Proper Orbifold Cohomotopy for M-Theory

Urs Schreiber on joint work with Hisham Sati

NYU AD Science Division, Program of Mathematics

& Center for Quantum and Topological Systems

New York University, Abu Dhabi

talk at:

String and M-Theory
The New Geometry of the 21^{st} Century
II

via NUS Singapore, Nov.-Dec. 2021

slides and further pointers available at: ncatlab.org/schreiber/show/Proper+Orbifold+Cohomotopy+for+M-Theory

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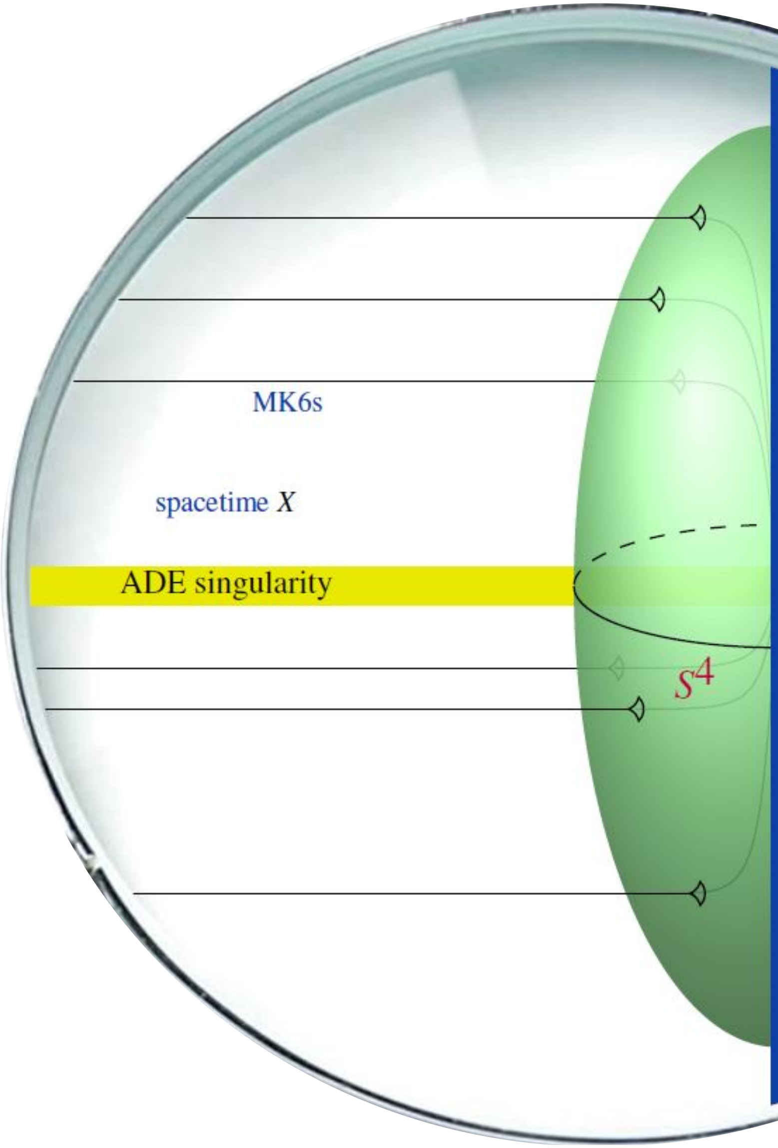
defining non-perturbative brane physics (M-theory) has remained an open problem.



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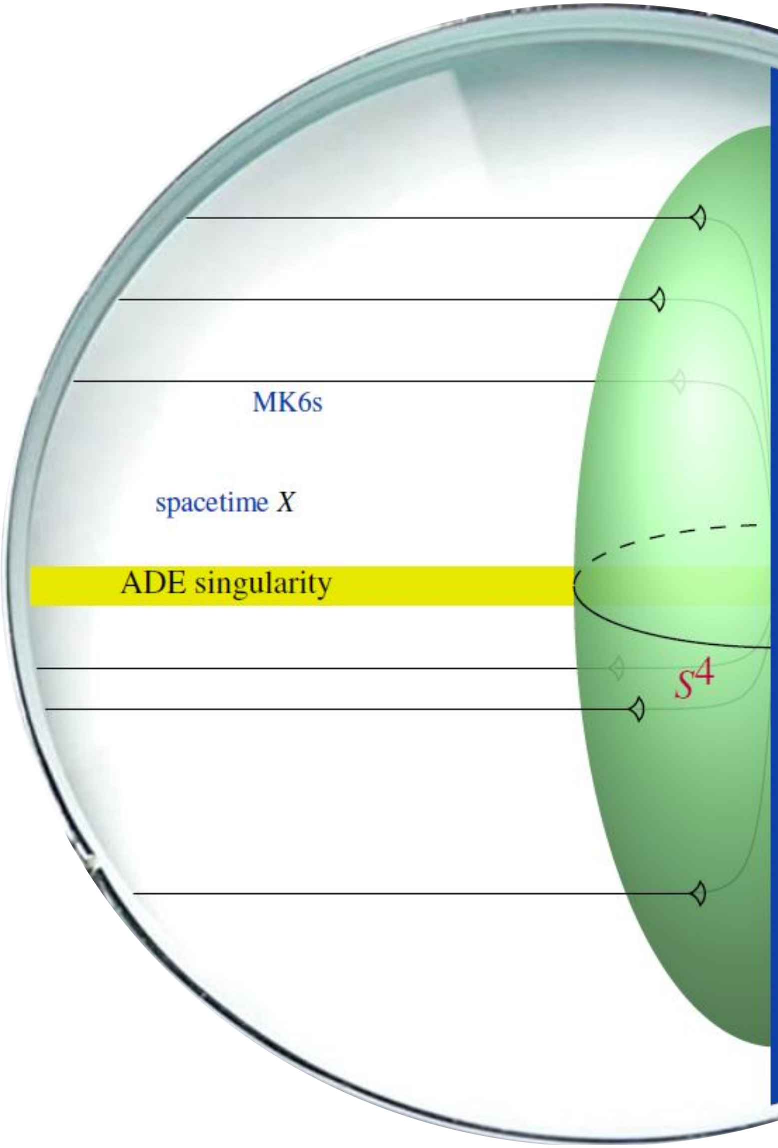
Hypothesis H is a new proposal for the mathematical definition of the quantum states/charges of M-theory



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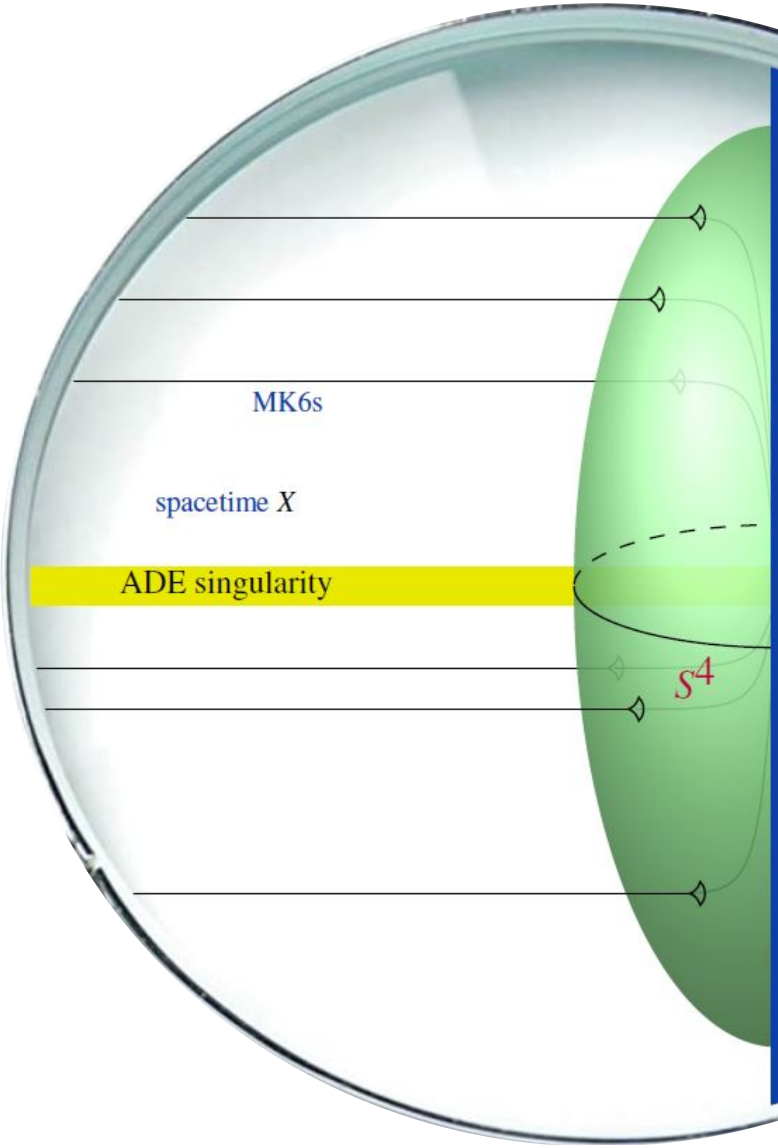

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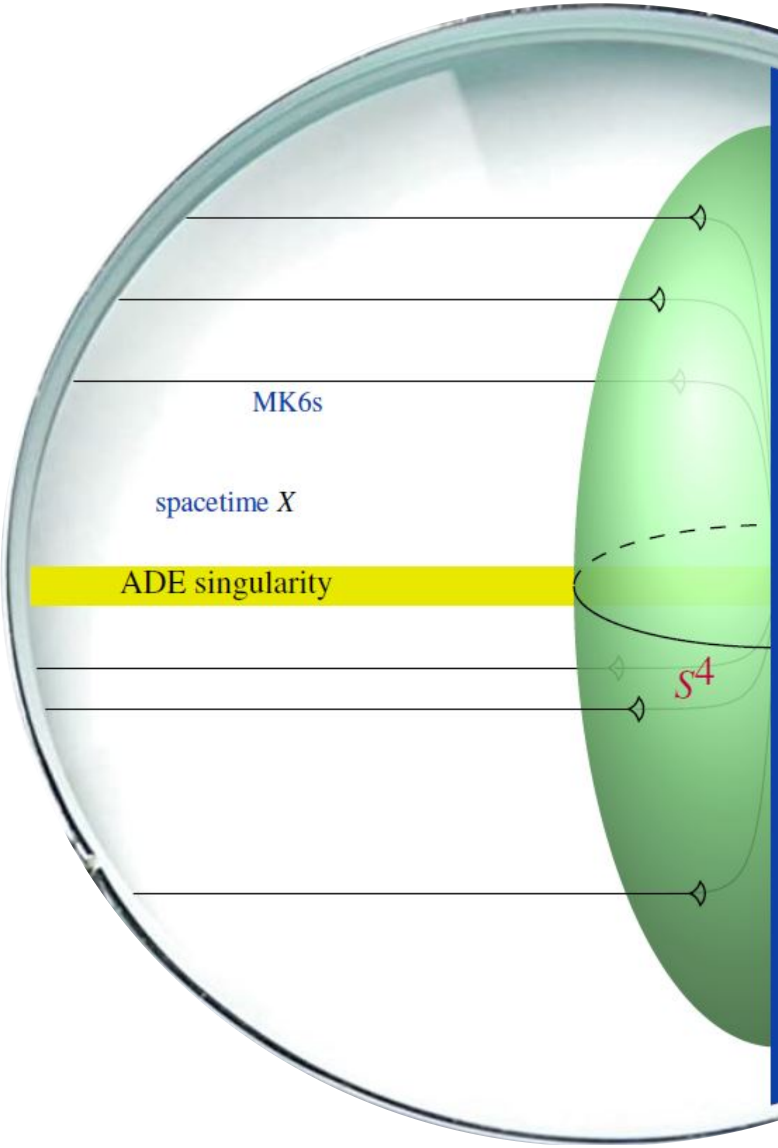

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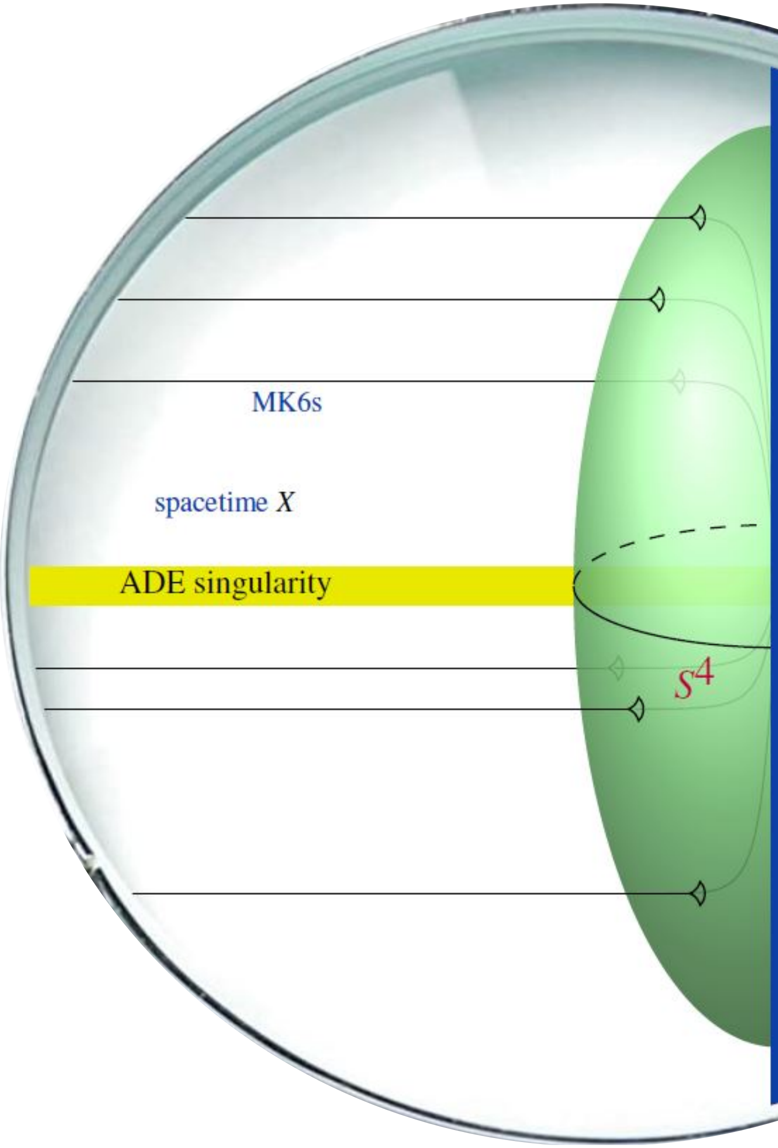

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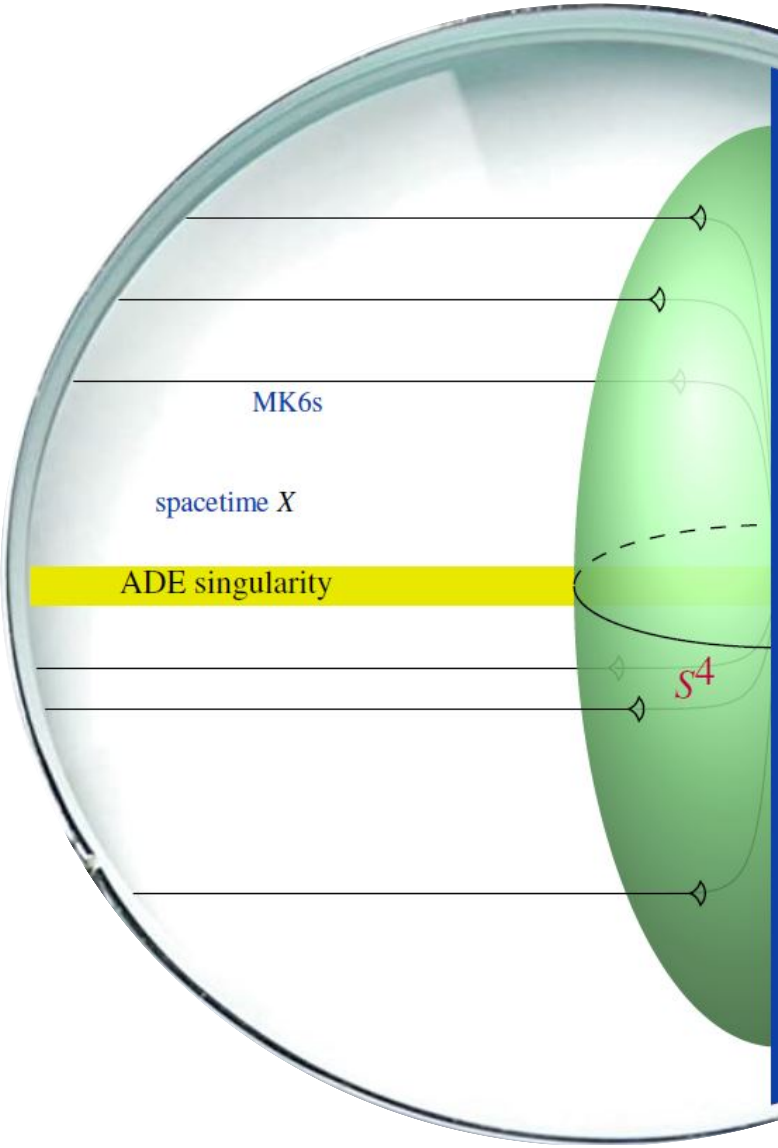

This talk surveys what this means and how this
- 1) is made precise over orbi-super-spacetimes;
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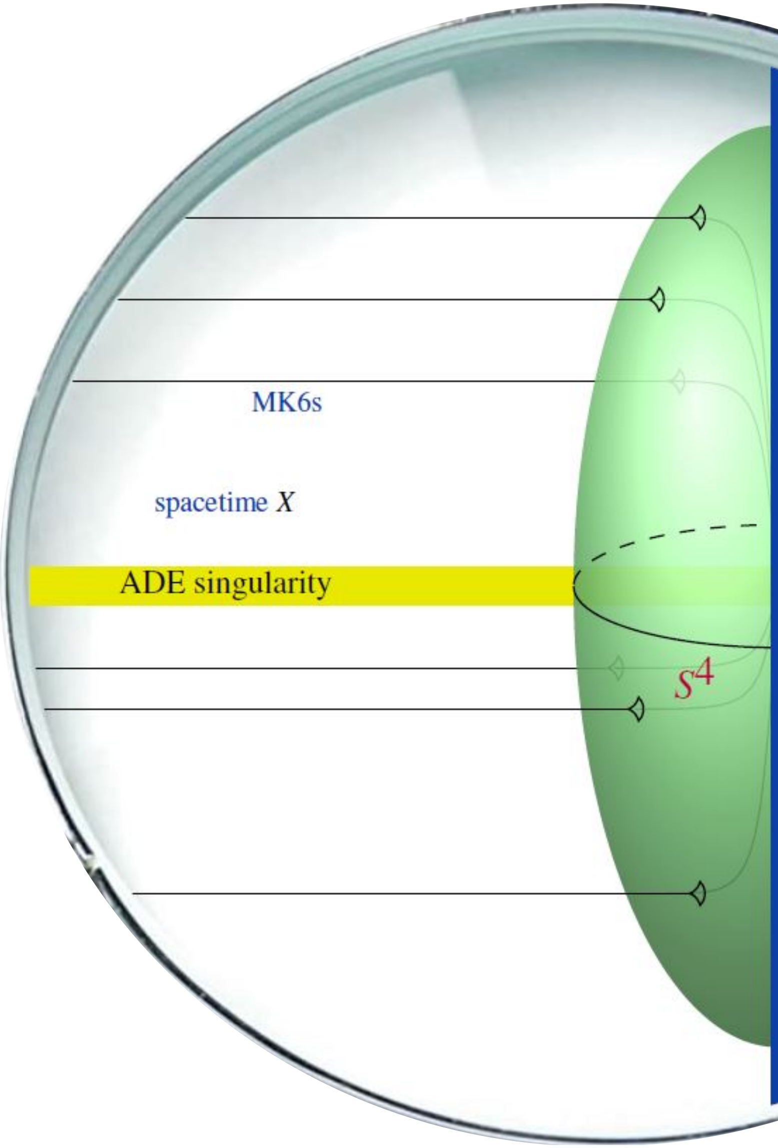

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For more see H. Sati's talks and see: `ncatlab.org/schreiber/show/Hypothesis+H` higher orbi-geometry

orbifold cohomology

I – Proper Orbifold Cohomotopy

II – for M-Theory

Hypothesis H

M5-brane charges in
flat orbi-orientifolds

String theory at its finest is, or should be, a new branch of geometry
... developed in the 21st century ... that fell by chance into the 20th century ...

To elucidate the proper generalization of geometry
[is] the central problem of string theory.

E. Witten (1988)

as quoted on p. 95, 102 in:

P C W Davis and J Brown (eds.)
Superstrings: A theory of everything?
Camb Univ Press 1988, 1991: Canto 1992

Non-perturbative quantum physics. [SS19-Quant][CSS21-Quant][Sc14-Quant]

What does it even *mean* to define a non-perturbative quantum theory?

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Lots of subtleties, but two general principles:

1) covariant phase space of *all* field histories (all instanton/soliton sectors)
2) quantum states are *global* sections of prequantum line bundle over phase space

Hence we need a geometry which makes sense of symbols like this:

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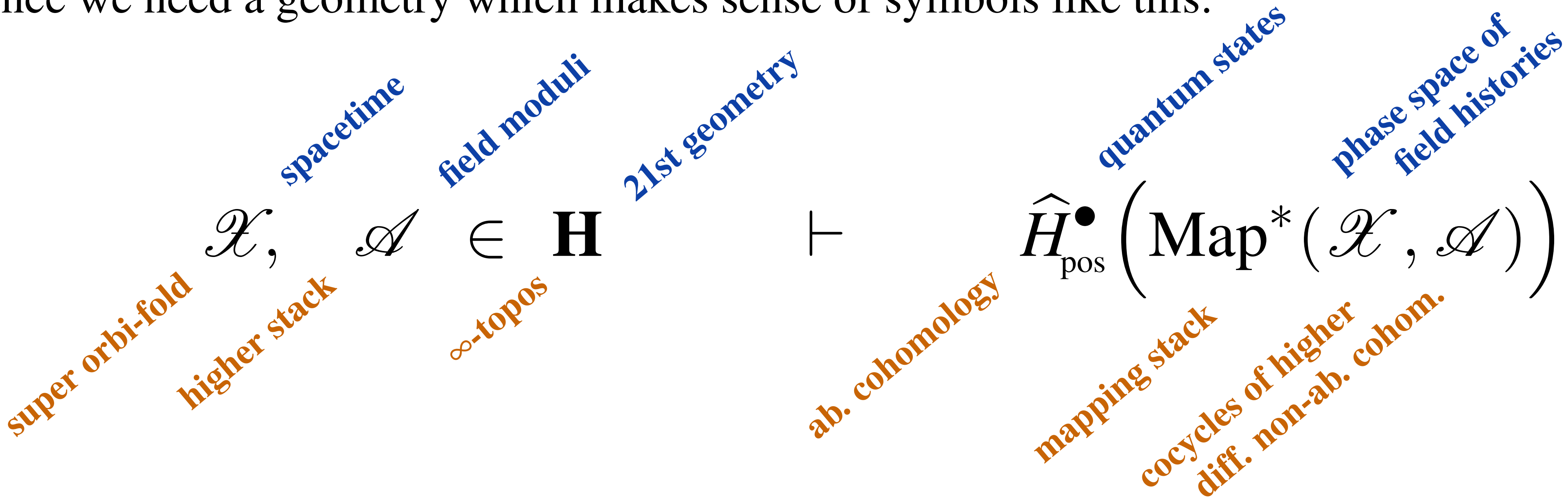

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Ex.: 3d CS theory with cpt gauge group:
[Hit90] [APW91]

$\mathscr{X}$ = surface
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$\widehat{H}^{\bullet}_{\text{pos}}$ = hol. sections of prequ. line bdl.

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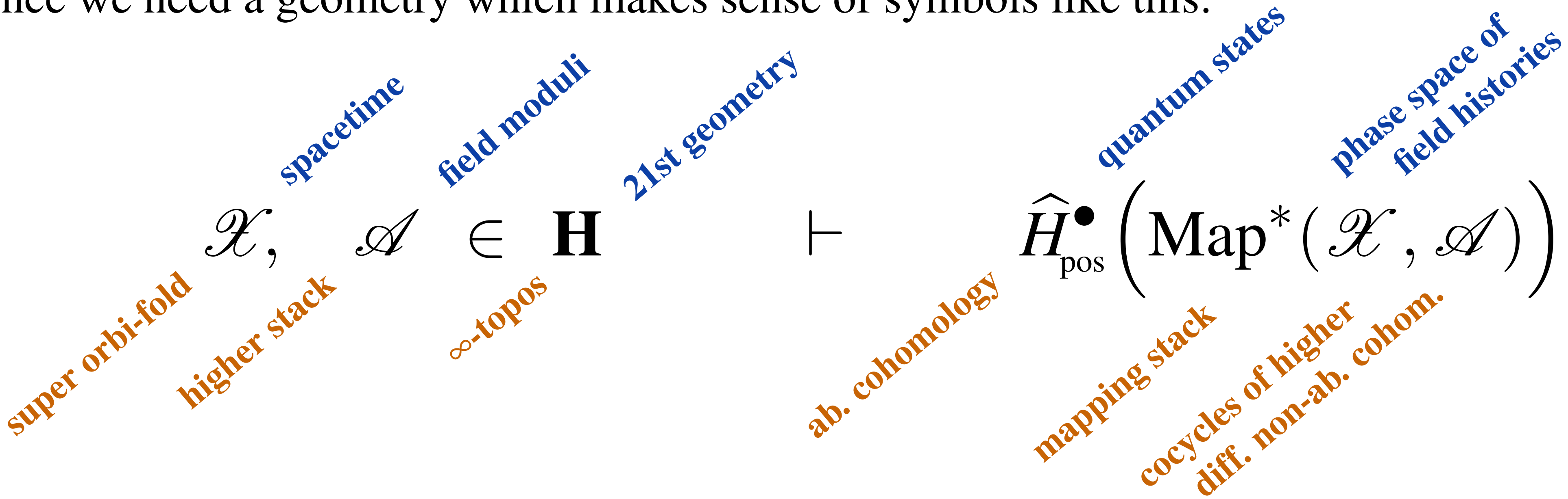

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Ex.: D6 ⊥ D8-branes via Hypothesis H
[SS19-Quant][CSS21-Quant]

$\mathscr{X}$ = transverse cptfd. space to branes
$\mathscr{A}$ = moduli stack of diff. 4-Cohomotopy
$\widehat{H}^{\bullet}_{\mathrm{pos}}$ = positive ordinary cohomology

The Higher Geometry of Physics. [JSSW18-HigStrc][FSS19-RatM][SS20-OrbCoh]

The dictionary:

	physics	**mathematics**
	geometry	topos theory
+	gauge principle	homotopy theory
=		∞-topos theory [Si99][Lu03,09][TV05][Re10]

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$$\mathscr{X}\,,\,\mathscr{A}\;\in\;\underset{\infty\text{-topos}}{\overset{\textbf{21st geometry}}{\mathbf{H}}}\;\simeq\;\underset{\infty\text{-stacks}}{\overset{\textbf{high. geom. spaces}}{\mathrm{Sh}_\infty}}\Big(\underset{\infty\text{-site}}{\overset{\textbf{local model geom.}}{\mathbf{S}}}\Big)$$

(examples follow)

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The key point:

stringy geometry	↔	**higher homotopy**
p-brane charges		$\pi_{p+1}(\mathscr{A}) \in \mathrm{Grp}(\mathbf{H}_0)$
$p_1 \perp p_2$-intersections		higher k-invariants

[FSS13-Bouq, §3]
[HSS18-ADE, §2]
[FSS19-RatM, §7]

Higher gauged sigma-model targets.

[JSSW18-HigStrc][FSS19-RatM][SS20-OrbCoh]

$$\underset{\textbf{representable}}{\overset{\textbf{probe worldvol.}}{\Sigma}} \in \mathbf{S}\,, \qquad \underset{\textbf{higher stack}}{\overset{\textbf{gauged target sp.}}{\mathscr{X}}} \in \mathbf{H}$$

The idea:

Just as an emergent target space $\mathscr{X}$ seen via probes by worldvolumes Σ,

so an ∞-**stack** $\mathscr{X}$ is a space bootstrapped by its gauged system of Σ-plots:

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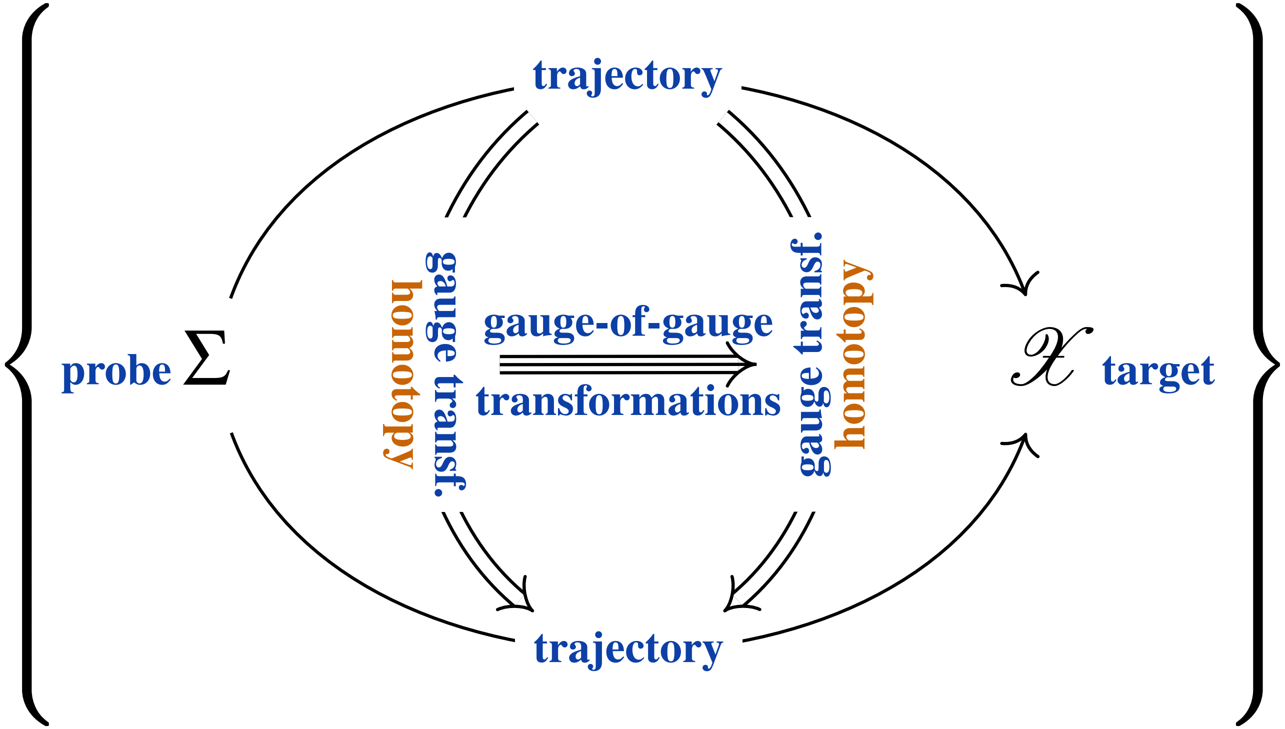
moduli space of
Σ-model fields

$$\mathrm{Map}\big(\Sigma, \mathscr{X}\big) \quad = \quad \Big\{ \cdots \Big\}$$

mapping ∞-stack



trajectory

probe Σ

gauge transf.
homotopy

gauge-of-gauge
transformations

gauge transf.
homotopy

$\mathscr{X}$ **target**

trajectory

Cohesive homotopy theory of Smooth ∞-stacks. [SSS09][Sc13][ScSh14][SS20-OrbCoh]

Key example: Higher geometry locally modeled on $\mathrm{CartSp} = \left\{ \mathbb{R}^n \xrightarrow{\text{smooth}} \mathbb{R}^{n'} \right\}$:

$$\mathbf{H} \;=\; \mathrm{SmthGrpd}_\infty \;:=\; \mathrm{Sh}_\infty\big(\mathrm{CartSp}\big) \;\simeq\; \mathrm{Sh}_\infty\big(\mathrm{SmthMfd}\big)$$

faithfully subsumes all differential topology:

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category of **topological spaces**

cartesian closed category of **compactly generated topological spaces**

cart. clsd. & locl. pres. category of **Δ-generated topological spaces**

quasi-topos of **diffeological spaces**

cohesive topos of **smooth sets**

cohesive ∞-topos of **smooth ∞-groupoids**

$$\begin{tikzcd}
\mathrm{TopSpc} \arrow[r, bend right, "k"'] \arrow[r, phantom, "\perp"] & \mathrm{kTopSpc} \arrow[l, hook', bend right] \arrow[r, bend right, "\mathrm{Cdfflg}"'] \arrow[r, phantom, "\perp"] & \mathrm{DTopSpc} \arrow[l, hook', bend right] \arrow[r, hook, bend right] \arrow[r, phantom, "\perp"] & \mathrm{DfflSpc} \arrow[l, bend right, "\mathrm{Dtplg}"'] \arrow[r, hook, "i_{\sharp 0}"'] & \mathrm{SmthGrpd}_0 \arrow[r, hook, "i_0"'] & \mathrm{SmthGrpd}_\infty \arrow[d, bend right, "\mathrm{Shp}"'] \arrow[d, phantom, "\int"] \\
& & & & & \mathrm{Grpd}_\infty \arrow[u, bend right, "\mathrm{Dsc}"']
\arrow[from=1-1, to=2-6] \arrow[from=1-2, to=2-6] \arrow[from=1-3, to=2-6] \arrow[from=1-4, to=2-6, "\mathrm{Pth}"']
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shape

underlying homotopy types are preserved

form path ∞-groupoids
(sing. simpl. compl.)

base ∞-topos of **bare ∞-groupoids**

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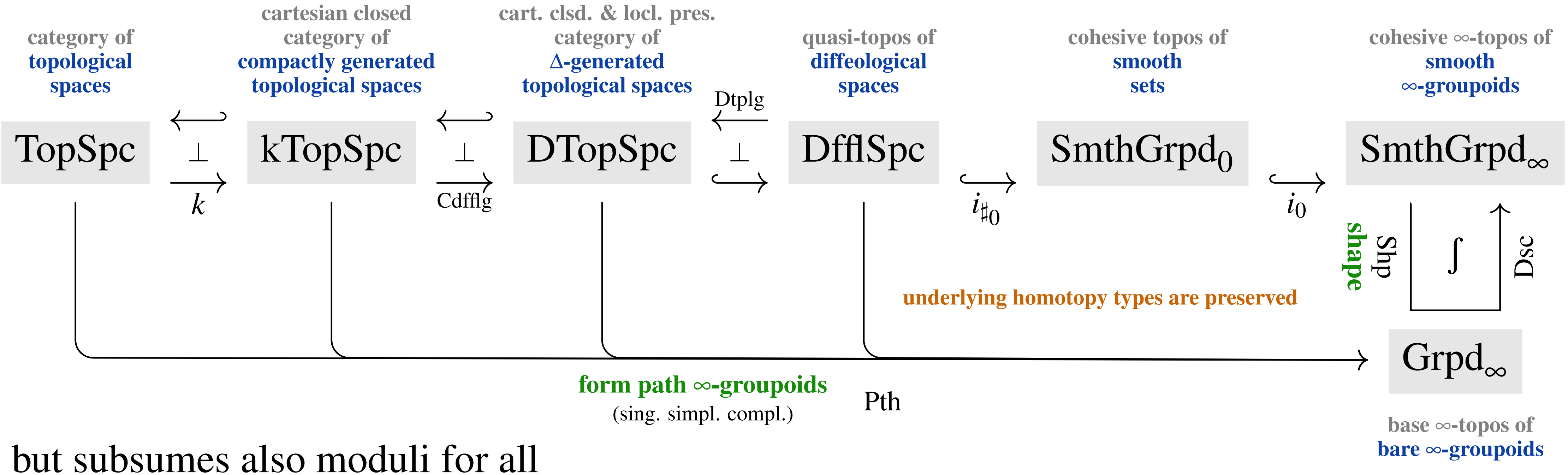

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but subsumes also moduli for all
higher gauge fields ↔ differential cohomology [FSS20-Char, §4.3]

in particular for *abelian* higher gauge fields:

$$\mathrm{Spectra}\big(\mathrm{SmthGrpd}_\infty\big) \;=\; \left\{ \begin{array}{c} \text{abelian generalized differential} \\ \text{cohomology theories} \end{array} \right\}$$

[Sc13] [BNV14],
review in [ADH21]

Higher symmetry. [NSS12-∞Bund][SS20-OrbCoh]

∞-Toposes $\mathbf{H}$ know all about *higher symmetries* – i.e. *n-group symmetries* for $n \in \mathbb{N} \sqcup \{\infty\}$):

higher groups

higher symmetry groups
group ∞-stacks

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\mathrm{Grp}(\mathbf{H}) \arrow[rr, shift right=1.5ex, "\mathbf{B}(-)"', "\text{delooping}"' near end] \arrow[rr, phantom, "\sim"] & & \mathbf{H}^{*/}_{\geq 1} \arrow[ll, shift right=1.5ex, "\Omega(-)"', "\text{looping}"' near end] \\
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higher symmetries

higher $\mathscr{G}$-symmetries
$\mathscr{G}$ ∞-actions

$\mathscr{G}$-gauge field sectors
$\mathscr{G}$-principal ∞-bundles

modulating maps
slice over delooping

$$\begin{tikzcd} \mathscr{G}\,\mathrm{Act}(\mathbf{H}) \arrow[rr, "\sim", "(-)/\!\!/G"'] & & \mathscr{G}\,\mathrm{PrnBdl}(\mathbf{H}) & & \mathbf{H}_{/\mathbf{B}\mathscr{G}} \arrow[ll, "\sim"', "\mathrm{fib}"] \\ G \circlearrowright P \arrow[rr, mapsto] & & \left(\begin{array}{c} P \\ \downarrow \\ P /\!\!/ \mathscr{G} \end{array}\right) & & \left(\begin{array}{c} P /\!\!/ \mathscr{G} \\ \downarrow \\ \mathbf{B}\mathscr{G} \end{array}\right) \arrow[ll, mapsto] \end{tikzcd}$$

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\end{tikzcd}$$

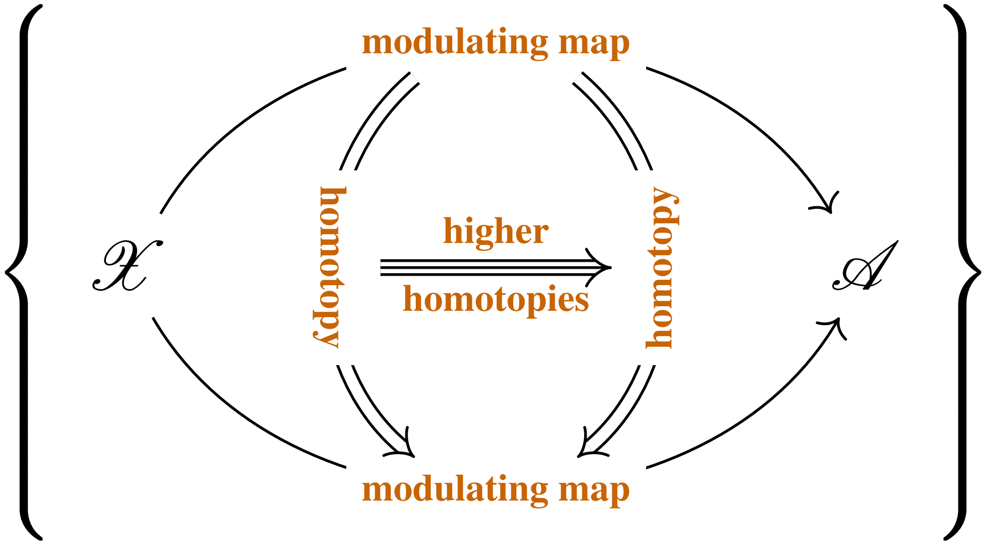
Technical side remark. – The correspondence is enacted by homotopy cartesian squares of this form:

$$\begin{tikzcd}
\overset{\substack{\mathscr{G}\text{-principal} \\ \text{bundle}}}{\text{action } \mathscr{G} \circlearrowright P} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \overset{\substack{\text{universal} \\ \mathscr{G}\text{-principal bundle}}}{* \simeq \mathscr{G} /\!\!/ \mathscr{G}} \arrow[d, "\mathrm{pt}_{\mathbf{B}\mathscr{G}}"] \\
\underset{\text{/ base-stack}}{\text{action quotient-}}\; P /\!\!/ \mathscr{G} \arrow[r, "\vdash P", "\text{cocycle}"'] & \mathbf{B}\mathscr{G} \;\underset{\text{moduli stack}}{\text{universal}}
\end{tikzcd}$$

Higher moduli stacks. [SS20-OrbCoh]

Maps *out* of an ∞-stack $\mathscr{X} \to \mathscr{A}$ encode $\mathscr{A}$-moduli on $\mathscr{X}$:

$$\overset{\text{higher moduli stack of } \mathscr{A}\text{-structures on } \mathscr{X}}{\underset{\text{mapping } \infty\text{-stack}}{\mathrm{Map}(\mathscr{X}, \mathscr{A})}} \quad =$$



Ex.: $\mathrm{Map}(\mathscr{X}, \mathbf{B}\mathscr{G})$ is the moduli ∞-stack of $\mathscr{G}$-principal ∞-bundles on $\mathscr{X}$ (high. gauge field sect.)

Higher moduli stacks.

[SS20-OrbCoh]

Maps *out* of an ∞-stack $\mathscr{X} \to \mathscr{A}$ encode $\mathscr{A}$-moduli on $\mathscr{X}$:

$$\underset{\textbf{mapping }\infty\textbf{-stack}}{\overset{\substack{\textbf{higher moduli stack}\\ \textbf{of } \mathscr{A}\textbf{-structures on } \mathscr{X}}}{\mathrm{Map}\big(\mathscr{X}, \mathscr{A}\big)}} \;=\; \left\{ \begin{tikzcd} \mathscr{X} \arrow[rr, bend left=60, "\textbf{modulating map}", ""{name=U, below}] \arrow[rr, bend right=60, "\textbf{modulating map}"', ""{name=D, above}] & & \mathscr{A} \arrow[from=U, to=D, Rightarrow, bend right=30, "\textbf{homotopy}"', ""{name=L}] \arrow[from=U, to=D, Rightarrow, bend left=30, "\textbf{homotopy}", ""{name=R}] \arrow[from=L, to=R, Rrightarrow, "\textbf{higher}", "\textbf{homotopies}"'] \end{tikzcd} \right\}$$

Ex.: $\mathrm{Map}(\mathscr{X}, \mathbf{B}\mathscr{G})$ is the moduli ∞-stack of $\mathscr{G}$-principal ∞-bundles on $\mathscr{X}$ (high. gauge field sect.)

Fund. Thm. [Lu09]: for $\mathscr{B} \in \mathbf{H}$ also $\mathbf{H}_{/\mathscr{B}}$ is an ∞-topos, whose objects are maps $\mathscr{X} \xrightarrow{c} \mathscr{B}$, with

$$\underset{\textbf{slice mapping }\infty\textbf{-stack}}{\mathrm{Map}\Big(\big(\mathscr{X}, c\big), \big(\mathscr{A}, c'\big)\Big)_{\mathscr{B}}} \;=\; \left\{ \begin{tikzcd} \mathscr{X} \arrow[rr, bend left=40, ""{name=U, below}] \arrow[rr, bend right=40, ""{name=D, above}] \arrow[dr, "c"', ""{name=C, right}] & & \mathscr{A} \arrow[dl, "c'"] \arrow[from=U, to=D, Rightarrow, bend right=40, ""{name=L}] \arrow[from=U, to=D, Rightarrow, bend left=40, ""{name=R}] \arrow[from=L, to=R, Rrightarrow] \arrow[from=U, to=C, Rightarrow, dashed] \arrow[from=D, to=C, Rightarrow, bend left] \\ & \mathscr{B} & \end{tikzcd} \right\}.$$

Ex.: $\mathrm{Map}(\mathscr{X}, \mathbf{B}\mathscr{G})_{\mathbf{B}\mathscr{Q}}$ is moduli ∞-stack of $\mathscr{G}$-*structures* on $\mathscr{Q}$-bundles (gravity/metrics, below)

Higher symmetry – Example: GS-mechanism. [SSS09][FSS12][FSS20-M5Str][SS20-M5GS]

Under
$$\begin{tikzcd} \textbf{Lie groups} & \textbf{diffeological groups} & \textbf{smooth } \infty\textbf{-groups} \\ \mathrm{Grp}(\mathrm{SmthMfd}) \arrow[r, hook] & \mathrm{Grp}(\mathrm{DfflSpc}) \arrow[r, hook, "\mathrm{Grp}(i_{0,\sharp_0})"] & \mathrm{Grp}(\mathrm{SmthGrpd}_\infty) \end{tikzcd}$$

mapping stack into delooping...	**is moduli stack of:**

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Under
$$\begin{tikzcd}
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mapping stack into delooping...	**is moduli stack of:**
$\mathrm{Map}(X, \mathbf{B}\mathrm{U}(1))$	circle bundles

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$$\text{Under}\quad \overset{\text{Lie groups}}{\mathrm{Grp}(\mathrm{SmthMfd})} \hookrightarrow \overset{\text{diffeological groups}}{\mathrm{Grp}(\mathrm{DfflSpc})} \xhookrightarrow{\mathrm{Grp}(i_{0,\sharp_0})} \overset{\text{smooth }\infty\text{-groups}}{\mathrm{Grp}(\mathrm{SmthGrpd}_\infty)}$$

mapping stack into delooping...	**is moduli stack of**:
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$\mathrm{Map}\big(X, \mathbf{B}^{p+1}\mathrm{U}(1)\big)$	bundle p-gerbes

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$\mathrm{Map}\big((\mathrm{X}, \mathbf{c}_1), (\mathbf{B}\mathrm{SO}(n), \mathbf{w}_2)\big)_{\mathbf{B}\mathrm{U}(1)}$ $\simeq \; \mathrm{Map}\big(\mathrm{X}, \mathbf{B}\mathrm{Spin}^{\mathbf{c}_1}(n)\big)$	Spin^c-bundles

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$\mathrm{Map}\big((\mathrm{X}, 0), (\mathbf{B}\mathrm{Spin}(n), \tfrac{1}{2}p_1)\big)_{\mathbf{B}^3\mathrm{U}(1)}$ $\simeq\ \mathrm{Map}\big(\mathrm{X}, \mathbf{B}\mathrm{String}(n)\big)$	String 2-bundles

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Rem.: Different smooth ∞-groups $\mathscr{G}$ may have same shape $\int\mathscr{G}$ discrete ∞-group, e.g:

$$\begin{tikzcd} \mathbf{B}\mathrm{PU}_\omega \arrow[r, "\eta^{\int}"] & B^3\mathbb{Z} \arrow[d, phantom, "\wr|"] & \mathbf{B}^2\mathrm{U}(1) \arrow[l, "\eta^{\int}"'] \\ & K(\mathbb{Z},3) & \end{tikzcd}$$

Higher symmetry – Example: Cohomotopy. Exs. 2.10, 2.16, 4.16 in [FSS20-Char]

Recall that *every connected* space is the classifying space of its loop ∞-group.

E.g., the 4-sphere encodes a rich ∞-higher symmetry

$$S^4 \;\simeq\; B(\Omega S^4) \;\;\in\; \mathrm{Grpd}_\infty\,, \qquad \Omega S^4 \;\in\; \mathrm{Grp}(\mathrm{Grpd}_\infty)\,.$$

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The corresponding moduli are classified by unstable/non-abelian **Cohomotopy**:

$$\underset{\text{4-Cohomotopy}}{\pi^4(\mathrm{X})} \;:=\; \pi_0 \mathrm{Map}\big(\mathrm{X}, S^4\big) \;\simeq\; \pi_0 \mathrm{Map}\big(\mathrm{X}, B(\Omega S^4)\big) \;\simeq\; \underset{\substack{\text{non-abelian}\\ \text{cohomology}}}{H^1}\underset{\substack{\text{with coefficients in}\\ \text{loop ∞-group}}}{\big(\mathrm{X};\, \Omega S^4\big)}$$

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Incidentally, on 10-manifolds X^{10}, 4-Cohomotopy is stably equivalent to tmf^4 (cf below):

$$\begin{tikzcd}
S^4 \arrow[r] & \Omega^\infty \Sigma^\infty S^4 = \Omega^\infty \mathbb{S}^4 \arrow[r, "\beta", "\sim_{\leq 10}"', "\text{Boardman homomorphism}"{yshift=12pt}] & \Omega^\infty \mathrm{tmf}^4 \\
\underbrace{\pi^4\big(\mathbb{R}^{0,1} \times \mathrm{X}^{10}\big)}_{\text{unstable/non-abelian 4-Cohomotopy}} \arrow[r, "\substack{\text{stabilization/}\\ \text{abelianization}}"] & \underbrace{\mathbb{S}^4\big(\mathbb{R}^{0,1} \times \mathrm{X}^{10}\big)}_{\text{stable/abelianized 4-Cohomotopy}} \arrow[r, "\sim"] & \underbrace{\mathrm{tmf}^4\big(\mathbb{R}^{0,1} \times \mathrm{X}^{10}\big)}_{\text{elliptic 4-cohomology}}
\end{tikzcd}$$

Cohesive homotopy theory of Super ∞-stacks. [KhSc17], §3.1.2-3 in [SS20-OrbCoh]

Higher geometry locally modeled on

$$\mathrm{SupCartSp} \;=\; \Big\{ \overset{\text{super-Cartesian space}}{\mathbb{R}^{n|q}} \times \overset{\text{infinitesimal disk}}{\mathbb{D}} \xrightarrow{\text{smooth}} \mathbb{R}^{n'|q'} \times \mathbb{D}' \Big\}$$

$$\begin{tikzcd} \mathrm{Grpd}_\infty \arrow[r, bend right, "\mathrm{Dsc}"'] \arrow[r, phantom, "\int"] & \mathrm{SmthGrpd}_\infty \arrow[l, bend right, "\mathrm{Shp}"'] \arrow[r, bend right, "\mathrm{InfDsc}"'] \arrow[r, phantom, "\Im"] & \underset{\text{super smooth } \infty\text{-groupoids}}{\mathrm{SupSmthGrpd}_\infty \;:=\; \mathrm{Sh}_\infty(\mathrm{SupCartSp})} \arrow[l, bend right, "\mathrm{InfShp}"'] \end{tikzcd}$$

shape **formally étale**

lifts all fundamentals of differential geometry to higher geometry of super ∞-stacks, e.g.:

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shape **formally étale** **super smooth ∞-groupoids**

lifts all fundamentals of differential geometry to higher geometry of super ∞-stacks, e.g.:

A morphism of super ∞-stacks is a **local diffeomorphism** or *formally étale* if its $\Im$-unit is homotopy cartesian:

$$\begin{tikzcd} X \arrow[r, "\eta^{\Im}"] \arrow[d, "f"', "\text{ét}"] \arrow[dr, phantom, "\text{(pb)}"] & \Im X \arrow[d, "\Im f"] \\ Y \arrow[r, "\eta^{\Im}"] & \Im Y \end{tikzcd}$$

The **infinitesimal neighbourhood** around point x in a super ∞-stack X is the x-fiber of the $\Im$-unit:

$$\begin{tikzcd} \mathbb{D}_x X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & * \arrow[d, "\Im x"] \\ X \arrow[r, "\eta^{\Im}"] & \Im X \end{tikzcd}$$

$\mathscr{G}$-Structured super étale ∞-stacks.

§4.2 in [SS20-OrbCoh]

For $V \in \mathrm{Grp}(\mathrm{SupSmthGrpd}_\infty)$ a super group stack such as super-Minkowski $V = \mathbb{R}^{d,1|\mathbf{N}}$:

Def.: A V**-fold** is an étale ∞-stack locally diffeomorphic to $V \overset{\text{ét}}{\twoheadleftarrow} \overset{\boldsymbol{V}\textbf{-atlas}}{U} \overset{\text{eff.epi}}{\twoheadrightarrow} \overset{\boldsymbol{V}\textbf{-fold}}{X}$.

(This cohesive **higher Cartan geometry** is formalized in Modal Homotopy Type Theory: [Sc15, §3][Cherubini (né Wellen) 17].)

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Thm.: A V-fold X naturally has a **frame ∞-bundle** $X \xrightarrow{\vdash \mathrm{Fr}(X)} \mathbf{B}\mathrm{Aut}(\mathbb{D}_{\mathrm{e}} V)$.

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Def.: For $\mathrm{str} : \mathscr{G} \longrightarrow \mathrm{Aut}(\mathbb{D}_{\mathrm{e}}V)$ we have moduli of $\mathscr{G}$**-structures** on V-folds X:

$$\mathrm{Map}\big(X, \mathbf{B}\mathscr{G}\big)_{\mathbf{B}\mathrm{Aut}(\mathbb{D}_{\mathrm{e}}V)} = \left\{ \begin{tikzcd} X \arrow[rr, dashed, "\tau", ""{name=T, below}] \arrow[dr, "\vdash \mathrm{Fr}(X)"', ""{name=F}] & & \mathbf{B}\mathscr{G} \arrow[dl, "\mathbf{B}\mathrm{str}"] \\ & \mathbf{B}\mathrm{Aut}(\mathbb{D}_{\mathrm{e}}V) & \arrow[Rightarrow, from=T, to=F, "\sim"', "\text{vielbein}"] \end{tikzcd} \right\}$$

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So for X a $\mathbb{R}^{d,1|\mathbf{N}}$-fold we have moduli of **super-vielbein fields**:

$$\mathrm{Map}\big(X, \mathbf{B}\mathrm{Spin}(d)\big)_{\mathbf{B}\mathrm{GL}(d,1|\mathbf{N})} = \left\{ \begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "\vdash \mathrm{Fr}(\mathscr{X})"'] & & \mathbf{B}\mathrm{Spin}(d) \arrow[dl] \arrow[from=1-3, to=2-1, Rightarrow, shorten <=20pt, shorten >=20pt, "\sim"', "\text{vielbein}"] \\ & \mathbf{B}\mathrm{GL}(d|\mathbf{N}) & \end{tikzcd} \right\}$$

(This cohesive **higher Cartan geometry** is formalized in Modal Homotopy Type Theory: [Sc15, §3][Cherubini (né Wellen) 17].)

Singular-Cohesive Homotopy Theory of orbi-∞-stacks. [Rezk14][SS20-OrbCoh]

Higher geometry locally modeled on orbi-singularities:

$$\mathrm{Snglrt} \;:=\; \Big\{ \overset{G}{\rotatebox[origin=c]{70}{$\prec$}} \;\Big\vert\; G \text{ fin. group} \Big\} \quad \text{with} \quad \mathrm{Map}\big(\overset{K}{\rotatebox[origin=c]{70}{$\prec$}} , \overset{G}{\rotatebox[origin=c]{70}{$\prec$}} \big) = \left\{ \begin{tikzcd} \overset{K}{\rotatebox[origin=c]{70}{$\prec$}} \arrow[rr, bend left=50, "\text{group homomorph.}"{name=U}] \arrow[rr, bend right=50, "\text{group homomorph.}"'{name=D}] && \overset{G}{\rotatebox[origin=c]{70}{$\prec$}} \arrow[from=U, to=D, Rightarrow, "\wr"', "\text{conjugation}"] \end{tikzcd} \right\}$$

$$\mathbf{H} \;=\; \underset{\text{orbi-singular super-}\infty\text{-stacks}}{\mathrm{GloSupSmthGrpd}_\infty} \;:=\; \mathrm{Sh}_\infty\big(\mathrm{SupCartSp} \times \mathrm{Snglrt}\big)$$

faithfully subsumes proper equivariant homotopy theory:

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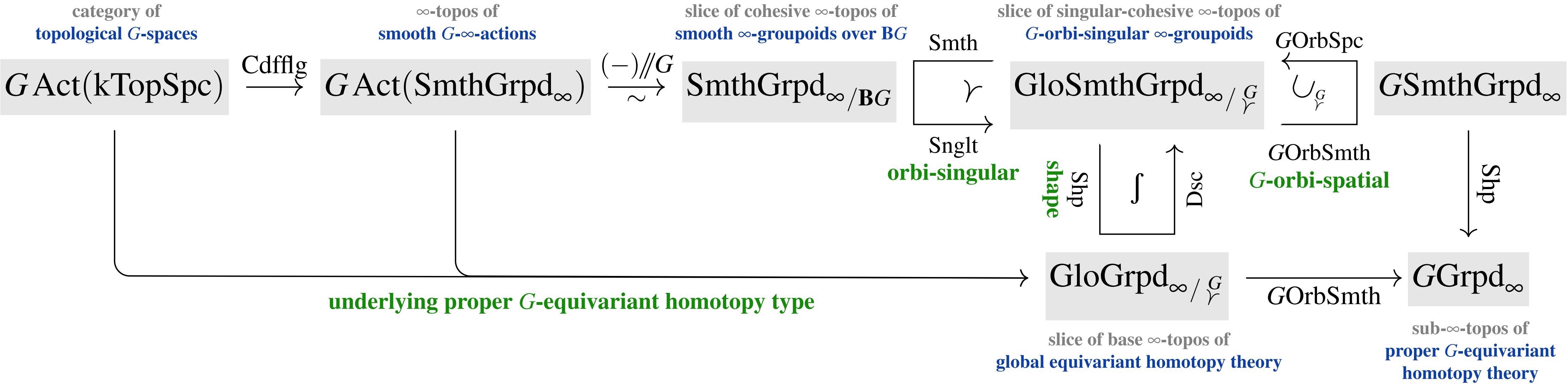
faithfully subsumes proper equivariant homotopy theory:



$$\begin{tikzcd}
G\,\mathrm{Act}(\mathrm{kTopSpc}) \arrow[r, "\mathrm{Cdfflg}"] \arrow[dddrrr, bend right=10] & G\,\mathrm{Act}(\mathrm{SmthGrpd}_\infty) \arrow[r, "(-)/\!\!/G", "\sim"'] \arrow[dddrr, bend right=10] & \mathrm{SmthGrpd}_{\infty/\mathbf{B}G} \arrow[r, bend right, "\mathrm{Snglt}"'] \arrow[r, phantom, "\rotatebox[origin=c]{70}{$\prec$}"] & \mathrm{GloSmthGrpd}_{\infty/\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}} \arrow[l, bend right, "\mathrm{Smth}"'] \arrow[r, bend right, "G\mathrm{OrbSmth}"'] \arrow[r, phantom, "\cup_{\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}}"] \arrow[ddd, bend right, "\mathrm{Shp}"'] \arrow[ddd, phantom, "\int"] & G\mathrm{SmthGrpd}_\infty \arrow[l, bend right, "G\mathrm{OrbSpc}"'] \arrow[ddd, "\mathrm{Shp}"] \\
\\
\\
& & & \mathrm{GloGrpd}_{\infty/\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}} \arrow[uuu, bend right, "\mathrm{Dsc}"'] \arrow[r, "G\mathrm{OrbSmth}"'] & G\mathrm{Grpd}_\infty
\end{tikzcd}$$

category of
topological G-spaces

∞-topos of
smooth G-∞-actions

slice of cohesive ∞-topos of
smooth ∞-groupoids over $\mathbf{B}G$

slice of singular-cohesive ∞-topos of
G-orbi-singular ∞-groupoids

orbi-singular

shape

G-orbi-spatial

underlying proper G-equivariant homotopy type

slice of base ∞-topos of
global equivariant homotopy theory

sub-∞-topos of
proper G-equivariant homotopy theory

Singular-Cohesive Homotopy Theory of orbi-∞-stacks. [Rezk14][SS20-OrbCoh]

Higher geometry locally modeled on orbi-singularities:

$$\mathrm{Snglrt} \;:=\; \left\{ \prec^{G} \;\middle|\; G \text{ fin. group} \right\} \quad \text{with} \quad \mathrm{Map}\big(\prec^{K}, \prec^{G}\big) \;=\; \left\{ \begin{tikzcd} \prec^{K} \arrow[rr, bend left=50, "\text{group homomorph.}"{name=U}] \arrow[rr, bend right=50, "\text{group homomorph.}"'{name=D}] && \prec^{G} \arrow[from=U, to=D, Rightarrow, "\sim"', "\text{conjugation}"] \end{tikzcd} \right\}$$

$$\mathbf{H} \;=\; \underset{\text{orbi-singular super-}\infty\text{-stacks}}{\underbrace{\mathrm{GloSupSmthGrpd}_\infty \;:=\; \mathrm{Sh}_\infty\big(\mathrm{SupCartSp} \times \mathrm{Snglrt}\big)}}$$

faithfully subsumes proper equivariant homotopy theory:

$$\begin{tikzcd}[column sep=large]
G\,\mathrm{Act}(\mathrm{kTopSpc}) \arrow[r, "\mathrm{Cdfflg}"] \arrow[rrrd, rounded corners] & G\,\mathrm{Act}(\mathrm{SmthGrpd}_\infty) \arrow[r, "(-)/\!\!/G", "\sim"'] \arrow[rrd, rounded corners, "\text{underlying proper } G\text{-equivariant homotopy type}"'] & \mathrm{SmthGrpd}_{\infty/\mathbf{B}G} \arrow[r, shift right, "\mathrm{Snglt}"', "\text{orbi-singular}"] \arrow[r, phantom, "\prec"] & \mathrm{GloSmthGrpd}_{\infty/\prec^{G}} \arrow[l, shift right, "\mathrm{Smth}"'] \arrow[d, shift right, "\mathrm{Shp}"', "\text{shape}"] \arrow[d, phantom, "\int"] \arrow[r, shift right, "G\mathrm{OrbSmth}"', "G\text{-orbi-spatial}"] \arrow[r, phantom, "\cup_{\prec^{G}}"] & G\mathrm{SmthGrpd}_\infty \arrow[l, hook', shift right, "G\mathrm{OrbSpc}"'] \arrow[d, "\mathrm{Shp}"] \\
& & & \mathrm{GloGrpd}_{\infty/\prec^{G}} \arrow[u, shift right, "\mathrm{Dsc}"'] \arrow[r, "G\mathrm{OrbSmth}"'] & G\mathrm{Grpd}_\infty
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Thm. (§4.1 in [SS20-OrbCoh])
Good orbifolds covered by $G \circlearrowright \mathrm{X}$ are equivalently $\prec(\mathrm{X}/\!\!/G) \;\in\; \mathbf{H} = \mathrm{GloSmthGrpd}_\infty$
and their proper-equivariant homotopy type is:

$$\textstyle\int \prec(\mathrm{X}/\!\!/G) \;\simeq\; G\mathrm{OrbSpc}\Big(\int\big(\mathrm{X}^{(-)}\big)\Big) \;\in\; G\mathrm{Grpd}_\infty \xhookrightarrow{\;\mathrm{Dsc}G\mathrm{OrbSpc}\;} \mathbf{H}_{/\prec^{G}}\,.$$

Equivariant classifying spaces. [SS22-EquBun]

Theorem.
If $G \circlearrowright \Gamma$ is a G-equivariant Hausdorff-topological group with $\int \Gamma$ truncated, then

$$\underset{\text{equivariant classifying shape}}{B_G\Gamma} \;:=\; \underset{G\text{-orbi-spatial}}{\cup_{\underset{\prec}{G}}} \; \underset{\text{shape}}{\int} \; \underset{\text{orbi-singular}}{\prec} \; \underset{\text{delooping}}{\mathbf{B}} \; \Gamma \quad \in \; \mathbf{H}\,.$$

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classifies G-equivariant Γ-principal bundles on G-orbifolds $\mathscr{X} \simeq \curlyvee(\mathrm{X} /\!\!/ G) \in \mathbf{H}_{/\overset{G}{\curlyvee}}$:

$$\big(G\,\mathrm{Equv}\,\Gamma\,\mathrm{PrnBdl}_{\mathrm{X}}\big)_{/\sim_{\mathrm{iso}}} \;\simeq\; \tau_0\,\mathrm{Map}\big(\mathscr{X}, B_G\Gamma\big)_{\overset{G}{\curlyvee}}$$

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Theorem.

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classifies G-equivariant Γ-principal bundles on G-orbifolds $\mathscr{X} \simeq \Upsilon(\mathrm{X} /\!\!/ G) \in \mathbf{H}_{/\underset{\Upsilon}{G}}$:

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and its equivariant homotopy groups are given by non-abelian group cohomology:

$$\underset{\text{equivariant homotopy groups}}{\pi_\bullet^H} \big(\underset{\text{equivariant classifying shape}}{B_G \Gamma}\big) \;\simeq\; \underset{\text{non-abelian group cohomology}}{H^{1-\bullet}_{\mathrm{Grp}}} \big(\underset{\text{isotropy group}}{H}; \underset{\text{shape of structure group}}{\int \Gamma}\big)$$

B-fields on orbi-orientifolds. [SS22-EquBun]

Specifically, for

$$1 \to N \hookrightarrow G \twoheadrightarrow \mathbb{Z}_2 \to 1$$

and

$$\mathbb{Z}_2 \circlearrowright \mathrm{PU}^{\mathrm{gr}}_{\boldsymbol{\omega}} \;\in\; G\,\mathrm{Act}\big(\mathrm{Grp}(\mathrm{kTopSpc})\big)$$

the graded projective unitary group acted on by complex conjugation, the G-orbi-space

$$B_G\big(\mathrm{PU}^{\mathrm{gr}}_{\boldsymbol{\omega}}\big) \;\in\; \mathbf{H}$$

classifies type **IIA** ***B*-fields on *G*-orbi-orientifolds**

$$\left\{ \begin{array}{c} \text{type IIA } B_2\text{-fields on} \\ G\text{-orbi-orientifold } \mathscr{X} \end{array} \right\}_{\sim_{\text{gauge}}} \;\simeq\; \tau_0 \mathrm{Map}\Big(\mathscr{X}\,, B_G\big(\mathrm{PU}^{\mathrm{gr}}_{\boldsymbol{\omega}}\big) \Big)_{\overset{G}{\curvearrowleft}}\,.$$

with $\quad \pi^H_n\Big(B_G\big(\mathrm{PU}_{\boldsymbol{\omega}}\big)\Big) \;\simeq\; H^{3-n}_{\mathrm{Grp}}\big(H;\mathbb{Z}\big)\,,\quad$ reproducing [UrLü14, Thm. 15.17].

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with $\pi^H_n\Big(B_G\big(\mathrm{PU}_\omega\big)\Big) \;\simeq\; H^{3-n}_{\mathrm{Grp}}\big(H;\mathbb{Z}\big)\,,$ reproducing [UrLü14, Thm. 15.17].

Philosophical question:

But *why* coefficients like $B_G\mathrm{PU}_\omega$? *Are there god-given coefficients?* $\rightsquigarrow$

God-given coefficients – Tate spheres.

For any line object $\mathbb{A}^1$ there are the *Tate spheres* (e.g. [VRO07, Rem. 2.22])

$$S^n_{\mathrm{Tate}} := \mathrm{cof}\big(\mathbb{A}^n \backslash \{0\} \hookrightarrow \mathbb{A}^n\big) \quad \in \mathbf{H}$$

God-given coefficients – Tate spheres. Def. 5.19 in [SS20-OrbCoh]

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Specifically for $\mathbb{A}^1 := \mathbb{R}^1 \in \mathrm{SmthGrpd}_\infty$ (incidentally $\int \simeq \mathrm{Loc}^{\mathbb{R}^1} \,:\, \mathbf{H} \to \mathbf{H}$)
we have the **smooth Tate spheres**

$$S^n_{\mathrm{Tate}} \;:=\; \mathrm{cof}\big(\mathbb{R}^n \backslash \{0\} \hookrightarrow \mathbb{R}^n\big) \quad \in \; \mathbf{H}\,.$$

Their shape is that of the ordinary *n*-spheres ([SS20-OrbCoh, Ex. 5.21]):

$$\int S^n_{\mathrm{Tate}} \;\simeq\; S^n \quad \in \; \mathrm{Grpd}_\infty \; \overset{\mathrm{Dsc}}{\hookrightarrow} \; \mathbf{H}$$

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More generally, for any

$$G \circlearrowright V \quad \in \; G\mathrm{Act}(\mathrm{VectorSpaces}_{\mathbb{R}}) \hookrightarrow G\mathrm{Act}(\mathrm{SmthMfd}) \hookrightarrow G\mathrm{Act}(\mathbf{H})$$

we have the **orbi-smooth V-Tate spheres** ([SS20-OrbCoh, Ex. 5.27])

$$\int \curlyvee \big(S^V_{\mathrm{Tate}} /\!\!/ G\big) \quad \in \; \mathbf{H}_{/\overset{G}{\curlyvee}}\,.$$

Example: The ADE-equivariant 4-sphere. §5.1 in [HSS18-ADE]; §3 in [SS19-TadCnc]

The G_{ADE}-equivariant Tate $\mathbf{4}$-sphere has equivariant homotopy type of the $\mathbf{4}$-representation sphere:

$$\textstyle\int \prec \big(S^{\mathbf{4}}_{\mathrm{Tate}} /\!\!/ G_{\mathrm{ADE}} \big) \underset{\substack{\text{stereogr.}\\ \text{project.}}}{\simeq} S\Big(\mathbb{R} \oplus \overset{\overset{G_{\mathrm{ADE}}}{\curvearrowright}}{\mathbb{H}} \Big) \;\in\; G_{\mathrm{ADE}}\mathrm{Grpd}_\infty \xhookrightarrow{\mathrm{Dsc}\; G_{\mathrm{ADE}}\mathrm{OrbSpc}} \mathbf{H}_{/ \prec G_{\mathrm{ADE}}}$$

Example: The ADE-equivariant 4-sphere. §5.1 in [HSS18-ADE]; §3 in [SS19-TadCnc]

Consider the left multiplication action of $\mathrm{Sp}(1) = S(\mathbb{H})$ on the quaternions $\mathbb{H}$:

$$\mathrm{Sp}(1) \circlearrowright \mathbb{H} \;\simeq\; \mathrm{SU}(2)_L \circlearrowright \mathbb{C}^2 \;\simeq\; \mathrm{Spin}(3)_L \circlearrowright \mathbb{R}^4 \,.$$

The finite subgroups have a famous ADE-classification:

Label	$G_{\mathrm{ADE}} \underset{\mathrm{fin}}{\subset} \mathrm{SU}(2)$	**Order**	**Name**
$\mathbb{A}_n$	$\mathbb{Z}_{n+1}$	n	Cyclic
$\mathbb{D}_{n+4}$	$2\mathrm{D}_{n+2}$	$4(n+2)$	Binary dihedral
$\mathbb{E}_6$	$2\mathrm{T}$	24	Binary tetrahedral
$\mathbb{E}_7$	$2\mathrm{O}$	48	Binary octahedral
$\mathbb{E}_8$	$2\mathrm{I}$	120	Binary icosahedral

Denote the restricted representation by $\mathbf{4} \;:=\; G_{\mathrm{ADE}} \circlearrowright \mathbb{R}^4 \quad \in \; \mathrm{RO}(G_{\mathrm{ADE}})\,.$

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Example: Super-Minkowski orbifolds.

Thm. 4.3 in [HSS18-ADE]

Theorem. Classification of subgroup actions of $\mathrm{Pin}^+(10,1) \circlearrowright \mathbb{R}^{10,1|\mathbf{32}}$ which fix $\geq 1/4$th of $\mathbf{32}$ such that all non-trivial subgroups have the same bosonic fixed locus:

Black brane species	**BPS**	**Fixed locus in $\mathbb{R}^{10,1\vert\mathbf{32}}$**	**Type of singularity in $\mathbb{R}^{10,1}$**	**Intersection law**							
				$\simeq$	$\mathbb{R}^{1,1}$	$\oplus$	$\mathbb{R}^4$	$\oplus$	$\mathbb{R}^4$	$\oplus$	$\mathbb{R}^1$
Elementary brane species			**Simple singularities**								
MO9	$1/2$	$\mathbb{R}^{9,1\vert\mathbf{16}}$	$\mathbb{Z}_2$	$=$	——		——		——		$(\mathbb{Z}_2)_{\mathrm{HW}}$
MO5	$1/2$	$\mathbb{R}^{5,1\vert 2\cdot\mathbf{8}}$	$\mathbb{Z}_2$	$\overset{\Delta}{\subset}$	——		——		$(\mathbb{Z}_2)_R$	$\times$	$(\mathbb{Z}_2)_{\mathrm{HW}}$
MO1	$1/2$	$\mathbb{R}^{1,1\vert 16\cdot\mathbf{1}}$	$\mathbb{Z}_2$	$\overset{\Delta}{\subset}$	——		$(\mathbb{Z}_2)_L$	$\times$	$(\mathbb{Z}_2)_R$	$\times$	$(\mathbb{Z}_2)_{\mathrm{HW}}$
MK6	$1/2$	$\mathbb{R}^{6,1\vert\mathbf{16}}$	$\mathbb{Z}_{n+1}, 2\mathbb{D}_{n+2},$ $2T, 2O, 2I$	$\subset$	——		——		$\mathrm{SU}(2)_R$		——
M2	$1/2 \;=\; 8/16$	$\mathbb{R}^{2,1\vert 8\cdot\mathbf{2}}$	$\mathbb{Z}_2$	$\overset{\Delta}{\subset}$	——		$\mathrm{SU}(2)_L$	$\times$	$\mathrm{SU}(2)_R$		——
M2	$6/16$	$\mathbb{R}^{2,1\vert 6\cdot\mathbf{2}}$	$\mathbb{Z}_{n+3}$	$\overset{\Delta}{\subset}$	——		$\mathrm{SU}(2)_L$	$\times$	$\mathrm{SU}(2)_R$		——
M2	$5/16$	$\mathbb{R}^{2,1\vert 5\cdot\mathbf{2}}$	$2\mathbb{D}_{n+2},$ $2T, 2O, 2I$	$\overset{\Delta}{\subset}$	——		$\mathrm{SU}(2)_L$	$\times$	$\mathrm{SU}(2)_R$		——
M2	$1/4 \;=\; 4/16$	$\mathbb{R}^{2,1\vert 4\cdot\mathbf{2}}$	$2\mathbb{D}_{n+2},$ $2O, 2I$	$\overset{(\mathrm{id},\tau)}{\subset}$	——		$\mathrm{SU}(2)_L$	$\times$	$\mathrm{SU}(2)_R$		——

$$\begin{tikzcd}
\mathbb{R}^{10,1} \arrow[r, phantom, "\simeq_{\mathbb{R}}"] & \mathbb{R}^{1,1} \oplus & \mathbb{R}^4 \arrow[loop above, "\mathrm{SU}(2)_L"] \oplus & \mathbb{R}^4 \arrow[loop above, "\mathrm{SU}(2)_R"] \oplus & \mathbb{R} \arrow[loop above, "(\mathbb{Z}_2)_{\mathrm{HW}}"]
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Example: Super-Minkowski orbifolds. Thm. 4.3 in [HSS18-ADE]

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MO1	$1/2$	$\mathbb{R}^{1,1\vert 16\cdot\mathbf{1}}$	$\mathbb{Z}_2$	$\overset{\Delta}{\subset}$	——		$(\mathbb{Z}_2)_L$	$\times$	$(\mathbb{Z}_2)_R$	$\times$	$(\mathbb{Z}_2)_{\mathrm{HW}}$
MK6	$1/2$	$\mathbb{R}^{6,1\vert\mathbf{16}}$	$\mathbb{Z}_{n+1}, 2\mathbb{D}_{n+2},$ $2T, 2O, 2I$	$\subset$	——				$\mathrm{SU}(2)_R$		——
M2	$1/2 = 8/16$	$\mathbb{R}^{2,1\vert 8\cdot\mathbf{2}}$	$\mathbb{Z}_2$	$\overset{\Delta}{\subset}$	——		$\mathrm{SU}(2)_L$	$\times$	$\mathrm{SU}(2)_R$		——
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M2	$5/16$	$\mathbb{R}^{2,1\vert 5\cdot\mathbf{2}}$	$2\mathbb{D}_{n+2},$ $2T, 2O, 2I$	$\overset{\Delta}{\subset}$	——		$\mathrm{SU}(2)_L$	$\times$	$\mathrm{SU}(2)_R$		——
M2	$1/4 = 4/16$	$\mathbb{R}^{2,1\vert 4\cdot\mathbf{2}}$	$2\mathbb{D}_{n+2},$ $2O, 2I$	$\overset{(\mathrm{id},\tau)}{\subset}$	——		$\mathrm{SU}(2)_L$	$\times$	$\mathrm{SU}(2)_R$		——

$$\begin{tikzcd}
\mathbb{R}^{10,1} & \simeq_{\mathbb{R}} & \mathbb{R}^{1,1} & \oplus & \mathbb{R}^4 \arrow[loop above, "\mathrm{SU}(2)_L"] & \oplus & \mathbb{R}^4 \arrow[loop above, "\mathrm{SU}(2)_R"] & \oplus & \mathbb{R} \arrow[loop above, "(\mathbb{Z}_2)_{\mathrm{HW}}"]
\end{tikzcd}$$

yields super-Minkowski orbifolds, e.g.:

$$\mathscr{X}_{\mathrm{MK6}\perp\mathrm{K3}} = \mathbb{R}^{6,1|\mathbf{16}} \times \Upsilon(\mathbb{T}^{\mathbf{4}} /\!\!/ \mathbb{Z}_2^A) \in \mathbf{H}_{/ \Upsilon \mathbb{Z}_2^A}$$

J-twisted proper orbifold Cohomotopy. Def. 5.28 in [SS20-OrbCoh]

Let $\mathscr{X}$ be an $\mathbb{R}^{d,1|\mathbf{N}}$-orbifold

with $\mathrm{Spin}(n)$-structure

$$\begin{tikzcd}
\mathscr{X} \arrow[rr, "\rotatebox[origin=c]{70}{$\prec$}\tau"] \arrow[dr, "\vdash \mathrm{Fr}(\mathscr{X})"'] & \arrow[d, phantom, "\overset{\sim}{\Swarrow}\ \text{vielbein}"] & \rotatebox[origin=c]{70}{$\prec$}\mathbf{B}\mathrm{Spin}(n) \arrow[dl] \\
& \rotatebox[origin=c]{70}{$\prec$}\mathbf{B}\mathrm{GL}(d,1|\mathbf{N}) &
\end{tikzcd}$$

Def. ([SS20-OrbCoh, Ex. 5.29]) **J-twisted proper orbifold Cohomotopy** of $(\mathscr{X}, \tau)$ is :

$$\overset{\substack{\text{tangentially J-twisted}\\ \text{proper orbifold Cohomotopy}}}{\pi^{\int \rotatebox[origin=c]{70}{$\prec$} \tau}(\mathscr{X})} \;:=\; \pi_0 \mathrm{Map}\Big(\mathscr{X}, \overbrace{\textstyle\int \rotatebox[origin=c]{70}{$\prec$}\big(S^{\mathbf{n}}_{\mathrm{Tate}} /\!\!/ \mathrm{Spin}(n)\big)}^{\substack{\text{proper equivariant homotopy type}\\ \text{of canonical n-sphere}}}\Big)_{\underset{\text{proper equivariant tangential twist}}{\int \rotatebox[origin=c]{70}{$\prec$}\mathbf{B}\mathrm{Spin}(d)}}$$

$$= \pi_0 \left\{ \begin{tikzcd}
\mathscr{X} \arrow[rr, dashed, "\substack{\text{cocycle}\\ c}"] \arrow[dr, "\substack{\eta^{\int} \circ \rotatebox[origin=c]{70}{$\prec$}\tau \\ \text{twist}}"'] & \arrow[d, phantom, "\overset{\sim}{\Swarrow}"] & \int \rotatebox[origin=c]{70}{$\prec$}\big(S^{n}_{\mathrm{Tate}} /\!\!/ \mathrm{Spin}(n)\big) \arrow[dl, "\text{local coeffs.}"] \\
& \int \rotatebox[origin=c]{70}{$\prec$}\mathbf{B}\mathrm{Spin}(n) &
\end{tikzcd} \right\}$$

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$$\begin{tikzcd}
\mathscr{X} \arrow[rr, "\prec\tau"] \arrow[dr, "\vdash\mathrm{Fr}(\mathscr{X})"'] \arrow[dr, phantom, shift left=3ex, "\overset{\sim}{\Leftarrow}\ \text{vielbein}"] & & \prec\mathbf{B}\mathrm{Spin}(n) \arrow[dl] \\
& \prec\mathbf{B}\mathrm{GL}(d,1|\mathbf{N}) &
\end{tikzcd}$$

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$$\underset{\substack{\text{tangentially J-twisted}\\ \text{proper orbifold Cohomotopy}}}{\pi^{\int\prec\tau}(\mathscr{X})} \;:=\; \pi_0\,\mathrm{Map}\Big(\mathscr{X}\,,\,\overbrace{\textstyle\int\prec\big(S^{\mathbf{n}}_{\mathrm{Tate}}/\!\!/\mathrm{Spin}(n)\big)}^{\substack{\text{proper equivariant homotopy type}\\ \text{of canonical n-sphere}}}\Big)_{\underset{\text{proper equivariant tangential twist}}{\int\prec\mathbf{B}\mathrm{Spin}(d)}}$$

$$= \pi_0\left\{\begin{tikzcd}
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& \int\prec\mathbf{B}\mathrm{Spin}(n) &
\end{tikzcd}\right\}$$

In the case that $\mathscr{X} = \mathrm{X}$ is smooth (i.e. a manifold), this reduces to **J-twisted Cohomotopy**: [FSS19-HypH, Def. 2.1][FSS20-Char, Ex. 2.41], see also [Cru03, Lem. 5.2].

Orbifold Cohomotopy at and away from the singularities. Thm. 5.16 in [SS20-OrbCoh]

$$\begin{tikzcd}
\mathscr{X} \arrow[rr, dashed, "\text{cocycle}"] \arrow[dr, swap, "\eta^{\int} \circ \rotatebox[origin=c]{70}{$\prec$}\tau"] & \arrow[d, phantom, "\Downarrow\wr"] & \int \rotatebox[origin=c]{70}{$\prec$}\big(S^{\mathbf{n}|\mathbf{N}}_{\mathrm{Tate}} /\!\!/ \mathrm{Spin}(n)\big) \arrow[dl] \\
& \int \rotatebox[origin=c]{70}{$\prec$}\mathbf{B}\mathrm{Spin}(n) &
\end{tikzcd}$$

Key Structure Theorem:

In limiting cases this reduces to:

(a) equiv Cohomotopy in RO-deg **n**

(b) tangent J-twisted Cohomotopy

tangentially J-twisted orbifold Cohomotopy

$$\begin{tikzcd}
& \pi^{\int \rotatebox[origin=c]{70}{$\prec$}\tau}(\mathscr{X}) \arrow[dl, rightsquigarrow, "\textbf{on flat orbifolds}", swap, "\mathscr{X} = \rotatebox[origin=c]{70}{$\prec$}(\mathbb{T}^d /\!\!/ G)"] \arrow[dr, rightsquigarrow, "\textbf{on smooth orbifolds}", swap, "\mathscr{X} = X"] & \\
\pi^{\mathbf{n}}_G(\mathbb{T}^d) & & \pi^{\tau}(\mathrm{X})
\end{tikzcd}$$

(a) **(b)**

[HSS18-ADE]
[SS19-TadCnc]
[BSS19-FrcBrn]
[SS20-M5GS]

equivariant Cohomotopy in RO-degree n

[FSS19-HypH]
[FSS19-M5WZ]
[FSS20-M5Str]
[SS20-M5Anom]
[FSS20-GSAnom]

tangentially J-twisted Cohomotopy

$$\begin{tikzcd}
\rotatebox[origin=c]{70}{$\prec$}(\mathbb{T}^d /\!\!/ G) \arrow[rr, dashed, "\text{cocycle}"] \arrow[dr, swap, "\rotatebox[origin=c]{70}{$\prec$}\tau"] & \arrow[d, phantom, "\Swarrow\wr"] & \int \rotatebox[origin=c]{70}{$\prec$}(S^{\mathbf{n}} /\!\!/ G) \arrow[dl] \\
& \underset{\rotatebox[origin=c]{70}{$\prec$}}{G} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{X} \arrow[rr, dashed, "\text{cocycle}"] \arrow[dr, swap, "\int\tau"] & \arrow[d, phantom, "\Swarrow\wr"] & S^n /\!\!/ \mathrm{Spin}(n) \arrow[dl] \\
& B\mathrm{Spin}(n) &
\end{tikzcd}$$

higher orbi-geometry

orbifold cohomology

I – Proper Orbifold Cohomotopy

II – for M-Theory

Hypothesis H

M5-brane charges in flat orbi-orientifolds

Future historians may judge the late 20th century as a time when theorists were like children playing on the seashore, diverting themselves with the smoother pebbles or prettier shells of superstrings

while the great ocean of M-theory lay undiscovered before them.

M. Duff (1998)

closing sentence in:

M. Duff:
The Theory Formerly Known as Strings
Scient Amer 1998

Flux-quantization in non-abelian cohomology theory. [FSS20-Char]

Non-perturbative completion of a theory of charged objects
with flux densities satisfying Bianchi identities
involves choosing a non-abelian cohomology theory A
whose character image enforces these Bianchi identities:

Flux-quantization in non-abelian cohomology theory.

[FSS20-Char]

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involves choosing a non-abelian cohomology theory A
whose character image enforces these Bianchi identities:

charge “lattice”
non-abelian cohomology

non-abelian
de Rham cohomology

Whitehead L_∞-algebra

$$\begin{tikzcd}
A(\mathrm{X}) \arrow[r, "\text{non-abelian character}", "\mathrm{ch}_A"'] & H_{\mathrm{dR}}\big(\mathrm{X};\, \mathfrak{l}A\big) \\
{[c]} \arrow[r, mapsto] & \left[\Big\{ F^{(a)}_{r_a} \in \Omega^{r_a}_{\mathrm{dR}}(X) \Big\}_{1 \leq a \leq \dim[\pi_\bullet(A), \mathbb{R}]} \,\middle|\, d\, F^{(a)}_{r_a} \;=\; P_{r_a}\Big(\{F^{(b)}_{r_b}\}_{b \leq a} \Big) \right]
\end{tikzcd}$$

quantized flux/charge
class in A-cohomology

field strengths/flux densities
L_∞-algebra valued diff forms

higher Bianchi identities
L_∞-flatness condition

Flux-quantization in non-abelian cohomology theory. [FSS20-Char]

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The choice of A is a **hypothesis** about the correct non-perturbative completion.

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quantized flux/charge
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field strengths/flux densities
L_∞-algebra valued diff forms

higher Bianchi identities
L_∞-flatness condition

The choice of A is a **hypothesis** about the correct non-perturbative completion.

Given such a choice, the moduli ∞-stack of fields is a differential refinement of A:

$$\mathscr{A} = \widehat{A} \in \mathrm{SmthGrpd}_\infty$$

Hypothesis H. [Sa13, §2.5][FSS16-RatCoh][FSS19-HypH][SS21-MF]

Fact: The Bianchi identity of the **type IIA RR/B-fields**
is that enforced by the Whitehead L_∞-algebra of twisted KU-theory

$$H_{\mathrm{dR}}\big(\mathrm{X}, \mathfrak{l}(\mathrm{KU} /\!\!/ B\mathrm{U}(1))\big) \;\simeq\; \left\{ \left(\begin{array}{c} \{F_{2k}\}_k \\ H_3 \end{array} \right) \in \Omega^{\bullet}_{\mathrm{dR}}(\mathrm{X}) \;\middle|\; \begin{array}{ll} d\,F_{2k} & = H_3 \wedge F_{2k-2} \\ d\,H_3 & = 0 \end{array} \right\}_{\sim_{\mathrm{conc}}}$$

The evident **hypothesis** here is the proposal by Minasian/Moore/Witten/Bouwknegt/Mathai:

The type IIA RR/B-field is flux-quantized in twisted K-theory.

It *must* be flux-quantized in something at least close, such as orbifold KR-theory.

Hypothesis H. [Sa13, §2.5][FSS16-RatCoh][FSS19-HypH][SS21-MF]

Fact: The Bianchi identity of the **type IIA RR/B-fields** is that enforced by the Whitehead L_∞-algebra of twisted KU-theory

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It *must* be flux-quantized in something at least close, such as orbifold KR-theory.

Fact: The Bianchi identity of the **M-theory *C*-field** is that enforced by the Whitehead L_∞-algebra of 4-Cohomotopy:

$$H_{\mathrm{dR}}\big(\mathrm{X}, \mathfrak{l}S^4\big) \;\simeq\; \left\{ \begin{pmatrix} G_7 \\ G_4 \end{pmatrix} \in \Omega^\bullet_{\mathrm{dR}}(\mathrm{X}) \;\middle|\; \begin{aligned} d\,G_7 &= -\tfrac{1}{2} G_4 \wedge G_4 \\ d\,G_4 &= 0 \end{aligned} \right\}_{\sim_{\mathrm{conc}}}$$

The evident **hypothesis** here [Sa13, §2.5] we called *Hypothesis H*:

The M-theory C-field is flux-quantized in 4-Cohomotopy.

It *must* be charge-quantized in something at least close, such as J-twisted orbifold Cohomotopy.

Cohomotopy unifies M5/M2-brane charge. [HSS18-ADE, p. 17][FSS19-HypH, §2.3]

Cohomotopy is dual to Homotopy:
$$\underset{\substack{\textbf{4-co-homotopy group}\\ \textbf{of spheres}}}{\pi^4\left(S^k\right)} \quad \simeq \quad \underset{\substack{\textbf{homotopy groups}\\ \textbf{of 4-sphere}}}{\pi_k\left(S^k\right)}$$

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Homotopy groups of the 4-sphere:

$k =$	1	2	3	4	5	6	7	8	9 $\cdots$
$\pi_k\left(S^4\right)$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	all torsion $\cdots\cdots$

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Homotopy groups of the 4-sphere:

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$\Longrightarrow$

4-Cohomotopy measures integer charges exactly around black BPS M2/M5-branes:

$$\pi^4\Big(\underset{\substack{\text{black M5-brane}\\ \text{spacetime}}}{\widehat{\mathrm{AdS}_7 \times S^4}}\Big) \simeq \pi^4\left(S^4\right) \simeq \pi_4\left(S^4\right) \simeq \mathbb{Z}$$

$$\pi^4\Big(\underset{\substack{\text{black M2-brane}\\ \text{spacetime}}}{\widehat{\mathrm{AdS}_4 \times S^7}}\Big) \simeq \pi^4\left(S^7\right) \simeq \pi_7\left(S^4\right) \simeq \mathbb{Z} \oplus \mathbb{Z}_{12}$$

Approximating Cohomotopy by K-theory.

[BSS19-FrcBrn][SS19-TadCnc][SS21-MF]

J-twisted orbifold Cohomotopy
around an orbi-singularity

$\pi_G^{\mathbf{4}}\left(\mathbb{R}^{\mathbf{4}}_{\mathrm{cpt}}\right)$

un-stable/
non-linear!

equivariant generalized
cohomologies in RO-degree 4

equivariant generalized
cohomologies of the point

representation
rings

Approximating Cohomotopy by K-theory. [BSS19-FrcBrn][SS19-TadCnc][SS21-MF]

J-twisted orbifold Cohomotopy around an orbi-singularity

un-stable/ non-linear!

stabilization/ linearization

equivariant stable Cohomotopy

$$\begin{tikzcd}
\pi^{\mathbf{4}}_G\big(\mathbb{R}^{\mathbf{4}}_{\mathrm{cpt}}\big) \arrow[d, "\Sigma^\infty"] & \\
\mathbb{S}^{\mathbf{4}}_G\big(S^{\mathbf{4}}\big) \arrow[r, equal] & \mathbb{S}^0_G
\end{tikzcd}$$

equivariant generalized cohomologies in RO-degree 4

equivariant generalized cohomologies of the point

representation rings

Approximating Cohomotopy by K-theory.

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J-twisted orbifold Cohomotopy around an orbi-singularity

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$$\begin{tikzcd}
\pi^{\mathbf{4}}_{G}\left(\mathbb{R}^{\mathbf{4}}_{\mathrm{cpt}}\right) \arrow[d, "\Sigma^{\infty}", "\text{stabilization/ linearization}"'] & & & \\
\mathbb{S}^{\mathbf{4}}_{G}\left(S^{\mathbf{4}}\right) \arrow[r, equal] & \mathbb{S}^{0}_{G} \arrow[r, equal, "\text{[BP72][Se74]}", "\text{[De06][Gui06]}"'] & \mathrm{R}_{\mathbb{F}_1}(G) \arrow[r, equal, "\text{[Se71][tD79]}"] & A_G
\end{tikzcd}$$

equivariant stable Cohomotopy

Burnside ring

equivariant generalized cohomologies in RO-degree 4

equivariant generalized cohomologies of the point

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Approximating Cohomotopy by K-theory. [BSS19-FrcBrn][SS19-TadCnc][SS21-MF]

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\mathbb{S}^{\mathbf{4}}_{G}\big(S^{\mathbf{4}}\big) \arrow[r, equal] \arrow[d] & \mathbb{S}^{0}_{G} \arrow[r, equal, "\text{[BP72][Se74]}", "\text{[De06][Gui06]}"'] \arrow[d, "\beta"] & \mathrm{R}_{\mathbb{F}_1}(G) \arrow[r, equal, "\text{[Se71][tD79]}"] \arrow[d, "\otimes_{\mathbb{F}_1}\mathbb{R}"] & A_G \\
\mathrm{KO}^{\mathbf{4}}_{G}\big(S^{\mathbf{4}}\big) \arrow[r, equal] & \mathrm{KO}^{0}_{G} \arrow[r, equal] & \mathrm{R}_{\mathbb{R}}(G) &
\end{tikzcd}$$

equivariant stable Cohomotopy

Burnside ring

Hurewicz-Boardman homomorphism
(initiality of $\mathbb{S}$)

equivariant orth. K-theory

equivariant generalized cohomologies in RO**-degree 4**

equivariant generalized cohomologies of the point

representation rings

Approximating Cohomotopy by K-theory. [BSS19-FrcBrn][SS19-TadCnc][SS21-MF]

$$\begin{tikzcd}
\text{J-twisted orbifold Cohomotopy around an orbi-singularity} & \pi^{\mathbf{4}}_G\big(\mathbb{R}^{\mathbf{4}}_{\mathrm{cpt}}\big) \arrow[d, "\Sigma^\infty", "\text{stabilization/ linearization}"'] & \text{un-stable/ non-linear!} & & & \\
\text{equivariant stable Cohomotopy} & \mathbb{S}^{\mathbf{4}}_G\big(S^{\mathbf{4}}\big) \arrow[r, equal] \arrow[d, "\text{Hurewicz-Boardman homomorphism (initiality of } \mathbb{S})"] & \mathbb{S}^0_G \arrow[r, equal, "\text{[BP72][Se74]}", "\text{[De06][Gui06]}"'] \arrow[d, "\beta"] & \mathrm{R}_{\mathbb{F}_1}(G) \arrow[r, equal, "\text{[Se71][tD79]}"] \arrow[d, "\otimes_{\mathbb{F}_1}\mathbb{R}"] & A_G & \text{Burnside ring} \\
\text{equivariant orth. K-theory} & \mathrm{KO}^{\mathbf{4}}_G\big(S^{\mathbf{4}}\big) \arrow[r, equal] \arrow[d, "\text{further extension of scalars}"] & \mathrm{KO}^0_G \arrow[r, equal] \arrow[d, "\otimes_{\mathbb{R}}\mathbb{C}"] & \mathrm{R}_{\mathbb{R}}(G) \arrow[d, "\otimes_{\mathbb{R}}\mathbb{C}"] & & \\
\text{equivariant complex K-theory} & \mathrm{KU}^{\mathbf{4}}_G\big(S^{\mathbf{4}}\big) \arrow[r, equal] & \mathrm{KU}^0_G \arrow[r, equal] & \mathrm{R}_{\mathbb{C}}(G) & & \\
 & \text{equivariant generalized cohomologies in RO-degree 4} & \text{equivariant generalized cohomologies of the point} & \text{representation rings} & &
\end{tikzcd}$$

Approximating Cohomotopy by K-theory.

[BSS19-FrcBrn][SS19-TadCnc][SS21-MF]

$$\begin{tikzcd}
\pi^{\mathbf{4}}_G\big(\mathbb{R}^{\mathbf{4}}_{\mathrm{cpt}}\big) \arrow[d, "\Sigma^\infty"] & & & \\
\mathbb{S}^{\mathbf{4}}_G\big(S^{\mathbf{4}}\big) \arrow[r, equal] \arrow[d] & \mathbb{S}^0_G \arrow[r, equal, "{[BP72][Se74]}", "{[De06][Gui06]}"'] \arrow[d, "\beta"] & \mathrm{R}_{\mathbb{F}_1}(G) \arrow[r, equal, "{[Se71][tD79]}"] \arrow[d, "\otimes_{\mathbb{F}_1}\mathbb{R}"] & A_G \arrow[ddl, rounded corners] \\
\mathrm{KO}^{\mathbf{4}}_G\big(S^{\mathbf{4}}\big) \arrow[r, equal] \arrow[d] & \mathrm{KO}^0_G \arrow[r, equal] \arrow[d, "\otimes_{\mathbb{R}}\mathbb{C}"] & \mathrm{R}_{\mathbb{R}}(G) \arrow[d, "\otimes_{\mathbb{R}}\mathbb{C}"] & \\
\mathrm{KU}^{\mathbf{4}}_G\big(S^{\mathbf{4}}\big) \arrow[r, equal] & \mathrm{KU}^0_G \arrow[r, equal] & \mathrm{R}_{\mathbb{C}}(G) &
\end{tikzcd}$$

J-twisted orbifold Cohomotopy around an orbi-singularity

un-stable/ non-linear!

stabilization/ linearization

equivariant stable Cohomotopy

Burnside ring

Hurewicz-Boardman homomorphism
(initiality of $\mathbb{S}$)

equivariant orth. K-theory

further extension of scalars

equivariant complex K-theory

linearize virtual G-sets of fractional M-branes to virtual G-representations of fractional D-branes

equivariant generalized cohomologies in RO-degree 4

equivariant generalized cohomologies of the point

representation rings

Approximating Cohomotopy by K-theory. [BSS19-FrcBrn][SS19-TadCnc][SS21-MF]

$$\begin{tikzcd}
\text{J-twisted orbifold Cohomotopy around an orbi-singularity} & \pi^{\mathbf{4}}_G\big(\mathbb{R}^{\mathbf{4}}_{\mathrm{cpt}}\big) \arrow[d, "\Sigma^\infty", "\text{stabilization/ linearization}"'] & \text{un-stable/ non-linear!} & & & \\
\text{equivariant stable Cohomotopy} & \mathbb{S}^{\mathbf{4}}_G\big(S^{\mathbf{4}}\big) \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\text{Hurewicz-Boardman homomorphism (initiality of } \mathbb{S})"] & \mathbb{S}^0_G \arrow[r, equal, "{[BP72][Se74]}", "{[De06][Gui06]}"'] \arrow[d, "\beta"] & \mathrm{R}_{\mathbb{F}_1}(G) \arrow[r, equal, "{[Se71][tD79]}"] \arrow[d, "\otimes_{\mathbb{F}_1}\mathbb{R}"] & A_G \arrow[ddl, bend left, "\text{linearize virtual } G\text{-sets of fractional M-branes to virtual } G\text{-representations of fractional D-branes}"] & \text{Burnside ring} \\
\text{equivariant orth. K-theory} & \mathrm{KO}^{\mathbf{4}}_G\big(S^{\mathbf{4}}\big) \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\text{further extension of scalars}"] & \mathrm{KO}^0_G \arrow[r, equal] \arrow[d, "\otimes_{\mathbb{R}}\mathbb{C}"] & \mathrm{R}_{\mathbb{R}}(G) \arrow[d, "\otimes_{\mathbb{R}}\mathbb{C}"] & & \\
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 & \text{equivariant generalized cohomologies in RO-degree 4} & \text{equivariant generalized cohomologies of the point} & \text{representation rings} & &
\end{tikzcd}$$

Rem. [FSS20-Char, (353)].

The Boardman homomorphism exhibits exactly the identification $G_4 \mapsto F_4$ of [DMW00]:

$$\begin{tikzcd}
\pi^4 \arrow[rr, "\Sigma^\infty", "\text{stabilization / linearization}"'] \arrow[d, bend right, "\mathrm{ch}"'] & & \mathbb{S}^4 \arrow[rr, "\beta", "\text{Boardman homomorphism}"'] & & \mathrm{KU}^4 \arrow[r, equal, "\text{Bott per.}"'] & \mathrm{KU} \\
(G_4, G_7) & \mapsto & G_4 & \mapsto & F_4 & \\
 & & \text{C-field flux} & & \text{RR-field flux} &
\end{tikzcd}$$

Approximating Cohomotopy by K-theory. [BSS19-FrcBrn][SS19-TadCnc][SS21-MF]

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\pi^{\mathbf{4}}_G\big(\mathbb{R}^{\mathbf{4}}_{\mathrm{cpt}}\big) \arrow[d, "\Sigma^\infty", "\text{stabilization/ linearization}"'] & & & \\
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\mathrm{KO}^{\mathbf{4}}_G\big(S^{\mathbf{4}}\big) \arrow[r, equal] \arrow[d] & \mathrm{KO}^0_G \arrow[r, equal] \arrow[d, "\otimes_{\mathbb{R}}\mathbb{C}"] & \mathrm{R}_{\mathbb{R}}(G) \arrow[d, "\otimes_{\mathbb{R}}\mathbb{C}"] & \\
\mathrm{KU}^{\mathbf{4}}_G\big(S^{\mathbf{4}}\big) \arrow[r, equal] & \mathrm{KU}^0_G \arrow[r, equal] & \mathrm{R}_{\mathbb{C}}(G) &
\end{tikzcd}$$

J-twisted orbifold Cohomotopy around an orbi-singularity

un-stable/ non-linear!

equivariant stable Cohomotopy

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(G_4, G_7) \arrow[r, mapsto] & G_4 \arrow[r, mapsto] & F_4 &
\end{tikzcd}$$

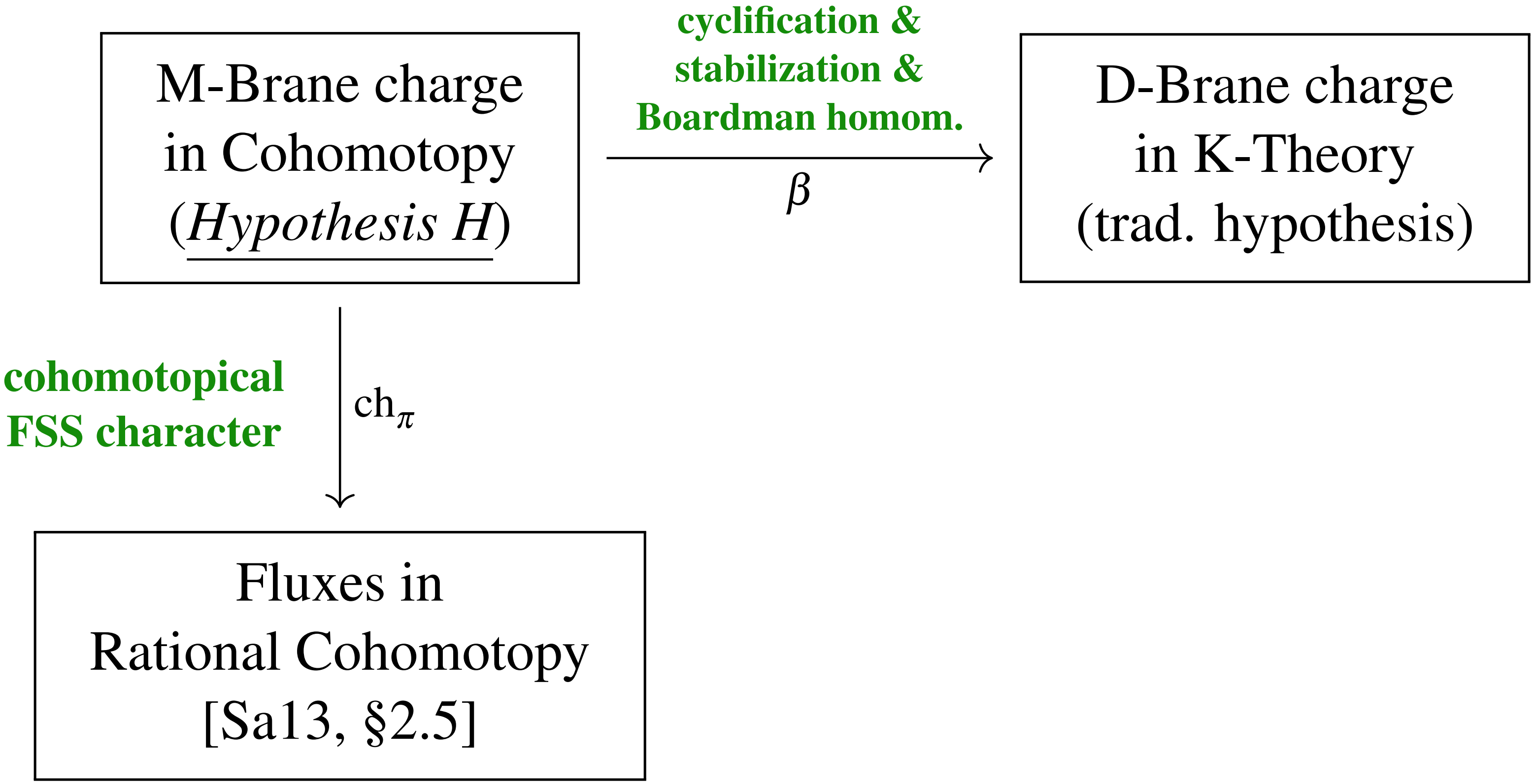
C-field flux **RR-field flux**

However, β (and [DMW00]) misses the double dimensional reductions $G_4 \mapsto H_3$ and $G_7 \mapsto F_6$; these do appear from Cohomotopy via *cyclification* ([FSS16-RatCoh][FSS16-TDual][BSS19-RatSt]).

Approximating Cohomotopy by K-theory.

[BSS19-FrcBrn][SS19-TadCnc][SS21-MF]

These two approximations...



M-Brane charge
in Cohomotopy
(*Hypothesis H*)

cyclification &
stabilization &
Boardman homom.

β

D-Brane charge
in K-Theory
(trad. hypothesis)

cohomotopical
FSS character

ch_π

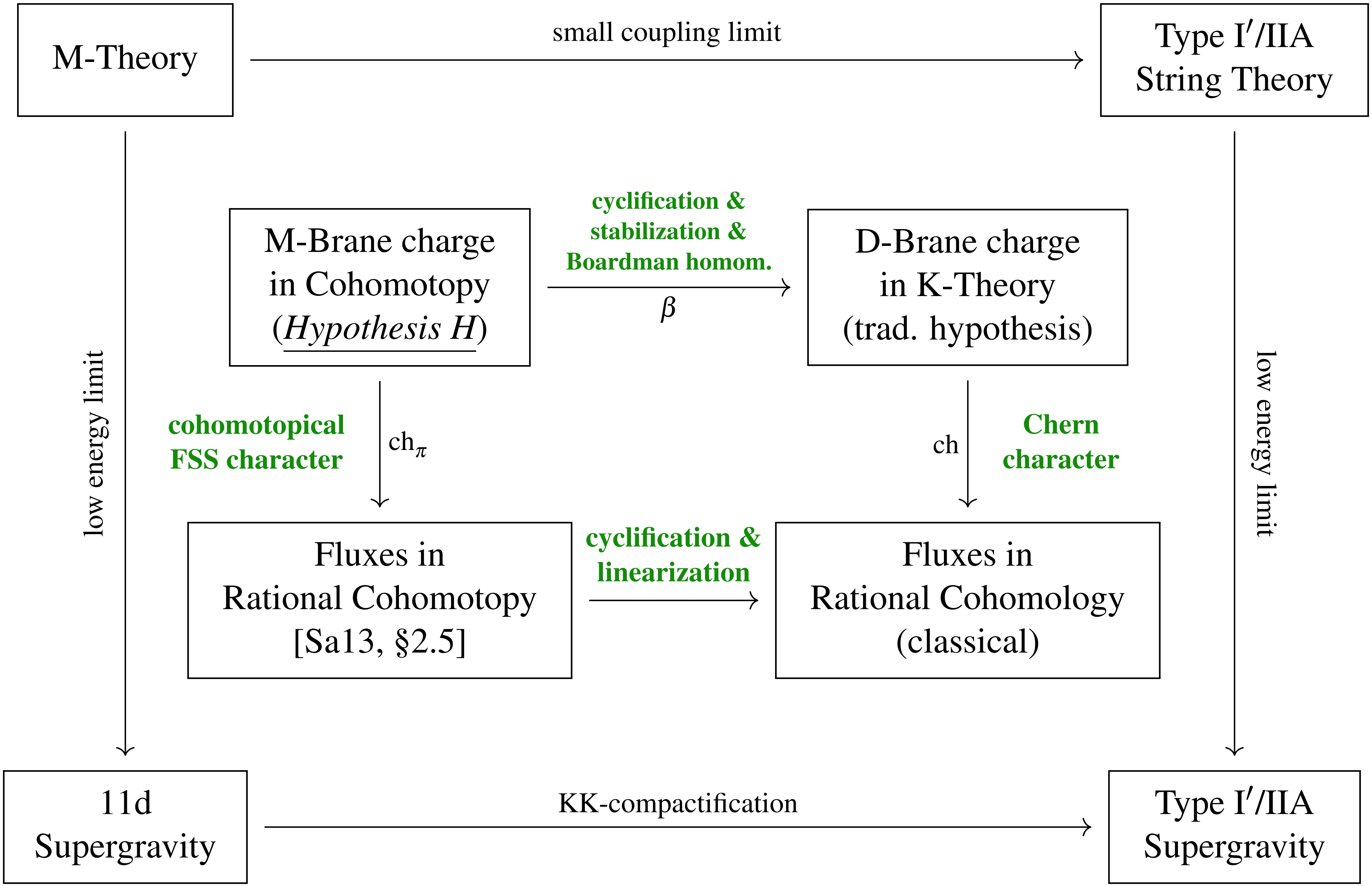
Fluxes in
Rational Cohomotopy
[Sa13, §2.5]

Approximating Cohomotopy by K-theory. [BSS19-FrcBrn][SS19-TadCnc][SS21-MF]

These two approximations are compatible with each other:

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{M-Brane charge}\\ \text{in Cohomotopy}\\ (\underline{\textit{Hypothesis H}})\end{array}} \arrow[r, "\begin{array}{c}\textbf{cyclification \&}\\ \textbf{stabilization \&}\\ \textbf{Boardman homom.}\end{array}", "\beta"'] \arrow[d, "\mathrm{ch}_\pi", "\begin{array}{c}\textbf{cohomotopical}\\ \textbf{FSS character}\end{array}"'] & \boxed{\begin{array}{c}\text{D-Brane charge}\\ \text{in K-Theory}\\ \text{(trad. hypothesis)}\end{array}} \arrow[d, "\mathrm{ch}"', "\begin{array}{c}\textbf{Chern}\\ \textbf{character}\end{array}"] \\
\boxed{\begin{array}{c}\text{Fluxes in}\\ \text{Rational Cohomotopy}\\ \text{[Sa13, §2.5]}\end{array}} \arrow[r, "\begin{array}{c}\textbf{cyclification \&}\\ \textbf{linearization}\end{array}"] & \boxed{\begin{array}{c}\text{Fluxes in}\\ \text{Rational Cohomology}\\ \text{(classical)}\end{array}}
\end{tikzcd}$$

Approximating Cohomotopy by K-theory. [BSS19-FrcBrn][SS19-TadCnc][SS21-MF]



M-Theory

small coupling limit

Type I′/IIA
String Theory

low energy limit

M-Brane charge
in Cohomotopy
(*Hypothesis H*)

cyclification &
stabilization &
Boardman homom.

β

D-Brane charge
in K-Theory
(trad. hypothesis)

low energy limit

cohomotopical
FSS character

ch_π

ch

Chern
character

Fluxes in
Rational Cohomotopy
[Sa13, §2.5]

cyclification &
linearization

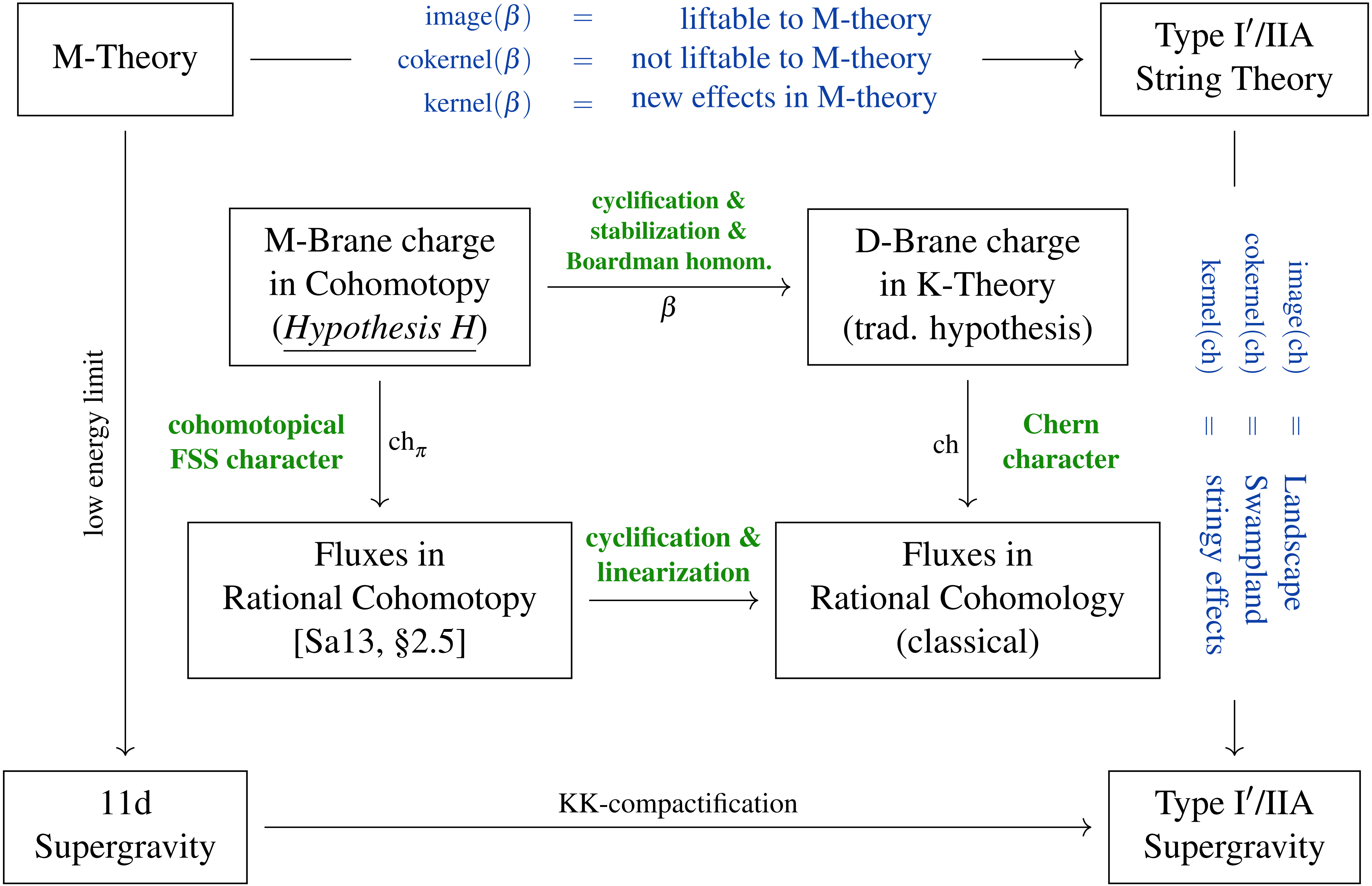
Fluxes in
Rational Cohomology
(classical)

11d
Supergravity

KK-compactification

Type I′/IIA
Supergravity

Approximating Cohomotopy by K-theory. [BSS19-FrcBrn][SS19-TadCnc][SS21-MF]



M-Theory

image(β) = liftable to M-theory
cokernel(β) = not liftable to M-theory
kernel(β) = new effects in M-theory

Type I$'$/IIA String Theory

low energy limit

M-Brane charge in Cohomotopy (*Hypothesis H*)

cyclification & stabilization & Boardman homom.

β

D-Brane charge in K-Theory (trad. hypothesis)

image(ch) = Landscape
cokernel(ch) = Swampland
kernel(ch) = stringy effects

cohomotopical FSS character

ch_π

ch

Chern character

Fluxes in Rational Cohomotopy [Sa13, §2.5]

cyclification & linearization

Fluxes in Rational Cohomology (classical)

11d Supergravity

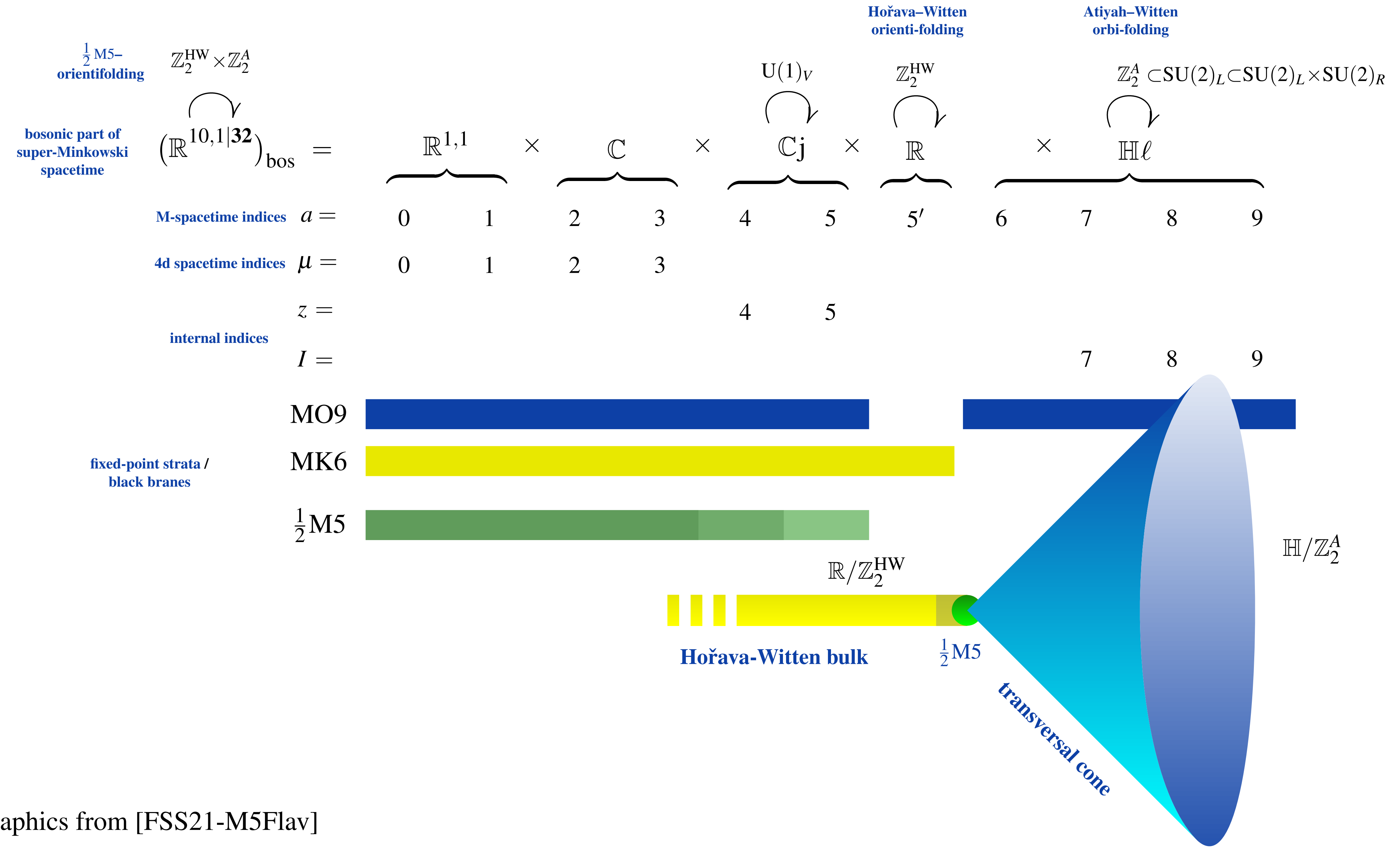
KK-compactification

Type I$'$/IIA Supergravity

Example: M5-tadpole cancellation. [SS19-TadCnc]

$$\pi^{\int \curlyvee (\vdash \mathrm{Fr})}\left(\mathbb{R}^{5,1|\mathbf{8}} \times \curlyvee\left(\mathbb{T}^{\mathbf{4}} /\!\!/ \mathbb{Z}_2^A\right)\right) \simeq \pi^{\mathbf{4}}_{\mathbb{Z}_2^A}\left(\mathbb{T}^{\mathbf{4}}\right) \simeq \left\{ N_{\mathrm{M5}}\cdot \mathbf{2} - N_{\mathrm{MO5}}\cdot \mathbf{1} \;\middle|\; \begin{array}{l} N_{\mathrm{MO5}} \in \{0,\cdots,16\} \leftarrow \text{non-additive!} \\ N_{\mathrm{M5}} \;\; \in \mathbb{Z}\end{array}\right\}$$

heterotic M5-charges according to Hypothesis H — **equivariant 4-Cohomotopy** — **coincides with informal folklore** [SS19-TadCnc, Cor. 4.6]



bosonic part of super-Minkowski spacetime

$$\left(\mathbb{R}^{10,1|\mathbf{32}}\right)_{\mathrm{bos}} = \underbrace{\mathbb{R}^{1,1}} \times \underbrace{\mathbb{C}} \times \underbrace{\mathbb{C}\mathrm{j}} \times \underbrace{\mathbb{R}} \times \underbrace{\mathbb{H}\ell}$$

M-spacetime indices	$a =$	0	1	2	3	4	5	$5'$	6	7	8	9
4d spacetime indices	$\mu =$	0	1	2	3							
internal indices	$z =$					4	5					
	$I =$									7	8	9

graphics from [FSS21-M5Flav]

Example: M5**-tadpole cancellation.** [SS19-TadCnc]

$$\pi^{\int \rotatebox[origin=c]{70}{$\prec$}(\vdash \mathrm{Fr})}\Big(\mathbb{R}^{5,1|\mathbf{8}} \times \rotatebox[origin=c]{70}{$\prec$}\big(\mathbb{T}^{\mathbf{4}} /\!\!/ \mathbb{Z}_2^A\big)\Big) \;\simeq\; \pi^{\mathbf{4}}_{\mathbb{Z}_2^A}\big(\mathbb{T}^{\mathbf{4}}\big) \;\simeq\; \left\{ N_{\mathrm{M5}} \cdot \mathbf{2} - N_{\mathrm{MO5}} \cdot \mathbf{1} \;\middle|\; \begin{array}{l} N_{\mathrm{MO5}} \in \{0, \cdots, 16\} \xleftarrow{\;} \text{non-additive!} \\ N_{\mathrm{M5}} \;\;\, \in \mathbb{Z} \end{array} \right\}$$

heterotic M5-charges according to Hypothesis H **equivariant 4-Cohomotopy** **coincides with informal folklore** [SS19-TadCnc, Cor. 4.6]

We now explain this example in more detail. $\rightsquigarrow$

Background: The black M5 in 11d SuGra.

Fact. [AFFH99, 5.2] [dMFF12, §8.3]:

All BPS black M5-brane solutions of 11D supergravity are $^1/_2$BPS of this form:

$$\begin{tikzcd}
& \boxed{\begin{array}{c}\text{black M5}\\ (^1/_2\ \text{BPS})\end{array}} \arrow[dl, "\text{near horizon limit}"'] \arrow[dr, "\text{far-horizon limit}"] & \\
\mathrm{AdS}_7 \times S^{\mathbf{4}} /\!\!/ G_{\mathrm{ADE}} & r \lesssim \ell_{\mathrm{Pl}} \sqrt[3]{N_{\mathrm{M5}}} \ll r & \mathbb{R}^{5,1} \times \mathbb{R}^1 \times \mathbb{R}^{\mathbf{4}} /\!\!/ G_{\mathrm{ADE}}
\end{tikzcd}$$

with $S^{\mathbf{4}} = S(\mathbb{R} \oplus \overset{G_{\mathrm{ADE}} \curvearrowright}{\mathbb{H}})$
as above

as in the
above Thm.

**throat geometry intersecting
orbifold ADE-singularity**

**flat orbifold
with ADE-singularity**

Background: The black M5 in 11d SuGra.

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near horizon limit

far-horizon limit

as in the above Thm.

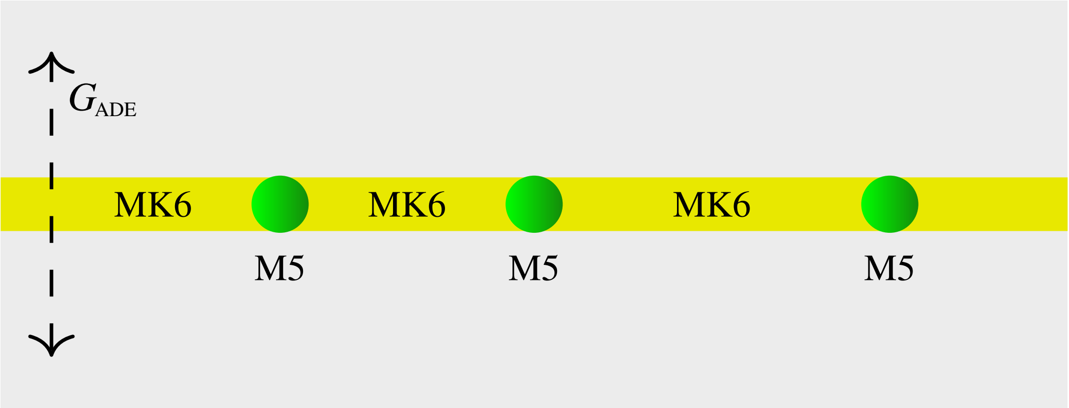
$$\begin{tikzcd}
 & \boxed{\text{black M5 } ({}^1\!/_2 \text{ BPS})} \arrow[dl, "\text{near horizon limit}"'] \arrow[dr, "\text{far-horizon limit}"] & \\
\mathrm{AdS}_7 \times S^4 /\!\!/ G_{\mathrm{ADE}} & & \mathbb{R}^{5,1} \times \mathbb{R}^1 \times \mathbb{R}^4 /\!\!/ G_{\mathrm{ADE}}
\end{tikzcd}$$

$$r \lesssim \ell_{\mathrm{Pl}} \sqrt[3]{N_{\mathrm{M5}}} \ll r$$

throat geometry intersecting orbifold ADE-singularity

flat orbifold with ADE-singularity

Consequence 1: Black BPS M5-branes are always domain walls inside an MK6-singularity:



E.g.: [ZHTV14, §3.1] [Fa17, §3.3.1]

Background: The black M5 in 11d SuGra.

Fact. [AFFH99, 5.2] [dMFF12, §8.3]:
All BPS black M5-brane solutions of 11D supergravity are $1/2$ BPS of this form:

$$\begin{tikzcd} & \boxed{\begin{matrix}\text{black M5}\\ (1/2\ \text{BPS})\end{matrix}} \arrow[dl, "r\ \lesssim"'] \arrow[dr, "\ll\ r"] & \\ \mathrm{AdS}_7 \times S^{\mathbf{4}} /\!\!/ G_{\mathrm{ADE}} & \ell_{\mathrm{Pl}}\sqrt[3]{N_{\mathrm{M5}}} & \mathbb{R}^{5,1} \times \mathbb{R}^1 \times \mathbb{R}^{\mathbf{4}} /\!\!/ G_{\mathrm{ADE}} \end{tikzcd}$$

with $S^{\mathbf{4}} = S(\mathbb{R}\oplus \overset{G_{\mathrm{ADE}}\curvearrowright}{\mathbb{H}})$ as above

near horizon limit

far-horizon limit

as in the above Thm.

throat geometry intersecting orbifold ADE-singularity

flat orbifold with ADE-singularity

Consequence 2:

Individual M5-branes $N_{\mathrm{M5}} \sim \mathscr{O}(1)^3$ have Planck scale thickness $r \sim \ell_{\mathrm{Pl}}$
hence their **near geometry make no sense** as solutions of M-theory
due to infinite + unknown tower of higher curvature quantum corrections $\sim \big(\overset{\text{Planck area}}{\ell_{\mathrm{Pl}}^2} \cdot \overset{\text{Riem. curvature}}{R}\big)^k$.

Conversely:
The **M-meaningful far geometry** yields flat super-orbifold spacetimes
where all curvature is crammed into orbi-singularities
so that also all quantum effects must be hiding inside orbi-singularities –
plausibly detected as charges measured in a proper orbifold cohomology theory!

Background: The black M5 in 11d SuGra.

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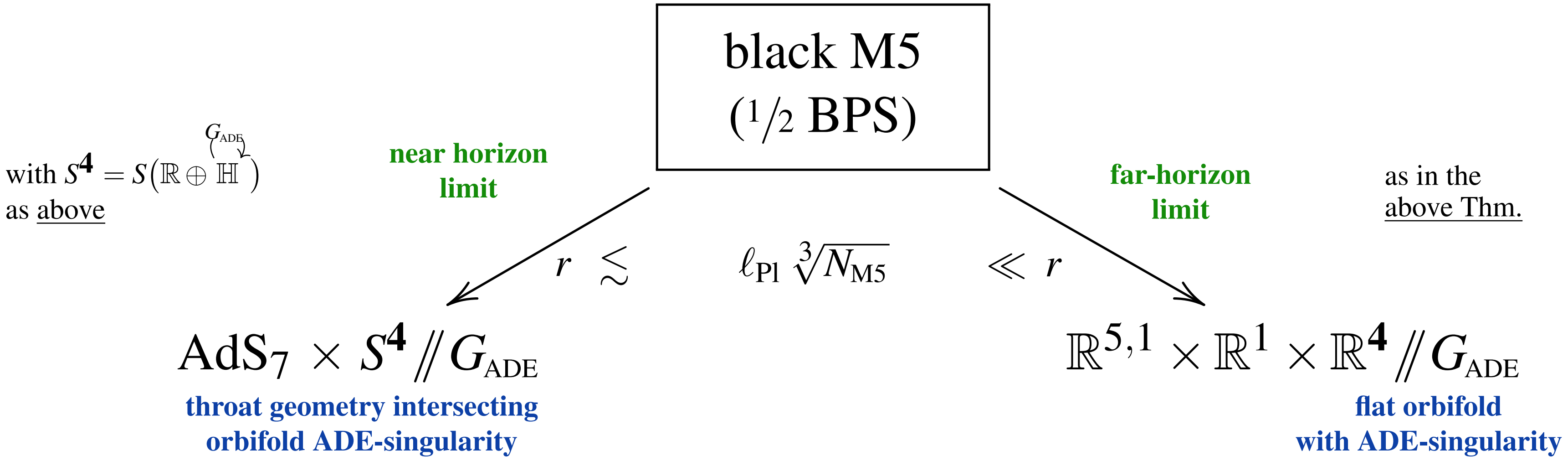


black M5
($1/2$ BPS)

with $S^{\mathbf{4}} = S(\mathbb{R} \oplus \overset{G_{\mathrm{ADE}}}{\overset{\curvearrowright}{\mathbb{H}}})$
as above

near horizon limit

far-horizon limit

as in the above Thm.

$r \lesssim \ell_{\mathrm{Pl}} \sqrt[3]{N_{\mathrm{M5}}} \ll r$

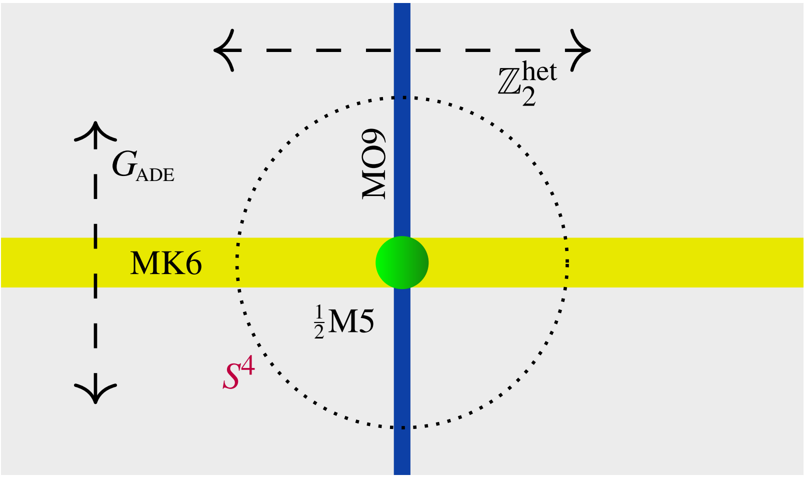
$\mathrm{AdS}_7 \times S^{\mathbf{4}} /\!\!/ G_{\mathrm{ADE}}$

throat geometry intersecting orbifold ADE-singularity

$\mathbb{R}^{5,1} \times \mathbb{R}^1 \times \mathbb{R}^{\mathbf{4}} /\!\!/ G_{\mathrm{ADE}}$

flat orbifold with ADE-singularity

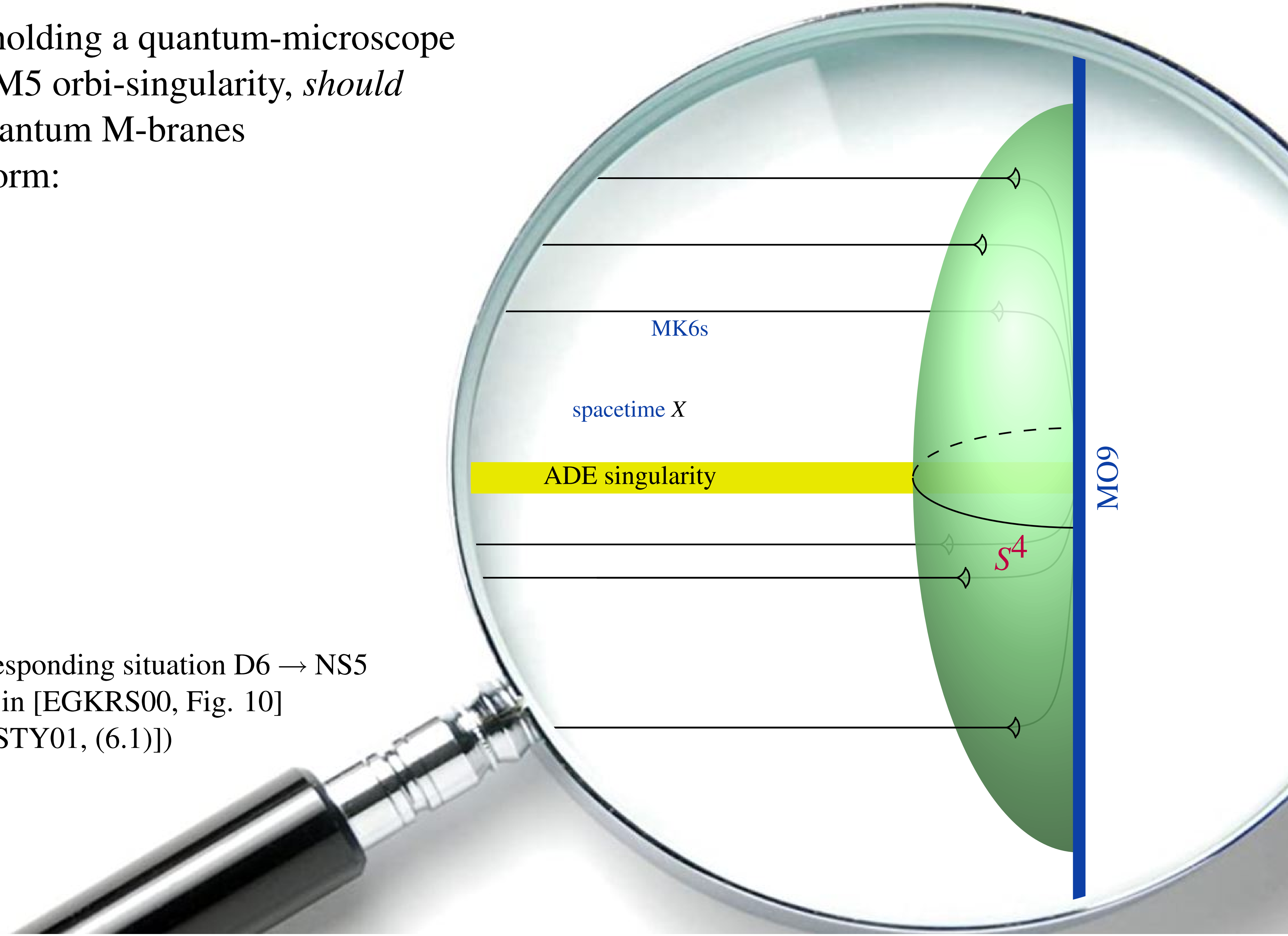
Consequence 3: An M5-shaped orbi-singularity must be $\underline{\mathrm{MK6} \perp \mathrm{MO9} =: \tfrac{1}{2}\mathrm{M5}}$:



E.g.: [GKSTY01, §6] [ZHTV14, §6] [GaTo14,§2.3]

Quantum M-branes?

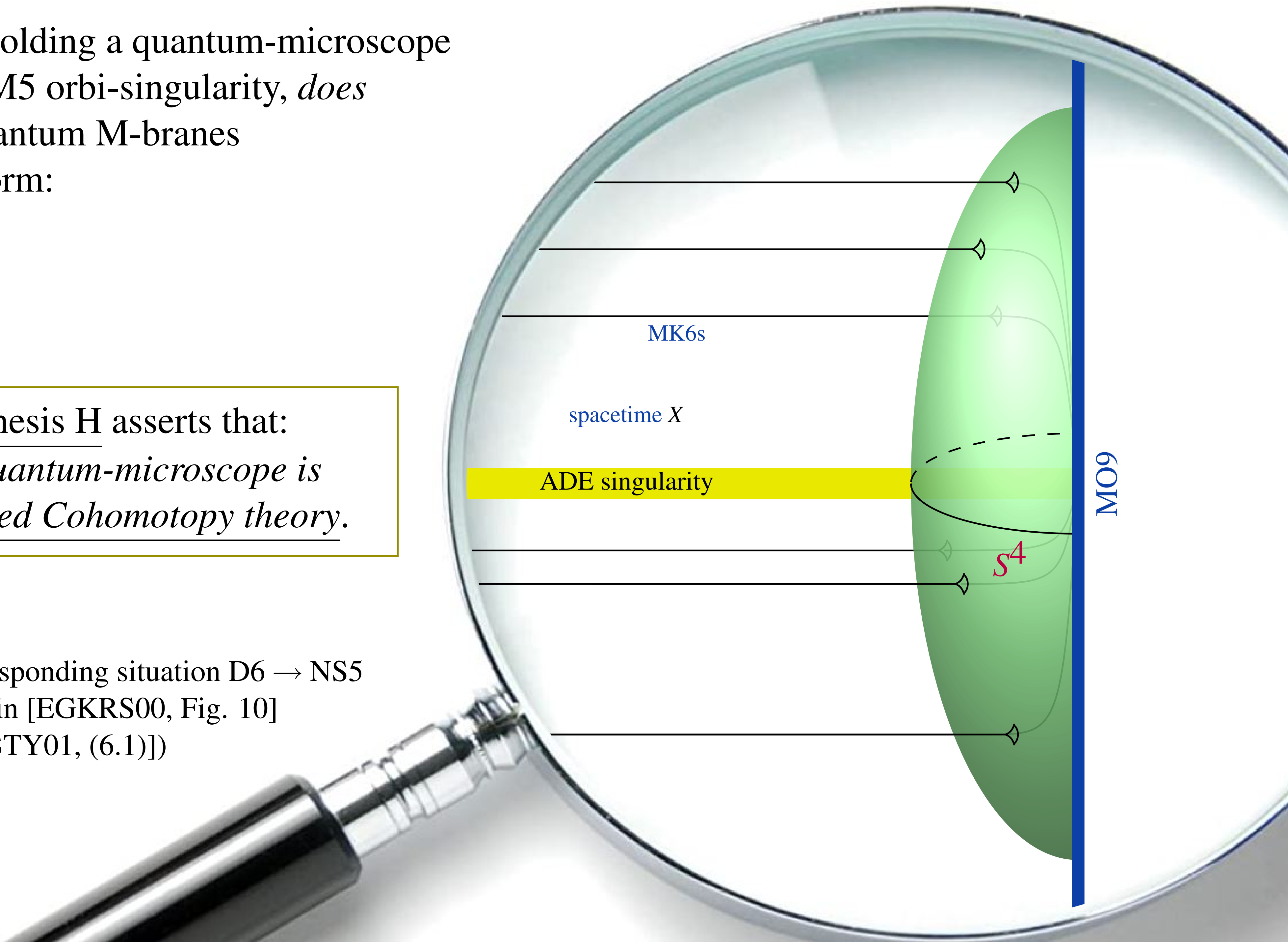
Hence, holding a quantum-microscope over a $\frac{1}{2}$M5 orbi-singularity, *should* show quantum M-branes of this form:



(the corresponding situation D6 → NS5 is shown in [EGKRS00, Fig. 10] and [GKSTY01, (6.1)])

Quantum M-branes!

Hence, holding a quantum-microscope over a $\frac{1}{2}$M5 orbi-singularity, *does* show quantum M-branes of this form:



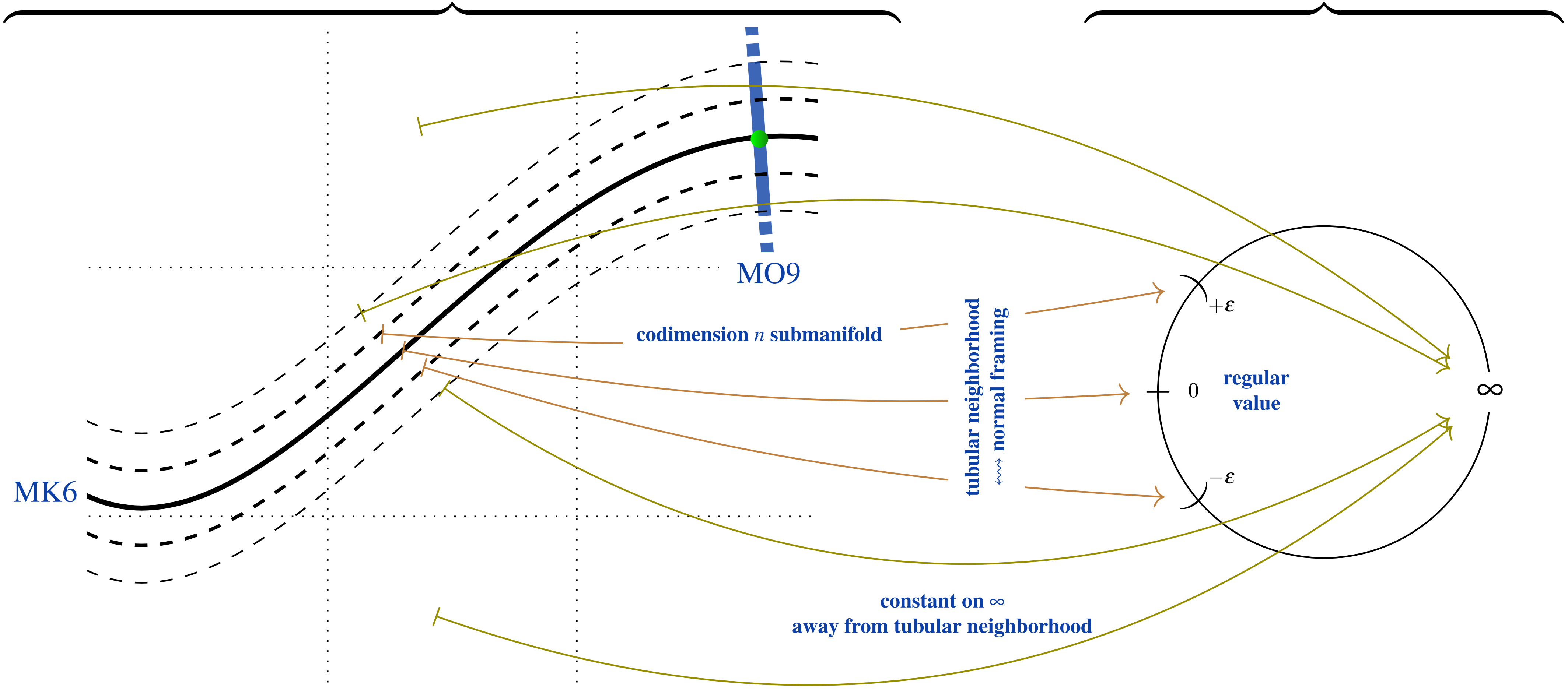
Hypothesis H asserts that:
This quantum-microscope is J-twisted Cohomotopy theory.

(the corresponding situation D6 ⟶ NS5 is shown in [EGKRS00, Fig. 10] and [GKSTY01, (6.1)])

Cohomotopy charge map. [SS19-TadCnc, §2.1] [SS21-MF, §2.2]

Namely, a small tubular neighbourhood of each MK6 carries *directed asymptotic transverse distance* from $\frac{1}{2}$M5 in MO9

$$\Rightarrow \underline{\textit{Cohomotopy charge}}: \underset{\substack{\text{spacetime}\\ \text{manifold}}}{X} \xrightarrow[\substack{\text{directed asymptotic transverse distance}\\ \text{from MK6 loci in MO9-planes}}]{\text{Cohomotopy charge}} \overset{\substack{\text{transversal space}\\ \text{1pt compactified}}}{(\mathbb{R}^4)^{\mathrm{cpt}}} = \underset{\substack{\text{4-sphere}\\ \text{Cohomotopy coefficient}}}{S^4}$$



This construction and its reverse is *Pontrjagin's construction* ([Pon38], long before [Thom54]).

Anti-branes and Cobordism. [SS19-TadCnc, §2.1] [SS21-MF, §2.2]

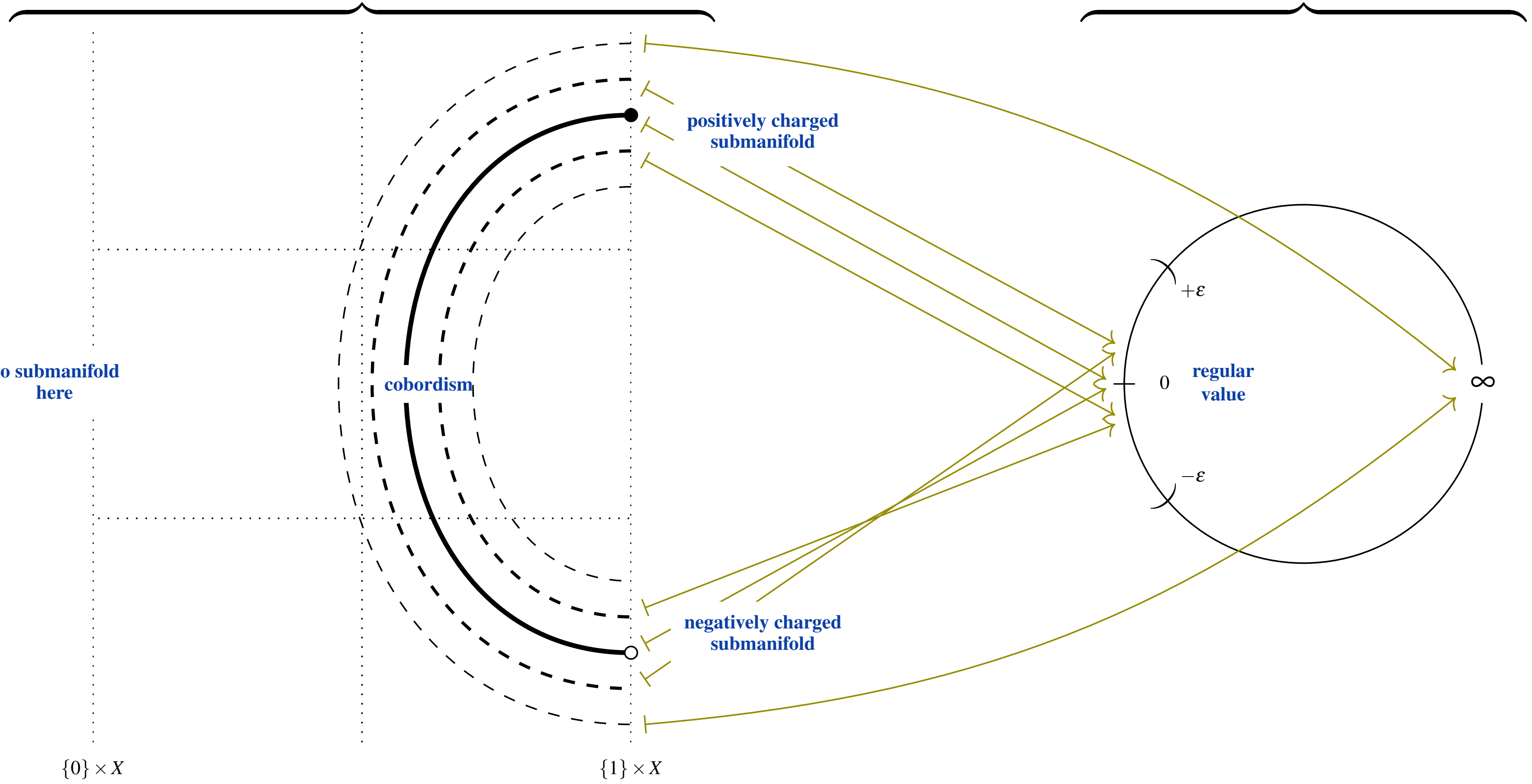
Under the above Pontrjagin construction one finds that:

cobordisms exhibiting brane/anti-brane reactions ⟷ **coboundaries in Cohomotopy theory**

$$[0,1] \times X \xrightarrow[\text{coboundary in Cohomotopy}]{0 \simeq (-1)+(+1)} (\mathbb{R}^n)^{\mathrm{cpt}} = S^n$$

product space of interval with manifold

n-sphere Cohomotopy coefficient



Cohomotopy is framed Cobordism. [SS19-TadCnc, §2.1] [SS21-MF, §2.2]

Pontrjagin's theorem says that
4-Cohomotopy is the conserved charge of these M-brane/anti-brane reactions
in that the Cohomotopy charge map yields a *bijection* on cobordism classes:

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quantized charges

$$\begin{tikzcd}
\text{unstable 4-Cohomotopy} & & \text{unstable framed 4-Cobordism} \\
\pi^4(\mathrm{X}) := \mathrm{Map}(\mathrm{X}, S^4)/_{\mathrm{htpy}} \arrow[rr, shift right=2, "\text{Pontrjagin constr.}"'] \arrow[rr, phantom, "\sim"] \arrow[d, "\text{1-step stabilization}"] & & \left\{\begin{array}{l}\text{normally framed (nmly:} \\ \text{Cohomotopy-charged)} \\ \text{codim-4 submfds of } X\end{array}\right\}/_{\mathrm{cob}} =: \mathrm{Cob}^4_{\mathrm{fr}}(\mathrm{X}) \arrow[ll, shift right=2, "\text{Cohomotopy charge}"'] \arrow[d, "\text{1-step stabilization}"', "\text{M5-brane moves away from MO9 into HW-bulk}"] \\
\pi^{4+1}(\mathrm{X} \times \mathbb{R}^1) \arrow[rr, leftrightarrow, "\sim"] \arrow[d, dotted, "\text{remaining stabilization}"] & & \mathrm{Cob}^{4+1}_{\mathrm{fr}}(\mathrm{X} \times \mathbb{R}^1) \arrow[d, dotted, "\text{remaining stabilization}"', "\text{often equivalence, e.g. for } \mathrm{X} \simeq S^7"] \\
\mathbb{S}^4(\mathrm{X}) \arrow[rr, leftrightarrow, "\sim", "\text{Pontrjagin-Thom theorem}"'] & & \mathrm{MFr}^4(\mathrm{X}) \\
\text{stable 4-Cohomotopy} & & \text{stable framed 4-Cobordism}
\end{tikzcd}$$

quantum M-branes

Cohomotopy is framed Cobordism.

[SS19-TadCnc, §2.1] [SS21-MF, §2.2]

Pontrjagin's theorem says that
4-Cohomotopy is the conserved charge of these M-brane/anti-brane reactions
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quantized charges

unstable 4-Cohomotopy

unstable framed 4-Cobordism

$$\begin{tikzcd}
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\pi^{4+1}(\mathrm{X} \times \mathbb{R}^1) \arrow[r, leftrightarrow, "\sim"] \arrow[d, dotted, "\text{remaining stabilization}"] & \mathrm{Cob}^{4+1}_{\mathrm{fr}}(\mathrm{X} \times \mathbb{R}^1) \arrow[d, dotted, "\text{remaining stabilization}"', "\text{often equivalence, e.g. for } \mathrm{X} \simeq S^7"] \\
\mathbb{S}^4(\mathrm{X}) \arrow[r, leftrightarrow, "\sim", "\text{Pontrjagin-Thom theorem}"'] & \mathrm{MFr}^4(\mathrm{X})
\end{tikzcd}$$

stable 4-Cohomotopy

stable framed 4-Cobordism

quantum M-branes

Rem. 1.
In particular this means that, in its stable = linearized approximation (cf above),
Hypothesis H says equivalently that M-brane charge is quantized in stable framed Cobordism.

This is reminiscent of discussion in [McNamara & Vafa 19], see [SS21-MF, §4] for more.

Cohomotopy is framed Cobordism.

[SS19-TadCnc, §2.1] [SS21-MF, §2.2]

Pontrjagin's theorem says that
4-Cohomotopy is the conserved charge of these M-brane/anti-brane reactions
in that the Cohomotopy charge map yields a *bijection* on cobordism classes:

quantized charges

unstable 4-Cohomotopy

unstable framed 4-Cobordism

$$\begin{tikzcd}
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\end{tikzcd}$$

stable 4-Cohomotopy

stable framed 4-Cobordism

quantum M-branes

Rem. 2.

equivariant Cohom. cocycles

plain Cohom. cocycles

$$\begin{tikzcd}
\mathrm{Map}(\mathrm{X}, S^{\mathbf{4}})^{G_{\mathrm{ADE}}} \arrow[r, "\text{forget equivariance}"] \arrow[d, two heads] & \mathrm{Map}(\mathrm{X}, S^4) \arrow[d, two heads] \\
\pi^{\mathbf{4}}_{G_{\mathrm{ADE}}}(\mathrm{X}) \arrow[r, "\mathrm{e}^*"] & \pi^4(\mathrm{X})
\end{tikzcd}$$

Via the forgetful map

the PT theorem allows to *see* the quantum M-branes around orb-singularities: $\rightsquigarrow$

Equivariant Cohomotopy of flat orbifolds.

Thm. 3.17 in [SS19-TadCnc]

Thm.: Charges in G_{ADE}-equivariant **4**-Cohomotopy of $\mathbb{T}^{\mathbf{4}}$ are labeled by:

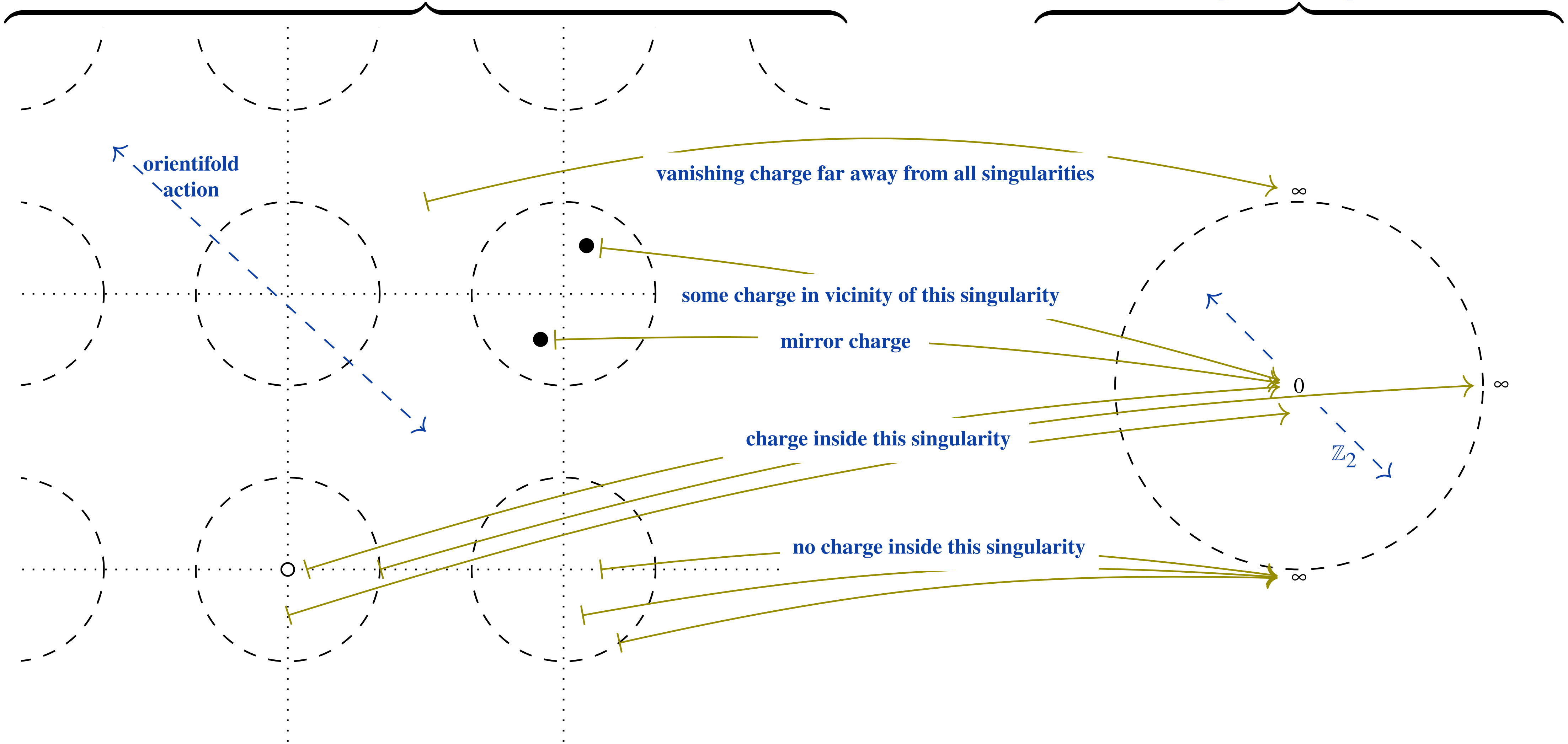
1. a choice of charge $\in \{0, -1\}$ *inside* each singularity;
2. an integer number of $|G_{\mathrm{ADE}}|$-tuples of mirror unit charges.

Proof.: Use tom Dieck's equivariant Hopf degree theorem. [tD79, §8.4]

$$\mathbb{T}^{\mathbf{4}} \xrightarrow[\mathbb{Z}_2\text{-equivariant Cohomotopy cocycle}]{\mathrm{c}} S^{\mathbf{4}}$$

toroidal orbifold

representation sphere



orientifold action

vanishing charge far away from all singularities

some charge in vicinity of this singularity

mirror charge

charge inside this singularity

no charge inside this singularity

Tadpole cancellation in Cohomotopy. [SS19-TadCnc]

Informal idea of tadpole cancellation in string-theory and in M-theory:

1) total brane charge in compact transversal space must vanish; and yet there

2) may/must be branes, countering opposite charges appearing inside O-plane singularities.

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Formalization in Cohomotopy theory according to Hypothesis H:

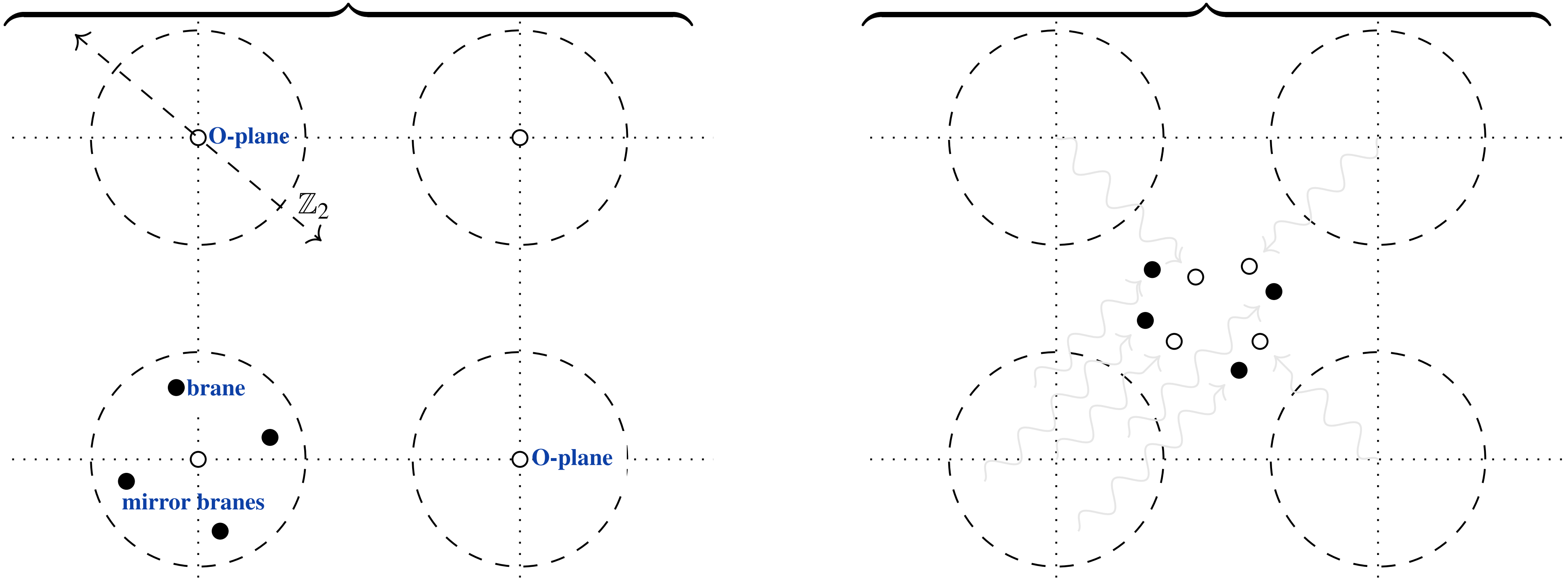
Rem. The condition tot $= 0$ is naturally understood from *super-differential refinement* of Cohomotopy [SS19-TadCnc, (58)].

tadpole-free charges
fiber of tot-operation

microscopic charges seen
in equivariant Cohomotopy

total charge seen
in plain Cohomotopy

$$\begin{tikzcd}
\pi^{\mathbf{4}}_{G_{\mathrm{ADE}}}\left(\mathbb{T}^{\mathbf{4}}\right)_{\mathrm{tot}=0} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "(\mathrm{pb})"] & \{0\} \arrow[d, hook] \\
\pi^{\mathbf{4}}_{G_{\mathrm{ADE}}}\left(\mathbb{T}^{\mathbf{4}}\right) \arrow[r, "\mathrm{tot}"] \arrow[r, swap, "\substack{\text{sum up floating \& stuck charges inside orbifold} \\ \text{non-abelian cohomology operation } (1 \hookrightarrow G_{\mathrm{ADE}})^*}"] & \pi^{4}\left(\mathbb{T}^{4}\right) \\
2 \cdot \mathbf{2} - 4 \cdot \mathbf{1} \arrow[r, mapsto] & 0
\end{tikzcd}$$



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fiber of tot-operation

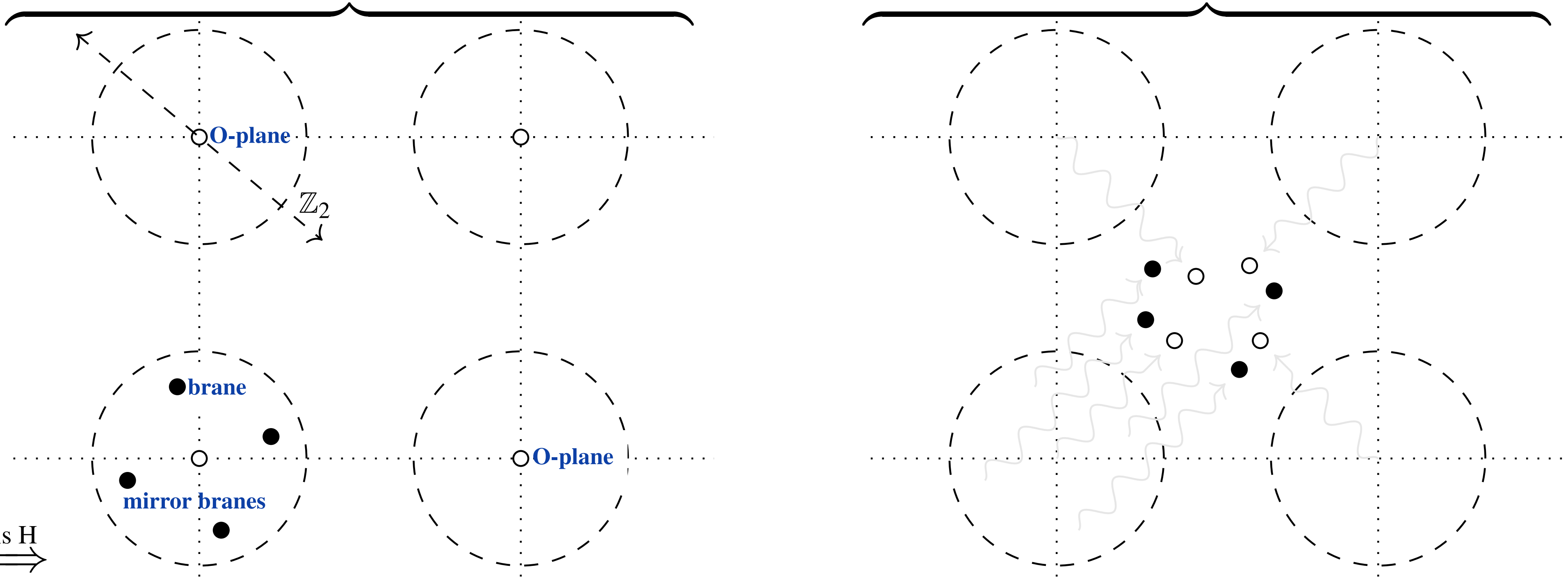
microscopic charges seen in equivariant Cohomotopy

$$\begin{tikzcd}
\pi^{\mathbf{4}}_{G_{\mathrm{ADE}}}\big(\mathbb{T}^{\mathbf{4}}\big)_{\mathrm{tot}=0} \arrow[rr] \arrow[d, hook] \arrow[drr, phantom, "\text{(pb)}"] & & \{0\} \arrow[d, hook] \\
\pi^{\mathbf{4}}_{G_{\mathrm{ADE}}}\big(\mathbb{T}^{\mathbf{4}}\big) \arrow[rr, "\mathrm{tot}"] & & \pi^{4}\big(\mathbb{T}^{4}\big) \\
2\cdot\mathbf{2}-4\cdot\mathbf{1} \arrow[rr, mapsto] & & 0
\end{tikzcd}$$

sum up floating & stuck charges inside orbifold
non-abelian cohomology operation $(1 \hookrightarrow G_{\mathrm{ADE}})^*$

total charge seen in plain Cohomotopy

Rem. The condition tot $= 0$ is naturally understood from *super-differential refinement* of Cohomotopy [SS19-TadCnc, (58)].



$\xrightarrow{\text{Hypothesis H}}$

MO5-planes carry $-\frac{1}{2}$M5-brane charge. Proves old conjecture: [DM95, §2][Wi 95, §3][Ho98, §2.1].

Outlook – Further predictions.

This concludes my survey of one prediction of Hypothesis H on flat orbifold spacetimes.

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This concludes my survey of one prediction of Hypothesis H on flat orbifold spacetimes.

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Among the predictions in the other limit, of smooth but curved spacetimes X, stands out the shifted 4-flux quantization [FSS19-HypH, Prop. 3.13][FSS20-Char, §5.3]:

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(where τ is $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$-structure on X and ω is a compatible connection/field of gravity).

Outlook – Further predictions.

This concludes my survey of one prediction of Hypothesis H on flat orbifold spacetimes.

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But various further consistency conditions on M-flux are expected, e.g.

Page charge quantization of G_7. Hypothesis H implies this, too: [FSS19-M5WZ, Thm. 4.8].

Outlook – Quantum M-Theory.

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$$\widehat{H}^{\bullet}_{\mathrm{pos}}\Big(\mathrm{Map}^{*}\big(\mathscr{X}\,,\,\textstyle\int \rotatebox[origin=c]{70}{$\prec$}\big(\widehat{S^{\mathbf{4}}_{\mathrm{Tate}} /\!\!/ \mathrm{Spin}(4)}\big)\big)_{\int \rotatebox[origin=c]{70}{$\prec$}\mathbf{B}\mathrm{Spin}(4)}\Big)$$

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We identified [CSS21-Quant, Thm. 1.2 & §4] such states in top. sector of D6 $\perp$ D8-branes and discovered [SS19-Quant] structures expected from DBI- and Hanany-Witten arguments.

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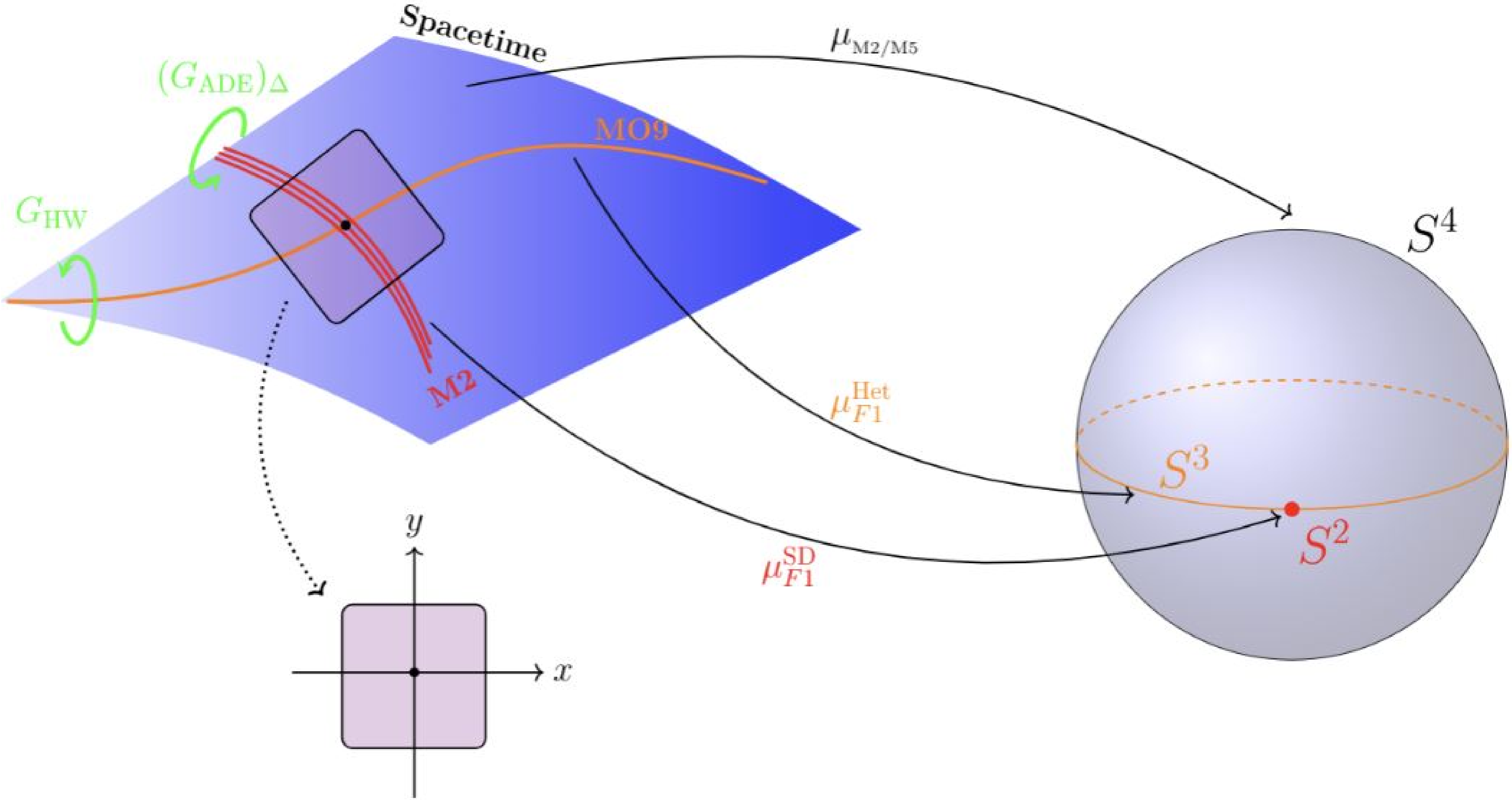
So it looks promising...

these slides and further pointers are available at:

ncatlab.org/schreiber/show/Proper+Orbifold+Cohomotopy+for+M-Theory



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