Topological Quantum Gates from M-theory

January 17, 2023

Abstract

The promising idea that strongly coupled quantum systems such as (confined QCD or) topologically ordered quantum materials are usefully modeled as worldvolume dynamics on intersecting branes in string theory has been suffering from the latter's lack of non-perturbative formulation (M-theory), necessary in the realistic regime of small numbers of individual branes (i.e. beyond the usual holographic large-N approximation). In this talk I will briefly review our "Hypothesis H" that brane charge in M-theory is quantized in a twisted equivariant non-abelian generalized cohomology theory called *Cohomotopy*, where M-brane quantum states are identified inside the twisted equivariant cohomology of the Cohomotopy moduli stack. Then I explain our recent derivation, from this assumption, of anyonic topological order in ground states of M5-brane intersections; and I close with an outlook on how this describes topological quantum logic gates via braiding of defect branes.

An exposition of key background and results of:

Anyonic defect branes in TED-K-theory
Rev. Math. Phys (2023, in print)
[`arXiv:2203.11838`]

with H. Sati

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The open question of fundamental quantum physics. The main embarrassment of contemporary fundamental quantum physics is not so much the lack of coherent quantum gravity theory as such, but the general lack of non-perturbative theory in general, due to which mundane phenomena such as room-temperature matter (hadron bound states in confined QCD) or heavily sought-after quantum materials – such as topologically ordered solid states needed for topological quantum computation – remain theoretically ill-understood.

The role of string theory. While string theory is, after some twists and turns, the result of understanding such strongly coupled quantum systems (notably QCD) as localized on *branes* in higher-dimensional bulk spacetimes, its currently available formulation is just as perturbative as available quantum field theory, which has restricted the analysis of physics on brane configurations to their classical "holographic" limit (the large-N limit).

But the crucial difference of perturbative string theory over perturbative quantum field theory is that string theory provides a tight web of *hints* towards its non-perturbative completion, enough so that non-perturbative string theory famously has a working title already (since 1995): "M-theory".

M-Theory and Algebraic Topology. After initial excitement, progress on actually formulating M-theory had stagnated and efforts had been largely abandoned, arguably due to a lack of appropriate mathematical tools. It was the vision of Hisham Sati to bring in the full force of modern algebraic/geometric topology, looking for homotopy-theoretic patterns in the available information on M-brane physics, deducing clues as to their fundamental mathematical nature.

This analysis eventually culminated in a formulation of a hypothesis – *Hypothesis H* – of what M-theory really is about [HpH0][HpH1][HpH2]:

1 Hypothesis-H on M-theory

What is a *quantum brane configuration*, really? We survey discussion and results laid out in the monograph [Chr].

To the classical asymptotic observer (like Faraday): A brane configuration is a tuple of *flux densities* ("sourced" by the brane configuration), given by differential r-forms[1] on a spacetime manifold X:

$$\Big(\overset{\text{flux densities}}{F^{(a)}_{r_a} \in \Omega^{r_a}_{\mathrm{dR}}(X)}\Big)_{1 \leq a \leq a_{\max}} \tag{1}$$

and satisfying filtered polynomial differential relations:

$$\underset{\text{higher “Bianchi identities”}}{\overset{\text{de Rham diff.}}{\mathrm{d}\, F^{(a)}_{r_a}} \;=\; \overset{\text{polynomial}}{P_{r_a}}\Big(\big\{F^{(b)}_{r_b}\big\}_{b \leq a}\Big)} \tag{2}$$

which constitute that sector of the *equations of motion* of the corresponding fields that is independent of dynamical gravity.

Modernizing this statement, we observe that the consistency condition $\mathrm{d}^2 = 0$ implies conditions on the P_{r_a} which equivalently characterize them as the structure constants of a nilpotent L_∞-algebra $\mathfrak{a}$, such that the above flux data is equivalently that of a flat $\mathfrak{a}$-valued differential form:

$$\cdots \;\Leftrightarrow\; \Big(F^{(a)}_{r_a}\Big)_{1 \leq a \leq a_{\max}} \in \overset{\substack{\text{flat } L_\infty\text{-algebra} \\ \text{valued diff. forms}}}{\Omega_{\mathrm{dR}}\big(X; \mathfrak{a}\big)_{\mathrm{flat}}}\,. \tag{3}$$

To the semi-classical observer (like Dirac): A brane configuration is such flux densities but equipped with *flux quantization*, reflecting a minimum *unit brane charge*.

Modernizing this statement, we observe that for every nilpotent L_∞-algebra $\mathfrak{a}$ we may *choose* a nilpotent classifying space A for a *non-abelian cohomology theory* $A(-)$

$$\overset{\substack{\text{non-abelian} \\ \text{cohomology}}}{A(X)} \;:=\; \pi_0 \, \mathrm{Map}(X, A) \;=\; \left\{ X \underset{\substack{\mathscr{F}' \\ \text{cocycle}}}{\overset{\substack{\text{cocycle (map)} \\ \mathscr{F}}}{\underset{\Downarrow\ \text{coboundary (homotopy)}}{\rightrightarrows}}} A \right\}_{/\sim} \tag{4}$$

[1] We assume that $r \geq 1$.

such that $\mathfrak{a}$ is the *Whitehead bracket L_∞-algebra* of A:

$$\overset{\substack{\text{Whitehead bracket} \\ L_\infty\text{-algebra}}}{\mathfrak{a}} \;=\; \mathfrak{l}A\,,$$

hence so that the flux densities give classes in the *$\mathfrak{l}A$-valued non-abelian de Rham cohomology* of X:

$$\Big[\big(F^{(a)}_{r_A}\big)_{1\leq a\leq \dim[\pi_\bullet(A),\mathbb{R}]}\Big] \;\in\; \overset{\substack{\mathfrak{l}A\text{-valued} \\ \text{non-abelian} \\ \text{de Rham cohomology}}}{H_{\mathrm{dR}}\big(X;\,\mathfrak{l}A\big)} \;:=\; \left\{ \Omega^\bullet_{\mathrm{dR}}(X) \;\underset{\underset{\text{cocycle}}{(F^{(a)}_{r_a})'}}{\overset{\overset{\text{cocycle (dga-hom)}}{(F^{(a)}_{r_a})}}{\xleftarrow[\quad]{\;\Downarrow\;\text{coboundary (concordance)}\;}}}\; \mathrm{CE}(\mathfrak{l}A) \right\}_{\big/\sim}.$$

Now the *non-abelian character map* ch_A approximates A-cohomology classes by $\mathfrak{l}A$-valued de Rham classes; and *flux quantization* in A-cohomology means to lift through this character map. Such **flux quantization gives the non-perturbative information** in brane confiurations.
It seems unlikely to formulate a non-perturbative theory of branes without first clarifying their flux quantization law.

$$\begin{array}{ccc} \overset{\substack{\text{non-abelian} \\ \text{cohomology}}}{A(X)} & \xrightarrow[\mathrm{ch}_A]{\text{non-abelian character}} & \overset{\substack{\text{non-abelian} \\ \text{de Rham cohomology}}}{H_{\mathrm{dR}}\big(X;\,\mathfrak{l}A\big)} \\ \underset{\substack{\text{class of} \\ A\text{-quantized flux}}}{\big[\mathscr{F}\big]} & \mapsto & \underset{\substack{\text{class of} \\ \text{underlying flux densities}}}{\Big[\big(F^{(a)}_{r_a}\big)_{1\leq a\leq \dim[\pi_\bullet(A),\mathbb{R}]}\Big]}\,. \end{array} \tag{5}$$

Given such choice of flux quantization law A, the corresponding moduli stack $\widehat{A}$ of *potentials* or *gauge fields* is the homotopy pullback of the sheaf of flux densities along the character map.

This classifies (non-abelian generalized) *differential cohomology*, in a way that takes care of all "Dirac strings" of gauge fields and of higher gauge fields ("conts. higher form symmetries").

$$\begin{array}{ccc} \overset{\substack{\text{moduli stack of higher gauge fields} \\ \text{with } A\text{-quantized flux densities}}}{\widehat{A}} & \xrightarrow{\text{flux densities}} & \Omega_{\mathrm{dR}}\big(-;\,\mathfrak{l}A\big)_{\mathrm{flat}} \\ {\scriptstyle \text{charges}}\big\downarrow & \overset{\text{potentials}}{\Swarrow}\;(\mathrm{pb}) & \big\downarrow \\ \underset{\text{flux quant. law}}{A} & \xrightarrow[\substack{\text{character map =} \\ \text{rationalization over the reals}}]{\mathrm{ch}_A} & \textstyle\int\Big(\Omega_{\mathrm{dR}}\big(-;\,\mathfrak{l}A\big)_{\mathrm{flat}}\Big) \simeq A^{\mathbb{R}}\,. \end{array} \tag{6}$$

Therefore the mapping stack into $\widehat{A}$ is the *integrated BRST complex* of $\widehat{A}$-valued fields with their (higher order) gauge transformations:

$$\overset{\substack{\text{integrated} \\ \text{BRST complex}}}{\mathrm{Map}\big(X,\widehat{A}\big)} \;=\; \left\{ X \;\underset{\underset{\text{gauge field}}{\widehat{\mathscr{F}}'}}{\overset{\overset{\text{gauge field (map)}}{\widehat{\mathscr{F}}}}{\xrightarrow[\quad]{\;\Downarrow\;\text{gauge transfo. (homotopy)}\;}}}\; \widehat{A} \right\}_{\big/\sim}. \tag{7}$$

Notice that this integrated BRST complex is *on-shell* for that sector (2) of the field equations which are independent of dynamical gravity. Therefore we say [Qnt1][Qnt2] that:

To a quantum observer, the *quantum states* of $\widehat{A}$-fields sourced by brane configurations are the *motive* of the integrated BRST complex (7), namely that aspect seen by its abelian twisted differential E-cohomology, for a "prequantum line bundle" τ of spectra E:

$$\underset{E\text{-quantum states of } A\text{-charged branes}}{\overset{\substack{\text{twisted abelian } E\text{-cohomology} \\ \text{of non-abelian } \widehat{A}\text{-moduli stack}}}{\widehat{E}^\tau\Big(\mathrm{Map}\big(X,\widehat{A}\big)\Big)}}\;, \qquad \underset{E\text{-quantum observables of } A\text{-charged branes}}{\overset{\substack{\text{twisted abelian } E\text{-homology} \\ \text{of non-abelian } \widehat{A}\text{-moduli stack}}}{\widehat{E}_\tau\Big(\mathrm{Map}\big(X,\widehat{A}\big)\Big)}} \tag{8}$$

This is an immensely rich object. In the following we will inspect but some of its shadows, e.g. by disregarding the differential enhancement of E, and considering only the first few choices of E (namely ordinary cohomology and K-cohomology).

Intersecting brane configurations. More generally, there may be a couple of different *brane species* which interact according to *brane intersection laws*, allowing a brane species with flux densities $\big(H^{(i)}_{r_i}\big)_{1\le i\le i_{\max}}$ as above to intersect or "end on" another brane species with fluxes $\big(F^{(a)}_{r_a}\big)_{1\le a\le a_{\max}}$.

In this case the Bianchi identities for the latter fluxes include polynomial "twists" by the former

$$\underbrace{\overset{\text{de Rham diff.}}{\mathrm{d}}F^{(a)}_{r_a} \;=\; \overset{\text{polynomial}}{P_{r_a}}\Big(\big\{F^{(b)}_{r_b}\big\}_{b\le a},\ \overset{\text{twisting fluxes}}{\big\{H^{(i)}_{r_i}\big\}_{1\le i\le i_{\max}}}\Big)}_{\text{twisted higher "Bianchi identities"}},$$

and the previous classifying spaces (4) generalize to *classifying fibrations*

$$\begin{array}{ccc} \multicolumn{3}{c}{\text{classifying spaces for...}} \\ \underset{\text{intersect}ed\text{ branes}}{A} & \longrightarrow & \underset{\text{brane intersections}}{A/\!\!/\mathscr{G}} \\ & & \Big\downarrow{\scriptstyle \text{classifying fibrations}} \\ & & \underset{\text{intersect}ing\text{ branes}}{B\mathscr{G}} \end{array}$$

which classify *twisted* non-abelian cohomology theories:

$$\overset{\substack{\text{twisted}\\\text{non-abelian}\\\text{cohomology}}}{A^{\tau}(X)} \;:=\; \overset{\substack{\text{vertical homotopy classes}\\\text{of slice maps}}}{\pi_0\,\mathrm{Map}\Big((X,\tau),\,A/\!\!/\mathscr{G}\Big)_{B\mathscr{G}}} \;=\; \left\{ \begin{array}{ccc} & \overset{\text{twist}ed\text{ cocycle}}{\mathscr{F}_\tau,\ \mathscr{F}'_\tau} & \\ X & \dashrightarrow & A/\!\!/\mathscr{G} \\ & {\scriptstyle \tau}\searrow \;\; \swarrow & \\ & \underset{\text{twist}ing\text{ cocycle}}{B\mathscr{G}} & \end{array} \right\}_{/\sim}$$

on which the *twisted non-abelian character map*

$$\begin{array}{ccc} \overset{\substack{\text{twisted}\\\text{non-abelian}\\\text{cohomology}}}{A^{\tau}(X)} & \xrightarrow[\mathrm{ch}_A]{\text{twisted non-abelian character}} & \overset{\substack{\text{twisted}\\\text{non-abelian}\\\text{de Rham cohomology}}}{H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\big(X;\,\mathfrak{l}A\big)} \\ \underset{\substack{\mathscr{H}\text{-twisted class of}\\A\text{-quantized flux}}}{\big[\mathscr{F}_{\mathscr{H}}\big]} & \longmapsto & \underset{\substack{\text{class of}\\\text{underlying flux densities}}}{\Big[\Big(F^{(a)}_{r_a}\Big)_{1\le a\le \dim[\pi_\bullet(A),\mathbb{R}]}\Big]} \end{array} \tag{9}$$

computes the classes of underlying flux densities satisfying twisted Bianchi identities:

$$\Big[\Big(F^{(a)}_{r_A}\Big)_{1\le a\le \dim[\pi_\bullet(A),\mathbb{R}]}\Big] \;\in\; \overset{\substack{\tau_{\mathrm{dR}}\text{-twisted}\\\mathfrak{l}A\text{-valued}\\\text{non-abelian}\\\text{de Rham cohomology}}}{H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\big(X;\,\mathfrak{l}A\big)} \;:=\; \left\{ \begin{array}{ccc} & \overset{\text{twist}ed\text{ cocycle (dga-hom)}}{\Big(F^{(a)}_{ra}\Big)} & \\ \Omega^\bullet_{\mathrm{dR}}(X) & \overset{\substack{\text{coboundary}\\\text{(concordance)}}}{\Downarrow} & \mathrm{CE}\big(\mathfrak{l}(A/\!\!/\mathscr{G})\big) \\ & \Big(F^{(a)}_{ra}\Big)' & \\ & {\scriptstyle \tau_{\mathrm{dR}}}\nwarrow \;\; \nearrow & \\ & \underset{\text{twist}ing\text{ cocycle}}{\mathrm{CE}(\mathfrak{l}B\mathscr{G})} & \end{array} \right\}_{/\sim}.$$

The key **examples** for our purpose are the Bianchi identities expected in type II string theory and in M-theory:

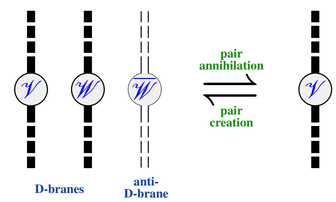
Hypothesis K — D/NS-brane flux quantization in K-CoHomology. For example, consider a classifying space for complex K-theory such as that of Fredholm operators $\mathrm{KU}_0 \simeq \mathrm{Fred}^{\mathbb{C}}$, and write $\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1)$ for its homotopy quotient by $B\mathrm{U}(1) \simeq \mathrm{PU}(H)$. Then the corresponding character map exhibits the flux densities and their Bianchi identities of D/NS-branes in type IIA string theory:

$$\begin{array}{ccc} \mathrm{KU}_0 \longrightarrow \mathrm{KU}_0 /\!\!/ B\mathrm{U}(1) & \Omega_{\mathrm{dR}}\Big(X, \mathfrak{l}\big(\mathrm{KU}_0/\!\!/B\mathrm{U}(1)\big)\Big) & = \left\{ \begin{array}{l|l} F_{2k} \in \Omega^{2k}_{\mathrm{dR}}(X), & \mathrm{d}F_{2k+2} = F_{2k} \wedge H_3, \\ H_3 \in \Omega^3_{\mathrm{dR}}(X) & \mathrm{d}H_3 = 0 \end{array} \right\} \\ \downarrow & & \\ B^2\mathrm{U}(1) & \Omega_{\mathrm{dR}}\big(X, \mathfrak{l}B\mathrm{U}(1)\big) & = \Big\{ H_3 \in \Omega^3_{\mathrm{dR}}(X) \,\Big|\, \mathrm{d}H_3 = 0 \Big\}. \end{array} \tag{10}$$

This is the central motivation for the traditional "*Hypothesis K*" that flux of D/NS-brane configurations is quantized in twisted K-theory:

$$\underbrace{\mathrm{KU}^{\mathscr{H}_3}(X)}_{\substack{\text{twisted}\\ \text{complex topological}\\ \text{K-cohomology}}} = \pi_0\,\mathrm{Map}\Big((X, \mathscr{H}_3), \mathrm{KU}_0/\!\!/B\mathrm{U}(1)\Big)_{B^2\mathrm{U}(1)}. \tag{11}$$

The other original motivation for *Hypothesis K* is the observation that the K-cohomology group $\mathrm{KU}(X)$ for a compact space X may equivalently be described as the equivalence classes of pairs of vector bundles and "anti-bundles" subject to a relation which annihilates equal but opposite vector bundles just as expected for brane/anti-brane annihiliation.



Various further consistency checks for *Hypothesis K* have been claimed, but unresolved issues remain, pointing to a need for a more refined description.

Now the evident underlying idea of *Hypothesis H* is that – with the non-abelian enhancement (9) of the twisted character map in hand – a directly analogous argument applies to M-branes (this is due to [HpH0, §2.5][HpH1]):

Hypothesis H — M-brane flux quantization in CoHomotopy. The simplest and yet richest candidates for classifying spaces are the *n-spheres*. In a few exceptional dimensions these arise as coset spaces, such as $S^4 \simeq \mathrm{Sp}(2)/\big(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\big)$, which canonically form fibrations such as this one:

$$\underbrace{\begin{array}{c} S^4 \longrightarrow S^4/\!\!/\mathrm{Sp}(2) \\ \downarrow \\ B\mathrm{Sp}(2) \end{array}}_{\substack{\text{universal 4-spherical fibration over}\\ \text{classifying space of } \mathrm{Sp}(2)\simeq\mathrm{Spin}(5)}} \quad \underbrace{\Omega_{\mathrm{dR}}\Big(X, \mathfrak{l}\big(S^4/\!\!/\mathrm{Sp}(2)\big)\Big)}_{\substack{\text{and its induced}\\ \text{Bianchi identities}}} = \left\{ \begin{array}{l|l} G_4 \in \Omega^4_{\mathrm{dR}}(X) & \mathrm{d}G_7 = -\tfrac{1}{2}\widetilde{G}_4 \wedge (\widetilde{G}_4 - \tfrac{1}{4}p_1) - 12\cdot I_8 \\ G_7 \in \Omega^7_{\mathrm{dR}}(X) & \mathrm{d}G_4 = 0 \\ p_1(\nabla) \in \Omega^4_{\mathrm{dR}}(X) & \mathrm{d}p_1(\nabla) = 0 \\ I_8(\nabla) \in \Omega^4_{\mathrm{dR}}(X)) & \mathrm{d}I_8(\nabla) = 0 \end{array} \right\} \tag{12}$$

Remarkably. these *non-linear* Bianchi identities $\boxed{\mathrm{d}G_4 = 0,\ \mathrm{d}G_7 = -\tfrac{1}{2}G_4 \wedge G_4}$ are those of M2/M5-brane fluxes in 11-dimensional supergravity, suggesting ([HpH0, §2.5][HpH1][HpH2]):

Hypothesis H: | *M-brane flux is quantized in tangentially twisted CoHomotopy.* |,

namely in the tangentially twisted (cf. [Orb3, Def. 5.13]) non-abelian cohomology theory represented by the spherical fibration (12):

$$\underbrace{\pi^\tau(X)}_{\substack{\text{tangentially twisted}\\ \text{4-CoHomotopy}}} := \left\{ \begin{array}{ccc} X & \overset{\text{cocycle}}{\dashrightarrow} & S^4/\!\!/\mathrm{Sp}(2) \\ {\scriptstyle \vdash Fr}\downarrow & \searrow^{\tau}_{\mathrm{Spin}(8)\text{-structure}} & \swarrow \\ B\mathrm{O}(10,1) & \longleftarrow & B\mathrm{Spin}(8) \end{array} \right\}_{/\sim} \xrightarrow{\mathrm{ch}^\tau_{S^4}} \underbrace{H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\big(X; \mathfrak{l}S^4\big) \hookrightarrow H_{\mathrm{dR}}\big(X; \mathfrak{l}(S^4/\!\!/\mathrm{Sp}(2))\big)}_{\text{underlying flux densities according to (12)}} \tag{13}$$

We checked that this quantization law reproduces effects expected in M-theory (e.g. [HpH1][M5a][M5b][M5c][Orb1][GS1][GS2]).

Notably, the flux quantization of G_4 in the resulting tangentially *twisted 4-CoHomotopy* theory enforces the notorious *shifted integral flux quantization* of the 4-flux density ([HpH1, §3.4]):

$$\underbrace{\big[\widetilde{G}_4\big] := \big[G_4 + \tfrac{1}{4}p_1\big]}_{\text{shifted 4-flux density}} \in \underbrace{H^4(X;\mathbb{Z})}_{\text{integral cohomology}} \to H^4(X;\mathbb{R}). \tag{14}$$

Last not least, if we forget the non-linear effects of non-abelian Cohomotopy by passing to its shadow in stable Cohomotopy (25), then the same plausibility check regarding brane/anti-brane annihiliation holds, in that stable CoHomotopy is also a form of K-theory, namely the algebraic K-theory of the absolute base "field" $\mathbb{F}_1$ (the BPQ-theorem, cf. [Orb2, p. 4]).

Therefore *Hypothesis H* is at least as compelling as *Hypothesis K*. Both may be wrong, but neither can be wrong by much.

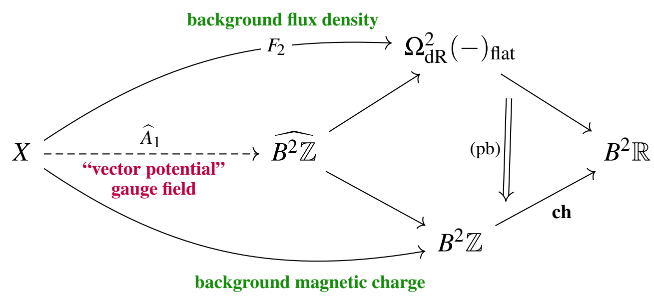
2 Resulting M5-brane model

We survey how *Hypothesis H* implies a globally consistent sigma-model for the topological sector of the M5-brane — at face value for a single M5, but *de facto* for two coincident but orientifolded M5s with a *nonabelian* $\mathrm{Sp}(1) \simeq \mathrm{SU}(2)$ gauge field on their worldvolume — this result is from [HpH1, §3.7][M5a][M5b].[2]

To warm up, we start by recalling, in modernized form, the corresponding issue for the 0-brane:

Topological 0-brane sigma-model. Dirac's original argument — initiating what led us to the above considerations of flux quantization — was the requirement on an electromagnetic background flux density $F_2 \in \Omega^2_{\mathrm{dR}}(X)$ (the *Faraday tensor*) to induce a consistent coupling to a quantum electron (a "0-brane") via an action functional that is locally the integral of a *vector potential* 1-form A_1 over the electron's *worldline* $\Sigma^{0+1} \xrightarrow{\phi} X$.

In modernized paraphrase, Dirac's conclusion was that F_2 must come from a cocycle in the differential refinement (6) of ordinary integral cohomology, via a dashed factorization in the following diagram:



(15)

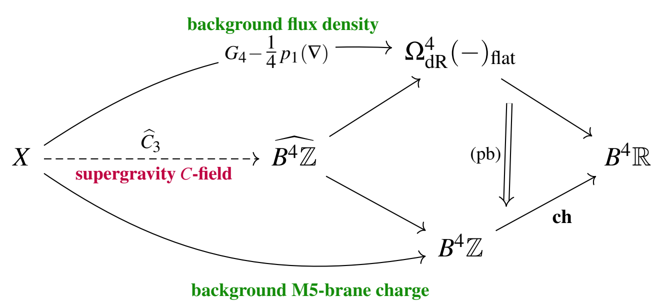
because then the required global action functional in question is given by the *holonomy* (for ordinary U(1)-connections):

$$
\begin{array}{ccc}
\mathrm{Map}(\Sigma^{0+1}, X) & \xrightarrow{\text{1-form holonomy}} & \mathrm{U}(1) \\
\left(\Sigma^{0+1} \xrightarrow{\phi} X\right) & \mapsto & \exp\left(2\pi \mathrm{i} \int\limits_{\Sigma^{0+1}} \phi^* \widehat{A}_1\right),
\end{array}
\tag{16}
$$

where the integral expression on the right is shorthand for the piecewise holonomy integrals of any Čech cocycle representative — the dependency on whose choices drops out under the exponentiation.

The expression (16) is the "topological" part of the action functional in the 0-brane sigma-model. We next consider the analogous topological sectors of the sigma-models for the M2-brane and the M5-brane.

Topological M2-brane sigma-model. The shifted integrality condition (14) implies in particular that we have choices of globally well-defined 3-form gauge fields C_3 ("2-gerbe connections") – the supergravity C-field –



(17)

with well-defined higher holonomy over closed wordvolumes $\Sigma^{2+1} \xrightarrow{X}$ of 2-branes

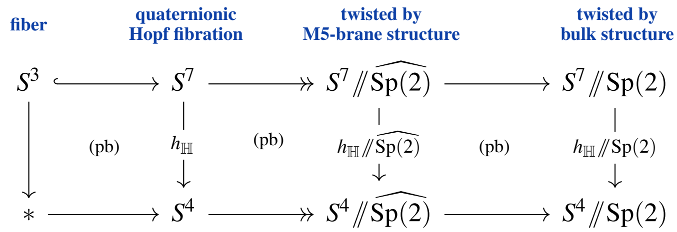
$$
\begin{array}{ccc}
\mathrm{Map}(\Sigma^{2+1}, X) & \xrightarrow{\text{3-form holonomy}} & \mathrm{U}(1) \\
\left(\Sigma^{2+1} \xrightarrow{\phi} X\right) & \mapsto & \exp\left(2\pi \mathrm{i} \int\limits_{\Sigma^{2+1}} \phi^* \widehat{C}_3\right),
\end{array}
\tag{18}
$$

This is the (globally corrected) "topological" factor in the action functional for the M2-brane sigma-model on X in a background with integral M5-brane charge.

[2] Hypothesis H also implies information about the *dynamical* (i.e. geometric, non-topological) sector of the $\mathrm{Sp}(1)$-gauged M5-brane sigma-model: this is discussed in [M5d][M5e]. The combination of the results [M5a][M5b][M5c]+[M5d][M5e] should serve to produce the full topological+geometric M5-brane model for two coincident super 5-branes with $\mathrm{Sp}(1)$ worldvolume gauge field — but we have not worked that out yet.

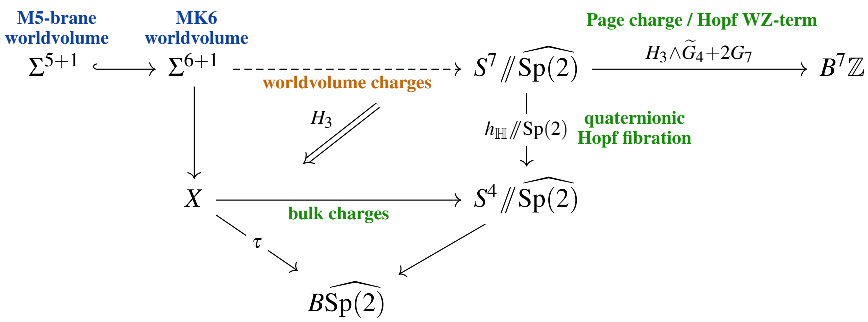
Topological M5-brane sigma-model. To obtain the analogous topological term of the sigma-model for M5-branes, we need an integral 7-flux density which measures the background charge of M2-branes, also known as the *Page charge*.

Postnikov theory shows that the 7-flux density of M2-branes under *Hypothesis H* is extracted from a cocycle in twisted 4-Cohomotopy by lifting it — over the M5-brane's worldvolume $\Sigma^{5,1}$, or rather over a codimension=1 extension $\Sigma^{6,1}$ — through the $\mathrm{Sp}(2)$-equivariantized quaternionic Hopf fibration — in the following ignore the hat on "$\widehat{\mathrm{Sp}(2)}$" until (22) –:



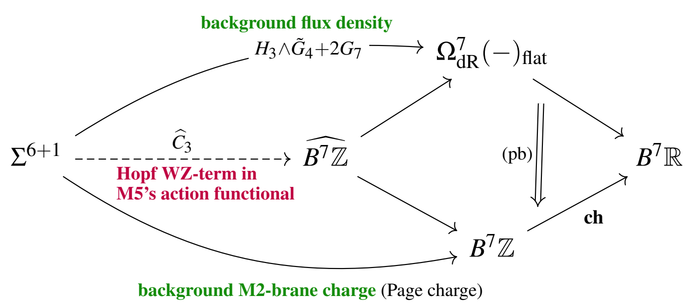
(19)

to a cocycle in tangentially twisted 7-Cohomotopy:



(20)

Such lifts turn out to correspond to choices of worldvolume H_3-fields — as expected to appear on MK6/M5-brane worldvolumes — automatically charge-quantized in twisted *3-CoHomotopy* (locally taking values in the S^3-fiber of (19)) and as such they appear in the "Hopf WZ term" of the M5-brane action functional:



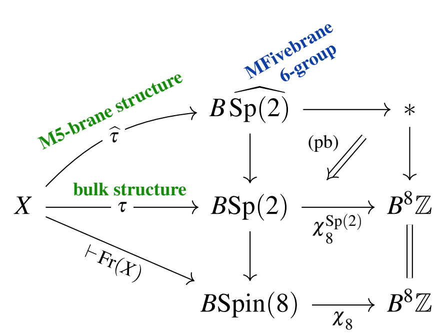
(21)

Level quantization of the 5-brane's Hopf WZ term. The *Page charge* or *Hopf WZ-term* appearing in (20) is not integral (violates "level quantization") without suitable assumptions – but if it is not then the would-be Hopf WZ-term (21) in the 5-brane sigma model is "anomalous", an issue that — prior to the discussion in [M5a], see the historical review given there — has received little attention.

But given $\mathrm{Sp}(2)$-structure as imposed by tangentially twisted 4-Cohomotopy (13), the obstruction to cancelling this anomaly is [M5a, Thm. 4.8] precisely the Euler 8-class χ_8, which for $\mathrm{Sp}(2) \hookrightarrow \mathrm{Spin}(8)$-structure, is the following combination of Pontrjagin classes [HpH1, (98)]:

$$\chi_8^{\mathrm{Sp}(2)} \;=\; \tfrac{1}{2}p_2 - \tfrac{1}{8}(p_1)^2 \;\in\; H^8\big(B\mathrm{Sp}(2);\mathbb{Z}\big)\,.$$

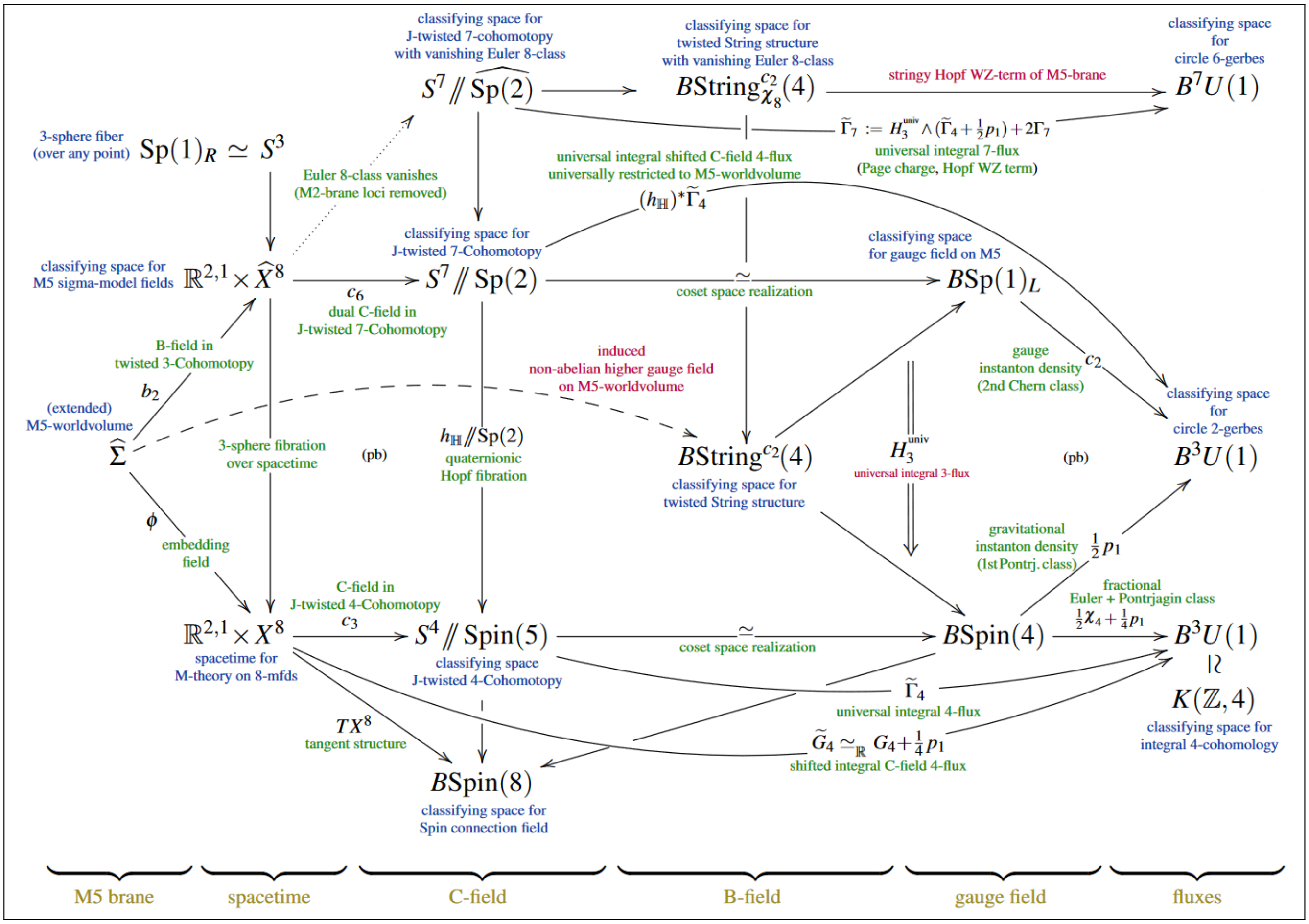
Lifting this obstruction – and hence cancelling the Hopf-WZ 5-brane anomaly – means to lift the bulk $\mathrm{Sp}(2)$-structure to the homotopy fiber $B\widehat{\mathrm{Sp}(2)}$ of χ_8 — an M-theoretic form of *Fivebrane structure* along the lines originally considered in [arXiv:0805.0564].



(22)

Non-abelian twisted string-structure on the M5. A careful homotopy-theoretic analysis of all the implications of the innocent-looking lift defining the H_3-field in (20) reveals a wealth of further structure that has long been expected on worldvolumes of coincident M5-branes; in particular it reveals a non-abelian gauge field for which H_3 serves a "non-abelian gerbe"-field that exhibits a worldvolume Green-Schwarz mechanism.

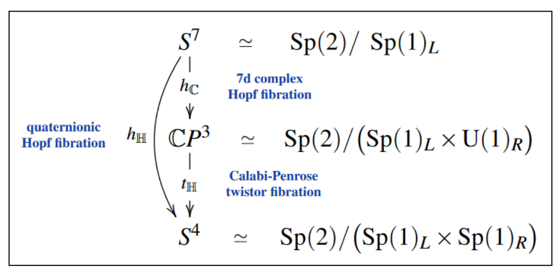
All of this follows by rigorous homotopy-theoretic unwinding of the following homotopy-commutative diagram[3], which is carefully and incrementally explained in [M5b] (but need not further concern us here):

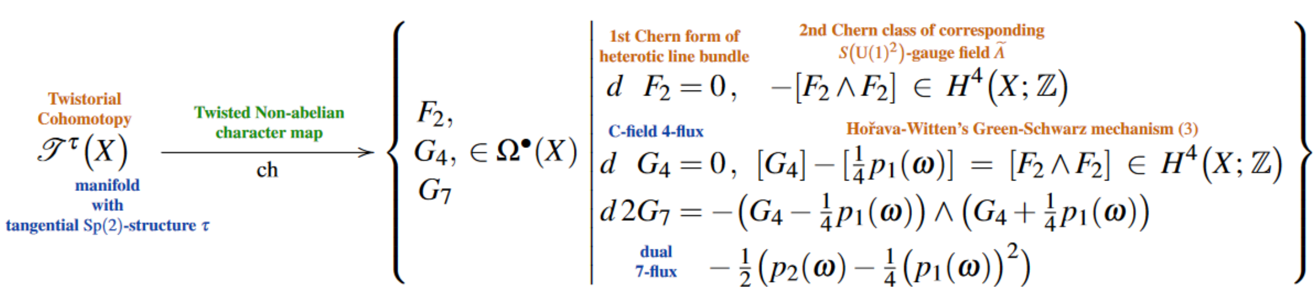


Hypothesis H for heterotic M-theory. In fact, such Sp(1)-gauged M5-branes are those expected in "heterotic M-theory" (Hořava-Witten theory).

Hypothesis H turns out to generalize to heterotic M-theory, where CoHomotopy is enhanced to "twistorial CoHomotopy", now represented by the "twistor space" $\mathbb{C}P^3$ covering the 4-sphere through the Calabi-Penrose fibration.

This is discussed in [GS1][GS2].





[3] This diagram became the logo of the meeting *M-Theory and Mathematics 2023*, that took place at NYU Abu Dhabi.

3 Resulting M5-probe branes

The classical sigma-model discussion in §2 confirms that the charges seen by twisted 4-CoHomotopy really do correspond to M5-branes of the kind expected in string/M-theory. We now survey how to pass from such classical sigma-model branes to *quantum brane probes* — as discussed in [Qnt1][Qnt2].

Branes: Solitonic, probes and webs. Notice the following precise formalizations of some general notions of "kinds of branes" that are ubiquitous in the informal string theory folklore. This follows [HpH2, §2.1].

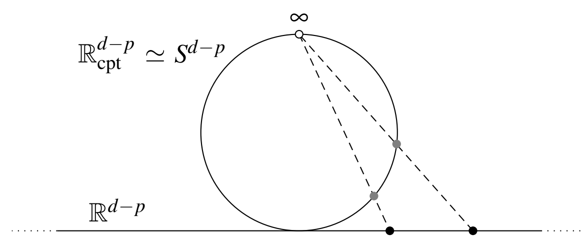
A spacetime manifold may carry background brane charge in several ways:[4]

- **Solitonic branes** have spacetime singularities which are *removed from spacetime*: the field flux sourced by the singularity is that through spheres in the normal bundle around these loci and *would diverge* at the singular brane locus.

$$\underbrace{\mathbb{R}^{d+1}}_{\text{bulk}} \setminus \underbrace{\mathbb{R}^{p+1}}_{\text{singular brane}} \underset{\text{homeomorphism}}{\simeq} \underbrace{\left(\mathbb{R}^{d-p} \setminus \{0\}\right) \times \mathbb{R}^{p+1}}_{\text{punctured transverse space}} \underset{\text{homotopy equivalence}}{\simeq} \underbrace{S^{d-p-1}}_{\text{encircling sphere}} \tag{23}$$

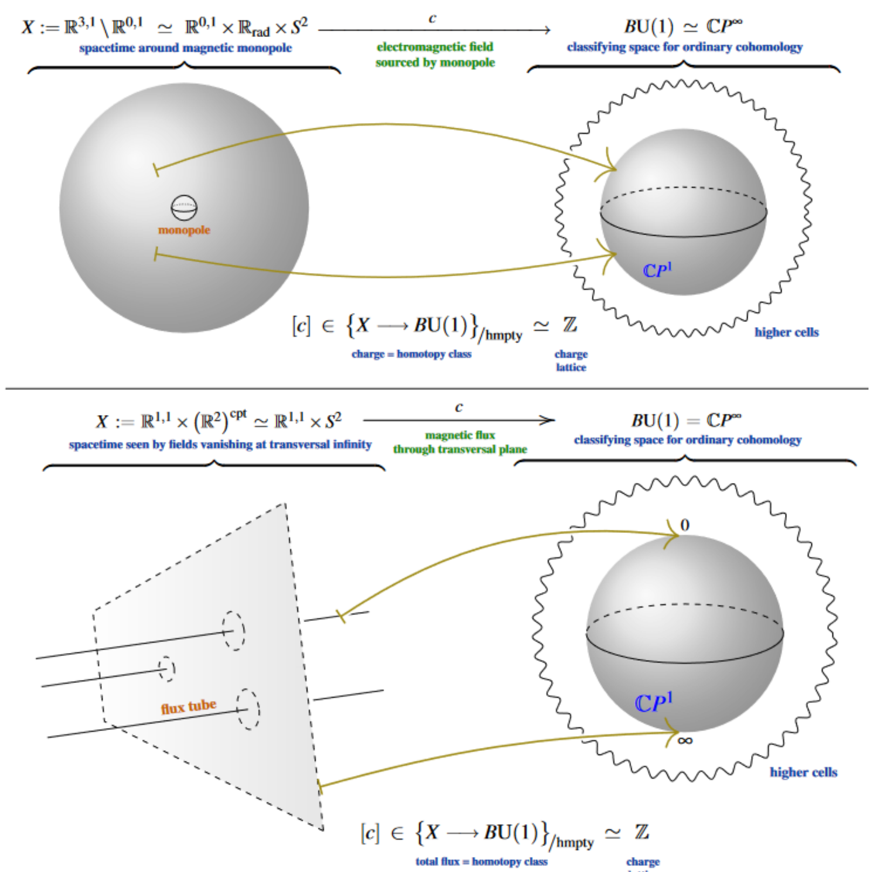
- **Probe branes** are witnessed by non-singular "local bumps" in the flux densities: Their flux *vanishes at infinity* which means that it is measured on the 1-point compactification of spacetime.

$$\underbrace{\mathbb{R}^{p+1}_{+}}_{\text{probe brane}} \wedge \underbrace{\mathbb{R}^{d-p}_{\text{cpt}}}_{\text{transv. space (with point at infinity)}} \simeq \mathbb{R}^{d-p}_{\text{cpt}} \underset{\text{homeomorphism}}{\simeq} \underbrace{S^{d-p}}_{\text{transverse sphere}} \tag{24}$$



Example: Magnetic charge in ordinary cohomology. The archetypical example is *Dirac's charge quantization* [Di31] (review in [Al85][Fr00, §2]), which is the observation that the quantum nature of electrons requires the magnetic charge carried by a magnetic monopole (say, a charged black hole) to be identified with a class in ordinary integral degree-2 cohomology of the spacetime surrounding the monopole.

While magnetic monopoles remain hypothetical, the same mechanism governs magnetic flux quantization in superconductors (e.g. [Cha00, §2]), which is experimentally observed. The difference here is that, instead of removing the worldline of a singular point source from spacetime, fields are required to *vanish at infinity* along some directions – here: along a plane perpendicular to the superconductor, e.g. [AGZ98, §IV.B]. The result is integer numbers of unit flux tubes: vortex strings [NO73] (review in [To09]).



[4] We are focusing on the simple case of "flat" Cartesian spacetimes/worldvolumes just for ease of exposition.

Probe branes seen in CoHomotopy are precisely cobordism classes of normally framed submanifolds:

Brane charge and Pontrjagin-Thom collapse. The above motivation of Cobordism cohomology as the natural home for p-brane charge, summarized in Table I, is in itself only a plausibility argument, just as is the traditional motivation ([Wi98, §3]) of K-theory as the natural home for D-brane charge. However, this physically plausible conclusion is rigorously *implied by Hypothesis H*, and hence is supported by and adds to the other evidence for that Hypothesis: This implication is the statement of *Pontrjagin's isomorphism*, which says that the operation of assigning to a normally framed closed submanifold its *asymptotic directed distance* function (traditionally known as the *Pontrjagin-Thom collapse construction*)

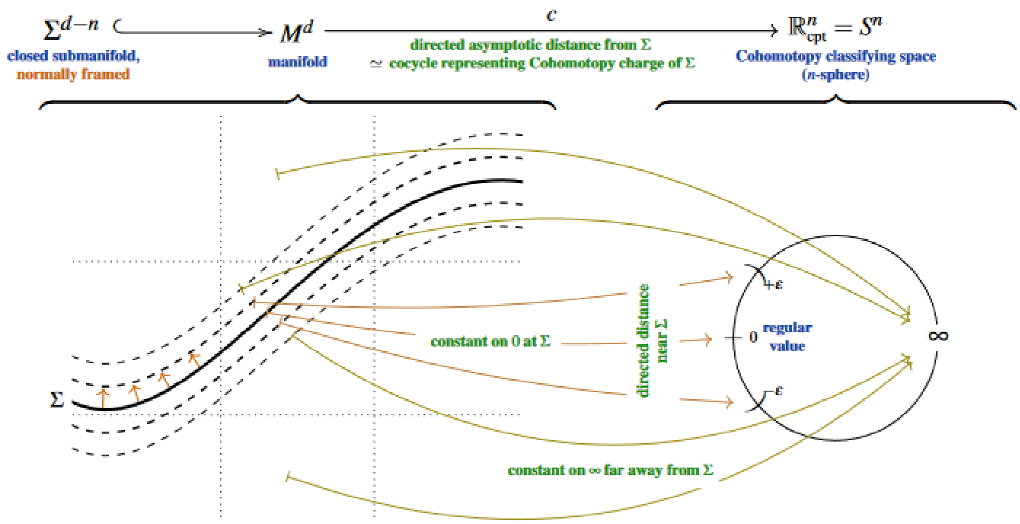


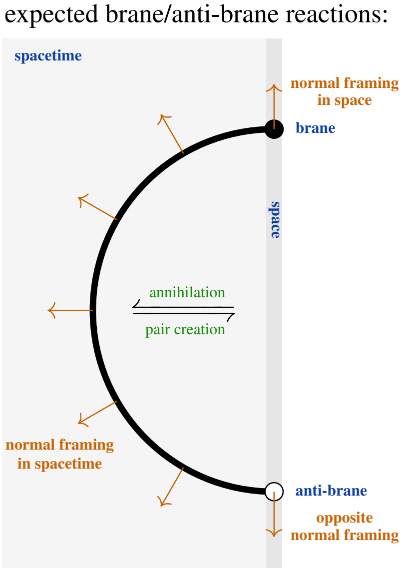
Figure D – The Pontrjagin construction. The *charge in Cohomotopy* of a manifold M^d, sourced by a normally framed closed submanifold Σ^{d-n}, is the homotopy class of the function that assigns directed asymptotic distance from Σ, measured along its normal framing.

identifies framed Cobordism with Cohomotopy, as non-abelian cohomology theories, over *closed manifolds* M^d:

$$\underset{\text{framed unstable } n\text{-Cobordism of } M^d}{\mathrm{Cob}^n_{\mathrm{Fr}}\big(M^d\big)} := \mathrm{NFramedSubmflds}_{d-n}\big(M^d\big)_{/\mathrm{brdsm}} \underset{\text{pre-image of regular value } 0}{\overset{\text{directed asymptotic distance}}{\simeq}} \mathrm{Maps}\big(M^d, S^n\big)_{/\mathrm{hmtpy}} =: \underset{\text{unstable } n\text{-Cohomotopy of } M^d}{\pi^n\big(\underset{\text{closed manifold}}{M^d}\big)}.$$

(assign Cohomotopy charge; reconstruct submanifold from its charge)

This equivalence of CoHomotopy with unstable framed Cobordism reflects exactly the expected brane/anti-brane reactions:



In fact, in linear approximation to the Bianchi identities, the resulting *stable CoHomotopy* is equivalent to *framed Cobordism Cohomology*

$$\underset{\text{non-abelian Cohomotopy}}{\pi^\bullet} \xrightarrow[\text{(i.e.: stabilize)}]{\text{linearize}} \underset{\text{abelian Cohomotopy}}{\mathbb{S}^\bullet} \overset{\text{Pontrjagin-Thom}}{=\!=\!=} \underset{\text{framed Cobordism}}{\mathrm{Mfr}^\bullet}, \qquad \mathbb{S}^\bullet \overset{\text{Barratt-Priddy-Quillen}}{=\!=} \underset{\text{algebraic K-theory of "field with one element"}}{K\mathbb{F}_1^\bullet} \tag{25}$$

Linearized Hypothesis H:

M-brane flux is quantized in (tangentially twisted) framed Cobordism.

(Possibly, this relates *Hypothesis H* to Vafa's *cobordism conjecture* cf. [HpH2, §4]).

In conclusion: the Pontrjagin theorem and its variants give, under Hypothesis H, a detailed description of *worldvolumes* of M-branes as (cobordism classes of normally framed) sub-manifolds of spacetime.

E.g. this allows to study exotic defect branes in low codimension ⟶

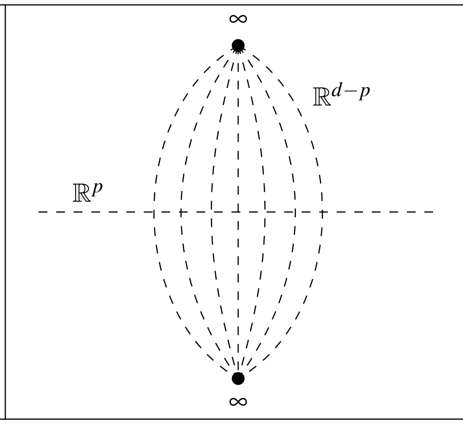
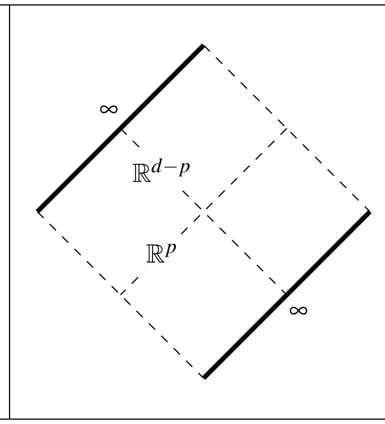
Low co-dimension probe branes tend to not exist semi-classically but, under Hypothesis H, as quantum states (8), in that:

$$\pi^4(X) := \pi_0 \operatorname{Map}(X, S^4) = 0 \qquad \text{but} \qquad \operatorname{Map}(X, S^4) \neq * .$$

For flat branes this is a consequence of the May-Segal theorem, which implies [Qnt1, Prop. 2.5] that (for $d > p \geq 1$):

$$\underbrace{\overbrace{\operatorname{Map}\big(\mathbb{R}^{d-p}_{\mathrm{cpt}} \wedge \mathbb{R}^p_+, S^d\big)}^{\text{CoHomotopy moduli}} \;\; \simeq \;\; \overbrace{\operatorname{Conf}\big(\mathbb{R}^{d-p}, \mathbb{R}^p_{\mathrm{cpt}}\big)}^{\text{configuration space of points}}}_{\text{homotopy equivalence}} \tag{26}$$

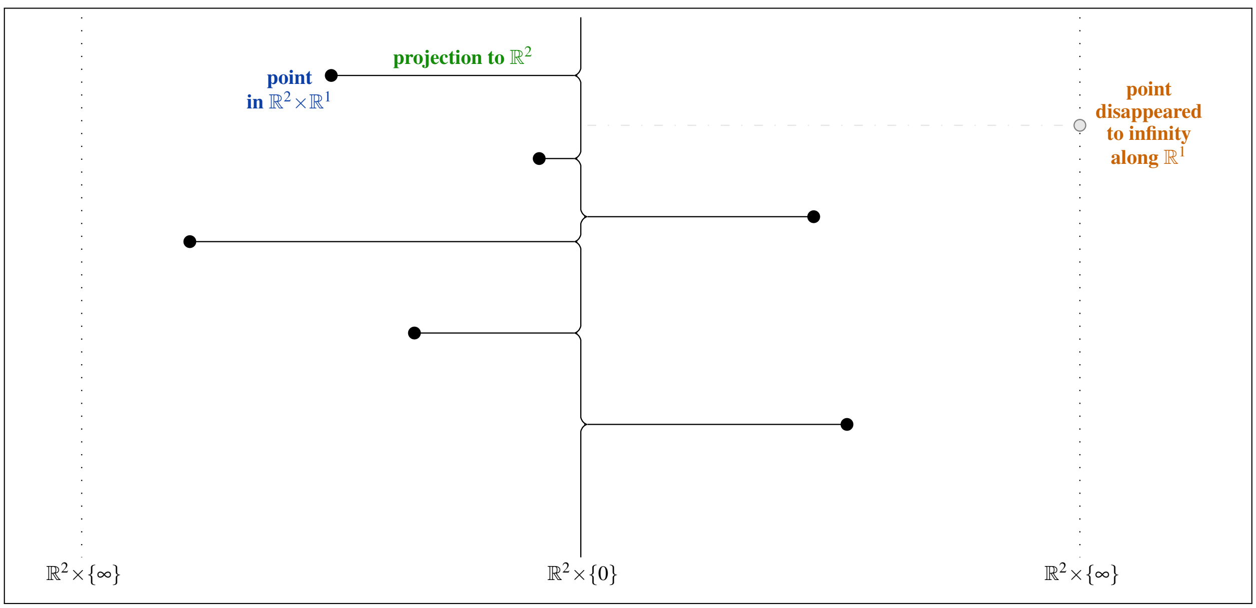
On the left of (26) we have maps out of $\mathbb{R}^d = \mathbb{R}^{d-p} \times \mathbb{R}^p$ required to vanish at infinity (only) along $\mathbb{R}^{n-p}$:

Smash product of pointed topological spaces	Visualization with point at infinity	as Penrose diagram
$\overbrace{\mathbb{R}^{d-p}_{\mathrm{cpt}}}^{\text{fluxes vanish at infinity along these directions}} \wedge \underbrace{\mathbb{R}^p_+}_{\text{...but not necessarily along these}}$		

On the right of (26) we have

$$\text{the moduli space} \quad \operatorname{Conf}\big(\mathbb{R}^{d-p}, \mathbb{R}^p_{\mathrm{cpt}}\big) \quad \text{of:} \quad \begin{array}{l} \text{- configurations of points in } \mathbb{R}^d = \mathbb{R}^{d-p} \times \mathbb{R}^p \\ \text{- which are distinct as points in } \mathbb{R}^{n-p} \\ \text{- and which may escape to infinity along } \mathbb{R}^p. \end{array} \tag{27}$$

Here is an illustration of an element in $\operatorname{Conf}\big(\mathbb{R}^{2-1}, \mathbb{R}^1_{\mathrm{cpt}}\big)$:



(28)

In words, (26) says that:

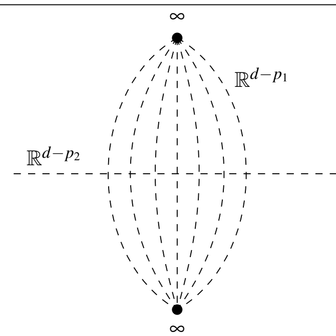
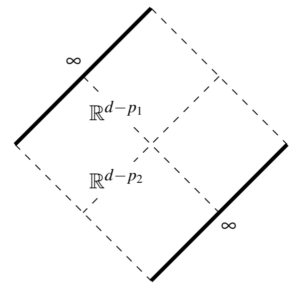
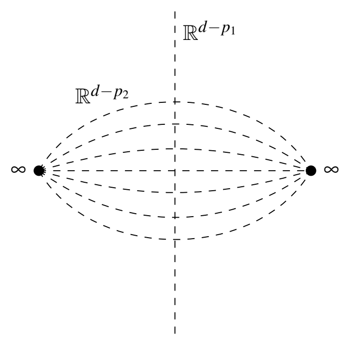
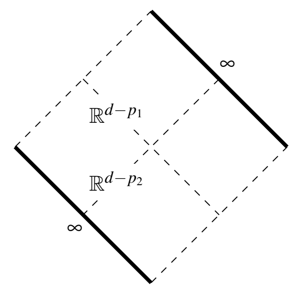
Flat low co-dimension probe branes always may and will disappear by escaping to infinity, together with their charge; but when several of them disappear jointly there are higher order topological charges in how they braid while doing so.

Next we see that *intersecting* such probe branes also prevents them from disappearing in the first place.

4 Resulting M5 $\perp$ M5-moduli

We discuss the formalization of *intersecting* probe branes from [Qnt1, §2].

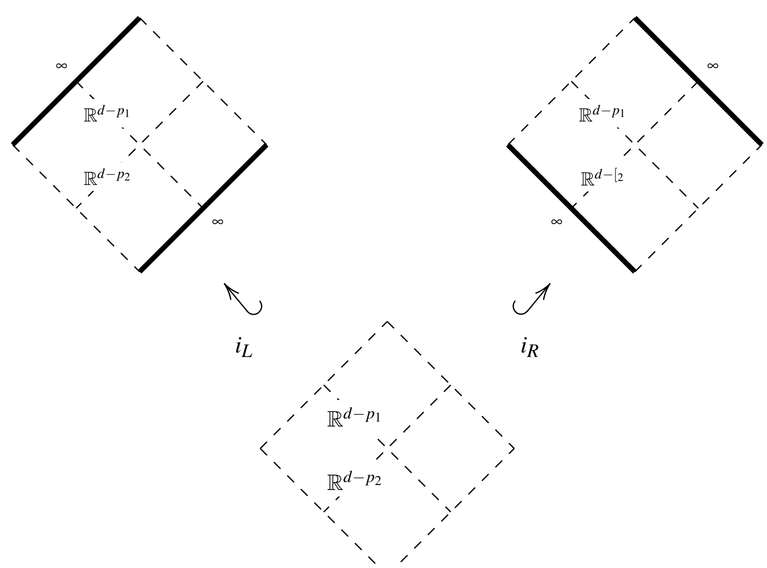
Above we have seen that the topological information encoding the potential presence of probe branes (or rather: their charges) is encoded in passing to the 1-point compactifications of their transverse space. This suggests that to measure the charges and hence the presence of *intersecting* branes we need to somehow amalgamate two different partial 1-point compactifications of the envelope of their transverse spaces:

	Smash product of pointed topological spaces	**Visualization with point at infinity**	**Visualization as Penrose diagram**
transverse space of p_1-probe	$\overbrace{\mathbb{R}^{d-p_1}_{\mathrm{cpt}}}^{\text{fluxes vanish at infinity along these directions}} \wedge \underbrace{\mathbb{R}^{d-p_2}_{+}}_{\text{...but not necessarily along these}}$		
transverse space of p_2-probe	$\underbrace{\mathbb{R}^{d-p_1}_{+}}_{\text{...but not necessarily along these}} \wedge \overbrace{\mathbb{R}^{d-p_2}_{\mathrm{cpt}}}^{\text{fluxes vanish at infinity along these direction}}$		

A sensible amalgamation of these transverse spaces does not exist *as a topological space*.

The topos theory of intersecting probe brane spaces. But we may pass to the universal mathematical context where it does exist: this is the *presheaf topos* over the category of "Penrose diagrams" of this form: In this topos, the amalgamation space transverse to flat intersecting branes is the "pushout" or "cofiber coproduct" of the two separately compactified transverse spaces over the uncompactified transverse space, hence their "gluing" according to the following diagram:

$$\left(\begin{array}{ccc} \mathbb{R}^{d-p_1}_{\mathrm{cpt}} \wedge \mathbb{R}^{d-p_2}_{+} & & \mathbb{R}^{d-p_1}_{\mathrm{cpt}} \wedge \mathbb{R}^{d-p_2}_{+} \\ & {}^{i_L}\nwarrow \quad \nearrow^{i_R} & \\ & \mathbb{R}^{2d-p_1-p_2}_{+} & \end{array}\right)$$



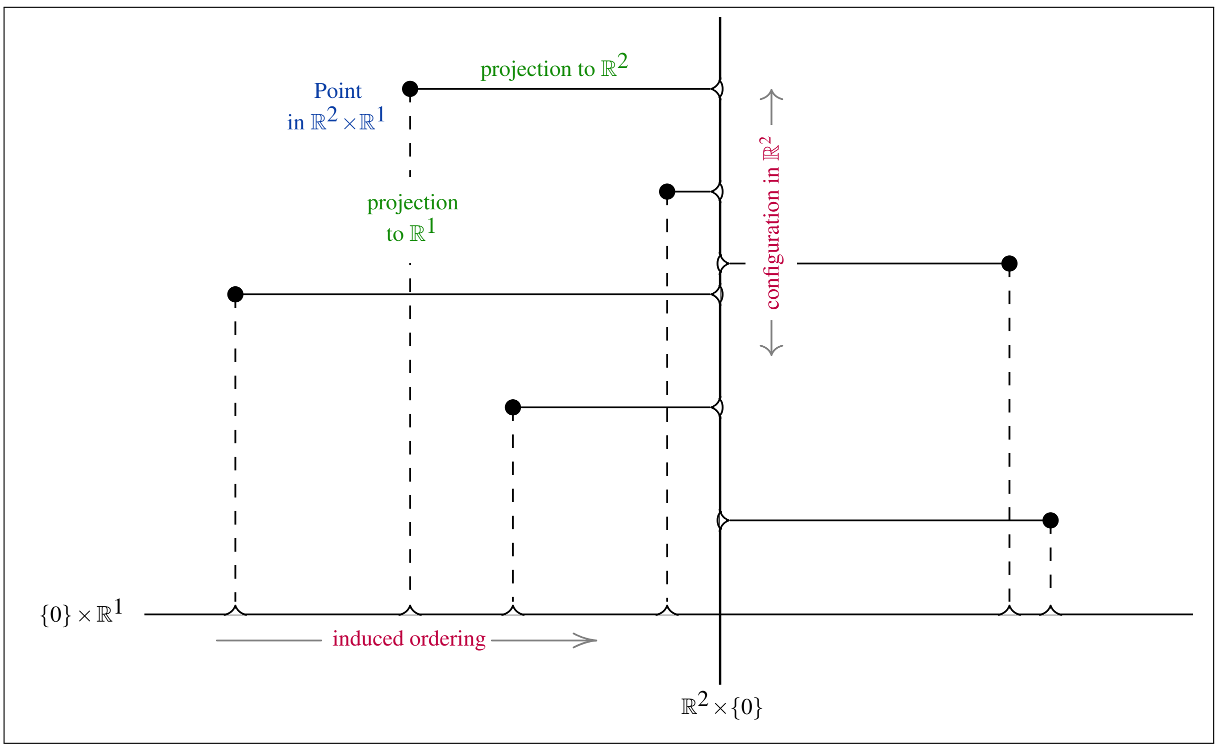
Intersecting brane charges in CoHomotopy. Furthermore, we may understand the May-Segal theorem (26) as providing a *differential refinement* of CoHomotopy theory on such spaces, in that the configuration space on the right of (26) canonically carries the structure of a manifold, which represents the homotopy type of the CoHomotopy space.

This provides a formalization of what it means to detect *intersecting* probe brane charges in CoHomotopy theory. Assuming this, the basic laws of topos theory imply that the (differential) CoHomotopy moduli space for intersecting probe branes as above is the *fiber product* of the two configuration spaces (27) for each probe brane separately.

For our example (28), the fiber product is as on the left here:

$$\underbrace{\overbrace{\mathrm{Conf}\big(\mathbb{R}^2, \mathbb{R}^1_{\mathrm{cpt}}\big)}^{\substack{\text{3-CoHomotopy moduli} \\ \text{of codim=1 probe branes}}} \underset{\mathrm{Conf}(\mathbb{R}^3)}{\times} \overbrace{\mathrm{Conf}\big(\mathbb{R}^1, \mathbb{R}^2_{\mathrm{cpt}}\big)}^{\substack{\text{3-CoHomotopy moduli} \\ \text{of codim=2 probe branes}}}}_{\text{their intersection}} \underset{\text{hmtp equvl}}{\simeq} \underbrace{\underset{n \in \mathbb{N}}{\sqcup} \underset{\{1,\cdots,n\}}{\mathrm{Conf}} \big(\mathbb{R}^2\big)}_{\substack{\text{ordered configurations of} \\ \text{probe brane intersections}}}, \tag{29}$$

where on the right we observe [Qnt1, Prop. 2.4. 2.11] that the configuration of the codimension=1 branes equips their intersecting branes with an *ordering*, as illustrated here:



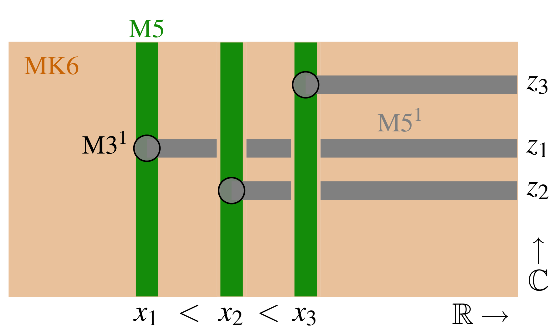
(30)

These *ordered configuration spaces* are

$$\text{the moduli spaces} \quad \underset{\{1,\cdots,n\}}{\mathrm{Conf}} \big(\mathbb{R}^2\big) \quad \text{of} \quad \begin{array}{l} \text{- configurations of } n \text{ points} \\ \text{- which are distinct} \\ \text{- and distinguishable} \end{array} \tag{31}$$

The homotopy type of such configuration spaces (31) is considerably richer than that of the configuration spaces (27) where points may escape to ∞. With Hypothesis H this implies that: There is *rich physics appearing on brane intersections*.

Moduli of the $\mathrm{M3} = \mathrm{M5} \perp \mathrm{M5}$**.** Concretely, applying this analysis to the 3-CoHomotopy fields in 7d which in §2 we saw appear on the MK6 locus ambient to M5-branes, we find a moduli space of 3-branes inside 5-branes, just as expected for $\mathrm{M5} \perp \mathrm{M5}$-brane intersections [Dfc1, pp. 28]:



(32)

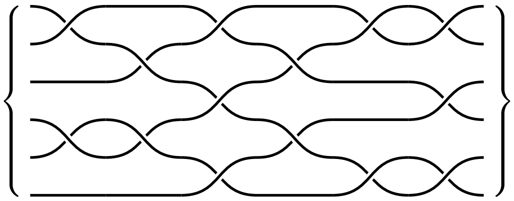
5 Resulting M5 ⊥ M5-states

In summary so far, Hypothesis H predicts that:

> *The quantum states* (8) *of* M5 ⊥ M5 *intersections* (32) *are*
> *the twisted cohomology of the configuration space* (31)
> *of points in their transverse plane* (23).

The spectral prequantum line bundle. These configuration spaces are non-simply connected: their fundamental group is the pure[5] *braid group* — being the *group of motions* of the M5 ⊥ M5-intersections around each other in the ambient M5-worldvolume:

$$\pi_1\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}^2\big)\Big) \;\simeq\; \mathrm{PBr}(n) \;=\; \Big\{ \quad \Big\} \tag{33}$$



Hence the corresponding twisted ordinary cohomology (aka: "local system cohomology") is that whose cocycles are sections of "Eilenberg-MacLane-spectrum line bundles" pulled back from the classifying space BC_κ of a cyclic group:

$$\underset{\substack{\text{twisted cohomology}\\ \text{of configuration space}}}{} H^{[\omega_1]}\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}^2\big)\Big)_{\text{quantum state space}} \;=\; \left\{ \begin{array}{ccc} & & \overset{\text{local coefficient bundle}}{B^k\mathbb{C}/\!\!/C_\kappa} \\ & \overset{\text{twisted cocyle}}{\underset{\text{quntm state}}{\nearrow}} & \downarrow \\ \underset{\text{phase space}}{\underset{\{1,\cdots,n\}}{\mathrm{Conf}}(\mathbb{C})} & \xrightarrow{\;\omega_1\;} & \underset{\text{prequantum line bundle}}{BC_\kappa} \end{array} \right\}_{/\mathrm{hmtp}} \tag{34}$$

(Closer analysis reveals [Dfc1, §3] that κ equals the order of the $\mathbb{A}_{\kappa-1}$-singularity at which dual D7/D3-branes are placed.)

In order to analyze these quantum states, we may decompose the problem by:

(1.) holding fixed N of the branes,
(2) letting n mobile branes move around them.

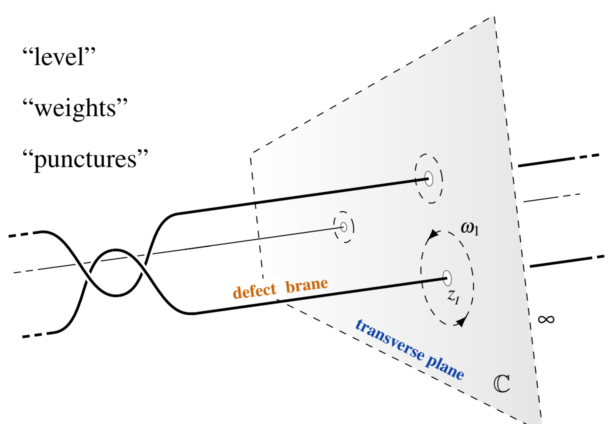
$$\begin{array}{ccc} \overset{\substack{n\text{-configuration space}\\ \text{of } N\text{-punctured plane}}}{\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{z_1,\cdots,z_N\}\big)} & \longrightarrow & \overset{\substack{\text{fibration of}\\ \text{configuration spaces}}}{\underset{\{1,\cdots,N+n\}}{\mathrm{Conf}}\big(\mathbb{C}^2\big)} \\ \downarrow & (\mathrm{pb}) & \big\downarrow{\scriptstyle \text{forget } n \text{ points}} \\ * & \xrightarrow[\text{pick } N\text{-configuration}]{(z_1,\cdots,z_N)} & \underset{\{1,\cdots,N\}}{\mathrm{Conf}}\big(\mathbb{C}^2\big) \end{array} \tag{35}$$

In the simple case of a single mobile brane moving – along a dashed line in (37) – among N fixed branes, we have

$$\underset{\{1,\cdots,1\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{z_1,\cdots,z_N\}\big) \;=\; \mathbb{C}\setminus\{z_1,\cdots,z_N\} \tag{36}$$

and the twist ω_1 (34) is fixed by:

$$\begin{array}{rcll} \kappa & := & k+2 & \text{"level"} \\ \mathrm{w}_I & \in & \{0,\cdots,k\} & \text{"weights"} \\ z_I & \in & \{z_1,\cdots,z_N\} & \text{"punctures"} \\ \text{as } \;\omega_1 & := & \sum_I -\frac{\mathrm{w}_I}{\kappa}\frac{\mathrm{d}z}{z-z_I} & \end{array} \tag{37}$$



[5]Our figures show im-pure braids just for ease of illustration.

Brane states identified with worldvolume correlators. Curiously, such sets of labels coincide with those of "conformal blocks" – namely chiral correlation functions – in the $\widehat{\mathfrak{su}_2}^k$-*conformal quantum field theory* on the punctured Riemann sphere

$$\mathbb{C}P^2 \setminus \{z_1, \cdots, z_N, \infty\} \;\simeq\; \mathbb{C}^2 \setminus \{z_1, \cdots, z_N\}\,. \tag{38}$$

And indeed, a well-but-not-widely known theorem called the *hypergeometric integral construction* identifies these conformal blocks of "degree=1" inside the twisted cohomology (34) of the punctured plane (38)

$$\begin{aligned}
\underset{\text{conformal blocks}}{\overset{\mathfrak{su}(2)\text{-affine deg=1}}{}} \mathrm{CnfBlck}^1_{\widehat{\mathfrak{sl}_2}^k}(\vec{\mathrm{w}}, \vec{z}) &\xhookrightarrow[\;]{\text{natural inclusion}} \overset{\text{1-twisted deg=1 de Rham cohomology}}{H^1\Big(\Omega^\bullet_{\mathrm{dR}}\big(\mathbb{C}\setminus\{\vec{z}\}\big),\, \mathrm{d} + \omega_1\wedge\Big)} \\
&\xhookrightarrow[\;]{\text{natural inclusion}} \underset{\text{inner local system-twisted deg=1 K-theory of } \mathbb{A}_{\kappa-1}\text{-singularity}}{\mathrm{KU}^{1+\omega_1}\Big(\big(\mathbb{C}\setminus\{\vec{z}\}\big) \times * /\!\!/ C_\kappa;\, \mathbb{C}\Big)} \quad \text{[Dfc1, Prop. 2.16]}
\end{aligned} \tag{39}$$

and generally the conformal blocks of any degree n inside n-configuration space of points, if we set

$$\omega_1 := \sum_{1\leq i\leq n} \sum_I -\frac{\mathrm{w}_I}{\kappa}\frac{\mathrm{d}z}{z - z_I} + \sum_{1\leq i<j\leq n} \frac{2}{\kappa}\frac{\mathrm{d}z}{z^i - z^j} \quad \text{on} \quad \underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{\vec{z}\}\big)\,. \tag{40}$$

namely:

$$\begin{aligned}
\underset{\text{conformal blocks}}{\overset{\mathfrak{su}(2)\text{-affine deg=}n}{}} \mathrm{CnfBlck}^n_{\widehat{\mathfrak{sl}_2}^k}(\vec{\mathrm{w}}, \vec{z}) &\hookrightarrow \overset{\text{1-twisted deg=}n\text{ de Rham cohomology of configuration space of } n \text{ points}}{H^n\Big(\Omega^\bullet_{\mathrm{dR}}\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{\vec{z}\}\big)\Big),\, \mathrm{d} + \omega_1\wedge\Big)} \\
&\hookrightarrow \underset{\text{inner local system-twisted deg=}n\text{ K-theory of configurations in } \mathbb{A}_{\kappa-1}\text{-singularity}}{\mathrm{KU}^{n+\omega_1}\Big(\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{\vec{z}\}\big)\Big) \times * /\!\!/ C_\kappa;\, \mathbb{C}\Big)} \quad \text{[Dfc1, Thm. 2.18]}
\end{aligned} \tag{41}$$

Concretely, this inclusion is given by sending the canonical basis elements of conformal blocks to "Slater-determinant"-like expressions, as follows:

$$\begin{array}{ccc}
\mathrm{CnfBlck}^n_{\widehat{\mathfrak{sl}_2}^k}(\vec{\mathrm{w}}, \vec{z}) & \longrightarrow & H^n\Big(\Omega^{\bullet,0}_{\mathrm{dR}}\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{\vec{z}\}\big)\Big)\big|_{\bar\partial=0},\, \partial + \omega_1(\vec{\mathrm{w}},\kappa)\wedge\Big) \\
f_{I_1}\cdots f_{I_n}|v_1^0\cdots,v_N^0\rangle & \underset{\text{generators}}{\longmapsto} & \left[\det\left(\left(\frac{\mathrm{w}_{I_j}}{\kappa}\frac{1}{z^i - z_{I_j}}\right)^n_{i,j=1}\right) dz^1\wedge\cdots\wedge dz^n\right] \\
\text{e.g.}\quad f_I f_J|v_1^0\cdots,v_N^0\rangle = [\cdots,(f\cdot v_I^0),\cdots,(f\cdot v_J^0),\cdots] & \longmapsto & \left[\frac{\mathrm{w}_I}{\kappa}\frac{\mathrm{d}z^1}{(z^1-z_I)}\wedge\frac{\mathrm{w}_J}{\kappa}\frac{\mathrm{d}z^2}{(z^2-z_J)} + \frac{\mathrm{w}_I}{\kappa}\frac{\mathrm{d}z^2}{(z^2-z_I)}\wedge\frac{\mathrm{w}_J}{\kappa}\frac{\mathrm{d}z^1}{(z^1-z_J)}\right].
\end{array} \tag{42}$$

In summary, we have derived, from Hypothesis H, that:

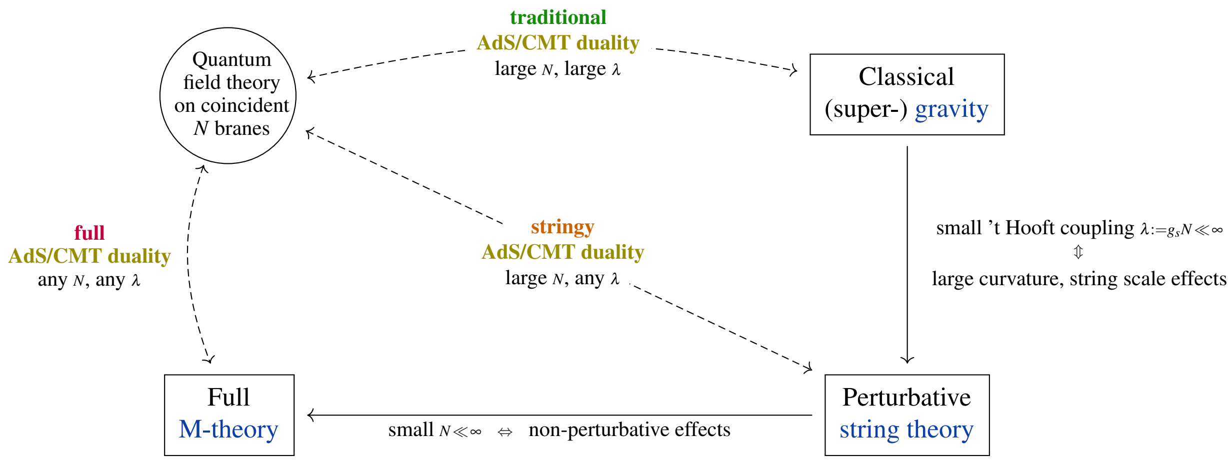
$$\boxed{\left.\begin{array}{c}\text{quantum states of}\\ \text{brane configurations}\\ \text{inside an M-theoretic bulk}\end{array}\right\} \text{ are identified with } \left\{\begin{array}{c}\text{quantum correlators of}\\ \text{a conformal field theory}\\ \text{on their worldvolume}\end{array}\right.} \tag{43}$$

This is just the form of "holographic duality" that is expected in string/M-theory, here specifically[6] in "Theory-$\mathscr{S}$"-compactifications of M5-branes on Riemann surfaces such as (38).

Strongly coupled holographic quantum materials. In [Dfc2] we give a detailed argument that the worldvolume CFT which we see here is that of *anyonic defects* in *topologically ordered ground states* of *crystalline quantum materials* which are in a *topological phase of matter*.

This being a strongly coupled QFT on a *small* number κ of branes, it is outside the realm of perturbative string theory and would indeed be expected to require M-theory for its holographic description:

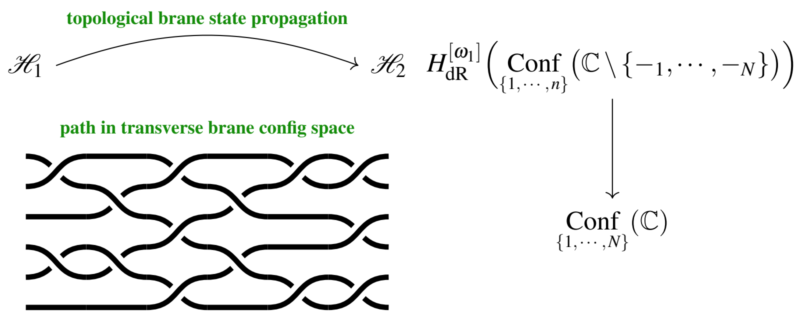
[6] As M. Ashwinkumar kindly reminded us during the talk, our prediction of $\widehat{\mathfrak{su}_2}^k$-conformal blocks for M5s compactified on a Riemann surface matches the conclusion in [Wi10, p. 22]

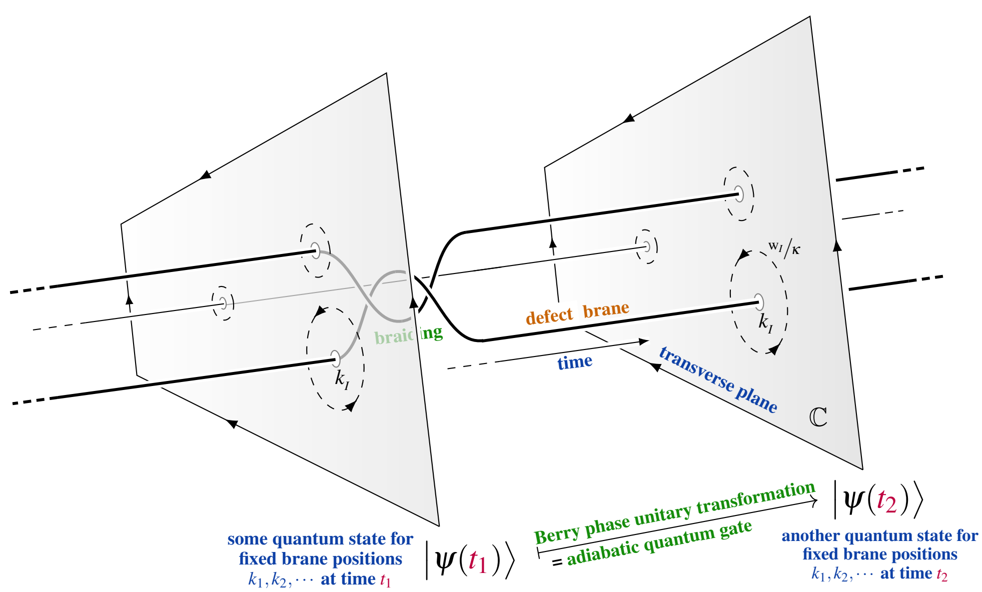


6 Resulting M5 ⊥ M5-braiding

We close by indicating how the "topopological dyanmics" of M5 ⊥ M5 (their adiabatic movement in moduli space) acts on their quantum states just as expected for *quantum logic gates* in *topological quantum computers* based on *anyon braiding* – as they should by the duality (43). Detailed discussion may be found in [TQC1][TQC2].

Modular functor of M5 ⊥ M5 **Hilbert spaces.** The fibration of configuration spaces (35) induces on its fiberwise twisted cohomology groups (41) a flat connection — called a *Gauss-Manin connection*, which on the spaces of conformal blocks restricts to the *Knizhnik-Zamolodchikov connection*.
The parallel transport of this connection computes the unitary transformations on the branes' quantum states induced by their adiabatic movement in moduli space:





Under the above holographic duality, such brane braiding translates to the braiding of anyonic defects in topologically ordered quantum materials, which is thought to potentially serve as quantum logic gates for topological quantum computers.

We list our publications that this review is based on, wherein the reader finds extensive pointers to the rest of literature:

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