*}_{\mathbb{C}}\,\mathcal{G})$, that is

$$\Omega^{1/2} := |\Omega|^{1/2}(\wedge^{max}(T^{t*}_{\mathbb{C}}\,\mathcal{G} \oplus T^{s*}_{\mathbb{C}}\,\mathcal{G}))$$

over $\mathcal{G}$.

REMARK 3.1.1. The fact that we use half-densities instead of half-forms is that for the case that $T^t\,\mathcal{G} \oplus T^s\,\mathcal{G}$ is orientable, then we have a choice of positivity for the square root, in which the half-forms are equivalent to half-densities. This orientability condition is satisfied if the Lie algebroid $A(\mathcal{G})$ is orientable or if $\mathcal{G}$ is s-simply connected. See [**39**].

Denote $\Gamma_c(\mathcal{G}, \Omega^{1/2})$ for the space of smooth compactly supported sections of $\Omega^{1/2}$, then the convolution product between two sections $f, g \in \Gamma_c(\mathcal{G}, \Omega^{1/2})$ should give us another section $f * g \in \Gamma_c(\mathcal{G}, \Omega^{1/2})$. This section is defined by integrating $pr_1^* f\; pr_2^* g$ over the fibers of m. Explicitly for $\gamma \in \mathcal{G}$ a fiber of m is given by $m^{-1}(\gamma) = \{(\eta, \eta^{-1}\gamma) | \eta \in t^{-1}(t(\gamma))\}$.

DEFINITION 3.1.2. The *convolution algebra* of a Lie groupoid $\mathcal{G}$ is the space $\Gamma_c(\mathcal{G}; \Omega^{1/2})$ of smooth compactly supported sections of $\Omega^{1/2}$, with the convolution product $f * g$ of $f, g \in \Gamma_c(\mathcal{G}; \Omega^{1/2})$ defined by

$$(f * g)(\gamma) = \int_{\eta \in t^{-1}(t(\gamma))} f(\eta) g(\eta^{-1}\gamma)$$

and the involution is given by

$$f^*(\gamma) = \overline{f(\gamma^{-1})}$$

.

To show that this definition is well-defined, we first we note that we have the isomorphism

$$T^m\,\mathcal{G}_2 \cong pr_1^* T^t\,\mathcal{G}$$

To see this we note that the sections of $T^m\,\mathcal{G}_2 = \ker Tm$ are tangent to fibers of m, that is for each $\gamma \in \mathcal{G}$ tangent to $m^{-1}(\gamma) = \{(\eta, \eta^{-1}\gamma) | \eta \in t^{-1}(t(\gamma))\}$. But this is parametrized by η and together with $pr_1^* T^t\,\mathcal{G}_{(\eta,\eta^{-1}\gamma)} = T^t_{pr_1(\eta,\eta^{-1}\gamma)}\,\mathcal{G} = T_\eta t^{-1}(t(\eta)) = T_\eta t^{-1}(t(\gamma))$ this gives us fiberwise an isomorphism of vector bundles. Note that we used that for each $\gamma \in \mathcal{G}$ we have $T^t_\gamma\,\mathcal{G} = T_\gamma t^{-1}(t(\gamma))$. Similarly, by the reparametrization $m^{-1}(\gamma) = \{(\gamma\eta^{-1}, \eta) | \eta \in s^{-1}(s(\gamma))\}$ and $pr_2^* T^s\,\mathcal{G}_{(\gamma\eta^{-1},\eta)} = T^s_{pr_2(\gamma\eta^{-1},\eta)}\,\mathcal{G} = T_\eta s^{-1}(s(\eta)) = T_\eta s^{-1}(s(\gamma))$ we have the isomorphism

$$T^m\,\mathcal{G}_2 \cong pr_2^* T^s\,\mathcal{G}$$

Furthermore have the isomorphism

$$m^* T^t\,\mathcal{G} \cong pr_2^* T^t\,\mathcal{G}$$

since we have fiberwise for each $(\gamma_1, \gamma_2) \in \mathcal{G}_2$ that $(m^* T^t\,\mathcal{G})_{(\gamma_1,\gamma_2)} = T^t_{m(\gamma_1,\gamma_2)}\,\mathcal{G} = T_{\gamma_1\gamma_2} t^{-1}(t(\gamma_1))$ and $(pr_2^* T^t\,\mathcal{G})_{(\gamma_1,\gamma_2)} = T^t_{pr_2(\gamma_1,\gamma_2)}\,\mathcal{G} = T_{\gamma_2} t^{-1}(s(\gamma_1))$ which are isomorphic by the tangent map of left multiplication. That is the left multiplication map $L_{\gamma_1} : t^{-1}(s(\gamma_1)) \to t^{-1}(t(\gamma_1))$ for $\gamma_1 \in \mathcal{G}$ induce the map $T_{\gamma_1\gamma_2}L_{\gamma_1^{-1}} : T_{\gamma_1\gamma_2}t^{-1}(t(\gamma_1)) \to T_{\gamma_2}t^{-1}(s(\gamma_1))$, which gives us the fiberwise isomorphism between $m^*T^t\,\mathcal{G}$ and $pr_2^*T^t\,\mathcal{G}$. Similarly we by using right multiplication we get the isomorphism

$$m^*T^s\,\mathcal{G} \cong pr_1^*T^s\,\mathcal{G}$$

Now remember that $\wedge^{max}(T^{t*}_{\mathbb{C}}\,\mathcal{G} \oplus T^{s*}_{\mathbb{C}}\,\mathcal{G}) = \wedge^{max}T^{t*}_{\mathbb{C}}\,\mathcal{G}) \otimes \wedge^{max}T^{s*}_{\mathbb{C}}\,\mathcal{G}$ and hence we can calculate the pullback of $\Omega^{1/2} = |\Omega|^{1/2}(\wedge^{max}(T^{t*}_{\mathbb{C}}\,\mathcal{G} \oplus T^{s*}_{\mathbb{C}}\,\mathcal{G})) = |\Omega|^{1/2}(\wedge^{max}T^{t*}_{\mathbb{C}}\,\mathcal{G}) \otimes |\Omega|^{1/2}(\wedge^{max}T^{s*}_{\mathbb{C}}\,\mathcal{G})$ along projection maps pr_1 and pr_2. That is

$$\begin{aligned}
pr_1^*\Omega^{1/2} &= pr_1^*|\Omega|^{1/2}(\wedge^{max}T^{t*}_{\mathbb{C}}\,\mathcal{G}) \otimes pr_1^*|\Omega|^{1/2}(\wedge^{max}T^{s*}_{\mathbb{C}}\,\mathcal{G}) \\
&= |\Omega|^{1/2}(\wedge^{max}(pr_1^*T^{t}_{\mathbb{C}}\,\mathcal{G})^*) \otimes |\Omega|^{1/2}(\wedge^{max}(pr_1^*T^{s}_{\mathbb{C}}\,\mathcal{G})^*) \\
&= |\Omega|^{1/2}(\wedge^{max}(T^{m}_{\mathbb{C}}\,\mathcal{G}_2)^*) \otimes |\Omega|^{1/2}(\wedge^{max}(m^*T^{s}_{\mathbb{C}}\,\mathcal{G})^*) \\
&= |\Omega|^{1/2}(\wedge^{max}T^{m*}_{\mathbb{C}}\,\mathcal{G}_2) \otimes m^*|\Omega|^{1/2}(\wedge^{max}T^{s*}_{\mathbb{C}}\,\mathcal{G})
\end{aligned}$$

and

$$\begin{aligned}
pr_2^*\Omega^{1/2} &= pr_2^*|\Omega|^{1/2}(\wedge^{max}T^{t*}_{\mathbb{C}}\,\mathcal{G}) \otimes pr_2^*|\Omega|^{1/2}(\wedge^{max}T^{s*}_{\mathbb{C}}\,\mathcal{G}) \\
&= |\Omega|^{1/2}(\wedge^{max}(pr_2^*T^{t}_{\mathbb{C}}\,\mathcal{G})^*) \otimes |\Omega|^{1/2}(\wedge^{max}(pr_2^*T^{s}_{\mathbb{C}}\,\mathcal{G})^*) \\
&= |\Omega|^{1/2}(\wedge^{max}(m^*T^{t}_{\mathbb{C}}\,\mathcal{G})^*) \otimes |\Omega|^{1/2}(\wedge^{max}(T^{m}_{\mathbb{C}}\,\mathcal{G}_2)^*) \\
&= m^*|\Omega|^{1/2}(\wedge^{max}T^{t*}_{\mathbb{C}}\,\mathcal{G}) \otimes |\Omega|^{1/2}(\wedge^{max}T^{m*}_{\mathbb{C}}\,\mathcal{G}_2)
\end{aligned}$$

Which gives us

$$\begin{aligned}
pr_1^*\Omega^{1/2} \otimes pr_2^*\Omega^{1/2} &\cong m^*\Omega^{1/2} \otimes |\Omega|^1(\wedge^{max}T^{m*}_{\mathbb{C}}\,\mathcal{G}_2) \\
&\cong m^*\Omega^{1/2} \otimes pr_1^*|\Omega|^1(\wedge^{max}T^{t*}_{\mathbb{C}}\,\mathcal{G})
\end{aligned}$$

But $|\Omega|^1(\wedge^{max}T^{m*}_{\mathbb{C}}\,\mathcal{G}_2) \cong pr_1^*|\Omega|^1(\wedge^{max}T^{t*}_{\mathbb{C}}\,\mathcal{G})$ is exactly the bundle of volume forms along the fibers of m in $\mathcal{G}_2$. With this isomorphism, the product $pr_1^*f\; pr_2^*g$ as a section of $m^*\Omega^{1/2} \otimes pr_1^*|\Omega|^1(\wedge^{max}T^{t*}_{\mathbb{C}}\,\mathcal{G})$, can be integrated over $t^{-1}(t(\gamma))$ as in the definition of the convolution algebra.

The involution is well-defined since we have the isomorphism

$$i^*\Omega^{1/2} \cong \overline{\Omega^{1/2}}$$

where i is the inversion morphism of $\mathcal{G}$.

3.2. Twisted convolution. In order to define the twisted convolution algebra of a Lie groupoid $\mathcal{G}$ we need a twist, that is a S^1-central extension of a $\mathcal{G}$. This twist determines a Hermitian line bundle $\mathbb{L}$ over the the groupoid $\mathcal{G}$ with a norm 1 cocycle σ with coefficients in $\mathbb{L}$.

Definition 3.2.1. Let $\mathbb{L}$ be a Hermitian line bundle over the groupoid $\mathcal{G}$ with a norm 1 cocycle $\sigma \in \Gamma(\mathcal{G}_2, \partial^*\mathbb{L}^*)$. The twisted convolution algebra is the space $\Gamma_c(\mathcal{G}; \mathbb{L} \otimes \Omega^{1/2})$ of smooth compactly supported sections of $\mathbb{L} \otimes \Omega^{1/2}$, with the twisted convolution product $f * g$ of $f, g \in \Gamma_c(\mathcal{G}; \mathbb{L} \otimes \Omega^{1/2})$ defined by

$$(f * g)(\gamma) = \int_{\eta \in t^{-1}(t(\gamma))} \sigma(\eta, \eta^{-1}\gamma) f(\eta) g(\eta^{-1}\gamma)$$

and involution given by

$$f^*(\gamma) = \overline{\sigma(\gamma, \gamma^{-1}) f(\gamma^{-1})}$$

Although this is constructed with a cocycle σ, the algebra only depends upon the cohomology class of $[\sigma] \in Tw(\mathcal{G})$ (See [**39**]).

4. Polarization of a symplectic groupoid

Now that we have defined the prequantization we can, similar to the case of polarizing a symplectic manifold, polarize the symplectic groupoid. This polarization, due to Hawkins [**39**], will give a polarized twisted convolution algebra, which can be completed to a C^*-algebra.

4.1. Polarization. Underlying this polarization of a symplectic groupoid lies a weaker notion and that is the polarization of Lie groupoid in general. Before we define a polarization of a Lie groupoid, let us first motivate its properties.

Let $\mathcal{G}$ be a Lie groupoid and $\mathcal{P} \subset T_{\mathbb{C}}\,\mathcal{G}$ an involutive distribution. Suppose that we have a $\mathcal{P}$-connection ∇ on the bundle $\mathbb{L} \otimes \Omega^{1/2}$ over $\mathcal{G}$. Later we will slightly modify this bundle together with its convolution product, but for the moment it suffice to motivate the definition of a polarization of a Lie groupoid. Take two polarized sections f and g of $\mathbb{L} \otimes \Omega^{1/2}$, that is for all $X \in \mathcal{P}$ we have $0 = \nabla_X f = \nabla_X g$. It would be natural to require that the convolution product $f * g$ is also a polarized section, since a C^*-algebra is closed under convolution.

The convolution product $f * g$ is defined by integrating $pr_1^* f\ pr_2^* g$ over the fibers of $m : \mathcal{G}_2 \to \mathcal{G}$. Taking the derivative $\nabla_X(f * g)$, for $X \in \mathcal{P}$, corresponds to taking the derivative of $pr_1^* f\ pr_2^* g$ by some vector $\tilde{X} \in T_{\mathbb{C}}\,\mathcal{G}_2$ such that $Tm(\tilde{X}) = X$, assuming here that there exists such a vector $\tilde{X}$. Actually it suffice to take $\tilde{X}$ modulo $T^m_{\mathbb{C}}\,\mathcal{G}_2$, since the integration absorbs differentiation along $T^m_{\mathbb{C}}\,\mathcal{G}_2$. The equality

$$0 = \nabla_{\tilde{X}}(pr_1^* f\ pr_2^* g) = pr_1^*(\nabla_{Tpr_1(\tilde{X})} f)\ pr_2^* g + pr_1^* f\ pr_2^*(\nabla_{Tpr_2(\tilde{X})} g)$$

is now satisfied if $Tpr_1(\tilde{X}), Tpr_2(\tilde{X}) \in \mathcal{P}$. This gives the condition that, if $\tilde{X}$ exists then it expresses $X \in \mathcal{P}$ as a product of two other vectors in $\mathcal{P}$, in other words $\tilde{X} = (Tpr_1(\tilde{X}), Tpr_2(\tilde{X})) \in (\mathcal{P} \times \mathcal{P}) \cap T_{\mathbb{C}}\,\mathcal{G}_2$ if $Tm(\tilde{X}) = X \in \mathcal{P}$.

On the other hand, let $\tilde{X} = (\tilde{X}_1, \tilde{X}_2) \in (\mathcal{P} \times \mathcal{P}) \cap T_{\mathbb{C}}\,\mathcal{G}_2$. Suppose that for any two polarized section f and g of $\mathbb{L} \otimes \Omega^{1/2}$ we have

$$\nabla_X(f * g) = 0$$

where $X = Tm(\tilde{X})$. Now since any element of a C^*-algebra is a finite sum of products, we have for any polarized section h of $\mathbb{L} \otimes \Omega^{1/2}$ that

$$\nabla_X h = 0$$

Hence this suggests that $X \in \mathcal{P}$. This gives the condition that any product of vectors from $\mathcal{P}$ is also in $\mathcal{P}$. These two conditions lead to the following definition for compatibility between $\mathcal{P}$ and the groupoid multiplication, which we call multiplicativity.

Definition 4.1.1. Let $\mathcal{P} \subset T_{\mathbb{C}}\,\mathcal{G}$ be a distribution and denote $\mathcal{P}_2 := (P \times P) \cap T_{\mathbb{C}}\,\mathcal{G}_2$, then $\mathcal{P}$ is *multiplicative* if for any $(\gamma, \eta) \in \mathcal{G}_2$ we have

$$Tm(\mathcal{P}_{2(\gamma,\eta)}) = \mathcal{P}_{\gamma\eta} \subset T_{\mathbb{C},\gamma\eta}\,\mathcal{G}$$

In order to construct a C^*-algebra we also need compatibility with the groupoid inverse. Rememeber the adjoint was defined by $f^* := i^* \bar{f}$, where $i : \mathcal{G} \to \mathcal{G}$ is the groupoid inverse map and $\bar{f}$ is the complex conjugate. For f a polarized section of $\mathbb{L} \otimes \Omega^{1/2}$, we require that f^* is also a polarized section. This implies that for all $X \in \mathcal{P}$ the equality

$$0 = \nabla_X f^* = i^* \overline{\nabla_{Ti\bar{X}} f}$$

is satisfied if $Ti(X) \in \overline{\mathcal{P}}$ and this condition lead to the following definition

Definition 4.1.2. A distribution $\mathcal{P} \subset T_{\mathbb{C}}\,\mathcal{G}$ is *Hermitian* if $Ti(\mathcal{P}) = \overline{\mathcal{P}}$.

With these two properties we can define the polarizatoin of a Lie groupoid and a symplectic groupoid.

DEFINITION 4.1.3. A *polarization of a Lie groupoid* $\mathcal{G}$ is an involutive, multiplicative, Hermitian distribution $\mathcal{P} \subset T_{\mathbb{C}}\,\mathcal{G}$. A *polarization of a symplectic groupoid* Σ is a polarization in both the symplectic and groupoid sense, that is, an involutive, multiplicative, Hermitian, Lagrangian distribution.

The existence and uniqueness of groupoid polarization is rather unknown territory. In [**39**] Hawkins investigate some of its properties. This definition of symplectic groupoid polarization is very restrictive and may not exist in sufficient generality to quantize all Poisson manifolds that should be quantizable. However this approach to quantization via polarization is the optimal scenario that one could get, in the sense that if this type of polarization doesn't work, then it will surely not work for more general polarization.

4.2. Real and strongly admissible polarizations. The manifold of morphisms of the symplectic groupoid is in particular a symplectic manifold and hence the polarization of a symplectic groupoid Σ is in particular a polarization of Σ treated as a symplectic manifold. In the previous chapter we saw already that a polarization for symplectic manifolds is not always well behaved. In order for such a polarization $\mathcal{P}$ to be well-behaved we needed to impose some further conditions. We called a polarization $\mathcal{P}$ strongly admissible, if for $D, E \subset T\Sigma$, defined as in definition 3.1.5, there exists manifolds and surjective submersions

$$\Sigma \overset{p}{\twoheadrightarrow} \Sigma/D \overset{q}{\twoheadrightarrow} \Sigma/E$$

such that D and E are the kernel foliations $D = \ker Tp$ and $E = \ker T(q \circ p)$. But in symplectic groupoid polarization Σ has besides a manifold structure also a groupoid structure, and hence this polarization can be seen as a foliation of a groupoid such that the leaf space Σ/D and Σ/E is also a groupoid. These quotients on a Lie groupoid (together with a groupoid structure) can be described by a certain equivalence relation, such that the equivalence classes form together the quotient Lie groupoid. To define this equivalence relation on a groupoid we need to consider the Cartesian product of this groupoid, which is naturally described by a double groupoid.

A *double groupoid* is a groupoid object in the category Grpd. It consists of a quadruple of sets $(D; H, V; B)$, together with groupoid structures on H and V, both with base B, and two groupoid structures on S, a horizontal structure with base V and a vertical structure with base H, such that the structure maps of each groupoid structure on S are morphisms with respect to the other. Such a double groupoid can be presented by the following diagram

$$\begin{tikzcd} D \arrow[r, shift left, "s_V^D"] \arrow[r, shift right, "t_V^S"'] \arrow[d, shift right, "t_H^D"'] \arrow[d, shift left, "s_H^D"] & V \arrow[d, shift right, "t_V"'] \arrow[d, shift left, "s_V"] \\ H \arrow[r, shift left, "s_H"] \arrow[r, shift right, "t_H"'] & B \end{tikzcd}$$

DEFINITION 4.2.1. [**59**] A *double Lie groupoid* is a double groupoid $(D; H, V; B)$, where D, H, V and H are manifolds and such that all the four groupoid structures are Lie groupoid structures and such that the double source map $(s_H^D, s_V^D) : D \to H \;{}_{s_H}\!\times_{s_V}\; V = \{(h, v) \mid s_H(h) = s_V(v)\}$.

The surjectivity condition on the double source map ensures us that given an $h \in H$ and $v \in V$ with matching sources, there exists an $d \in D$ having these sources h and v. The submersion condition on the double source map guarantees that $D_2^H \rightrightarrows V_2$ and $D_2^V \rightrightarrows H_2$ are both Lie groupoids, where D_2^H, V_2, D_2^V and H_2 are the domain of the corresponding multiplication maps of the four Lie groupoids. This makes D a Lie groupoid in the category of Lie groupoids, see [**59**].

EXAMPLE 4.2.2. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Then the pair groupoid of $\mathcal{G}$ is a double Lie groupoid $Pair(\mathcal{G}) := \mathcal{G} \times \mathcal{G}$, which can be presented by the diagram

$$\begin{tikzcd} \mathcal{G} \times \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\ M \times M \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

A *double Lie subgroupoid* of a double Lie groupoid $(D; H, V; B)$ is a double Lie groupoid $(D'; H', V'; B')$ such that $D' \rightrightarrows H'$, $D' \rightrightarrows V'$, $H' \rightrightarrows B'$, $V' \rightrightarrows B'$ are respectively Lie subgroupoids of $D \rightrightarrows H$, $D \rightrightarrows V$, $H \rightrightarrows B$, $V \rightrightarrows B$.

DEFINITION 4.2.3. A *smooth congruence* of a Lie groupoid $\mathcal{G}$ is a closed, embedded sub double Lie groupoid of $Pair(\mathcal{G})$ that is wide over $\mathcal{G}$.

Hence a smooth congruence on $\mathcal{G}$ consisting of a pair (S, R) can be described by the following diagram

$$\begin{tikzcd} S \arrow[d, shift left] \arrow[d, shift right] & \subseteq & \mathcal{G} \times \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\ R & \subseteq & M \times M \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

For any such smooth congruence (S, R) on a Lie groupoid $\mathcal{G} \rightrightarrows M$, there exist a unique Lie groupoid structure on the the quotients sets $\mathcal{G}' = \mathcal{G}/S$, $M' = M/R$, such that the natural projections $F : \mathcal{G} \to \mathcal{G}'$, $f : M \to M'$ form a morphism of Lie groupoids. In fact (F, f) form a fibration.

DEFINITION 4.2.4. A *fibration* of Lie groupoids is a morphism of Lie groupoids $F : \mathcal{G} \to \mathcal{G}'$, $f : M \to M'$ such that the base map f and the map

$$(F, s) : \mathcal{G} \to \mathcal{G}' \;{}_t\!\times_f M = \{(g, x) \mid t(g) = f(x)\}$$

are surjective submersions.

In this way the smooth congruence (S, R) defines a quotient Lie groupoid $\mathcal{G}/S \rightrightarrows M/R$ such that the natural projections (F, f) from the Lie groupoid $\mathcal{G} \rightrightarrows M$ is a fibration. Conversely every fibration $F : \mathcal{G} \to \mathcal{G}'$, $f : M \to M'$ of Lie groupoids determines a smooth congruence $(R(F), R(f))$ on $\mathcal{G}$. Hence the notion of a smooth congruence and fibration are equivalent, which is proven in detail in [**41**].

DEFINITION 4.2.5. A *real polarization* (of a symplectic manifold, groupoid, or symplectic groupoid) is a real distribution $\mathcal{P}$ whose complexification $\mathcal{P}_{\mathbb{C}}$ is a polarization.

Remember any polarization that satisfies $\bar{\mathcal{P}} = \mathcal{P}$ is the complexification of a real polarization and hence we regard a real polarization as a special case of complex polarization.

A real polarization $\mathcal{P} \subset T\mathcal{G}$ is in particular a foliation of $\mathcal{G}$ and we want that the leaf space of this foliation has beside a manifold structure also a groupoid structure. Now we first notice that $T\mathcal{G}$ can be seen as a Lie algebroid-groupoid associated to a the double Lie groupoid $Pair(\mathcal{G})$. A Lie algebroid-groupoid arises from a single application of the Lie functor to a double Lie groupoid.

DEFINITION 4.2.6. [**60**] A *Lie algebroid-groupoid* is given by the following data

$$
\begin{tikzcd}
\Omega \arrow[rr, shift left, "t_\Omega"] \arrow[rr, shift right, "s_\Omega"'] \arrow[dr, "\rho_\Omega"] \arrow[dd, "q_\Omega"'] & & A \arrow[dr, "\rho_A"] \arrow[dd, "q_A"] & \\
& T\mathcal{G} \arrow[rr, shift left, "Tt_\mathcal{G}"] \arrow[rr, shift right, "Ts_\mathcal{G}"'] \arrow[dl, "\pi_\mathcal{G}"] & & TM \arrow[dl, "\pi_M"] \\
\mathcal{G} \arrow[rr, shift left, "t_\mathcal{G}"] \arrow[rr, shift right, "s_\mathcal{G}"'] & & M &
\end{tikzcd}
$$

where $\mathcal{G}$ is a Lie groupoid on M and A is a Lie algebroid on M, and where Ω has both a Lie algebroid structure on base $\mathcal{G}$ and a Lie groupoid structure on base A, such that the structure maps of the Lie groupoid structure on Ω are Lie algebroid morphisms and such that the double source map $(q_\Omega, s_\Omega) : \Omega \to \mathcal{G}\ {}_{q_A}\!\times_{s_A} A$ is a surjective submersion.

Now consider a double Lie groupoid $(D; H, V, B)$ then we can apply the Lie functor A to the vertical groupoid structure on D and V to get a *vertical Lie algebroid-groupoid*

$$
\begin{tikzcd}
A(D) \arrow[rr, shift left, "s_{A(D)}"] \arrow[rr, shift right, "t_{A(D)}"'] \arrow[dr, "\rho_{A(D)}"] \arrow[dd, "q_{A(D)}"'] & & A(V) \arrow[dr, "\rho_{A(V)}"] \arrow[dd, "q_{A(V)}"] & \\
& TH \arrow[rr, shift left, "Ts_H"] \arrow[rr, shift right, "Tt_H"'] \arrow[dl, "\pi_H"] & & TB \arrow[dl, "\pi_B"] \\
H \arrow[rr, shift left, "s_H"] \arrow[rr, shift right, "t_H"'] & & B &
\end{tikzcd}
$$

Similarly we can apply the Lie functor A to the horizontal groupoid structure on D and H to get a *horizontal Lie algebroid-groupoid*[**60**]

$$
\begin{tikzcd}
A(D) \arrow[rr, "q_{A(D)}"] \arrow[dr, "\rho_{A(D)}"] \arrow[dd, shift right, "t_{A(D)}"'] \arrow[dd, shift left, "s_{A(D)}"] & & V \arrow[dd, shift right, "t_V"'] \arrow[dd, shift left, "s_V"] \\
& TV \arrow[ur, "\pi_V"'] \arrow[dd, shift right, "Ts_V"'] \arrow[dd, shift left, "Tt_V"] & \\
A(H) \arrow[rr, "q_{A(H)}"] \arrow[dr, "\rho_{A(H)}"'] & & B \\
& TB \arrow[ur, "\pi_B"'] &
\end{tikzcd}
$$

EXAMPLE 4.2.7. Let $\mathcal{G}$ be a Lie groupoid on M and consider the double Lie groupoid $Pair(\mathcal{G})$ as in the example above. Then the vertical Lie algebroid-groupoid and horizontal Lie algebroid-groupoid are diagrammatically given by,

$$\begin{tikzcd}
A(\mathcal{G}) \times A(\mathcal{G}) \arrow[rr, Rightarrow] \arrow[dd] \arrow[dr] & & A(\mathcal{G}) \arrow[dd] \arrow[dr] & \\
& TM \times TM \arrow[rr, Rightarrow] \arrow[dl] & & TM \arrow[dl] \\
M \times M \arrow[rr, Rightarrow, "t_H"'] & & M &
\end{tikzcd}
\qquad
\begin{tikzcd}
T\,\mathcal{G} \arrow[rr] \arrow[dd, shift left] \arrow[dd, shift right] \arrow[dr] & & \mathcal{G} \arrow[dd, shift left] \arrow[dd, shift right] \\
& T\,\mathcal{G} \arrow[ur] \arrow[dd, shift left] \arrow[dd, shift right] & \\
TM \arrow[rr] \arrow[dr] & & M \\
& TM \arrow[ur] &
\end{tikzcd}$$

respectively. Remember $A(\mathcal{G} \times \mathcal{G}) = T^s(\mathcal{G} \times \mathcal{G})|_{\mathcal{G}} \cong T\,\mathcal{G}$ and similarly $A(M \times M) \cong TM$. Hence the tangent bundle $T\,\mathcal{G}$ can be seen as a Lie algebroid-groupoid from the double groupoid $Pair(\mathcal{G})$.

A *sub Lie algebroid-groupoid* of a Lie algebroid-groupoid is a subset that is both a Lie subalgebriod and a Lie subgroupoid and subsequently is automatically itself a Lie algebroid-groupoid. Recall that a Lie subalgebroid $A' \to M'$ of a Lie algebroid $A \to M$ is a morphism of Lie algebroids such that $M' \subset M$ is a submanifold and $A' \to M'$ is a subbundle of the bundle of the restriction of $A \to M$ to M'. A *wide* Lie subalgebroid is a one that shares the same base manifold.

THEOREM 4.2.8. *A real polarization of a Lie groupoid $\mathcal{G}$ is precisely a wide sub Lie algebroid-groupoid $\mathcal{P} \subset T\,\mathcal{G}$.*

PROOF. A real polarization is a real involutive multiplicative, Hermitian, distribution $\mathcal{P} \subset T\,\mathcal{G}$. The property that it is a real involutive distribution corresponds precisely with a wide Lie subalgebroid of the tangent bundle $T\,\mathcal{G}$. The property of multiplicativity and Hermiticity corresponds with the fact that $\mathcal{P} \subset T\,\mathcal{G}$ is also a Lie subgroupoid. $\square$

A real polarization $\mathcal{P} \subset T\,\mathcal{G}$ is in particular a foliation of $\mathcal{G}$ and hence if $\mathcal{P}$ as a Lie algebroid-groupoid integrates to a smooth congruence, then the leaf space of this foliation is precisely a quotient Lie groupoid, which we denote by $\mathcal{G}\,/\,\mathcal{P}$. This suggest a definition of a strongly admissible polarization in terms of Lie groupoids.

DEFINITION 4.2.9. [**39**] For a polarization $\mathcal{P} \subset T_{\mathbb{C}}\,\mathcal{G}$ of a Lie groupoid $\mathcal{G}$, the distributions $\mathcal{D}, \mathcal{E} \subset T\,\mathcal{G}$ are defined by $\mathcal{D}_{\mathbb{C}} := \mathcal{P} \cap \bar{\mathcal{P}}$ and $\mathcal{E}_{\mathbb{C}} := \mathcal{P} + \bar{\mathcal{P}}$. The polarization $\mathcal{P}$ is *strongly admissible* if there exist Lie groupoids and fibrations

$$\mathcal{G} \overset{p}{\twoheadrightarrow} \mathcal{G}\,/\,\mathcal{D} \overset{q}{\twoheadrightarrow} \mathcal{G}\,/\,\mathcal{E}$$

such that $\mathcal{D}$ and $\mathcal{E}$ are $\mathcal{D} = \ker Tp$ and $\mathcal{E} = \ker T(q \circ p)$.

In particular, if $\mathcal{D}$ has constant rank, then it is a real polarization of $\mathcal{G}$. Furthermore, if $\mathcal{P}$ is a strongly admissible polarization, then $\mathcal{D}$ and $\mathcal{E}$ are themselves strongly admissible real polarizations. But for the case where $\mathcal{P}$ is a symplectic groupoid polarization we have that $\mathcal{D}$ and $\mathcal{E}$ will not be symplectic groupoid polarizations, unless $\mathcal{D} = \mathcal{E} = \mathcal{P}$.

The main motivation for considering strongly admissible polarization is that this condition makes the polarization well-behaved, in the sense that we can consider the space of leaves of the underlying real polarization $\mathcal{D}$ as a quotient groupod. In the case that we quantized a ordinary symplectic manifold M, we considered not the polarized section of the prequantum line bundle over M but over the space M/D of leaves of the polarization. Similarly we want to consider here sections of the multiplicative prequantum line bundle over the quotient groupoid $\mathcal{G}\,/\,\mathcal{D}$. This is what we will explore in more detail in the next sections. We like to emphasize that not all polarizations are real and strongly admissible, but a lot of polarization are, since any kernel foliation $\mathcal{P} = \ker TF$ of a fibration F is a strongly admissible real polarization.

5. Twisted polarized convolution algebra

In section 3 we constructed a twisted convolution algebra of a general Lie groupoid $\mathcal{G}$, using half-densities. The convolution algebra was defined for certain sections of the half-density bundle $\Omega^{1/2}$ over $\mathcal{G}$ and the twisted convolution algebra was defined for certain section of the bundle $\mathbb{L} \otimes \Omega^{1/2}$ where $\mathbb{L}$ was the Hermitian line bundle of the twist of the Lie groupoid. In the presence of a strongly admissible polarization of $\mathcal{G}$, we can restrict this to sections that are polarized, which we will turn to now.

Let $\mathcal{G}$ be a Lie groupoid and $\mathcal{P} \subset T\,\mathcal{G}$ a strongly admissible real polarization of $\mathcal{G}$. Then we have that $\mathcal{P}$ is the kernel foliation of the fibration

$$p : \mathcal{G} \to \mathcal{G} \,/\, \mathcal{P}$$

Instead of considering the convolution algebra of $\mathcal{G}$ we want to consider the convolution algebra of the quotient Lie groupoid $\mathcal{G} \,/\, \mathcal{P}$. In [**39**] Hawkins defines the *polarized convolution algebra* of any strongly admissible real polarization as the convolution algebra

$$C^*_{\mathcal{P}}(\mathcal{G}) := C^*(\mathcal{G} \,/\, \mathcal{P})$$

An element of this polarized convolution algebra is a section of the half-density bundle $\Omega^{1/2}$ over $\mathcal{G} \,/\, \mathcal{P}$, which should correspond to a polarized section of $p^*\Omega^{1/2}$. Remember that the half-density bundle on $\mathcal{G} \,/\, \mathcal{P}$ was defined by $\Omega^{1/2} = |\Omega|^{1/2}(\wedge^{max}(T^{t*}_{\mathbb{C}}(\mathcal{G} \,/\, \mathcal{P}) \oplus T^{s*}_{\mathbb{C}}(\mathcal{G} \,/\, \mathcal{P})))$. The pullback of this bundle along p gives

$$p^*\Omega^{1/2} \cong \Omega^{1/2}_{\mathcal{P}} := |\Omega|^{1/2}(\wedge^{max}(T^t\,\mathcal{G} \,/ (T^t\,\mathcal{G} \cap \mathcal{P}) \oplus T^s\,\mathcal{G} \,/ (T^s\,\mathcal{G} \cap \mathcal{P}))^*_{\mathbb{C}})$$

Where we used that $p^*T^t(\mathcal{G} \,/\, \mathcal{P}) \cong T^t\,\mathcal{G} \,/ (T^t\,\mathcal{G} \cap \mathcal{P})$ and $p^*T^s(\mathcal{G} \,/\, \mathcal{P}) \cong T^s\,\mathcal{G} \,/ (T^s\,\mathcal{G} \cap \mathcal{P})$. Similarly to the result in section 3 we have that

$$pr_1^*\Omega^{1/2}_{\mathcal{P}} \otimes pr_2^*\Omega^{1/2}_{\mathcal{P}} \cong m^*\Omega^{1/2}_{\mathcal{P}} \otimes pr_1^*|\Omega|^1(\wedge^{max}(T^t\,\mathcal{G} \,/ (T^t\,\mathcal{G} \cap \mathcal{P}))^*_{\mathbb{C}})$$

where on the left we tensor with the pullback $pr1^*$ of the bundle of volume forms along the t-fibers that are transverse to $\mathcal{P}$. Hence for two global polarized sections $f, g \in \Gamma(\mathcal{G}, \Omega^{1/2}_{\mathcal{P}})$, meaning the pullbacks of sections over $\mathcal{G} \,/\, \mathcal{P}$, that are compactly supported modulo $\mathcal{P}$, the polarized convolution can be defined similarly to the integral of definition 3.1.2, only we do not integrate over the fiber $t^{-1}(t(\gamma))$ but over the quotient space, since the integrand is constant along $\mathcal{P}$-fibers.

We can extend this definition of $\Omega^{1/2}_{\mathcal{P}}$ to an arbitrary strongly admissible polarization $\mathcal{P}$, by defining the half-density bundle as

$$\Omega^{1/2}_{\mathcal{P}} := |\Omega|^{1/2}(\wedge^{max}(T^t_{\mathbb{C}}\,\mathcal{G} \,/ (T^t_{\mathbb{C}}\,\mathcal{G} \cap \mathcal{P}) \oplus T^s_{\mathbb{C}}\,\mathcal{G} \,/ (T^s_{\mathbb{C}}\,\mathcal{G} \cap \mathcal{P}))^*)$$

provided that $T^t\,\mathcal{G} \cap \mathcal{P}$ has constant rank. If it does not have constant rank then $T^t\,\mathcal{G} \cap \mathcal{P}$ is not a bundle, and hence $\Omega^{1/2}_{\mathcal{P}}$ is not defined. Furthermore we require that $\Omega^{1/2}_{\mathcal{P}}$ satisfies

$$i^*\Omega^{1/2}_{\mathcal{P}} \cong \overline{\Omega^{1/2}_{\mathcal{P}}}$$

where i is the groupoid inverse.

REMARK 5.0.10. This half-density bundle $\Omega^{1/2}_{\mathcal{P}}$ may not be unique, since there is a choice of square root involved. Any other choice is obtained by tensoring the bundle $\Omega^{1/2}_{\mathcal{P}}$ with a real line bundle which is isomorphic to its pullback by the groupoid inverse (see [**39**]).

This half-density bundle $\Omega^{1/2}_{\mathcal{P}}$ also carries a natural flat $\mathcal{P}$-connection and our previous definition, for the case of a strongly admissible real polarization, of polarized section as the pullback by p of a section of $\mathcal{G} \,/\, \mathcal{P}$ can be expressed in terms of this connection. Due to Hawkins we have the following

THEOREM 5.0.11. [**39**] *If $\mathcal{P}$ is a polarization of a Lie groupoid $\mathcal{G}$, then the $\mathcal{P}$-Bott connection induces a natural flat $\mathcal{P}$-connection ∇ on $\Omega^{1/2}_{\mathcal{P}}$ whenever this bundle is defined.*

With this connection we can speak of *polarized sections* of $\Omega^{1/2}_{\mathcal{P}}$ in the ordinary sense, that is if $U \subset \mathcal{G}$ is an open subset, then $f \in \Gamma(U, \Omega^{1/2}_{\mathcal{P}})$ is *polarized* if $\nabla_X f = 0$ for any $X \in \Gamma(U, \mathcal{P})$. This definition recovers our previous definition in terms of pullback and extends to arbitrary strongly admissible polarizations.

However, for arbitrary strongly admissible polarizations there is a pitfall in defining the polarized convolution of two polarized sections, since we do not have in general that $\mathcal{P} = \mathcal{D} = \mathcal{E}$, as in the real case. For two polarized sections $f, g \in \Gamma(U, \Omega^{1/2}_{\mathcal{P}})$ the polarized convolution should be defined by some integral over the integrand of definition 3.1.2

$$f(\eta)g(\eta^{-1}\gamma)$$

where $\eta \in t^{-1}(t(\gamma))$. Since the sections f and g are polarized, the integrand is covariantly constant along the foliation $\mathcal{D}$ restricted to $t^{-1}(t(\gamma))$, and hence we should integrate over the quotient by $\mathcal{D}$, instead of the quotient by $\mathcal{P}$ as in the real case. In order to integrate over the quotient one would like to have an isomorphism

$$pr_1^*\Omega^{1/2}_{\mathcal{P}} \otimes pr_2^*\Omega^{1/2}_{\mathcal{P}} \cong m^*\Omega^{1/2}_{\mathcal{P}} \otimes pr_1^*|\Omega|^1(\wedge^{max}(T^t\,\mathcal{G}\,/(T^t\,\mathcal{G} \cap \mathcal{D}))^*_{\mathbb{C}})$$

Instead one has the natural isomorphism[**39**]

$$pr_1^*\Omega^{1/2}_{\mathcal{P}} \otimes pr_2^*\Omega^{1/2}_{\mathcal{P}} \cong m^*\Omega^{1/2}_{\mathcal{P}} \otimes pr_1^*|\Omega|^1(\wedge^{max}(T^t\,\mathcal{G}\,/(T^t\,\mathcal{G} \cap \mathcal{D}))^*_{\mathbb{C}}) \otimes pr_1^*|\Omega|^1(\wedge^{max}(T^t\,\mathcal{G}\,/(T^t\,\mathcal{G} \cap \mathcal{E}))^*_{\mathbb{C}})$$

This may not always reduce to the isomorphism we would like to have, but in the case that we have a symplectic structure on the Lie groupoid it does reduce, by the following result.

THEOREM 5.0.12. [**39**] *If $\mathcal{P}$ is a polarization of a symplectic groupoid Σ and $T^t_{\mathbb{C}}\Sigma \cap \mathcal{P}$ has a constant rank, then for some $k \in \mathbb{N}$ the contraction with $\frac{\omega^k}{k!}$ gives an isomorphism*

$$\wedge^{max}(T^t\Sigma \cap \mathcal{E}) \xrightarrow{\sim} \wedge^{max}(T^t\Sigma \cap \mathcal{D})$$

This isomorphism will induce the isomorphism

$$pr_1^*\Omega^{1/2}_{\mathcal{P}} \otimes pr_2^*\Omega^{1/2}_{\mathcal{P}} \cong m^*\Omega^{1/2}_{\mathcal{P}} \otimes pr_1^*|\Omega|^1(\wedge^{max}(T^t\Sigma/(T^t\Sigma \cap \mathcal{D}))^*_{\mathbb{C}})$$

which would make the integration meaningful.

However in order to define the polarized convolution on the symplectic groupoid, we need to have suitable global polarized sections, which in general may not exist, this is illustrated by the following example.

EXAMPLE 5.0.13. Consider the pair groupoid $Pair(\mathbb{T}^2)$ of the complex torus $\mathbb{T}^2$ where we take the reversed complex structure on the second factor. Take for the polarization the antiholomorphic tangent bundle, that is $\mathcal{P} = T_{0,1}\mathbb{T}^2 \oplus T_{1,0}\mathbb{T}^2 \subset T_{\mathbb{C}}Pair(\mathbb{T}^2)$. Explicitly we have $T^t_{\mathbb{C}}Pair(\mathbb{T}^2) = T_{\mathbb{C}}\mathbb{T}^2 \oplus 0$ and $T^s_{\mathbb{C}}Pair(\mathbb{T}^2) = 0 \oplus T_{\mathbb{C}}\mathbb{T}^2$ and hence $T^t_{\mathbb{C}}Pair(\mathbb{T}^2) \cap \mathcal{P} = T_{0,1}\mathbb{T}^2 \oplus 0$ and $T^s_{\mathbb{C}}Pair(\mathbb{T}^2) \cap \mathcal{P} = 0 \oplus T_{1,0}\mathbb{T}^2$, which gives

$$\Omega^{1/2}_{\mathcal{P}} = |\Omega|^{1/2} \wedge^{max} (T_{1,0}\mathbb{T}^2 \oplus T_{0,1}\mathbb{T}^2)^*$$

A global section of this bundle is equivalent to a holomorphic function on $\mathbb{T}^4$. By an implication of Liouville's theorem, there are no non-constant global section of a holomorphic functions on a compact complex manifold, hence every global sections of $\Omega^{1/2}_{\mathcal{P}}$ has to be a constant, which implies that $\Omega^{1/2}_{\mathcal{P}}$ is the trivial bundle. Thus global polarized sections of $\Omega^{1/2}_{\mathcal{P}}$ are not suitable for defining a convolution algebra. However if $\Omega^{1/2}_{\mathcal{P}}$ is twisted by a line bundle with positive curvature, there will exist many suitable holomorphic sections[**39**].

This lead to the twisted polarized convolution algebra, which we denote by $C^*_{\mathcal{P}}(\Sigma, \sigma)$. Let $\mathcal{P}$ be a strongly admissible polarization of a symplectic groupoid Σ and $(\sigma, \mathbb{L}, \nabla)$ a prequantization. The connection on $\mathbb{L}$ with the partial connection on $\Omega^{1/2}_{\mathcal{P}}$ give together a flat $\mathcal{P}$-connection on $\mathbb{L} \otimes \Omega^{1/2}_{\mathcal{P}}$, which defines locally the polarized sections. In the definition 2.1.7 of a prequantization of Σ we added the compatibility condition of the connection ∇ with the twist σ. This compatibility condition was needed in order for

$$\sigma(\eta, \eta^{-1}\gamma) f(\eta) g(\eta^{-1}\gamma)$$

to be covariantly $\mathcal{D}$-constant as a function of $\eta \in t^{-1}(t(\gamma))$, where $f, g \in \Gamma(\Sigma, \mathbb{L} \otimes \Omega^{1/2}_{\mathcal{P}})$ are polarized sections. The *twisted convolution* $f * g$ is now defined as the integration of $\sigma(\eta, \eta^{-1}\gamma) f(\eta) g(\eta^{-1}\gamma)$ over the quotient, whenever it make sense. The integration only make sense by choosing a suitable fall-off condition. Remember that without a polarization the convolution algebra was defined with compactly supported sections. For a strongly admissible polarization, compactly supported sections will only exist if the leaves of the foliation $\mathcal{E}$ are compact. In some cases a weaker fall-off condition may also be suitable for producing the same C^*-algebra, but a general prescription for this weaker fall-off condition is still not at hand. Furthermore the space of polarized sections for a strongly admissible polarization may be too small in some cases. This may be resolved by taking the total sheaf cohomology of polarized sections of $\mathbb{L} \otimes \Omega^{1/2}_{\mathcal{P}}$ into account as is suggested by the idea of "cohomological wave functions", however the lack of the inner product of cohomological wave functions has to be resolved first. In any case if $\mathcal{P}$ is strongly admissible with $\Omega^{1/2}_{\mathcal{P}}$ a bundle such that the higher degree cohomology of polarized sections of $\mathbb{L} \otimes \Omega^{1/2}_{\mathcal{P}}$ vanishes, then the convolution algebra should consist of global polarized sections[**39**].

6. Real polarization and Bohr-Sommerfeld condition

Let $\mathcal{P}$ be a strongly admissible real polarization of a symplectic groupoid Σ, with $p : \Sigma \to \Sigma/\mathcal{P}$ the quotient fibration and let $(\sigma, \mathbb{L}, \nabla)$ be a prequantization of Σ. In the case that the leaves of $\mathcal{P}$ are simply connected we have that the connection trivializes $\mathbb{L}$ along these leaves, which means that there is a line bundle $\mathbb{L}_0$ over $\Sigma/\mathcal{P}$ such that $\mathbb{L} \cong p^*\mathbb{L}_0$. Since the line bundle $\mathbb{L}$ carries a twist σ the line bundle $\mathbb{L}_0$ carries also a twist σ_0 such that

$$C^*_{\mathcal{P}}(\Sigma, \sigma) \cong C^*(\Sigma/\mathcal{P}, \sigma_0)$$

This cocycle with coefficients in $\mathbb{L}_0$ we call the *reduced cocycle*. If furthermore the line bundle $\mathbb{L}$ is trivializable, which happens in a lot of examples, then the connection can be given by a 1-form $\theta \in \Omega^1(\Sigma)$ as $\nabla = d + i\theta$. There exists a particular interesting class of 1-forms which makes it very easy to compute the reduced cocycles for the corresponding prequantization.

Definition 6.0.14. [**39**] A *symplectic potential* is a 1-form $\theta \in \Omega^1(\Sigma)$ such that $d\theta = -\omega$ and $1^*\theta = 0$. It is *adapted* if it is conormal to the polarization, i.e. $\theta \in \Gamma(\Sigma, \mathcal{P}^\perp)$.

Lemma 6.0.15. [**39**] *Let $\mathcal{P}$ be a strongly admissible real polarization of a symplectic groupoid Σ, with $p : \Sigma \to \Sigma/\mathcal{P}$ the quotient fibration. If θ is an adapted symplectic potential, then $\mathbb{L}_0$ is trivial and the reduced cocycle $\sigma_0 \in \Gamma([\Sigma/\mathcal{P}]_2, U(1))$ is given (up to a locally constant phase) by*

$$p^*(\sigma_0^{-1} d\sigma_0) = i\partial^*\theta$$

Usually there exists a real function $\phi \in C^\infty([\Sigma/\mathcal{P}]_2)$ such that

$$dp^*\phi = \partial^*\theta$$

and $\sigma_0 = e^{i\phi}$.

Remark 6.0.16. To compute the algebra up to isomorphism we only need the cohomology class of $[\sigma_0] \in Tw(\Sigma/\mathcal{P})$ such that $Tw(p)[\sigma_0] \simeq [\sigma]$.

Similar to the situation encountered in the construction of a Hilbert space, if the leaves of the polarizations are not simply connected, then $\mathbb{L}$ will typically have non-trivial holonomy around non-contractable loops in these leaves. In this case, there is an open set over which the smooth polarized sections of $\mathbb{L} \otimes \Omega^{1/2}_{\mathcal{P}}$ must vanish. This leads to the following definition.

DEFINITION 6.0.17. The *Bohr-Sommerfeld subgroupoid* $\Sigma_{B-S} \subset \Sigma$ is the set of points through which $\mathbb{L} \otimes \Omega^{1/2}_{\mathcal{P}}$ has trivial holonomy. The *reduced groupoid* it the quotient $\Sigma_{B-S}/\mathcal{P}$.

More generally, if $\mathcal{P}$ is a strongly admissible polarization, that is not necessarily real, then the Bohr-Sommerfeld conditions come from holonomy around the leaves of $\mathcal{D}$. We have also a Bohr-Sommerfeld subgroupoid Σ_{B-S} and a reduced groupoid $\Sigma_{B-S}/\mathcal{D}$.

7. C^*-algebra

The final step is to complete the twisted polarized convolution algebra $C^*_{\mathcal{P}}(\Sigma, \sigma)$ to a C^*-algebra. This algebra is a generalization of the convolution algebra of a groupoid, and hence we expect that there will be more than one natural way of completing it. We explain here that we can construct both the maximal and the reduced C^*-algebras from the twisted polarized convolution algebra, which might give a suitable quantization.

Assume from now on that the algebras are defined over the field $\mathbb{C}$.

DEFINITION 7.0.18. A *Banach algebra* is a Banach space $(\mathcal{A}, \|\|)$ with an associative algebra structure, such that for all $x, y \in \mathcal{A}$ one has: $\|xy\| \leq \|x\|\|y\|$.

A morphism of Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is a bounded linear map $\phi : \mathcal{A} \to \mathcal{B}$ satisfying $\phi(ab) = \phi(a)\phi(b)$.

DEFINITION 7.0.19. Let $\mathcal{A}$ be an algebra. An *involution* on $\mathcal{A}$ is an antilinear operator $^* : \mathcal{A} \to \mathcal{A}$ (i.e. for $x, y \in \mathcal{A}$, $\lambda, \mu \in \mathbb{C}$, one has $(\lambda x + \mu y)^* = \bar{\lambda} x^* + \bar{\mu} y^*$), such that for all $x, y \in \mathcal{A}$, one has $(xy)^* = y^* x^*$, and $x^{**} = x$. An algebra equipped with an involution is called a **-algebra.*

DEFINITION 7.0.20. A *C^*-algebra* is a Banach algebra $\mathcal{A}$ with an involution *, such that for all $x \in \mathcal{A}$, $\|x\|^2 = \|x^* x\|$. This norm $\|\|$ satisfying this condition is also called a C^*-norm.

A morphism of C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is a morphism of Banach algebras $\phi : \mathcal{A} \to \mathcal{B}$ satisfying $\phi(a^*) = \phi(a)^*$, which is also called a **-homomorphism.*

EXAMPLE 7.0.21. Suppose X is a locally compact Hausdorff space. The space $C_0(X)$ of continuous functions on X that vanish at infinity, form a commutative C^*-algebra under pointwise multiplication of functions and involution given by $f^*(x) := \overline{f(x)}$. The norm is the supremum norm of functions (i.e. $\|f\| = \sup\{|f(x)| : x \in X\}$). Conversely, it is known that if $\mathcal{A}$ is a commutative C^*-algebra then there is a locally compact Hausdorff space X, such that $\mathcal{A}$ is isometrically isomorphic to the algebra $C_0(X)$. In fact it can be shown that the correspondence $X \mapsto C(X)$ is a faithful contravariant functor from the category of compact Hausdorff spaces (with morphisms being continuous maps) to that of commutative C^*-algebras with unit.

EXAMPLE 7.0.22. An example of noncommutative C^*-algebra is $\mathcal{B}(\mathcal{H})$, the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. The multiplication is given by composition, the involution is given by taking the the adjoint, and the norm is the operator norm (i.e. $\|P\| := \sup\{\|P(h)\| : h \in \mathcal{H}, \|h\| = 1\}$ for all $P \in \mathcal{B}(\mathcal{H})$).

DEFINITION 7.0.23. For A a *-algebra and $\mathcal{H}$ a Hilbert space, a **-representation* of A on $\mathcal{H}$ is a *-homomorphism $\pi : A \to \mathcal{B}(\mathcal{H})$. A *-representation π is said to be *faithful* if $\ker(\pi) = 0$.

For A a C^*-algebra we call a *-representation of A just a *representation.*

Theorem 7.0.24. *(The Gelfand-Naimark representation theorem) Any C^*-algebra has a faithful representation.*

Example 7.0.25. Suppose $\mathcal{H}$ is a Hilbert space. For elements $h, h' \in \mathcal{H}$ one can define the operator $h\langle h', .\rangle : \mathcal{H} \to \mathcal{H}$. All the operators of this form generate a *-subalgebra of $\mathcal{B}(\mathcal{H})$. The norm-closure of this algebra is the C^*-algebra $\mathcal{K}(\mathcal{H})$ of compact operators. A given operator $T \in \mathcal{B}(\mathcal{H})$ is compact iff T maps the unit ball in $\mathcal{H}$ to a set with compact closure.

Let $C^*_{\mathcal{P}}(\Sigma, \sigma)$ be the twisted polarized convolution algebra, which is in particular a *-algebra. To make it a C^*-algebra we need to define a C^*-norm on $C^*_{\mathcal{P}}(\Sigma, \sigma)$. Let $\mathcal{R}$ be the set of all *-representations π for which π is continuous when $C^*_{\mathcal{P}}(\Sigma, \sigma)$ has the inductive limit topology and $\mathcal{B}(\mathcal{H})$ the weak operator topology and such that the linear span $\{\pi(f)\xi \,:\, f \in C^*_{\mathcal{P}}(\Sigma, \sigma), \xi \in \mathcal{H}\}$ is dense in $\mathcal{H}$. One defines the maximal C^*-norm of $f \in C^*_{\mathcal{P}}(\Sigma, \sigma)$ by [**69**]

$$\|f\|_{max} := \sup_{\pi \in \mathcal{R}} \|\pi(f)\|$$

Definition 7.0.26. The completion of $C^*_{\mathcal{P}}(\Sigma, \sigma)$ with respect to the norm $\|\|_{max}$ is the *maximal C^*-algebra*, which we denote also by $C^*_{\mathcal{P}}(\Sigma, \sigma)$.

The reduced C^*-algebra should be defined using a (left) regular representation of the twisted polarized convolution algebra $C^*_{\mathcal{P}}(\Sigma, \sigma)$. Since $\mathcal{P}$ is a strongly admissible polarization of the symplectic groupoid Σ we have a fibration $p : \Sigma \to \Sigma/\mathcal{D}$. For each point in the leaf in the base of $\Sigma/\mathcal{D}$ one has a natural *-representation of the *-algebra. Fix a point $x \in \Sigma_0$ such that $p(x) \in (\Sigma/\mathcal{D})_0$, we define a representation π_x of $C^*_{\mathcal{P}}(\Sigma, \sigma)$ on the Hilbert space $L^2(s^{-1}(x) \cap \Sigma/\mathcal{D}, \sigma)$ of square integrable (twisted) half-densities on the s-fiber of x intersected with $\Sigma/\mathcal{D}$ by

$$(\pi_x(f)\xi)(\gamma) = \int_{\eta \in t^{-1}(t(\gamma)) \cap \Sigma/\mathcal{D}} \sigma(\eta, \eta^{-1}\gamma) f(\eta) \xi(\eta^{-1}\gamma)$$

where $f \in C^*_{\mathcal{P}}(\Sigma, \sigma)$ and $\xi \in L^2(s^{-1}(x) \cap \Sigma/\mathcal{D}, \sigma)$. This representation is called a (left) regular representation with respect to x, since $\pi_x(f)\xi(\gamma) = f * \xi(\gamma)$.The inner-product on the Hilbert space $L^2(s^{-1}(x) \cap \Sigma/\mathcal{D}, \sigma)$ is defined by

$$\begin{aligned}
\langle \xi, \xi \rangle = \xi^* * \xi(x) &= \int_{y \in t^{-1}(t(x)) \cap \Sigma/\mathcal{D}} \sigma(y, y^{-1}x) \xi^*(y) \xi(y^{-1}x) \\
&= \int_{y \in s^{-1}(x) \cap \Sigma/\mathcal{D}} \overline{\sigma(y^{-1}, y)} \sigma(y^{-1}, y) \overline{\xi(y)} \xi(y) < \infty
\end{aligned}$$

Definition 7.0.27. [**14**] The completion of $C^*_{\mathcal{P}}(\Sigma, \sigma)$ with respect to the norm $\|f\|_{red} = \sup_{x \in \Sigma_0} \|\pi_x(f)\|$is the *reduced C^*-algebra*, which we denote by $C^*_{\mathcal{P}}(\Sigma, \sigma)_{red}$.

Hawkins studied several examples and it is not yet known which completion will fit a reasonable definition of quantization. Some examples in [**39**] suggest that both the maximal and reduced C^*-algebras are quantizations, some suggest that only the reduced C^*-algebra is a natural quantization and the results in [**43**] suggests that the maximal C^*-algebra is most suitable.

8. Examples

At last we will give two examples to illustrate the above quantization procedure for Poisson manifolds, which can also be found in [**39**]. We will start with the motivating example where we consider the quantization of symplectic manifold and show how we reproduce the C^*-algebra of compact operators. The second example concerns the quantization of a linear Poisson manifold, and we will see how this reproduce the usual Moyal quantization of Poisson vector spaces. For more examples we refer the reader to [**39**].

8.1. Symplectic manifold. Let (M,ω_M) be a symplectic manifold with a polarization $\mathcal{F} \subset T_{\mathbb{C}}M$ and a prequantization $(\mathbb{L},\langle,\rangle,\nabla)$. Then we have the pair groupoid $\Sigma := Pair(M)$ as the integrated symplectic groupoid, with multiplicative symplectic form $\omega := t^*\omega_M - s^*\omega_M$, as is explained in example 1.0.23 of appendix A. The polarization $\mathcal{F}$ of M induces a symplectic groupoid polarization $\mathcal{P} := \mathcal{F} \times \bar{\mathcal{F}}$ on $Pair(M)$. To see this we mention first that the set of composable pairs can be identified as $\Sigma_2 \cong M \times M \times M$ and the multiplication map $m : \Sigma_2 \to \Sigma$ simply forgets the middle factor. Hence $\mathcal{P}_2 = (\mathcal{P} \times \mathcal{P}) \cap T_{\mathbb{C}}\Sigma_2 = \mathcal{F} \times (\mathcal{F} \cap \bar{\mathcal{F}}) \times \bar{\mathcal{F}}$ from which one sees that $\mathcal{P}$ is multiplicative and $Ti(\mathcal{P}) = Ti(\mathcal{F} \times \bar{\mathcal{F}}) = \bar{\mathcal{F}} \times \mathcal{F} = \bar{\mathcal{P}}$ implies that $\mathcal{P}$ is Hermitian, the fact that $\mathcal{P}$ is an involutive Lagrangian distribution is immediate and hence $\mathcal{P}$ is a symplectic groupoid polarization of $Pair(M)$. In particular, if $\mathcal{F}$ is a real polarization of M, the polarization $\mathcal{P} = \mathcal{F} \times \mathcal{F}$ is just a pairs of vectors from $\mathcal{F}$ where two such pairs are composable if the second of the first pair and the first of the second pair coincide, that is $\mathcal{P}_2 = \mathcal{F} \times \mathcal{F} \times \mathcal{F}$.

Example 8.1.1. (Vertical polarization) Let Q be a compact manifold, and $M = T^*Q$ be its cotangent bundle, with as polarization the vertical polarization which, as is explained in example 3.1.7, is just the kernel foliation of the projection down to Q. Similarly the pair groupoid $Pair(T^*Q) \cong T^*(Pair(Q))$ and hence the induced polarization $\mathcal{P}$ is just the kernel foliation of the projection down to $Pair(Q)$, which is itself the reduced groupoid. As in example 3.1.7 one has also a tautological 1-form $\theta = t^*\tau - s^*\tau$ on the cotangent bundle $T^*(Pair(Q))$ which is a symplectic potential. This symplectic potential is adapted since for any element of the polarization $\mathcal{P}$ the form vanishes and it is multiplicative, that is $\partial^*\theta = 0$, since $\theta = \partial^*\tau$. But multiplicativity means that $dp^*\phi = 0$ and hence we have that $\sigma_0 = e^{i\phi}$ is locally constant from which can conclude that the reduced cocycle is trivial. This gives $C^*_{\mathcal{P}}(Pair(T^*Q),\sigma) \cong C^*(Pair(Q),\sigma_0)$, which is precisely equivalent to the untwisted groupoid convolution algebra $C^*(Pair(Q))$. The completion of this to the reduced C^*-algebra is equivalent to the C^*-algebra of compact operators $\mathcal{K}(L^2(Q))$. For every $x \in M$ one has that $s^{-1}(x) \cap \Sigma/\mathcal{P} \cong T^*Q \oplus 0 \cap Pair(Q) \cong Q$, which gives precisely the Hilbert space $L^2(Q)$ of square integrable half-densities on Q. The (left) regular representation is given by

$$\begin{aligned}(\pi_x(f)\xi)(\gamma) &= \int_{\eta \in Q} f(\eta)\xi(\eta^{-1}\gamma)\\ &= \int_{\eta_1 \in Q} f(\eta_1\eta_2^{-1})\xi(\eta_2)\end{aligned}$$

where $f \in C^*_{\mathcal{P}}(Pair(T^*Q),\sigma)$ and $\xi \in L^2(Q)$. But if we define $k(\eta_1,\eta_2) = f(\eta_1\eta_2^{-1})$ for every $\eta_1,\eta_2 \in Q$, then it is immediate that k is the smoothing kernel of the operator $\pi_x(f)$ with a finite L^2-norm and hence $\pi_x(f)$ defines a compact operator on $L^2(Q)$[**14**]. Which gives the expected quantization[**39**]

$$C^*_{\mathcal{P}}(Pair(T^*Q),\sigma) \cong \mathcal{K}(L^2(Q))$$

Example 8.1.2. (Kähler polarization) Let (M,J,ω) be a compact Kähler manifold together with the antiholomorphic tangent bundle $\mathcal{F} := T_{0,1}M$ as the natural Kähler polarization, as in example 3.1.4. The induced polarization $\mathcal{P} := \mathcal{F} \times \bar{\mathcal{F}}$ on $Pair(M)$ is just a pair of vectors, where two such pairs are composable if the second of the first and the first of the second pair both vanish, that is $\mathcal{P}_2 = \mathcal{F} \times 0 \times \bar{\mathcal{F}}$. This polarization $\mathcal{P}$ of $Pair(M)$ is equivalent to a Kähler structrure on $Pair(M)$ itself. For a prequantization $(\mathbb{L},\langle,\rangle,\nabla)$ on M we had that the polarized sections of $\mathbb{L}$ are precisely the holomorphic sections. Similarly one finds that the convolution product between polarized sections of the line bundle over $Pair(M)$ is precisely the convolution product between holomorphic sections of this line bundle.

More generally, let M be a (compact) symplectic manifold with polarization $\mathcal{F}$ and prequantization $(\mathbb{L},\langle,\rangle,\nabla)$. We have $\Sigma := Pair(M)$ and the symplectic groupoid polarization $\mathcal{P} := \mathcal{F} \times \bar{\mathcal{F}}$. In the case that $\mathcal{P}$ is strongly admissible we can construct the half-density bundle $\Omega^{1/2}_{\mathcal{P}}$. First note that

$T^t_{\mathbb{C}}\Sigma = \ker T_{\mathbb{C}}t = 0 \times T_{\mathbb{C}}M = s^*T_{\mathbb{C}}M$ and $T^t_{\mathbb{C}}\Sigma \cap \mathcal{P} = 0 \times \bar{\mathcal{F}} = s^*\bar{\mathcal{F}}$, and hence

$$T^t_{\mathbb{C}}\Sigma/(T^t_{\mathbb{C}}\Sigma \cap \mathcal{P}) = s^*(T_{\mathbb{C}}M/\bar{\mathcal{F}}) = s^*(\bar{\mathcal{F}}^{\perp})^*$$

Remember $\mathcal{F}^{\perp} := \{\xi \in T^*M : \forall X \in \mathcal{F}, \ \langle X, \xi\rangle = 0\}$. Similarly we find

$$T^s_{\mathbb{C}}\Sigma/(T^s_{\mathbb{C}}\Sigma \cap \mathcal{P}) = t^*(\mathcal{F}^{\perp})^*$$

which gives

$$\begin{aligned}\Omega^{1/2}_{\mathcal{P}} &= |\Omega|^{1/2}(\wedge^{max}(T^t_{\mathbb{C}}\Sigma/(T^t_{\mathbb{C}}\Sigma \cap \mathcal{P}) \oplus T^s_{\mathbb{C}}\Sigma/(T^s_{\mathbb{C}}\Sigma \cap \mathcal{P}))^*) \\ &= |\Omega|^{1/2} \wedge^{max} (t^*\, \mathcal{F}^{\perp} \oplus s^* \bar{\mathcal{F}}^{\perp}) \\ &= |\Omega|^{1/2} \wedge^{max} (t^*\, \mathcal{F}^{\perp}) \otimes |\Omega|^{1/2} \wedge^{max} (s^* \bar{\mathcal{F}}^{\perp})\end{aligned}$$

For the prequantization of Σ we consider the line bundle $\partial^*\mathbb{L} = t^*\mathbb{L} \otimes s^*\bar{\mathbb{L}}$ together with a twist σ. Then the algebra $C^*_{\mathcal{P}}(\Sigma, \sigma)$ is constructed from the polarized sections of

$$t^*(\mathbb{L} \otimes |\Omega|^{1/2} \wedge^{max} \mathcal{F}^{\perp}) \otimes s^*(\bar{\mathbb{L}} \otimes |\Omega|^{1/2} \wedge^{max} \bar{\mathcal{F}}^{\perp})$$

This space of polarized sections over $\Sigma = M \times M$ is just a tensor product of the space of polarized sections of $\mathbb{L} \otimes |\Omega|^{1/2} \wedge^{max} \mathcal{F}^{\perp}$ over M with its complex conjugate. The completion of this twisted polarized convolution algebra $C^*_{\mathcal{P}}(\Sigma, \sigma)$ to the reduced C^*-algebra is equivalent to the C^*-algebra of compact operators $\mathcal{K}(L^2(M/\mathcal{F}, \sigma))$. For every $x \in M$ one has that $s^{-1}(x) \cap \Sigma/\mathcal{P} \cong M \oplus 0 \cap (M \times M)/(\mathcal{F} \times \bar{\mathcal{F}}) \cong M/\mathcal{F}$, which gives precisely the Hilbert space $L^2(M/\mathcal{F}, \sigma)$ of twisted square integrable half-densities on $M/\mathcal{F}$. The (left) regular representation is given by

$$(\pi_x(f)\xi)(\gamma) = \int_{\eta \in M/\mathcal{F}} \sigma(\eta, \eta^{-1}\gamma) f(\eta)\xi(\eta^{-1}\gamma)$$

where $f \in C^*_{\mathcal{P}}(\Sigma, \sigma)$ and $\xi \in L^2(M/\mathcal{F}, \sigma)$. These operators precisely form the algebra of bounded smoothing kernels on $M/\mathcal{F}$[**14**]. This algebra can be completed to the C^*-algebra $\mathcal{K}(L^2(M/\mathcal{F}, \sigma))$ of compact operators[**39**]. This example shows that Hawkins' strict C^*-deformation quantization recovers the standard geometric quantization of symplectic manifolds, which in this particular case gives explicitly the Hilbert space of square integrable sections. From the physics perspective we have in well-behaved cases that, the Hilbert space $L^2(M/\mathcal{F}, \sigma)$ is densely spanned by polarized sections of $\mathbb{L} \otimes |\Omega|^{1/2} \wedge^{max} \mathcal{F}^{\perp}$. The algebra $\mathcal{K}(L^2(M/\mathcal{F}, \sigma))$ is densely spanned by the tensor product over $Pair(M)$ of a section and a complex conjugate section, which in physics is denoted by "ket-bras". But not all cases are well-behaved and the issues that one must be consider in constructing the inner product over M translates into issues in the construction of the convolution product over the groupoid Σ[**39**].

8.2. Poisson vector space with constant Poisson bivector. Here we consider a first non-trivial example of the strict C^*-deformation quantization. Let our Poisson manifold be a vector space V with a constant Poisson bivector $\pi \in \wedge^2 V$. The cotangent bundle T^*V is turned into a Lie algebroid as discussed in appendix A. This Lie algebroid can be integrated to the the symplectic groupoid $\Sigma = T^*V \cong V \oplus V^*$, where all the structure maps are linear. To see this let x^i be coordinates on V and y_i coordinates on V^*, and let the symplectic 2-form ω in these coordinates be given by $\omega = dx^i \wedge dp_i$. The groupoid structure on Σ depends on the constant Poisson bivector π, which in these coordinates is just a matrix (π^{ij}) and can be regarded as a map $\pi : V^* \to V$. The inclusion map of $i : V \to \Sigma$ is given by $i(x) : (x^i) \mapsto (x^i, p_i)$. A morphism of $\gamma \in \Sigma$ is given by a pair (x^i, p_i) and can be described using Bopp shifts

$$x^i - \tfrac{1}{2}\pi^{ij}p_j \xrightarrow{\ \gamma\ } x^i + \tfrac{1}{2}\pi^{ij}p_j$$

Which gives the source and target maps on Σ by

$$s(x^i, p_j) = x^i - \frac{1}{2}\pi^{ij}p_j$$
$$t(x^i, p_j) = x^i + \frac{1}{2}\pi^{ij}p_j$$

One can check that the target map $t : \Sigma \to V$ is a Poisson map and from this it follows that Σ has to be an integrating groupoid of V, see [**78**]. The set of composable pairs can be identified with $\Sigma_2 \cong V \oplus V^* \oplus V^*$, where an element (x^i, p_i, p'_i) represents the concatenation $\gamma \circ \eta$ of two arrows γ and η given by

$$x^i - \tfrac{1}{2}\pi^{ij}(p_j + p'_j) \xrightarrow{\eta} x^i - \tfrac{1}{2}\pi^{ij}(p_j - p'_j) \xrightarrow{\gamma} x^i + \tfrac{1}{2}\pi^{ij}(p_j + p'_j)$$

Which gives the projections $pr_1(\gamma, \eta) = \gamma$ and $pr_2(\gamma, \eta) = \eta$ and the multiplication $m(\gamma, \eta) = \gamma \circ \eta$ on Σ_2 by

$$pr_1(x^i, p_i, p'_i) = (x^i + \frac{1}{2}\pi^{ij}p'_j, p_i)$$
$$pr_2(x^i, p_i, p'_i) = (x^i - \frac{1}{2}\pi^{ij}p_j, p'_i)$$
$$m(x^i, p_i, p'_i) = (x^i, p_i + p'_i)$$

This construction gives all the relevant structures on the Lie groupoid Σ. In order for Σ to be a symplectic groupoid we still need to verify that $\omega = dx^i \wedge dp_i$ is multiplicative for this groupoid structure, i.e. $\partial^*\omega = 0$. This can now easily be checked

$$pr_1^*\omega = dx^i \wedge dp'_i + \frac{1}{2}\pi^{ij}dp_j \wedge dp'_i$$
$$pr_2^*\omega = dx^i \wedge dp_i + \frac{1}{2}\pi^{ij}dp'_j \wedge dp_i$$
$$m^*\omega = dx^i \wedge dp_i + dx^i \wedge dp'_i$$

and thus $\partial^*\omega = pr_1^*\omega - m^*\omega + pr_2^*\omega = 0$.

The projection p from the groupoid $\Sigma \cong V \oplus V^*$ to the additive group V^* gives precisely a fibration of groupoids, that is

$$\begin{tikzcd} V \oplus V^* \arrow[r, "p"] \arrow[d, shift left] \arrow[d, shift right] & V^* \arrow[d, shift left] \arrow[d, shift right] \\ V \arrow[r, "p_0"'] & * \end{tikzcd}$$

Remember that the kernel foliation $\mathcal{P} = \ker Tp$ of the fibration p is always a strongly admissible real polarization. This gives that $\mathcal{P} \cong TV$, which gives in particular that $\mathcal{P}$ is a Lagrangian distribution which makes this a polarization of the symplectic groupoid. Now let $(\sigma, \mathbb{L}, \nabla)$ be a prequantization of Σ. In order to determine the convolution algebra we can instead look at the twisted line bundle $\mathbb{L}_0$ over $\Sigma/\mathcal{P}$ with twist σ_0. This σ_0 can be obtained by an adapted symplectic potential θ. The simplest choice is

$$\theta = -x^i dp_i$$

This symplectic potential is adapted since it is perpendicular to the polarization $\mathcal{P}$. The twist σ_0 can be obtained by computing

$$\begin{aligned}\partial^*\theta &= (x^i + \frac{1}{2}\pi^{ij}p'_j)dp_i - x^i(dp_i + dp'_i) + (x^i - \frac{1}{2}\pi^{ij}p_j)dp'_i \\ &= -\frac{1}{2}\pi^{ij}p'_j dp_i - \frac{1}{2}\pi^{ij}p_i dp'_j \\ &= d(-\frac{1}{2}\pi^{ij}p_i p'_j)\end{aligned}$$

Hence

$$dp^*\phi = d(-\frac{1}{2}\pi^{ij}p_i p'_j)$$

implies precisely that the reduced group cocycle $\sigma_0 : V^* \times V^* \to U(1)$ is given by $\sigma_0(p, p') = e^{-\frac{i}{2}\pi(p,p')}$. Alltogether, the twisted polarized convolution algebra $C^*_{\mathcal{P}}(\Sigma, \sigma)$ is equivalent to the twisted convolution algebra $C^*(V^*, \sigma_0)$, which is determined by the convolution product

$$(f * g)(p) = \int_{p' \in V^*} e^{-\frac{i}{2}\pi(p', p-p')} f(p')g(p - p')$$

for functions f, g on V^*, which is precisely the Moyal $\star$-product, which is usually written as a power series expansion of the exponential. The strict C^*-deformation quantization procedure of the Poisson vector space (V, π) gives precisely the twisted group algebra $C^*(V^*, \sigma_0)$ and hence reproduces the famous Moyal quantization of Poisson vector spaces, see [**39, 72**].

CHAPTER 4

Higher geometric perspective

Traditional geometric quantization applies to symplectic manifolds and not to Poisson manifolds. We saw in the previous chapter that there is a similar geometric quantization route via polarization for Poisson manifolds. Both geometric quantization procedures constructed a prequantum bundle. Over the symplectic manifold we constructed a prequantum circle bundle and over the symplectic groupoid we constructed a multiplicative prequantum circle bundle, which is a higher analog of a circle bundle with connection and can be interpreted as a circle 2-bundle with connection. This is an instance of higher geometric prequantization, where the notion of prequantum circle bundle is refined to that of a prequantum circle n-bundle with connection for all $n \in \mathbb{N}$.

To motivate higher geometric prequantization, it is helpful to look first at one of the fundamental examples of quantum field theory which is the 3-dimensional Chern-Simons theory as introduced in [**92**]. We will give a short review in the perpsective of higher stacks and treat the basic constructions in classical Chern-Simons theory. This will give a blueprint along which we will interpreted the prequantization of the symplectic groupoid in terms of higher stacks. At the same time, this stacky perspective of the prequantization of a symplectic groupoid shows how it can be interpreted in higher symplectic geometry. This makes the geometric quantization of symplectic groupoids a good test case against which to check notions of higher geometric quantization.

It turns out that the geometric prequantization of a Poisson manifold can be seen as the geometric prequantization of the 2d Chern-Simons theory, which here specifically is the case induced by a non-degenerate binary invariant polynomial, namely the Lie integrated version of the Poisson σ-model. This statement that there should be such a relation was already contained in Cattaneo and Felder [**8**], which identified the construction by Kontsevich of the algebraic deformation quantization of any Poisson manifold, with the limiting case of the 3-point function in the perturbative quantization of the corresponding 2d Poisson σ-model. But at that time the geometric quantization of symplectic groupoids as in [**39**] had yet to be fully understood. This shows that the geometric prequantization of a Poisson manifold can be seen as the boundary of the Poisson σ-model.

The higher geometric quantization of a 2d field theory yields a 2-vector space of quantum 2-states. We will see that the 2-basis of this space of quantum 2-states is, up to Morita equivalence, an algebra. This algebra is precisely the algebra produced via strict C^*-deformation quantization by Hawkins. In the case that the Poisson manifold is a symplectic manifold, this algebra was the C^*-algebra of compact operators. But the C^*-algebra of compact operators are Morita equivalent to the ground field, which precisely reflects the fact that in higher geometry Lie groupoids are considered up to Morita equivalence and that the pair groupoid is Morita equivalent to the point. This shows that Hawkins' strict C^*-deformation quantization is not Morita faithful, in the sense that it distinguish Morita equivalent Lie groupoids and Mortia equivalent algebras.

1. Motivating examples

1.1. 3d Chern-Simons theory. To motivate higher geometric prequantization, it is helpful to look first at one of the fundamental examples of quantum field theory which is the 3-dimensional Chern-Simons theory as introduced in [**92**]. See [**23**] for a comprehensive account and [**29**] for a higher stacky perspective. The reader who is uncomfortable reading this section, may read first section 2 about smooth higher stacks.

First consider a compact connected and simply connected simple Lie group G and a 3-dimensional smooth (paracompact) manifold Σ_3. Then there exist a classifying space BG, such that gauge equivalence classes of principal G-bundles over Σ_3 are in natural bijective correspondence with the set $H(\Sigma_3, BG)$ of homotopy classes of maps from Σ_3 to BG. Since BG is homotopically trivial in degree less or equal to 3, any principal G-bundle on Σ_3 can be trivialized. Now similarly write $H(\Sigma_3, \mathbf{B}G_{conn})$ for the set of gauge equivalence classes of principal G-bundles with connection on Σ_3. Since any principal G-bundle on Σ_3 can be trivialized, for any gauge equivalence class of connections there exists a representative, which is given by a smooth $\mathfrak{g}$-valued 1-form A on Σ_3. The action functional of 3d Chern-Simons theory over Σ_3 is a function of sets

$$\exp(iS(-)) : H(\Sigma_3, \mathbf{B}G_{conn}) \to U(1)$$
$$A \mapsto \exp(2\pi i \int_{\Sigma_3} CS(A))$$

Where $CS(A) \in \Omega^3(\Sigma_3)$ is the Chern-Simons 3-form of A, where CS is called the Lagrangian of the theory. This action functional is well-defined since for every gauge transformation $g : A \to A^g$ for $g \in C^\infty(\Sigma_3, G)$ both A and its gauge transform A^g are mapped to the same element of $U(1)$.

There are two important properties that is not immediatly seen from this action functional in terms of sets. It has the property of being invariant under gauge transformation and it has the property of being smooth. A natural way to express the gauge invariance is to consider the groupoid $\mathbf{H}(\Sigma_3, \mathbf{B}G_{conn})$, whose objects are gauge fields A and whose morphisms are gauge transformations g as above. The gauge invariance of the action functional can then be expressed by functoriality, that is by a morphism of groupoids $\exp(iS(-)) : \mathbf{H}(\Sigma_3, \mathbf{B}G_{conn}) \to U(1)$, where here $U(1)$ is regarded as a groupoid with only identity morphisms. Furthermore the property of being smooth can be formulated in terms of stacks, which we will explain in the next section in more detail. A stack on the site of Cartesian spaces maps to every Cartesian space U a groupoid, as above, of smooth U-families of gauge fields and a smooth U-family of gauge transformations between, in a consistent way. We will denote the groupoid $\mathbf{H}(\Sigma_3, \mathbf{B}G_{conn})$ as a smooth stack by the same symbol and call it the smooth moduli stack of gauge fields on Σ_3. In fact Σ_3 and $\mathbf{B}G_{conn}$ can both be interpreted as stacks. Alltogether the Chern-Simons action functional refines to a morphism of smooth stacks

$$\exp(i\mathbf{S}(-)) : \mathbf{H}(\Sigma_3, \mathbf{B}G_{conn}) \to U(1)$$

Where the groupoid $U(1)$ is here regarded as a smooth stack. This refined action functional makes it explicit that Chern-Simons theory is actually a gauge theory.

Remark 1.1.1. In the literature one usually distinguish between "external hom" and "internal hom". Where one use the external hom for denoting $\mathbf{H}(\Sigma_3, \mathbf{B}G_{conn})$ as a groupoid and where one use the internal hom , which is denoted by $[\Sigma, \mathbf{B}G_{conn}]$, for the actual stack of fields. This stack is defined by the assignment

$$[\Sigma, \mathbf{B}G_{conn}] : U \mapsto \mathbf{H}(\Sigma \times U, \mathbf{B}G_{conn})$$

We will for simplicity not distinguish between these notations and we will use throughout this thesis only the notation for the external hom. From the context it should be clear whether one needs to interpreted it is a external or internal hom.

The smooth structure on the action functional allows us to define the differential $d\exp(i\mathbf{S}(-))$ of the action functional and hence its critical locus, which is characterized by the Euler-Lagrange equations of motion. In Chern-Simons theory this critical locus happens to be $\mathbf{H}(\Sigma_3, \flat\mathbf{B}G)$ the groupoid of principal G-bundles with flat connections on Σ_3, interpreted as a stack. A special case of interest is the product manifold $\Sigma_3 = \Sigma_2 \times [0,1]$, which can be thought of as a 3-dimensional worldvolume swept out by a 2-dimensional surfaces Σ_2. The groupoid of critical configurations on Σ_3 is equivalent to the groupoid on the initial boundary, i.e. of their restriction to $\Sigma_2 \times \{0\}$, since the equation of motions uniquely determine the extension of this boundary to the whole cylinder Σ_3. Hence the *phase space* of the theory is given by the substack

$$\mathbf{H}(\Sigma_2, \flat\mathbf{B}G) \hookrightarrow \mathbf{H}(\Sigma_2, \mathbf{B}G_{conn})$$

consisting of flat connections on principal G-bundles over Σ_2. This phase space has a natural symplectic form and is the restriction of a presymplectic 2-form defined on the whole of $\mathbf{H}(\Sigma_2, \mathbf{B}G_{conn})$. This presymplectic 2-form on the moduli stack of field configurations $\mathbf{H}(\Sigma_2, \mathbf{B}G_{conn})$ can be formulated as a morphism of smooth stacks

$$\omega : \mathbf{H}(\Sigma_2, \mathbf{B}G_{conn}) \to \Omega^2_{cl}$$

Where Ω^2_{cl} is the smooth stack of closed 2-forms. To see more explicitly what this form ω is, consider a Cartesian space $U \in \mathrm{CartSp}$. Over this U the map of stacks ω is a function which sends a connection $A \in \Omega^1(U \times \Sigma_2)$ to the closed 4-form $\langle F_A \wedge F_A \rangle \in \Omega^4_{cl}(U \times \Sigma_2)$. Under suitable condition we can use the *fiber integration* $\int_{\Sigma_2} : \Omega^4(U \times \Sigma_2) \to \Omega^2(U)$ to get the 2-form

$$\int_{\Sigma_2} \langle F_A \wedge F_A \rangle \in \Omega^2(U)$$

REMARK 1.1.2. Technically speaking the element A representing a field in the phase space, should be taken as an element in the concretification of the mapping stack, i.e. it is a $\mathfrak{g}$-valued 1-form on $U \times \Sigma_2$ with no "leg" along U, meaning that it vanishing on tangent vectors to U and can be thought of as a $\mathfrak{g}$-valued 1-form on Σ_2 that depends smoothly on the parameter U. Accordingly its curvature 2-form $F_A = F_A^{\Sigma_2} + d_U A$ where $F_A^{\Sigma_2} = d_{\Sigma_2} A + \frac{1}{2}[A \wedge A]$ is the U-parametrized collection of curvature forms on Σ_2. The fiber integration $\int_{\Sigma_2} : \Omega^4(U \times \Sigma_2) \to \Omega^2(U)$ picks out the component of $\langle F_A \wedge F_A \rangle$ with two legs along Σ_2 and two along U. See [**29, 79**] for more details.

Actually it is the invariant polynomial $\langle -, - \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ that induces the map $\langle F_{(-)} \wedge F_{(-)} \rangle : \mathbf{B}G_{conn} \to \Omega^4_{cl}$ and on Σ_2 this map together with the composition of the fiber integration gives us what is called a *transgression* of the map $\langle F_{(-)} \wedge F_{(-)} \rangle : \mathbf{B}G_{conn} \to \Omega^4_{cl}$ to a 2-form in Ω^2_{cl} on the mapping stack via

$$\omega : \mathbf{H}(\Sigma_2, \mathbf{B}G_{conn}) \xrightarrow{\mathbf{H}(\Sigma_2, \langle F_{(-)} \wedge F_{(-)} \rangle)} \mathbf{H}(\Sigma_2, \Omega^4_{cl}) \xrightarrow{\quad \int_{\Sigma_2} \quad} \Omega^2_{cl}$$

Hence this invariant polynomial $\langle - \rangle$ induces a map that sends a connection A to a cocycles $[\langle F_A \wedge F_A \rangle] \in H^4_{dR}(\Sigma_2)$, which is precisely the *Chern-Weil homomorphism* in Chern-Weil theory. We say that the invariant polynomial is in transgression with this cocycle via the Chern-Simons element CS, that is the Lagrangian of the theory.

Now consider the moduli stack $\mathbf{B}U(1)_{conn}$ of principal circle bundles with connection, we have the natural morphism of smooth stacks

$$F_{(-)} : \mathbf{B}U(1)_{conn} \to \Omega^2_{cl}$$

that sends every connection ∇ of a principal circle bundle over a parameter manifold U to its curvature 2-form $F_\nabla \in \Omega^2_{cl}(U)$, and since this 2-form is gauge invariant this morphism is well-defined. Regarding the above morphism $\omega : \mathbf{H}(\Sigma_2, \mathbf{B}G_{conn}) \to \Omega^2_{cl}$ as a (pre)symplectic form, then a choice of lift given by the dashed morphism in the diagram below, is a choice of refinement of the 2-form by a circle bundle with connection and hence the choice of a prequantum circle bundle in the language of prequantization.

$$\begin{tikzcd} & \mathbf{B}U(1)_{conn} \arrow[d, "F_{(-)}"] \\ \mathbf{H}(\Sigma_2, \mathbf{B}G_{conn}) \arrow[ur, dashed] \arrow[r, "\omega"'] & \Omega^2_{cl} \end{tikzcd}$$

In summary we have the following table

dimension	**description**	**moduli stack**
$k=3$	action functional	$\mathbf{H}(\Sigma_3, \mathbf{B}G_{conn}) \to U(1)$
$k=2$	prequantum circe bundle	$\mathbf{H}(\Sigma_2, \mathbf{B}G_{conn}) \to \mathbf{B}U(1)_{conn}$

In dimension k the Chern-Simons theory appears as a *circle* $(3-k)$*-bundle with connection* at least for $k = 2, 3$. Indeed for the case $k = 3$, the action functional can be interpreted as a circle 0-bundle with connection and for the case $k = 2$ the prequantum circle bundle is precisely a circle 1-bundle with connection. In ordinary geometric quantization of Chern-Simons theory we construct a vector space from this prequantum circle bundle over a closed manifold Σ_2 by considering the *polarized sections* (holomorphic sections) of the line bundle associated to the this circle 1-bundle on Σ_2. Hence this gives an assignment of a vector space to a closed manifold Σ_2. According to the definition of a *extended* topological quantum field theory of dimension n, due to Lurie in [**55**], we can roughly assign a $(n-k)$-categorical analog of a vector space of quantum states to every closed k dimensional manifold after quantization. Interpreting 3d Chern-Simons theory as an extended topological quantum field theory suggests that for every closed oriented manifold of dimension $0 \le k \le 3$ we can assigns a prequantum circle $(3-k)$-bundle on the moduli stack of field configuration over Σ_k, that is a morphism

$$\mathbf{H}(\Sigma_k, \mathbf{B}G_{conn}) \to \mathbf{B}^{(3-k)}U(1)_{conn}$$

In particular for the case $k = 0$, we can take Σ_0 the space $*$ consisting of a single point together with the fact that $\mathbf{H}(*, \mathbf{B}G_{conn}) \simeq \mathbf{B}G_{conn}$, the geometric prequantization of the morphism $\langle F_{(-)} \wedge F_{(-)} \rangle : \mathbf{B}G_{conn} \to \Omega^4_{cl}$ is a choice of refinement of the 4-form given by the dashed morphism

$$\begin{tikzcd} & \mathbf{B}^3U(1)_{conn} \arrow[d] \\ \mathbf{B}G_{conn} \arrow[ur, dashed, "\hat{\mathbf{c}}"] \arrow[r, "\langle F_{(-)} \wedge F_{(-)} \rangle"'] & \Omega^4_{cl} \end{tikzcd}$$

The vertical arrow is the higher analog of the curvature morphism and $\hat{\mathbf{c}}$ is called the *universal characteristic* morphism and we call it also the *extended Lagrangian.* This morphism of smooth higher stacks is the differential refinement of a smooth refinement of the topological characteristic map, which determine what is called the level of the theory. By forgetting the connections and only remembering the underlying higher bundles, we have a morphism of smooth higher stacks $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^3U(1)$, which is the smooth refinement of the continuous map of topological spaces $c : BG \to B^3U(1) \simeq K(\mathbb{Z}, 4)$, where $K(\mathbb{Z}, 4)$ is the Eilenberg-Maclane space, which represents the level as a class of the integral cohomology $H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}$. So $\mathbf{c}$ is the *smooth refinement* of c and $\hat{\mathbf{c}}$ is the *differential refinement* of $\mathbf{c}$, where $[c] \in H^4(BG, \mathbb{Z})$ determines the level.

Now for a closed (i.e., compact and without boundary) oriented manifold Σ_k of dimension $k \le 3$ we can form the morphism of mapping stacks

$$\mathbf{H}(\Sigma_k, \hat{\mathbf{c}}) : \mathbf{H}(\Sigma_k, \mathbf{B}G_{conn}) \to \mathbf{H}(\Sigma_k, \mathbf{B}^3U(1)_{conn})$$

Which assigns to every trajectory of a brane Σ_k in $\mathbf{B}G_{conn}$ a circle 3-bundle with connection over Σ_k. Just like an ordinary circle bundle with connection assigns holonomy (or parallel transport) to curves, so a circle n-bundle with connection assigns holonomy to k-dimensional trajectories for $k \leq n$. This higher holonomy (or higher parallel transport) of circle n-bundles with connection is precisely given by fiber integration in ordinary differential cohomology (see [**79, 30**]). That is we have the natural morphism of smooth higher stacks

$$hol_{\Sigma_k} := \exp(2\pi i \int_{\Sigma_k} (-)) : \mathbf{H}(\Sigma_k, \mathbf{B}^n U(1)_{conn}) \to \mathbf{B}^{(n-k)} U(1)_{conn}$$

Which we call the *k-dimensional holonomy* (or k-dimensional parallel transport) along Σ_k. Now composing these two maps gives the morphism of higher stacks

$$\exp(2\pi i \textstyle\int_{\Sigma_k} \hat{\mathbf{c}}(-)) : \mathbf{H}(\Sigma_k, \mathbf{B}G_{conn}) \xrightarrow{\mathbf{H}(\Sigma_k, \hat{\mathbf{c}})} \mathbf{H}(\Sigma_k, \mathbf{B}^3 U(1)_{conn}) \xrightarrow{\exp(2\pi i \int_{\Sigma_k}(-))} \mathbf{B}^{(3-k)} U(1)_{conn}$$

and is called the *extended action functional*. In this sense for Σ_k a closed oriented manifold of dimension $k \leq 3$, this morphism sends a field configuration $\nabla : \Sigma_k \to \mathbf{B}G_{conn}$ in $\mathbf{H}(\Sigma_k, \mathbf{B}G_{conn})$ to

$$\nabla \mapsto \mathrm{hol}_{\Sigma_k}(\hat{\mathbf{c}}(\nabla)) := \exp(2\pi i \int_{\Sigma_k} \hat{\mathbf{c}}(\nabla)) \in \mathbf{B}^{3-k} U(1)_{conn}$$

For the case $k = 3$ the 3-dimensional holonomy gives precisely the Cherns-Simons action functional and in fact we recover the above table, which can now naturally be extended to all cases of $k \leq 3$, that is

dimension	**description**	**prequantum $(3-k)$-bundle**
$k = 3$	action functional	$\mathbf{H}(\Sigma_3, \mathbf{B}G_{conn}) \xrightarrow{\mathbf{H}(\Sigma_3, \hat{\mathbf{c}})} \mathbf{H}(\Sigma_3, \mathbf{B}^3 U(1)_{conn}) \xrightarrow{\exp(2\pi i \int_{\Sigma_3}(-))} U(1)$
$k = 2$	prequantum circe bundle	$\mathbf{H}(\Sigma_2, \mathbf{B}G_{conn}) \xrightarrow{\mathbf{H}(\Sigma_2, \hat{\mathbf{c}})} \mathbf{H}(\Sigma_2, \mathbf{B}^3 U(1)_{conn}) \xrightarrow{\exp(2\pi i \int_{\Sigma_2}(-))} \mathbf{B}U(1)_{conn}$
$k = 1$	WZW	$\mathbf{H}(S^1, \mathbf{B}G_{conn}) \xrightarrow{\mathbf{H}(S^1, \hat{\mathbf{c}})} \mathbf{H}(S^1, \mathbf{B}^3 U(1)_{conn}) \xrightarrow{\exp(2\pi i \int_{S^1}(-))} \mathbf{B}^2 U(1)_{conn}$
$k = 0$	universal characteristic	$\hat{\mathbf{c}} : \mathbf{B}G_{conn} \longrightarrow \mathbf{B}^3 U(1)_{conn}$

The case $k = 2$ gives a bundle gerbe with connection inducing the Wess-Zumino-Witten bundle gerbe on G, which is explained in [**29**] in detail.

Furthermore we can compose the extended action functional on the right with the (higher analog) curvature morphism to get the underlying closed $(4-k)$-form

$$\mathbf{H}(\Sigma_k, \mathbf{B}G_{conn}) \to \Omega^{(4-k)}_{cl}$$

on this moduli stack. In other words, the moduli stack of principal G-bundles with connection over Σ_k carries a canonical *pre-*$(3-k)$*-plectic structure*, which is the higher order generalization of a symplectic structure (see [**73**]). This structure is equipped with a higher geometric prequantization, namely the above circle $(3-k)$-bundle with connection. This means that higher geometric prequantization needs a higher analog of symplectic geometry which goes under the name higher symplectic geometry, which we will explain in more detail in 3.

1.2. σ-models. A σ-model is supposed to be a type of model for quantum field theory. The basic idea is that we have some kind of space X, which we call the *target space*, and a space Σ, called the *worldvolume*, mapping into this target space. These *fields* $\phi : \Sigma \to X$ form together the *configuration space of fields* or the *trajectories* of the brane Σ in X, which is the *mapping stack* $\mathbf{H}(\Sigma, X)$. The σ-model describes the propagation of the worldvolume on the target space by a gauge field on X under which the worldvolume is charged. This is called the *background gauge field* of the σ-model.

The example of 3d Chern-Simons theory is a particular example of a σ-model, where the target space is $\mathbf{B}G_{conn}$ and the worldvolume is closed oriented manifold Σ_k for $k \leq 3$. The configuration space of fields on Σ_k is the mapping stack $\mathbf{H}(\Sigma_k, \mathbf{B}G_{conn})$ and can be seen as the trajectories of the brane Σ_k in $\mathbf{B}G_{conn}$. The propagation of the worldvolume on the target space is given by the background gauge field of the 3d Chern Simons theory, which is here given by the extended Lagrangian via the holonomy map

$$\hat{\mathbf{c}} : \mathbf{B}G_{conn} \to \mathbf{B}^3 U(1)_{conn}$$

For example we have the notion of higher parallel transport over a worldvolume Σ_3, which is given by the holonomy map

$$\nabla \mapsto \mathrm{hol}_{\hat{\mathbf{c}}(\nabla)}(\Sigma_3) := \exp(2\pi i \int_{\Sigma_3} \hat{\mathbf{c}}(\nabla)) \in U(1)$$

Which sends the trajectory $\nabla : \Sigma_3 \to \mathbf{B}G_{conn}$ to elements in $U(1)$ and which corresponds precisely to the (exponentiated) action functional of 3d Chern-Simons theory.

This example of 3d Chern-Simons theory as a σ-model is also a blueprint for a more general construction. For this one takes for the *target space* the universal moduli stack of field configurations itself which we denote by **Fields**. For every closed oriented *worldvolume* Σ_k we have the mapping stack $\mathbf{H}(\Sigma_k, \mathbf{Fields})$ which is the *configuration space of fields* on Σ_k and the *background gauge field* is given by the *extended Lagrangian*, that is the map

$$\mathbf{L} : \mathbf{Fields} \to \mathbf{B}^n U(1)_{conn}$$

of the universal higher stack to the n-stack of prinicpal circle n-bundles with connections. The Lagrangian $\mathbf{L}$ induces Lagrangian data in arbitrary codimension, that is for every closed oriented worldvolume Σ_k of dimension $k \leq n$ there is a *transgressed* Lagrangian

$$\mathbf{H}(\Sigma_k, \mathbf{Fields}) \xrightarrow{\mathbf{H}(\Sigma_k, \mathbf{L})} \mathbf{H}(\Sigma_k, \mathbf{B}^n U(1)_{conn}) \xrightarrow{\exp(2\pi i \int_{\Sigma_k}(-))} \mathbf{B}^{(n-k)} U(1)_{conn}$$

defining the prequantum circle $(n-k)$-bundle of the given field theory. The curvature of these bundles induce the canonical pre-$(n-k)$-plectic structure on the moduli stack of field configurations on Σ_k. In codimension 0, that is for $k = n$ one has the holonomy map which is given by the morphism of stacks

$$\exp(2\pi i \int_{\Sigma_n} (-)) : \mathbf{H}(\Sigma_n, \mathbf{Fields}) \to U(1)$$

which reduce to the (exponentiated) *action functional* by taking global sections over the point and passing to equivalence classes

$$\exp(2\pi i \int_{\Sigma_n} (-)) : \text{Field configurations/equiv} \to U(1)$$

Furthermore by postcomposing with the curvature morphism we get

$$\omega : \mathbf{Fields} \xrightarrow{\mathbf{L}} \mathbf{B}^n U(1)_{conn} \xrightarrow{F_{(-)}} \Omega^{n+1}_{cl}$$

which shows that the stack of field configurations is naturally equipped with a pre-n-plectic structure, hence these are examples of σ-models with (pre)-n-plectic targets.

We saw above that in 3d Chern-Simons theory as a σ-model, we had an action functional with as target the stack $\mathbf{B}G_{conn}$. This target space is equipped with the pre-3-plectic form

$$\langle F_{(-)} \wedge F_{(-)} \rangle : \mathbf{B}G_{conn} \to \Omega^4_{cl}$$

This binary dependence of the invariant polynomial $\langle -, - \rangle$ is an important feature of a whole tower of σ-models. This tower of σ-models goes under the name AKSZ σ-models (see [**26**]) and are determined by binary non-degenerate invariant polynomials. These AKSZ σ-models form a large class of σ-models and include ordinary 3d Chern-Simons theory, the Poisson σ-model, the Courant σ-model and higher dimensional abelian Chern-Simons theory. The Poisson σ-model can be seen as the perturbative part of the 2d Chern-Simons theory, which we will see later in this chapter.

2. Higher prequantum geometry

In the first section we indicated how higher prequantum geometry has a natural formulation in terms of stacks. This section will give you a brief outline of the basic construction and facts about higher geometry, which can be read in full detail in [**79**]. We will start by describing the fields as smooth higher stacks.

2.1. Fields as smooth higher stacks. A field theory associates to a spacetime Σ the configuration space of fields on Σ. We saw already in the above case that this configuration space of fields is richer than just a plain set. It needs to carry some smooth structure, which allows us to consider smooth maps from certain test spaces into **Fields**. This smooth structure allows us to perform variantional calculus on the configuration space of fields in order to find the critical locus of the action functional. But actually there is more structure and that is that the action functional should also be invariant under gauge transformation of the field configurations. One would like to have a configuration space of fields that also contains all the information of the gauge transformations. This is achieved by considering the configuration space of fields on Σ as a smooth (Lie) groupoid, that is a groupoid with smooth structure, instead of a manifold. A field theory where the configurations spaces of fields carry such a notion of gauge transformations is called a *gauge theory.* It is precisely this extra structure which allows us to treat the symplectic groupoid as a configuration space of fields over the point, instead of just the Poisson manifold. The objects of a symplectic groupoid can be seen as the field configurations on the point and the morphisms between these objects are the gauge transformations between the fields.

More generally, there may also be gauge transformations between gauge transformations, and so on, meaning that there are higher groupoids. In mathematical terms, these data define an ∞-category, where the objects are the fields, the 1-morphisms are the gauge transformations, the 2-morphisms are the gauge of gauge transfomrations, and so on, and since gauge transformations are always invertible we have that every k-morphism is invertible for $k \geq 1$ and hence one has that the ∞-category of fields is an ∞-groupoid. These ∞-groupoids can be seen as the particular simplicial sets, known as *Kan complexes*[**24**]. This gives us the following assignment

$$\textbf{Fields} : \text{Smooth manifolds} \to \infty\text{-groupoids}$$

The fields and their gauge transformations between them can be restricted to smaller regions of spacetime and more generally, they can be pulled back along smooth maps between different spacetimes. This means that this assignment is contravariant, which makes **Fields** a *simplicial presheaf* on the site of smooth manifolds taking values in Kan complexes.

These ∞-groupoids we described so far are all discrete and do not have any smooth structure. In the previous case we treated the smooth manifold M secretly as the (simplicial) presheaf **Fields** which assigns to a smooth manifold Σ the set of smooth maps from Σ into M. In this way a smooth manifold M can be realized as a (simplicial) presheaf on the category of smooth manifolds. Now suppose that Σ is a manifold with a open cover $\{U_i\}$, then by definition the smooth map $\Sigma \to M$ can be obtained by gluing smooth maps $U_i \to M$ that agrees on the overlap. This makes the presheaf

$$M : \mathrm{SmoothMfd}^{op} \to \mathrm{Set}$$
$$U \mapsto C^\infty(U, M)$$

into a sheaf on the category of smooth manifolds. Moreover, since every manifold can be obtained by gluing Cartesian spaces, i.e. those smooth manifolds diffeomorphic to $\mathbb{R}^n$ for some n, we see that all the information about M is in fact already encoded in the restriction of the sheaf to the category of Cartesian spaces and smooth maps between them

$$M : \mathrm{CartSp}^{op} \to \mathrm{Set}$$

In the case where **Fields** is a groupoid or a higher groupoid , we want a similar gluing construction. In the groupoid case, we should really glue by specifying gauge transformations on overlapping regions, and in the case of 2-groupoids, we should glue by specifying also gauge transformation between gauge transformations on triple overlaps, and so on. A presheaf of ∞-groupoids satisfying such a gluing law is called a simplicial sheaf or a ∞-stack. This gluing condition precisely means that a field theory can be completely described in terms of *local data*, since **Fields** can be probed by local patches diffeomorphic to Cartesian spaces.

2.2. Smooth ∞-stacks. Actually the precise formulation of the intuitive notion of a ∞-stack on the site of Cartesian space requires a bit of work and can be found in [**79, 28**], we will briefly recall here the important concepts to place us in the right setting. To begin, we recall that a sheaf on the site of Cartesian spaces is a presheaf $M : \mathrm{CartSp}^{op} \to \mathrm{Set}$ such that for each Čech nerve $\check{C}(\{U_i\}) \to U$ the morphism $A(U) \simeq [\mathrm{CartSp}^{op}, \mathrm{Set}](U, A) \mapsto [\mathrm{CartSp}^{op}, \mathrm{Set}](C(\{U_i\}, A)$ is an isomorphism, where $C(\{U_i\})$ is the colimit of $\check{C}(\{U_i\})$, this is called the *descent condition* or *sheaf condition.* The higher analog of this should give us the simplicial sheaf or the ∞-stack.

This may be achieved by equipping the category of simplicial presheaves $[\mathrm{CartSp}^{op}, \mathrm{sSet}]$, that is the category whose objects are simplicial presheaves over Cartesian spaces, and whose morphisms are natural transformations between them, with a model category structure. The notion of a *model category* structure provide a way to study ∞-categories. A model category structure is a category equipped with three classes of morphisms, the weak equivalences, fibrations and cofibrations, which satisfy certain conditions (see [**54**]). The fundamental example is the presentation of ∞-groupoids in terms of Kan complexes by the standard Quillen model structure $\mathrm{sSet}_{Quillen}$ on the category of simplicial sets. We are interested in the so called *simplicial model categories*, which are the $\mathrm{sSet}_{Quillen}$-enriched model categories. The simplicial localization of a simplicial model category gives us a ∞-category. Note that not every ∞-category comes from a simplicial model category. Now the category of simplicial presheaves $[\mathrm{CartSp}^{op}, \mathrm{sSet}]$ is naturally a sSet-enriched category and together with a model structure it becomes a (combinatorial) simplicial model category, which is called the *global projective model category structure* $[\mathrm{CartSp}^{op}, \mathrm{sSet}]_{proj}$, that is

(i) the *fibrations* are those morphisms whose components over each object $U \in \mathrm{CartSp}$ is a Kan fibration of simplicial sets;
(ii) the *weak equivalences* are those morphisms whose component over each object is a weak equivalence in the Quillen model structure on simplicial sets;
(iii) the *cofibrations* are the morphisms having the right lifting property against the morphisms that are both fibrations as well as weak equivalences.

This model structure presents the ∞-category of ∞-presheaves on the category of Cartesian spaces. We impose now an ∞-sheaf condition, in terms of another model structure on $[\mathrm{CartSp}^{op}, \mathrm{sSet}]$. Write

$[\mathrm{CartSp}^{op}, \mathrm{sSet}]_{proj,loc}$ for the model category structure on $[\mathrm{CartSp}^{op}, \mathrm{sSet}]$, which is the *left Bousfield localization* of $[\mathrm{CartSp}^{op}, \mathrm{sSet}]_{proj}$ at the set of morphisms of the form $\check{C}(\{U_i\}) \to U$ for every differentiably good open cover of U, this is called the *local projective model structure* on simplicial presheaves. The fibrant objects of $[\mathrm{CartSp}^{op}, \mathrm{sSet}]_{proj,loc}$ are precisely those simplicial presheaves A that are objectwise Kan complexes and such that for all differentiably good open covers $\{U_i\}$ of a Cartesian space U the induced morphims

$$A(U) \xrightarrow{\simeq} [\mathrm{CartSp}^{op}, \mathrm{sSet}](U, A) \to [\mathrm{CartSp}^{op}, \mathrm{sSet}](\check{C}(\{U_i\}), A)$$

is a weak equivalence of Kan complexes, which is stated in [**28**]. This is the *descent condition* or ∞-*sheaf/*∞-*stack condition* on A. The simplicial localization of $[\mathrm{CartSp}^{op}, \mathrm{sSet}]_{proj,loc}$ gives us the ∞-category, called the ∞-topos of simplicial sheaves or ∞-stacks, which we denote by $\mathbf{H} := \mathrm{Sh}_\infty(\mathrm{CartSp})$. This simplicial localization $\mathbf{H}$ of $[\mathrm{CartSp}^{op}, \mathrm{sSet}]_{proj,loc}$ is equivalent to the ∞-category $[\mathrm{CartSp}^{op}, \mathrm{sSet}]^{\circ}_{proj,loc}$, which is the full sSet-subcategory of $[\mathrm{CartSp}^{op}, \mathrm{sSet}]_{proj,loc}$ on the fibrant and cofibrant objects. Since $\mathbf{H}$ is a localization of $[\mathrm{CartSp}, \mathrm{sSet}]_{proj}$, every simplicial presheaf A in $[\mathrm{CartSp}, \mathrm{sSet}]_{proj}$ has a corresponding image $\mathbf{A}$ in $\mathbf{H}$, called its ∞-*stackification.*This ∞-stackification is given by a fibrant replacement of A in the model category, i.e. we need to find any fibrant resolution of A, since the fibrant replacement satisfy the descent condion. For X, A two simplicial presheaves, we consider the hom-space of the ∞-topos of ∞-stacks $\mathbf{H}(X, A) := [\mathrm{CartSp}^{op}, \mathrm{sSet}](Q(X), P(A))$, which is defined up to equivalence, where $Q(X)$ is any local cofibrant resolution of X, i.e. $Q(X)$ is cofibrant and it has a local weak equivence of X out of it, and $P(A)$ any local fibrant resolution of A, i.e. $P(A)$ is fibrant and has a local weak equivalence into it (See [**54, 79, 20**]).

REMARK 2.2.1. Remember that for the descent condition of sheaves we needed to take the colimit of $\check{C}(\{U_i\})$, which just identifies the points $x_i \in U_i,\ x_j \in U_j$ if they correspond to the same point x in U, but in the ∞-categorical case we need to take the homotopy colimit. We need to remember the equivalence between the two points, which can be depicted as follow

$$(x,i) \xrightarrow{(x,i,j)} (x,j)$$

This indicates that the objects are pair (x, i) consist of a point $x \in U$ and a patch $U_i \subset U$ that contains x, a morphism is a triple (x, i, j) consisting of a point and two patches, that both contain the point, in that $x \in U_i \cap U_j$. Similarly we need to remember the 2-morphisms for each triple intersection containing x

$$\begin{tikzcd}
& (x,j) \arrow[dr, "{(x,j,k)}"] \arrow[d, phantom, "\Downarrow (x,i,j,k)"] & \\
(x,i) \arrow[ur, "{(x,i,j)}"] & & (x,k) \arrow[ll, "{(x,k,i)}"]
\end{tikzcd}$$

and so on for higher n-morphisms. The object $\check{C}(\{U_i\})$ is this sense equivalent to the homotopy colimit in $[\mathrm{CartSp}^{op}, \mathrm{sSet}]_{proj}$ over the simplicial diagram of its components

$$\check{C}(\{U_i\}) \simeq \operatorname{hocolim}\left(\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \coprod_{i,j,k} U_{ijk} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i \right)$$

Where we suppressed the degeneracy maps for readability, see [**79**] for more details.

In practice the above boils down to the following simple recipe, for computing a map from a smooth manifold Σ into a simplicial presheaf. A smooth manifold is not cofibrant, but in case that the smooth manifold is paracompact it has a cofibrant resolution. The Čech nerve $\check{C}(\{U_i\}) \to \Sigma$ of a differentiably good open cover over a paracompact smooth manifold Σ is a cofibrant resolution of Σ in $[\mathrm{CartSp}^{op}, \mathrm{sSet}]_{proj,loc}$ and so we write

$$\check{C}(\{U_i\}) \xrightarrow{\sim_{loc}} \Sigma$$

This follows from a cofibrancy criterion by Dugger[**20**]. From now on, we assume that the smooth manifold Σ is paracompact and has a cofibrant resolution, which leaves us with the task of finding a version **Fields** of a simplicial presheaf that is fibrant over CartSp. For describing the morphisms of simplicial sheaves from the smooth manifold Σ to **Fields**, we need to choose a differentiably good open cover $\{U_i\}$ of Σ, form the Čech nerve simplicial presheaf $\check{C}(\{U_i\})$ and then consider spans of ordinary morphisms of simplicial presheaves of the form

$$\begin{tikzcd} \check{C}(\{U_i\}) \arrow[r, "g"] \arrow[d, "\sim_{loc}"] & \mathbf{Fields} \\ \Sigma & \end{tikzcd}$$

This diagram of simplicial presheaves presents an object in $\mathbf{H}(\Sigma, \mathbf{Fields}) \simeq \mathbf{Fields}(\Sigma)$, the hom-space of the ∞-topos of ∞-stacks. We call $g : \check{C}(\{U_i\}) \to \mathbf{Fields}$ also a **Fields**-cocycle on Σ and on local data it is given by a diagram

$$\begin{tikzcd} \vdots \arrow[d, shift left=4] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \arrow[d, shift right=4] & \vdots \arrow[d, shift left=4] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \arrow[d, shift right=4] \\ \coprod_{i,j,k} U_{ijk} \arrow[r, "g^{(2)}"] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & \mathbf{Fields}_2 \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\ \coprod_{i,j} U_{ij} \arrow[r, "g^{(1)}"] \arrow[d, shift left=1] \arrow[d, shift right=1] & \mathbf{Fields}_1 \arrow[d, shift left=1] \arrow[d, shift right=1] \\ \coprod_i U_i \arrow[r, "g^{(0)}"] & \mathbf{Fields}_0 \end{tikzcd}$$

of simplicial sets, where $\mathbf{Fields}_k$ denote the set of k-simplices of **Fields**. This diagram gives a collection $(\{g_i\}, \{g_{ij}\}, \{g_{ijk}\}, \cdots)$, where

- g_i is a 0-simplex in $\mathbf{Fields}(U_i)_0$ for any i;
- g_{ij} is an 1-simplex in $\mathbf{Fields}(U_{ij})_1$ for any i, j, whose boundary 0-simplices are the restrictions of g_i and g_j to U_{ij}

$$g_i|_{U_{ij}} \xrightarrow{g_{ij}} g_j|_{U_{ij}}$$

- g_{ijk} is an 2-simplex in $\mathbf{Fields}(U_{ijk})_2$ for any i, j, k, whose boundary 1-simplices are the restrictions of g_{ij},g_{jk} and g_{kl} to U_{ijk}

$$\begin{tikzcd} & g_j|_{U_{ijk}} \arrow[dr, "g_{jk}|_{U_{ijk}}"] \arrow[d, phantom, "\Downarrow g_{ijk}"] & \\ g_i|_{U_{ijk}} \arrow[ur, "g_{ij}|_{U_{ijk}}"] & & g_k|_{U_{ijk}} \arrow[ll, "g_{kl}|_{U_{ijk}}"] \end{tikzcd}$$

- g_{ijkl} is a 3-simplex in $\mathbf{Fields}(U_{ijkl})_3$ for any i, j, k, l, which is of the form

$$\begin{tikzcd}
g_j|_{U_{ijkl}} \arrow[r, "g_{jk}|_{U_{ijkl}}"] \arrow[dr, phantom, "\Downarrow g_{ijk}|_{U_{ijkl}}"] & g_k|_{U_{ijkl}} \arrow[d, "g_{kl}|_{U_{ijkl}}"] & & g_j|_{U_{ijkl}} \arrow[r, "g_{jk}|_{U_{ijkl}}"] \arrow[dr, "g_{jl}|_{U_{ijkl}}"] & g_k|_{U_{ijkl}} \arrow[d, "g_{kl}|_{U_{ijkl}}"] \arrow[dl, phantom, "\Downarrow g_{jkl}|_{U_{ijkl}}"] \\
 & & \xrightarrow{g_{ijkl}} & & \\
g_i|_{U_{ijkl}} \arrow[uu, "g_{ij}|_{U_{ijkl}}"] \arrow[uur, "g_{ik}|_{U_{ijkl}}"] \arrow[r, "g_{il}|_{U_{ijkl}}"'] \arrow[ur, phantom, "\Downarrow g_{ikl}|_{U_{ijkl}}"] & g_l|_{U_{ijkl}} & & g_i|_{U_{ijkl}} \arrow[uu, "g_{ij}|_{U_{ijkl}}"] \arrow[r, "g_{il}|_{U_{ijkl}}"'] \arrow[ur, phantom, "\Downarrow g_{ijl}|_{U_{ijkl}}"] & g_l|_{U_{ijkl}}
\end{tikzcd}$$

- and so on

This description only gives the objects (i.e. the 0-morphism) of the ∞-groupoid $\mathbf{Fields}(\Sigma)$.

The description of 1-morphisms in $\mathbf{Fields}(\Sigma)$ is straightforward. Let g and g' be two objects of $\mathbf{Fields}(\Sigma)$, then a 1-morphism h between them is the data of

- h_i is a 1-simplex in $\mathbf{Fields}(U_i)_1$ for any i, whose boundary 0-simplices are g_i and g'_i, respectively

$$g_i \xrightarrow{h_i} g'_i$$

- h_{ij} is a "square", which can be thought of as pairs of 2-simplices in $\mathbf{Fields}(U_{ij})_2$ with a common edge, whose boundary 1-simplices are as in the following diagram

$$\begin{tikzcd}
g_i|_{U_{ij}} \arrow[r, "h_i"] \arrow[d, "g_{ij}"] \arrow[dr, phantom, "\Downarrow h_{ij}"] & g'_i|_{U_{ij}} \arrow[d, "g'_{ij}"] \\
g_j|_{U_{ij}} \arrow[r, "h_j"'] & g'_j|_{U_{ij}}
\end{tikzcd}$$

- and so on.

Similarly, one describes k-morphisms for any $k \geq 1$.

2.3. Smooth higher stacks presented by Lie groupoids. A large class of examples of ∞-stacks are given by a presheaf of Kan complexes which sends a Cartesian space to the nerve of some Lie groupoid.

EXAMPLE 2.3.1. Let M be a smooth manifold. As said before M induces a sheaf, and in particular a presheaf, on CartSp, mapping a Cartesian space U to the set of smooth functions from U to M. This sheaf can be seen as a stack or even more a ∞-stack in the above sense. To see this we may regard this smooth manifold M as a Lie groupoid $\mathbf{M}$ with precisely one identity morphism for every object in M, that is

$$\mathbf{M} := \left(M\ {}_t\times_s M \rightrightarrows\!\!\!\!\rightarrow M \rightrightarrows M \right)$$

which can be depicted by

$$x \xrightarrow{Id} x$$

for all $x \in M$. Now consider another smooth manifold Σ with differentiably good open cover $\{U_i\}$ then we can associate canonically the Čech-groupoid $C(\{U_i\})$ to it, that is

$$C(\{U_i\}) = \left(\coprod_{i,j,k} U_{ijk} \rightrightarrows\!\!\!\!\rightarrow \coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i \right)$$

which can be depicted by

$$\begin{tikzcd} & (x,j) \arrow[dr, "{(x,j,k)}"] \arrow[d, phantom, "\|"] & \\ (x,i) \arrow[ur, "{(x,i,j)}"] \arrow[rr, "{(x,i,k)}"'] & & (x,k) \end{tikzcd}$$

This Čech-groupoid inherits a smooth structure from the fact that U_i are smooth manifolds and the inclusions $U_i \hookrightarrow \Sigma$ are smooth functions, hence $C(\{U_i\})$ is a Lie groupoid. Then we have the canonical projection functor

$$C(\{U_i\}) \to \Sigma$$
$$(x,i) \mapsto x$$

The notion of *good* open cover is needed in order to make this smooth functor a weak equivalence of Lie groupoids, in the sense that $C(\{U_i\})$ is cofibrant in a suitable model category structure on the category of Lie groupoids, this will be discussed later in more detail. To see that the Lie groupoid $\mathbf{M}$ presents a smooth stack follows from the fact that $\mathbf{M}$ give rise to a presheaf of groupoids on CartSp given by

$$U \mapsto C^\infty(U, \mathbf{M})$$

To see that this serves as a presentation of the stack associated to this presheaf of groupoids, we consider the functor $C(\{U_i\}) \to \mathbf{M}$ which gives following diagram

$$\begin{tikzcd} \coprod_{i,j,k} U_{ijk} \arrow[r, "g^{(2)}"] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & M \; {}_t\times_s M \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\ \coprod_{i,j} U_{ij} \arrow[r, "g^{(1)}"] \arrow[d, shift left] \arrow[d, shift right] & M \arrow[d, shift left] \arrow[d, shift right] \\ \coprod_i U_i \arrow[r, "g^{(0)}"] & M \end{tikzcd}$$

Specifically this diagram gives a collection $(\{g_i\}, \{g_{ij}\})$ where we have local maps $g_i : U_i \to M$ for all i such that $g_{ij} : U_{ij} \to M$ for every i, j is subject to the condition $g_i|_{U_{ij}} = sg_{ij} = tg_{ij} = g_j|_{U_{ij}}$ (remember $s = Id = t$). This can be visualized by the simplicial diagram

$$g_i|_{U_{ij}} \xrightarrow{=} g_j|_{U_{ij}}$$

The condition for the 2-simplices are completely determined by this condition and hence the smooth manifold M can be seen as a smooth stack $\mathbf{M}$. This presentation of this Lie groupoid as a smooth stack can naturally be seen as a ∞-stacks, by taking the nerve over the presheaf of groupoids, since every groupoid is mapped by the nerve functor N : Grpd $\to$ KanCplx to a Kan complex (see [**54**]). This gives us the presheaf of Kan complexes

$$U \mapsto N(C^\infty(U, \mathbf{M}))$$

that sends each Cartesian space U to the ∞-groupoid $N(C^\infty(U, \mathbf{M}))$. To describe the ∞-stackification, which is also denoted by $\mathbf{M}$, of this simplicial presheaf we need to say what the ∞-groupoid associated to a smooth manifold Σ is. By the above recipe we need take the Čech nerve as cofibrant resolution of Σ, but this Čech nerve is precisely the nerve of the Čech-groupoid, hence applying the nerve functor

N to the morphism $C(\{U_i\}) \to \mathbf{M}$ gives us an object of the ∞-groupoid $\mathbf{M}(\Sigma)$, which is given by a diagram

$$\begin{tikzcd}
\vdots \arrow[d, shift left=3] \arrow[d, shift left=1] \arrow[d, shift right=1] \arrow[d, shift right=3] & \vdots \arrow[d, shift left=3] \arrow[d, shift left=1] \arrow[d, shift right=1] \arrow[d, shift right=3] \\
\coprod_{i,j,k} U_{ijk} \arrow[r, "g^{(2)}"] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & M \; {}_t\!\times_s M \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
\coprod_{i,j} U_{ij} \arrow[r, "g^{(1)}"] \arrow[d, shift left] \arrow[d, shift right] & M \arrow[d, shift left] \arrow[d, shift right] \\
\coprod_i U_i \arrow[r, "g^{(0)}"] & M
\end{tikzcd}$$

Since the objects of the ∞-groupoid of $\mathbf{M}$-cocyles on Σ are completely determined by the local smooth maps $g_i : U_i \to M$ such that $g_i|_{U_{ij}} = g_j|_{U_{ij}}$ it is immediate that these objects are just smooth maps from Σ to M. Moreover the morphisms between these objects are the precisely the identities. This gives the map

$$\mathbf{M} : \Sigma \mapsto C^\infty(\Sigma, M)$$

Which is just the image of M via the Yoneda lemma and hence we will denote the ∞-stack $\mathbf{M}$ just by the symbol M. Now for any smooth ∞-stack **Fields** we have by the Yoneda lemma the natural equivalence

$$\mathbf{Fields}(M) \simeq \mathbf{H}(M, \mathbf{Fields})$$

where on the right M is identified with the ∞-stack it defines.

EXAMPLE 2.3.2. Let Ω^n, for $n \in \mathbb{N}$, be the presheaf on CartSp, mapping a Cartesian space U to the set $\Omega^n(U; \mathbb{R})$ of degree n smooth differential forms with real coefficients on U. This presheaf Ω^n can just like example 2.3.1 be seen as a sheaf, a stack and even an ∞-stack and the Ω^n-cocycles on a smooth manifold Σ are just the smooth differential forms of degree n with real coefficients on Σ.

EXAMPLE 2.3.3. Let G be a Lie group. Consider the action groupoid $*//G$, consisting of a single point and the manifold G as space of morphisms. This is a Lie groupoid since its collection of objects and of morphisms each form a smooth manifold, and all the structure maps are smooth functions. We will write $\mathbf{B}G$ for this Lie groupoid and it is called the one-object *delooping groupoid*, it can be depicted by the following diagram

$$\begin{tikzcd}
 & * \arrow[dr, "g_j"] \arrow[d, phantom, "\Vert"] & \\
* \arrow[ur, "g_i"] \arrow[rr, "g_j g_i"'] & {} & *
\end{tikzcd}$$

Where the $g_i \in G$ are the elements of the group and the bottom morphism is labeled by forming the product in the group. This Lie groupoid $\mathbf{B}G$ give rise to a presheaf of groupoids on CartSp given by

$$U \mapsto C^\infty(U, \mathbf{B}G)$$

and it serves as a presentation of the stack associated to this presheaf of groupoids. To see this observe that the functor $g : C(\{U_i\}) \to \mathbf{B}G$, for $\{U_i\}$ an differentiably good open cover of a smooth manifold Σ, is given by the collection $(\{g_{ij}\}, \{g_{ijk}\})$, where we have local smooth maps $g_{ij} : U_{ij} \to G$ and $g_{ijk} : U_{ijk} \to G \times G$ such that $g_{ij} = pr_1 g_{ijk}$, $g_{jk} = pr_2 g_{ijk}$ and $g_{ik} = m g_{ijk}$ on U_{ijk}. But this data precisely means that the functor $g : C(\{U_i\}) \to \mathbf{B}G$ is given by the smooth functions $g_{ij} : U_{ij} \to G$ such that $g_{ij}g_{jk}g_{ki} = Id_G$ on U_{ijk}, which is called the *cocycle* constraint. This can be visualized by a similar simplicial diagram

$$\begin{tikzcd} & * \arrow[dr, "g_{jk}|_{U_{ijk}}"] & \\ * \arrow[ur, "g_{ij}|_{U_{ijk}}"] \arrow[rr, "g_{ij}g_{jk}|_{U_{ijk}}"'] \arrow[rr, phantom, bend left, "\|"] & & * \end{tikzcd}$$

This is precisely the data defining a principal G-bundle over Σ. To see this consider another Lie groupoid associated to G, that is the action groupoid $G//G$. We will write $\mathbf{E}G$ for this Lie groupoid and it can be depicted by

$$\begin{tikzcd} & g_j \arrow[dr, "g_kg_j^{-1}"] & \\ g_i \arrow[ur, "g_jg_i^{-1}"] \arrow[rr, "g_kg_i^{-1}"'] \arrow[rr, phantom, bend left, "\|"] & & g_k \end{tikzcd}$$

Together with the forgetful functor, which is called the *universal G-principal bundle*

$$\mathbf{E}G \to \mathbf{B}G$$
$$(g_i \overset{g_jg_i^{-1}}{\to} g_j) \mapsto (* \overset{g_jg_i^{-1}}{\to} *)$$

we can consider the pullback diagram in the category of Lie groupoids

$$\begin{tikzcd} \bar{P} \arrow[r] \arrow[d] & \mathbf{E}G \arrow[d] \\ C(\{U_i\}) \arrow[r] \arrow[d, "\simeq"] & \mathbf{B}G \\ \Sigma & \end{tikzcd}$$

Where $\bar{P}$ is the Lie groupoid which can be depicted as follow

$$(x, i, g_i) \to (x, j, g_j = g_{ij}(x)g_i)$$

Whenever such a morphism exists it is unique. Due to this uniqueness, this Lie groupoid is weakly equivalent to a manifold P regarded as a Lie groupoid, that is $\bar{P} \overset{\simeq}{\to} P$. This P can be written as

$$P = (\coprod_i U_i \times G)/\sim$$

where two points in $\coprod_i U_i \times G$ are equivalent precisely if there exist a corresponding morphism in $\bar{P}$. We immediatly recognize here the traditional construction of principal G-bundle from a cocycle function $\{g_{ij}\}$. Indeed it is easy to see in components that P obtained this way does have a principal G-action. One basic feature of principal bundles is that they are locally trivializable, which can already be seen in the construction of P. Namely, the local trivializability is described by the following commutative diagram

$$\begin{tikzcd} & & Y \times G \arrow[r] \arrow[d] & P \arrow[d] \\ Y^{[3]} \arrow[r, shift left=2, "\pi_{12}"] \arrow[r, "\pi_{13}"] \arrow[r, shift right=2, "\pi_{23}"'] & Y^{[2]} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & Y \arrow[r, "\pi"'] & \Sigma \end{tikzcd}$$

where $Y := \coprod_i U_i$ and π is the covering map of Σ. The smooth functions $g_{ij} : U_{ij} \to G$ define a transition function $g : Y^{[2]} \to G$, where $Y^{[k]}$ denotes the k-fold fibre product of Y with itself, that is the disjoint union of all k-fold intersections of the open sets U_i. The cocycle condition $g_{ij}g_{jk} = g_{ik}$ on U_{ijk} is equivalently expressed as

$$\pi_{12}^* g \cdot \pi_{23}^* g = \pi_{13}^* g$$

over $Y^{[3]}$, where π_{ij} are the projections on the repspective components. We will later give a similar description of higher bundles, called bundle gerbes.

All together this shows that the objects in $\mathbf{B}G(\Sigma)$ are precisely the principal G-bundles on Σ. The morphisms in $\mathbf{B}G(\Sigma)$ are precisely the gauge transformations between the G-principal bundles, and hence we have reproduced the groupoid of G-principal bundles, that is $\mathbf{H}(\Sigma, \mathbf{B}G) \simeq GBund(\Sigma)$. We call $\mathbf{B}G$ the *moduli stack of G-principal bundles*. We will especially be interested in the case where $G = U(1)$ where $\mathbf{B}U(1)$ is the classifying stack of principal $U(1)$-bundles or equivalently of Hermitian line bundles.

Remark 2.3.4. Actually the above construction of the principal G-bundle via the pullback diagram of a morphism $\Sigma \to \mathbf{B}G$ is just the homotopy pullback of the point along this morphism, i.e.

$$\begin{tikzcd} P \arrow[r] \arrow[d] & * \arrow[d] \\ \Sigma \arrow[r] & \mathbf{B}G \end{tikzcd}$$

In others words this says that the cocycle $\Sigma \to \mathbf{B}G$ pulled back to the bundle $P \to \Sigma$ that it classifies becomes $P \to \Sigma \to \mathbf{B}G$, which is homotopic to the trivial cocycle on P (the one that factors through the point). In the above we computed the homotopy pullback as an ordinary pullback after replacing one of the maps with an equivalent fibration. We used the fibrant replacement of the pullback diagram, by replacing $* \to \mathbf{B}G$ by $\mathbf{E}G \to \mathbf{B}G$, with $\mathbf{E}G$ weakly equivalent to the point.

To see this, we note that the morphism $\mathbf{E}G \to \mathbf{B}G$ is defined by the pullback diagram

$$\begin{tikzcd} \mathbf{E}G \arrow[r] \arrow[d] & * \arrow[d] \\ \mathbf{B}G^I \arrow[r] \arrow[d] & \mathbf{B}G \\ \mathbf{B}G & \end{tikzcd}$$

Where $\mathbf{E}G$ is defined as the pullback $\mathbf{B}G^I \times_{\mathbf{B}G} *$. By the factorization lemma, see [**7**], we have that the left vertical morphism $\mathbf{E}G \to \mathbf{B}G$ is a fibration and since $\mathbf{B}G^I \to \mathbf{B}G$ is both a weak equivalence and a fibration and thus preserves pullbacks we have that $\mathbf{E}G \to *$ is a weak equivalence. Now together with the cocycle $\Sigma \overset{\simeq}{\twoheadleftarrow} \check{C}(\{U_i\}) \to \mathbf{B}G$ on Σ the homotopy pullback is computed as the two consecutive pullbacks

$$\begin{tikzcd}
\bar{P} \arrow[r] \arrow[dd] & \mathbf{E}G \arrow[r] \arrow[d] & * \arrow[d] \\
 & \mathbf{B}G^I \arrow[r] \arrow[d] & \mathbf{B}G \\
\check{C}(\{U_i\}) \arrow[r] \arrow[d, "\simeq"] & \mathbf{B}G & \\
\Sigma & &
\end{tikzcd}$$

To see that this moduli stack of G-principal bundles $\mathbf{B}G$ is actually a ∞-stack, we can apply again the nerve functor. Indeed by taking the nerve over the presheaf of groupoids we get the presheaf of Kan complexes

$$U \mapsto N(C^\infty(U, \mathbf{B}G))$$

Its ∞-stackification, which we denote also by $\mathbf{B}G$, can be described by the ∞-groupoid of $\mathbf{B}G$-cocycles on a smooth manifold Σ. For example let $\{U_i\}$ be a differentiably good open cover of Σ then applying the nerve functor N to the morphism $g : C(\{U_i\}) \to \mathbf{B}G$ gives us an object of the ∞-groupoid $\mathbf{B}G(\Sigma)$.

These examples are in fact a special case of the following example.

Example 2.3.5. Let $\mathcal{G}$ be a Lie groupoid. Then $\mathcal{G}$ give rise to a presheaf of groupoids on CartSp given by

$$U \mapsto C^\infty(U, \mathcal{G})$$

To see that this gives a presentation of $\mathcal{G}$ as a stack we let $\{U_i\}$ be a differentiably good open cover of a smooth manifold Σ. Then a functor $g : C(\{U_i\}) \to \mathcal{G}$ is given by a collection $(\{g_i\}, \{g_{ij}\}, \{g_{ijk}\})$ such that

- g_i is a smooth function in $C^\infty(U_i, \mathcal{G}_0)$ for any i;
- g_{ij} is a smooth function in $C^\infty(U_{ij}, \mathcal{G}_1)$ for any i, j such that $g_i = sg_{ij}$ and $g_j = tg_{ij}$ on U_{ij};
- g_{ijk} is a smooth function in $C^\infty(U_{ijk}, \mathcal{G}_2)$ for any i, jk such that $g_{ij} = pr_1 g_{ijk}$, $g_{jk} = pr_2 g_{ijk}$ and $g_{ik} = mg_{ijk}$.

Which completely determines a presentation of $\mathcal{G}$ as a stack. This Lie groupoid can also be presented as an ∞-stack, by applying the nerve functor. We have that $\mathcal{G}$ induces a presheaf of Kan complexes on CartSp by

$$U \mapsto N(C^\infty(U, \mathcal{G}))$$

We will denote its ∞-stackification by $\mathcal{G}_\bullet$ and we can describe the ∞-groupoid of $\mathcal{G}_\bullet$-cocycles on a smooth manifold Σ as follow. Let $\{U_i\}$ be an differentiably good open cover of Σ then the nerve of a morphism $g : C(\{U_i\}) \to \mathcal{G}$ gives us an object of the ∞-groupoid $\mathcal{G}_\bullet(\Sigma)$ and it is completely determined by the above relations. In this way every Lie groupoid $\mathcal{G}$ can be seen as a ∞-stack $\mathcal{G}_\bullet$. The fact that it is a Lie groupoid, that is a groupoid with some additional smooth structure, and not a bare groupoid, makes it eligible to be a stack or ∞-stack. In particular every symplectic groupoid can be treated as a stack, which will be the point of view for interpreting geometric quantization in terms of higher geometry.

So far we considered only examples of ∞-stacks **Fields** which come from a presheaf of Kan complexes which sends a Cartesian space to the nerve of some Lie groupoid, that is they are presented by Lie groupoids. This depended on the fact that the nerve of a groupoid is equal to a Kan complex. Stacks that can be presented by Lie groupoids are also called *differentiable stacks*. They form an important class of stacks and contains many examples of interest, to which we will come back later.

The construction above of principal G-bundles was based on the delooping groupoid $\mathbf{B}G$, that was canonically induced by a Lie group G. We say that $\mathbf{B}G$ is the delooping of G. In the case where G is a Lie 2-group, the construction of $\mathbf{B}G$ go through essentially verbatim, only that we pick up 2-morphisms everywhere instead of 1-morphisms. This is the first step towards higher geometry, with resulting a generalization of the notion of prinicipal bundle, namely that of principal 2-bundle. This can be further generalized to principal n-bundles. The classifying stacks of these principal n-bundles can be described using the *Dold-Kan correspondence*, which associate to every chain complex of abelian groups a Kan complex.

2.4. Smooth higher stacks presented by the Dold-Kan correspondence. The Dold-Kan correspondence is a useful tool for producing a large class of examples of ∞-stacks induced from chain complexes of sheaves of abelian groups. The classical Dold-Kan correspondence asserts that there is an equivalence of categories between non-negatively graded chain complexes and simplicial abelian groups (see [**34**] for a comprehensive treatment), that is

$$\mathrm{Ch}_\bullet^+ \xrightarrow[\simeq]{\Gamma} \mathrm{sAb}\ ,$$

For our purpose we will give here an explicit description of the functor Γ. So given a chain complex

$$A_\bullet = \cdots \xrightarrow{\partial} A_3 \xrightarrow{\partial} A_2 \xrightarrow{\partial} A_1 \xrightarrow{\partial} A_0$$

in $\mathrm{Ch}_\bullet^+$, the corresponding simplicial abelian group $\Gamma(A_\bullet)$ is defined as follows:

- the abelian group of 0-simplices of $\Gamma(A_\bullet)$ is the abelian group A_0;
- the abelian group of n-simplices of $\Gamma(A_\bullet)$ is the abelian group whose element are standard n-simplices decorated by an element x in the abelian group A_n such ∂x equals the (oriented) sum of the decorations on the boundary $(n-1)$-simplices.

For example a 2-simplex in $\Gamma(A_\bullet)$ is given by

$$\begin{tikzcd} & a_j \arrow[dr, "b_{jk}"] \arrow[d, phantom, "\Downarrow c_{ijk}"] & \\ a_i \arrow[ur, "b_{ij}"] \arrow[rr, "b_{ik}"'] & & a_k \end{tikzcd}$$

where

- $a_i \in A_0$
- $b_{ij} \in A_1$ and $\partial b_{ij} = a_j - a_i$;
- $c_{ijk} \in A_2$ and $\partial c_{ijk} = b_{jk} - b_{ik} + b_{ij}$.

By forgetting the group structure of this simplicial abelian group, we obtain just a bare simplical set. A result by Moore [**64**] tells us that any underlying simplicial set of a simplicial group is a Kan complex. Hence we obtain a forgetful functor, which maps a simplicial abelian group to a Kan complex

$$F : \mathrm{sAb} \to \mathrm{sSet}_{\mathrm{fib}} \hookrightarrow \mathrm{sSet}$$

The composition of Γ and F we denote by DK and is what we call the *Dold-Kan correspondence*

$$DK : \mathrm{Ch}_\bullet^+ \xrightarrow[\simeq]{\Gamma} \mathrm{sAb} \xrightarrow{F} \mathrm{sSet}$$

This correspondence can directly be extended to presheaves of chain complexes and presheaves of abelian groups on Cartesian spaces, which we will denote by the same symbol

$$DK : [\mathrm{CartSp}^{op}, \mathrm{Ch}_\bullet^+] \xrightarrow{\Gamma} [\mathrm{CartSp}^{op}, \mathrm{sAb}] \xrightarrow{F} [\mathrm{CartSp}^{op}, \mathrm{sSet}] \; .$$

EXAMPLE 2.4.1. Let A be an abelian Lie group and consider for any non-negative integer n the presheaf of chain complexes

$$U \mapsto C^\infty(U, A)[n] := [\cdots \to 0 \to C^\infty(U, A) \to 0 \to \cdots \to 0],$$

with $C^\infty(-, A)$ placed in degree n and U a Cartesian space. Under the Dold-Kan correspondence one get the simplicial presheaf

$$U \mapsto DK(C^\infty(U, A) \to 0 \to \cdots \to 0)$$

whose stackification is the n-stack $\mathbf{B}^n A$ of principal A-n-bundles. For $n = 0$ this is the sheaf of smooth functions with values in A; for $n = 1$ this reproduces the usual stack $\mathbf{B}A$ of principal A-bundles, which is the abelian case of example 2.3.1. For $n = 2$ this is the 2-stack of principal A-2-bundles, where we only pick up the 2-morphisms instead of the 1-morphisms in the case of prinicpal A-bundles. We see that every time we deloop A once more we shift the morphisms to one degree higher.

These classifying stacks for principal A-n-bundles allows us to talk about higher circle bundles. To interpret the prequantization of a Poisson manifolds in terms of a prequantum circle 2-bundle, we will work out the case where $A = U(1)$ in more detail, that is the 2-stack $\mathbf{B}^2 U(1)$ of principal $U(1)$-2-bundles.

EXAMPLE 2.4.2. The Lie groupoid $\mathbf{B}U(1)$ has the special property that it has itself the structure of a group object, but a Lie groupoid that is at the same time a group object is precisely a Lie 2-group. We can perform a delooping once more on this Lie 2-group to get the Lie 2-groupoid $\mathbf{B}^2 U(1)$, which can be depicted by

$$\begin{tikzcd} & * \arrow[dr, "Id"] \arrow[d, phantom, "\Downarrow g"] & \\ * \arrow[ur, "Id"] \arrow[rr, "Id"'] & {} & * \end{tikzcd}$$

where $g \in U(1)$ and both the horizontal as the vertical composition of the 2-morphisms is given by a product in $U(1)$. To see that this Lie 2-groupoid is actual a 2-stack we consider the smooth manifold Σ with differentiably good open cover $\{U_i\}$ and we may think of the Čech groupoid as a Lie 2-groupoid by looking at the 2-groupoid part of the full Čech nerve

$$C(\{U_i\}) = \left(\coprod_{i,j,k,l} U_{ijkl} \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; \coprod_{i,j,k} U_{ijk} \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; \coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i \right)$$

Now a smooth 2-functor $g : C(\{U_i\}) \to \mathbf{B}^2 U(1)$ is given by the following data. The 2-morphisms are given by the assignment

$$\left(\begin{tikzcd} & (x,j) \arrow[dr, "{(x,j,k)}"] \arrow[d, phantom, "\Downarrow (x,i,j,k)"] & \\ (x,i) \arrow[ur, "{(x,i,j)}"] \arrow[rr, "{(x,i,k)}"'] & {} & (x,k) \end{tikzcd} \right) \mapsto \left(\begin{tikzcd} & * \arrow[dr, "Id"] \arrow[d, phantom, "\Downarrow g_{ijk}(x)"] & \\ * \arrow[ur, "Id"] \arrow[rr, "Id"'] & {} & * \end{tikzcd} \right)$$

which is nothing more than a collection of smooth functions $g_{ijk} : U_{ijk} \to U(1)$ for all i, j, k. On these function a constraint is given by the 3-morphisms, since the only 3-morphism is the identity element, which is depicted by

$$\left(\begin{tikzcd} (x,j) \arrow[r] & (x,k) \arrow[d] \\ (x,i) \arrow[u] \arrow[ur] \arrow[r] & (x,l) \end{tikzcd} \Rightarrow \begin{tikzcd} (x,j) \arrow[r] \arrow[dr] & (x,k) \arrow[d] \\ (x,i) \arrow[u] \arrow[r] & (x,l) \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} * \arrow[r] \arrow[dr, phantom, "g_{ijk}(x)"] & * \arrow[d] \\ * \arrow[u] \arrow[ur] \arrow[r, phantom, "g_{ikl}(x)"'] \arrow[r] & * \end{tikzcd} = \begin{tikzcd} * \arrow[r] \arrow[dr] \arrow[r, phantom, "g_{jkl}(x)"'] & * \arrow[d] \\ * \arrow[u] \arrow[r] \arrow[r, phantom, "g_{ijl}(x)"] & * \end{tikzcd}\right)$$

This gives the relation

$$g_{ijk} \cdot g_{ikl} = g_{ijl} \cdot g_{jkl}$$

which is called the degree 2-cocycle constraint. These cocycles classify principal circle 2-bundles and to find such a principal circle 2-bundle, we need to construct the 2-functor $\mathbf{EB}U(1) \to \mathbf{B}^2U(1)$, which is called the *universal principal circle 2-bundle*. Analogous to the case of example 2.3.3, we have that $\mathbf{EB}U(1)$ can be depicted by

$$\begin{tikzcd} & * \arrow[dr, "c_{jk}"] \arrow[d, phantom, "\Downarrow g"] & \\ * \arrow[ur, "c_{ij}"] \arrow[rr, "c_{ik}=c_{ij}c_{jk}g"'] & & * \end{tikzcd}$$

where $c_{ij}, c_{jk}, c_{ik}, g \in U(1)$ and all possible composition operations are given by the group product of $U(1)$. The universal principal circle 2-bundle is just the forgetful functor $\mathbf{EB}U(1) \to \mathbf{B}^2U(1)$, which forgets the labels c_i of the 1-morphisms and just remembers the label g of the 2-morphism. Now consider the pullback diagram in the category of Lie 2-groupoids

$$\begin{tikzcd} \hat{P} \arrow[r] \arrow[d] & \mathbf{EB}U(1) \arrow[d] \\ C(\{U_i\}) \arrow[r] \arrow[d, "\simeq"] & \mathbf{B}^2U(1) \\ \Sigma & \end{tikzcd}$$

Where $\hat{P}$ is the Lie 2-groupoid whose objects are that of $C(\{U_i\})$, whose morphisms are morphisms in $C(\{U_i\})$ each equipped with a label $c \in U(1)$, and whose 2-morphisms can be depicted as follow

$$\begin{tikzcd} & (x,j) \arrow[dr, "c_{jk}"] \arrow[d, phantom, "\Downarrow g_{ijk}(x)"] & \\ (x,i) \arrow[ur, "c_{ij}"] \arrow[rr, "c_{ik}"'] & & (x,k) \end{tikzcd}$$

which is subject to the condition

$$c_{jk} = g_{ijk}(x) c_{ij} c_{jk}$$

REMARK 2.4.3. More generally for arbitrary $n \in \mathbb{N}$, this constuction models the homotopy pullback of the point along a morphism $\Sigma \to \mathbf{B}^n U(1)$, i.e.

$$\begin{tikzcd} \hat{P} \arrow[r] \arrow[d] & * \arrow[d] \\ \Sigma \arrow[r] & \mathbf{B}^n U(1) \end{tikzcd}$$

Which together with the cofibrant replacement $\Sigma \overset{\simeq}{\leftarrow} \check{C}(\{U_i\}) \to \mathbf{B}^n U(1)$ can be presented by the ordinary pullback diagram

$$\begin{tikzcd} \hat{P} \arrow[r] \arrow[d] & \mathbf{EB}^{n-1}U(1) \arrow[r, "\simeq"] \arrow[d] & * \\ \check{C}(\{U_i\}) \arrow[r] \arrow[d, "\simeq"] & \mathbf{B}^n U(1) & \\ \Sigma & & \end{tikzcd}$$

Where $\mathbf{EB}^{n-1}U(1)$ is given by the image under DK of the complex of sheaves of abelian groups

$$U \mapsto [\cdots \to 0 \to C^\infty(U, U(1)) \overset{Id}{\to} C^\infty(U, U(1)) \to 0 \to \cdots \to 0]$$

and we have the obvious morphism $\mathbf{EB}^{n-1}U(1) \to \mathbf{B}^n U(1)$. To see that $\mathbf{EB}^{n-1}U(1) \overset{\simeq}{\to} *$ is a weak equivalence, we note that the Dold-Kan correspondence takes quasi-isomorphisms to weak equivalences in $[\mathrm{CartSp}^{op}, \mathrm{sSet}]_{proj}$, see[**28**], where a chain map is a quasi-isomorphism if it induces isomorphisms on all homology groups. Since the above chain-complex has only trivial homology groups it is quasi-isomorphic to the trivial chain-complex. Hence we have that $\mathbf{EB}^{n-1}U(1) \overset{\simeq}{\to} *$ (remember $*$ is given by the image of DK of the complex of sheaves of trivial groups). For the case $n = 2$, this gives precisely the above construction.

We saw before in example 2.3.3, that the pullback Lie 1-groupoid $\bar{P}$ was equivalent to the the principal 1-bundle P, as a Lie 0-groupoid, since whenever a morphism existed it was unique. Here we have a similar situation, where every 2-morphism is unique if it exist, and hence here the Lie 2-groupoid $\hat{P}$ is equivalent to the Lie 1-groupoid

$$\bar{P} = \Big(C(\{U_i\})_2 \times U(1) \Rrightarrow C(\{U_i\})_1 \times U(1) \rightrightarrows C(\{U_i\})_0 \Big)$$

where the multiplication is given by

$$(x,i) \overset{c_{ij}}{\to} (x,j) \overset{c_{jk}}{\to} (x,k) = (x,i) \overset{c_{ik} = g_{ijk}(x) c_{ij} c_{jk}}{\to} (x,k)$$

But this precisely defines a groupoid central extension

$$\mathbf{B}U(1) \to \bar{P} \to C(\{U_i\}) \simeq \Sigma$$

which are known in the literature as *bundle gerbes* over Σ with surjective submersion $Y = \coprod_i U_i \to \Sigma$. A bundle gerbe is defined analogous to what remains of the locally trivialized principal $U(1)$-bundle in example 2.3.3, only we move up one step since we have delooped once more. Instead of a transition function on $Y^{[2]}$ we take a principal $U(1)$-bundle $P \to Y^{[2]}$ and since we cannot multiply $U(1)$-bundles like the pullback of the transition function, the cocycle condition has to be relaxed to an isomorphism $\pi_{12}^* P \otimes \pi_{23}^* P \to \pi_{13}^* P$ of $U(1)$-bundles over $Y^{[3]}$, which capture the groupoid multiplication on $\bar{P}$, which we demand to be associative. Explicitly, the above groupoid central extension is equivalently a bundle gerbe over a smooth manifold Σ together with a surjective submersion $\pi : Y \to \Sigma$, which is defined by a (principal) $U(1)$-bundle

$$\begin{tikzcd} & & P \arrow[d, "p"] & \\ Y^{[4]} \arrow[r, shift left=3, "\pi_{123}"] \arrow[r, shift left=1, "\pi_{124}"] \arrow[r, shift right=1, "\pi_{134}"] \arrow[r, shift right=3, "\pi_{234}"'] & Y^{[3]} \arrow[r, shift left=2, "\pi_{12}"] \arrow[r, "\pi_{13}"] \arrow[r, shift right=2, "\pi_{23}"'] & Y^{[2]} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & Y \arrow[d, "\pi"] \\ & & & \Sigma \end{tikzcd}$$

over the the fiber product $Y^{[2]} := Y \times_\Sigma Y = C(\{U_i\})_1$ of morphisms, together with a *bundle gerbe multiplication* which is an isomorphism

$$\sigma_g : \pi_{12}^* P \otimes \pi_{23}^* P \to \pi_{13}^* P$$

of $U(1)$-bundles on $Y^{[3]} = C(\{U_i\})_2$, such that it is associative, in the sense that on $Y^{[4]} = C(\{U_i\})_3$ the diagram

$$\begin{tikzcd} \pi_{12}^* P \otimes \pi_{23}^* P \otimes \pi_{34}^* P \arrow[r, "\pi_{123}^*\sigma_g \otimes Id"] \arrow[d, "Id\otimes\pi_{234}^*\sigma_g"'] & \pi_{13}^* P \otimes \pi_{34}^* P \arrow[d, "\pi_{134}^*\sigma_g"] \\ \pi_{12}^* P \otimes \pi_{24}^* P \arrow[r, "\pi_{124}^*\sigma_g"'] & \pi_{14}^* P \end{tikzcd}$$

commutes. This bundle gerbe multiplication can be rephrased as a section σ_g of the $U(1)$-bundle $\partial^* P^* \to Y^{[3]}$ (remember that $\partial^* P^* = \pi_{12}^* P^* \otimes \pi_{13}^* P \otimes \pi_{23}^* P^*$). Moreover $\partial^* \sigma_g$ is a section of $\partial^{*2} P^* \to Y^{[4]}$, but $\partial^{*2} P^* = \pi_{123}^* \partial^* P^* \otimes \pi_{124}^* \partial^* P \otimes \pi_{134}^* \partial^* P^* \otimes \pi_{234}^* \partial^* P$ is canonically trivial so we have that $\partial^* \sigma_g = 1$, which is precisely the condition of associativity, see [**65**] . This bundle gerbe multiplication σ_g is equivalent to the multiplication on P and the associativity condition is precisely the 2-cocycle condition on g. So we find that bundle gerbes are presentations that are the total spaces of principal circle 2-bundles.

In this example we considered a bundle gerbe over a smooth manifold Σ by considering $\Sigma \simeq C(\{U_i\}) \to \mathbf{B}^2 U(1)$ as a central extension of the Čech groupoid over Σ by $\mathbf{B}U(1)$. But we are interested in the case where we have a bundle gerbe over a symplectic groupoid Σ, which can be considered as a central extension of the symplectic groupoid Σ by $\mathbf{B}U(1)$. Let Σ be a symplectic groupoid, which as in example 2.3.5 can be presented as a 2-stack. A morphism $\chi : \Sigma \to \mathbf{B}^2 U(1)$ of 2-stack can similarly be described by a central groupoid extension

$$\mathbf{B}U(1) \to \bar{P} \to \Sigma$$

And since $U(1)$-bundles are equivalent to Hermitian line bundles we have that this is eqiuvalent to a Hermitian line bundle

$$\begin{tikzcd} & L \arrow[d, "p"] & \\ \Sigma_2 \arrow[r, shift left=2, "pr_1"] \arrow[r, "m"] \arrow[r, shift right=2, "pr_2"'] & \Sigma_1 \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & \Sigma_0 \end{tikzcd}$$

over the space Σ_1 of morphisms, together with an isomorphism

$$\sigma : pr_1^* L \otimes pr_2^* L \to m^* L$$

of line bundles on Σ_2, that is $\sigma \in \Gamma(\Sigma_2, \partial^* L^*)$ of norm 1 at every point, such that it satisfies the associativity condition, which precisely is the cocycle condition on σ, as is stated in the definition 2.1.7 of a prequantization of symplectic groupoid, that is $\partial^*\sigma = 1$.

2.5. Prequantization in terms of smooth higher stacks. So far we have classified the principal circle n-bundles by the ∞-stack $\mathbf{B}^n U(1)$. For higher geometric prequantization we are more interested in principal n-bundles with the additional structure of a connection, in order to describe the the prequantization of a symplectic groupoid. We will begin with the case of ordinary prequantization of symplectic manifolds, where we need a differential refinement of $\mathbf{B}U(1)$ in order to describe the principal circle-bundle with connection.

Example 2.5.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Consider the presheaf of groupoids which sends a Cartesian space U to the action groupoid $\Omega^1(U;\mathfrak{g})//C^\infty(U,G)$, i.e. the 1-groupoid with the set $\Omega^n(U;\mathfrak{g})$ of $\mathfrak{g}$-valued 1-forms on U as objects and the group of smooth functions $C^\infty(U,G)$ acting on $\Omega^1(U;\mathfrak{g})$ via the gauge transformation

$$g : A \mapsto g^{-1}Ag + g^{-1}dg$$

as set of morphisms, where here $g^{-1}Ag$ denotes pointwise the adjoint action of G on $\mathfrak{g}$ and $g^{-1}dg$ is the pullback $g^*(\theta)$ of the Maurer-Cartan form $\theta \in \Omega^1(G,\mathfrak{g})$. This serves as a presentation of a stack, which we denote by $\mathbf{B}G_{conn}$, since for a smooth manifold Σ with a differentiably good open cover $\{U_i\}$ it can be described by a collection $(\{A_i\},\{g_{ij}\})$ consisting of

- A_i is a 1-form in $\Omega^1(U_i,\mathfrak{g})$ for any i;
- g_{ij} is a smooth function in $C^\infty(U_{ij},G)$ for any i,j such that $A_j = g_{ij}^{-1}A_i g_{ij} + g_{ij}^{-1}dg_{ij}$ on U_{ij};
- we have the cocycle constraint $g_{ij}g_{jk}g_{ki} = Id_G$ on U_{ijk} for any i,j,k.

Which corresponds to the simplicial diagram

$$\begin{tikzcd}
& A_j|_{U_{ijk}} \arrow[dr, "g_{jk}|_{U_{ijk}}"] \arrow[d, phantom, "\Vert"] & \\
A_i|_{U_{ijk}} \arrow[ur, "g_{ij}|_{U_{ijk}}"] \arrow[rr, "g_{ij}g_{jk}|_{U_{ijk}}"'] & {} & A_k|_{U_{ijk}}
\end{tikzcd}$$

These are readily seen to be the data defining a g-connection on a principal G-bundle over Σ. We find that $\mathbf{B}G_{conn}$ is the *moduli stack of G-principal bundles with connection* and we have $\mathbf{H}(\Sigma, \mathbf{B}G_{conn}) \simeq GBund_{conn}(\Sigma)$. Furthermore we have an evident morphism of presheaves of groupoids

$$\Omega^1(-;\mathfrak{g})//C^\infty(-;G) \to *//C^\infty(-;G)$$

which induces a forgetful morphism of stacks $\mathbf{B}G_{conn} \to \mathbf{B}G$ which just forgets the connection.
As a particular case of interest we consider $G = U(1)$. In this case the ∞-stack $\mathbf{B}U(1)_{conn}$ classifies principal $U(1)$-bundles with connection. The presheaf of groupoids defining $\mathbf{B}U(1)_{conn}$ can then be identified with the presheaf of groupoids $\Omega^1//C^\infty(-;U(1))$ where the gauge transformation are given by

$$g : A \mapsto A + \frac{1}{2\pi i} d\log g$$

As in example 2.3.3 this describes precisely the connection ∇ on a $U(1)$-bundle $P \to \Sigma$. For a covering map $\pi : Y := \coprod_i U_i \to \Sigma$, this connection ∇ defines a 1-from $A \in \Omega^1(Y)$, which is related to the transition function $g : Y^{[2]} \to U(1)$ by

$$\pi_2^* A - \pi_1^* A = \frac{1}{2\pi i} d\log g$$

Again the stacks $\mathbf{B}G_{conn}$ and $\mathbf{B}U(1)_{conn}$ can be seen as ∞-stacks by using the nerve functor.

EXAMPLE 2.5.2. We can associate a curvature to the connections induced by $\mathbf{B}U(1)_{conn}$. The de Rham differential $d : \Omega^1 \to \Omega^2$ induces a morphism of simplicial presheaves

$$d : \Omega^1//C^\infty(-; U(1)) \to \Omega^2_{cl}$$

The (∞-)stackification of this morphism induces the morphism of (∞-)stacks

$$F_{(-)} : \mathbf{B}U(1)_{conn} \to \Omega^2_{cl}$$

mapping a $U(1)$-bundle with connection to its curvature 2-form.

For a smooth manifold M a map $\nabla : M \to \mathbf{B}U(1)_{conn}$ into the moduli stack $\mathbf{B}U(1)_{conn}$ is equivalent to a principal $U(1)$-bundle with connection and the map of universal moduli stacks $F_{(-)}$: $\mathbf{B}U(1)_{conn} \to \Omega^2_{cl}$ which sends a principal circle connection to its universal curvature 2-form, which characterizes traditional prequantization of symplectic manifolds. That is for $\omega \in \Omega^2_{cl}(M)$ a symplectic form, a prequantization of (M, ω) is equivalently a lift ∇ in the diagram

$$\begin{tikzcd} & \mathbf{B}U(1)_{conn} \arrow[d, "F_{(-)}"] \\ M \arrow[ur, dashed, "\nabla"] \arrow[r, "\omega"'] & \Omega^2_{cl} \end{tikzcd}$$

where the commutativity of the diagram expresses the traditional prequantization condition $F_\nabla = \omega$.

We would like to have an analog of such a lift in the case of prequantizing symplectic groupoids, for which we need to go to higher stacks. In the above example we saw that $\mathbf{B}G_{conn}$ is the differential refinement of $\mathbf{B}G$, now there is a straightforward generalization of this construction to the case where we have the circle n-group $G = \mathbf{B}^{n-1}U(1)$, that is to principal circle n-bundles with connection for all $n \in \mathbb{N}$. A famous model for describing these n-stacks is due to Deligne and Beilinson, and is called the *Deligne complex*.

For $n \in \mathbb{N}$ the *Deligne complex* in degree $(n + 1)$ is the chain complex of sheaves on the Cartesian spaces of abelian groups

$$U(1)[n+1]^\infty_D := \left(C^\infty(-, U(1)) \overset{dlog}{\to} \Omega^1(-) \overset{d}{\to} \cdots \overset{d}{\to} \Omega^n(-) \right)$$

Where *dlog* is as in example 2.5.1 and d is the de Rham differential on the sheaves of differential forms. The n-stack of principal circle-n-bundles with connection $\mathbf{B}^n U(1)$ is the n-stack presented by the stackification of the Deligne complex $U(1)[n+1]^\infty_D$ via the Dold-Kan correspondence, that is

$$\mathbf{B}^n U(1)_{conn} := DK \left(C^\infty(-, U(1)) \overset{dlog}{\to} \Omega^1(-) \overset{d}{\to} \cdots \overset{d}{\to} \Omega^n(-) \right)$$

We have the canonical forgetful morphism

$$\begin{tikzcd} \mathbf{B}^n U(1)_{conn} \arrow[d] & := DK(C^\infty(-, U(1)) \arrow[r, "dlog"] \arrow[d] & \Omega^1(-) \arrow[r, "d"] \arrow[d] & \cdots \arrow[r, "d"] \arrow[d] & \Omega^n(-)) \arrow[d] \\ \mathbf{B}^n U(1) & := DK(C^\infty(-, U(1)) \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0) \end{tikzcd}$$

which forgets the n-connections. More generally, we may consider all the intermediate stages

$$\begin{tikzcd}
\mathbf{B}^nU(1)_{conn} \arrow[d] & := DK(C^\infty(-,U(1)) \arrow[r, "dlog"] \arrow[d] & \Omega^1(-) \arrow[r, "d"] \arrow[d] & \cdots \arrow[r, "d"] \arrow[d] & \Omega^k(-) \arrow[r, "d"] \arrow[d] & \Omega^{k+1}(-) \arrow[r, "d"] \arrow[d] & \cdots \arrow[r, "d"] \arrow[d] & \Omega^n(-)) \arrow[d] \\
\mathbf{B}^nU(1)_{conn^k} & := DK(C^\infty(-,U(1)) \arrow[r, "dlog"] & \Omega^1(-) \arrow[r, "d"] & \cdots \arrow[r, "d"] & \Omega^k(-) \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0)
\end{tikzcd}$$

for $k \leq n$. In other words, $\mathbf{B}^nU(1)_{conn^k} := \mathbf{B}^{n-k}(\mathbf{B}^kU(1)_{conn})$, since delooping $\mathbf{B}^kU(1)_{conn}$ $(n-k)$-times is just shifting the underlying chain complex of $\mathbf{B}^kU(1)_{conn}$ up in degree by $(n-k)$. In this sense we have that $\mathbf{B}^nU(1)_{conn^n} = \mathbf{B}^nU(1)_{conn}$ and $\mathbf{B}^nU(1)_{conn^0} = \mathbf{B}^nU(1)$.

Furthermore we have the morphism of stacks $F_{(-)} : \mathbf{B}^nU(1) \to \Omega^{n+1}_{cl}$ which is given by

$$\begin{tikzcd}
\mathbf{B}^nU(1)_{conn} \arrow[d] & := DK(& C^\infty(-,U(1)) \arrow[r, "dlog"] \arrow[d] & \Omega^1(-) \arrow[r, "d"] \arrow[d] & \cdots \arrow[r, "d"] \arrow[d] & \Omega^n(-)) \arrow[d, "d"] \\
\Omega^{n+1}_{cl} & := DK(& 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & \Omega^{n+1}_{cl}(-))
\end{tikzcd}$$

and maps a circle n-bundle with connection to the curvature $(n+1)$-form of its connection.

Remark 2.5.3. For Σ a smooth manifold, the set of connected components $\pi_0\mathbf{H}(\Sigma, \mathbf{B}^nU(1)_{conn})$, that is the equivalence classes of circle n-bundles with connection, is naturally identified with the $(n+1)^{\text{st}}$ differential cohomology group of Σ, that is

$$\hat{H}^{n+1}(\Sigma;\mathbb{Z}) \simeq \pi_0\mathbf{H}(\Sigma, \mathbf{B}^nU(1)_{conn})$$

Using the Dold-Kan correspondence, this is precisely equivalent to the Deligne cohomology of the Čech-Deligne double complex, since for a good open cover $\{U_i\}$ of Σ the chain complex

$$Tot(\check{C}(\{U_i\}), U(1)[n+1]^\infty_D)_0 \xrightarrow{\delta} \cdots \xrightarrow{\delta} Tot(\check{C}(\{U_i\}), U(1)[n+1]^\infty_D)_1 \xrightarrow{\delta} Tot(\check{C}(\{U_i\}), U(1)[n+1]^\infty_D)_{cl,n}$$

where δ the total differential of the double complex, and this is under the Dold-Kan correspondence a presentation of $\mathbf{H}(\Sigma, \mathbf{B}^nU(1)_{conn})$. This goes along the lines of the procedure for constructing the simplicial de Rham cohomology from chain complexes which is described in section 3.1.

We similarly have that the equivalence classes of circle n-bundles are in natural bijection with

$$H^{n+1}(\Sigma;\mathbb{Z}) \simeq \pi_0\mathbf{H}(\Sigma, \mathbf{B}^nU(1))$$

Hence the "forget the connection" morphism $\mathbf{B}^nU(1)_{conn} \to \mathbf{B}^nU(1)$ induces at the level of equivalence classe the natural morphism

$$\hat{H}^{n+1}(\Sigma;\mathbb{Z}) \to H^{n+1}(\Sigma;\mathbb{Z})$$

from differential cohomology to integral cohomology. See [**79**] for more details.

Example 2.5.4. The differential refinement of $\mathbf{B}^2U(1)$ can by the above construction written out as follows. Consider the 2-stack

$$\mathbf{B}^2U(1)_{conn} := DK\left(C^\infty(-,U(1)) \overset{dlog}{\to} \Omega^1(-) \overset{d}{\to} \Omega^2(-)\right)$$

and let $\{U_i\}$ be a differentiably good open cover of a smooth manifold Σ. A smooth 2-functor $g : C(\{U_i\}) \to \mathbf{B}^2U(1)_{conn}$ can be described by a collection $(\{B_i\}, \{A_{ij}\}, \{g_{ijk}\})$ consisting of

- B_i is a 2-form in $\Omega^2(U_i)$ for any i;
- A_{ij} is a 1-form in $\Omega^1(U_{ij})$ for any i,j such that $dA_{ij} = B_j|_{U_{ij}} - B_i|_{U_{ij}}$;
- g_{ijk} is a smooth function in $C^\infty(U_{ijk}, U(1))$ for any i,j,k such that $dlog g_{ijk} = A_{ij}|_{U_{ijk}} - A_{ik}|_{U_{ijk}} + A_{jk}|_{U_{ijk}}$;
- we have the 2-cocycle constraint $Id_G = g_{ijk}g^{-1}_{ijl}g_{ikl}g^{-1}_{jkl}$ on U_{ijkl} for any i,j,k,l.

Which can be depicted by the simplicial diagram

$$\begin{tikzcd} & B_j \arrow[dr, "A_{jk}"] \arrow[d, phantom, "\Downarrow g_{ijk}"] & \\ B_i \arrow[ur, "A_{ij}"] \arrow[rr, "A_{ik}"'] & & B_k \end{tikzcd}$$

This data defines what is called a *connective structure* on the bundle gerbe from example 2.4.2. It consists of a connection on the $U(1)$-bundle $P \to Y^{[2]}$ which defines a 1-form $A \in \Omega^1(Y^{[2]})$ such that the isomorphism σ_g respects the connection, and of a 2-form $B \in \Omega^2(Y)$, called the *curving*, which has to be related to the connection on P by $F_A = \pi_2^* B - \pi_1^* B$, where F_A is the curvature of A. More precisely the connection A is called a *bundle gerbe connection* if it repsects the bundle gerbe multplication, that is if the section $\sigma_g \in \Gamma(Y^{[3]}, \partial^* P^*)$ satisfies $\sigma_g^*(\partial^* A) = 0$. If A is a bundle gerbe connection, which always exists for bundle gerbes, then the curvature $F_A \in \Omega^2(Y^{[2]})$ satisfies $\partial^* F_A = 0$ and by exactness it follows that there must be a $B \in \Omega^2(Y)$ such that $F_A = \partial^* B = \pi_2^* B - \pi_1^* B$. The collection $(\{B_i\}, \{A_{ij}\}, \{g_{ijk}\})$ satisfying the above conditions gives precisely the local description of this. Consider a good open cover $\{U_i\}$ of Σ with local sections $s_i : U_i \to Y$ and sections over the double overlaps c_{ij} of $(s_i, s_j)^*(P) \to U_{ij}$. Over triple overlaps we have that the bundle gerbe multiplication determines the smooth functions g_{ijk} by $\sigma_g(c_{ij}(x), c_{jk}(x)) = g_{ijk}(x) c_{ik}(x)$. The 2-cocycle constraint corresponds to the associativity condition on σ_g. The 1-form A_{ij} is defined by $A_{ij} = (s_i, s_j)^*(A)$ and the above condition on the A_{ij}'s is the fact that σ_g preserves the connection. The 2-form B_i is defined by $B_i = s_i^*(B)$ and the condition on the B_i's is precisely the curvature condition. (See [**65**]).

EXAMPLE 2.5.5. Consider the 2-stack

$$\mathbf{B}^2 U(1)_{conn^1} := DK\left(C^\infty(-, U(1)) \stackrel{dlog}{\to} \Omega^1(-) \to 0\right)$$

By the forgetful morphims $\mathbf{B}^2 U(1)_{conn} \to \mathbf{B}^2 U(1)_{conn^1}$ we see that it gives the same data as the previous example, only we need to forget everything about the 2-forms $\{B_i\}$, that is we just need to forget the curving. That is for a differentiably good open cover $\{U_i\}$ of a smooth manifold Σ we have that the smooth 2-functor $g : C(\{U_i\}) \to \mathbf{B}^2 U(1)_{conn^1}$ can be described by a collection $(\{A_{ij}\}, \{g_{ijk}\})$ consisting of

- A_{ij} is a 1-form in $\Omega^1(U_{ij})$ for any i, j
- g_{ijk} is a smooth function in $C^\infty(U_{ijk}, U(1))$ for any i, j, k such that $dlog g_{ijk} = A_{ij}|_{U_{ijk}} - A_{ik}|_{U_{ijk}} + A_{jk}|_{U_{ijk}}$;
- we have the 2-cocycle constraint $Id_G = g_{ijk} g_{ijl}^{-1} g_{ikl} g_{jkl}^{-1}$ on U_{ijkl} for any i, j, k, l.

In this last example we recognize all the data that is needed for defining the prequantization of a symplectic groupoid as defined in definition 2.1.7. Consider the symplectic groupoid Σ as a 2-stack and the morphism $\nabla^1 : \Sigma \to \mathbf{B}^2 U(1)_{conn^1} := \mathbf{B}(\mathbf{B}U(1)_{conn})$ of 2-stacks, which can be described by a Hermitian line bundle

$$\begin{tikzcd} & L \arrow[d, "p"] & \\ \Sigma_2 \arrow[r, shift left=2, "pr_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "pr_2"'] & \Sigma_1 \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & \Sigma_0 \end{tikzcd}$$

over the space Σ_1 of morphisms, together with an element $\sigma \in \Gamma(\Sigma_2, \partial^* L^*)$, such that it is a cocycle on σ and has a norm 1 at every point. The bundle gerbe connection gives a Hermitian connection ∇ on the line bundle L such that it satisfies $\sigma^*(\partial^* \nabla) = 0$, which precisely means that the cocycle σ is a covariantly constant section of $\partial^* L^*$. The fact that the curvature of ∇ equals the symplectic form can be reformulated by a diagram similar to the case expressing the prequantization condition of a symplectic manifold, that is it is given by the diagram

$$\begin{tikzcd} & \mathbf{B}(\mathbf{B}U(1)_{conn}) \arrow[d, "\mathbf{B}F_{(-)}"] \\ \Sigma \arrow[ur, "\nabla^1"] \arrow[r, "\omega^1"'] & \mathbf{B}\Omega^2_{cl} \end{tikzcd}$$

To see this, we note that the curvature morphism $F_{(-)} : \mathbf{B}U(1)_{conn} \to \Omega^2_{cl}$ in example 2.5.2 can be presented under $DK(-)$ by the chain map

$$\begin{tikzcd} C^\infty(-, U(1)) \arrow[r, "dlog"] \arrow[d] & \Omega^1(-) \arrow[d, "d"] \\ 0 \arrow[r] & \Omega^2_{cl}(-) \end{tikzcd}$$

Delooping this curvature morphism $F_{(-)}$ gives us the morphism

$$\begin{tikzcd} \mathbf{B}(\mathbf{B}U(1)_{conn}) \arrow[d, "\mathbf{B}F_{(-)}"'] & := DK(& C^\infty(-, U(1)) \arrow[r, "dlog"] \arrow[d] & \Omega^1(-) \arrow[r, "d"] \arrow[d, "d"] & 0) \arrow[d] \\ \mathbf{B}\Omega^2_{cl} & := DK(& 0 \arrow[r] & \Omega^2_{cl}(-) \arrow[r] & 0) \end{tikzcd}$$

Which sends precisely the connection ∇ of the line bundle $L \to \Sigma_1$ to its curvature $F_\nabla \in \Omega^2_{cl}(\Sigma_1)$ which needs to equal the multiplicative symplectic form $\omega \in \Omega^2(\Sigma_1)$ in order for the symplectic groupoid to be prequantizable. Now let Σ be a symplectic groupoid with multiplicative symplectic form $\omega \in \Omega^2(\Sigma_1)$, which as we will see in the next section, can precisely be encoded in the morphism ω^1. Then a prequantization of (Σ, ω) according to definition 2.1.7 is equivalently a lift ∇^1 of ω^1 such that the above diagram is commutative. This show that the prequanization of a Poisson manifold and thus of a symplectic groupoid is an instance of higher geometric prequantization.

3. Higher symplectic geometry

In the example of 3d Chern-Simons theory we mentioned that for higher geometric prequantization we needed a higher analog of symplectic geometry, which is called higher symplectic geometry. In order to interpret the prequantization of a symplectic groupoid, we need that the symplectic groupoid as the moduli stack of fields can naturally be interpreted in terms of higher symplectic geometry.

Higher symplectic geometry is a generalization of symplectic geometry to the context of higher geometry. The first generalization is the generalization of the manifolds with a symplectic form to a manifold equipped with a closed non-degenerate form of arbitrary degree. For example, a 1-plectic manifold, or just a symplectic manifold, M is equipped with a closed, non-degenerate two-form, and a 2-plectic manifold is equipped with a closed, non-degenerate three-form, etc. The second generalization is the generalization of the base manifold to a smooth ∞-groupoid or Lie ∞-algebroid.

First we will show how the symplectic groupoid can be seen as an object in higher symplectic geometry, namely 2-plectic geometry. We will show how the multiplicative symplectic form of the symplectic groupoid can be seen as a degree 3-cocycle in the simplicial de Rham cohomology. We explain how the non-degeneracy of this cocycle is encoded in the symplectic Lie algebroid associated to the Poisson Lie algebroid, which is the infinitesimal approximation of the symplectic groupoid. More generally, these symplectic Lie n-algebroids house a large class of topological field theories, known as the AKSZ σ-models, and in particular this symplectic Lie algebroid gives the Poisson σ-model. This Poisson σ-model can be Lie integrated to a 2d Poisson-Chern-Simons theory. The moduli stack of the 2d Poisson-Chern-Simons theory is precisely the Lie integration of the Poisson Lie algebroid associated to the Poisson manifold.

3.1. Simplicial de Rham cohomology. We saw earlier that the prequantization of a symplectic manifold (M, ω) is equivalently described by a lift ∇ in the diagram

$$\begin{tikzcd} & \mathbf{B}U(1)_{conn} \arrow[d, "F_{(-)}"] \\ M \arrow[ur, dashed, "\nabla"] \arrow[r, "\omega"'] & \Omega^2_{cl} \end{tikzcd}$$

The symplectic form ω as a closed 2-form gives a cocycle in the ordinary de Rham cohomology group of degree 2. In terms of stacks this symplectic form $\omega : M \to \Omega^2_{cl}$ can naturally be seen as a cocycle in the simplicial de Rham cohomology, where the manifold M in terms of a ∞-stack can be represented by a simplicial manifold, which is just the nerve of the Lie groupoid M. We will see that this means that ω can be seen as a degree 2-cocycle in the simplicial de Rham cohomology.

For $n \in \mathbb{N}$ the *de Rham complex* in degree $(n+1)$ is the chain complex of sheaves on the Cartesian spaces of abelian groups

$$\flat U(1)[n+1]^{\infty}_{\mathrm{dR}} := \left(\Omega^1(-) \xrightarrow{d} \Omega^2(-) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n+1}_{cl}(-)\right)$$

The n-stack $\flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)$ is presented by the stackification of the de Rham complex $\flat U(1)[n+1]^{\infty}_{\mathrm{dR}}$ via the Dold-Kan correspondence, that is

$$\flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1) := DK\left(\Omega^1(-) \xrightarrow{d} \Omega^2(-) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n+1}_{cl}(-)\right)$$

We have the canonical morphism

$$\begin{tikzcd} \Omega^{n+1}_{cl} \arrow[d] & := DK(& 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \Omega^{n+1}_{cl}(-)) \arrow[d] \\ \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1) & := DK(& \Omega^1(-) \arrow[r, "d"] & \Omega^2(-) \arrow[r, "d"] & \cdots \arrow[r, "d"] & \Omega^{n+1}_{cl}(-)) \end{tikzcd}$$

For Σ a smooth manifold, for the case where $n \geq 1$ the set of connected components $\pi_0\mathbf{H}(\Sigma, \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1))$ is naturally identified with the ordinary de Rham cohomology of Σ, that is

$$H^{n+1}_{\mathrm{dR}}(\Sigma; \mathbb{R}) \simeq \pi_0\mathbf{H}(\Sigma, \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1))$$

To see this let $\{U_i\}$ be a differentiably good open cover. A element of the ∞-groupoid $\mathbf{H}(\check{C}(\{U_i\}), \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1))$ corresponds to a collection

$$(Z_{i_1, \cdots, i_{n+1}}, \cdots, C_{ijk}, B_{ij}, A_i)$$

of differential forms with $A_i \in \Omega^{n+1}_{cl}(U_i)$, $B_{ij} \in \Omega^n(U_{ij})$, $C_{ijk} \in \Omega^{n-1}(U_{ijk})$, and so on, such that they satisfy the cocycle condition

$$((-1)^n d + \partial^*)(Z_{i_1, \cdots, i_{n+1}}, \cdots, C_{ijk}, B_{ij}, A_i) = 0$$

where ∂^* is the alternating sum of the pullback of forms along the face maps of the Čech nerve $\check{C}(\{U_i\})$.

We only need to show that such a cocycle is equivalent to one given by a globally defined differential form, that is one of the form

$$(0, \cdots, 0, F_i)$$

We will show explicitly that there exist a coboundary by which these two forms differ. For this we begin by using the partition of unity $\rho_i \in C^\infty(\Sigma, [0,1])$ subordinate to the cover $\{U_i\}_{i\in I}$, i.e. we have that $\sum_I \rho_i(x) = 1$ for all $x \in \Sigma$ and for each $x \in \Sigma$ there is a finite number $i \in I$ such that $\rho_i(x) \neq 0$, and add the following coboundary to the first cocycle

$$(Z_{i_1,\cdots,i_{n+1}}, Y_{i_1,\cdots,i_n}, \cdots, C_{ijk}, B_{ij}, A_i) + \delta(\sum_{i_0} \rho_{i_0} Z_{i_0,\cdots,i_n}, 0, \cdots, 0)$$

Where $\sum_{i_0} \rho_{i_0} Z_{i_0,\cdots,i_n} \in \Omega^1(U_{i_1,\cdots,i_n})$. The cocycle condition in particular means that $\partial^* Z_{i_1,\cdots,i_{n+1}} = 0$ which induce the following identity

$$\begin{aligned}
\partial^*\left(\sum_{i_0} \rho_{i_0} Z_{i_0,\cdots,i_n}\right) &= \sum_{i_0} \rho_{i_0} \partial^* Z_{i_0,\cdots,i_n} \\
&= \sum_{i_0} \rho_{i_0} \sum_{k=1}^{n+1} (-1)^k Z_{i_0,i_1,\cdots,\hat{i}_k,\cdots,i_{n+1}} \\
&= -\sum_{i_0} \rho_{i_0} Z_{i_1,\cdots,i_{n+1}} \\
&= -Z_{i_1,\cdots,i_{n+1}} \in \Omega^1(U_{i_1,\cdots,i_{n+1}})
\end{aligned}$$

Where $Z_{i_0,i_1,\cdots,\hat{i}_k,\cdots,i_{n+1}}$ is the pullback of $Z_{i_0,\cdots,i_n}$ along the face map $\pi_{i_1,\cdots,\hat{i}_k,\cdots,i_{n+1}} : U_{i_1,\cdots,i_{n+1}} \to U_{i_1,\cdots,i_{k-1},i_{k+1},\cdots,i_{n+1}}$ and where we used in the third equality that $\partial^* Z_{i_1,\cdots,i_{n+1}} = 0$, which is equivalent to

$$Z_{i_1,\cdots,i_{n+1}} = -\sum_{k=1}^{n+1} (-1)^k Z_{i_0,i_1,\cdots,\hat{i}_k,\cdots,i_{n+1}}$$

Taken together we get

$$\begin{aligned}
&(Z_{i_1,\cdots,i_{n+1}}, Y_{i_1,\cdots,i_n}, \cdots, C_{ijk}, B_{ij}, A_i) + \delta(\sum_{i_0} \rho_{i_0} Z_{i_0,\cdots,i_n}, 0, \cdots, 0) \\
&= (Z_{i_1,\cdots,i_{n+1}}, Y_{i_1,\cdots,i_n}, \cdots, C_{ijk}, B_{ij}, A_i) + (\partial^*\left(\sum_{i_0} \rho_{i_0} Z_{i_0,\cdots,i_n}\right), (-1)^n d \sum_{i_0} \rho_{i_0} Z_{i_0,\cdots,i_n}, 0, \cdots, 0) \\
&= (0, Y_{i_1,\cdots,i_n} + (-1)^n d \sum_{i_0} \rho_{i_0} Z_{i_0,\cdots,i_n}, \cdots, C_{ijk}, B_{ij}, A_i)
\end{aligned}$$

Now by recursively adding such coboundaries we will eventually end up with a cocycle of the form $(0, \cdots, 0, F_i)$. This cocycle is δ-closed, which precisely means that $F_i = F|_{U_i}$ for $F \in \Omega^{n+1}_{cl}(\Sigma)$ a globally defined closed differential form. Moreover, for two such cocycles that differ by a coboundary, that is

$$(0, \cdots, 0, F_i) = (0, \cdots, 0, F'_i) + \delta(\cdots, \kappa_{ij}, \kappa_i)$$

The element $(\cdots, \kappa_{ij}, \kappa_i)$ is itself necessarily of the form $(0, \cdots, 0, \lambda_i)$ with $\lambda_i = \lambda|_{U_i}$ for $\lambda \in \Omega^{n-1}(\Sigma)$ a globally defined differential n-form and thus we have $F = F' + d\lambda$. From this we can conclude that cocycles of $\mathbf{H}(\Sigma, \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1))$ represents classes in de Rham cohomology for $n \geq 1$.

Remark 3.1.1. In [**79, 28**] it is shown that there exist a "universal curvature characteristic" morphim $\mathrm{curv} : \mathbf{B}^n U(1) \to \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)$ and a homotopy pullback diagram

$$\begin{tikzcd}
\mathbf{B}^n U(1)_{conn} \arrow[r, "F_{(-)}"] \arrow[d] & \Omega^{n+1}_{cl} \arrow[d] \\
\mathbf{B}^n U(1) \arrow[r, "\mathrm{curv}"] & \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)
\end{tikzcd}$$

of higher moduli stacks, which induces in cohomology the commutative diagram

$$\begin{tikzcd} \hat{H}^{n+1}(\Sigma;\mathbb{Z}) \arrow[r, "F_{(-)}"] \arrow[d] & \Omega^{n+1}_{cl} \arrow[d] \\ H^{n+1}(\Sigma;\mathbb{Z}) \arrow[r, "\text{curv}"] & H^{n+1}_{\text{dR}}(\Sigma;\mathbb{R}) \end{tikzcd}$$

Which shows that an integral cohomology class together with a closed differential form representing the same class in de Rham cohomology can naturally be refined to a single object in differential cohomology.

Analogous to the de Rham complex of differential forms of an ordinary manifold we have the *simplicial de Rham complex* of a simplicial manifold. A *simplicial manifold* is a simplicial object in the category of smooth manifolds. We will here particularly be interested in simplicial manifolds that come from the nerve of a Lie groupoid $\mathcal{G}$, which we will denote by $\mathcal{G}_\bullet$. The de Rham complex of ordinary manifolds can naturally be generalized to simplicial manifolds. Consider a simplicial manifold, say $\mathcal{G}_\bullet$, then we have the double complex $\Omega^\bullet(\mathcal{G}_\bullet)$, which is given by

$$\begin{tikzcd} \cdots & \cdots & \cdots & \\ \Omega^1(\mathcal{G}_0) \arrow[r, "\partial^*"] \arrow[u, "d"] & \Omega^1(\mathcal{G}_1) \arrow[r, "\partial^*"] \arrow[u, "d"] & \Omega^1(\mathcal{G}_2) \arrow[r, "\partial^*"] \arrow[u, "d"] & \cdots \\ \Omega^0(\mathcal{G}_0) \arrow[r, "\partial^*"] \arrow[u, "d"] & \Omega^0(\mathcal{G}_1) \arrow[r, "\partial^*"] \arrow[u, "d"] & \Omega^0(\mathcal{G}_2) \arrow[r, "\partial^*"] \arrow[u, "d"] & \cdots \end{tikzcd}$$

The boundary maps $d : \Omega^k(\mathcal{G}_n) \to \Omega^{k+1}(\mathcal{G}_n)$ are the usual derivatives of differential forms and $\partial^* : \Omega^k(\mathcal{G}_n) \to \Omega^k(\mathcal{G}_{n+1})$, is the alternating sum of the pullback maps along the face maps of the simplicial manifold. The total complex $Tot(\Omega^\bullet(\mathcal{G}_\bullet))$ of the double complex $\Omega^\bullet(\mathcal{G}_\bullet)$ is the chain complex

$$Tot(\Omega^\bullet(\mathcal{G}_\bullet))_0 \xrightarrow{\delta} Tot(\Omega^\bullet(\mathcal{G}_\bullet))_1 \xrightarrow{\delta} \cdots$$

whose components are the direct sums $Tot(\Omega^\bullet(\mathcal{G}_\bullet))_n = \bigoplus_{k+l=n} \Omega^k(\mathcal{G}_l)$ and whose total differential is given by $\delta = (-1)^n d + \partial^*$, is called the *simplicial de Rham complex*. Under the Dold-Kan correspondence, this simplicial de Rham complex gives precisely a presentation of $\mathbf{H}(\mathcal{G}_\bullet, \flat_{\text{dR}}\mathbf{B}^n U(1))$[**79**]. The cohomology groups of this total complex are

$$H^n_{\text{dR}}(\mathcal{G}_\bullet;\mathbb{R}) = H^n(Tot(\Omega^\bullet(\mathcal{G}_\bullet)))$$

and are called the *simplicial de Rham cohomology* groups of $\mathcal{G}$. A cocycle $[\omega] \in H_{\text{dR}}(\mathcal{G}_\bullet;\mathbb{R})$ of degree n is a collection $\omega = (\omega^0_n, \omega^1_{n-1}, \cdots, \omega^n_0)$ with $\omega^k_l \in \Omega^k(\mathcal{G}_l)$ for $k+l=n$, such that $\delta\omega = 0$. In particular this means that $d\omega^n_0 = 0$ and $\partial^*\omega^0_n = 0$, which means that similar to the argument above we can add a coboundary such that we get a collection of the form $\nu = (0, \nu^1_{n-1}, \cdots, \nu^n_0)$ with $\nu^k_l \in \Omega^k(\mathcal{G}_l)$ for $k+l=n$, which represents the same cocycle $[\omega]$ in the simplicial de Rham cohomology and we have that ν is naturally an element in $\Omega^1(\mathcal{G}_{n-1}) \oplus \cdots \oplus \Omega^{n-1}(\mathcal{G}_1) \oplus \Omega^n_{cl}(\mathcal{G}_0)$. Here we recognize that a cocycle of the simplicial de Rham cohomology $H^n_{\text{dR}}(\mathcal{G}_\bullet;\mathbb{R})$ is precisely a cocycle in the ∞-groupoid $\mathbf{H}(\mathcal{G}_\bullet, \flat_{\text{dR}}\mathbf{B}^n U(1))$, where we interpreted the simplicial manifold $\mathcal{G}_\bullet$ as a ∞-stack. From which we can conclude that

$$H^n_{\text{dR}}(\mathcal{G}_\bullet;\mathbb{R}) \simeq \pi_0 \mathbf{H}(\mathcal{G}_\bullet, \flat_{\text{dR}}\mathbf{B}^n U(1))$$

In this sense for a symplectic manifold (M, ω), we can represent the symplectic form by the morphism of stacks $\omega : M \to \Omega^2_{cl} \hookrightarrow \flat_{\text{dR}}\mathbf{B}^2 U(1)$ and this morphism defines obviously a degree 2-cocycle in the simplicial de Rham cohomology.

Similarly, the prequantization of a symplectic groupoid (Σ, ω) was equivalently a lift ∇^1 of ω^1 such that the following diagram commutes

$$\begin{tikzcd} & \mathbf{B}(\mathbf{B}U(1)_{conn}) \arrow[d, "\mathbf{B}F_{(-)}"] \\ \Sigma \arrow[ur, "\nabla^1"] \arrow[r, "\omega^1"'] & \mathbf{B}\Omega^2_{cl} \end{tikzcd}$$

This ω^1 can be seen as a degree-3 cocycle in the simplicial de Rham cohomology. Since we have obviously the map

$$\omega^1 : \Sigma \to \mathbf{B}\Omega^2_{cl} \hookrightarrow \flat_{\mathrm{dR}}\mathbf{B}^3U(1)$$

which is equivalently an element

$$(0, \omega, 0) \in \bigoplus_{n=0,1,2} \Omega^{3-n}(\Sigma_n)$$

where $\omega \in \Omega^2(\Sigma_1)$ is a symplective form, which is multiplicative, that is $\partial^*\omega = 0$, but this precisely means that this element is a cocycle of degree 3 in the simplicial de Rham complex of Σ, instead of a degree 2 cocycle for the case of a symplectic manifold. This observation shows that the symplectic groupoid is really an object in higher symplectic geometry, namely 2-plectic geometry.

In order for these cocycles to be symplectic cocycles, we need to have a non-degeneracy condition. The non-degeneracy condition of a symplectic form ω on a smooth manifold M means that the contraction map $\iota_{(-)}\omega : \Gamma(TM) \to \Omega^1(M)$ is injective. Similarly a n-plectic form on a smooth manifold M is a closed $(n+1)$-form $\omega \in \Omega^{n+1}(M)$, which is non-degenerate, which means that the contraction map $\iota_{(-)}\omega : \Gamma(TM) \to \Omega^n(M)$ is injective (see [**73**]). This is the first generalization of higher symplectic geometry, where we generalize a 1-plectic manifold, that is a symplectic manifold, to a n-plectic manifold for arbitrary $n \in \mathbb{N}$. The second generalization is the generalization of the base manifold to a smooth ∞-stack or a Lie ∞-algebroid.

Remember that the infinitesimal approximation of a symplectic groupoid is the Poisson Lie algebroid associated to the Poisson manifold which integrates the symplectic groupoid. The cocycle from above has also an infinitesimal variant which is an invariant polynomial, which together form a symplectic Lie algebroid. Conversely, this invariant polynomial of the symplectic Lie algebroid can be integrated to the above cocycle of the symplectic groupoid, see [**79**]. This is what we will treat in the next sections.

3.2. Lie ∞-algebroids. A Lie algebroid serve the same role in the theory of Lie groupoids that Lie algebras serve in the theory of Lie groups. A Lie groupoid can be thought of as a Lie group with many objects, similarly a Lie algebroid is like a Lie algebra with many objects. It is the infinitesimal approximation to the Lie groupoid. We described a way to integrate a Lie algebroid to a particular smooth groupoid, called a Lie groupoid. This integration can naturally extended to smooth ∞-groupoids or smooth ∞-stacks, which we call the differential integration of an Lie ∞-algebroid to a smooth ∞-stack.

In these terms a n-plectic form on a Lie ∞-algebroid $\mathfrak{a}$ can be described as an invariant polynomial ω on $\mathfrak{a}$ which is $(n+1)$-linear, that is $\omega \in W^{n+1}(\mathfrak{a})$, and non-degenerate, which means that $\iota_{(-)}\omega : T\,\mathfrak{a} \to W^n(\mathfrak{a})$ is injective, where $W^n(\mathfrak{a})$ are the elements of the Weil algebra of $\mathfrak{a}$ of degree n. In the case where $\mathfrak{a}$ is a Lie 0-algebroid, $\mathfrak{a}$ is just a smooth manifold X and $W(\mathfrak{a}) = \Omega^\bullet(X)$ the de Rham algebra and an invariant polynomial is just a closed differential of positive degree n. Hence a n-plectic form on $\mathfrak{a}$ is just a closed $(n+1)$-form on X, such that $\iota_{(-)}\omega : \Gamma(TX) \to \Omega^n(X)$ is injective, which recovers our previous definition of a n-plectic form on a smooth manifold X.

A Lie ∞-algebroid is an infinitesimal approximation of a smooth ∞-groupoid.

DEFINITION 3.2.1. The category of Lie ∞-algebroids is the opposite category of the full subcategory of $\mathrm{cdgAlg}_{\mathbb{R}}^{op}$

$$\mathrm{CE} : L_\infty\mathrm{Algd} \hookrightarrow \mathrm{cdgAlg}_{\mathbb{R}}^{op}$$

on graded-commutative cochain dg-algebras in non-negative degree whose underlying graded algebra is an exterior algebra over its degree-0 algebra, and this degree-0 algebra is the algebra of smooth functions on a smooth manifold.

REMARK 3.2.2. This definition is in fact that of an *affine* $C^\infty(X)$ Lie ∞-algebroid over a smooth manifold X. But for this introductory discussion the above definition will suffice for the cases we treat in this thesis and we don't need to refine this to something more encompassing. For a full comprehensive account see [**79**].

In practice an object $\mathfrak{a} \in L_\infty\mathrm{Algd}$ may be identified (non-canonically) with a pair $(\mathrm{CE}(\mathfrak{a}), X)$,

(i) X is a smooth manifold, called the *base space* of the Lie ∞-algebroid;
(ii) $\mathfrak{a}$ is a non-positively graded $C^\infty(X)$-module degreewise of finite rank
(iii) $\mathrm{CE}(\mathfrak{a}) = (\wedge^\bullet_{C^\infty(X)} \mathfrak{a}^*, d_{\mathrm{CE}(\mathfrak{a})})$ is a differential graded commutative algebra, called the *Chevalley-Eilenberg algebra* of the Lie ∞-algebroid, where

$$\wedge^\bullet_{C^\infty(X)} \mathfrak{a}^* = C^\infty(X) \oplus \mathfrak{a}_0^* \oplus ((\mathfrak{a}_0^* \wedge_{C^\infty(X)} \mathfrak{a}_0^*) \oplus \mathfrak{a}_1^*) \oplus \cdots$$

with the kth summand on the right being in degree k and

$$d_{\mathrm{CE}(\mathfrak{a})} : \wedge^\bullet_{C^\infty(X)} \mathfrak{a}^* \to \wedge^\bullet_{C^\infty(X)} \mathfrak{a}^*$$

is a degree $+1$ derivation linear over the ground field such that $d^2_{\mathrm{CE}(\mathfrak{a})} = 0$.

If $\mathfrak{a}$ is concentrated in degree 0 throught $-(n-1)$, then we speak of a *Lie n-algebroid* and if the base space $X = *$ the point we speak of a *Lie n-algebra*.

REMARK 3.2.3. An Lie ∞-algebroid with base space $X = *$ the point is an Lie ∞-algebra, or rather is the delooping of an *Lie ∞-algebra*. We write $b\,\mathfrak{g}$ for Lie ∞-algebroids over the point and they form the full subcategory

$$b : L_\infty\mathrm{Alg} \hookrightarrow L_\infty\mathrm{Algd}$$

of the traditional category of Lie ∞-algebras into that of Lie ∞-algebroids.

EXAMPLE 3.2.4. For $X = *$ and $\mathfrak{a}$ concentrated in degree 0 the finite dimensional Lie algebra $\mathfrak{g}$, we have that $\mathrm{CE}(\mathfrak{a})$, where the underlying graded commutative algebra is the Grassmann algebra on $\mathfrak{g}^*$, that is

$$\wedge^\bullet \mathfrak{g}^* = \mathbb{R} \oplus \mathfrak{g}^* \oplus (\mathfrak{g}^* \wedge \mathfrak{g}^*) \oplus \cdots$$

and where the Chevalley-Eilenberg differential $d_{\mathrm{CE}(\mathfrak{g})}$ of degree $+1$ is on $\mathfrak{g}^*$ the dual of the Lie bracket

$$d_{\mathrm{CE}(\mathfrak{g})} := [-,-]^* : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$$

extended uniquely as a graded derivation on $\wedge^\bullet \mathfrak{g}^*$. The fact that the Lie bracket satisfies the Jacobi identity means precisely that the differential $d_{\mathrm{CE}(\mathfrak{g})}$ squares to 0, i.e. $d^2_{\mathrm{CE}(\mathfrak{g})} = 0$. This $\mathrm{CE}(\mathfrak{a}) = (\wedge^\bullet \mathfrak{g}^*, d_{\mathrm{CE}(\mathfrak{g})})$ is precisely the ordinary Chevalley-Eilenberg algebra. The Lie ∞-algebroid arising this way can be written as $\mathfrak{a} = b\,\mathfrak{g}$, which is the *delooping* of $\mathfrak{g}$. This notation is the infinitesimal analog of the notation $\mathbf{B}G$ for the one-object delooping groupoid corresponding to the Lie group G.

EXAMPLE 3.2.5. For X an arbitrary smooth manifold and $\mathfrak{a}$ concentrated in degree 0, this is equivalent to the usual definition of a Lie algebroids as vector bundle $E \to X$ with anchor map $\rho : E \to TX$, where we have $\mathrm{CE}(\mathfrak{a}) = (\wedge^\bullet_{C^\infty(X)}\Gamma(E)^*, d_{\mathrm{CE}(\mathfrak{a})})$ and the anchor is encoded in the map $d_{\mathrm{CE}(\mathfrak{a})} : C^\infty(X) \to \Gamma(E)^*$ which sends $f \mapsto \rho(-)(f)$ and can uniquely be extended to a graded derivation on $\wedge^\bullet_{C^\infty(X)}\Gamma(E)^*$. For $X = *$ this definition indeed reproduces the previous example.

EXAMPLE 3.2.6. For $n \in \mathbb{N}$ the delooping of the *line Lie n-algebra* is the Lie ∞-algebroid $b^{n-1}\mathbb{R}$ defined by the fact that $\mathrm{CE}(b^{n-1}\mathbb{R})$ is generated over $\mathbb{R}$ from a single generator in degree n with vanishing differential.

EXAMPLE 3.2.7. For X a smooth manifold and TX the *tangent Lie algebroid*, the corresponding Chevalley-Eilenberg algebra is precisely the *de Rham algebra* of X, that is

$$\mathrm{CE}(TX) = (\Omega^\bullet(X), d_{\mathrm{dR}})$$

Notice that $\Omega^\bullet(X) := \wedge^\bullet_{C^\infty(X)}\Gamma(T^*X)$ and the anchor map $\rho = Id$.

EXAMPLE 3.2.8. For (M,π) a Poisson manifold, where $\pi \in \wedge^2\Gamma(TM)$ is the Poisson bivector, we have the corresponding Poisson Lie algebroid $\mathfrak{B}(M,\pi)$, see appendix A.0.1.13. The $\mathrm{CE}(\mathfrak{B}(M,\pi))$, where the underlying graded commutative algebra is the Grassmann algebra on $\Gamma(TM)$, that is $\wedge^\bullet\Gamma(TM)$ the multivectors on M. Notice that the Poisson bivector π is an element of degree 2 in $\wedge^\bullet\Gamma(TM)$. The Lie bracket on tangent vectors in $\Gamma(TM)$ extends uniquely to a bracket $[-,-]_{\mathrm{Sch}}$ on multivector fields, called the *Schouten bracket*. It can be checked, see [**75**], that the Poisson bracket π satisfies the Jacobi identity precisely if and only if π satisfies

$$[\pi,\pi]_{\mathrm{Sch}} = 0$$

This makes the Chevalley-Eilenberg differential

$$d_{\mathrm{CE}(\mathfrak{B}} := [\pi,-]_{\mathrm{Sch}} : \wedge^\bullet\Gamma(TM) \to \wedge^\bullet\Gamma(TM)$$

into a differential of degree $+1$ on multivector fields, that squares to 0. Hence the Poisson Lie algebroid is defined by

$$\mathrm{CE}(\mathfrak{B}(M,\pi)) = (\wedge^\bullet\Gamma(TM), [\pi,-]_{\mathrm{Sch}})$$

where π is the Poisson bivector.

For simplicity we assume for the moment that $\mathfrak{g}$ is a Lie ∞-algebra. The generalization to Lie ∞-algebroids should be straightforward. For X a smooth manifold, we can think of $Hom_{\mathrm{dgAlg}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(X))$ as the set of Lie ∞-algebroid valued differential 1-forms, whose curvature form vanishes. We can see this by first forgetting the differential structure and denoting

$$\Omega^\bullet(X,\mathfrak{g}) := Hom_{\mathrm{grAlg}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(X)) \subset Hom_{\mathrm{grVect}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(X))$$

for the space of graded algebra homomorphisms from $\mathrm{CE}(\mathfrak{g})$ to $\Omega^\bullet(X)$, which is a subspace of linear maps (in particular of degree 0) from $\mathrm{CE}(\mathfrak{g})$ to $\Omega^\bullet(X)$ as graded vector spaces. By definition $\mathrm{CE}(\mathfrak{g})$ is freely generated generated algebra and is degreewise of finite rank and hence this is isomorphic to the space of linear grading preserving maps $Hom_{\mathrm{grVect}}(\mathfrak{g}^*, \Omega^\bullet(X))$ from graded vector space $\mathfrak{g}^*$ of dual generators to $\Omega^\bullet(X)$ as a graded vector space. Since these maps are of degree 0 and recalling that $\mathfrak{g}$ is non-positvely graded, this is is isomorphic to the space of elements of total degree 1 in elements of $\Omega^\bullet(X)$ tensored with $\mathfrak{g}$

$$Hom_{\mathrm{grVect}}(\mathfrak{g}^*, \Omega^\bullet(X)) \simeq (\Omega^\bullet(X) \otimes \mathfrak{g})_1$$

The dg-algebra homomorphisms form a subspace of this space on those elements that respect the differential and is denoted by

$$\Omega^\bullet(X,\mathfrak{g})_{\mathrm{flat}} := Hom_{\mathrm{dgAlg}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(X)) \hookrightarrow Hom_{\mathrm{grAlg}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(X)) \simeq (\Omega^\bullet(X) \otimes \mathfrak{g})_1$$

Under this equivalence these elements A in $(\Omega^\bullet(X) \otimes \mathfrak{g})_1$ satisfy a certain condition. By a simple computations, one finds that this condition is precisely the flatness constraint, namely we have the *Mauer-Cartan equation*

$$dA + \partial A + [A \wedge A] + [A \wedge A \wedge A] + \cdots = 0$$

where the differential d and $\wedge$ are the operations in $\Omega^\bullet(X)$ and where $[.,.,\cdots]$ are the n-ary brackets in the Lie ∞-algebra $\mathfrak{g}$ and ∂ is the differential in the chain complex $\mathfrak{g}$. For $\mathfrak{g}$ an ordinary Lie algebra we only have the binary bracket and $A \in \Omega^1(X) \otimes \mathfrak{g}$ satisfies the ordinary Maurer-Cartan equation $dA + [A \wedge A] = 0$.

In order to describe the non-flat $\mathfrak{g}$-valued differential 1-forms by homomorphism of differential graded algebras we need to pass from the Chevalley-Eilenberg algebra to the Weil algebra, which we will construct now. Consider the Lie ∞-algebroid $\mathfrak{a}$ with base space X, then we can form the Grassman algebra over $C^\infty(X)$ on the graded $C^\infty(X)$-module $\Gamma(T^*X) \oplus \mathfrak{a}^* \oplus \mathfrak{a}^*[1]$, where $\mathfrak{a}^*[1]$ is the shifted copy of $\mathfrak{a}^*$, that is

$$\wedge^\bullet(\Gamma(T^*X) \oplus \mathfrak{a}^* \oplus \mathfrak{a}^*[1])$$

This is equipped with the differential $\mathbf{d}$ defined on generators as follows

(i) $\mathbf{d}|_{C^\infty(X)} = d_{\mathrm{dR}}$ is the ordinary de Rham differential with values in $\Omega^1(X) := \Gamma(T^*X)$;
(ii) $\mathbf{d}|_{\mathfrak{a}^*} : \mathfrak{a}^* \to \mathfrak{a}^*[1]$ is the degree-shift isomorphism;
(iii) and $\mathbf{d}$ vanishes on all remaining generators.

which can be extended uniquely as a graded derivation. This defines the tangent Lie ∞-algebroid $\mathfrak{T}\,\mathfrak{a}$ of $\mathfrak{a}$ by $\mathrm{CE}(\mathfrak{T}\,\mathfrak{a}) = (\wedge^\bullet(\Gamma(T^*X) \oplus \mathfrak{a}^* \oplus \mathfrak{a}^*[1]), \mathbf{d})$.

Example 3.2.9. Let X be a smooth manifold, as a Lie 0-algebroid it can be seen as a Lie ∞-algebroid. The tangent Lie ∞-algebroid $\mathfrak{T}X$, that is

$$\mathrm{CE}(\mathfrak{T}X) = (\wedge^\bullet(\Gamma(T^*X), d_{\mathrm{dR}})$$

corresponds precisely to the tangent Lie algebroid TX as defined in example 3.2.7.

Example 3.2.10. For $\mathfrak{g}$ an ordinary finite dimensional Lie algebra. The tangent Lie ∞-algebroid $\mathfrak{T}\,\mathfrak{g}$ is given by

$$\mathrm{CE}(\mathfrak{T}\,\mathfrak{g}) = (\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1]), \mathbf{d})$$

where $\mathbf{d} : \mathfrak{g}^* \to \mathfrak{g}^*[1]$ is the grade shifting isomorphism on the generators, which can be extended as a graded derivation. The notion of the ordinary Weil algebra $W(\mathfrak{g})$, see [**12**], can be defined as the Chevalley-Eilenberg algebra of this tangent Lie ∞-algebroid, it has

- as underlying graded algebra the graded algebra of the tangent Lie ∞-algebroid $\mathfrak{T}\,\mathfrak{g}$, that is

$$\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1])$$

- equipped with a differential on the copy $\mathfrak{g}^*$

$$d_{W(\mathfrak{g})}|_{\mathfrak{g}^*} = d_{\mathrm{CE}(\mathfrak{g})} + \mathbf{d}$$

  where $d_{\mathrm{CE}(\mathfrak{g})}$ acts on $\mathfrak{g}^*$ as the differential of the Chevalley-Eilenberg algebra of $\mathfrak{g}$ and is extended uniquely to the shifted generators $\mathfrak{g}^*[1]$ by the graded commutativity

$$d_{\mathrm{CE}(\mathfrak{g})}\mathbf{d} = -\mathbf{d}d_{\mathrm{CE}(\mathfrak{g})}$$

  which in turn can be extended as a graded derivation to $\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1])$.

This defines the ordinary Weil algebra

$$W(\mathfrak{g}) = \mathrm{CE}(\mathfrak{T}\,\mathfrak{g}) = (\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1]), d_{W(\mathfrak{g})})$$

Remark 3.2.11. This abstract construction of the Weil algebra $W(\mathfrak{g})$ looks a bit arbitrary, but in fact it is the unique dg-algebra free on the underlying graded vector space such that the projection morphism $i^* : \mathfrak{g}^* \oplus \mathfrak{g}^*[1] \to \mathfrak{g}^*$ of graded vector spaces extends to a dg-homomophism $i^* : W(\mathfrak{g}) \to \mathrm{CE}(\mathfrak{g})$ (see [**28, 76, 77**]).

This notion of the ordinary Weil algebra for Lie algebras can be extended to Lie ∞-algebras and Lie ∞-algebroids. Analogous to the previous example we define the Weil algebra of a Lie ∞-algebroid $\mathfrak{a}$ as the Chevalley-Eilenberg algebra of the tangent Lie ∞-algebroid $\mathfrak{T}\,\mathfrak{a}$.

DEFINITION 3.2.12. The *Weil algebra* of an Lie ∞-algebroid $\mathfrak{a}$ with base space X, is the differential graded commutative algebra

$$W(\mathfrak{a}) := (\wedge^\bullet(\Gamma(T^*X) \oplus \mathfrak{a}^* \oplus \mathfrak{a}^*[1]), d_{W(\mathfrak{a})})$$

where the differential is the sum

$$d_{W(\mathfrak{a})} = d_{\mathrm{CE}(\mathfrak{a})} + \mathbf{d}$$

of two degree $+1$ graded derivations, where $\mathbf{d}$ acts on the generators as above and $d_{\mathrm{CE}(\mathfrak{a})}$ acts on the unshifted elements in $\mathfrak{a}^*$ as the differential of the Chevalley-Eilenberg algebra of $\mathfrak{a}$ and is extended uniquely to shifted generators by the graded commutativity

$$d_{\mathrm{CE}(\mathfrak{g})}\mathbf{d} = -\mathbf{d}d_{\mathrm{CE}(\mathfrak{g})}$$

Furthermore $d_{\mathrm{CE}(\mathfrak{a})}$ vanishes on $C^\infty(X)$

REMARK 3.2.13. The correct definition of the Weil algebra of a Lie ∞-algebroid, as stated in [**26**], should be over Lie ∞-algebroids with in degree 0 an arbitrary $\mathbb{R}$-algebra A. The Weil algebra is defined as a representative of the free smooth dg-algebra on the underlying graded A-modules such that the projection morphism $i^* : \Omega^\bullet(A) \oplus \mathfrak{a}^* \oplus \mathfrak{a}^*[1] \to \mathfrak{a}^*$ of graded A-modules extends to a dg-homomorphism $i^* : W(\mathfrak{a}) \to \mathrm{CE}(\mathfrak{a})$ (see [**28, 76, 77**]). We will focus here on the special case where the $\mathbb{R}$-algebra is the smooth algebra $C^\infty(X)$ of smooth functions over a smooth manifold X, for which we gave the explicit definition.

EXAMPLE 3.2.14. Let $\mathfrak{g}$ be a Lie algebra, then the definition of $W(b\,\mathfrak{g})$ recovers precisely the ordinary definition of the Weil algebra as in example 3.2.10.

EXAMPLE 3.2.15. Let $\mathfrak{a} = X$ be an ordinary smooth manifold, then $W(X) = \Omega^\bullet(X)$, that is the ordinary de Rham algebra of X.

EXAMPLE 3.2.16. Let $\mathfrak{a} = b^{n-1}\mathbb{R}$ be the delooping of the line Lie n-algebra, then $W(b^{n-1}\mathbb{R})$ is the free dg-algebra on a single generator c in degree n. In other words it is the graded algebra of two generators c and γ, with c in degree n and γ in degree $n+1$ together with a differential $d_{W(b^{n-1}\mathbb{R})}$ defined by sending c to γ.

For $\mathfrak{g}$ a Lie ∞-algebra, $W(\mathfrak{g})$ is the unique dg-algebra free on the underlying graded vector space $\mathfrak{g}^*$ such that the canonical projection $i^* : W(\mathfrak{a}) \to \mathrm{CE}(\mathfrak{a})$ is a dg-homomorphism. Due to the freeness of $W(\mathfrak{g})$ we have an isomorphism

$$\Omega^\bullet(X, \mathfrak{g}) = Hom_{\mathrm{grAlg}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(X)) \simeq Hom_{\mathrm{dgAlg}}(W(\mathfrak{g}), \Omega^\bullet(X))$$

from which we conclude that $Hom_{\mathrm{dgAlg}}(W(\mathfrak{g}), \Omega^\bullet(X))$ is the collection of total degree 1 differential form with values in the Lie ∞-algebra $\mathfrak{g}$. Consider a morphism

$$(A, F_A) : W(\mathfrak{g}) \to \Omega^\bullet(X)$$

Then by similar computation as for finding the Maurer-Cartan equations one finds that

$$F_A = dA + \partial A + [A \wedge A] + [A \wedge A \wedge A] + \cdots$$

and precisely if the curvature vanish, that is $F_A = 0$, then this morphism factors through the Chevalley-Eilenberg algebra

$$\begin{tikzcd} & \mathrm{CE}(\mathfrak{g}) \arrow[dl, "\exists A_{\mathrm{flat}}"'] \\ \Omega^\bullet(X) & W(\mathfrak{g}) \arrow[l, "A"] \arrow[u] \end{tikzcd}$$

in which case we call A flat.

3.3. Cocycles, invariant polynomials and Chern-Simons elements.

DEFINITION 3.3.1. For $\mathfrak{a}$ a Lie ∞-algebroid and $n \in \mathbb{N}$, a *cocycle* in degree n on $\mathfrak{a}$ is an element $\mu \in \mathrm{CE}(\mathfrak{a})$ which is $d_{\mathrm{CE}(\mathfrak{a})}$-closed, i.e. $d_{\mathrm{CE}(\mathfrak{a})}\mu = 0$.

Since $b^{n-1}\mathbb{R}$ is the Lie ∞-algebroid whose Chevalley-Eilenberg algebra has a single generator in degree n and a trivial differential, a cocycle μ in degree n on $\mathfrak{a}$ is presicely given by a morphism of dg-algebras $\mu : \mathrm{CE}(b^{n-1}\mathbb{R}) \to \mathrm{CE}(\mathfrak{a})$ or dually by a morphism of Lie ∞-algebroids $\mu : \mathfrak{a} \to b^{n-1}\mathbb{R}$.

EXAMPLE 3.3.2. Let $\mathfrak{a} = b\,\mathfrak{g}$ be the delooping of a Lie algebra $\mathfrak{g}$, a cocycle on $\mathfrak{a}$ of degree n corresponds precisely with a traditional Lie algebra cocycle on $\mathfrak{g}$ of degree n.

EXAMPLE 3.3.3. Let X be a smooth manifold, a cocycle in degree n of the tangent Lie ∞-algebroid $\mathfrak{T}X$ is precisely a closed n-form on X.

DEFINITION 3.3.4. An *invariant polynomial* on $\mathfrak{a}$ is an $d_{W(\mathfrak{a})}$-closed element $\langle - \rangle$ in $\wedge^\bullet(\Gamma(T^*X) \oplus \mathfrak{a}^*[1]) \subset W(\mathfrak{a})$.

The property $\langle - \rangle \in \wedge^\bullet(\Gamma(T^*X) \oplus \mathfrak{a}^*[1])$ says that for any $v \in \mathfrak{a}$ and $\iota_v : W(\mathfrak{a}) \to W(\mathfrak{a})$ the contraction derivation, we have

$$\iota_v\langle - \rangle = 0 \quad \text{(horizontality)}$$

Together with the second property $d_{W(\mathfrak{a})}\langle - \rangle = 0$ this implies that for the Lie derivative

$$\mathcal{L}_v := [d_{W(\mathfrak{a})}, \iota_v]$$

in $W(\mathfrak{a})$ along $v \in \mathfrak{a}$, which encodes the coadjoint action of $\mathfrak{a}$ on $W(\mathfrak{a})$, we have

$$\mathcal{L}_v\langle - \rangle = 0 \quad \text{(ad-invariance)}$$

EXAMPLE 3.3.5. Let $\mathfrak{a} = b\,\mathfrak{g}$ be the delooping of an ordinary Lie algebra $\mathfrak{g}$. The above encodes precisely the classical definition of $ad_{\mathfrak{g}}$-invariant polynomials. Indeed, for a Lie algebra $\mathfrak{g}$, the condition $d_{W(\mathfrak{a})}\langle - \rangle = 0$ implies precisely the $ad_{\mathfrak{g}}$-invariance of an element $\langle - \rangle \in \wedge^\bullet(\mathfrak{g}^*[1])$.

EXAMPLE 3.3.6. Let $\mathfrak{a} = X$ be a smooth manifold, seen as a Lie 0-algebroid, an invariant polynomial of degree n is a closed differential form of degree n.

DEFINITION 3.3.7. Let $\langle - \rangle \in W(\mathfrak{a})$ be an invariant polynomial on a Lie ∞-algebroid, we say a cocycle $\mu \in \mathrm{CE}(\mathfrak{a})$ is *in transgression* with $\langle - \rangle$ if there exists an element $cs \in W(\mathfrak{a})$ such that

(i) $d_{W(\mathfrak{a})}cs = \langle - \rangle$;
(ii) $i * cs = \mu$

We say that cs is a *Chern-Simons element* witnessing this transgression.

The above ingredients can be summarized in the following diagram

$$\begin{tikzcd}
\mathrm{CE}(\mathfrak{a}) & \mathrm{CE}(b^{n-1}\mathbb{R}) \arrow[l, "\mu"'] \\
W(\mathfrak{a}) \arrow[u] & W(b^{n-1}\mathbb{R}) \arrow[l, "(cs,\langle - \rangle)"'] \arrow[u] \\
\mathrm{inv}(\mathfrak{a}) \arrow[u] & \mathrm{inv}(b^{n-1}\mathbb{R}) \arrow[l, "\langle - \rangle"'] \arrow[u]
\end{tikzcd}$$

For the full construction of this commuting diagram we refer the reader to [**28, 79**].

Remember from the motivating example 1 of 3d Chern-Simons theory that we talked about transgressing an cocycle via a Chern-Simons element to an invariant polynomial. There the Chern-Simons element played the role of the Lagrangian of the theory, which maps a $\mathfrak{g}$-valued 1-form A to a differential form $CS(A)$. Actually the above transgression encodes precisely this information, remember that the dg-algebra morphisms $\Omega^\bullet(\Sigma) \leftarrow W(\mathfrak{g}) : (A, F_A)$ are in natural bijection with the degree 1 $\mathfrak{g}$-valued differential forms. Hence we write

$$\Omega^\bullet(\Sigma) \xleftarrow{A} W(\mathfrak{g}) \xleftarrow{cs} W(b^{n+1}\mathbb{R}) : cs(A)$$

for the differential form associated by the Chern-Simons element cs to the degree 1 $\mathfrak{g}$-valued differential form A, and call this the Chern-Simons differential form associated to A. This differential form $cs(A) \in \Omega^\bullet(\Sigma)$ can be integrated to the corresponding (higher) Chern-Simons action functional

$$S_{\langle - \rangle} : A \mapsto \int_\Sigma cs(A)$$

Similarly, for $\langle - \rangle$ an invariant polynomial on $\mathfrak{g}$, we write

$$\Omega^\bullet(\Sigma)_{cl} \xleftarrow{A} W(\mathfrak{g}) \xleftarrow{\langle - \rangle} \mathrm{inv}(b^{n+1}\mathbb{R}) : \langle F_A \rangle$$

Which we call the *curvature characteristic form* of A with repspect to $\langle - \rangle$.

Since we can pick $\mathfrak{g}$ any Lie ∞-algebroid, the above constitute not only 3d Chern-Simons theory, but a whole class of action functionals which go under name of ∞-Chern-Simons theory. We will see in the next section how this construction secretly encodes the generalization of the ordinary Chern-Weil homomorphism to the full ∞-Chern-Weil homomorphism in ∞-Chern-Weil theory.

Remark 3.3.8. The degree 1 $\mathfrak{g}$-valued differential form on Σ should be thought of a (non-trivial) $\mathfrak{g}$-valued connection on a trivial principal ∞-bundle on Σ. Remember in our example of 3d Chern-Simons theory the principal G-bundle on a 3-dimensional smooth manifold Σ can always be trivialized. This principal G-bundle on Σ can be obtained via universal integration of $\mathfrak{g}$, which we will treat in the section 3.5.

In the case of 3d Chern-Simons theory we transgressed a cocycle to a binary invariant polynomial. For our purpose it suffice to consider only a certain class of binary invariant polynomials, namely the class of binary invariant polynomials that constitute what we call a symplectic Lie n-algebroid.

Definition 3.3.9. A *symplectic Lie n-algebroid* $(\mathfrak{B}, \omega)$ is a Lie n-algebroid $\mathfrak{B}$ that is equipped with a quadratic non-degenerate invariant polynomial $\omega \in W(\mathfrak{B})$ of degree $n+2$.

This definition means that for each chart $U \to X$ where X is the base manifold of $\mathfrak{B}$, there is a basis $\{x^i\}$ for $\mathrm{CE}(\mathfrak{B}\,|_U)$ such that

$$\omega = \frac{1}{2}\omega_{ij}\mathbf{d}x^i \wedge \mathbf{d}x^j$$

with $\{\omega_{ij} \in \mathbb{R} \hookrightarrow C^\infty(X)\}$ and $\deg(x^i) + \deg(x^j) = n$. The non-degeneracy condition precisely means that the coefficient matrix $\{\omega_{ij}\}$ has an inverse and furthermore we have that $d_{W(\mathfrak{B})}\omega = 0$. These symplectic Lie n-algebroids $(\mathfrak{B}, \omega)$ are important for us, since these invariant polynomials ω are non-degenerate and hence they are $(n+1)$-plectic forms on the Lie n-algebroid $\mathfrak{B}$. We will give now two examples that are important for us.

Example 3.3.10. (Symplectic manifold: $n = 0$) A Lie 0-algebroid is just a smooth manifold M. A quadratic non-degenerate invariant polynomial of degree 2 on M is precisely a non-degenerate closed 2-form $\omega \in \Omega^2(M)$, that is a symplectic 2-form. Hence a symplectic manifold being a pair (M, ω) is a symplectic Lie 0-algebroid.

Example 3.3.11. (Poisson manifold: $n = 1$) For a Poisson manifold (M, π), with Poisson bivector $\pi \in \wedge^2\Gamma(TX)$, we have the corresponding Poisson Lie algebroid $\mathfrak{B}(M, \pi)$. The Chevalley-Eilenberg algebra of the Poisson Lie algebroid is given by

$$\mathrm{CE}(\mathfrak{B}(M, \pi)) = (\wedge^\bullet\Gamma(TM), d_{\mathrm{CE}(\mathfrak{B})} = [\pi, -]_{\mathrm{Sch}})$$

which is explained in example 3.2.8. Now if we work locally, that is if we consider a chart $U \to M$, then the underlying graded algebra of $\mathrm{CE}(\mathfrak{P}(M,\pi)|_U)$ is generated from degree 0 elements $\{x^i\}$ and degree 1 elements $\{\partial_i\}$, since the underlying graded algebra is free over $C^\infty(U)$ (remember it is a Grassmann algebra).The Poisson bivector can be written as

$$\pi = -\frac{1}{2}\pi^{ij}\partial_i \wedge \partial_j$$

which is evidently a Lie algebroid cocycle since

$$d_{\mathrm{CE}(\mathfrak{P})}\pi = [\pi,\pi]_{\mathrm{Sch}} = 0$$

Acting with the differential $d_{\mathrm{CE}(\mathfrak{P})}$ on the generators gives us

$$\begin{aligned} d_{\mathrm{CE}(\mathfrak{P})}x^i &= [\pi, x^i]_{\mathrm{Sch}} = -\frac{1}{2}[\pi^{jk}\partial_j \wedge \partial_k, x^i]_{\mathrm{Sch}} \\ &= \frac{1}{2}\partial_j \wedge \partial_k \wedge [\pi^{jk}, x^i] - \frac{1}{2}\pi^{jk}\partial_k \wedge [\partial_j, x^i] + \frac{1}{2}\pi^{jk}\partial_j \wedge [\partial_k, x^i] \\ &= -\pi^{ij}\partial_j \end{aligned}$$

Where we used that $[\partial_j, x^i] = \mathcal{L}_{\partial_j} x^i = dx^i(\partial_j)$ is the Lie derivative of ∂_j along the smooth function x^i. Similarly

$$\begin{aligned} d_{\mathrm{CE}(\mathfrak{P})}\partial_i &= [\pi, \partial_i]_{\mathrm{Sch}} = -\frac{1}{2}[\pi^{jk}\partial_j \wedge \partial_k, \partial_i]_{\mathrm{Sch}} \\ &= \frac{1}{2}\partial_j \wedge \partial_k \wedge [\pi^{jk}, \partial_i] - \frac{1}{2}\pi^{jk}\partial_k \wedge [\partial_j, \partial_i] + \frac{1}{2}\pi^{jk}\partial_j \wedge [\partial_k, \partial_i] \\ &= \frac{1}{2}\frac{\partial \pi^{jk}}{\partial x^i}\partial_j \wedge \partial_k = 0 \end{aligned}$$

Where $[\partial_j, \partial_i] = 0$. By definition, the Weil algebra $W(\mathfrak{P}(M,\pi))$ is generated from elements x^i and ∂_i, together with their shifted partners $\mathbf{d}x^i$ and $\mathbf{d}\partial_i$, and it has the differential

$$d_{W(\mathfrak{P})} = [\pi, -]_{\mathrm{Sch}} + \mathbf{d}$$

Which acts on the generators x^i and ∂_i as

$$\begin{aligned} d_{W(\mathfrak{P})}x^i &= -\pi^{ij}\partial_j + \mathbf{d}x^i \\ d_{W(\mathfrak{P})}\partial_i &= \mathbf{d}\partial_i \end{aligned}$$

Now consider element

$$cs = \partial_i \wedge \mathbf{d}x^i + \pi \in W(\mathfrak{P})$$

We will show that this element cs is a Chern-Simons element transgressing the cocycle π to an invariant polynomial, say ω. First we have obviously the property that

$$i^* cs = cs|_{\wedge^\bullet \Gamma(TM)} = \pi$$

For the second property of a Chern-Simons element, lets rewrite cs in terms of $d_{W(\mathfrak{P})}$ instead of $\mathbf{d}$, this is

$$cs = \partial_i \wedge d_{W(\mathfrak{P})}x^i - \pi$$

Then

$$\begin{aligned} d_{W(\mathfrak{P})}cs &= d_{W(\mathfrak{P})}\partial_i \wedge d_{W(\mathfrak{P})}x^i - d_{W(\mathfrak{P})}\pi \\ &= \mathbf{d}\partial_i \wedge (-\pi^{ij}\partial_j) + \mathbf{d}\partial_i \wedge \mathbf{d}x^i - \mathbf{d}\pi \\ &= \mathbf{d}\partial_i \wedge \mathbf{d}x^i \\ &= \mathbf{d}x^i \wedge \mathbf{d}\partial_i =: \omega \end{aligned}$$

Where in the second equality we used that $d_{W(\mathfrak{B})}\pi = \mathbf{d}\pi$, since π is cocycle, in the third equality we used that

$$\mathbf{d}\pi = -\frac{1}{2}\frac{\partial \pi^{jk}}{\partial x^i}\partial_j \wedge \partial_k \wedge \mathbf{d}x^i - \pi^{jk}\mathbf{d}\partial_j \wedge \partial_k$$

And in the last equality we used that $W(\mathfrak{B})$ has an underlying commutative algebra and since the degree of $\mathbf{d}\partial_i$ and $\mathbf{d}x^i$ is 2 and 1 respectively, they have to commute like this. Furthermore this ω is an non-degenerate invariant polynomial of degree 3, since it is a $d_{W(\mathfrak{B})}$-closed element $\omega \in \wedge^\bullet(\Gamma(T^*M) \oplus \Gamma(TM)[1])$ that is obviously non-degenerate. We conclude that for a Poisson manifold (M, π) with Poisson bivector $\pi = \pi^{ij}\partial_i \wedge \partial_j$, the pair

$$(\mathfrak{B}(M, \pi), \omega = \mathbf{d}x^i \wedge \mathbf{d}\partial_i)$$

consisting of the Poisson Lie algebroid $\mathfrak{B}(M, \pi)$ and of the invariant polynomial ω that is in transgression with the cocycle π, is a symplectic Lie algebroid.

3.4. Poisson σ-model. We saw in the example of previous section that there are certain symplectic Lie algebroids where the invariant polynomial is in transgression with a cocycle, via a Chern-Simons element. More generally, if a symplectic Lie n-algebroid $(\mathfrak{B}, \omega)$ give rise to a triple (π, cs, ω), consisting of a Chern-Simons element transgressing an invarient polynomial ω to a cocycle π, then this defines an AKSZ σ-model action, see for the full proof of this result [**26**]. The AKSZ constuction is a mathematical formulation to unify a large class of topological field theories, known as *AKSZ σ-models* [**1**].

This AKSZ σ-model has as target space the tangent Lie ∞-algebroid $\mathfrak{T}\mathfrak{B}$. The configuration space of fields is the space of maps $Maps(\mathfrak{T}\Sigma, \mathfrak{T}\mathfrak{B})$ from the worldsheet Σ to $\mathfrak{T}\mathfrak{B}$. Dually, this is space of dg-morphisms from $W(\mathfrak{B})$ to $\Omega^\bullet(\Sigma)$, i.e. the space of degree 1 $\mathfrak{B}$-valued differential forms on Σ. As we saw before, a degree 1 $\mathfrak{B}$-valued differential form A on Σ maps the Chern-Simons element $cs \in W(\mathfrak{B})$ to a differential form $cs(A)$ on Σ. Integrating this differential form on Σ will give an AKSZ σ-model action

$$\begin{aligned} Maps(\mathfrak{T}\Sigma, \mathfrak{T}\mathfrak{B}) &\to \mathbb{R} \\ A &\mapsto \int_\Sigma cs(A) \end{aligned}$$

EXAMPLE 3.4.1. (Poisson σ-model) Let (M, π) be a Poisson manifold, with Poisson bivector $\pi = \pi^{ij}\partial_i \wedge \partial_j$ and let $(\mathfrak{B}(M, \pi), \omega = \mathbf{d}x^i \wedge \mathbf{d}\partial_i)$ be the corresponding symplectic Lie algebroid. Then the invariant polynomial ω is in transgression with the cocycle π via the Chern-Simons element

$$cs = \partial_i \wedge d_{W(\mathfrak{B})}x^i - \pi$$

Let Σ a 2-dimensional oriented compact manifold and consider the dg-morphism

$$\Omega^\bullet(\Sigma) \leftarrow W(\mathfrak{B}) : (X, \eta)$$

which is a Poisson Lie algebroid valued differential form on Σ.This is just given by a vector bundle morphism $T\Sigma \to T^*M$ from the tangent bundle $T\Sigma$ to the cotangent bundle T^*M. Such a map is in components a smooth function $X : \Sigma \to M$ and a 1-form $\eta \in \Omega^1(\Sigma, X^*(T^*M))$ with values in the pullback of the tangent bundle of M along X. The AKSZ action is

$$\int_\Sigma cs(X, \eta) = \int_\Sigma \eta \wedge d_{\mathrm{dR}}X + \frac{1}{2}\pi^{ij}(X)\eta_i \wedge \eta_j$$

where $\eta_i = \eta(\partial_i)$ and we used that $X(d_{W(\mathfrak{B})}x^i) = d_{\mathrm{dR}}X^i$, since (X, η) is a dg-algebra homomorphism. This gives us precisely the Lagrangian of the Poisson σ-model (see [**46, 8, 26**]). The Poisson σ-model is a 2-dimensional topological field theory with target a Poisson manifold (M, π), or rather the Poisson Lie algebroid $\mathfrak{B}(M, \pi)$ corresponding to that.

Similar to the construction of the Poisson σ-model Lagrangian on a 2-dimensional manifold, the ordinary Chern-Simons theory action functional on a 3-dimensional manifold can also be constructed this way. Here one takes the symplectic Lie 2-algebroid $(b\,\mathfrak{g}, \langle -,- \rangle)$ where $\mathfrak{g}$ is a semisimple Lie algebra and $\langle -,- \rangle$ the Killing form invariant polynomial. This invariant polynomial is in transgression with the cocycle $\mu = -\frac{1}{6}\langle -, [-,-] \rangle$, via a Chern-Simons element $cs \in W(b\,\mathfrak{g})$. Under a $\mathfrak{g}$-valued form $\Omega^\bullet(\Sigma_3) \leftarrow W(b\,\mathfrak{g}) : A$ this element cs maps to the ordinary degree 3 Chern-Simons form

$$cs(A) = \langle A \wedge dA \rangle + \frac{1}{3}\langle A \wedge [A \wedge A] \rangle$$

For the full construction we refer the reader to [**26, 23**]. This ordinary Chern-Simons theory action functional is in fact a special case of the Courant σ-model Lagrangian where the Lie 2-algebroid has as base manifold the point. Besides these Poisson σ-model and Courant σ-model Lagrangians this AKSZ σ-model construction contains higher abelian Chern-Simons functionals and many more examples of interest.

As said before these AKSZ σ-model actions form a special case of a larger class of action functionals which goes under the name of ∞-Chern-Simons theory. The AKSZ σ-model actions arise only from those triples (π, cs, ω) that come from a symplectic Lie n-algebroid. For the more general case, these triples form the infinitesimal data of the construction of the ∞-Chern-Weil homomorphism, which generalizes the ordinary Chern-Weil homomorphism for ordinary Chern-Simsons theory. In the next section we show, how this infinitesimal data can be integrated to a morphisms of higher stacks, in the same way as ordinary Chern-Simons theory is enhanced to a morphism from the stacks of principal G-bundles with connection to the 3-stack of circle 3-bundles with connections. In this way we will see that the Poisson σ-model is the infinitesimal (or perturbative) versions of the Lie integrated 2d Chern-Simons theory.

3.5. Lie ∞-integration. Ordinary Lie integration assigns to a Lie algebra $\mathfrak{g}$ a Lie group G that is infinitesimally modelled by $\mathfrak{g}$. We explained in appendix A how this is generalized to Lie algebroids and Lie groupoids. In this section we discuss briefly the construction of Lie ∞-integration, that sends a Lie ∞-algebroid to a smooth ∞-groupoid of which it is a infinitesimal approximation. For a more comprehensive account, see [**79, 28**].

Definition 3.5.1. For $\mathfrak{a}$ a Lie algebroid, let $\exp(\mathfrak{a}) \in \mathbf{H}(\mathrm{CartSp}^{op}, \mathrm{sSet})$ be the simplicial presheaf, which for $U \in \mathrm{CartSp}$ and $k \in \mathbb{N}$ is given by the assignment

$$\exp(\mathfrak{a}) : (U, [k]) \mapsto \{\Omega^\bullet(U \times \Delta^k)_{\mathrm{vert,si}} \overset{A_{vert}}{\longleftarrow} \mathrm{CE}(\mathfrak{a})\}$$

where Δ^k is the standard realization of the k-simplex as a smooth manifold with boundary and corners and where $\Omega^\bullet(U \times \Delta^\bullet)_{\mathrm{vert,si}}$ is the dg-algebra of vertical differential forms on $U \times \Delta^k \to U$, that have sitting instants toward the boundary faces of the simplex. See [**28**] for details.

This simplicial presheaf presents the *universal Lie integration* of the Lie algebroid $\mathfrak{a}$. This Lie integration of $\mathfrak{a}$ always exists and in fact it is a smooth ∞-groupoid as in 2.2.

Proposition 3.5.2. [**26, 28**] *For $\mathfrak{a}$ an Lie ∞-algebroid, the simplicial presheaf* $\exp(\mathfrak{a})$ *is a smooth ∞-groupoid (is objectwise a Kan complex).*

Example 3.5.3. Let G be the simply-connected Lie group integrating the Lie algebra $\mathfrak{g}$ and $\mathbf{B}G$ its delooping Lie groupoid as in example 2.3.3. The simplicial presheaf $\exp(\mathfrak{g})$ that we get by universal integrating $\mathfrak{g}$ can be truncated to the presheaf of groupoids $\mathbf{B}G$. We denote by $\tau_1(-)$ the truncation operation that quotients out 2-morphisms in a simplicial presheaf to obtain a preseheaf of groupoids. Then we have an isomorphism

$$\mathbf{B}G = \tau_1 \exp(\mathfrak{g})$$

To see this remember that the dg-algebra morphisms $\Omega^\bullet(\Delta^k) \leftarrow \mathrm{CE}(\mathfrak{g})$ are in natural bijection with the $\mathfrak{g}$-valued 1-forms that are flat, that is they have a curvature form that vanish. Now the 1-morphisms in

$\exp(\mathfrak{g})$ are U-parametrized families of flat $\mathfrak{g}$-valued 1-forms A_{vert} on the interval Δ^1, and 2-morphisms are U-parametrized families of flat $\mathfrak{g}$-valued 1-forms on the disk Δ^2, interpolating between these 1-morphisms. By identifying theses 1-forms with the pullback of the Mauer-Cartan form θ on G we can equivalently think of a 1-morphism as a based smooth path in G, i.e. there is a smooth path $\gamma : \Delta^1 \to G$ such that $A_{\mathrm{vert}} = \gamma^*\theta \in \Omega^1(\Delta^1, \mathfrak{g})$. In this way we can think of 2-morphisms as smooth homotopies relative endpoints between these smooth paths. Since G is simply-connected this means that after dividing out 2-morphisms only the endpoints of these paths γ remain, which we identify with the point in G.

EXAMPLE 3.5.4. Let $\mathfrak{a}$ be a Lie algebroid A over a smooth manifold M, then this construction applied to $\mathfrak{a}$ reproduces the integration method by A-paths of Crainic and Fernanded (see [**17**]). The simplicial presheaf $\exp(\mathfrak{a})$ that we get by univeral integrating $\mathfrak{a}$ can be truncated to the presheaf of groupoids $\mathcal{G}(A)$. We denote by $\tau_1(-)$ again the truncation operation that quotients out 2-morphisms in a simplicial presheaf to obtain a presheaf of groupoids. Then we want to show that

$$\mathcal{G}(A) = \tau_1 \exp(\mathfrak{a})$$

A 1-morphism in $\exp(\mathfrak{a})$ is just an U-parametrized family of dg-algebra morphisms between the Chevalley-Eilenberg algebra $\mathrm{CE}(\mathfrak{a})$ and $\Omega^\bullet(\Delta^1)$. But since $\Omega^\bullet(\Delta^1) = \mathrm{CE}(T\Delta^1)$, where $T\Delta^1$ is the tangent Lie algebroid of Δ^1, we have that a dg-algebra morphism $\Omega^\bullet(\Delta^1) \leftarrow \mathrm{CE}(\mathfrak{a})$ is dually a Lie algebroid morphism $T\Delta^1 \to \mathfrak{a}$. Now such a morphism of Lie algebroids is precisely equivalent to what Crainic and Fernandes defined as an A-path $a : \Delta^1 \to A$ in the Lie algebroid $\mathfrak{a}$, see [**16, 17**] and appendix A for more details.

Analogous to the above situation for 1-morphisms, a 2-morphism in $\exp(\mathfrak{a})$ is an U-parametrized family of dg-algebra morphisms $\Omega^\bullet(\Delta^2) \leftarrow \mathrm{CE}(\mathfrak{a})$, which is precisely a variation of A-paths, that interpolates between these 1-morphisms. The 2-morphims are precisely homotopies relative endpoints between these 1-morphisms. By applying the trunction $\tau_1(-)$ to $\exp(\mathfrak{a})$ we divide out the 2-morphisms and thus we divide out the homotopies between A-paths, which is precisely equivalent to the construction of the Weinstein groupoid $\mathcal{G}(A)$. Furthermore the Weinstein groupoid $\mathcal{G}(A)$ is a topological groupoid and we need to impose the condition integrability on $\mathfrak{a}$ in order for $\mathcal{G}(A)$ to be a s-simply connected Lie groupoid.

REMARK 3.5.5. There are some technicalities involved concerning composition of A-paths. In Crainic and Fernandes approach in [**17**] we needed to introduce some cut-off function in order for the composition of A-paths to be smooth instead of piecewise smooth. By universal Lie integration of the Lie algebroid, this is accomplished by concerning differential forms on $U \times \Delta^1$ that have sitting instants, as is explained in [**28**].

REMARK 3.5.6. Although $\exp(\mathfrak{a})$ is a smooth ∞-groupoid, it is in general not degreewise a smooth manifold. In order for this to be true, we need impose some extra integrability condition on $\mathfrak{a}$, which in general are not known. For the case that $\mathfrak{a}$ is a Lie algebroid, the integrability condition to integrate it to a Lie groupoid are studied by Crainic and Fernandes in [**17**], and we call the Lie algebroid integrable if it integrates to a Lie groupoid.

EXAMPLE 3.5.7. Let $n \geq 1$ and consider the line Lie n-algebra $b^{n-1}\mathbb{R}$. Then fiber integration over simplices induces an equivalence

$$\int_{\Delta^\bullet} : \exp(b^{n-1}\mathbb{R}) \xrightarrow{\simeq} \mathbf{B}^n\mathbb{R}$$

Which is proven in [**28**].

For a $\mathfrak{a}$ a Lie ∞-algebroid, a dg-algebra homomorphism $\mathrm{CE}(\mathfrak{a}) \leftarrow \mathrm{CE}(b^{n-1}\mathbb{R}) : \mu$ is precisely a cocycle, i.e. an element $\mu \in \mathrm{CE}(\mathfrak{a})$ that is $d_{\mathrm{CE}(\mathfrak{a})}$-closed. Such a cocycle μ induces a morphism of simplicial presheaves

$$\exp(\mu) : \exp(\mathfrak{a}) \to \exp(b^{n-1}\mathbb{R})$$

given by the postcomposition

$$\Omega^\bullet(U \times \Delta^k)_{\mathrm{vert,si}} \overset{A_{\mathrm{vert}}}{\leftarrow} \mathrm{CE}(\mathfrak{a}) \overset{\mu}{\leftarrow} \mathrm{CE}(b^{n-1}\mathbb{R})$$

Such a morphism of simplicial presheaves we call a *characteristic map* representing the cocycle μ.

PROPOSITION 3.5.8. *Let G be a compact connected and simply connected Lie group with Lie algebra $\mathfrak{g}$ and $\mu : \mathfrak{g} \to b^2\mathbb{R}$ a degree 3 Lie algebra cocycle. Then there is a smallest subgroup Λ_μ of $(\mathbb{R}, +)$ such that we have a commuting diagram*

$$\begin{tikzcd} \exp(\mathfrak{g}) \arrow[r, "\exp(\mu)"] \arrow[d, "\tau_1"'] & \exp(b^2\mathbb{R}) \arrow[r, "\int_{\Delta^\bullet}", "\sim"'] & \mathbf{B}^3\mathbb{R} \arrow[d] \\ \mathbf{B}G \arrow[rr] & & \mathbf{B}^3\mathbb{R}/\Lambda_\mu \end{tikzcd}$$

PROOF. We will give here the sketch of the proof, for more detail see [**26**]. In this diagram the vertical map $\mathbf{B}^3\mathbb{R} \to \mathbf{B}^3\mathbb{R}/\Lambda_\mu$ is the obvious quotient map of simplicial abelian groups and it is sufficient to define this map on 3-cells. For this diagram to commute, the bottom morphism must send a form $A_{\mathrm{vert}} \in \Omega^1_{\mathrm{si,vert}}(U \times \Delta^3, \mathfrak{g})$ to the image of $\int_{\Delta^3} \mu(A_{\mathrm{vert}}) \in \mathbb{R}$ under the quotient map. Since this morphism is a morphism of simplicial sets, it must be true that for all $A_{\mathrm{vert}} \in \Omega^1_{si,vert}(U \times \partial\Delta^4, \mathfrak{g})$ the integral $\int_{\partial\Delta^4} \mu(A_{\mathrm{vert}}) \in \mathbb{R}$ lands in $\Lambda_\mu \subset \mathbb{R}$. Remember from example 2.3.3 that we may identify flat $\mathfrak{g}$-valued forms on $\partial\Delta^4$ with based smooth maps $\partial\Delta^4 \to G$. If we have two such 3-spheres A_{vert} and A'_{vert} that are homotopic, then there exist a smooth homotopy interpolating between them and hence a uniqeu extension $\hat{A}_{\mathrm{vert}}$. Since this extension is closed, the fiber integral $\int_{\partial\Delta^4}$ of A_{vert} and A'_{vert} has to coincide and hence the fiber integral $\int_{\partial\Delta^4}$ depends only on the homotopy class of maps $\partial\Delta^4 \to G$. Hence we have a group homomorphism

$$\int_{\partial\Delta^4} : \pi_3(G, x) \to \mathbb{R}$$

Now the minimal subgroup of $\mathbb{R}$ that makes this diagram commutative is precisely the subgroup of $\mathbb{R}$ generated by the image of this map, which we denote by Λ_μ. Note that since G is compact and simply connected its homotopy groups are finitely generated and so Λ_μ is also finitely generated. $\square$

In the case G is compact simple and simply connected Lie group, we have that the homotopy of G is trivial up to degree 3 and $\pi_3(G) \simeq H^3(G; \mathbb{Z}) \simeq \mathbb{Z}$ by the Hurewicz isomorphism (see [**29**]). In this case, in the above proposition we have $\Lambda_\mu \simeq \mathbb{Z}$ and hence presents a morphism of smooth ∞-groupoids $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^3U(1)$. Hence if we have the triple $(\mu, cs, \langle -, - \rangle)$ with cocycle $\mu = -\frac{1}{6}\langle -, [-, -]\rangle$, this morphism gives the underlying circle 3-bundle of the extended Lagrangian of the 3d Chern-Simons theory.

PROPOSITION 3.5.9. *Let (M, π) be a Poisson manifold, with Poisson bivector $\pi \in \wedge^2\Gamma(TM)$, and corresponding Poisson Lie algebroid $\mathfrak{B}(M, \pi)$. We have that $\pi : \mathfrak{B}(M, \pi) \to b^1\mathbb{R}$ is a degree 2 cocycle and there is a smallest subgroup Λ_μ of $(\mathbb{R}, +)$ such that we have a commuting diagram*

$$\begin{tikzcd} \exp(\mathfrak{B}(M, \pi)) \arrow[r, "\exp(\pi)"] \arrow[d, "\tau_1"'] & \exp(b^1\mathbb{R}) \arrow[r, "\int_{\Delta^\bullet}", "\sim"'] & \mathbf{B}^2\mathbb{R} \arrow[d] \\ \tau_1 \exp(\mathfrak{B}(M, \pi)) \arrow[rr] & & \mathbf{B}^2\mathbb{R}/\Lambda_\pi \end{tikzcd}$$

PROOF. First of all the Poisson bivector $\pi \in \wedge^2\Gamma(TM)$ can be seen as an element of degree 2 in $\mathrm{CE}(\mathfrak{B}(M, \pi))$ such that it is $d_{\mathrm{CE}(\mathfrak{B})}$-closed, and hence it is a degree 2 cocycle. In this diagram the vertical map $\mathbf{B}^2\mathbb{R} \to \mathbf{B}^2\mathbb{R}/\Lambda_\pi$ is again the obvious quotient map of simplicial abelian groups and it is sufficient to define this map on 2-cells. Remember a 2-morphism in $\exp(\mathfrak{B}(M,\pi))$ is a U-parametrized family of dg-algebra morphisms $\Omega^\bullet(\Delta^2) \leftarrow \mathrm{CE}(\mathfrak{B}) : (X,\eta)$ .In the order for the diagram of the proposition to commute, the bottom morphism must send a dg-algebra morphism (X,η) to the image $\int_{\Delta^2} \pi(X,\eta) \in \mathbb{R}$ under the quotient map. Since this morphism is a morphism of simplicial sets, it must send every dg-algebra morpism $\Omega^\bullet(\partial\Delta^3) \leftarrow \mathrm{CE}(\mathfrak{B}) : (X,\eta)$ to $\int_{\partial\Delta^3} \pi(X,\eta) \in \Lambda_\pi \subset \mathbb{R}$. Now since $\partial\Delta^3$ may be identified with the 2-sphere S^2, we have that a dg-algebra morphism $\Omega^\bullet(S^2) \leftarrow \mathrm{CE}(\mathfrak{B}) : (X,\eta)$ is dually just a Lie algebroid morphism $(X,\eta) : TS^2 \to T^*M$, where

$$\begin{tikzcd} TS^2 \arrow[r, "\eta"] \arrow[d] & T^*M \arrow[d] \\ S^2 \arrow[r, "X"] & M \end{tikzcd}$$

Now as is proven in [**5**], the base map X of such a Lie algebroid morphism, maps S^2 to a symplectic leaf $L \subset M$. The tangent space of the leaf L is spanned by the Hamiltonian vectors $X_f = \pi(df, -)$ and is endowed with a symplectic form defined by

$$\omega_L(X_f, X_g) = \pi(df, dg)$$

The integral $\int_{\partial\Delta^3} \pi(X,\eta)$ evaluated on a dg-algebra morphism (X,η) can be rewritten as the pullback of this symplectic form ω_L on the leaf $L \subset X(S^2)$ as

$$\int_{S^2} X^*(\omega_L)$$

These are precisely the periods of the Poisson manifold M as was defined in definition 2.1.9 of the previous chapter. By the same arguments that were used in the previous example, the fiber integral $\int_{S^2}$ depends only on the homotopy class of maps $S^2 \to M$. Hence we have a group homomorphism

$$\int_{S^2} : \pi_2(M, x) \to \mathbb{R}$$

Take Λ_π to be again the subgroup of $\mathbb{R}$ generated by its image, which is the minimal subgroup of $\mathbb{R}$ that makes the diagram commutative. $\square$

THEOREM 3.5.10. *For a Poisson manifold (M,π) that is integrable and prequantizable we have a commuting diagram*

$$\begin{tikzcd} \exp(\mathfrak{B}(M,\pi)) \arrow[r, "\exp(\pi)"] \arrow[d, "\tau_1"'] & \exp(b^1\mathbb{R}) \arrow[r, "\int_{\Delta^\bullet}", "\sim"'] & \mathbf{B}^2\mathbb{R} \arrow[d] \\ \mathrm{SymplGpd}(M,\pi) \arrow[rr] & & \mathbf{B}^2 U(1) \end{tikzcd}$$

where $\mathrm{SymplGpd}(M,\pi)$ *is a Lie groupoid.*

PROOF. This follows from the previous proposition together with the fact that if the Poisson manifold is integrable, then $\mathrm{SymplGpd}(M,\pi)$ is a s-simply connected Lie groupoid by the reasoning of example 3.5.4 and if furthermore the Poisson manifold is prequantizable then by theorem 2.1.10 of the previous chapter, we find that $\Lambda_\pi = \mathbb{Z}$ and hence the bottom morphism reduces to the morphism of highers stacks $\mathrm{SymplGpd}(M,\pi) \to \mathbf{B}^2 U(1)$. $\square$

REMARK 3.5.11. Remember that the classical solutions of the 3d Chern-Simons action functional are those $\mathfrak{g}$-valued connections that are flat. This corresponds precisely to the fact that we are working in the Chevalley-Eilenberg algebra instead of the Weil algebra, where we have a restriction on the connections via the Mauer-Cartan equation. For the Poisson σ-model Lagrangian we have a similar situation. The classical solutions of the Poisson σ-model Lagrangian are those vector bundle morphisms

$T\Sigma \to T^*M$ that are flat, where the flatness criterion here precisely means that the vector bundle morphisms are Lie algebroid homomorphisms. This statement is proven in [**5**].

This prequantization condition on the Poisson manifold was called the *integrality condition*, since the periods of the cocycle needs to be integral. This integrality condition is the one that appears in the traditional literature [**5**] and is based on theorem due to Crainic and Zhu, which is stated in theorem 2.1.10 of the previous chapter. Hence for a Poisson manifold that is integrable and satisfy the integrality condition, there exist canonically a (principal) circle 2-bundle, or bundle gerbe, over the integrated Poisson manifold. This bundle gerbe precisely coincides with the one of the traditional literature as one can see for example in [**5**].

Next we will show how this circle 2-bundle $\mathrm{SymplGpd}(M,\pi) \to \mathbf{B}^2U(1)$ is actually the underlying bundle of a circle 2-bundle with connection, which is needed for the full prequantization of a Poisson manifold. Remember in the definition of $\exp(\mathfrak{a})$ we only used the Chevalley-Eilenberg algebra of $\mathfrak{a}$. By using the Weil algebra instead of the Chevalley-Eilenberg algebra, we can describe the differential refinement of $\exp(\mathfrak{a})$ and hence differential refine the circle 2-bundle to a circle 2-bundle with connection.

Definition 3.5.12. For $\mathfrak{a}$ a Lie algebroid, let $\exp(\mathfrak{a})_{conn} \in \mathbf{H}(\mathrm{CartSp}^{op}, \mathrm{sSet})$ be the simplicial presheaf, which for $U \in \mathrm{CartSp}$ and $k \in \mathbb{N}$ is given by the assignment

$$\exp(\mathfrak{a})_{conn} : (U,[k]) \mapsto \{\Omega^\bullet(U\times\Delta^k)_{\mathrm{si}} \overset{(A,F_A)}{\hookleftarrow} \mathrm{CE}(\mathfrak{a}) \mid \forall v \in \Gamma(T\Delta^k) : \iota_v F_A = 0\}$$

where k-morphisms are $\mathfrak{a}$-valued forms A on $U \times \Delta^k$ with sitting instants and with the property that their curvature vanishes on vertical vectors.

This extra condition that the curvature needs to vanish on vertical vectors is needed in order to make the curvature characteristic form $\langle F_A\rangle$ descend to the base space U. This condition that $\iota_v F_A = 0$ for all vertical vectors v, is in fact an analogue of the horizontality condition of an ordinary Ehresmann connection, and can easily be deduced from the conditions of an ordinary Ehresmann connection (see [**21**]).

This definition can equivalently be rewritten as the simplicial presheaf

$$\exp(\mathfrak{a})_{conn} : (U,[k]) \mapsto \left(\begin{tikzcd} \Omega^\bullet_{\mathrm{vert,si}}(U\times\Delta^k) & \mathrm{CE}(\mathfrak{a}) \arrow[l, "A_{\mathrm{vert}}"'] \\ \Omega^\bullet_{\mathrm{si}}(U\times\Delta^k) \arrow[u] & W(\mathfrak{a}) \arrow[l, "(A,F_A)"'] \arrow[u] \\ \Omega^\bullet_{cl}(U) \arrow[u] & \mathrm{inv}(\mathfrak{a}) \arrow[l, "F_A"'] \arrow[u] \end{tikzcd}\right)$$

Example 3.5.13. Let G be the simply-connected Lie group integrating the Lie algebra $\mathfrak{g}$. The simplicial presheaf $\exp(\mathfrak{g})_{conn}$ is precisely the differential refinement of $\exp(\mathfrak{g})$ and can be truncated to the presheaf of groupoids $\mathbf{B}G_{conn}$. We have an isomorphism

$$\mathbf{B}G_{conn} = \tau_1 \exp(\mathfrak{g})_{conn}$$

For a 1-morphism, that is a form $\Omega^\bullet(U\times\Delta^1) \leftarrow W(\mathfrak{g}) : A$ which may be decomposed into its vertical and horizontal components

$$A = \lambda dt + A_U$$

where $\lambda \in C^\infty(U\times\Delta^1)$ and A_U in the image of $\Omega^1(U,\mathfrak{g})$. The horizontality condition $\iota_{\partial_t} F_A = 0$ is given by the differential equation

$$\frac{\partial}{\partial t} A_U = d_U\lambda + [A_U, \lambda]$$

Take as initial condition $A_U(t=0) = A_0$ then this gives the unique solution

$$A_U(t) = g(t)^{-1} A_0 g(t) + g(t)^{-1} d_U g(t)$$

where $g(t) \in G$ is the parallel transport for the connection λdt along the path $[0,t]$ in the 1-simplex Δ^1. This is easily seen since

$$\frac{\partial}{\partial t} A_U(t) = g(t)^{-1}(A_0 + d_U)\lambda g(t) - g(t)^{-1}\lambda(A_0 + d_U)g(t) = d_U\lambda + [A_U(t), \lambda]$$

Evaluating this solution at $t = 1$, and writing $g(1) = g$ and $A_U(t=1) = A_1$ we find

$$A_1 = g^{-1} A_0 g + g^{-1} dg$$

which are precisely the gauge transformation between 1-forms $A_0, A_1 \in \Omega^1(U, \mathfrak{g})$. These computation carry on without substantial modification to higher simplices. The claim then follows from the previous statement of Lie integration that $\tau_1 \exp(\mathfrak{g}) = \mathbf{B}G$. Furthermore we the obvious forgetful morpism $\exp(\mathfrak{g})_{conn} \to \exp(\mathfrak{g})$ can be truncated to the forgetful morphism of stack $\mathbf{B}G_{conn} \to \mathbf{B}G$.

Example 3.5.14. Let $n \geq 1$ and consider the line Lie n-algebra $b^{n-1}\mathbb{R}$. Then fiber integration over simplices induces an equivalence

$$\int_{\Delta^\bullet}^{conn} : \exp(b^{n-1}\mathbb{R})_{conn} \to \mathbf{B}^n\mathbb{R}_{conn}$$

Which is proven in [**26**].

Remember that for a cocycle $\mu : \mathfrak{a} \to b^{n-1}\mathbb{R}$ we obtained the morphism of presheaves $\exp(\mu)$ by postcomposition of k-cells in $\exp(\mathfrak{a})$ with this cocycle μ. Similarly we can postcompose k-cells in $\exp(\mathfrak{a})_{conn}$ by postcomposing it with a diagram, which in our case can precisely be done by a diagram of a transgressive cocycle

$$\begin{tikzcd}
\mathrm{CE}(\mathfrak{a}) & \mathrm{CE}(b^{n-1}\mathbb{R}) \arrow[l, "\mu"'] \\
W(\mathfrak{a}) \arrow[u] & W(b^{n-1}\mathbb{R}) \arrow[l, "{(cs, \langle - \rangle)}"'] \arrow[u] \\
\mathrm{inv}(\mathfrak{a}) \arrow[u] & \mathrm{inv}(b^{n-1}\mathbb{R}) \arrow[l, "\langle - \rangle"'] \arrow[u]
\end{tikzcd}$$

where $\langle - \rangle$ is an invariant polynomial in transgression with the cocycle μ and cs is a Chern-Simons element witnessing the transgression, which we will also denote by the triple $(\mu, cs, \langle - \rangle)$.

Definition 3.5.15. For every triple $(\mu, cs, \langle - \rangle)$ we can define the morphism of simplicial presheaves

$$\exp(cs) : \exp(\mathfrak{a})_{conn} \to \exp(b^{n-1}\mathbb{R})_{conn}$$

degreewise by pasting composition with the transgression diagram

$$
\exp(cs)(U)_k : \left(\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert,si}}(U\times\Delta^k) & \mathrm{CE}(\mathfrak{a}) \arrow[l, "A_{\mathrm{vert}}"'] \\
\Omega^\bullet_{\mathrm{si}}(U\times\Delta^k) \arrow[u] & W(\mathfrak{a}) \arrow[l, "(A,F_A)"'] \arrow[u] \\
\Omega^\bullet_{cl}(U) \arrow[u] & \mathrm{inv}(\mathfrak{a}) \arrow[l, "F_A"'] \arrow[u]
\end{tikzcd}\right)
$$

$$
\mapsto \left(\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert,si}}(U\times\Delta^k) & \mathrm{CE}(\mathfrak{a}) \arrow[l, "A_{\mathrm{vert}}"'] & \mathrm{CE}(b^{n-1}\mathbb{R}) \arrow[l, "\mu"'] & : \mu(A_{vert}) \\
\Omega^\bullet_{\mathrm{si}}(U\times\Delta^k) \arrow[u] & W(\mathfrak{a}) \arrow[l, "(A,F_A)"'] \arrow[u] & W(b^{n-1}\mathbb{R}) \arrow[l, "(cs,\langle-\rangle)"'] \arrow[u] & : cs(A) \\
\Omega^\bullet_{cl}(U) \arrow[u] & \mathrm{inv}(\mathfrak{a}) \arrow[l, "F_A"'] \arrow[u] & \mathrm{inv}(b^{n-1}\mathbb{R}) \arrow[l, "\langle-\rangle"'] \arrow[u] & : \langle F_A\rangle
\end{tikzcd}\right)
$$

Such a morphism of simplicial presheaves is a *differential characteristic map* representing the triple $(\mu, cs, \langle-\rangle)$ and this is what we call the presentation of the ∞-*Chern-Weil homomorphism* induced by the invariant polynomial $\langle-\rangle$. The commutativity of the lower part of the diagram encodes the classical equation

$$dcs(A) = \langle F_A\rangle$$

stating that the curvature of the connection $cs(A)$ is the horizontal differential form $\langle F_A\rangle$ in $\Omega^\bullet(U)$.

For Σ a n-dimensional compact smooth manifold, an $\mathfrak{a}$-valued ∞-connection is a morhism

$$\nabla : \Sigma \to \exp(\mathfrak{a})_{conn}$$

and the composite

$$\Sigma \to \exp(\mathfrak{a})_{conn} \xrightarrow{\exp(cs)} \exp(b^{n-1}\mathbb{R})_{conn}$$

is a $b^{n-1}\mathbb{R}$-valued connections whose higher parallel transport over Σ is locally given by the integral $\int_\Sigma cs(\nabla)$ of the Chern-Simons form $cs(\nabla)$ over Σ. The assignment $\nabla \mapsto \int_\Sigma cs(\nabla)$ is the action functional for the ∞-Chern-Simons theory defined by the invariant polynomial $\langle-\rangle \in W(\mathfrak{a})$. Hence we may regard $\exp(cs)$ as being the Lagrangian of this ∞-Chern-Simons theory.

We said in a previous section that the AKSZ σ-model action is an instance of the ∞-Chern-Weil homomorphism. To see this we restrict the ∞-Chern-Weil homomorphism to the case of the trivial ∞-bundles with $\mathfrak{a}$-valued ∞-connections. In this case the ∞-Chern-Weil homomorphism simplifies drastically, since only the 0-cells are involved and the AKSZ σ-model Lagrangian corresponds to the following commutative diagram

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert,si}}(U) & \mathrm{CE}(\mathfrak{a}) \arrow[l, "A_{\mathrm{vert}}"'] & \mathrm{CE}(b^{n-1}\mathbb{R}) \arrow[l, "\pi"'] & : \pi(A_{vert}) \\
\Omega^\bullet_{\mathrm{si}}(U) \arrow[u] & W(\mathfrak{a}) \arrow[l, "(A,F_A)"'] \arrow[u] & W(b^{n-1}\mathbb{R}) \arrow[l, "(cs,\langle-\rangle)"'] \arrow[u] & : cs(A) \\
\Omega^\bullet_{cl}(U) \arrow[u] & \mathrm{inv}(\mathfrak{a}) \arrow[l, "F_A"'] \arrow[u] & \mathrm{inv}(b^{n-1}\mathbb{R}) \arrow[l, "\omega"'] \arrow[u] & : \omega(F_A)
\end{tikzcd}$$

EXAMPLE 3.5.16. Consider G a compact simple and simply connected Lie group, with Lie algebra $\mathfrak{g}$. Let $(b\,\mathfrak{g}, \langle -,-\rangle)$ be the symplectic Lie 2-algebroid with $\langle -,-\rangle$ the Killing form invariant polynomial, transgressing the cocycle $\mu = -\frac{1}{6}\langle -,[-,-]\rangle$ via the Chern-Simons element $cs \in W(b\,\mathfrak{g})$. We recall that for a $\mathfrak{g}$-valued form $\Omega^\bullet(\Sigma_3) \leftarrow W(b\,\mathfrak{g}) : A$ this element cs maps to

$$cs(A) = \langle A \wedge dA\rangle + \frac{1}{3}\langle A \wedge [A \wedge A]\rangle$$

Now similar to proposition 3.5.8 we have that the triple $(\mu, cs, \langle -,-\rangle)$ defines a commuting diagram

$$\begin{tikzcd}
\exp(\mathfrak{g})_{conn} \arrow[r, "\exp(cs)"] \arrow[d, "\tau_1"'] & \exp(b^2\mathbb{R})_{conn} \arrow[r, "\int^{conn}_{\Delta^\bullet}", "\sim"'] & \mathbf{B}^3\mathbb{R}_{conn} \arrow[d] \\
\mathbf{B}G_{conn} \arrow[rr, "\hat{\mathbf{c}}"'] & & \mathbf{B}^3U(1)_{conn}
\end{tikzcd}$$

Where the bottom morphism $\hat{\mathbf{c}} : \mathbf{B}G_{conn} \to \mathbf{B}^3U(1)_{conn}$ is the *universal characteristic* morphism, which is the extended Lagrangian of the 3d Chern-Simons theory. Now take as worldvolume a 3-dimensional closed oriented smooth manifold Σ and consider the mapping stack $\mathbf{H}(\Sigma, \mathbf{B}G_{conn})$ as the configuration space, which can be seen as the trajectories of the brane Σ in the target space $\mathbf{B}G_{conn}$. The background gauge field is precisely given by the extended Lagrangian $\hat{\mathbf{c}} : \mathbf{B}G_{conn} \to \mathbf{B}^3U(1)_{conn}$, which can be transgressed to the morphism

$$\mathbf{H}(\Sigma, \mathbf{B}G_{conn}) \xrightarrow{\mathbf{H}(\Sigma,\hat{\mathbf{c}})} \mathbf{H}(\Sigma, \mathbf{B}^3U(1)_{conn}) \xrightarrow{\exp(2\pi\int_\Sigma(-))} U(1)$$

Since the morphisms in $\mathbf{H}(\Sigma, \mathbf{B}G_{conn})$ are gauge transformations between field configurations, and since $U(1)$ has no non-trivial morphisms, this map gives necessarily gauge invariant $U(1)$-valued function on field configurations. Evaluating over the point and passing to equivalence classes, this induces the Chern-Simons actions functional

$$\exp(2\pi \int_\Sigma \hat{\mathbf{c}}(-)) : \{\text{principal } G\text{-bundles with connection on } \Sigma\}/\text{equiv} \to U(1)$$

Now since $\pi_3(G) \simeq H^3(G;\mathbb{Z}) \simeq \mathbb{Z}$ we have that the principal G-bundle $P \to \Sigma$ is trivializable and from the construction of $\hat{\mathbf{c}}$ we have

$$\exp(2\pi \int_\Sigma \hat{\mathbf{c}}(P,\nabla)) = \exp(2\pi \int_\Sigma cs(A))$$

where $A \in \Omega^1(\Sigma, \mathfrak{g})$ is the $\mathfrak{g}$-valued 1-form on Σ representing the connection ∇ in a chosen trivialization of P. See [**29**] for more details.

This morphism $\hat{\mathbf{c}}$ is the differential refinement of $\mathbf{c}$, and by forgetting the connection we have obviously the commuting diagram

$$\begin{tikzcd}
\mathbf{B}G_{conn} \arrow[r, "\hat{\mathbf{c}}"] \arrow[d] & \mathbf{B}^3U(1)_{conn} \arrow[d] \\
\mathbf{B}G \arrow[r, "\mathbf{c}"'] & \mathbf{B}^3U(1)
\end{tikzcd}$$

which gives precisely the underlying circle 3-bundle of the extended Lagrangian. By forgetting the connection on both sides of the morphism we loose more information than needed. Actually the morphism $\hat{\mathbf{c}}$ does descend according to the commuting diagram (See [**87, 27**])

$$\begin{tikzcd}
\mathbf{B}G_{conn} \arrow[r, "\hat{\mathbf{c}}"] \arrow[d] & \mathbf{B}^3U(1)_{conn} \arrow[d] \\
\mathbf{B}G \arrow[r, "\bar{\mathbf{c}}"'] & \mathbf{B}(\mathbf{B}^2U(1)_{conn})
\end{tikzcd}$$

This kind of phenomenon we will see later again.

3.6. 2d Poisson-Chern-Simons theory. Let (M, π) be an integrable Poisson manifold that satisfy the integrality condition, and let $(\mathfrak{B}(M, \pi), \omega)$ be the corresponding symplectic Lie algebroid. In local coordinates the Poisson bivector $\pi = \pi^{ij}\partial_i \wedge \partial_j$, the invariant polynomial $\omega = \mathbf{d}x^i \wedge \mathbf{d}\partial_i$ and the Chern-Simons element that transgresses the cocycle to the invariant polynomial is given by

$$cs = \partial_i \wedge d_{W(\mathfrak{B})}x^i - \pi$$

. Let Σ be a 2-dimensional oriented compact manifold and recall that for a dg-morphism $\Omega^\bullet(\Sigma) \leftarrow W(\mathfrak{B}) : (X, \eta)$ the elements cs maps to

$$cs(X, \eta) = \eta \wedge d_{\mathrm{dR}}X + \frac{1}{2}\pi^{ij}(X)\eta_i \wedge \eta_j$$

Which gives the Lagrangian of the Poisson σ-model. The Lie integration of this data gives the extended Lagrangian of an 2d Poisson-Chern-Simons theory. Since similar to proposition 3.5.9 we have that this triple (π, cs, ω) defines the commuting diagram

$$\begin{tikzcd}
\exp(\mathfrak{B})_{conn} \arrow[r, "\exp(cs)"] \arrow[d, "\tau_1"'] & \exp(b^1\mathbb{R})_{conn} \arrow[r, "\int_{\Delta^\bullet}^{conn}", "\sim"'] & \mathbf{B}^2\mathbb{R}_{conn} \arrow[d] \\
\mathrm{SymplGpd}_{conn} \arrow[rr, "\nabla"'] & & \mathbf{B}^2U(1)_{conn}
\end{tikzcd}$$

Where $\mathrm{SymplGpd}_{conn}$ can be seen as the smooth moduli stack of fields of the 2d Poisson-Chern-Simons theory and the bottom morphism $\nabla : \mathrm{SymplGpd}_{conn} \to \mathbf{B}^2U(1)_{conn}$ is the extended Lagrangian of the 2d Poisson-Chern-Simons theory. The non-degenerate and binary invariant polynomial ω, induces by Lie integration a pre-2-plectic structure on this moduli stack $\mathrm{SymplGpd}_{conn}$ of the 2d Poisson-Chern-Simons theory

$$\omega(F_{(-)}, F_{(-)}) : \mathrm{SymplGpd}_{conn} \to \Omega^3_{cl}$$

This curvature morphism we also denote by ω and gives the following prequantization[**27**]

$$\begin{tikzcd}
 & \mathbf{B}^2U(1)_{conn} \arrow[d, "F_{(-)}"] \\
\mathrm{SymplGpd}_{conn} \arrow[ur, "\nabla"] \arrow[r, "\omega"'] & \Omega^3_{cl}
\end{tikzcd}$$

For Σ be a closed oriented 2-dimensional smooth manifold, the extended Lagrangian ∇ can be transgressed to the morphism

$$\mathbf{H}(\Sigma, \mathrm{SymplGpd}_{conn}) \xrightarrow{\mathbf{H}(\Sigma,\nabla)} \mathbf{H}(\Sigma, \mathbf{B}^2 U(1)_{conn}) \xrightarrow{\exp(2\pi \int_\Sigma(-))} U(1)$$

The restriction of this morphism to the canonical inclusion $\Omega(\Sigma, \mathfrak{B}) \hookrightarrow \mathbf{H}(\Sigma, \mathrm{SymplGpd}_{conn})$ of globally defined $\mathfrak{B}$-valued differential forms, gives the action functional of the Poisson σ-model (see [**27**]).

The restriction of the moduli stack $\mathrm{SymplGpd}_{conn}$ to just SymplGpd, obtained by forgetting the connection data, will give us the bare Lie groupoid SymplGpd of $\mathfrak{B}$. By forgetting the connection data on both sides of the morphism ∇, we loose again to much information. Similar to the case of 3d Chern-Simons theory, the morphism ∇ actually does descend according to the commuting diagram (See [**27**])

$$\begin{tikzcd}
\mathrm{SymplGpd}_{conn} \arrow[r, "\nabla"] \arrow[d] & \mathbf{B}^2 U(1)_{conn} \arrow[d] \\
\mathrm{SymplGpd} \arrow[r, "\nabla^1"'] & \mathbf{B}(\mathbf{B}U(1)_{conn})
\end{tikzcd}$$

This bottom morphism is precisely a bundle gerbe with connective structure but without curving, which precisely the datum that we need in order to define the prequantization of the Poisson manifold according to definition 2.1.7. Similarly the curvature morphism ω does descend to its curvature ω^1 which maps to $\mathbf{B}\Omega^2_{cl}$ instead of Ω^3_{cl} via the following diagram

$$\begin{tikzcd}
 & \mathbf{B}(\mathbf{B}U(1)_{conn}) \arrow[d, "\mathbf{B}F_{(-)}"] \\
\mathrm{SymplGpd} \arrow[ur, "\nabla^1"] \arrow[r, "\omega^1"'] & \mathbf{B}\Omega^2_{cl}
\end{tikzcd}$$

The morphism ω^1 can be seen as a degree 3-cocycle in the simplicial de Rham cohomology, since we have the map $\omega^1 : \mathrm{SymplGpd} \to \mathbf{B}\Omega^2_{cl} \hookrightarrow \flat_{\mathrm{dR}}\mathbf{B}^3 U(1)$. If this ω^1 represents a globally defined 2-form on the manifold of morphisms of the Lie groupoid SymplGpd then this local data is called a *(pre-)symplectic groupoid*. In the case that this $\omega^1 : \mathrm{SymplGpd} \to \mathbf{B}\Omega^2_{cl}$ is represented by a multiplicative symplectic 2-form on the manifold of morphisms of the Lie groupoid SymplGpd is a *symplectic groupoid*. This is the situation which, according to definition 2.1.7, is called the prequantization condition of a symplectic groupoid. In the literature this is also known as the underlying instanton sector, since we forget the connection on the moduli stack of fields and consider only the underlying bundle structure.

In summary, the following table indicates the relation between the prequantum theory of the Poisson σ-model and that of ordinary 3d Chern-Simons theory.

	2d Poisson-Chern-Simons theory	**3d Chern-Simons theory**
Field Configurations	$\mathfrak{B}$-valued differential form $\{X, \eta\}$ on a symplectic groupoid	$\mathfrak{g}$-valued differential form $\{A\}$ on a principal G-bundle
(Perturbative) Lagrangian	$\eta \wedge d_{\mathrm{dR}} X + \frac{1}{2}\pi^{ij}\eta_i \wedge \eta_j$	$\langle A \wedge dA \rangle + \frac{1}{3}\langle A \wedge [A \wedge A] \rangle$
Extended (non-perturbative) Lagrangian	$\mathrm{SymplGpd}_{conn} \to \mathbf{B}^2 U(1)_{conn}$	$\mathbf{B}G_{conn} \to \mathbf{B}^3 U(1)_{conn}$
Underlying bundle on the instanton sector	$\mathrm{SymplGpd} \to \mathbf{B}^2 U(1)_{conn}$[1]	$\mathbf{B}G \to \mathbf{B}^3 U(1)_{conn}$[2]

3.7. Boundary field theory. A (codimension 1) *boundary theory* to a prequantum field theory $\mathbf{L} : \mathbf{Fields} \to \mathbf{B}^n U(1)_{conn}$ is given by a diagram of the form

$$\begin{tikzcd}
& \mathbf{Fields}^{\partial} \arrow[dl] \arrow[dr, "i"] & \\
* \arrow[dr] & \arrow[phantom, "\Swarrow"] & \mathbf{Fields} \arrow[dl, "\mathbf{L}"] \\
& \mathbf{B}^n U(1)_{conn} &
\end{tikzcd}$$

That is a choice of boundary fields $\mathbf{Fields}^{\partial}$, a choice of map from boundary fields into bulk fields $\mathbf{Fields}$, and a choice of trivialization of the extended Lagrangian after restriction to the boundary fields, see for more details [**80, 67**]. A famous example of a boundary theory is the Wezz-Zumino-Witten theory as a boundary theory of the 3d Chern-Simons theory. Often these boundary theories of topological field theories are themselves not topological, but require some extra geometric structure, for example the Wezz-Zumino-Witten theory is a conformal field theory and hence need the choice of a conformal structure. The phenomenon that the bulk fields may be identified with the boundary fields in the correlation functions of the boundary theory, is known as the *holographic principle*. One instance of the holographic principle in quantum field theory is the relation between the space of quantum states of 3d Chern-Simons theory may be identified with the correlators of Wezz-Zumino-Witten theory (see [**31**]).

If we ignore in the above diagram the connection data and write

$$\begin{tikzcd}
& \mathbf{Fields}^{\partial} \arrow[dl] \arrow[dd, "i^*\mathbf{L}"] \arrow[dr, "i"] & \\
* \arrow[dr] \arrow[r, phantom, "\overset{\Leftarrow}{\xi}"] & {} \arrow[r, phantom, "\Leftarrow"] & \mathbf{Fields} \arrow[dl, "\mathbf{L}"] \\
& \mathbf{B}^n U(1) &
\end{tikzcd}$$

where the right 2-cell is the trivial cell that witness the composition of i and $\mathbf{L}$ and the left 2-cell ξ is the trivialization of the circle n-bundle $i^*\mathbf{L} : \mathbf{Fields}^{\partial} \to \mathbf{B}^n U(1)$. The 2-cell ξ that trivializes the diagram, is in fact a circle $(n-1)$-bundle on the boundary fields $\mathbf{Fields}^{\partial}$. This follows from the fact that $i^*\mathbf{L}$ is homotopic to the trivial circle n-bundle on M, and morphisms between trivial circle n-bundles are equivalent to circle $(n-1)$-bundles.

To see this we first note that the loop space object of the pointed object $\mathbf{B}^n U(1)$ with point $* \to \mathbf{B}^n U(1)$ is equivalent to $\mathbf{B}^{n-1} U(1)$, where the loop space object is defined as the homotopy pullback $\Omega\mathbf{B}^nU(1)$ of this point along itself

$$\begin{tikzcd}
\mathbf{B}^{n-1}U(1) \simeq \Omega\mathbf{B}^nU(1) \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & \mathbf{B}^nU(1)
\end{tikzcd}$$

This follows from the fact that we have the weak equivalence $* \xrightarrow{\simeq} \mathbf{EB}^{n-1}U(1)$ and the homotopy pullback can be computed as an ordinary pullback after replacing one of the morphism by an equivalent fibration

$$\begin{tikzcd}
DK(0 \to C^\infty(-,U(1)) \to 0 \cdots \to 0) \arrow[r] \arrow[d] & DK(C^\infty(-,U(1)) \to C^\infty(-,U(1)) \to 0 \cdots \to 0) \arrow[d] \\
DK(0 \to 0 \to 0 \cdots \to 0) \arrow[r] & DK(C^\infty(-,U(1)) \to 0 \to 0 \cdots \to 0)
\end{tikzcd}$$

which shows that $\Omega\mathbf{B}^nU(1) \simeq \mathbf{B}^{n-1}U(1)$. Now since $i^*\mathbf{L} : \mathbf{Fields}^\partial \to \mathbf{B}^nU(1)$ is a trivial circle n-bundle it has to factor through the point and by the universal property of the homotopy pullback

$$\begin{tikzcd}
 & \mathbf{Fields}^\partial \arrow[d, dashed, "\xi"] \arrow[ddl] \arrow[ddr] & \\
 & \mathbf{B}^{n-1}U(1) \arrow[dl] \arrow[dr] & \\
* \arrow[dr] & & * \arrow[dl] \\
 & \mathbf{B}^nU(1) &
\end{tikzcd}$$

we have that $\xi : \mathbf{Fields}^\partial \to \mathbf{B}^{n-1}U(1)$ is a unique circle $(n-1)$-bundle on the boundary fields. The extended Lagrangian on the bulk fields determines the equation of motion of the fields $\phi : \Sigma_n \to \mathbf{Fields}$, where Σ_n can be seen as as a n-dimensional closed oriented manifold that propagates through the target space **Fields**. If Σ_n has a boundary $\partial\Sigma_n$ such that $\phi(\partial\Sigma_n) \subseteq \mathbf{Fields}^\partial$, then the equation of motion of the boundary fields is determined by the extended Lagrangian on the boundary, which is precisely given by this circle $(n-1)$-bundle $\xi : \mathbf{Fields}^\partial \to \mathbf{B}^{n-1}U(1)$.

The extended Lagrangian of the 2d Poisson-Chern-Simons theory given by the prequantum circle 2-bundle ∇^1 can be interpreted in these terms. Notice that the integration of the Poisson manifold M also gave us the inclusion $i : M \to \mathrm{SymplGpd}$, which we call the *atlas*. The pullback of this prequantum circle 2-bundle ∇^1 along this atlas gives the following trivialization

$$\begin{tikzcd}
 & M \arrow[dl] \arrow[dr, "i"] & \\
* \arrow[dr] & \arrow[l, phantom, "\swarrow"] & \mathrm{SymplGpd} \arrow[dl, "\nabla^1"] \\
 & \mathbf{B}^2U(1)_{conn^1} &
\end{tikzcd}$$

Which means that the pullback of the underlying circle 2-bundle is trivializable, and that the corresponding 2-connection is trivial. To see this remember that ∇^1 can be described by a line bundle L over the manifold of morphisms of SymplGpd together with a connection ∇ and a cocycle σ that has norm 1 and which is compatible with the connection. The cocycle σ makes the pullback i^*L over M into a trivial line bundle over M. Hence there exist a unit section of i^*L over M, and since σ is compatible with the connection this unit section has to be covariantly constant from which it follows that the connection has to be trivial.

The instanton sector SymplGpd can be seen as the phase space of the open string 2d Poisson-Chern-Simons theory. An open string moving in SymplGpd can have some of its endpoints confined to the boundary M. These confined endpoints are forced to move inside the boundary M, which in the physics literature is also called a *D-brane*. These endpoints behave as particles moving in M. The motion of these open strings in SymplGpd are determined by the equation of motion of the action functional of the extended Lagrangian ∇^1. By the above this extended Lagrangian ∇^1 also give a circle bundle on ξ on M, which in particular determines the equation of motion of the particles that move in M. In this way, the geometric quantization procedure of Hawkins[**39**] secretly quantizes not only the Poisson manifold with its prequantum circle bundle, but als the 2d Poisson-Chern-Simons theory with its prequantum circle 2-bundle.

REMARK 3.7.1. This point of view was already suggested by Cattaneo and Felder [**8**], where they studied the Poisson σ-model via the Hamiltonian formalism. They constructed a canonical moment map via which the symplectic groupoid as the phase space of the Poisson σ-model is obtained by symplectic reduction.

4. Higher geometric quantization

The geometric quantization of the prequantum circle 2-bundle ∇^1 over the symplectic groupoid, was done by interpreting ∇^1 as the multiplicative prequantum line bundle over the space of morphisms of the symplectic groupoid. We constructed the convolution C^*-algebra $\mathcal{A}(\nabla^0)$ of sections of the underlying multiplicative prequantum line bundle ∇_0 (tensored with half-densities) and we considered the subalgebra

$$\mathcal{A}(\nabla^0)_P \hookrightarrow \mathcal{A}(\nabla^0)$$

of polarized sections, after choosing a suitable polarization. This convolution subalgebra of the prequantum circle 2-bundle ∇^1 can naturally be interpreted in higher geometry by considering bundle gerbe modules.

For every ordinary central extension of Lie groups $U(1) \to U(n) \to PU(n)$ we have the delooping sequence $\mathbf{B}U(1) \to \mathbf{B}U(n) \to \mathbf{B}PU(n) \xrightarrow{\mathbf{dd}_n} \mathbf{B}^2U(1)$, where the last map is the universal Dixmier-Douady class of smooth $PU(n)$-principal bundles (see [**27**]). For ∇^0 : SymplGpd $\to \mathbf{B}^2U(1)$ the underlying bundle of ∇^1, a section σ of the associated $\mathbf{B}U(n)$-fiber 2-bundle is a dashed lift

$$\begin{tikzcd} & \mathbf{B}PU(n) \simeq \mathbf{B}U(n)//\mathbf{B}U(1) \arrow[d, "\mathbf{dd}_n"] \\ \mathrm{SymplGpd} \arrow[ur, dashed, "\sigma"] \arrow[r, "\nabla^0"'] & \mathbf{B}^2U(1) \end{tikzcd}$$

The lift σ of the bundle gerbe ∇^0 is equivalent to what is called a *bundle gerbe module* or rank n *twisted unitary bundle*. These sections σ, as ∇^0-twisted unitary bundles, are equivalent to the (pre-)quantum 2-states of ∇^1 regarded as a prequantum 2-bundle (see [**11**]). By section 5 of [**11**], we find that a bundle gerbe module of the bundle gerbe ∇^0 is precisely a module over C^*-algebra $\mathcal{A}(\nabla^0)_P$. These modules over $\mathcal{A}(\nabla^0)_P$ form together the category $\mathrm{Mod}_{\mathcal{A}(\nabla^0)_P}$ of modules, which can be interpreted as a 2-vector space. Notice that the category of k-vector spaces is the category of k-modules $\mathrm{Vect}_k = k\text{-Mod}$ and the 2-category of 2-vector spaces is

$$2\mathrm{Vect} = \mathrm{Vect}_{\mathrm{Vect}} = \mathrm{Vect}\text{-Mod}$$

that is the 2-category of abelian categories equipped with a $(\mathrm{Vect}, \otimes)$-action. The strict 2-category of Algebras of algebras, algebra homomorphisms and intertwiners (that is the 2-category for algebras regarded as one-object Vect-enriched categories) are sitting inside 2Vect according to (see appendix of [**81**])

$$\mathrm{Algebras} \hookrightarrow 2\mathrm{Vect}$$

$$\begin{tikzcd} A \arrow[r, bend left, "N"] \arrow[r, bend right, "N'"'] \arrow[r, phantom, "\Downarrow\phi"] & B \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \mathrm{Mod}_A \arrow[r, bend left, "-\otimes_A N"] \arrow[r, bend right, "-\otimes_A N'"'] \arrow[r, phantom, "\Downarrow -\otimes_A \phi"] & \mathrm{Mod}_B \end{tikzcd}$$

Hence the quantum 2-states of the prequantum 2-bundle ∇^1, that are the ∇^0-twisted unitary bundles are naturally interpreted as an instance of a 2-vector space

$$\left\{ \begin{array}{c} \text{quantum 2-states of} \\ \text{prequantum 2d Poisson-Chern-Simons theory} \end{array} \right\} \simeq \mathrm{Mod}_{\mathcal{A}(\nabla^0)_P} \in 2\mathrm{Vect}$$

In this way the geometric quantization of the Poisson manifold M can be seen as the higher geometric quantization of the 2d Poisson-Chern-Simons theory. We can think of the Poisson manifold as a quantum mechanical system, hence a 1-dimensional quantum field theory, which in the case it is a symplectic manifold yields a vector space of quantum states. Where similarly the higher geometric quantization of the 2d Poisson-Chern-Simons theory yields a 2-vector space of quantum 2-states. This indicates a holographic relation between higher geometric quantization of the 2d Poisson-Chern-Simons theory and quantum mechanics as a 1d quantum field theory.

These category of modules $\mathrm{Mod}_{\mathcal{A}(\nabla^0)_P}$ can naturally be interpreted as 2-vector spaces with a 2-basis. Just like an ordinary k-vector space V has as basis a set S such that $V \simeq \mathrm{Hom}_{\mathrm{Set}}(S, k)$, a 2-basis for a 2-vector space V has to be a category S such that $V \simeq \mathrm{Hom}(S, \mathrm{Vect})$. If S is a Vect-enriched category and we see Vect as a Vect-enriched over itself, then $V \simeq \mathrm{Hom}(S, \mathrm{Vect})$ corresponds to the Vect-enriched functors from S to Vect. A Vect-enriched category S is just a linear category (or algebroid) and if it has a single object it is an algebra and we have

$$\mathrm{Mod}_{\mathcal{A}(\nabla^0)_P} = \mathrm{Hom}(\mathbf{B}\mathcal{A}(\nabla^0)_P, \mathrm{Vect})$$

where the algebra $\mathcal{A}(\nabla^0)_P$ is here regarded as the one-object Vect-enriched category $\mathbf{B}\mathcal{A}(\nabla^0)_P$. Hence the linear category $\mathbf{B}\mathcal{A}(\nabla^0)_P$ can be regarded as the 2-basis of the 2-vector space $\mathrm{Mod}_{\mathcal{A}(\nabla^0)_P}$. In this sense the we can interpreted the algebra $\mathcal{A}(\nabla^0)_P$ in higher geometry as the 2-basis for the 2-space of quantum 2-states.

From the point of view of ordinary geometric quantization, we produced a space of states from the space of sections of the prequantum bundle, and the constructions of the algebra of operators acting on this space of states required some work. From the point of view of the geometric quantization or strict C^*-deformation quantization of Hawkins[**39**], we produced immediately the algebra. The 2-space of 2-states we got from taking sections of the associated $\mathbf{B}U(n)$-fiber 2-bundle associated to the prequantum 2-bundle, which has the algebra as 2-basis. The construction of the 2-algebra of higher operators acting on these 2-states is still an open question.

This strict C^*-deformation quantization approach of Hawkins[**39**] has a infinitesimal analogue, which is the formal deformation quantization of the Poisson manifold, which was studied by Konstevich, Cattaneo and Felder. In [**8**] Cattaneo and Felder compute Kontsevich's formula for the star-product as the 3-point function of the corresponding 2d Poisson σ-model. This 3-point function is a certain correlator for three points on the boundary of a closed disc, where the fields on the boundary take value in the Poisson Lie algebroid associated to the Poisson manifold. This shows a similar holographic relation between the Poisson manifold and its holographically related 2d Poisson σ-model. Similarly, Gukov and Witten showed in [**38**] that the quantization of a symplectic manifold can be formulated in terms of the quantization of a 2d quantum field theory, called the A-model, for which the symplectic manifold is a boundary. This A-model is a special case of the Poisson σ-model. Alltogether we will summarize in the following table the state of affairs of quantization procedures of a Poisson manifold in terms of higher geometry.

	(perturbative) formal algebraic quantization	**(non-perturbative) geometric quantization**
quantization of Poisson manifold	formal deformation quantization	strict C^*-deformation quantization
"holographically" related 2d field theory	Poisson σ-model	2d Poisson-Chern-Simons theory
moduli stack of fields of the 2d field theory	Poisson Lie algebroid	symplectic groupoid
quantization of the 2d field theory	perturbative quantization of Poisson σ-model	higher geometric quantization of 2d Poisson-Chern-Simons theory

4.1. Higher geometric quantization of a symplectic manifold. We consider the special case where the Poisson manifold (M, ω^{-1}) happens to be a symplectic manifold (M, ω), which we assume to be simply connected. In this case the Poisson manifold integrates to the pair groupoid $Pair(M)$. The symplectic form is given by $t^*\omega - s^*\omega$. If M is prequantizable we saw in section 2.5 that we have the prequantum bundle $\xi : M \to \mathbf{B}U(1)_{conn}$ and the pair groupoid $Pair(M)$ can be prequantized by the multiplicative circle bundle $t^*\xi - s^*\xi$, which gives rise to the circle 2-bundle $\nabla^1 : Pair(M) \to \mathbf{B}^2U(1)_{conn^1}$.

Remember that the stacks that were represented by Lie groupoids, where called *differentiable stacks*. From the higher geometric perspective we considered stacks up to weak equivalence. This weak equivalence for differentiable stacks captures precisely the notion of *Morita equivalence* of their presenting Lie groupoids. Remember the morphisms in the bicategory of Lie groupoids LieGpd are precisely the left principal bibundles and it is a Morita equivalence if it is both a left and right principal bibundle, see appendix A for more details. More precisely, we have the well-known fact, that the bicategory of Lie groupoids LieGpd presents the bicategory of differentiable stacks DiffStack, that is

Proposition 4.1.1. [**4**] *There is a equivalence of* $(2,1)$*-categories*

$$LieGpd \xrightarrow{\sim} DiffStack$$

which sends each Lie groupoid to the associated stack.

Hence from the higher geometric perspective it suffice to consider the pair groupoid $Pair(M)$ up to Morita equivalence. In example 2.0.31 in appendix A, we saw that the pair groupoid $Pair(M)$ is Morita equivalent to the point. This means that $Pair(M)$ gives nothing more than a presentation of the trivial stack $*$. Now forgetting the connection data for a moment, we see that the boundary condition for the 2d Poisson-Chern-Simons theory induced by the symplectic manifold is given by the diagram

$$\begin{tikzcd}
 & M \arrow[dl] \arrow[dr, "i"] & \\
* \arrow[dr] & \arrow[l, phantom, "\Leftarrow \xi"] & Pair(M) \simeq * \arrow[dl, "\nabla^0"] \\
 & \mathbf{B}^2 U(1) &
\end{tikzcd}$$

Where we immediately see that ξ is given by the unique map to the loop space object of $\mathbf{B}^2U(1)$, that is

$$\begin{tikzcd}
 & M \arrow[d, dashed, "\xi"] \arrow[ddl] \arrow[ddr] & \\
 & \mathbf{B}U(1) \arrow[dl] \arrow[dr] & \\
* \arrow[dr] & & * \arrow[dl] \\
 & \mathbf{B}^2U(1) &
\end{tikzcd}$$

and so is equivalent to the prequantum circle bundle $\xi : M \to \mathbf{B}U(1)$. Remember from the example of section 8.1, that the strict C^*-deformation quantization of a symplectic manifold together with a certain polarization P is taken to be the C^*-algebra of compact operators on the leaf space M/P. We found that the C^*-algebra of compact operators is the twisted polarized convolution algebra of the the symplectic groupoid $Pair(M/P)$ carrying a circle 2-bundle. The fact that we consider Lie groupoids up to Morita equivalence is also reflected in their associated convolution algebras. In [**50, 52**] Landsman proved that Morita equivalent Lie groupoids have Morita equivalent convolution algebras. Hence the Morita equivalence of the pair groupoid $Pair(M/P)$ and the point $*$ is reflected by the well-known fact that the compact operators are Morita equivalent to the base algebra of complex numbers.

In this sense Hawkins' strict C^*-deformation quantization is not *Morita faithful*, meaning that it distinguishes Morita equivalent Lie groupoids and Morita equivalent C^*-algebras. The above example shows us that Hawkins' solution of the strict C^*-deformation problem does not have intrinsic meaning under Morita equivalence, since for the symplectic manifold case, one just arrives at a trivial quantization. Hence in this higher geometric perspective Hawkins' strict C^*-deformation quantization of symplectic manifolds contains a conundrum: either one breaks with the Morita equivalence or else one arrives at trivial quantization.

CHAPTER 5

Outlook

In the previous chapter we saw that in the higher geometric perspective Hawkins' strict C^*-deformation quantization of symplectic manifolds contains a conundrum, that is either one breaks with the Morita equivalence or else one arrives at trivial quantization. This conundrum is resolved by quantizing more than just the symplectic groupoid itself. From the Lie groupoid perspective one sees that the convolution algebra is only sensitive to the underlying differentiable stack and the information missing in the latter is precisely the induced *atlas*, given by the epimorphism of smooth stacks $M \to \mathrm{SymplGpd}$, which is given by the inclusion of the manifold of objects into the Lie groupoid. Together with the trivial stack $*$ as terminal object, this is equivalent to the datum of a correspondence

$$* \longleftarrow M \xrightarrow{i} \mathrm{SymplGpd}$$

which exhibits the original as a boundary of the corresponding 2d Chern-Simons theory if there exists a lift to the prequantum field theory given by the diagram

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr, "i"] & \\ * \arrow[dr] \arrow[rr, phantom, "\Swarrow \xi"] & & \mathrm{SymplGpd} \arrow[dl, "\nabla^0"] \\ & \mathbf{B}^2U(1) & \end{tikzcd}$$

We do not only assign to the differentiable stack SymplGpd carrying a circle 2-bundle $\mathrm{SymplGpd} \to \mathbf{B}^2U(1)$ a twisted convolution algebra as a C^*-algebra, but we can assign to a morphism of differentiable stacks carrying circle 2-bundles a Hilbert bimodule of C^*-algebras. In [**67**] Nuiten shows that forming groupoid twisted convolution algebras naturally extends to a functor of (2,1)-categories

$$(\mathrm{DiffStacks}_{prop})^{op}_{/\mathbf{B}^2U(1)} \xrightarrow{C^*(-)} C^*\mathrm{Alg}_{prop}$$

from differentiable stacks over $\mathbf{B}^2U(1)$ to C^*-algebras with (proper) Hilbert bimodules between them. Appyling this functor to the above span of differentiable stacks over $\mathbf{B}^2U(1)$ gives us a co-span of Hilbert bimodules

$$\mathbb{C} \xrightarrow{\xi} C^*_{i^*\nabla^0}(M) \xleftarrow{i^*} C^*_{\nabla^0}(\mathrm{SymplGpd})$$

These are Hilbert bimodules of twisted convolution algebras and for Hawkins' strict C^*-deformation quantization we needed a twisted *polarized* convolution algebra, which were constructed from a connection on the circle 2-bundles over the groupoid together with the choice of a certain polarizition. In the symplectic manifold case, this connection and polarization were induced from the connection and polarization of its traditional geometric quantization scheme via polarization. But remember that for traditional geometric quantization of symplectic manifolds, we also had the Spin^c-geometric quantization scheme. In this quantization scheme we only needed to choose a Spin^c-structrure such that its associated determinant line bundle is a prequantum line bundle. The actual Spin^c-quantization was the index of its corresponding Spin^c-Dirac operator. This index is precisely described by a pushforward map in K-theory, which was already observed by [**61**].

The description of maps in K-theory via elliptic operators is nicely captured in KK-theory, which is a bivariant generalization of operator K-theory and K-homology. The KK-groups in KK-theory are natural homotopy equivalence classes of Hilbert bimodules equipped with some Fredholm operator and the famous Kasparov product gives the composition operation between these KK-groups and hence induces the KK-category. Furthermore there is a natural functor

$$C^*\mathrm{Alg}_{prop} \longrightarrow KK$$

from the bicategory of C^*-algebras with (proper) bimodules to the category KK. Hence the above cospan of Hilbert bimodules can be regarded as a cospan in KK-theory. And in order to push-forward in K-theory we need to turn the right map around, that is instead of pulling back along the map i, we push along it. For the full construction we refer the reader to [**67**], but we will give here the main idea. We begin by noting that the choice of Spin^c-structure in Spin^c-quantization, precisely corresponds to a choice of what is called a *K-orientation* of the atlas, which is in our case the inclusion map i. This choice of Spin^c-structure does only not restrict to symplectic manifold and in our case can be defined for any compact Poisson manifold M. Assuming that $C^*_{\nabla_0}(\mathrm{SymplGpd})$ is dualizable in KK then we have a canonical dual map

$$(i^*)^\vee : C^*_{i^*\nabla_0}(M)^\vee \longrightarrow C^*_{\nabla_0}(\mathrm{SymplGpd})^\vee$$

The choice of a K-orientation determines a *Thom isomorphism*

$$Th(M) : C^*_{i^*\nabla_0}(M) \longrightarrow C^*_{i^*\nabla_0}(M)^\vee$$

and composing these two maps gives what is called the *Umkehr map*

$$i_! : C^*_{i^*\nabla_0}(M) \xrightarrow{Th} C^*_{i^*\nabla_0}(M)^\vee \xrightarrow{(i^*)^\vee} C^*_{\nabla_0}(\mathrm{SymplGpd})^\vee$$

Now the *pull-push quantization* in KK-theory is given by the composition

$$\mathbb{C} \xrightarrow{\xi} C^*_{i^*\nabla_0}(M) \xrightarrow{i_!} C^*_{\nabla_0}(\mathrm{SymplGpd})^\vee \in KK(\mathbb{C}, C^*_{\nabla_0}(\mathrm{SymplGpd})^\vee) = K_0(C^*_{\nabla_0}(\mathrm{SymplGpd})^\vee)$$

Which describes precisely a K-theory class of $C^*_{\nabla_0}(\mathrm{SymplGpd})^\vee$. Notice that the symplectic groupoid SymplGpd of a Poisson manifold behaves roughly like the symplectic leaf space of the Poisson manifold. In particular if the Poisson manifold happens to be a symplectic manifold, then the symplectic groupoid is equivalent to the point. In this case the above yields an element of the K-theory of the point and as is proven in [**67**] we get precisely the traditional Spin^c-quantization for symplectic manifolds, that is $i_!(\mathbb{L}) = \mathrm{index}(D_{\mathbb{L}}) \in KK(\mathbb{C}, \mathbb{C}) \in \mathbb{Z}$, where $\mathbb{L}$ is here ξ. In this sense a class in $K_0(C^*_{\nabla_0}(\mathrm{SymplGpd})^\vee)$ may be thought of as providing one (virtual) vector space for each symplectic leaf, to be thougth of as the space of quantum states.

That KK-theory is a natural codomain for quantization of Poisson manifolds has long been amplified by Landsman[**53**], but Landsman focused mainly on the issue of symplectic reduction, while here the point is to define the quantization of Poisson manifolds. This pull-push quantization thought of as a 2-geometric quantization of Poisson manifolds should fix the issue that the strict C^*-deformation quantization of Hakwins is not Morita faithful.

The general framework for pull-push quantization is treated in [**67**] where several other applications are covered. For example the extension of geometric quantization of symplectic manifold that carry a Hamiltonian G-action by pushing forward in equivariant K-theory, the D-brane charges as the quantization of a particle at the boundary of a string, the M9-brane charge as the quantization of a string moving at the boundary of a 2-brane, to name a few. This pull-push quantization is a first step in the direction of a full description of geometric quantization for every prequantum field theory. We only treated the case where we lifted a correspondence of the form

$$* \longleftarrow M \xrightarrow{i} \mathrm{SymplGpd}$$

to a diagram

$$\begin{tikzcd}
 & M \arrow[dl] \arrow[dr, "i"] & \\
* \arrow[dr] & \Swarrow & \mathrm{SymplGpd} \arrow[dl] \\
 & \mathbf{B}^2U(1) &
\end{tikzcd}$$

But in the full n-dimensional prequantum field theory we want to consider more generally any n-fold correspondence by a monoidal functor

$$\mathrm{Bord}_n \xrightarrow{\mathbf{Fields}} \mathrm{Corr}_n(\mathbf{H})$$

from the (∞, n)-category of cobordisms to the (∞, n)-category of n-fold correspondences in $\mathbf{H}$ and an action functional on this field as a lift of monoidal functors

$$\begin{tikzcd}
 & \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{B}^nU(1)}) \arrow[d] \\
\mathrm{Bord}_n \arrow[ur, dashed, "\exp(iS)"] \arrow[r, "\mathbf{Fields}"] & \mathrm{Corr}_n(\mathbf{H})
\end{tikzcd}$$

where the morphism $\exp(iS)$ maps to n-fold correspondences in the slice topos over the moduli stack of circle n-bundles. This datum together is called a n-dimensional prequantum field theory (see [**80, 67**]). We have only treated the quantization of a single correspondence diagram of the above specific form. A single correspondence diagram of a more general form is treated in [**67**]. Since the n-fold correspondence diagrams are built up from single spans, one hopes that this is a first step in right direction for the quantization of n-fold correspondences. Furthermore the pull-push quantization works mainly for the case where one considers spans of differentiable stacks over $\mathbf{B}^2U(1)$, where one can apply the twisted K-theory for differentiable stacks. But for arbitrary stacks over $\mathbf{B}^nU(1)$ there is still no general procedure. The well-understood geometric quantization of 3d Chern-Simons theory should at least give a blueprint for the case $n = 3$. Eventually one would like to have a fully extended geometric quantization of all prequantum field theories, but this requires still some work.

APPENDIX A

Lie groupoids, Lie algebroids and integrability

Recall that a groupoid can diagrammatically be defined as

$$\begin{tikzcd}
\mathcal{G}_2 = \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \arrow[d, shift right=6, "pr_1"'] \arrow[d, "m"] \arrow[d, shift left=6, "pr_2"] \\
\mathcal{G}_1 \arrow[loop right, "i"] \arrow[d, shift right=6, "t"'] \arrow[d, leftarrow, "1"] \arrow[d, shift left=6, "s"] \\
\mathcal{G}_0
\end{tikzcd}$$

such that the morphisms satisfy the conditions of composition, associativity, identity and inverses.

If $x \in M$, then the sets

$$\mathcal{G}(x, -) = s^{-1}(x), \quad \mathcal{G}(-, x) = t^{-1}(x)$$

are called the *s-fibers* at x, and the *t-fibers* at x, respectively. The inverse map induces a natural bijection between these two sets

$$i : \mathcal{G}(x, -) \to \mathcal{G}(-, x)$$

For every $g : x \to y$ the right multiplication by g defines the bijection

$$R_g : \mathcal{G}(y, -) \to \mathcal{G}(x, -)$$

Similarly the left multiplication by g induces a bijection

$$L_g : \mathcal{G}(-, x) \to \mathcal{G}(-, y)$$

Then intersection of the s and t-fiber at $x \in M$, that is $\mathcal{G}_x = \mathcal{G}(x, -) \cap \mathcal{G}(-, x)$, together with the restriction of the groupoid multiplication give a group, which we call the *isotropy group at x*.
Furthermore we have also an equivalence relation $\sim_{\mathcal{G}}$ on the base manifold M, which is defined by: $x, y \in M$ are said to be equivalent if there exist a morphism $g \in \mathcal{G}$ such that $s(g) = x$ and $t(g) = y$. The equivalence class of $x \in M$ is called the *orbit through* x, that is $\mathcal{O}_x = \{t(g) : g \in s^{-1}(x)\}$ and the quotient set $M/\mathcal{G} := M/\sim_{\mathfrak{g}} = \{\mathcal{O}_x : \ x \in M\}$, is called the *orbit set of* $\mathcal{G}$.

Definition 0.1.2. A *Lie groupoid* is a groupoid $\mathcal{G}$ whose set of objects $M = \mathcal{G}_0$ and set of morphisms $\mathcal{G}_1$, which we denote also by $\mathcal{G}$, are manifolds, whose structure maps $s, t, m, 1, i$ are all smooth maps and such that s, t are submersion.

The condition that s and t are submersions ensure that the s and t-fibers are manifolds. They also ensure that the space $\mathcal{G}_2$ of composable morphims is a submanifold of $\mathcal{G} \times \mathcal{G}$.

Remark 0.1.3. A *topological groupoid* is a groupoid $\mathcal{G}$ whose set of objects and set of morphisms are both topological spaces, whose structure maps are all continuous, and such that s and t are open maps.

We call $\mathcal{G}$ *s-connected* if the s-fibers are connected, $\mathcal{G}$ is called *s-simply connected* if they are simply connected and $\mathcal{G}$ is called *s-locally trivial* if they form a bundle. Any Lie groupoid $\mathcal{G}$ has an associated s-connected groupoid $\mathcal{G}^0$, the s-connected component of the identities.

A morphism between two Lie groupoids $\mathcal{G}$ over M and $\mathcal{H}$ over N is a morphism between groupoids, i.e. it consists of a map $F : \mathcal{G} \to \mathcal{H}$ between the set of arrows and a map $f : M \to N$ between the set of objects, which are compatible with all the structure maps, and in addition these two morphism needs to be smooth. The Lie groupoids together with there morphisms form the category of Lie groupoids.

A *Lie subgroupoid* of a groupoid $\mathcal{G}$ is a pair $(\mathcal{H}, i)$, where $\mathcal{H}$ is a groupoid and $i : \mathcal{H} \to \mathcal{G}$ is an injective immersive groupoid morphism. A *wide Lie subgoupoid* is a Lie subgroupoid $\mathcal{H} \subset \mathcal{G}$ which has the same space of units as $\mathcal{G}$.

Let $F : \mathcal{G} \to \mathcal{H}$, $f : M \to N$ be a morphism of groupoids. Then the *kernel of* F is the set $ker(F) := \{g \in \mathcal{G} \ : \ F(g) = 1(x), \ \exists x \in N\}$.

Example 0.1.4. (Groups) A group is nothing more then a groupoid for which the set of objects contains a single element. Obviously Lie groups are examples of Lie groupoids.

Example 0.1.5. (The pair groupoid) Let M be any set. The Cartesian product $Pair(M) := M \times M$ is the *pair groupoid* over M where the set of objects is M and where we take a pair $(x, y) \in M \times M$ as a morphism $x \to y$. Composition is defined by:

$$(z, y) \circ (y, x) = (z, x)$$

If M is a manifold, then the pair groupoid is a Lie groupoid.

Example 0.1.6. (The fundamental groupoid) Let M be a manifold. The *fundamental groupoid* of M, denoted by $\Pi_1(M)$ is the groupoid whose set of objects is M and whose morphisms from x to y are the homotopy classes $[\gamma]$ of continuous maps $\gamma : [0, 1] \to M$ with $\gamma(0) = 1$ and $\gamma(1) = y$. Composition is defined by concatenation and reparametrization of representative maps. The homotopy-equivalence relation makes it a groupoid. Note also that, when M is connected and simply-connected, $\Pi_1(M)$ is isomorphic to the pair groupoid $Pair(M)$, since a homotopy class of a path is determined by its end points. In general the fundamental groupoid $\Pi_1(M)$ is the connected cover of $Pair(M)$.

Example 0.1.7. (The gauge groupoid) Let G be a Lie group and $\pi : P \to M$ be a left principal G-bundle. Remember that a smooth map of principal G-bundle $\phi : P \to P$ that commutes with the G-action is called a gauge transformation. Then we define $Gauge(P)$ to be the *gauge groupoid* that is the groupoid $P \times_G P$ over M, where $P \times_G P$ is the orbit space of the diagonal action of G on $P \times P$. It obtains a groupoid structure from the pair groupoid $Pair(P)$ over P, that is $s := \pi \circ pr_2$, $t := \pi \circ pr_1$ and composition is given by $[p, q][q', r] = [p, q][q, g \cdot r] = [p, g \cdot r]$ where $p, q, q', r \in P$ such that $\pi(q) = \pi(q')$ and $g \in G$ is unique such that $g \cdot q' = q$. The unit is defined by $1(x) =: [p, p]$ for any $p \in \pi^{-1}(x)$ and the inverse is given by $[p, q]^{-1} := [q, p]$ for $p, q \in P$.

We saw that a Lie groupoid is a generalization of a Lie group, and we know that the infinitesimal objects corresponding to a Lie group is a Lie algebra. Now similarly the infinitesimal object that correspond to a Lie groupoid is a Lie algebroid. A Lie algebra consists of a vector space that corresponds to the tangent space of the group at the unit element, hence we expect that a Lie algebroid is a vector bundle over M, since we have a unit for each point in M. Secondly the vector space of a Lie algebra can be identified with the space of left invariant vector field on the group and this space of left invariant vector fields is closed under the Lie bracket of vector fields. Now since left multiplication on a Lie groupoid $\mathcal{G}$ is only defined on the t-fibers, the left invariant vector fields on $\mathcal{G}$ should be those vector fields which are tangent to the t-fibers, i.e. the sections of the subbundle $T^t\,\mathcal{G}$ of $T\,\mathcal{G}$ defined by

$$T^t\,\mathcal{G} = \ker(dt) \subset T\,\mathcal{G}$$

Hence the corresponding vector bundle A of a $\mathcal{G}$ should have for each fiber at $x \in M$ the tangent space at the unit of the t-fiber at x, i.e. $A := T^t\,\mathcal{G}\,|_M$. It turns out the there exists here also an isomorphism between the space of left invariant vector fields and the space of sections of A, which we denote by $\Gamma(A)$. The Lie bracket $[\cdot,\cdot])_A$ on A is the Lie bracket on $\Gamma(A)$ obtained from the Lie bracket on the space of left invariant vector fields under this isomorphism. To describe the entire structure underlying A we need also an anchor map of A, which is a bundle map $\rho_A : A \to TM$ obtained by restricting $ds : T\mathcal{G} \to TM$ to $A \subset T\mathcal{G}$. This defined Lie bracket and anchor map are related by the Leibniz identity. Let $\alpha, \beta \in \Gamma(A)$ and $f \in C^\infty(M)$, then

$$[\alpha, f\beta]_A = f[\alpha,\beta]_A + \rho_A(\alpha)(f)\beta$$

For more details we refer the reader to [**16**]. Summarizing the above gives us the following definition:

Definition 0.1.8. The *Lie algebroid of the Lie groupoid* $\mathcal{G}$ is the vector bundle $A := T^t\mathcal{G}|_M$, together with the anchor $\rho_A : A \to TM$ and the Lie bracket on $[\cdot,\cdot]_A$ on $\Gamma(A)$.

Proposition 0.1.9. *For any Lie groupoid, $\mathcal{G}^0$ is an open subgroupoid of $\mathcal{G}$. Hence, $\mathcal{G}^0$ is the s-connected Lie groupoid that has the same Lie algebroid as $\mathcal{G}$.*

We arrive at the abstract notion of a Lie algebroid:

Definition 0.1.10. A *Lie algebroid* over a manifold M consists of a vector bundle A together with a bundle map $\rho_A : A \to TM$, called the anchor map, and a Lie bracket $[\cdot,\cdot]_A$ on the space of sections $\Gamma(A)$, satisfying the Leibniz identity:

$$[\alpha, f\beta]_A = f[\alpha,\beta]_A + \rho_A(\alpha)(f)\beta$$

for $\alpha, \beta \in \Gamma(A)$ and $f \in C^\infty(M)$.

A morphism between two Lie algebroids $A_1 \to M_1$ and $A_2 \to M_2$ is a vector bundle map

$$\begin{tikzcd} A_1 \arrow[r, "F"] \arrow[d] & A_2 \arrow[d] \\ M_1 \arrow[r, "f"'] & M_2 \end{tikzcd}$$

which is compatible with the anchors, that is

$$df(\rho_{A_1}(a)) = \rho_{A_2}(F(a))$$

and is compatible with the Lie bracket, that is

$$F([\alpha,\beta]_{A_1}) = [\alpha',\beta']_{A_2} \circ f$$

for $\alpha, \beta \in \Gamma(A_1)$ and where $\alpha', \beta' \in \Gamma(A_2)$ are defined by $F(\alpha) = \alpha' \circ f$ and $F(\beta) = \beta' \circ f$.

The Lie algebroids together with there morphisms form the category of Lie algebroids. Every morphism of Lie groupoids induce a morphism between their associated Lie algebras, hence there exists a Lie functor A from the category of Lie groupoids to the category of Lie algebroids (see [**58**]).

Example 0.1.11. (Lie algebras) Any Lie algebra $\mathfrak{g}$ is a Lie algebroid over the singleton. The above functor A restricts to the classical functor from Lie groups to Lie algebras.

Example 0.1.12. (Tangent bundles) The tangent bundle $A = TM$ is an example of a Lie algebroid, with the identity map as anchor, and the usual Lie bracket of vector fields.

Example 0.1.13. (Poisson manifolds) Let M be a manifold. The space of smooth functions on M, denoted by $C^\infty(M)$ is a vector space and forms an algebra by pointwise multiplication of functions. A Poisson bracket on a manifold M is a Lie bracket $\{\cdot,\cdot\}$ on the algebra $C^\infty(M)$ satisfying the derivation property:

$$\{fg, h\} = f\{f,h\} + g\{f,h\} \quad \forall f,g,h \in C^\infty(M)$$

We call the algebra $C^\infty(M)$ together with the Poisson bracket $\{\cdot,\cdot\}$ a Poisson algebra, which we denote by $(C^\infty(M),\{\cdot,\cdot\})$. A Poisson manifold is a manifold M equipped with a Poisson bracket $\{\cdot,\cdot\}$ on the algebra $C^\infty(M)$ such that $(C^\infty(M),\{\cdot,\cdot\})$ is a Poisson algebra. We call this Poisson algebra also the Poisson structure on M. Alternatively a Poisson structure on a manifold M is given by a choice of smooth antisymmetric bivector, called a Poisson bivector, $\pi \in \Gamma(\wedge^2 TM)$ such that

$$\{f,g\}_\pi = \pi(df,dg)$$

and $\{\cdot,\cdot\}_\pi$ satisfy Jacobi identity. Let π be a Poisson bivector on a manifold M. This Poisson structure induces a Lie algebroid structure on T^*M as follows. Define the map

$$\pi^\sharp : T^*M \to TM$$

by $\beta(\pi^\sharp(\alpha)) = \pi(\alpha,\beta)$. The Lie algebroid structure on T^*M has $\pi^\sharp$ as anchor map, and the Lie bracket, also called the Koszul bracket, is defined by

$$[\alpha,\beta] = \mathcal{L}_{\pi^\sharp(\alpha)}(\beta) - \mathcal{L}_{\pi^\sharp(\beta)}(\alpha) - d(\pi(\alpha,\beta))$$

We call this the Poisson Lie algebroid of the Poisson manifold (M,π).

We saw that any Lie groupoid has an associated Lie algebroid.

Definition 0.1.14. A Lie algebroid A is called *integrable* if it is isomorphic to the Lie algebroid of a Lie groupoid $\mathcal{G}$. For such a $\mathcal{G}$, we say that $\mathcal{G}$ integrates A.

We introduce this definition since not every Lie algebroid is integrable, the obstructions to integrate a Lie algebroid are discussed in detail in [**17**]. Suppose A is a Lie algebroid over M, then we can construct a topological groupoid $\mathcal{G}(A)$. It turns out that whenever A is integrable, this topological groupoid $\mathcal{G}(A)$ admits smooth structure, which makes it a Lie groupoid, such that it integrates A. The main idea is as follows: Suppose $\pi : A \to M$ is an integrable Lie algebroid. Define an A-path a to be a curve $a : [0,1] \to A$ with the property that $\rho_A(a(t)) = \frac{d}{dt}\pi(a(t))$. Denote by $P(A)$ the space of A-paths. We can compose two A-paths if they have the same end-points. In order for these composition to be smooth we need to introduce some cut-off function and reparametrization which defines the multiplication of composable A-paths. Under the equivalence relation of homotopy of A-paths we can construct our groupoid $\mathcal{G}(A)$ as $\mathcal{G}(A) = P(A)/\sim$, which is called the *Weinstein groupoid*.

Theorem 0.1.15. *If A is integrable, then there exists an unique (up to isomorphism) s-simply connected Lie groupoid $\mathcal{G}(A)$ integrating A.*

Theorem 0.1.16. *Let $\phi : A \to B$ be a morpism of integrable Lie algebroids, and let $\mathcal{G}(A)$ and $\mathcal{H}(B)$ be integrations of A and B. If $\mathcal{G}(A)$ is s-simply connected, then there exists a (unique) morphism of Lie groupoids $\Phi : \mathcal{G}(A) \to \mathcal{H}(H)$ integrating ϕ.*

Hence there exists an inverse functor G of A, which has as domain the full subcategory of integrable Lie algebroids. For more details see [**17**].

Remark 0.1.17. The notion of integrability does not include the assumption of Hausdorfness. In the theory of Lie groupoids, it is often necessary to consider non-Hausdorff groupoids. We will assume that the base manifold, algebroids, and the s- and t-fibers are Hausdorff. The non-Huasdorff case will not be addressed in this thesis.

1. Poisson Geometry

We are especially interested in the integration of Poisson manifolds, since a Lie algebroid integrating a Poisson manifold has a natural symplectic structure, which will be usefull for developing various quantization schemes.

Definition 1.0.18. Let $\mathcal{G}$ be a Lie groupoid and $\omega \in \Omega^*(\mathcal{G}_1)$. Let $\partial^*\omega := pr_1^*\omega - m^*\omega + pr_2^*\omega$, then we call ω *multiplicative* if

$$\partial^*\omega = 0$$

A *symplectic groupoid* is a groupoid Σ with a multiplicative symplectic form $\omega \in \Omega^2(\Sigma_1)$.

Remark 1.0.19. The nerve $\mathcal{G}_*$ of a groupoid $\mathcal{G}$ has a natural structure of a simplicial manifold. This structure gives a boundary operator ∂^* on differential forms on the double complex $\Omega^*(\mathcal{G}_*)$, which is the simplicial de Rham complex of the nerve $\mathcal{G}_*$. That is for $\theta \in \Omega^*(M)$ we have $\partial^*\theta := t^*\theta - s^*\theta \in \Omega^*(\mathcal{G})$ and for $\omega \in \Omega^*(\mathcal{G})$ we have $\partial^*\omega := pr_1^*\omega - m^*\omega + pr_2^*\omega \in \Omega^*(\mathcal{G}_2)$. The simplicial coboundary extends to line bundle, since for a line bundle L over M we have the line bundle $\partial^*L := t^*L \otimes s^*L^*$ over $\mathcal{G}$. Similarly a line bundle $\mathbb{L}$ over $\mathcal{G}$ gives a line bundle $\partial^*\mathbb{L} := pr_1^*\mathbb{L} \otimes m^*\mathbb{L}^* \otimes pr_2^*\mathbb{L}$ over $\mathcal{G}_2$, which continuous this way for higher $\mathcal{G}_k$. The coboundary of a coboundary, that is $\partial^*\partial^*L$ is canonically a trivial line bundle, since $curv\ \partial^*\partial^*L = \partial^*\partial^*(curv\ L)$. A section of a line bundle $\sigma \in \Gamma(\mathcal{G}_k, L)$ has a multiplicative coboundary $\partial^*\sigma \in \Gamma(\mathcal{G}_k, \partial^*L)$.

The base manifold of a symplectic groupoid has a canonical Poisson bracket.

Proposition 1.0.20. *Let (Σ, ω) be a symplectic groupoid over M. Then there exists a unique Poisson structure on M such that s is Poisson and t is anti-Poisson and the Lie algebroid of Σ is canonically isomorphic to T^*M.*

Remark 1.0.21. Recall a map $\phi : M \to N$ between Poisson manifolds is called a *Poisson map* if it preserves the Poisson brackets.

We saw in example 0.1.13 that Poisson manifolds form a nice class of Lie algebroid structures. We can integrate a Poisson manifold by integrating their associated Poisson Lie algebroid. In this case the Poisson Lie algebroid $A = T^*M$ and we will denote $\mathcal{G}(T^*M) =: \Sigma(M)$. Here the of T^*M-paths and T^*M-homotopies are known as cotangent paths, and cotangent homotopies. Then one should think of $\Sigma(M)$ as the fundamental groupoid of the Poisson manifold and it can be described as the quotient

$$\Sigma(M) = \text{cotangent paths/cotangent homotpies.}$$

Theorem 1.0.22. *Let M be a Poisson manifold. If the associated Poisson Lie algebroid T^*M is integrable, then there exists an unique (up to isomorphism) s-simply connected symplectic groupoid $\Sigma(M)$ integrating M.*

Example 1.0.23. (Symplectic manifolds) Let M be a symplectic manifold with symplectic form ω. For each smooth function $f \in C^\infty(M)$, we have an associated *Hamiltonian vector field* $X_f \in Vect(M)$ such that

$$df = \omega(X_f, -)$$

This determines X_f uniquely, since ω is non-degenerate. The *Poisson bracket* is defined by

$$\{f, g\} := X_f(g) = \omega(X_g, X_f) = -X_g(f)$$

for $f, g \in C^\infty(M)$ and hence a symplectic manifold is a particular example of a Poisson manifold. The symplectic form induce the isomorphism $\omega^\sharp : TM \to T^*M$ with inverse, the anchor map $\pi^\sharp : T^*M \to TM$ of the Poisson Lie algebroid associated to the Poisson manifold. Hence the Poisson (or cotangent) Lie algebroid T^*M is isomorphic to the tangent Lie algebroid TM and a cotangent path is determined a path in the base manifold M. Hence the symplectic groupoid is the fundamental groupoid of M, that is $\Sigma(M) \simeq \Pi_1(M)$. When M is simply connected, the $\Sigma(M)$ is just the pair groupoid $Pair(M)$, and in this case the symplectic form on $Pair(M)_1$ is just $\omega \oplus -\omega$. The minus sign comes from the simplicial coboundary operator ∂^* acting on ω, that is $\partial^*\omega = t^*\omega - s^*\omega = \omega \oplus -\omega$.

Remark 1.0.24. Note that any symplectic groupoid integrating M is a quotient of $\Sigma(M)$.

2. Morita equivalent Lie groupoids

In higher geometry we need to consider Lie groupoids only up to Morita equivalence. In this section we will define this equivalence and give some examples of interest.

REMARK 2.0.25. Most of the constructions in this sections can be adapted to locally compact Hausdorff groupoids.

DEFINITION 2.0.26. Let M be a manifold and $\mathcal{G}$ a Lie groupoid. A *left action* of $\mathcal{G}$ on M consists of a smooth map $\tau : M \to \mathcal{G}_0$, called the *moment map*, and a smooth map $\mathcal{G}\ {}_s\!\times_\tau M \to M$, $(g, x) \mapsto gx$ such that

(i) $\tau(gx) = t(g)$,
(ii) $\tau(x)x = x$ and
(iii) $(gh)x = g(hx)$ for all $(g, h) \in \mathcal{G}_2$ and $(g, x) \in \mathcal{G} \times_{\mathcal{G}_0} M$.

Similarly the notion of a *right action* is obtained with the smooth map $\sigma : M \to \mathcal{G}_0$ and by switching s and t and considering $M\ {}_\sigma\!\times_t \mathcal{G}$.

EXAMPLE 2.0.27. (The action groupoid) Let $\mathcal{G}$ be a Lie groupoid, acting smoothly from the left on a manifold M. The corresponding *action groupoid* $\mathcal{G} \ltimes M$ over M in which $(\mathcal{G} \ltimes M)_1 = \mathcal{G}\ {}_s\!\times_\tau M$. For a morphism $(g, x) \in \mathcal{G}\ {}_s\!\times_\tau M$ we have for the source map $s(g, x) = x$, for the target map $t(g, x) = g \cdot x$ and the multiplication map is defined by $(h, y)(g, x) = (hg, x)$. This groupoid is a Lie groupoid .We define the quotient $\mathcal{G} \backslash M$ as the space of orbits of the groupoid $\mathcal{G} \ltimes M$ but this space is in general not a manifold. Similarly the right action of $\mathcal{G}$ on M is defined analogously which gives the action groupoid $M \rtimes \mathcal{G}$ and space of orbits $M/\mathcal{G}$.

The notion of principal G-bundle for Lie groups can with this definition easily be extended to the case of Lie groupoids.

DEFINITION 2.0.28. Let $\mathcal{G}$ be a Lie groupoid. A *left $\mathcal{G}$-bundle* over a manifold M is a manifold P equipped with a smooth map $\pi : P \to M$ and a smooth left $\mathcal{G}$-action on P that is invariant under the $\mathcal{G}$-action, i.e. $\pi(gp) = \pi(g)$ for any $(g, p) \in \mathcal{G}\ {}_s\!\times_\tau P$. The notion of a *right $\mathcal{G}$-bundle* is defined similarly.

The left $\mathcal{G}$-bundle is called *left principal* if τ is a surjective submersion and the map

$$\begin{aligned} \mathcal{G}\ {}_s\!\times_\tau P &\to P\ {}_\pi\!\times_\pi P \\ (g, p) &\mapsto (gp, p) \end{aligned}$$

is a diffeomorphism. In other words, the action is free, that is $xp = p$ iff $x \in \mathcal{G}_0$, and transitive along the fibers of π, and one has $\mathcal{G} \backslash P \simeq X$ through π (See [**51**]). Similarly we have the notion of *right principal* $\mathcal{G}$-bundle.

Note that the case where $\mathcal{G}$ is a Lie group we recover the usual notion of principal $\mathcal{G}$-bundle.

Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groupoids. Then we can define a principal $\mathcal{G}$-bundle over $\mathcal{H}$ as a left principal $\mathcal{G}$-bundle $\pi : P \to \mathcal{H}_0$ over the manifold $\mathcal{H}_0$,

$$\mathcal{G}_0 \xleftarrow{\tau} P \xrightarrow{\pi} \mathcal{H}_0$$

which is equipped with a right $\mathcal{H}$-action on P along the moment map τ, which commutes with the left $\mathcal{G}$-action, i.e. $\tau(ph) = \tau(p)$ and $(gp)h = g(ph)$ for all $(g, p) \in \mathcal{G}\ {}_s\!\times_\tau P$, $(p, h) \in P\ {}_\pi\!\times_t \mathcal{H}$.

These principal bundles can be thought of as objects which represent abstract morphisms between Lie groupoids, which are called *generalized morphisms* or also sometimes called *Hilsum-Skandalis maps*, which can be written as $P : \mathcal{G} \to \mathcal{H}$.

A morphism between two generalized morphisms (See [**4, 63**]) $P, P' : \mathcal{G} \to \mathcal{H}$, that is a map $P \to P'$ between the two principal $\mathcal{G}$-bundles over $\mathcal{H}$, is a smooth biequivariant map $f : P \to P'$ such that

(i) $\pi'(f(p)) = \pi(p)$
(ii) $\tau'(f(p)) = \tau(p)$
(iii) $f(gp) = gf(p)$ for all $(g,p) \in \mathcal{G}\ {}_s\!\times_\tau P$
(iv) $f(ph) = f(p)h$ for all $(p,h) \in P\ {}_\pi\!\times_t \mathcal{H}$

These generalized morphisms can also be formulated in the language of bibundles, which we will turn to now.

Definition 2.0.29. A $\mathcal{G}$-$\mathcal{H}$-*bibundle* P carries a left $\mathcal{G}$- and a right $\mathcal{H}$-action such that they commute, that is

(i) $(gz)h = g(zh)$ for all $(g,z) \in \mathcal{G}\ {}_s\!\times_\tau Z$, $(z,h) \in Z\ {}_\sigma\!\times_t \mathcal{H}$,
(ii) $\tau(gz) = \tau(z)$ for all $(g,z) \in \mathcal{G}\ {}_s\!\times_\tau Z$,
(iii) $\sigma(zh) = \sigma(z)$ for all $(z,h) \in Z\ {}_\sigma\!\times_t \mathcal{H}$.

A $\mathcal{G}$-$\mathcal{H}$-bibundle P is called *left principal* when it is left principal for the $\mathcal{G}$-action with respects to $M = \mathcal{H}_0$ and $\pi = \sigma$. Similarly it is called *right principal* when it is right principal for the $\mathcal{H}$-action with respect to $M = \mathcal{G}_0$ and $\pi = \tau$. With this terminology a generalized morphism $P : \mathcal{G} \to \mathcal{H}$ is equivalently a left principal $\mathcal{G}$-$\mathcal{H}$-bibundle P.

Generalized morphisms have identity morphisms and can be composed. Let $\mathcal{G}$ be a Lie groupoid, then the identity morphism is the natural left principal $\mathcal{G}$-bundle $Id(\mathcal{G}) := \mathcal{G}_1$ over $\mathcal{G}$ and it has a natural right $\mathcal{G}$-action, given by composition, and it is a pricipal $\mathcal{G}$-bundle over $\mathcal{G}$. Here we have $P = \mathcal{G}_1$, $\pi = s$ and $\tau = t$. We denote this bundle by $Id(\mathcal{G}) : \mathcal{G} \to \mathcal{G}$.

Let $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{K}$ be Lie groupoids. Suppose $P : \mathcal{G} \to \mathcal{H}$ and $Q : \mathcal{H} \to \mathcal{K}$ are two generalized morphisms, which we write as $\mathcal{G}_0 \overset{\tau_P}{\leftarrow} P \overset{\pi_P}{\to} \mathcal{H}_0$ and $\mathcal{H}_0 \overset{\tau_Q}{\leftarrow} Q \overset{\pi_Q}{\to} \mathcal{K}_0$. There exits a pullback over $\mathcal{H}_0$, which we denote by $P\ {}_{\pi_P}\!\times_{\tau_Q} Q$ and it carries a diagonal left $\mathcal{H}$ action, given by $h(m,n) \mapsto (mh, h^{-1}n)$ whenever defined. The composition of the generalized morphisms P and Q is defined as the tensor product over $\mathcal{H}$, which is simply the orbit space

$$P \otimes_{\mathcal{H}} Q = P\ {}_{\pi_P}\!\times_{\tau_Q} Q/\mathcal{H}$$

Since Q is a left principal $\mathcal{H}$-bundle over K we have that $P \otimes_{\mathcal{H}} Q$ is a smooth manifold (see [**62**]). Moreover the pullback $P\ {}_{\pi_P}\!\times_{\tau_Q} Q$ carries a left $\mathcal{G}$-action and right $\mathcal{K}$-action, which respects the $\mathcal{H}$-action, this induce a well-defined commuting action on $P \otimes_{\mathcal{H}} Q$ by $\mathcal{G}$ from the left and $\mathcal{K}$ from the right

$$g(p \otimes q)k = gp \otimes qk$$

The left $\mathcal{G}$-action is principal over $\mathcal{K}_0$ since the left $\mathcal{G}$-bundle over $\mathcal{H}_0$ is principal and the left $\mathcal{H}$-bundle over $\mathcal{K}_0$ is left principal. (See[**63**]).

In terms of bibundles we have thus a identity left principal $\mathcal{G}$-$\mathcal{G}$-bibundle $Id(\mathcal{G})$ and we can compose a left principal $\mathcal{G}$-$\mathcal{H}$-bibundle P with a left principal $\mathcal{H}$-$\mathcal{K}$-bibundle Q to form the left principal $\mathcal{G}$-$\mathcal{K}$-bibundle $P \otimes_{\mathcal{H}} Q$. This tensor product $P \otimes_{\mathcal{H}} Q$ is also called the *Hilsum-Skandalis tensor product.*

Alltogether we have the following proposition

Proposition 2.0.30. [**4**] *Lie groupoids, generalized morphisms, and smooth biequivariant maps of generalized morphisms form a weak 2-category denoted by LieGpd.*

Note that a weak 2-category is also called a bicategory. Now let $P : \mathcal{G} \to \mathcal{H}$ be a generalized morphim, it is called a *Morita equivalence* if $\tau : P \to \mathcal{G}_0$ is right principal as an $\mathcal{G}$-bundle over $\mathcal{H}_0$. The name Morita equivalence stems from the fact that in this case there exist a generalized morphism $Q : \mathcal{H} \to \mathcal{G}$ such that $P \otimes_{\mathcal{H}} Q \simeq Id(\mathcal{G})$ and $Q \otimes_{\mathcal{G}} P \simeq Id(\mathcal{H})$ (see [**63**]). If such a P exists, we say that

$\mathcal{G}$ and $\mathcal{H}$ are *Morita equivalent*. In terms of bibundles we say that a $\mathcal{G}$-$\mathcal{H}$-bibundle is a *equivalence bibundle*, if it is both left- and right-principal. Hence two groupoids $\mathcal{G}$, $\mathcal{H}$ are *Morita equivalent* if there exists such an equivalence $\mathcal{G}$-$\mathcal{H}$-bibundle.

Here are some examples

EXAMPLE 2.0.31. Let $\pi : X \to Y$ be a smooth map between smooth manifolds and consider the groupoid $X\ {}_{\pi}\!\times_{\pi} X$ over X which has the same structure maps as the pair groupoid of X, only here $X \times X$ is restricted to $X\ {}_{\pi}\!\times_{\pi} X$, and consider the unit groupoid Y over Y, where all structure maps, except m, are identities. This smooth map $\pi : X \to Y$ induces a $(X\ {}_{\pi}\!\times_{\pi} X)$-$Y$-bibundle X, since X carries a left $(X\ {}_{\pi}\!\times_{\pi} X)$-action and a right Y-action such that they commute. Indeed, we can take the obvious left $(X\ {}_{\pi}\!\times_{\pi} X)$-action on X and for the right Y-action on X we can take the trivial one $X\ {}_{\pi}\!\times_{id} Y \to X$. This $(X\ {}_{\pi}\!\times_{\pi} X)$-$Y$-bibundle X is left- and right- principal, and thus a equivalence bibundle, if and only if $\pi : X \to Y$ is a surjective submersion. Hence for the map smooth map $\pi : X \to *$, which maps everything to the point, we have an equivalence $Pair(X)$-$*$-bibundle and hence $Pair(X)$ is Morita equivalent to the point.

EXAMPLE 2.0.32. Let G be a Lie group and $\pi : P \to M$ be a left principal G-bundle and consider the gauge groupoid $Gauge(P)$ as defined in example 0.1.7. Then we have a G-$Gauge(P)$-bibundle P, since P carries an obvious left G-action and a right $Gauge(P)$-action such that they commute. The right $Gauge(P)$-action on P is defined by $p \cdot [p, q] = q$. This G-$Gauge(P)$-bibundle P is left- and right-principal, and thus a equivalence bibundle. Hence $Gauge(P)$ is Morita equivalent to G (See [**6**]).

EXAMPLE 2.0.33. (Čech groupoid) For X a smooth manifold and $\{U_i\}_{i\in I}$ an open cover of X. Let $\coprod_i U_i$ be the disjoint union with the surjective submersion $\pi : \coprod_i U_i \to X$. By the previous example, we have the groupoid $\coprod_i U_i\ {}_{\pi}\!\times_{\pi} \coprod_i U_i$, the object of this groupoid are the covering patches of $\coprod_i U_i$, and the morphisms is the interstection $U_i \cap U_j$ of these patches. This is precisely the definition of the *Čech groupoid*, which we denote by

$$C(\{U_i\}_{i\in I}) := \coprod_{I\times I} U_{ij} \rightrightarrows \coprod_{I} U_i$$

Since π is an surjective submersion we have that the Čech groupoid $C(\{U_i\}_{i\in I})$ is Morita equivalent to the unit groupoid X

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