The Fundamental Theorem of dg-Algebraic Rational Homotopy Theory (reviewed as [FSS23-Char, Prop. 5.6]) says that the homotopy theory of rationalization of simply connected spaces with fin-dim rational cohomology is all encoded by their Whitehead L_{∞} -algebra (40) over the rational numbers.

In particular, for X a CW-complex one gets

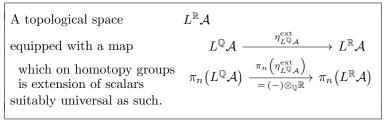
$$\operatorname{Map}(X, L^{\mathbb{Q}}\mathcal{A})_{/\operatorname{homotopy}} \simeq \operatorname{Hom}_{\operatorname{dgAlg}}(\operatorname{CE}(\mathbb{I}^{\mathbb{Q}}\mathcal{A}), \Omega^{\bullet}_{\operatorname{PLdR}}(X))_{/\operatorname{concordance}},$$
(51)

where on the right we have something called the "piecewise linear de Rham complex" of the topological space X.

Notice that the right-hand side looks close to the definition of $\mathcal{I}\mathcal{A}$ -valued de Rham cohomology in (32). In order to actually connect to such smooth differential forms, we need to extend the scalars from the rational to the real numbers:

Rational homotopy theory over the Reals. [FSS23-Char, Def. 5.7, Rem. 5.2] The construction (51) also works over \mathbb{R} (but is then not a "localization") to give the \mathbb{R} -rationalization [FSS23-Char, Def. 5.7, Prop. 5.8]:

With this "derived extension of scalars" [FSS23-Char, Lem 5.3] and for X a smooth manifold, the fundamental theorem (51) does relate to smooth differential forms (32) [FSS23-Char, Lem. 6.4] via a *non-abelian de Rham theorem* [FSS23-Char, Thm. 6.5]: The \mathbb{R} -rationalization of \mathcal{A} :



$$\begin{array}{c} \begin{array}{c} \begin{array}{c} \operatorname{non-abelian} \\ \operatorname{real cohomology} \\ H(X; L^{\mathbb{Q}}\mathcal{A}) & \xrightarrow{\operatorname{derived extension of scalars}} \\ H(X; L^{\mathbb{Q}}\mathcal{A}) & \xrightarrow{\operatorname{derived extension of scalars}} \\ \end{array} \\ \begin{array}{c} H(X; L^{\mathbb{Q}}\mathcal{A}) \\ & \downarrow \\ \end{array} \\ \pi_{0} \operatorname{Map}(X, L^{\mathbb{Q}}\mathcal{A}) & \xrightarrow{\pi_{0} \operatorname{Map}(X, \eta_{L^{\mathbb{Q}}\mathcal{A}})} \\ \xrightarrow{\pi_{0} \operatorname{Map}(X, \eta_{L^{\mathbb{Q}}\mathcal{A}})} \\ \xrightarrow{\pi_{0} \operatorname{Map}(X, \mu^{ext})} \\ \xrightarrow{\pi_{0} \operatorname{Map}$$

In abelian (ie. Whitehead-generalized) cohomology theories both the rationalization step and the subsequent extension of scalars to \mathbb{R} can be more easily described as forming the smash product of the coefficient spectrum with $H\mathbb{Q}$ or $H\mathbb{R}$, respectively [FSS23-Char, Ex. 5.7]. This is how the Chern-Dold character map over \mathbb{R} is tacitly used in all the literature on Whitehead-generalized differential cohomology theory (e.g. [BN19, Def. 4.2]):

The point of the non-abelian de Rham theorem (52) from [FSS23-Char] is to generalize the real-iffication of spectra (53) to non-abelian cohomology, such as to Cohomotopy; and the key result that makes this work is the fundamental theorem of dg-algebraic homotopy theory (51). This, ultimately, is the 'reason' why L_{∞} -valued differential forms connect fluxes to their flux-quantization laws.

The general non-abelian character map is now immediate [FSS23-Char, Def. IV.2]: It is the cohomology operation induced by \mathbb{R} -rationalization of classifying spaces, seen under the non-abelian de Rham theorem (52):