12.4 The rank of $R_K(G)$

We return now to the case of an arbitrary field K of characteristic zero. We shall determine the rank of $R_K(G)$, or equivalently, *the number of irreducible representations of* G *over* K.

Choose an integer m which is a multiple of the orders of the elements of G (for example, their least common multiple or the order g of G), and let L be the field obtained by adjoining to K the mth roots of unity. We know (cf. for example Bourbaki, *Alg.* V, §11) that the extension L/K is Galois and that its Galois group Gal(L/K) is a subgroup of the multiplicative group $(\mathbf{Z}/m\mathbf{Z})^*$ of invertible elements of $\mathbf{Z}/m\mathbf{Z}$. More precisely, if $\sigma \in \mathrm{Gal}(L/K)$, there exists a unique element $t \in (\mathbf{Z}/m\mathbf{Z})^*$ such that

$$\sigma(\omega) = \omega^t \quad \text{if } \omega^m = 1.$$

We denote by Γ_K the image of Gal(L/K) in $(\mathbf{Z}/m\mathbf{Z})^*$, and if $t \in \Gamma_K$, we let σ_t denote the corresponding element of Gal(L/K). The case considered in the preceding section was that where $\Gamma_K = \{1\}$.

Let $s \in G$, and let n be an integer. Then the element s^n of G depends only on the class of n modulo the order of s, and so *a fortiori* modulo m; in particular s^t is defined for each $t \in \Gamma_K$. The group Γ_K acts as a *permutation group* on the underlying set of G. We will say that two elements s, s' of G are Γ_K-*conjugate* if there exists $t \in \Gamma_K$ such that s' and s^t are conjugate by an element of G. The relation thus defined is an equivalence relation and does not depend upon the choice of m; its classes are called the Γ_K-*classes* (or the K-*classes*) *of* G.

Theorem 25. *In order that a class function f on* G, *with values in* L, *belong to* $K \otimes_{\mathbf{Z}} R(G)$, *it is necessary and sufficient that*

$$(*) \qquad \sigma_t(f(s)) = f(s^t) \quad \textit{for all } s \in G \textit{ and all } t \in \Gamma_K.$$

(In other words, we must have $\sigma_t(f) = \Psi^t(f)$ for all $t \in \Gamma_K$, cf. 11.2.)

Let ρ be a representation of G with character χ. For $s \in G$, the eigenvalues ω_i of $\rho(s)$ are mth roots of unity, and the eigenvalues of $\rho(s^t)$ are the ω_i^t. Thus we have

$$\sigma_t(\chi(s)) = \sigma(\sum \omega_i) = \sum \omega_i^t = \chi(s^t),$$

which shows that χ satisfies the condition $(*)$. By linearity, the same is true for all the elements of $K \otimes R(G)$.

Conversely, suppose f is a class function on G satisfying condition $(*)$. Then

$$f = \sum c_\chi \chi, \quad \text{with } c_\chi = \langle f, \chi \rangle,$$

where χ runs over the set of irreducible characters of G. We have to show that the c_χ belong to K, which, according to Galois theory, is equivalent to showing that they are invariant under the σ_t, $t \in \Gamma_K$. But, if φ and χ are two class functions on G, then we have

$$\langle \Psi^t \varphi, \Psi^t \chi \rangle = \langle \varphi, \chi \rangle,$$

as can be easily verified. Whence

$$c_\chi = \langle f, \chi \rangle = \langle \Psi^t f, \Psi^t \chi \rangle = \langle \sigma_t(f), \sigma_t(\chi) \rangle = \sigma_t(\langle f, \chi \rangle) = \sigma_t(c_\chi),$$

which finishes the proof. □

Corollary 1. *In order that a class function f on* G *with values in* K *belong to* $K \otimes R_K(G)$, *it is necessary and sufficient that it be constant on the* Γ_K-*classes of* G.

If $f \in K \otimes R_K(G)$, then $f(s) \in K$ for all $s \in G$, and formula (∗) shows that $f(s) = f(s^t)$ for all $t \in \Gamma_K$. Hence f is constant on the Γ_K-classes of G.

Conversely, suppose that f has values in K, and is constant on the Γ_K-classes of G. Then condition (∗) is satisfied, and we can write

$$f = \sum \langle f, \chi \rangle \chi, \quad \text{with } \langle f, \chi \rangle \in K$$

as above. Moreover, the fact that f is invariant under the σ_t, $t \in \Gamma_K$, shows that $\langle f, \chi \rangle = \langle f, \sigma_t(\chi) \rangle$, so the coefficients of the two conjugate characters χ and $\sigma_t(\chi)$ are the same. Collecting characters in the same conjugacy class, we can write f as a linear combination of characters of the form $\mathrm{Tr}_{L/K}(\chi)$. Since the latter belong to $R_K(G)$, cf. 12.1, this proves the corollary.

[*Alternately*: Let Γ_K act on $K \otimes R(G)$ by $f \mapsto \sigma_t(f) = \Psi^t(f)$, and observe that the set of fixed points is $K \otimes R_K(G)$.] □

Corollary 2. *Let χ_i be the characters of the distinct irreducible representations of* G *over* K. *Then the χ_i form a basis for the space of functions on* G *which are constant on* Γ_K-*classes, and their number is equal to the number of* Γ_K-*classes.*

This follows from cor. 1. □

Remark. In cor. 1, we can replace $R_K(G)$ by $\overline{R}_K(G)$. Indeed prop. 34 shows that

$$\mathbf{Q} \otimes R_K(G) = \mathbf{Q} \otimes \overline{R}_K(G), \quad \text{whence } K \otimes R_K(G) = K \otimes \overline{R}_K(G).$$

12.5 Generalization of Artin's theorem

If H is a subgroup of G, it is clear that

$$\mathrm{Res}_H \colon R(G) \to R(H) \qquad \text{and} \qquad \mathrm{Ind}_H \colon R(H) \to R(G)$$