The M-theory brane charge super Lie 6-algebra

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famous fact 1 (Azcárraga et al 89, Sorokin-Townsend 97): Rationally, charges of super p-branes are the spacetime supersymmetry Noether charges of their Green-Schwarz-type sigma-models.

$$\underbrace{\{Q,Q\} = P}_{\text{supersymmetry}} \;+\; \underbrace{Z_{p_1} + Z_{p_2} + \cdots}_{\text{Noether charge extensions}}$$

"type II supersymmetry algebra"
"M-theory supersymmetry algebra"
etc.

open question 1: Since...

a) ...these Noether currents have higher order symmetries-of-symmetries (ghosts-of-ghosts);

b) ...D-brane and M5-brane sigma-models are defined on higher stacky extension of spacetime (due to tensor multiplets);

...what is the full M-theory super L_∞-algebra?

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famous fact 2: Beyond rational approximation, D-brane charge is refined through Chern character from ordinary cohomology to twisted K-theory, twist being magnetic F1-brane charge $[H_3]$.

open question 2: Lifting to M-theory, what is the true twisted cohomological nature of M5-brane charge (twist now being the magnetic M2-brane charge $[G_4]$)?

Definition: For $\omega \in \Omega^{p+2}_{\mathrm{cl}}(X)$, write

$$\mathrm{Ham}^p(X,\omega) := \{(v,J) | \iota_v \omega + dJ = 0\} \subset \mathrm{Vect}(X) \oplus \Omega^p(X)$$

for the pairs of vector fields with Hamiltonian forms (currents) and

$$\mathrm{HamVect}(\mathrm{X}) \subset \mathrm{Vect}(X)$$

for the vector fields for which there is a Hamiltonian form.

Proposition (C.L.Rogers 10): The chain complex

$$(\Omega^0(X) \stackrel{d}{\to} \Omega^1(X) \to \cdots \to \Omega^{p-1}(X) \stackrel{(0,d)}{\to} \mathrm{Ham}^p(X,\omega))$$

carries an L_∞-structure $\boxed{\mathfrak{pois}(X,\omega)}$ with non-trivial k-ary brackets proportional to contraction of k vector fields into ω.

Theorem ([FRS13b], [FSS15b]):

1. $\mathfrak{pois}(X,\omega)$ is equivalently the Lie $(p+1)$-algebra of Noether currents of infinitesimal target-space symmetries of the WZW-type sigma-model on X with WZW-curvature ω.
2. There is a homotopy fiber sequence of L_∞-algebras
$$\Omega^\bullet[p] \to \mathfrak{pois}(X,\omega) \to \mathrm{HamVect}(X).$$
3. On 0-homology this is a central extension of Lie algebras
$$0 \to H^p_{\mathrm{dR}}(X) \longrightarrow \tau_0 \mathfrak{pois}(X,\omega) \longrightarrow \mathrm{HamVect}(X,\omega) \to 0$$
where the bracket on $\tau_0\mathfrak{pois}(X,\omega)$ is the Dickey bracket on equivalence classes of Noether currents.

All this goes through verbatim also for supergeometry and super-L_∞-algebras, see [FSS13], [dcct].

For $\mathbb{R}^{d-1,1|\mathbf{N}}$ a super-Minkowski spacetime, consider the left-invariant $(p+2)$-form built from the super-vielbein (e^a, ψ^α):

$$\mu_{p+2} := \overline{\psi} \wedge \Gamma^{a_1 \cdots a_p} \psi \wedge e_{a_1} \wedge \cdots e_{a_p} \, .$$

This is closed and non-trivial precisely for $(d, \mathbf{N}, p)$ an element in the "old brane scan" classifying super-p-branes on which no other branes may end, those without (higher) gauge fields on their worldvolume.

Equations of motion of supergravity on super-spacetime $(X, \mathbf{g})$ locally modeled on $\mathbb{R}^{d-1,1|\mathbf{N}}$ imply a definite globalization $\omega \in \Omega^{p+2}_{\mathrm{cl}}(X)$ of μ_{p+2}.

This is the curvature of the WZW-term for the p-brane on $(X, \mathbf{g})$.

Proposition ([FSS15b]): In this case the extension

$$0 \to H^p_{\mathrm{dR}}(X) \longrightarrow \tau_0 \mathfrak{pois}(X, \omega) \longrightarrow \mathfrak{iso}(X, \mathbf{g}) \to 0$$

is the brane-charge extension of the super-isometry super Lie algebra ($\sim$ Azcárraga et al 89).

$(X, \mathbf{g})$ is $\frac{1}{k}$-BPS if there is a super Lie algebra embedding

$$\mathbb{R}^{0|\dim(\mathbf{N})/k} \hookrightarrow \tau_0 \mathfrak{pois}(X, \omega)$$

Fact ([FSS13]): μ_{p_1+2} classifies super-Lie (p_1+1)-algebra extension

$$\widehat{\mathbb{R}^{d-1,1|\mathbf{N}}} \to \mathbb{R}^{d-1,1|\mathbf{N}} \;;$$

gives p_2-branes with tensor multiplets from consecutive cocycles:

- $D_{p_2=2k+1}$-brane comes from cocycle on $\widehat{\mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}}}$;
- $D_{p_2=2k}$-brane comes from cocycle on $\widehat{\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}}$;
- M-theory $(p_2 = 5)$-brane comes from cocycle on $\widehat{\mathbb{R}^{10,1|\mathbf{32}}}$.

Proposition ([dcct]): This globalizes to super-p_1-stack extension

of super-spacetime
$$\begin{array}{ccc} \mathbf{B}^{p_1}U(1)_{\mathrm{conn}} & \longrightarrow & \widetilde{\hat{X}} \\ & & \downarrow \\ & & X \end{array}$$
equipped with WZW term on $\widetilde{\hat{X}}$.

Consequence: A map $\Sigma_{p_2+1} \to \widetilde{\hat{X}}$ is a pair consisting of

1. a sigma-model field $\phi : \Sigma_{p_2+1} \longrightarrow X$;
2. a ϕ-twisted p_1-form gauge field on Σ_{p_2+1}.

Proposition ([FRS13a]): Previous extension result generalizes to such higher super-stacky spacetimes;

yields brane charge extension by $H^{p_2}(\widetilde{\hat{X}}, \mathbb{R})$.

Proposition ([dcct]+D.Pavlov): Homotopy type of $\widetilde{\hat{X}}$ is that of underlying $K(\mathbb{Z}, p_1 + 1)$-fibration $\hat{X}$ over the bosonic body of X.

Corollary: Running the Serre spectral sequence gives that, rationally, M5-brane charge on spacetime X is in middle cohomology of

$$H^1(X) \stackrel{(0,d_4)}{\longrightarrow} H^2(X) \oplus H^5(X) \stackrel{(d_4,0)}{\longrightarrow} H^6(X).$$

- When the C-field class is torsion, this reduces to
$$H^2(X) \oplus H^5(X).$$
- When the C-field class is not torsion, there are corrections by M1 charges and by KK-monopole charges.
- Non-rationally, there are many more corrections.

Hence M5-brane charge is in *some* degree-4-twisted differential generalized cohomology theory.

Plausible candidates include (Sati10, Sati13):

1. tmf, 2. degree-7 cohomotopy.

Proposition ([FSS15a]): M5-brane WZW term descends along $\widehat{\mathbb{R}^{10,1|\mathbf{32}}} \to \mathbb{R}^{10,1|\mathbf{32}}$ to cocycle in degree-4-twisted degree-7 (differential) cohomotopy.

Thank you!

Further details and pointers are in these course notes:

ncatlab.org/schreiber/show/Structure+Theory+for+Higher+WZW+Terms

minicourse at:
H. Sati (org.), *Flavours of cohomology*, Pittsburgh, June 2015

D. Fiorenza, H. Sati, U. Schreiber,
The WZW term of the M5-brane and differential cohomotopy,
2015
ncatlab.org/schreiber/show/The+WZW+term+of+the+M5-brane

D. Fiorenza, H. Sati, U. Schreiber,
Lie n-algebras of BPS charges,
2015

D. Fiorenza, C. L. Rogers, U. Schreiber,
Higher geometric prequantum theory,
arXiv:1304.0236

D. Fiorenza, C. L. Rogers, U. Schreiber,
L_∞-algebras of local observables from higher prequantum bundles,
Homology, Homotopy and Applications,
Volume 16 (2014) Number 2, p. 107-142
arXiv:1304.6292

D. Fiorenza, H. Sati, U. Schreiber,
Super Lie n-algebra extensions, higher WZW models and super p-branes with tensor multiplet fields,
Intern. J. of Geom. Meth, in Mod. Phys.,
Volume 12, Issue 02 (2015) 1550018
arXiv:1308.5264

U. Schreiber,
Differential cohomology in a cohesive topos,
arXiv:1310.7930,
expanded version at:
dl.dropboxusercontent.com/u/12630719/dcct.pdf