The Fundamental Theorem of dg-Algebraic Rational Homotopy Theory (reviewed as [FSS23-Char, Prop. 5.6]) says that the homotopy theory of rationalization of simply connected spaces with fin-dim rational cohomology is all encoded by their Whitehead L_∞-algebra (40) over the rational numbers.

In particular, for X a CW-complex one gets

$$\mathrm{Map}\big(X,\, L^{\mathbb{Q}}\mathcal{A}\big)_{/\mathrm{homotopy}} \;\simeq\; \mathrm{Hom}_{\mathrm{dgAlg}}\Big(\mathrm{CE}\big(\mathfrak{l}^{\mathbb{Q}}\mathcal{A}\big),\, \Omega^\bullet_{\mathrm{PLdR}}(X)\Big)_{/\mathrm{concordance}}, \tag{51}$$

where on the right we have something called the "piecewise linear de Rham complex" of the topological space X.

Notice that the right-hand side looks close to the definition of $\mathfrak{l}\mathcal{A}$-valued de Rham cohomology in (32). In order to actually connect to such smooth differential forms, we need to extend the scalars from the rational to the real numbers:

Rational homotopy theory over the Reals. [FSS23-Char, Def. 5.7, Rem. 5.2] The construction (51) also works over $\mathbb{R}$ (but is then not a "localization") to give the $\mathbb{R}$-rationalization [FSS23-Char, Def. 5.7, Prop. 5.8]:
With this "derived extension of scalars" [FSS23-Char, Lem 5.3] and for X a smooth manifold, the fundamental theorem (51) does relate to smooth differential forms (32) [FSS23-Char, Lem. 6.4] via a *non-abelian de Rham theorem* [FSS23-Char, Thm. 6.5]:

The $\mathbb{R}$-rationalization of $\mathcal{A}$:

A topological space	$L^{\mathbb{R}}\mathcal{A}$
equipped with a map	$L^{\mathbb{Q}}\mathcal{A} \xrightarrow{\eta^{\mathrm{ext}}_{L^{\mathbb{Q}}\mathcal{A}}} L^{\mathbb{R}}\mathcal{A}$
which on homotopy groups is extension of scalars	$\pi_n\big(L^{\mathbb{Q}}\mathcal{A}\big) \xrightarrow[=(-)\otimes_{\mathbb{Q}}\mathbb{R}]{\pi_n\left(\eta^{\mathrm{ext}}_{L^{\mathbb{Q}}\mathcal{A}}\right)} \pi_n\big(L^{\mathbb{R}}\mathcal{A}\big)$
suitably universal as such.	

$$\begin{tikzcd}
\underset{\text{rational cohomology}}{\overset{\text{non-abelian}}{}} H\big(X;\, L^{\mathbb{Q}}\mathcal{A}\big) \arrow[rr, "\text{derived extension of scalars}"] \arrow[d, equal] & & \underset{\text{real cohomology}}{\overset{\text{non-abelian}}{}} H\big(X;\, L^{\mathbb{R}}\mathcal{A}\big) \arrow[d, equal] \\
\pi_0\,\mathrm{Map}\big(X,\, L^{\mathbb{Q}}\mathcal{A}\big) \arrow[rr, "\pi_0\,\mathrm{Map}\left(X,\, \eta^{\mathrm{ext}}_{L^{\mathbb{Q}}\mathcal{A}}\right)"] \arrow[d, leftrightarrow, "\wr", "\substack{\text{fundamental theorem}\\ \text{of dg-algebraic}\\ \text{rational homotopy}}"'] & & \pi_0\,\mathrm{Map}\big(X,\, L^{\mathbb{R}}\mathcal{A}\big) \arrow[d, leftrightarrow, "\wr"] \\
\mathrm{Hom}_{\mathrm{dgAl}}\Big(\mathrm{CE}\big(\mathfrak{l}^{\mathbb{Q}}\mathcal{A}\big),\, \Omega^\bullet_{\mathrm{PLdR}}(X)\Big)_{/\mathrm{cncd}} \arrow[rr, "\text{extension of}", "\text{scalars}"'] & & \mathrm{Hom}_{\mathrm{dgAl}}\Big(\mathrm{CE}\big(\mathfrak{l}\mathcal{A}\big),\, \Omega^\bullet_{\mathrm{dR}}(X)\Big)_{/\mathrm{cncd}} \;\equiv\; \underset{\substack{\text{non-abelian}\\ \text{de Rham cohomology}}}{H_{\mathrm{dR}}\big(X;\, \mathfrak{l}\mathcal{A}\big)} \arrow[uu, "\substack{\text{non-abelian}\\ \text{de Rham theorem}}", "\sim"']
\end{tikzcd} \tag{52}$$

In abelian (ie. Whitehead-generalized) cohomology theories both the rationalization step and the subsequent extension of scalars to $\mathbb{R}$ can be more easily described as forming the smash product of the coefficient spectrum with $H\mathbb{Q}$ or $H\mathbb{R}$, respectively [FSS23-Char, Ex. 5.7]. This is how the Chern-Dold character map over $\mathbb{R}$ is tacitly used in all the literature on Whitehead-generalized differential cohomology theory (e.g. [BN19, Def. 4.2]):

$$\begin{tikzcd}
\mathrm{Spectra} \arrow[r, "(-)\wedge H\mathbb{Q}", "\substack{\text{rationalization}\\ \text{localization}}"'] \arrow[rr, bend left, "\substack{\text{rationalization over } \mathbb{R}\\ (-)\wedge H\mathbb{R}}"] & \mathrm{Spectra} \arrow[r, "(-)\wedge_{H\mathbb{Q}} H\mathbb{R}", "\substack{\text{extension}\\ \text{of scalars}}"'] & \mathrm{Spectra}
\end{tikzcd} \tag{53}$$

The point of the non-abelian de Rham theorem (52) from [FSS23-Char] is to generalize the real-ification of spectra (53) to non-abelian cohomology, such as to Cohomotopy; and the key result that makes this work is the fundamental theorem of dg-algebraic homotopy theory (51). This, ultimately, is the 'reason' why L_∞-valued differential forms connect fluxes to their flux-quantization laws.

The general non-abelian character map is now immediate [FSS23-Char, Def. IV.2]: It is the cohomology operation induced by $\mathbb{R}$-rationalization of classifying spaces, seen under the non-abelian de Rham theorem (52):

$$\begin{tikzcd}
\mathcal{A}(X) \arrow[r, "\text{rationalization}"] \arrow[d, equal] \arrow[rrr, bend left, "\text{character map on } \mathcal{A}\text{-cohomology}"] & H\big(X;\, L^{\mathbb{Q}}A\big) \arrow[r, "\text{extension}", "\text{of scalars}"'] \arrow[d, equal] & H\big(X;\, L^{\mathbb{R}}A\big) \arrow[r, "\text{nonabelian}", "\text{de Rham theorem}"'] \arrow[d, equal] & H_{\mathrm{dR}}\big(X;\, \mathfrak{l}\mathcal{A}\big) \arrow[d, equal] \\
\pi_0\,\mathrm{Map}\big(X,\, \mathcal{A}\big) \arrow[r, "(\eta^{\mathbb{Q}}_{\mathcal{A}})_*"] & \pi_0\,\mathrm{Map}\big(X,\, L^{\mathbb{Q}}\mathcal{A}\big) \arrow[r, "(\eta^{\mathrm{ext}}_{L^{\mathbb{Q}}\mathcal{A}})_*"] & \pi_0\,\mathrm{Map}\big(X,\, L^{\mathbb{R}}\mathcal{A}\big) \arrow[r, "\substack{\text{fundamental theorem}\\ \text{of dg-algebraic RHT}}"'] & \mathrm{Hom}_{\mathrm{dgAlg}}\big(\mathrm{CE}(\mathfrak{l}\mathcal{A}),\, \Omega^\bullet_{\mathrm{dR}}(X)\big)_{/\mathrm{cncrd}}
\end{tikzcd} \tag{54}$$