1.3 Hypothesis H on M-theory

With a general understanding of flux-quantization in hand (§1.2) we are in position to motivate and state *Hypothesis H* (§1.3.2) on M-brane charge quantization. In order to put this in perspective, we first review (§1.3.1) the widely accepted *Hypothesis K* that D-brane charges are quantized in twisted K-theory.

In the special case of *flat* spacetimes X possibly with a point at infinity adjoined (35), Hypothesis H postulates the following, in direct analogy, with Dirac's EM-charge quantization (p. 20):

Hypothesis H on flat spacetimes says that the non-perturbative completion of the C-field in 11d supergravity (9) involves a map χ from spacetime to the homotopy type of the 4-sphere, so that the C-field gauge potentials $(\widehat{C}_3, \widehat{C}_6)$ exhibit the flux densities (G_4, G_7) as $\mathbb{R}$-rational representatives of χ.

In other words, on flat spacetimes Hypothesis H postulates that the non-perturbative C-field is a cocycle in canonical *differential non-abelian 4-Cohomotopy* [FSS15-M5WZW, §4][GrS20, §3.1][Char, Ex. 9.3].

As an immediate plausibility check: This implies, from the well-known homotopy groups of spheres in low degrees, cf. (67) below:

integral quantization of charges carried by singular M5-brane branes *and*

integral quantization of charges carried by singular M2-branes... plus a torsion-contribution (a first prediction of Hypothesis H).

$$\begin{array}{ccc}
 & & \mathrm{Maps}(X;\, S^4) \\
 & & \downarrow{\scriptstyle \mathrm{ch}} \\
\Omega_{\mathrm{dR}}(X;\, \iota S^4) & \xrightarrow{\eta^{\int}} & \mathrm{Maps}(X,\, L^{\mathbb{R}} S^4) \\
(G_4,\, G_7) & \mapsto & \eta^{\int}(G_4, G_7)
\end{array}$$

χ — Cohomotopical charge sector

$\mathrm{ch}(\chi)$

(G_4, G_7): C-field flux densities

$(\widehat{C}_3, \widehat{C}_6)$: gauge potentials

$$(\widehat{C}_3,\, \widehat{C}_6) \;\in\; \widehat{\pi}^4(X) \xrightarrow{\chi} \pi^4(X)$$

$(\widehat{C}_3, \widehat{C}_6)$: full nonperturbative 11d SuGra C-field

$\widehat{\pi}^4(X)$: canonical differential non-abelian (unstable) 4-Cohomotopy

$\pi^4(X)$: plain non-abelian (unstable) 4-Cohomotopy

χ: topological sector

(G_4, G_7): flux densities

$$H_{\mathrm{dR}}(X;\, \iota S^4)$$

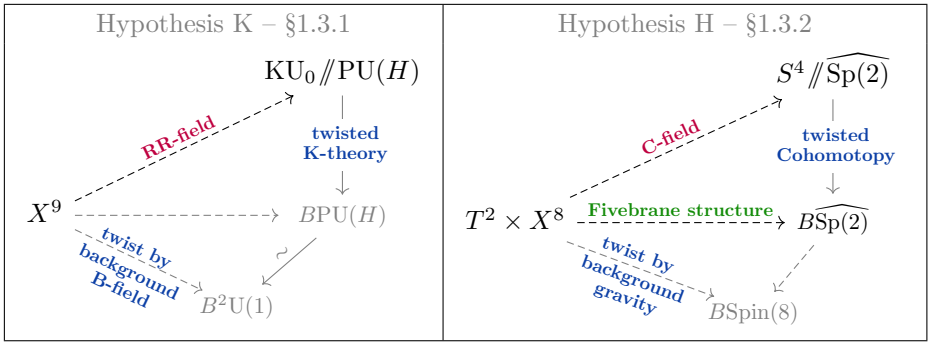
de Rham cohomology with coefficients in Whitehead L_∞-algebra of the 4-sphere

$$\pi^4\big(\mathbb{R}^{10,1} \setminus \mathbb{R}^{5,1}\big) \;=\; \pi^4\big(\mathbb{R}^{5,1} \times \mathbb{R}_+ \times S^4\big)$$
$$=\; \pi^4(S^4) \;=\; \pi_4(S^4) \;=\; \mathbb{Z}$$

$$\pi^4\big(\mathbb{R}^{10,1} \setminus \mathbb{R}^{2,1}\big) \;=\; \pi^4\big(\mathbb{R}^{2,1} \times \mathbb{R}_+ \times S^7\big)$$
$$=\; \pi^4(S^7) \;=\; \pi_7(S^4) \;=\; \mathbb{Z} \;\oplus \mathbb{Z}_{12}$$

To generalize this to non-flat spacetimes, it remains to discuss the twisting (according to §1.2.4) of Cohomotopy received by the gravitational background field. To appreciate this it may be helpful to recall that also the B-field in 10d may be understood as part of the "generalized geometric" gravitational background flux.

Hypothesis H on gravitational backgrounds. In the following we explain this gravitationally coupled twisted version of Hypthesis H, in parallel to traditional Hypothesis K:



To distinguish M2/M5-charge, the tangential twisting needs to preserve the $\mathbb{H}$-Hopf fibration $\Rightarrow$ tangential $\mathrm{Sp}(2) \hookrightarrow \mathrm{Spin}(8)$-structure [FSS20-HpH1, §2.3]. With this, integrality of M2's Page charge & anomaly-cancellation of the M5's Hopf-WZ term follows from trivialization of the Euler 8-class, which means lift to the *Fivebrane* 6-group $\widehat{\mathrm{Sp}(2)} \to \mathrm{Sp}(2)$ [FSS21-M5a, §4].

This implies [FSS20-HpH1][FSS21-M5a]:
(1.) half-integrally shifted quantization (72) of M5-brane charge in curved backgrounds, *and*
(2.) integral quantization of the "Page charge" of M2-branes (82).

$$[\widetilde{G}_4] \;:=\; \underbrace{[G_4]}_{\substack{\text{C-field} \\ \text{4-flux}}} + \tfrac{1}{2}\Big(\underbrace{\tfrac{1}{2} p_1(TX^8)}_{\substack{\text{integral Spin-} \\ \text{Pontrjagin class}}}\Big) \;\in\; H^4\big(X^8;\, \mathbb{Z}\big)$$

$$2[\widetilde{G}_7] \;:=\; 2\big([G_7] + \tfrac{1}{2}[H_3 \wedge \widetilde{G}_4]\big) \;\in\; H^7\big(\widehat{X}^8; \mathbb{Z}\big)$$

Both of these quantization conditions on M-brane charge are thought to be crucial for M-theory to make any sense. Previously the first had remained enigmatic and the second had remained wide open.