Urs Schreiber

(New York University, Abu Dhabi & Czech Academy of Science, Prague)

Cohomotopy Theory and Branes

talk at

Geometry, Topology & Physics at Abu Dhabi

NYU AD 2020

based on joint work with

H. Sati and D. Fiorenza

0) Nonabelian Differential Cohomology

Brane charge in...

1) Twisted Cohomotopy theory

2) Equivariant Cohomotopy theory

3) Differential Cohomotopy theory

implies...

4) Hanany-Witten Theory

5) Chan-Paton Data

6) BMN Matrix Model States

7) M2/M5 Brane Bound States

(0)

Nonabelian Differential Cohomology

in

Cohesive Homotopy Theory

Sec. 4 of Fiorenza-Sati-Schreiber 15 [arXiv:1506.07557]

based on Sec. 6.4.14.3 of Schreiber dcct

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Geometry

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Physics

Geometry

Topology

topological
space
X

Physics

smooth manifold
Σ

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topological space
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Physics

smooth
manifold

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topological
space

$\mathrm{Maps}(\Sigma, X)$

mapping space =
cocycle space of X-cohomology on Σ

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smooth
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loop
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$\mathfrak{l}X$

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L_∞-algebra valued
differential forms
(flat/closed)

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$X_{\mathbb{R}}$
rationalized
(real-ified)
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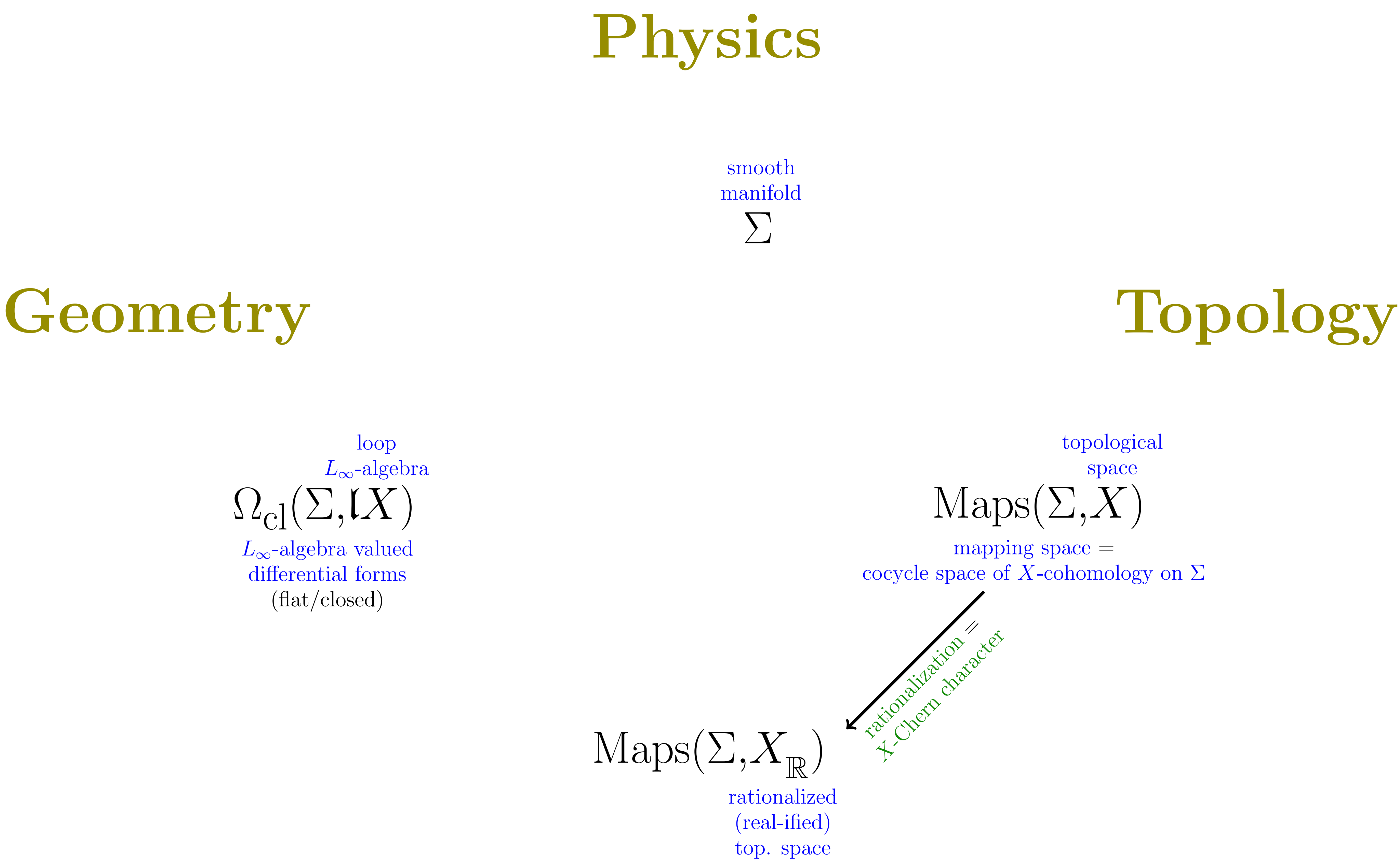
rationalized (real-ified) top. space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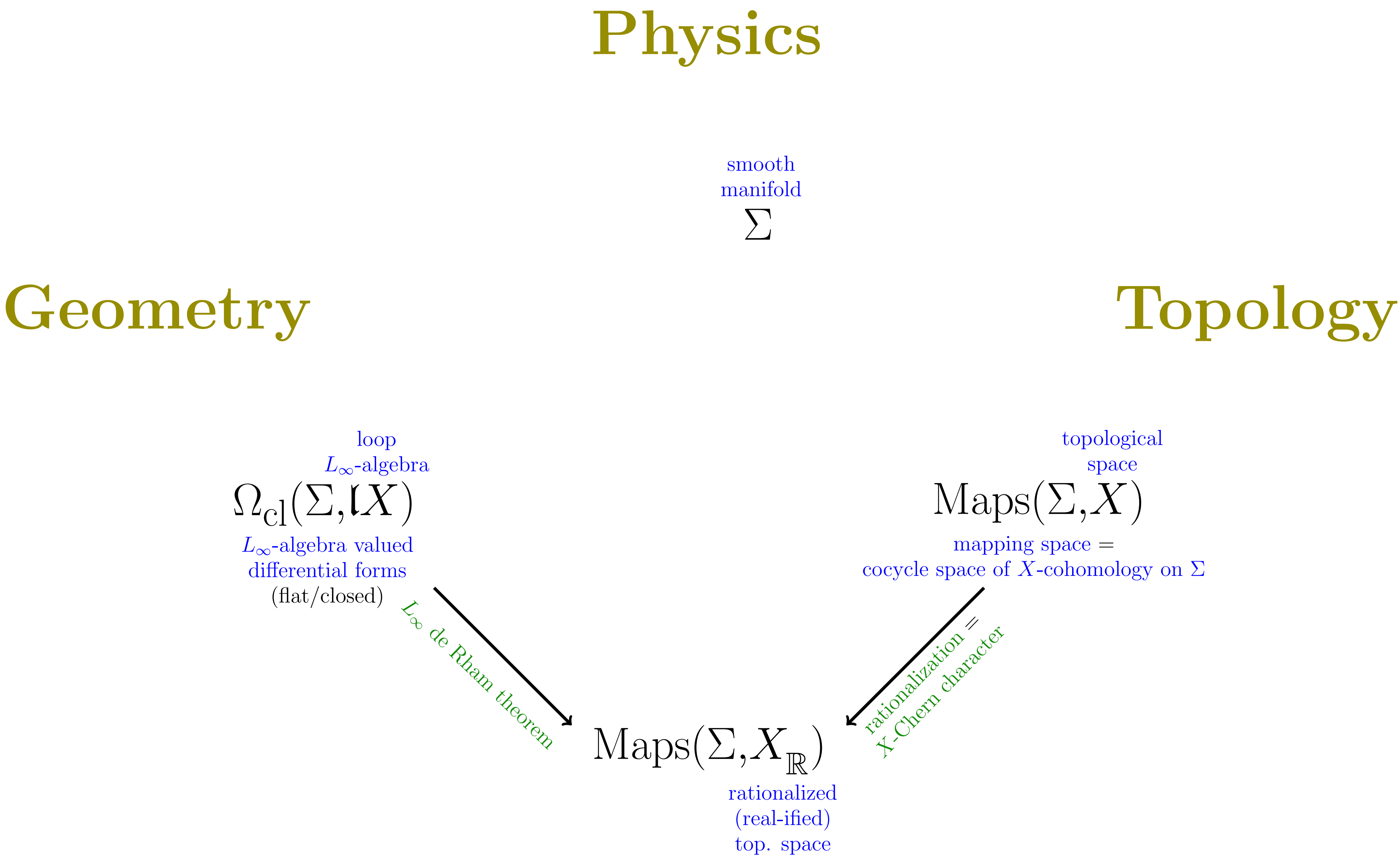
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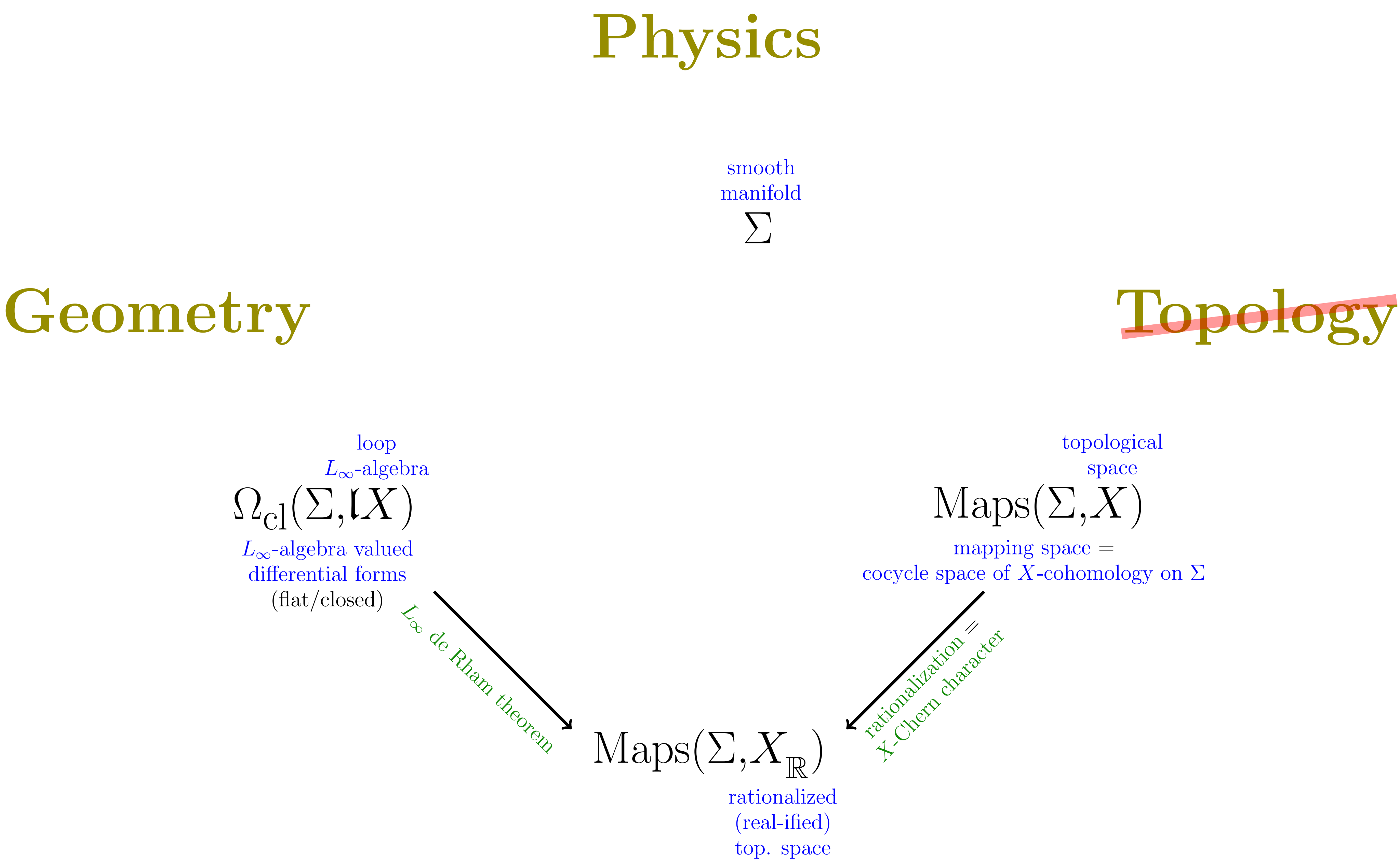
rationalized (real-ified) top. space



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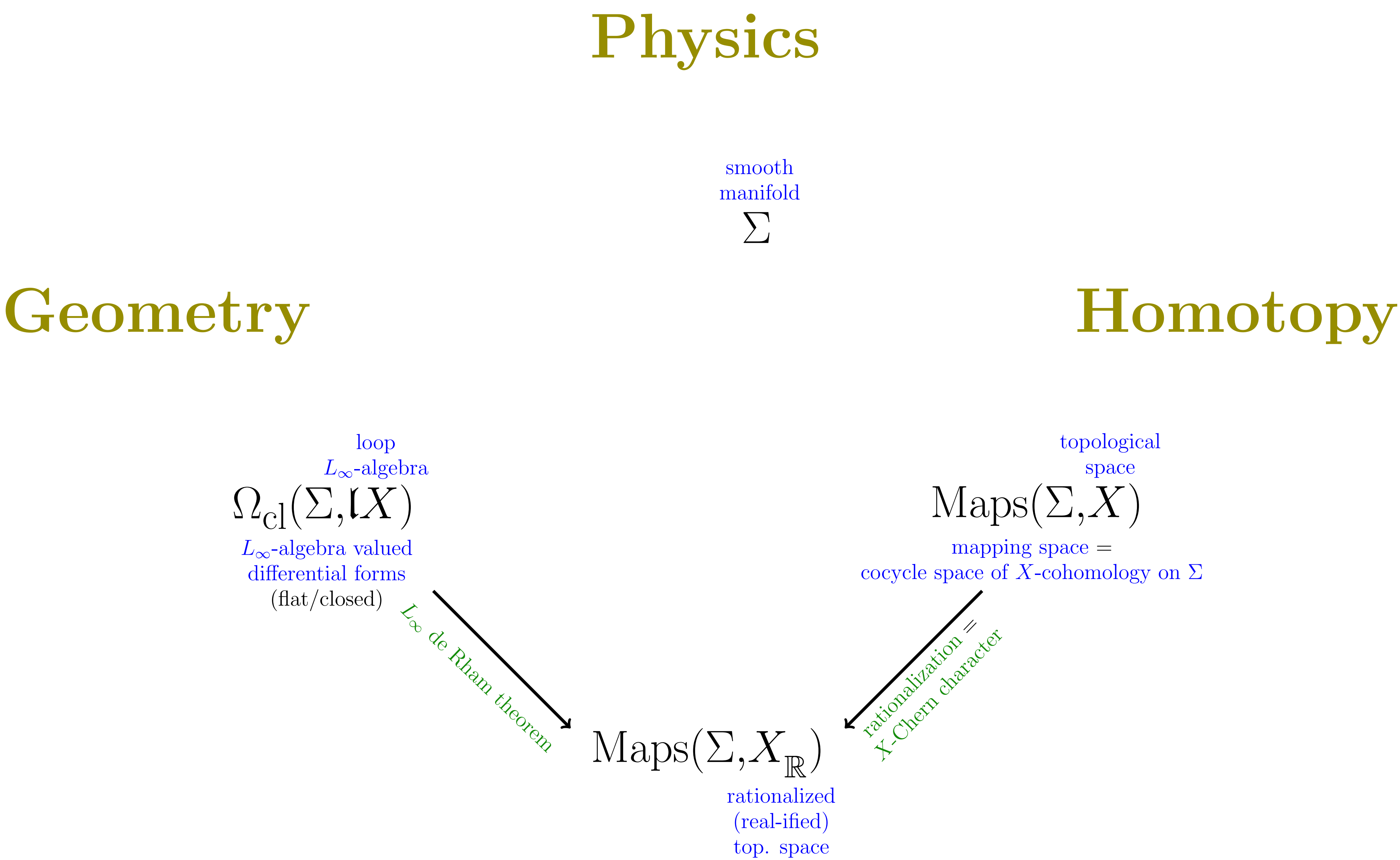
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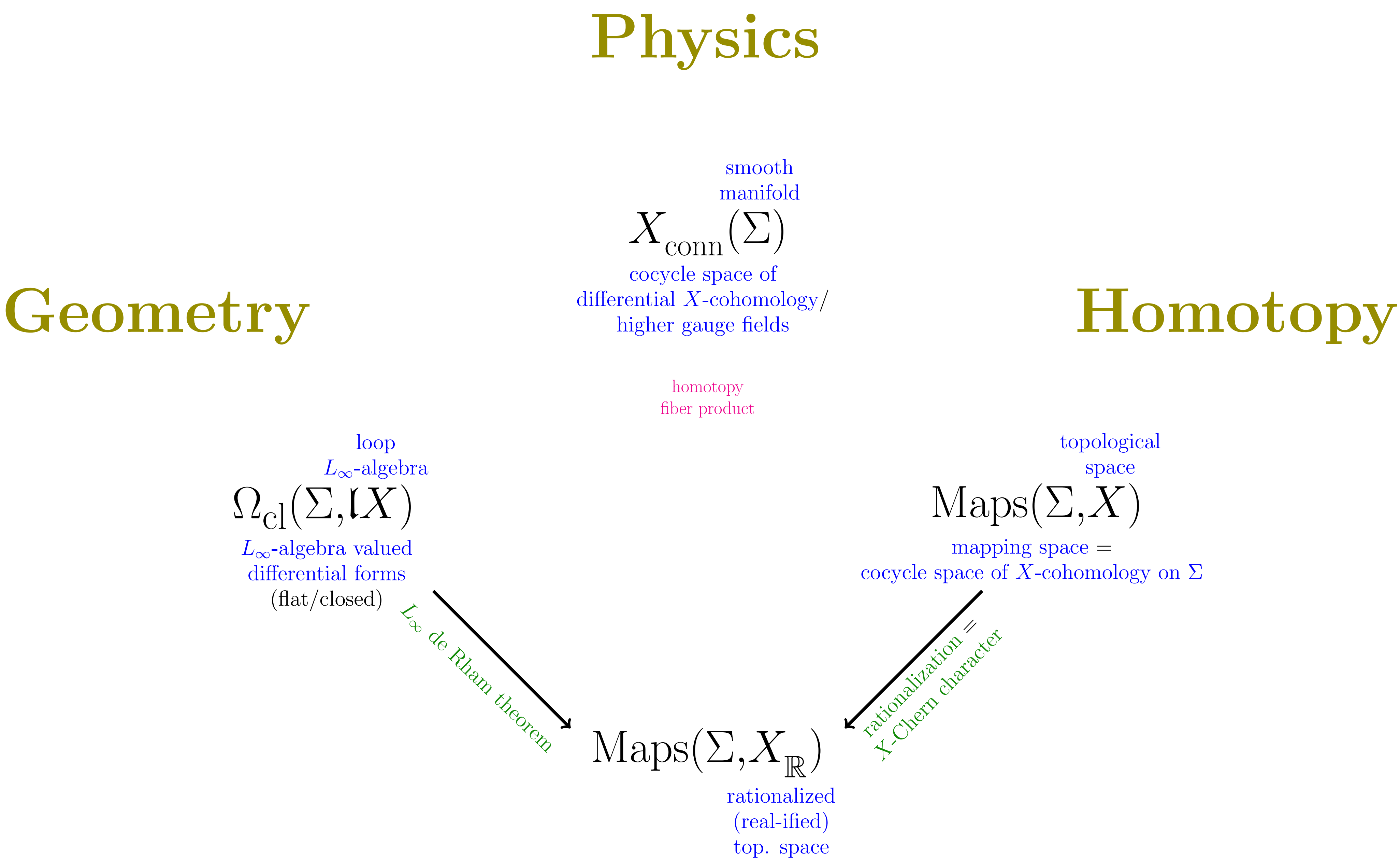
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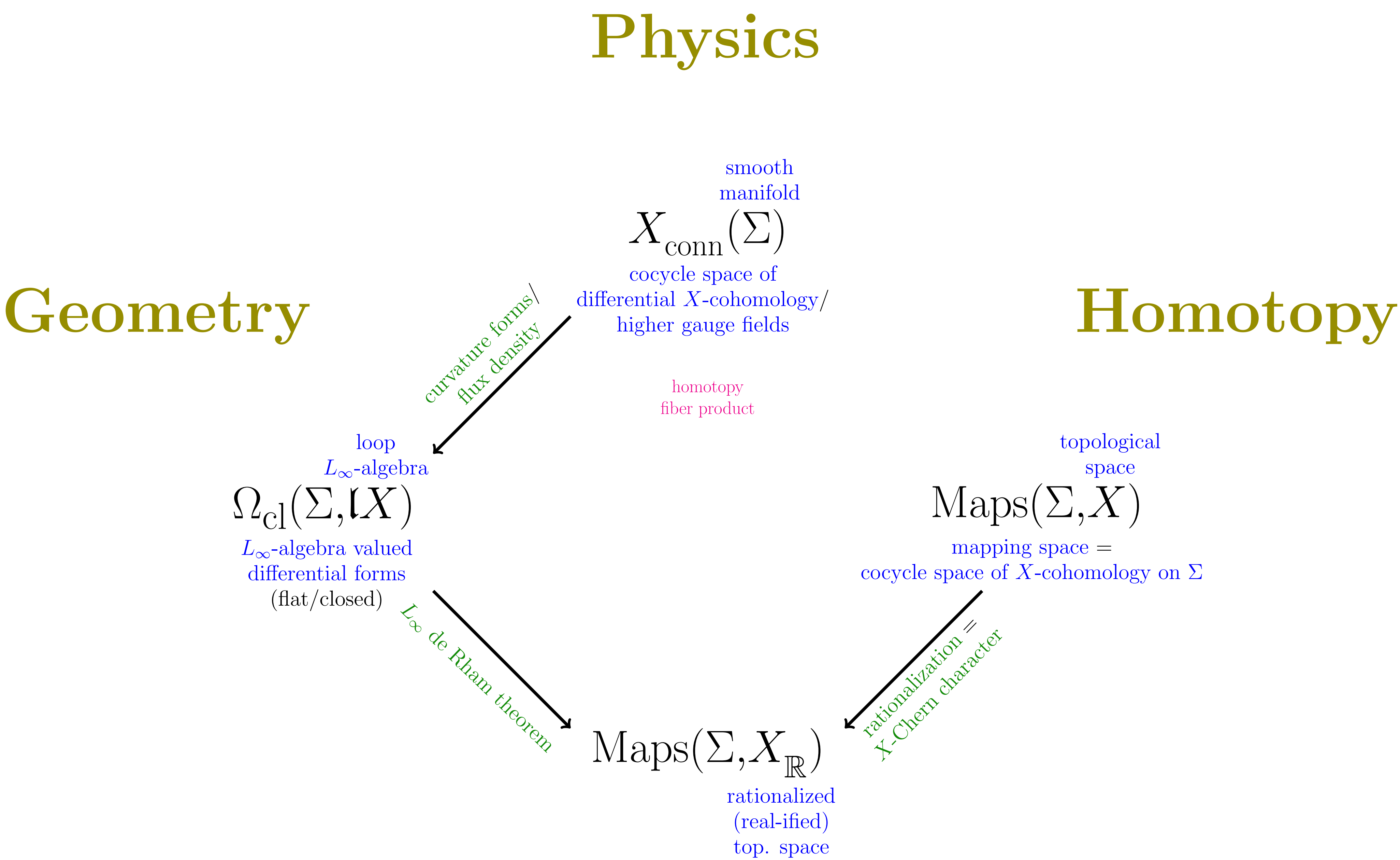
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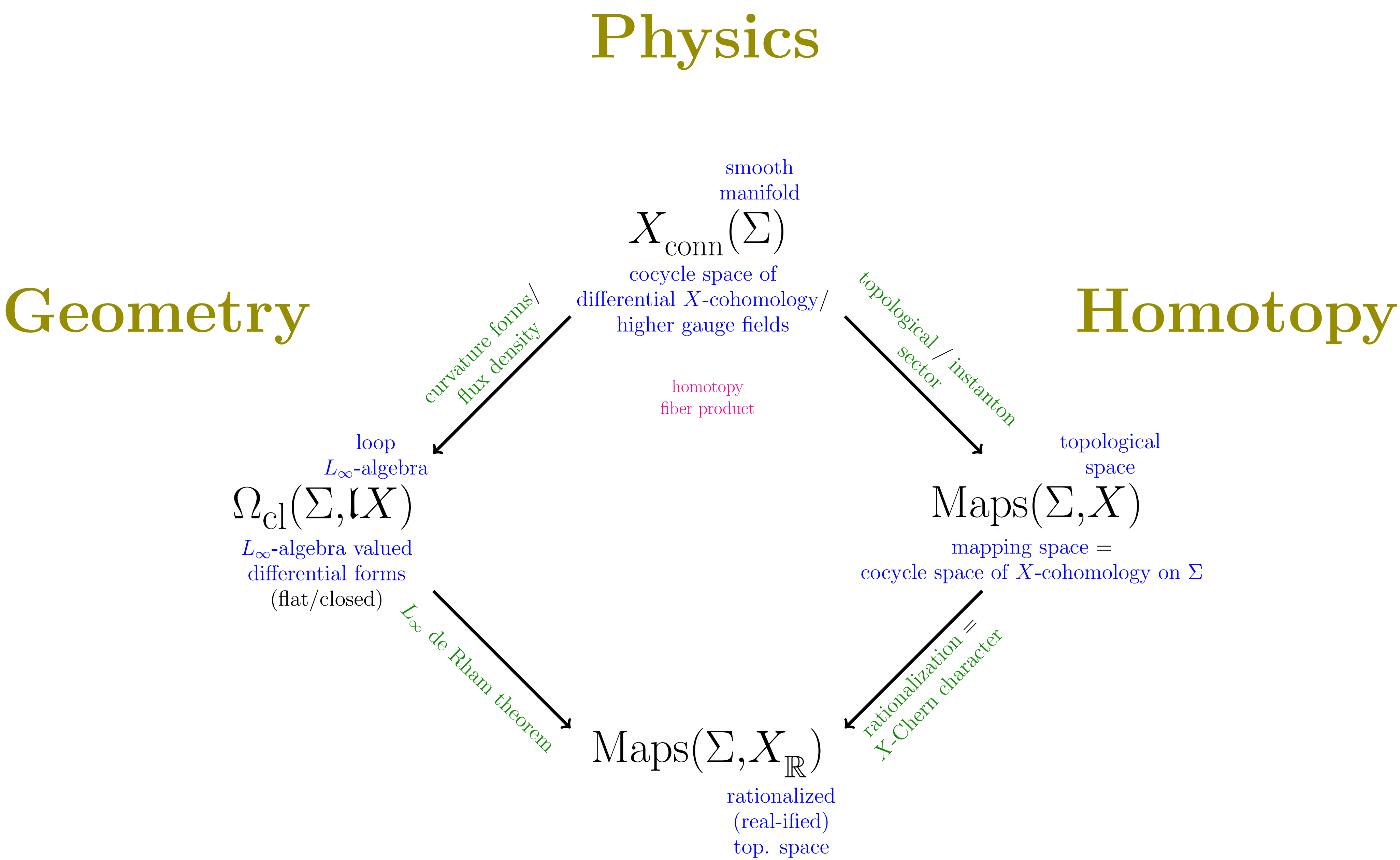
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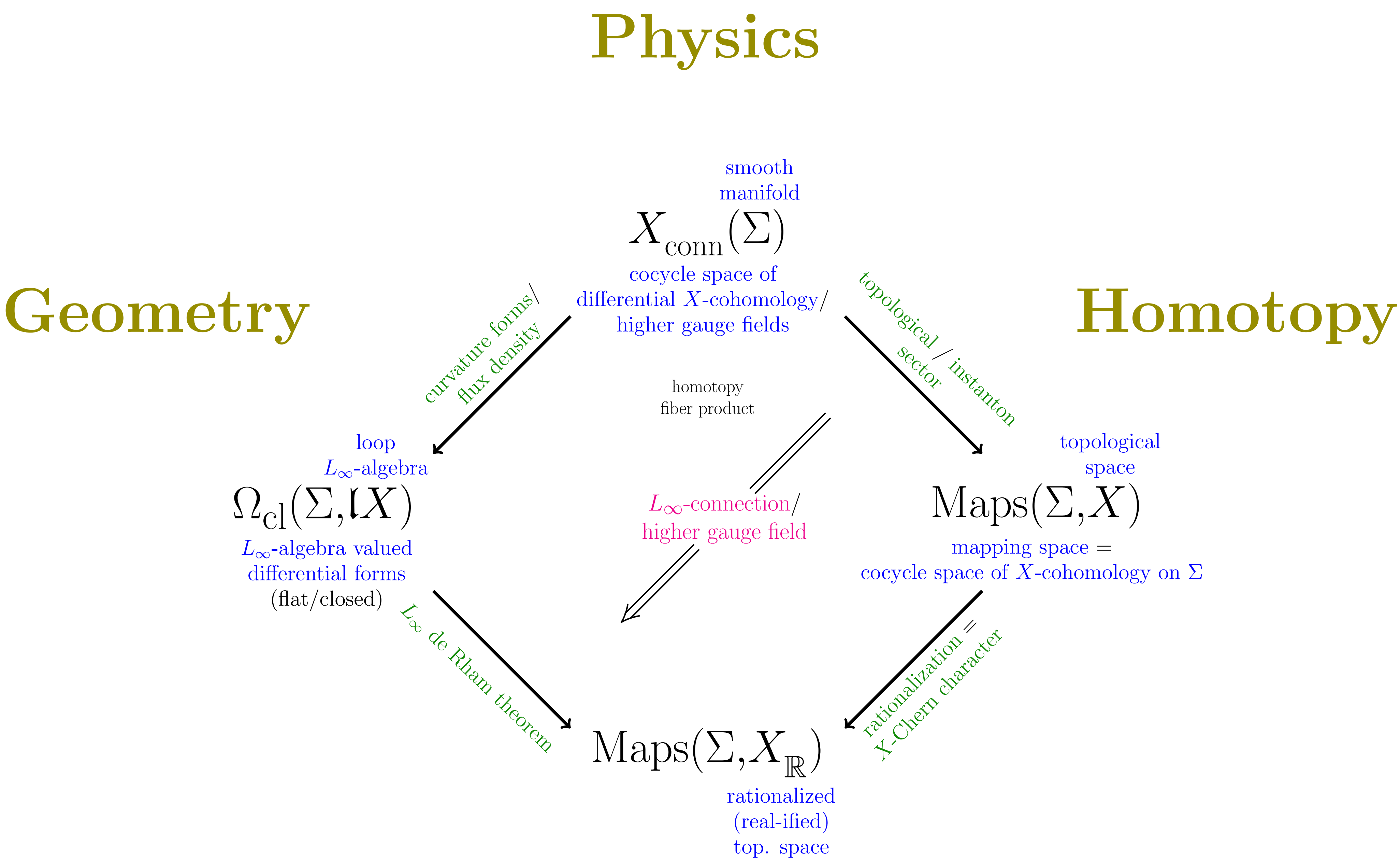

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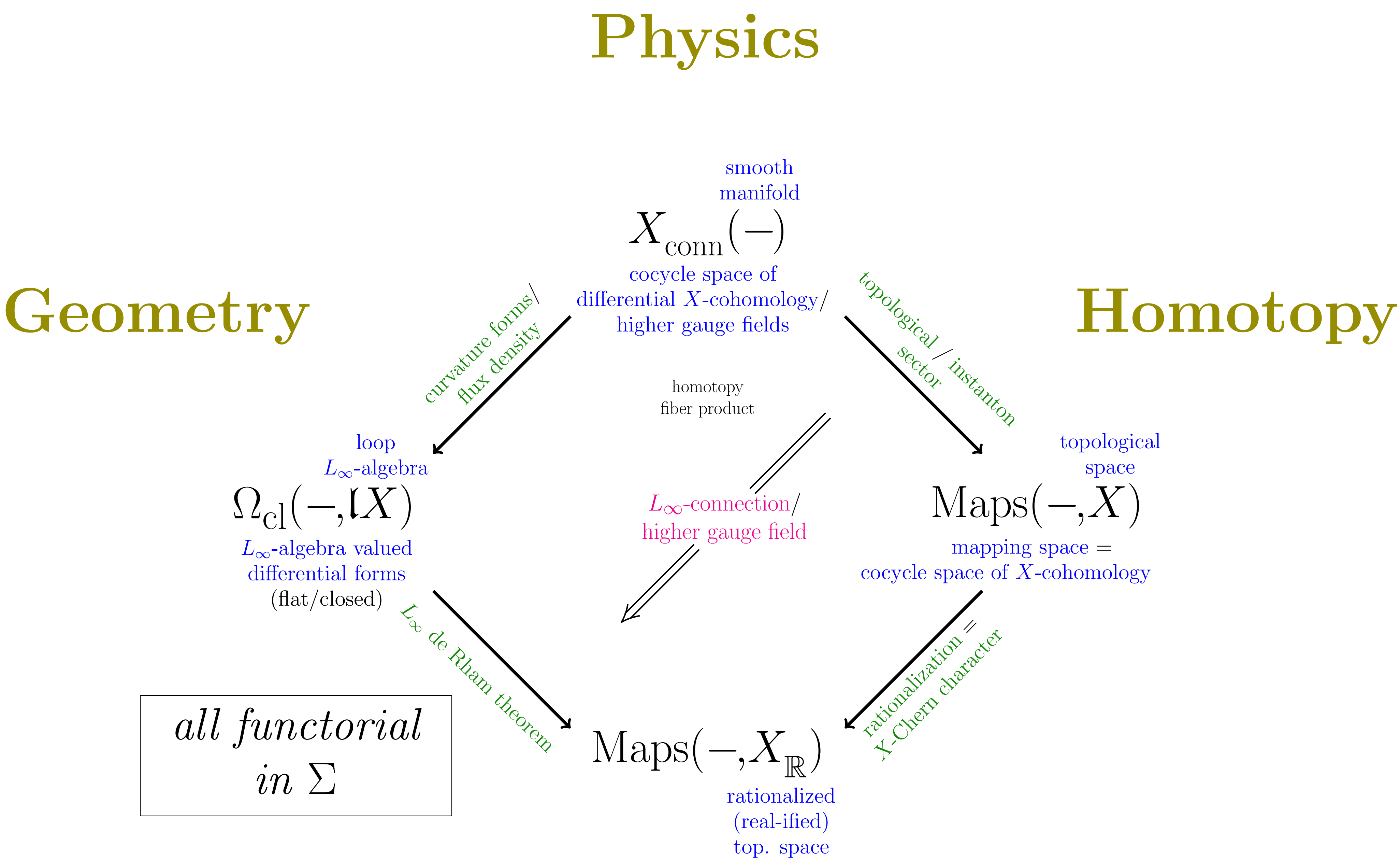

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rationalized (real-ified) top. space
all functorial in Σ

Physics

X_{conn}

moduli stack of
differential X-cohomology/
higher gauge fields

Geometry

Homotopy

X

classifying space of
X-cohomology

Physics

Geometry

Homotopy

$\mathbf{X}_{\mathrm{conn}}$

moduli stack of
differential $\mathbf{X}$-cohomology/
higher gauge fields

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X

classifying space of
X-cohomology

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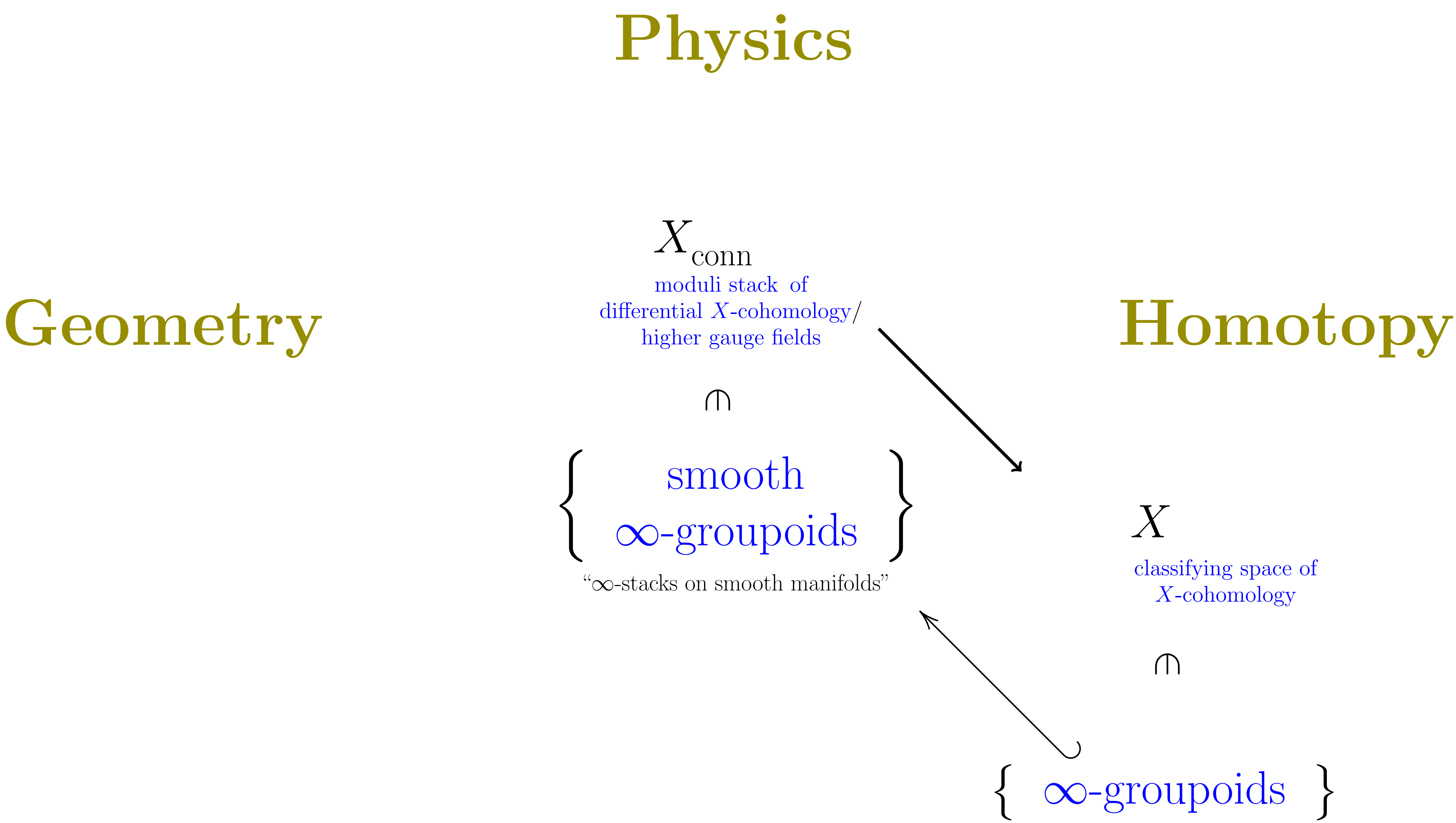
$\{$ ∞-groupoids $\}$



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higher gauge fields
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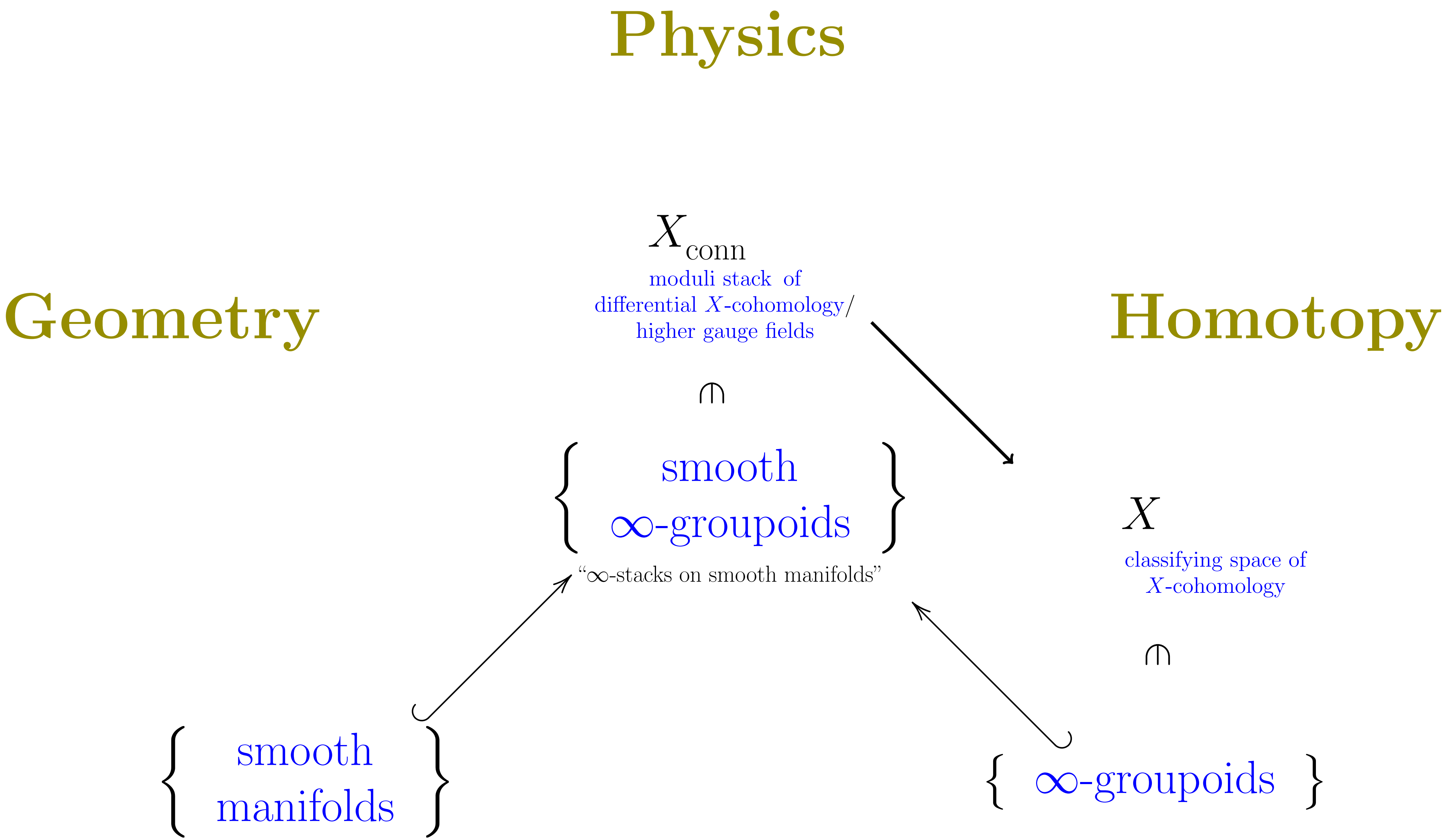
∞-groupoids
"∞-stacks on smooth manifolds"
X
classifying space of
X-cohomology
∞-groupoids



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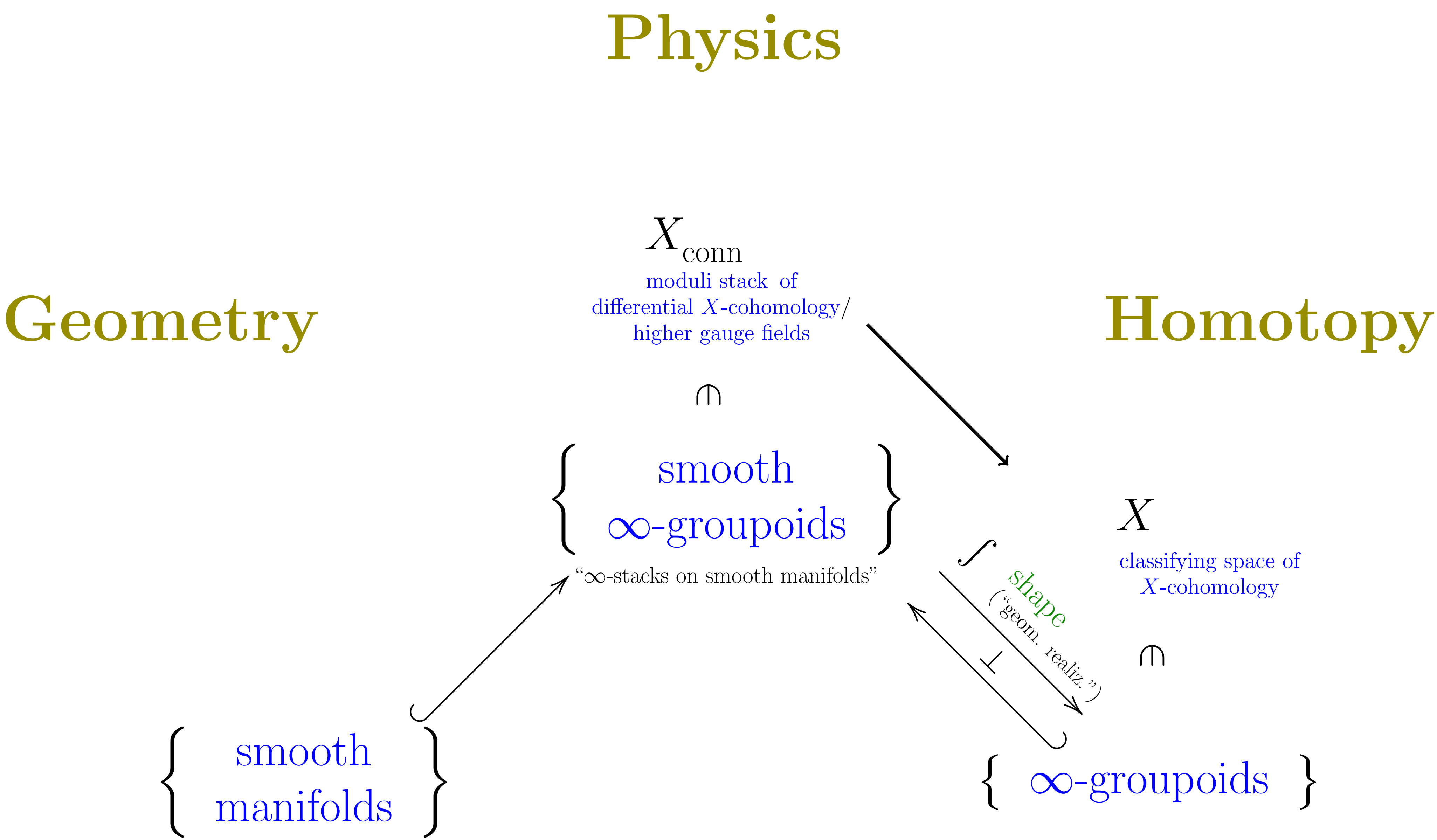

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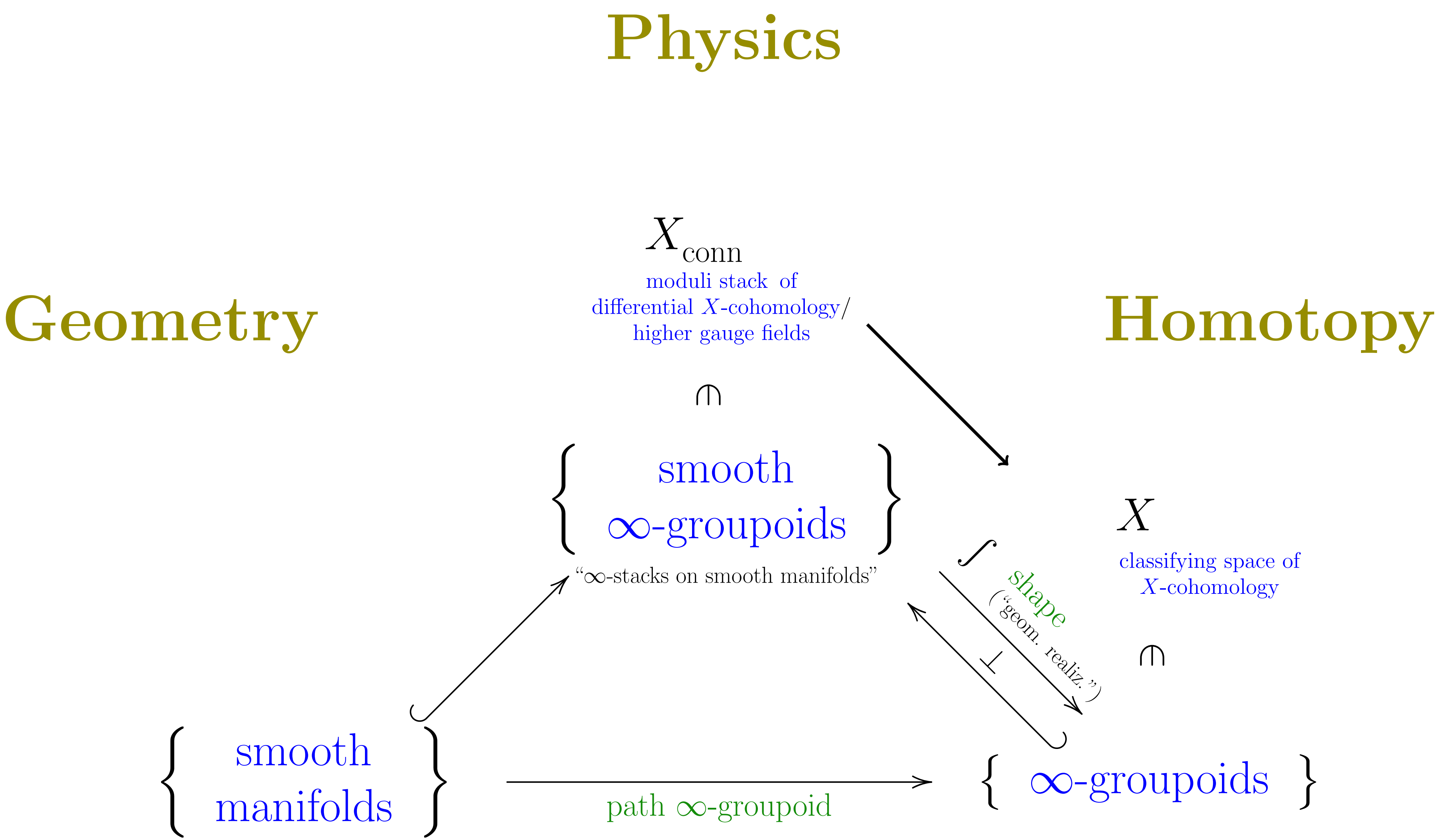
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∞-groupoids

(1)

Brane Charge in Cohomotopy

implied by

Hypothesis H with Pontrjagin-Thom Theorem

Sati-Schreiber 19a [arXiv:1909.12277]

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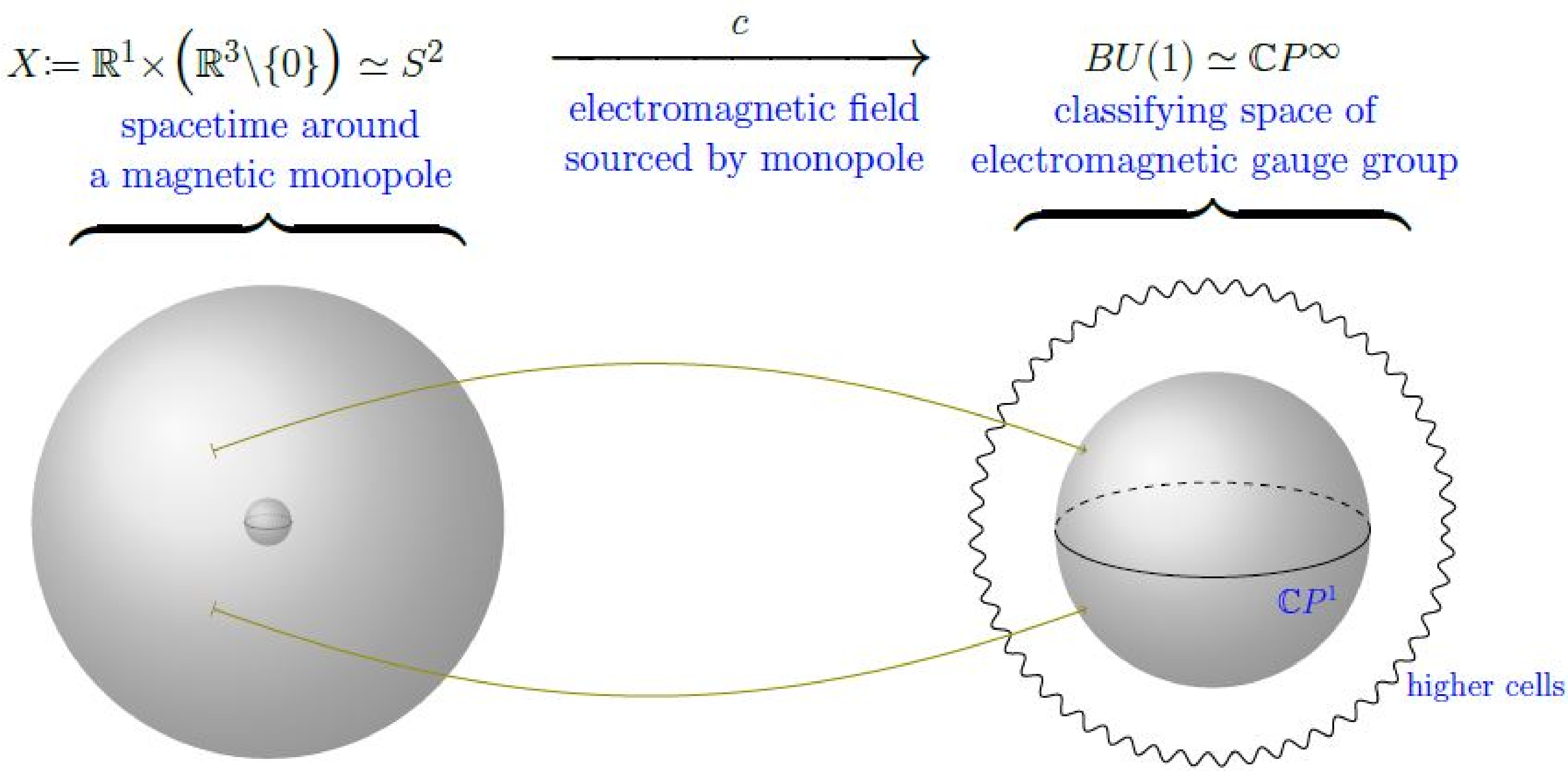
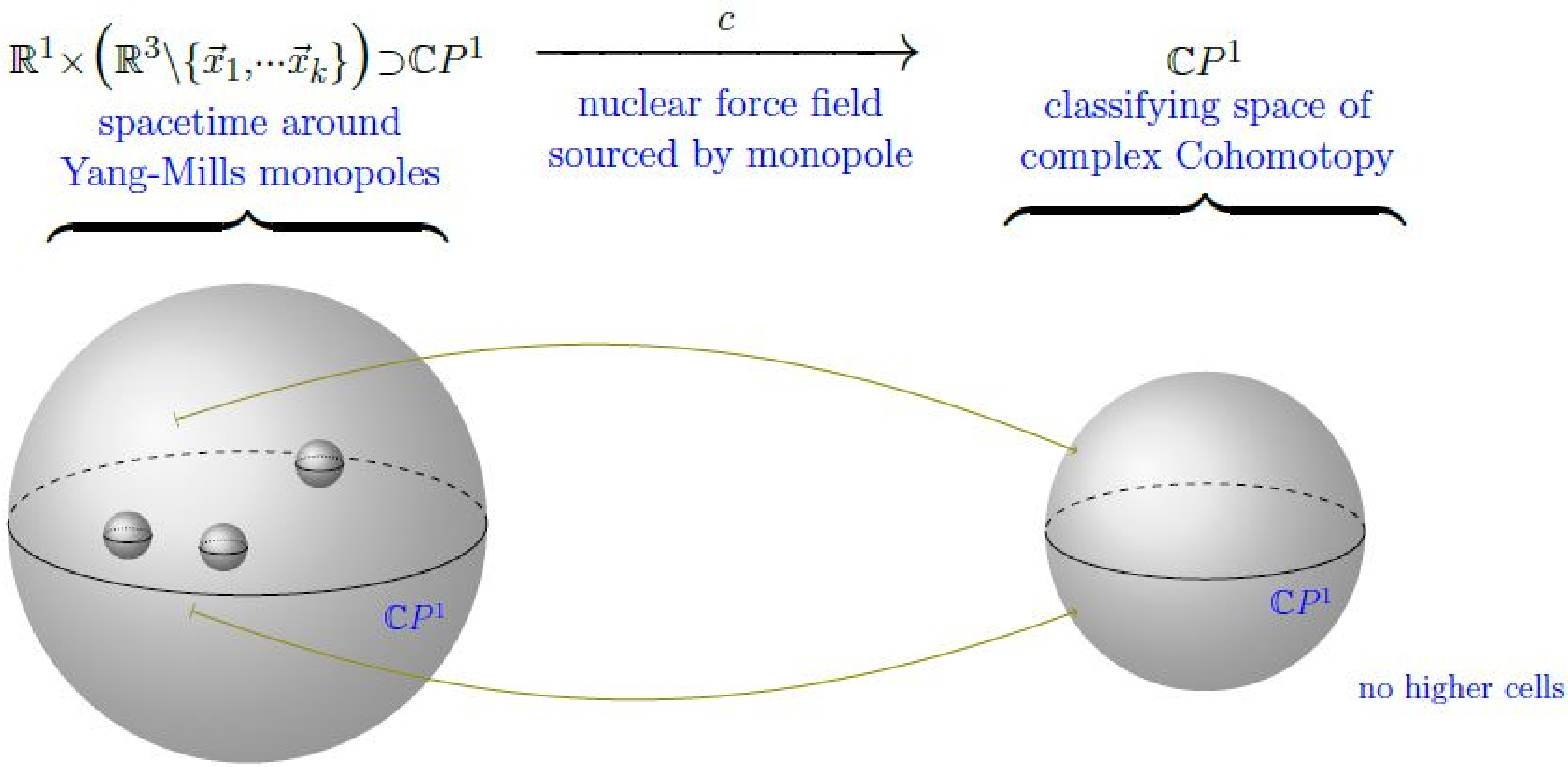


$$[c] \in \left\{ X \longrightarrow BU(1) \right\} \big/ \sim_{\text{homotopy}} \simeq \mathbb{Z}$$

charge = homotopy class

charge lattice

Dirac charge quantization – The topological sector of the electromagnetic field is a cocycle in degree-2 ordinary cohomology, with classifying/coefficient space $BU(1)$.



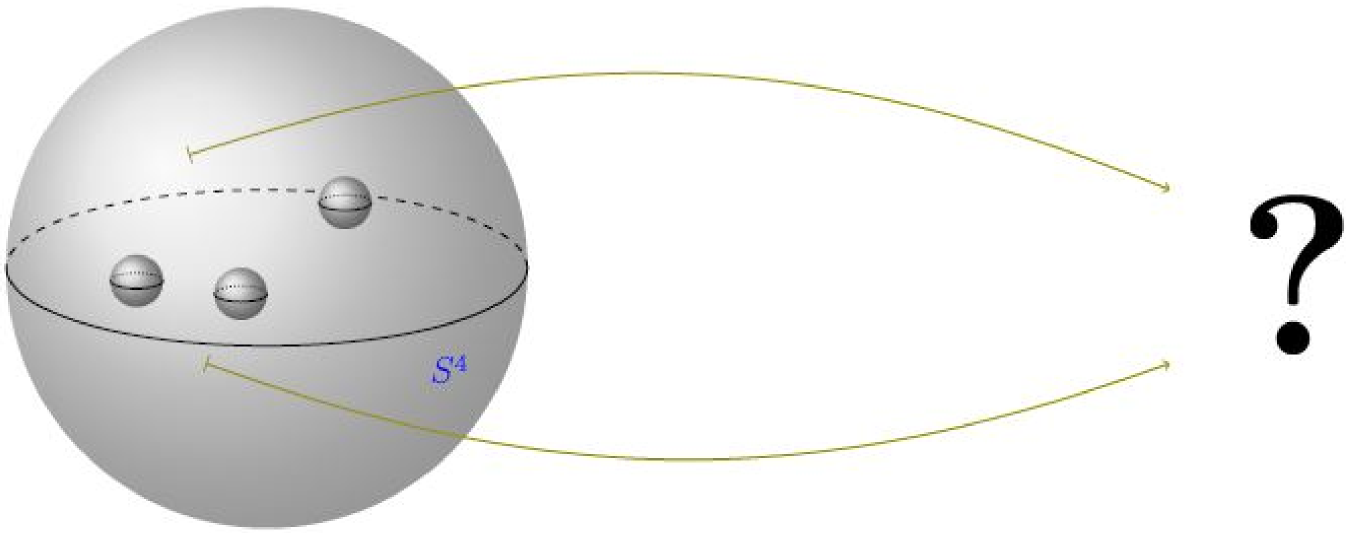
$$[c] \in \left\{ \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1 \right\} \big/_{\sim \text{homotopy}} \simeq \underset{\text{charge lattice}}{\mathbb{Z}}$$

charge = homotopy class

Atiyah-Hitchin charge quantization – The moduli space of SU(2) Yang-Mills monopoles is the cocycle space of complex-rational Cohomotopy of any sphere enclosing them.

$$\mathbb{R}^{5,1} \times \left(\mathbb{R}^4 \setminus \{\vec{x}_1, \cdots \vec{x}_k\}\right) \times \mathbb{R}^1_{/\mathbb{Z}_2^{\mathrm{HW}}} \xrightarrow{\;\; c \;\;} ??$$

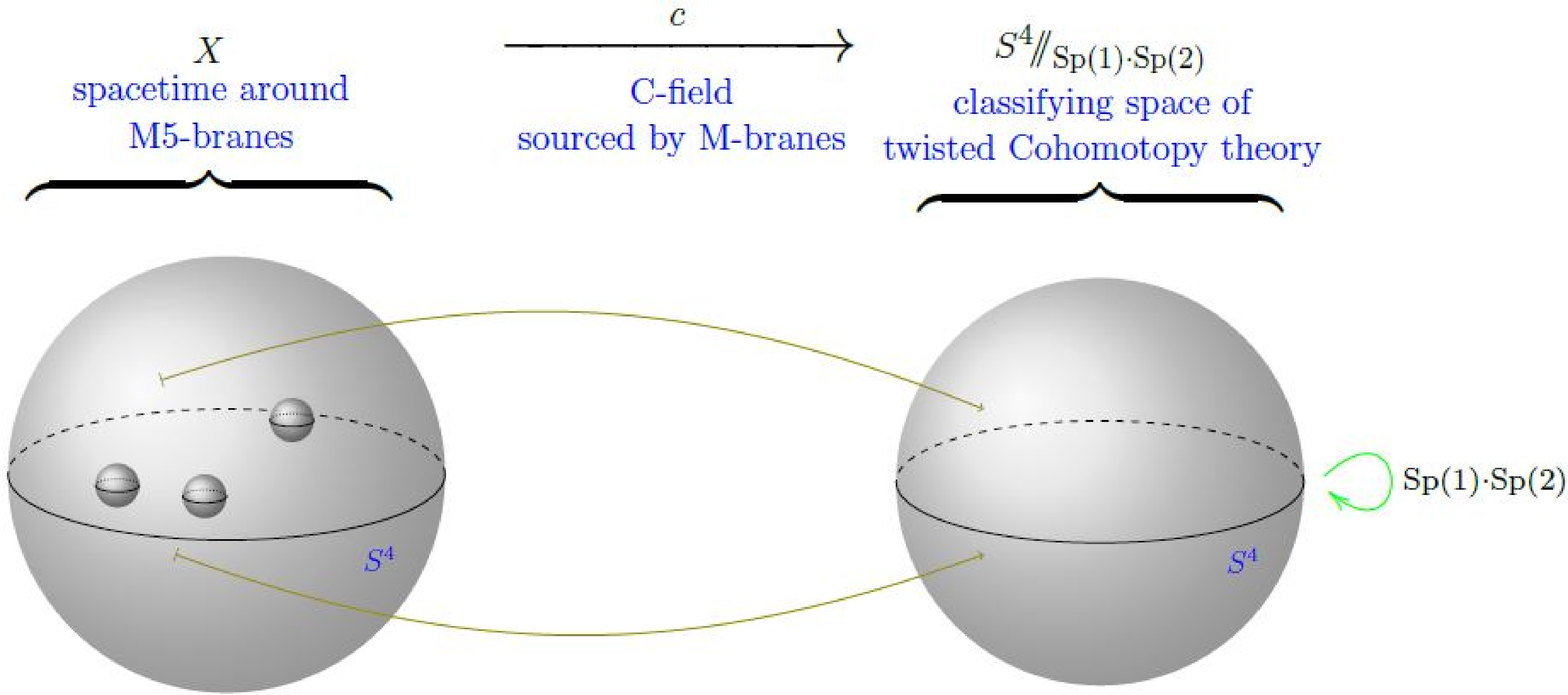
spacetime around M5-branes — C-field sourced by M-branes — classifying space of which cohomology theory??



$$[c] \in \left\{ X \longrightarrow ?? \right\} \big/ \sim_{\text{homotopy}} \simeq \underset{\text{charge lattice}}{??}$$

charge = homotopy class

Strominger-Witten: Monopoles are wrapped M5-branes
and the elusive non-perturbative Yang-Mills theory is in M-theory.
$\rightsquigarrow$ Open problem: *Wherein is M5-brane charge quantization?*



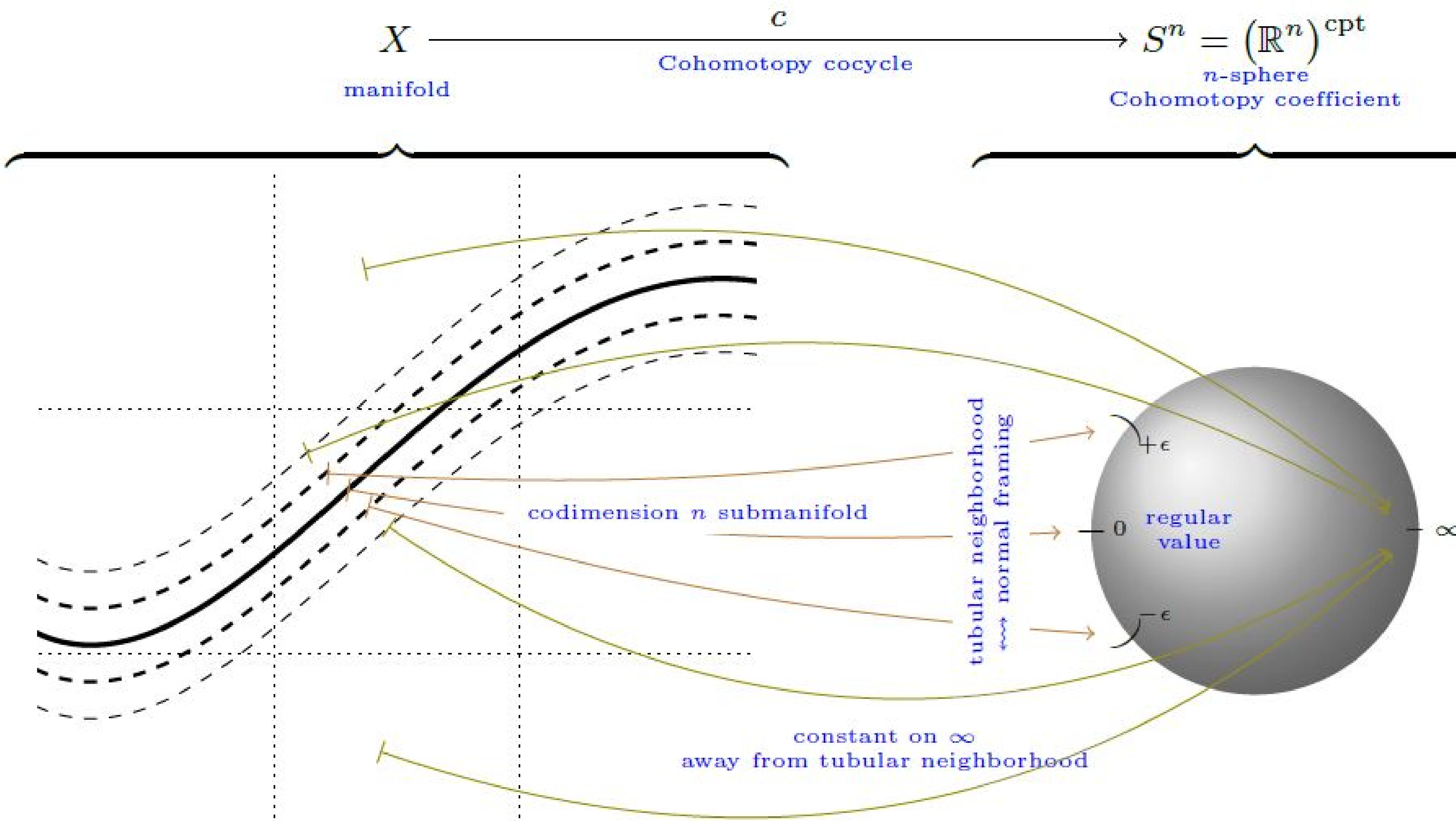
$$[c] \;\in\; \left\{ \begin{array}{ccc} X & \longrightarrow & S^4 /\!\!/_{\mathrm{Sp}(1)\cdot\mathrm{Sp}(2)} \\ & {\scriptstyle TX}\searrow \quad \swarrow & \\ & B(\mathrm{Sp}(1)\cdot\mathrm{Sp}(2)) & \end{array} \right\} \Big/ \sim_{\text{homotopy}} \;\simeq\; \mathrm{Cob}_{\mathrm{fr}}^{TX}(X)$$

charge = homotopy class

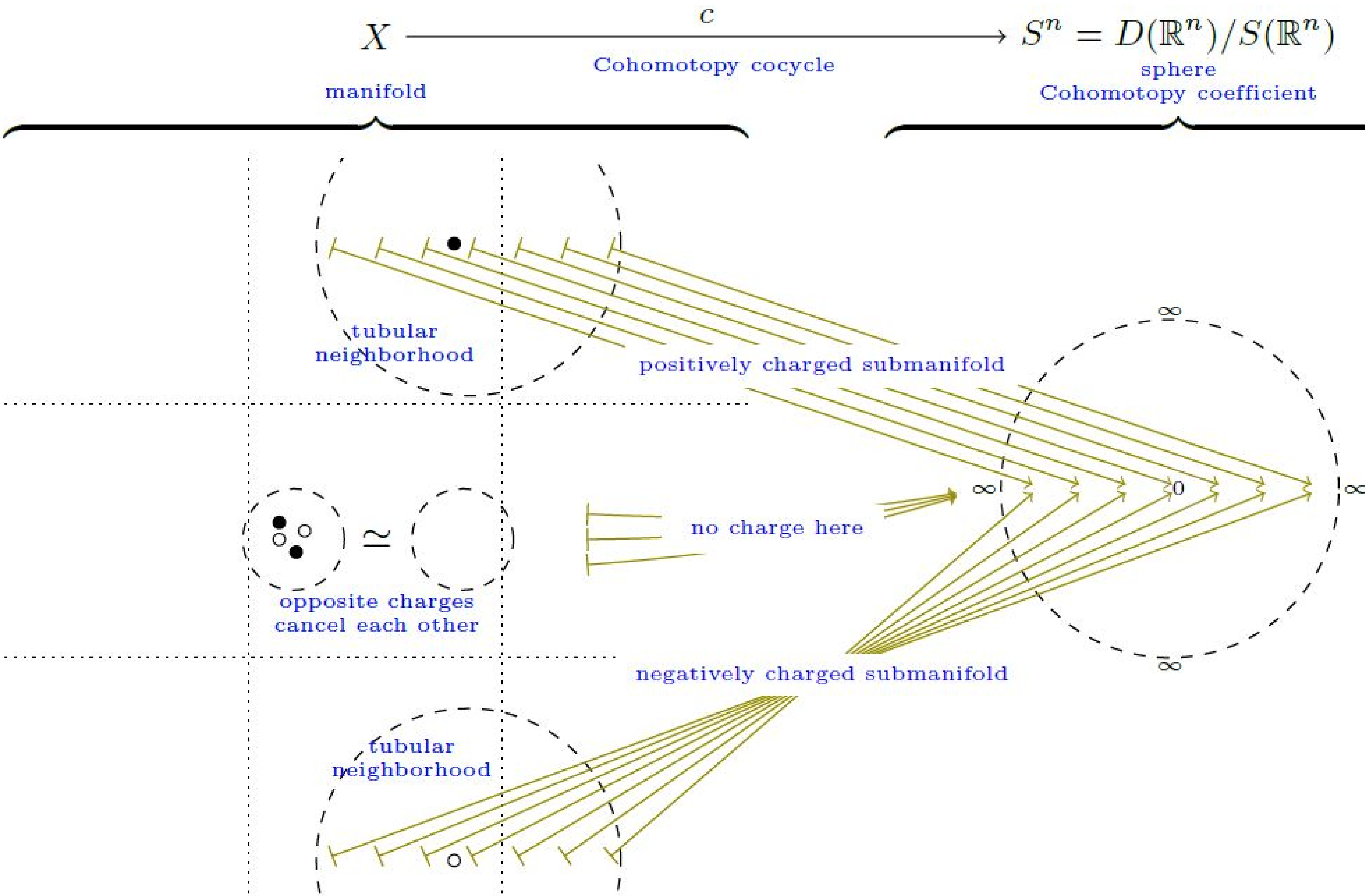
charge "lattice"

Hypothesis H (Fiorenza-Sati-Schreiber 19):

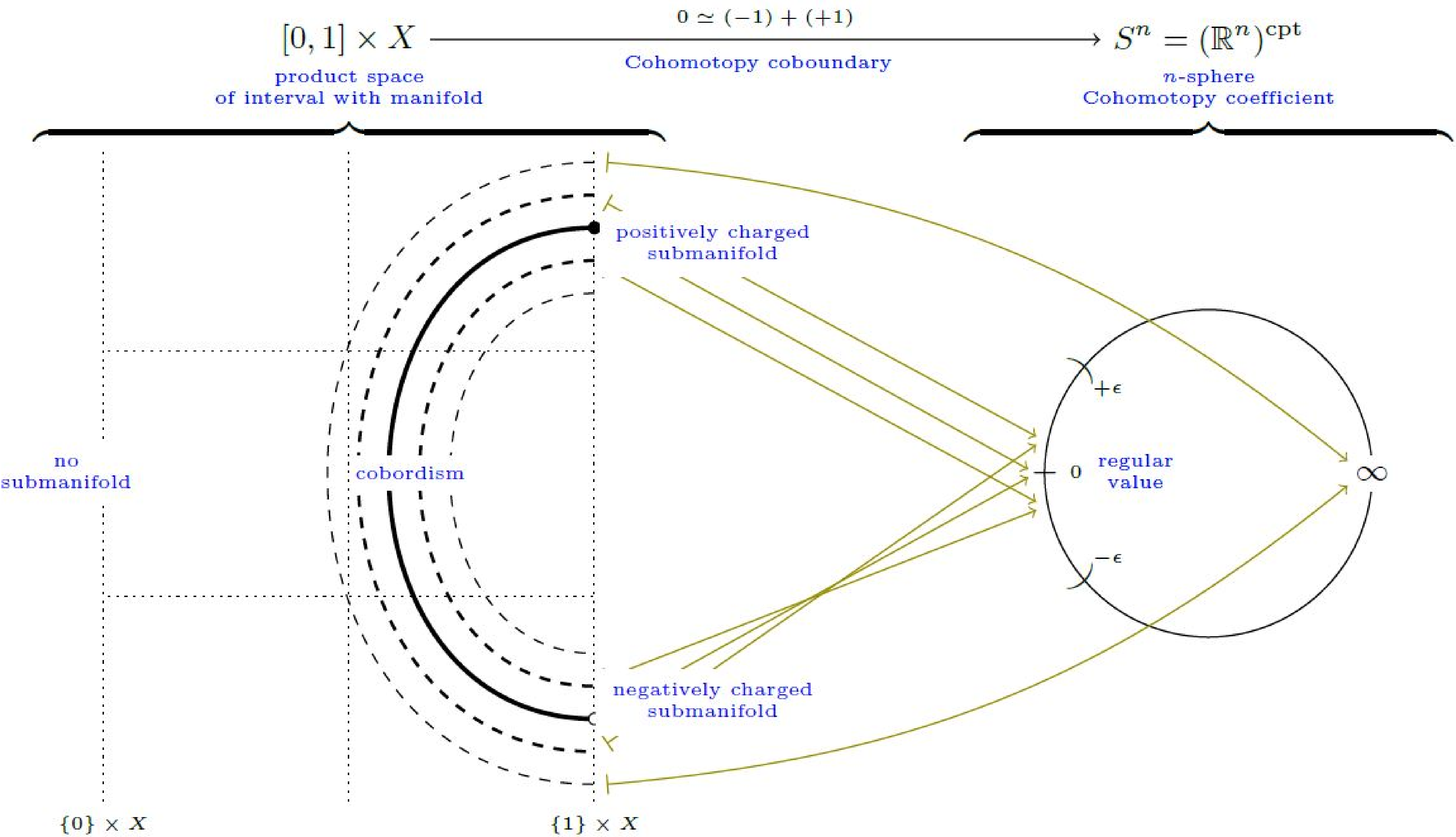
C-field is charge-quantized in J-twisted Cohomotopy theory.



Cohomotopy charge of normally framed submanifolds is represented by the submanifold's *asymptotic distance function*, traditionally known as the *Pontrjagin-Thom collapse*.

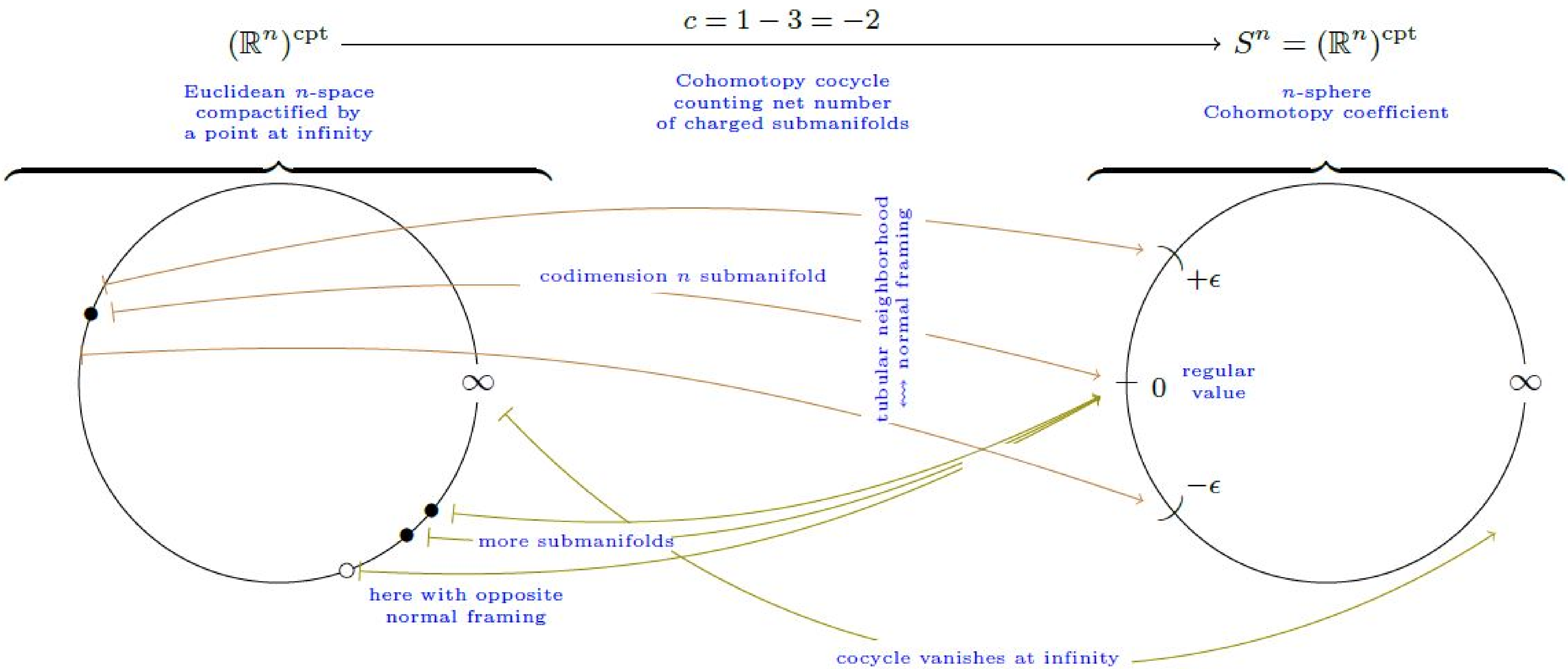


Cohomotopy charge of 0-dimensional submanifolds
(traditionally known as “electric field map” or *scanning map*)
exhibits net brane/anti-brane charge in $\mathbb{Z}$.



Brane/anti-brane pair creation & annihilation

is exhibited, under Hypothesis H, by normally framed cobordism.



Cohomotopy charge vanishing at ∞ on Euclidean n-space
is equivalently the Cohomotopy charge of the n-sphere
and hence takes values in homotopy groups of spheres.

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Brane charge in Equivariant Cohomotopy

implied by

Hypothesis H with Equivariant Hopf Degree Theorem

Sati-Schreiber 19a [arXiv:1909.12277]

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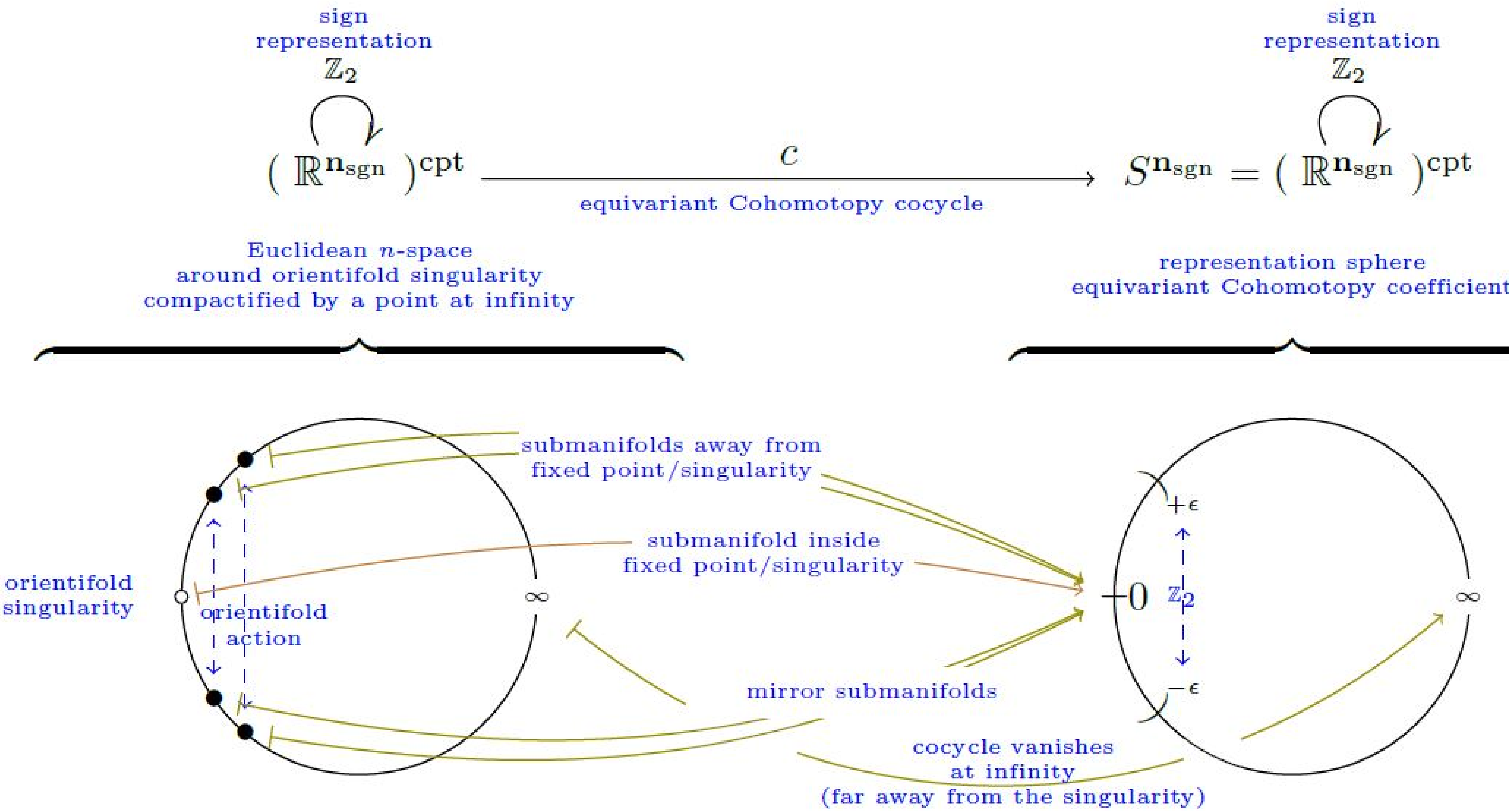
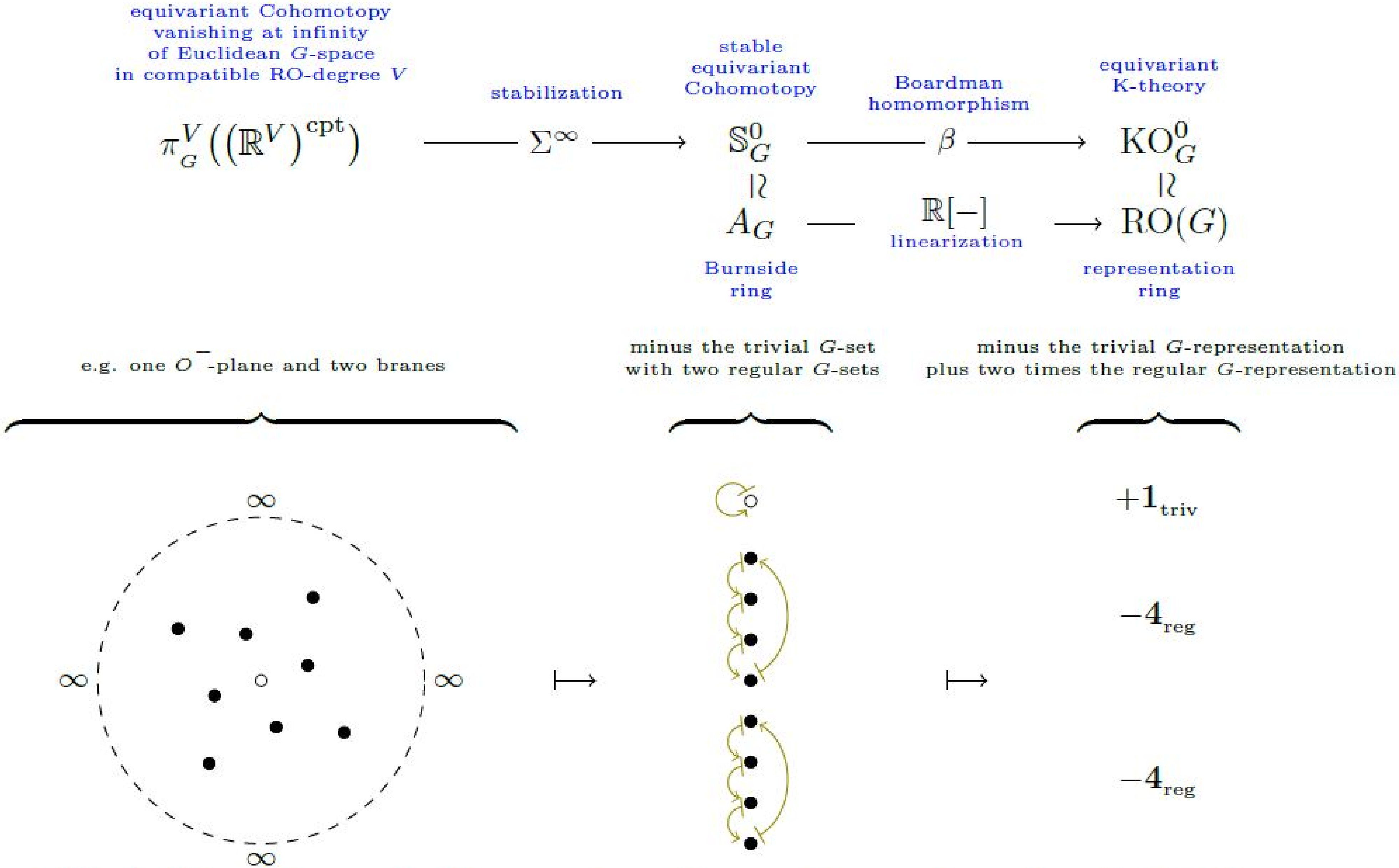
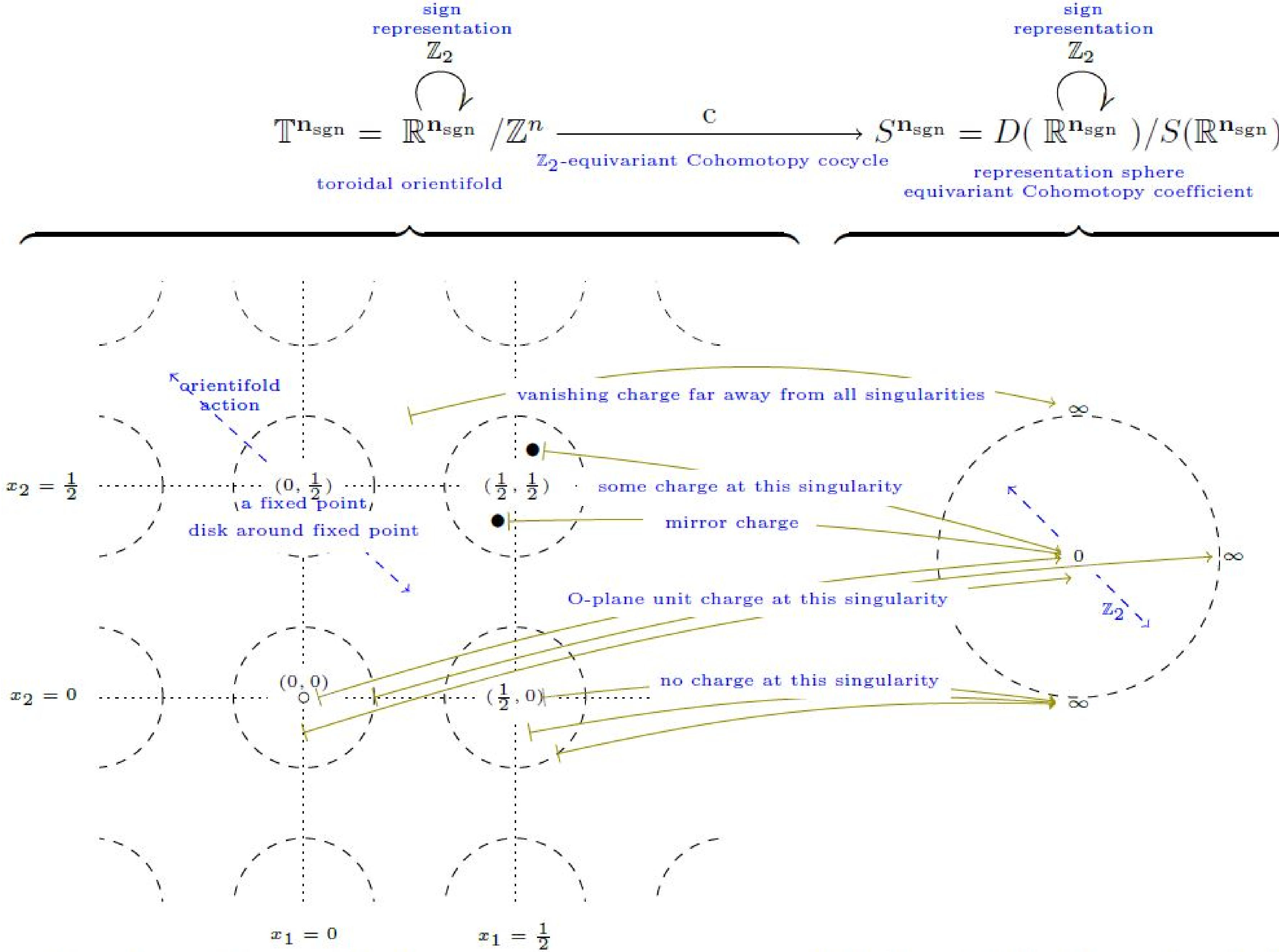


The equivariant Hopf degree theorem

says that $\mathbb{Z}_2$-equivariant Cohomotopy charges near singularities are sourced by, possibly, a charge attached to the singularity and any integer number of twice this charge located nearby.



Stabilization & linearization of equivariant Cohomotopy lands in equivariant K-theory. In this approximation virtual G-sets of (anti-)branes map to virtual permutation representations.



Equivariant Cohomotopy on toroidal orbifolds glued from local cocycles in the vicinity of singularities. By the equivariant Hopf degree theorem, all global cocucles are obtained this way.

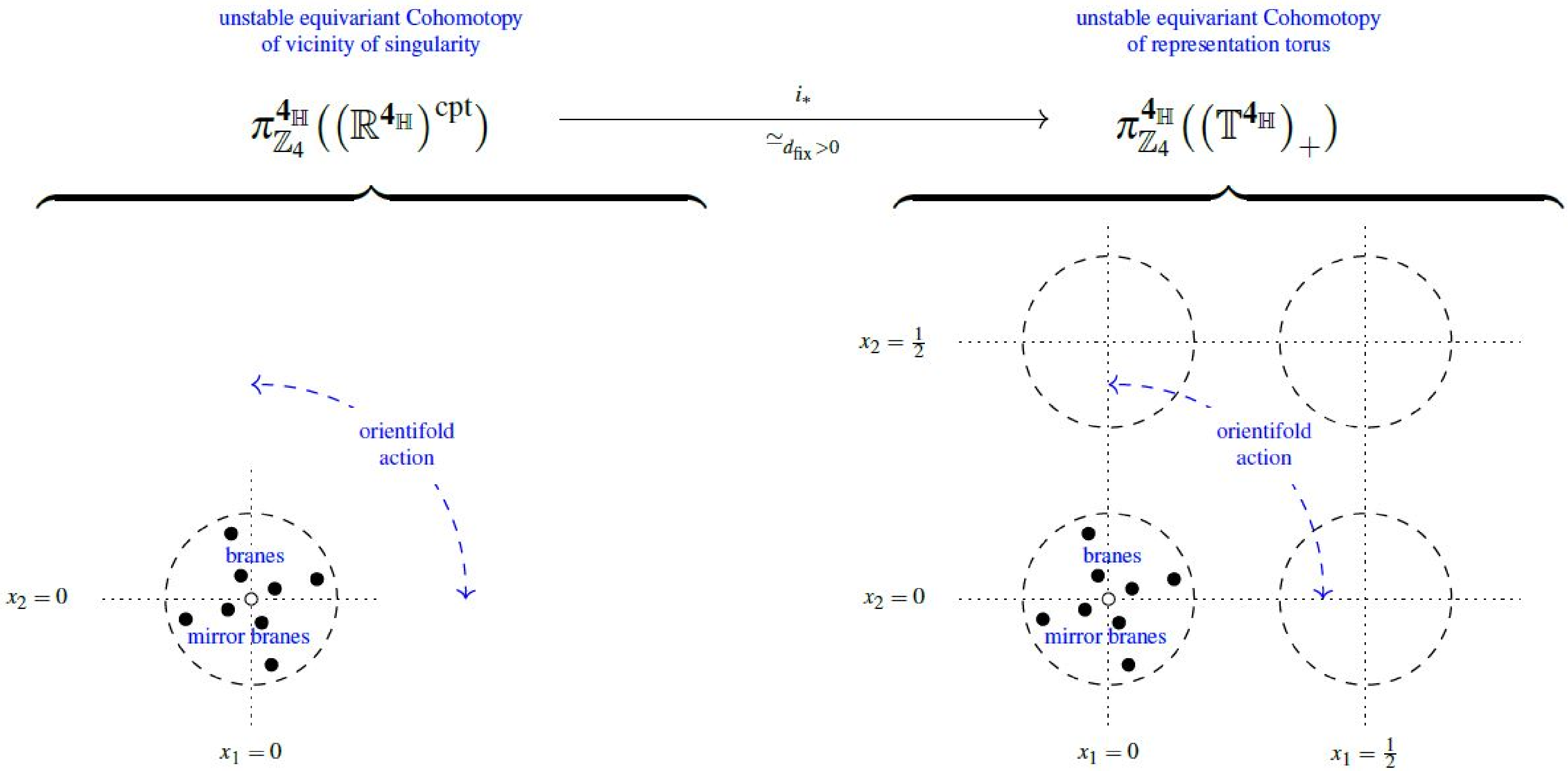
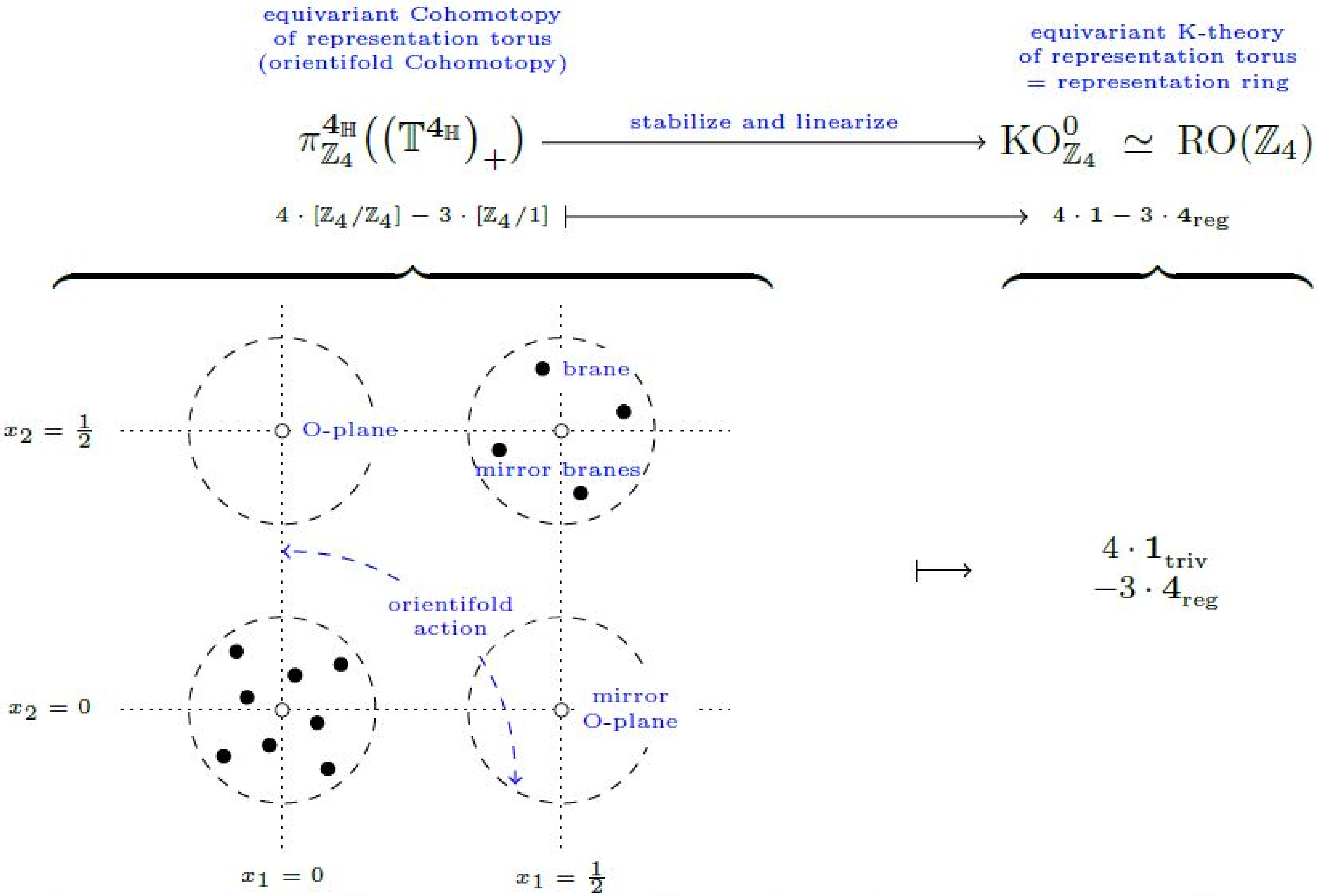


Figure O – Pushforward in equivariant Cohomotopy from the vicinity of a singularity to the full toroidal orientifold is an isomorphism on brane charges and an injection on O-plane charges, by Prop. 3.18. Shown is a case with $G = \mathbb{Z}_4$, as in *Figure M*.



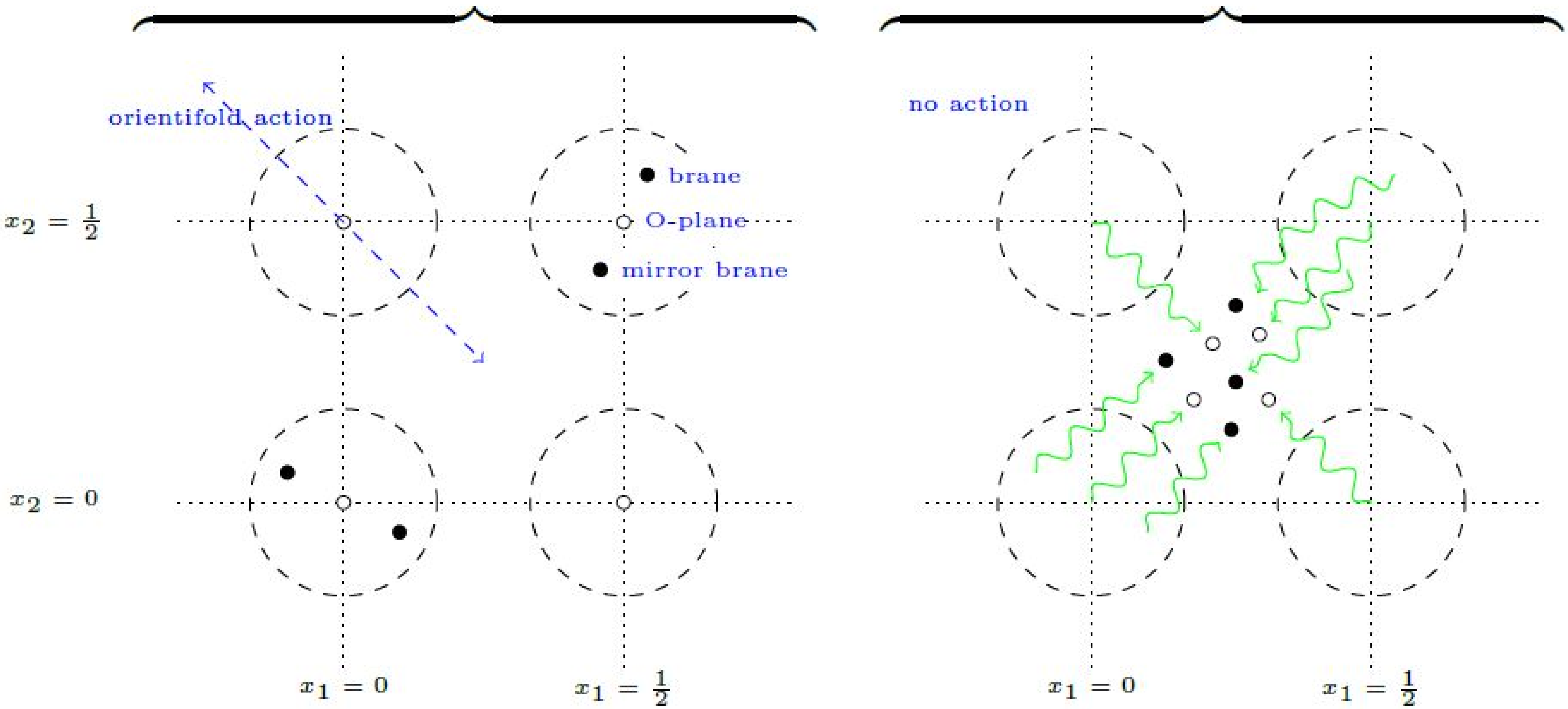
Equivariant Cohomotopy implies local tadpole cancellation

by the combined unstable and stable version of the equivariant Hopf degree theorem.

equivariant Cohomotopy of representation torus (orientifold Cohomotopy) $\xrightarrow{\text{forget equivariance}}$ plain Cohomotopy of plain torus

$$\pi^{\mathbf{n}_{\mathrm{sgn}}}_{\mathbb{Z}_2}\Big(\big(\mathbb{T}^{\mathbf{n}_{\mathrm{sgn}}}\big)_+\Big) \xrightarrow{\text{forget equivariance}} \pi^{n}\Big(\big(\mathbb{T}^{n}\big)_+\Big)_{\mathbb{R}}$$

$$4\cdot[\mathbb{Z}_2/\mathbb{Z}_2] - 2\cdot[\mathbb{Z}_2/1] \longmapsto \begin{array}{l}4\cdot 1 - 2\cdot 2\\ = 0\end{array}$$



Super-differential equivariant Cohomotopy implies global tadpole cancellation by forcing the charge to vanish at global Elmendorf stage, and only there.

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Brane charge in Differential Cohomotopy

implied by

Hypothesis H with May-Segal Theorem

Sati-Schreiber 19c [arXiv:1912.10425]

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Cohomotopy cocycle space

pointed mapping space

$$\boldsymbol{\pi}^4(X) \;:=\; \mathrm{Maps}^{*/}(X, S^4)$$

boldface!

$$\pi_0\big(\boldsymbol{\pi}^4(X)\big) = \left\{ \begin{array}{c} \text{Cohomotopy} \\ \text{cohomology} \\ \text{classes} \end{array} \right\} = \pi^4(X)$$

not boldface

Cohomotopy set

$$\pi_1\big(\boldsymbol{\pi}^4(X)\big) = \left\{ \begin{array}{c} \text{Cohomotopy} \\ \text{gauge} \\ \text{transformations} \end{array} \right\}$$

$$\pi_2\big(\boldsymbol{\pi}^4(X)\big) = \left\{ \begin{array}{c} \text{Cohomotopy} \\ \text{gauge of gauge} \\ \text{transformations} \end{array} \right\}$$

$\vdots$

Cohomotopy cocycle space

vanishing at ∞ on Euclidean 3-space

May-Segal theorem

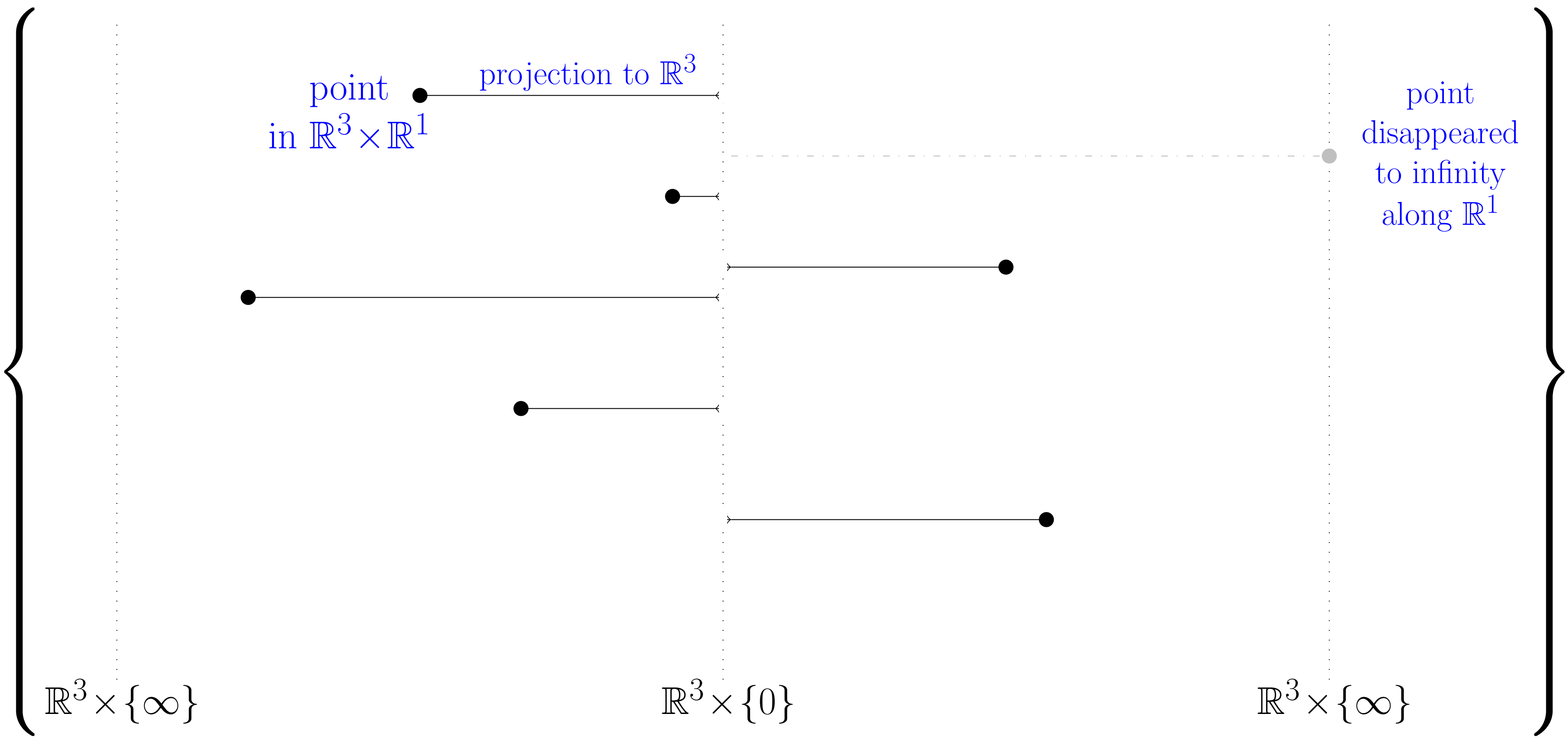
$$\boldsymbol{\pi}^4\big((\mathbb{R}^3)^{\mathrm{cpt}}\big) \xleftarrow{\ \mathrm{hmtpy}\ \simeq\ } \mathrm{Conf}\big(\mathbb{R}^3, \mathbb{D}^1\big)$$

assign unit charge
in Cohomotopy
to each point

configuration space of points

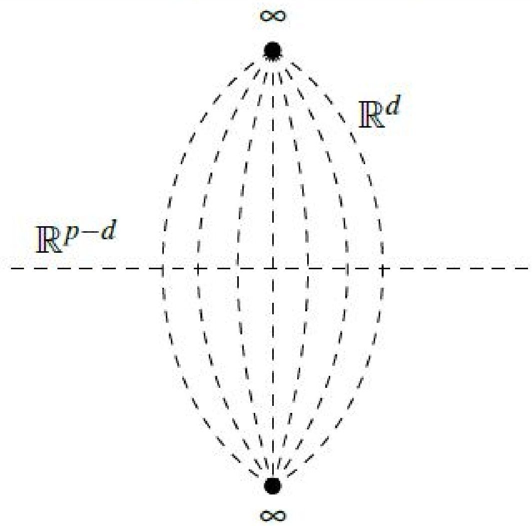
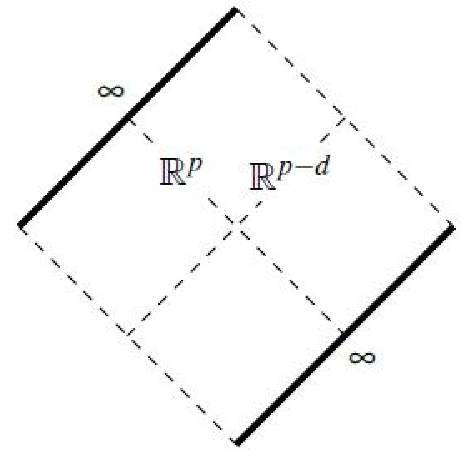
in $\mathbb{R}^3 \times \mathbb{R}^1$
which are:
1) unordered
2) distinct after projection to $\mathbb{R}^3$
3) allowed to vanish to ∞ along $\mathbb{R}^1$

$=$



hence: a form of differential Cohomotopy assigns configuration spaces:

$$\boldsymbol{\pi}^4\big((\mathbb{R}^d)^{\mathrm{cpt}} \wedge (\mathbb{R}^{4-d})_+\big) \xleftarrow[\simeq]{\mathrm{hmtpy}} \boldsymbol{\pi}^4_{\mathrm{diff}}\big((\mathbb{R}^d)^{\mathrm{cpt}} \wedge (\mathbb{R}^{4-d})_+\big) := \mathrm{Conf}\big(\mathbb{R}^d, \mathbb{D}^{4-d}\big)$$

<table>
<tr><th rowspan="2">Smash product of pointed topological spaces</th><th colspan="2">Visualization</th></tr>
<tr><th>with point at infinity</th><th>as Penrose diagram</th></tr>
<tr><td>cocycles vanish at infinity along these direction
$(\mathbb{R}^d)^{\mathrm{cpt}} \wedge (\mathbb{R}^{p-d})_+$
...but not necessarily along these</td><td>
</td><td>
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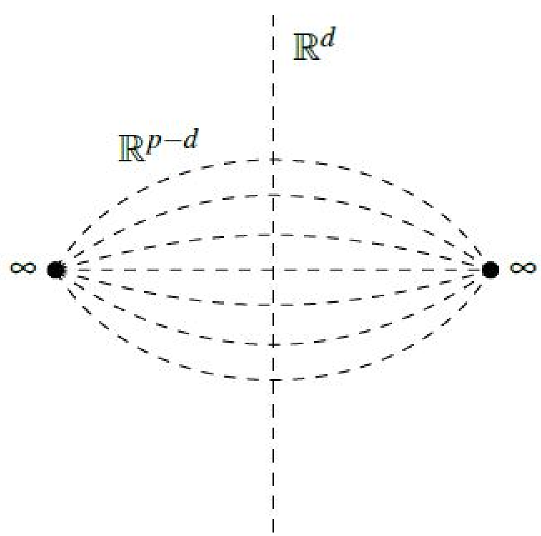
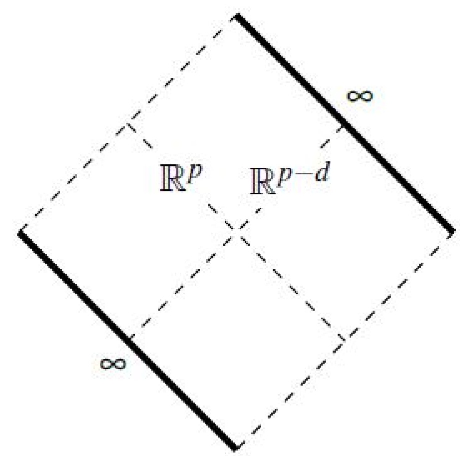
<tr><td>cocycles vanish at infinity along these direction
$(\mathbb{R}^d)_+ \wedge (\mathbb{R}^{p-d})^{\mathrm{cpt}}$
...but not necessarily along these</td><td>
</td><td>
</td></tr>
</table>

Lemma:

Un-ordered configurations of points in $\mathbb{R}^D$ with labels in $\mathbb{D}^1$

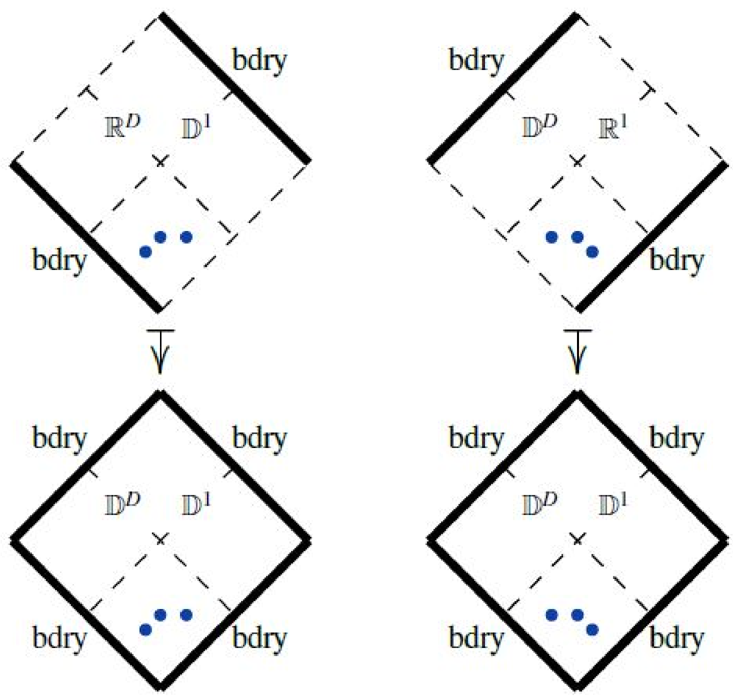
Un-ordered configurations of points in $\mathbb{R}^1$ with labels in $\mathbb{R}^D$

$$\underset{n\in\mathbb{N}}{\sqcup} \mathrm{Conf}_{\{1,\cdots,n\}}\big(\mathbb{R}^D\big) \underset{hmtpy}{\simeq} \mathrm{Conf}\big(\mathbb{R}^D,\mathbb{D}^1\big) \underset{\mathrm{Conf}(\mathbb{D}^{D+1})}{\times} \mathrm{Conf}\big(\mathbb{R}^1,\mathbb{D}^D\big)$$

Ordered configurations of points in $\mathbb{R}^D$

Un-ordered configurations of points in $\mathbb{D}^{D+1}$

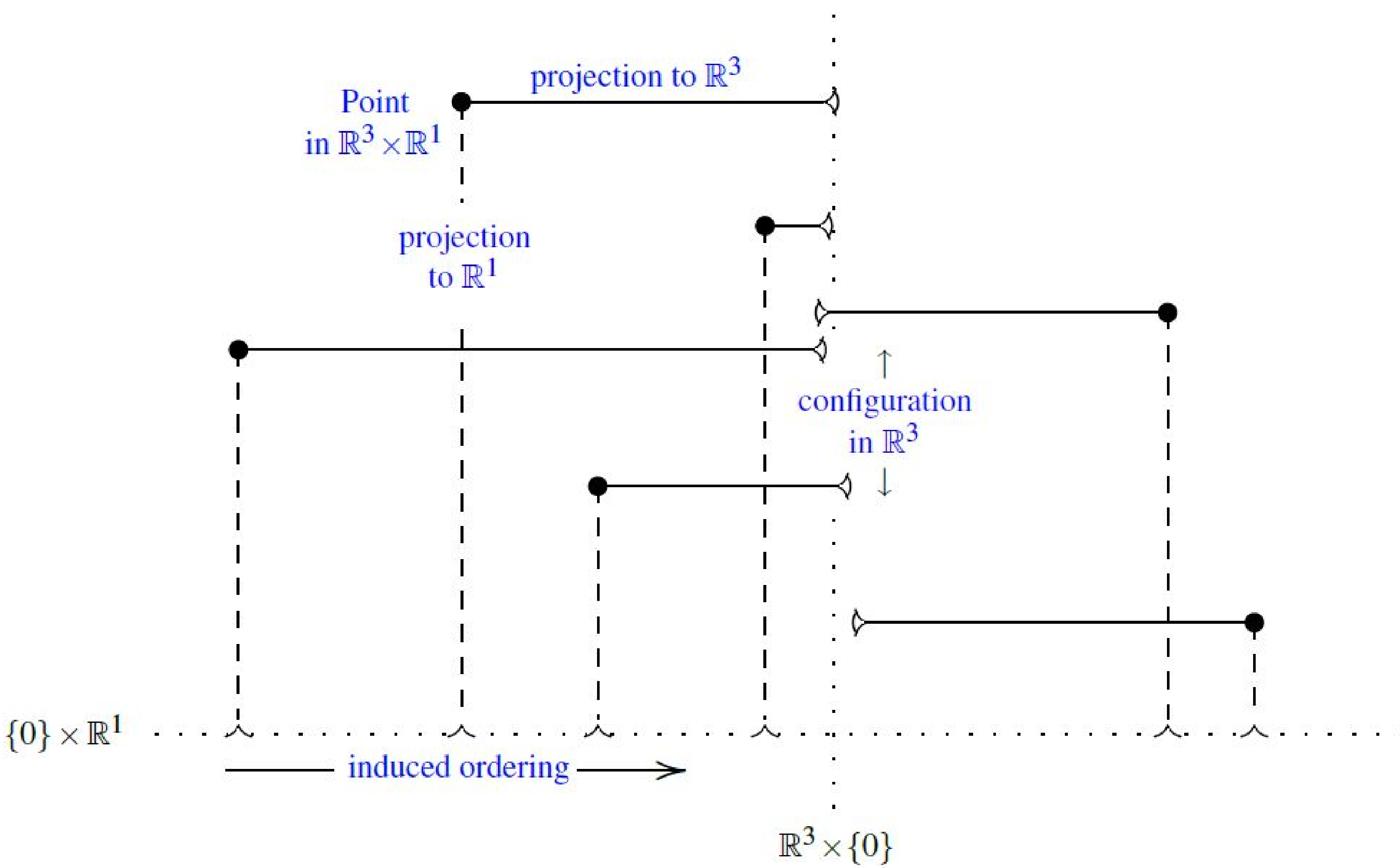
$$\left(\begin{array}{ccc} \mathrm{Conf}\big(\mathbb{R}^D,\mathbb{D}^1\big) & & \mathrm{Conf}\big(\mathbb{R}^1,\mathbb{D}^D\big) \\ & {\scriptstyle ((i_L)^*)_*} \searrow \quad \swarrow {\scriptstyle ((i_R)^*)_*} & \\ & \mathrm{Conf}\big(\mathbb{D}^{D+1}\big) & \end{array}\right)$$



Lemma:

$$\underset{n\in\mathbb{N}}{\sqcup}\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{R}^D\big) \underset{hmtpy}{\simeq} \mathrm{Conf}\big(\mathbb{R}^D,\mathbb{D}^1\big) \underset{\mathrm{Conf}(\mathbb{D}^{D+1})}{\times} \mathrm{Conf}\big(\mathbb{R}^1,\mathbb{D}^D\big)$$

- $\underset{n\in\mathbb{N}}{\sqcup}\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{R}^D\big)$: *Ordered configurations of points in $\mathbb{R}^D$*
- $\mathrm{Conf}\big(\mathbb{R}^D,\mathbb{D}^1\big)$: *Un-ordered configurations of points in $\mathbb{R}^D$ with labels in $\mathbb{D}^1$*
- $\mathrm{Conf}(\mathbb{D}^{D+1})$: *Un-ordered configurations of points in $\mathbb{D}^{D+1}$*
- $\mathrm{Conf}\big(\mathbb{R}^1,\mathbb{D}^D\big)$: *Un-ordered configurations of points in $\mathbb{R}^1$ with labels in $\mathbb{R}^D$*

$=$ {  }

Consequence:

assuming
Hypothesis H:

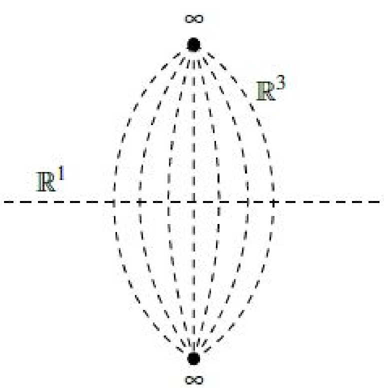
Transversal space to 3-codim branes hence to D6-branes

Transversal space to 1-codim branes hence to D8-branes

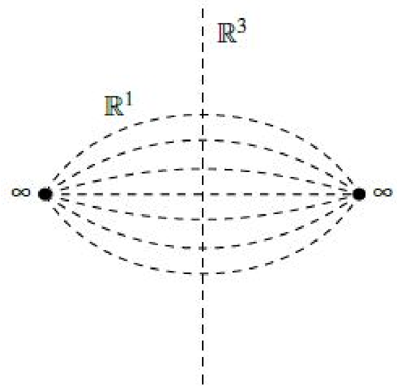
Differential Cohomotopy

differential Cohomotopy cocycle space reflecting D6 ⊥ D8 -charges

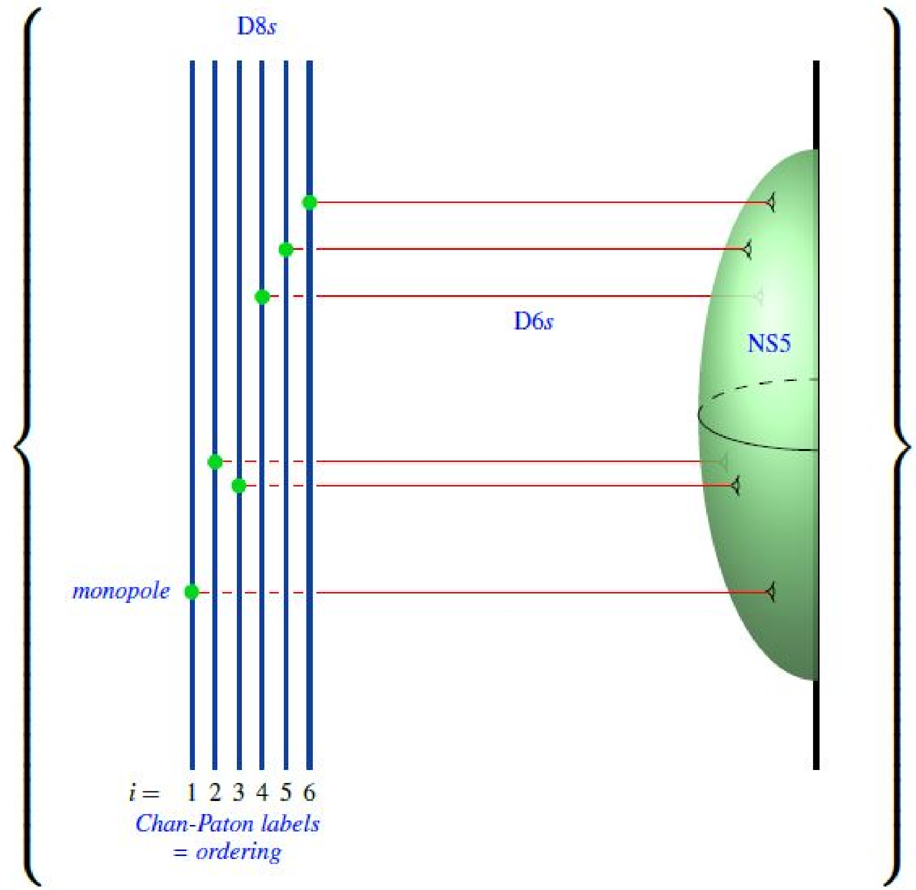
$$(\mathbb{R}^3)^{\mathrm{cpt}} \wedge (\mathbb{R}^1)_+ \;\cup\; (\mathbb{R}^3)_+ \wedge (\mathbb{R}^1)^{\mathrm{cpt}} \;\xmapsto{\;\boldsymbol{\pi}^4_{\mathrm{diff}}\;}\; \bigsqcup_{n \in \mathbb{N}} \mathrm{Conf}_{\{1,\cdots,n\}}\big(\mathbb{R}^3\big)$$



$\cup$



$\longmapsto$



0) Nonabelian Differential Cohomology

Brane charge in...

1) Twisted Cohomotopy theory

2) Equivariant Cohomotopy theory

3) Differential Cohomotopy theory

implies...

4) Hanany-Witten Theory

5) Chan-Paton Data

6) BMN Matrix Model States

7) M2/M5 Brane Bound States

(4)

Hanany-Witten Theory

implied by

Hypothesis H with Fadell-Husseini Theorem

Sati-Schreiber 19c [arXiv:1912.10425]

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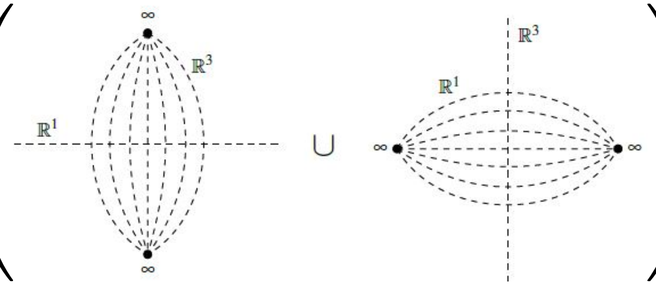
higher co-observables on D6 ⊥ D8-intersections

topological phase space

$$H_\bullet\left(\underset{[c]}{\bigsqcup} \Omega_c\, \boldsymbol{\pi}^4_{\mathrm{diff}} \left(\cdots \right) \right)$$

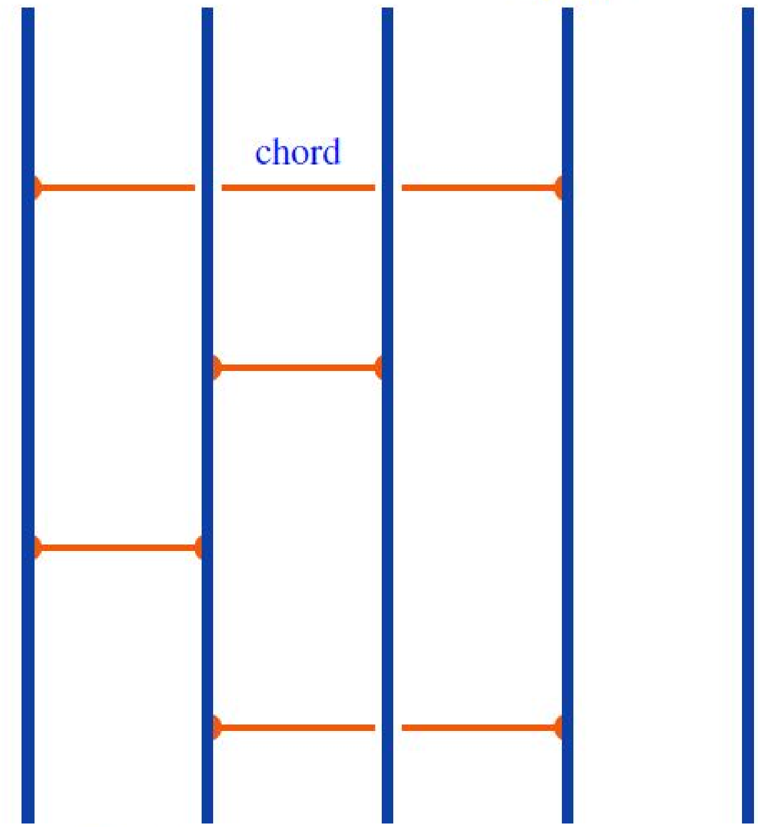


$$\simeq H_\bullet\left(\underset{N_{\mathrm{f}}\in\mathbb{N}}{\bigsqcup} \Omega \underset{\{1,\cdots,N_{\mathrm{f}}\}}{\mathrm{Conf}} (\mathbb{R}^3) \right) \quad \text{(by the above)}$$

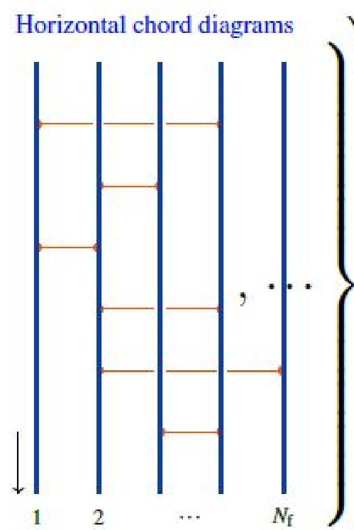
$$\simeq \underset{N_{\mathrm{f}}\in\mathbb{N}}{\bigoplus} \mathcal{A}^{\mathrm{pb}}_{N_{\mathrm{f}}}$$

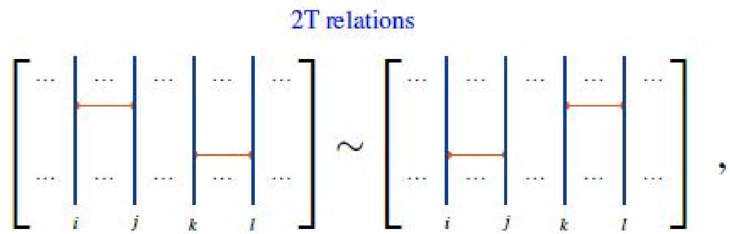
Fadell-Husseini theorem

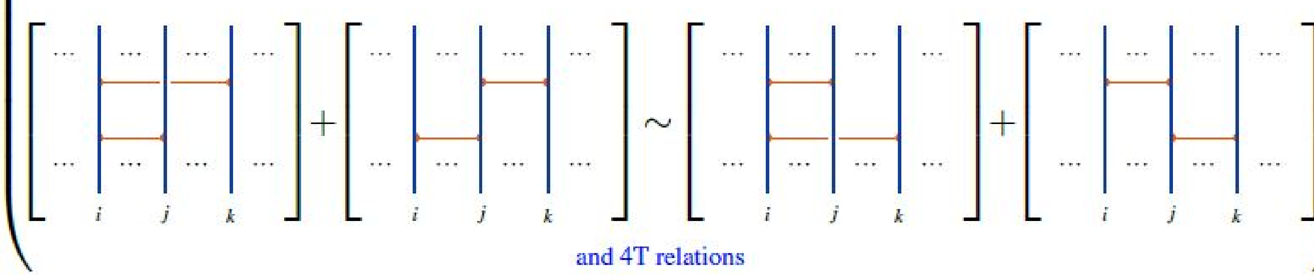
are algebra of **horizontal chord diagrams**:



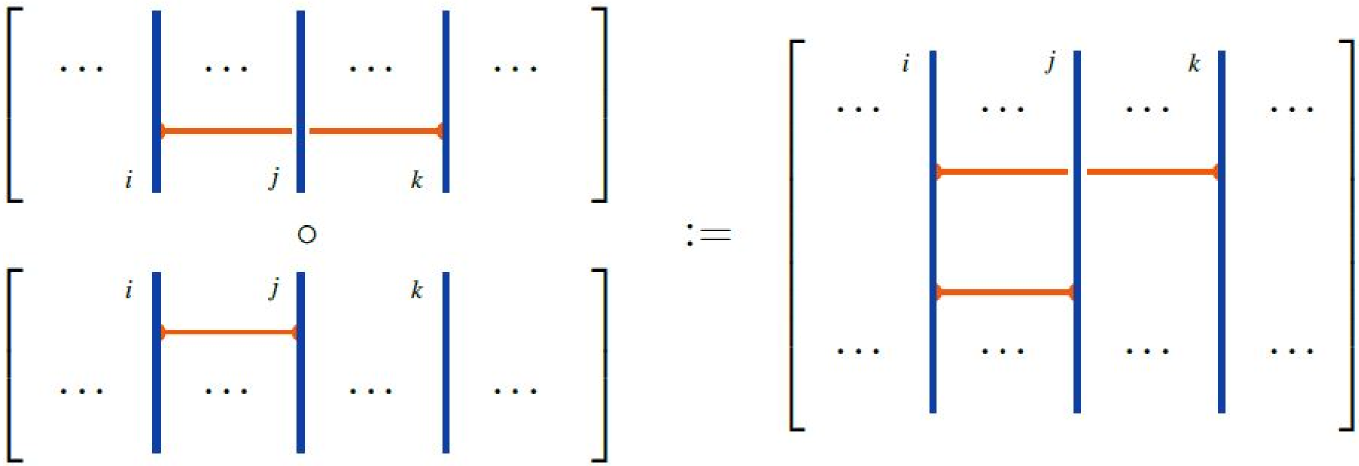
$$\mathcal{A}^{\mathrm{pb}}_{N_{\mathrm{f}}} := \mathrm{Span}\left(\left\{ \cdots , \cdots \right\} \right) \Big/ \left(\cdots \right)$$







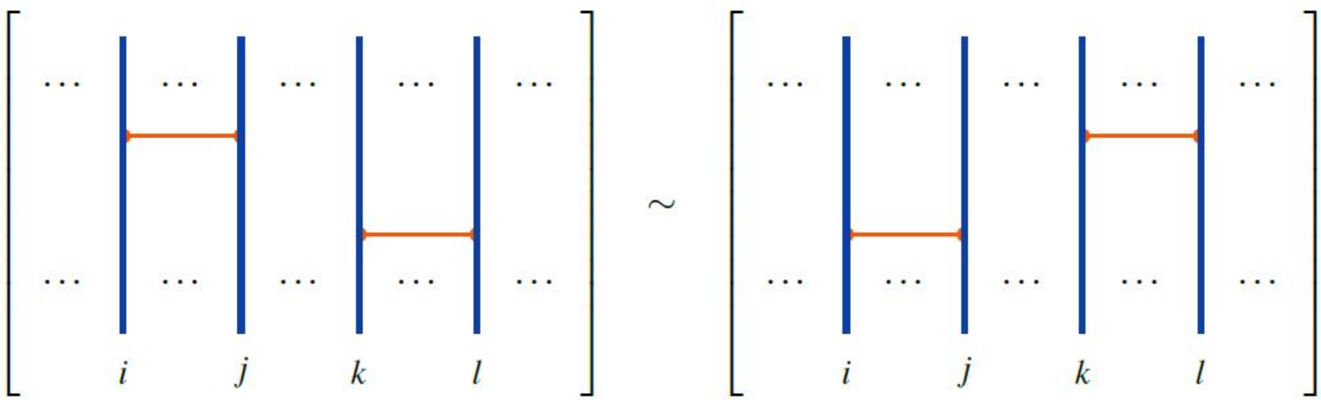
Horizontal chord diagrams form **algebra under concatenation of strands**.



$$t_{ik} \circ t_{ij} \;=\; t_{ik} t_{ij}$$

This is universal enveloping algebra of the infinitesimal braid Lie algebra (Kohno):

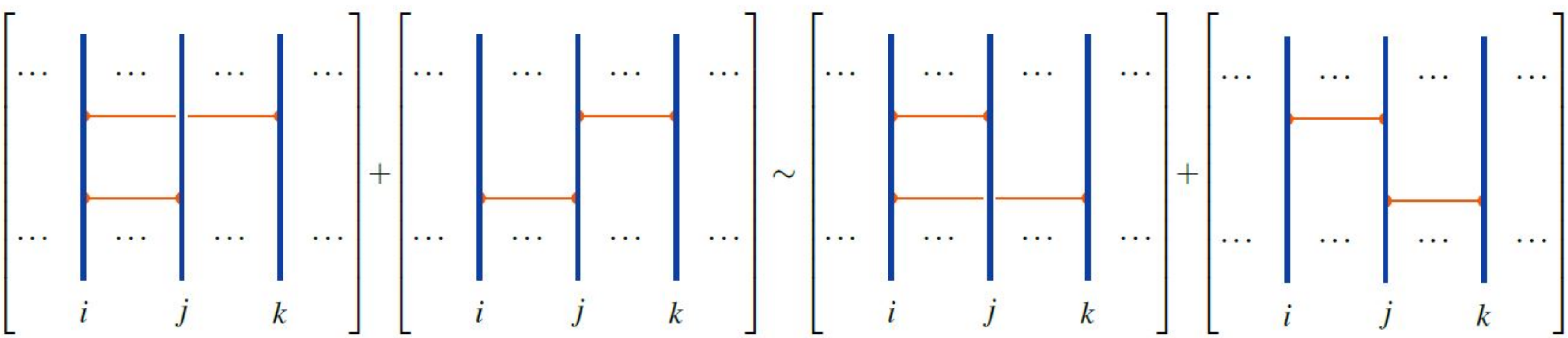
(i) the *2T relations*:



$$[t_{ij}, t_{kl}] = 0$$

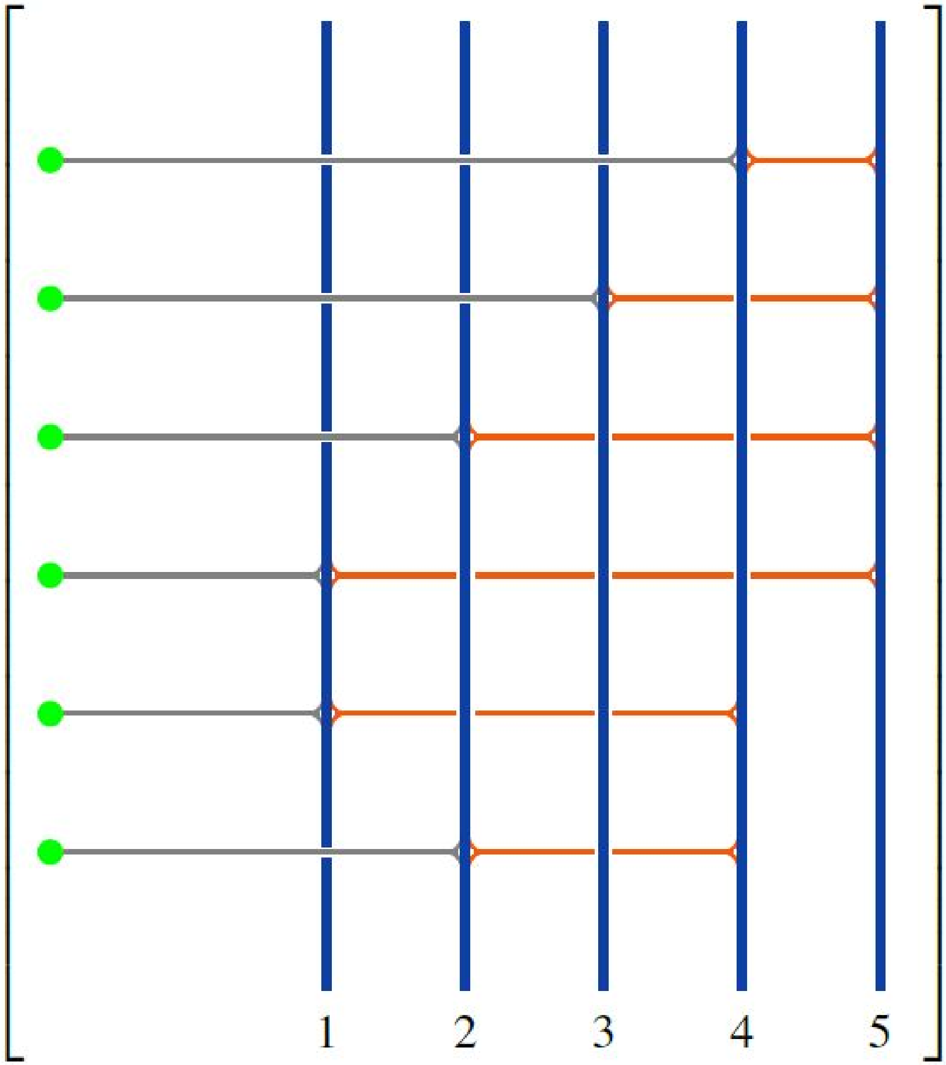
(ii) the *4T relations*

$$[t_{ij}, t_{ik} + t_{jk}] = 0$$



Consider the subspace of skew-symmetric co-observables,

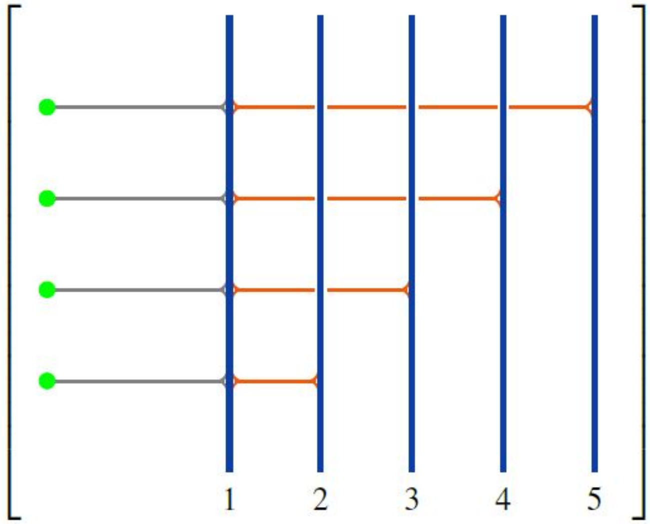
denote elements as follows:



$$= t_{45} \wedge t_{35} \wedge t_{25} \wedge t_{15} \wedge t_{14} \wedge t_{24}$$

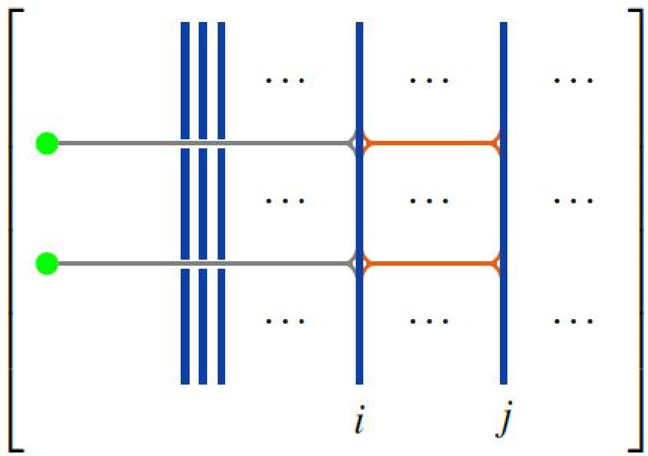
In the subspace of skew-symmetric co-observables we find:

the 2T relations become the *ordering constraint*



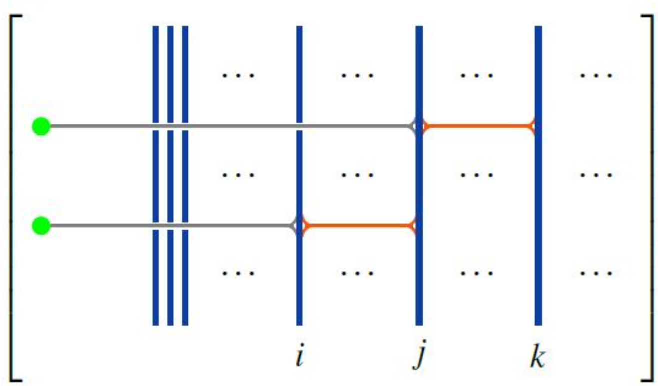
= form of any non-vanishing element

skew-symmetry becomes the *s-rule*



= 0

the 4T relations become the *breaking rule*



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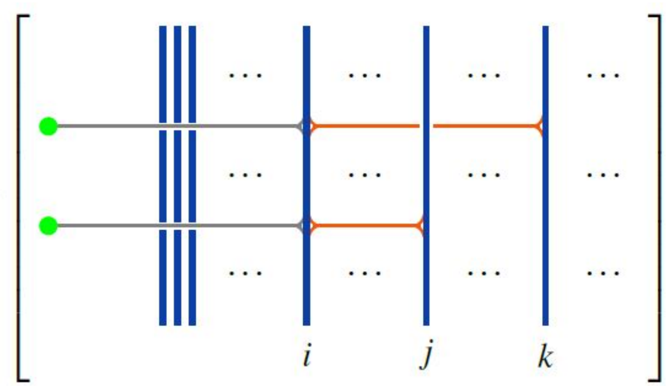


these are the rules of Hanany-Witten theory
for NS5 $\perp$ Dp $\perp$ D$(p+2)$-brane intersections

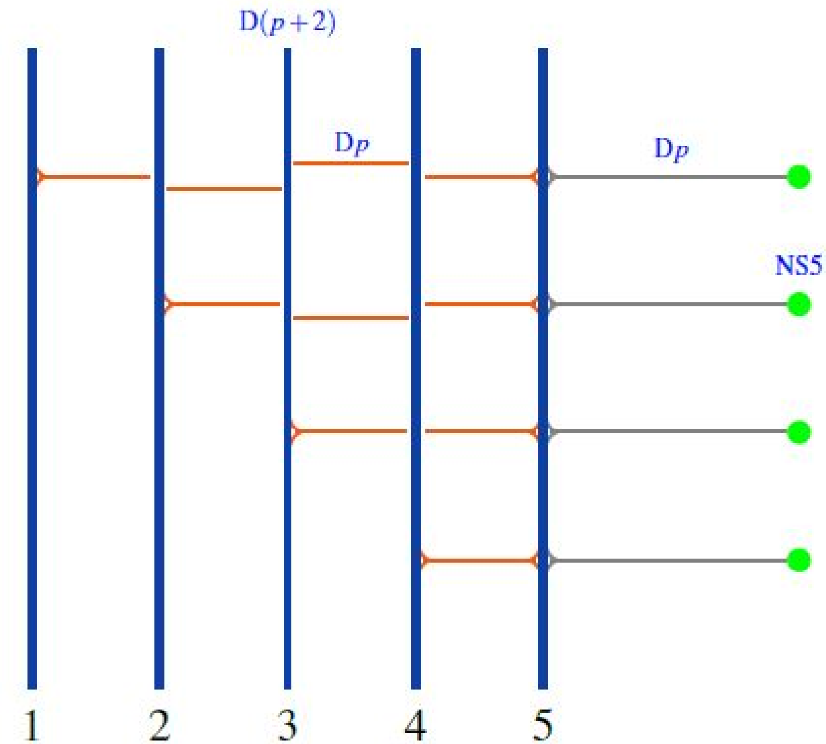
if we identify horizontal chord diagrams as follows:

(i) strands as D$(p+2)$-branes;

(ii) chords as Dp-branes,
stretching between D$(p+2)$s;

(iii) green dots as NS5-branes;

(iv) gray lines as Dp-branes,
stretching from NS5 to D$(p+2)$.



(5)

Chan-Paton data

implied by

Hypothesis H with Bar-Natan theorem

Sati-Schreiber 19c [arXiv:1912.10425]

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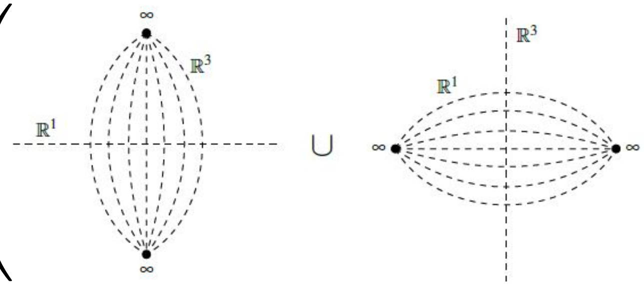
higher co-states on D6 ⊥ D8 -intersections

$$H^\bullet\left(\underset{\text{phase space}}{\overset{\text{topological}}{\coprod_{[c]}\Omega_c\,\boldsymbol{\pi}^4_{\mathrm{diff}}}}\left(\right.\right.$$



$$\left.\left.\right)\right)$$

$$\simeq H^\bullet\Big(\coprod_{N_{\mathrm{f}}\in\mathbb{N}} \mathrm{Conf}_{\{1,\cdots,N_{\mathrm{f}}\}}(\mathbb{R}^3)\Big) \quad \text{(by the above)}$$

$$\simeq \bigoplus_{N_{\mathrm{f}}\in\mathbb{N}} \mathcal{W}^{\mathrm{pb}}_{N_{\mathrm{f}}}$$

Kohno & Cohen-Gitler theorem

are **horizontal weight systems**:

Cohomology — weight systems

Higher observables $H^\bullet\big(\underbrace{\coprod_{N_{\mathrm{f}}\in\mathbb{N}}\Omega\,\mathrm{Conf}_{\{1,\cdots,N_{\mathrm{f}}\}}(\mathbb{R}^3)}\big) \simeq \mathscr{W} = (\mathscr{A})^*$ Higher co-states

phase space — dualization

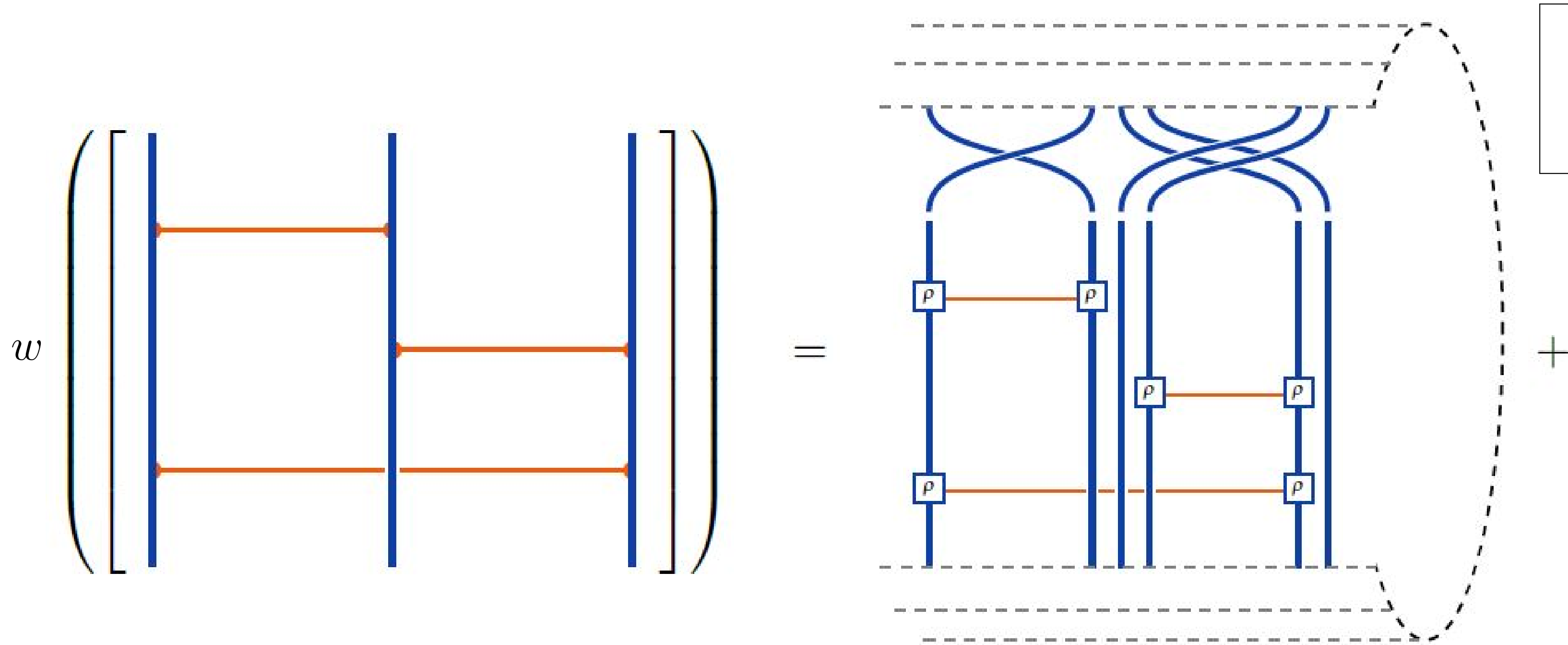
Higher co-observables $H_\bullet\big(\overbrace{\coprod_{N_{\mathrm{f}}\in\mathbb{N}}\Omega\,\mathrm{Conf}_{\{1,\cdots,N_{\mathrm{f}}\}}(\mathbb{R}^3)}\big) \simeq \mathscr{A} = (\mathscr{W})^*$ Higher states

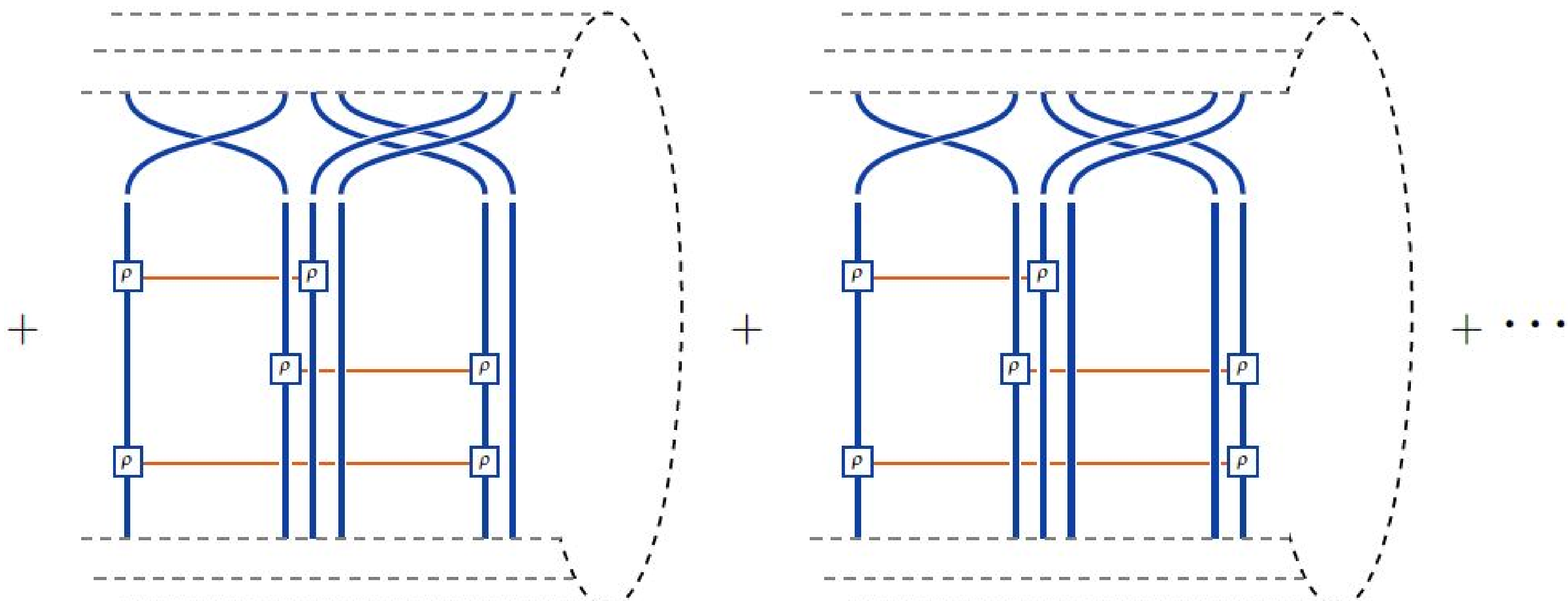
Homology — chord diagrams

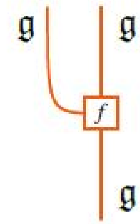
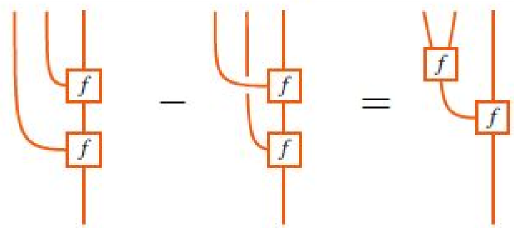
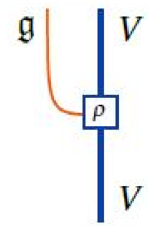
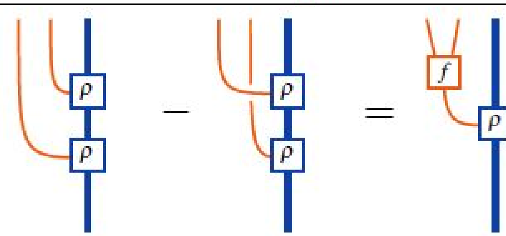
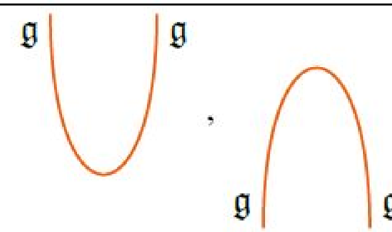
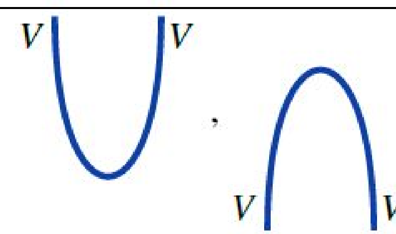
All **horizontal weight systems** $w : \mathcal{A}^{\text{pb}} \to \mathbb{C}$ **come from Chan-Paton data**:

1) metric Lie representations ρ	2) stacks of coincident strands	3) winding monodromies:

Bar-Natan theorem





Data of metric Lie representation	Category notation	Penrose notation	Index notation
Lie bracket	$\mathfrak{g}\otimes\mathfrak{g} \xrightarrow{f} \mathfrak{g}$		$f_{ab}{}^{c}$
Jacobi identity	$\begin{array}{ccc} \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} & \xrightarrow{\mathrm{id}\otimes f - f\otimes\mathrm{id}} & \mathfrak{g}\otimes\mathfrak{g} \\ \downarrow{\scriptstyle \sigma_{213}\circ(\mathrm{id}\otimes f)} & & \downarrow{\scriptstyle f} \\ \mathfrak{g}\otimes\mathfrak{g} & \xrightarrow[f]{} & \mathfrak{g} \end{array}$		$f_{ae}{}^{d} f_{bc}{}^{e} - f_{be}{}^{d} f_{ac}{}^{e} = f_{ec}{}^{d} f_{ab}{}^{e}$
Lie action	$\mathfrak{g}\otimes V \xrightarrow{\rho} V$		$\rho_{a}{}^{i}{}_{j}$
Lie action property	$\begin{array}{ccc} \mathfrak{g}\otimes\mathfrak{g}\otimes V & \xrightarrow{\mathrm{id}\otimes \rho - f\otimes\mathrm{id}} & \mathfrak{g}\otimes V \\ \downarrow{\scriptstyle \sigma_{213}\circ(\mathrm{id}\otimes \rho)} & & \downarrow{\scriptstyle \rho} \\ \mathfrak{g}\otimes V & \xrightarrow[\rho]{} & V \end{array}$		$\rho_{a}{}^{j}{}_{l}\, \rho_{b}{}^{l}{}_{i} - \rho_{b}{}^{j}{}_{l}\, \rho_{a}{}^{l}{}_{i} = f_{ab}{}^{c}\, \rho_{c}{}^{j}{}_{i}$
Metric	$\mathfrak{g}\otimes\mathfrak{g} \xrightarrow{g} \mathbf{1}$, $\mathbf{1} \xrightarrow{g^{-1}} \mathfrak{g}\otimes\mathfrak{g}$		g_{ab} , g^{ab}
	$V\otimes V \xrightarrow{k} \mathbf{1}$, $\mathbf{1} \xrightarrow{k^{-1}} V\otimes V$		k_{ij} , k^{ij}

(6)

BMN Matrix Model States

implied by

Hypothesis H

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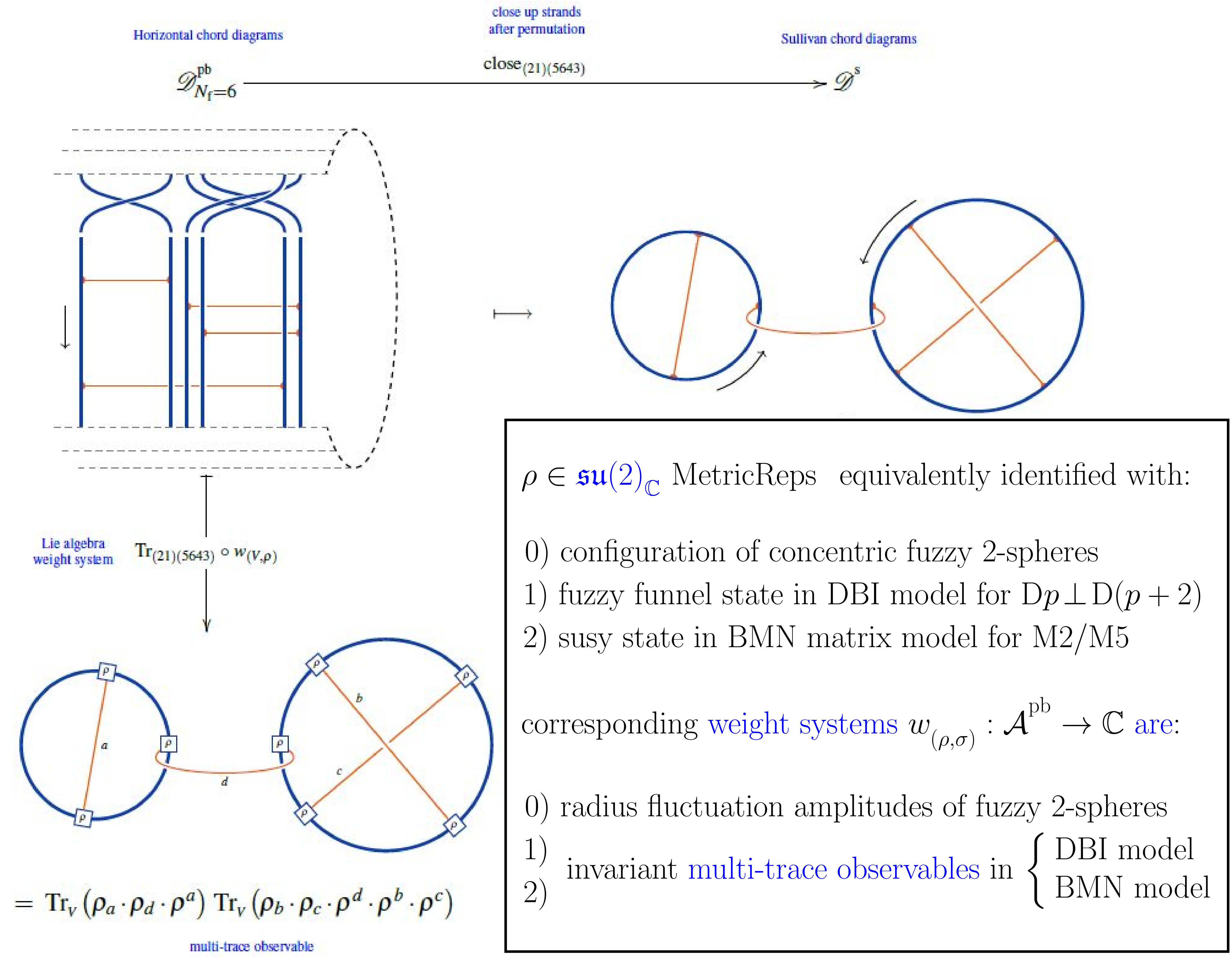


$\rho \in \mathfrak{su}(2)_{\mathbb{C}}$ MetricReps equivalently identified with:

0) configuration of concentric fuzzy 2-spheres
1) fuzzy funnel state in DBI model for $\mathrm{D}p \perp \mathrm{D}(p+2)$
2) susy state in BMN matrix model for M2/M5

corresponding weight systems $w_{(\rho,\sigma)} : \mathcal{A}^{pb} \to \mathbb{C}$ are:

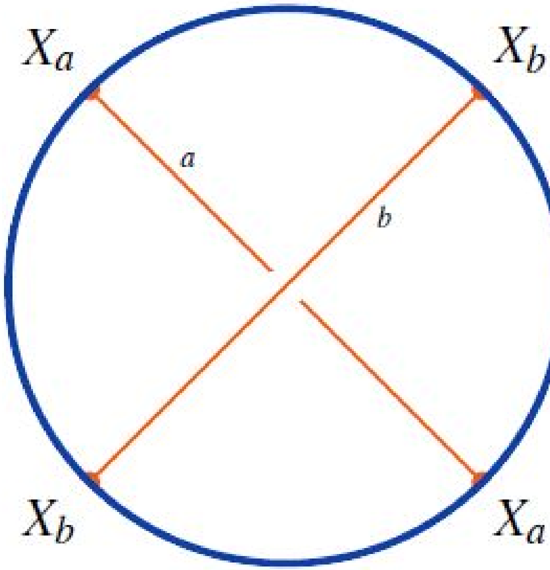
0) radius fluctuation amplitudes of fuzzy 2-spheres
1)
2) invariant multi-trace observables in $\left\{ \begin{array}{l} \text{DBI model} \\ \text{BMN model} \end{array} \right.$

’

0) Radius fluctuation observables on N-bit fuzzy 2-spheres S^2_N are $\mathbf{N} \in \mathfrak{su}(2)_{\mathbb{C}}$MetricReps weight systems on chord diagrams:

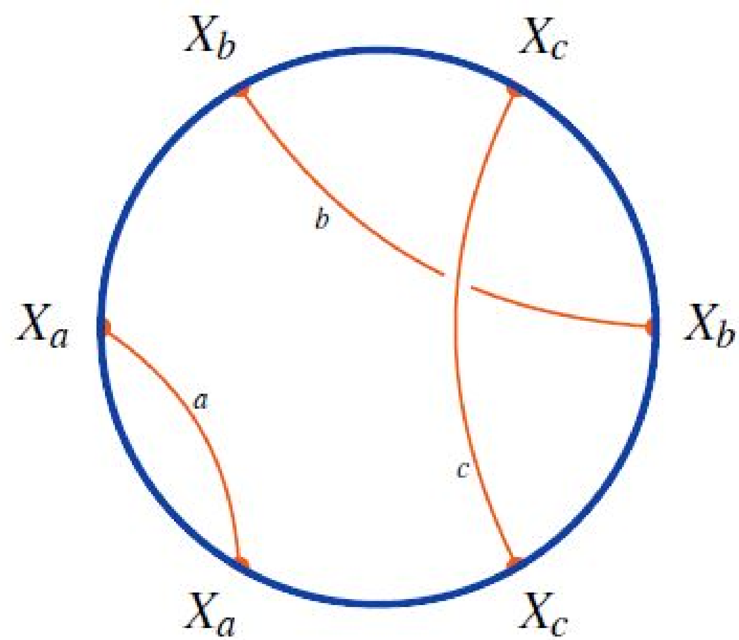
$$\int_{S^2_N} (R^2)^2_{\otimes}$$ 

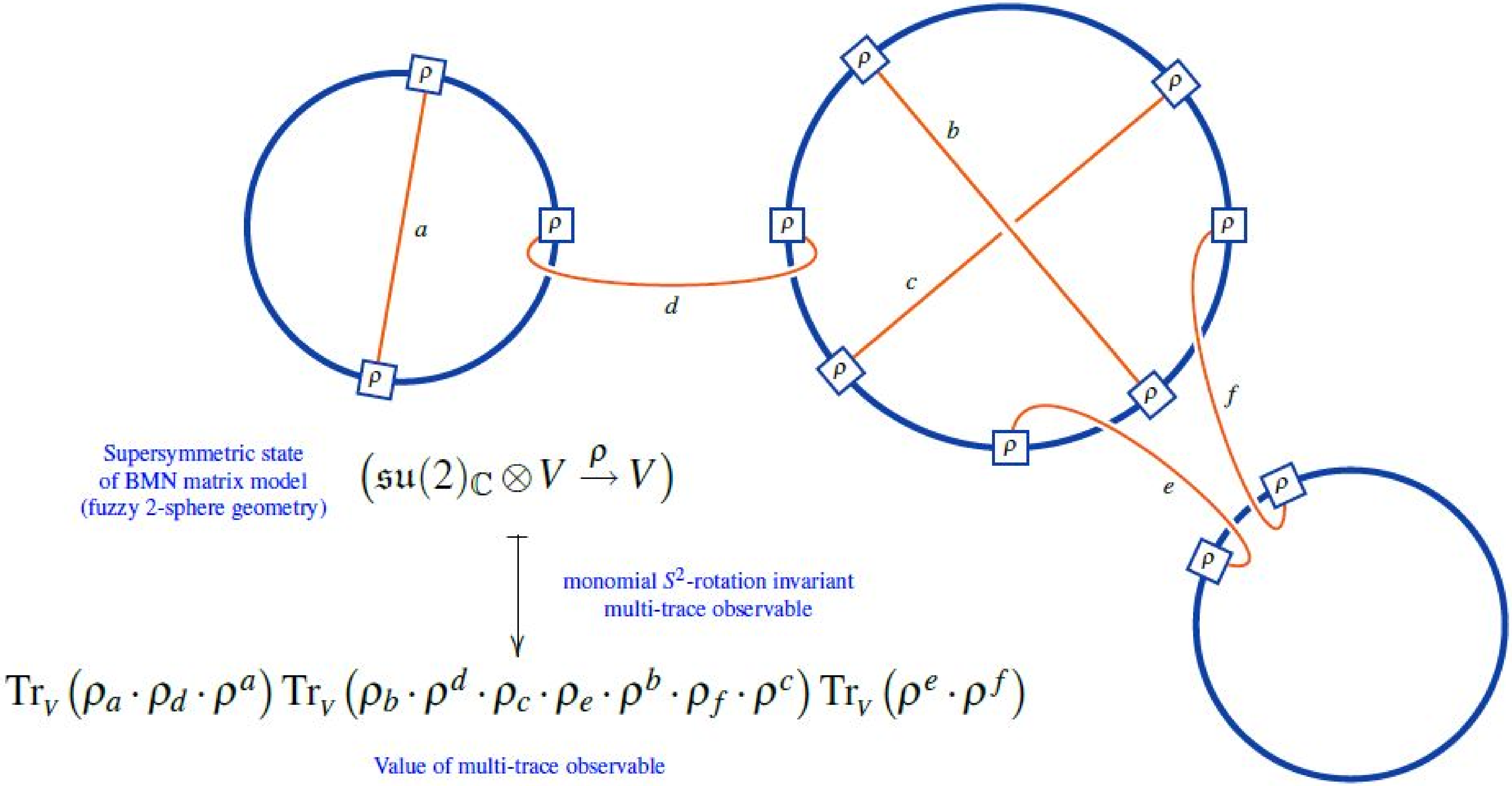
$$= \frac{4\pi}{\sqrt{N^2-1}} \mathrm{Tr}\left(X_a \cdot X_b \cdot X^a \cdot X^b\right)$$



$$\int_{S^2_N} (R^2)^3$$ 

$$= \frac{4\pi}{\sqrt{N^2-1}} \mathrm{Tr}\left(X_a \cdot X^a \cdot X_b \cdot X_c \cdot X^b \cdot X^c\right)$$





1,2) weight system w_ρ is the observable aspect of matrix model state ρ:

linear combinations of
finite-dim $\mathfrak{su}_{\mathbb{C}}$-representations

weight systems on
horizontal chord diagrams

$$\mathrm{Span}\big(\mathfrak{su}(2)_{\mathbb{C}}\mathrm{MetricReps}\big) \xrightarrow{\rho \mapsto w_\rho} \mathcal{W}^{\mathrm{pb}}$$

naive funnel- / susy-states of
DBI model / BMN matrix model

states of DBI model / BMN matrix mode
as observed by invariant multi-trace observables

(7)

M2/M5 Brane Bound States

implied by

Hypothesis H

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Given a *sequence* of susy states in the BMN matrix model

$$\overbrace{(V,\rho)}^{\substack{\text{M2/M5-brane state} \\ (\text{finite-dim } \mathfrak{su}(2)_{\mathbb{C}}\text{-rep})}} \;:=\; \underbrace{\bigoplus_i}_{\substack{\text{stacks of coincident branes} \\ (\text{direct sum over irreps})}} \overbrace{\left(N_i^{(\mathrm{M2})} \cdot \mathbf{N}_i^{(\mathrm{M5})}\right)}^{\substack{\text{M2/M5-brane charge in } i\text{th stack} \\ (i\text{th irrep with multiplicity})}} \;\in\; \mathfrak{su}(2)_{\mathbb{C}}\mathrm{MetricReps}_{/\sim}$$

---

this is argued to converge to macroscopic M2- *or* M5-branes depending on how the sequence behaves in the large N limit:

Stacks of macroscopic...

	M2-branes	*M5-branes*	
If for all i	$N_i^{(\mathrm{M5})} \to \infty$	$N_i^{(\mathrm{M2})} \to \infty$	(the relevant large N limit)
with fixed	$N_i^{(\mathrm{M2})}$	$N_i^{(\mathrm{M5})}$	(the number of coincident branes in the ith stack)
and fixed	$N_i^{(\mathrm{M2})}/N$	$N_i^{(\mathrm{M5})}/N$	(the charge/light-cone momentum carried by the ith stack)

Given a *sequence* of susy states in the BMN matrix model

$$\overbrace{(V,\rho)}^{\substack{\text{M2/M5-brane state} \\ \text{(finite-dim } \mathfrak{su}(2)_{\mathbb{C}}\text{-rep)}}} := \underbrace{\bigoplus_i}_{\substack{\text{stacks of coincident branes} \\ \text{(direct sum over irreps)}}} \overbrace{\Big(N_i^{(\mathrm{M2})} \cdot \mathbf{N}_i^{(\mathrm{M5})}\Big)}^{\substack{\text{M2/M5-brane charge in } i\text{th stack} \\ (i\text{th irrep with multiplicity})}} \in \mathfrak{su}(2)_{\mathbb{C}}\mathrm{MetricReps}_{/\sim}$$

---

the large N

limit does *not* exist

here:

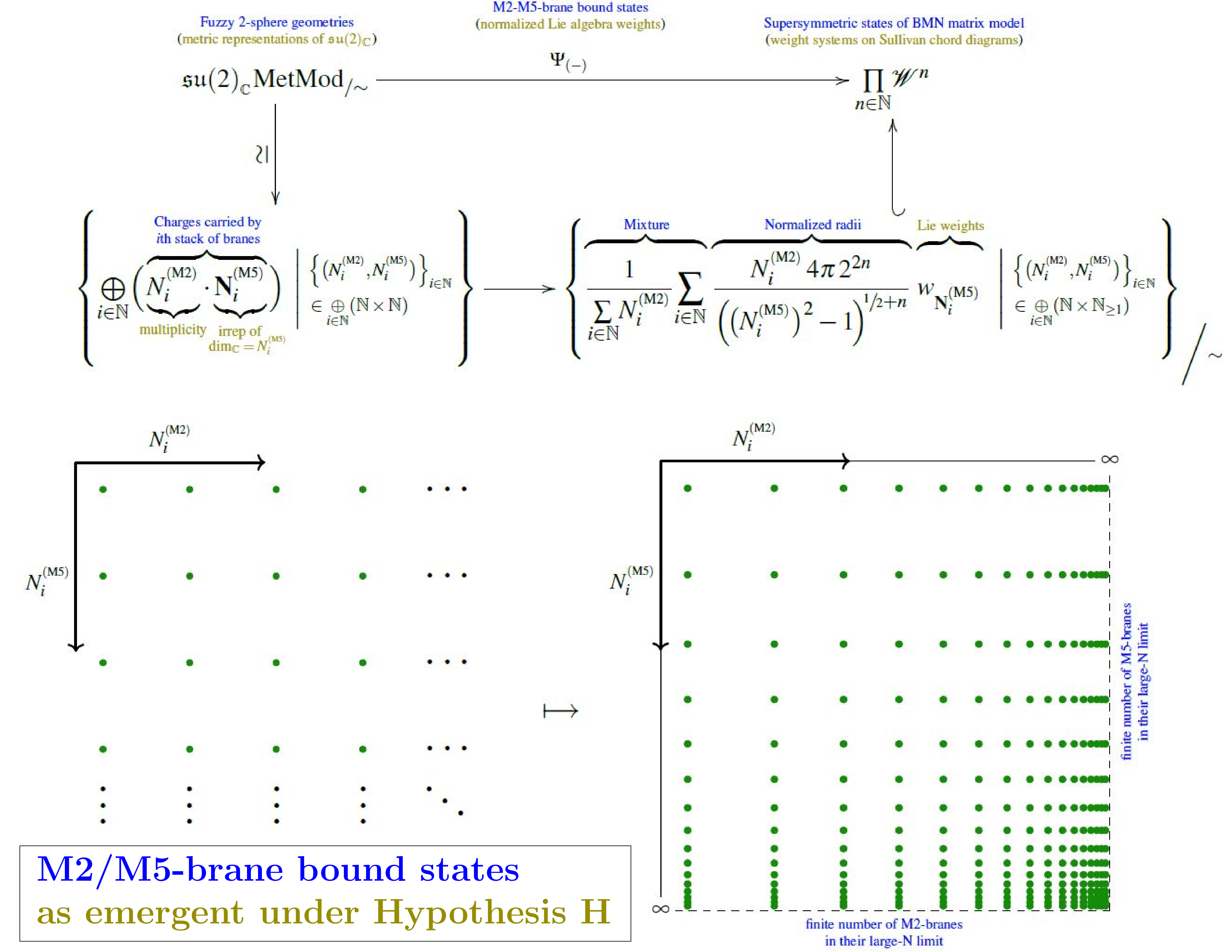
but

does exist in weight systems

$$\mathrm{Span}\big(\mathfrak{su}(2)_{\mathbb{C}}\mathrm{MetricReps}\big) \xrightarrow{\rho \mapsto w_\rho} \mathcal{W}^{\mathrm{pb}}$$

if we normalize by the scale of the fuzzy 2-sphere geometry:

$$\underbrace{\frac{4\pi\, 2^{2n}}{\Big(\big(N^{(\mathrm{M5})}\big)^2 - 1\Big)^{1/2+n}}\, w_{\mathbf{N}^{(\mathrm{M5})}}}_{\substack{\text{Single M2-brane state in BMN model} \\ \text{(multiple of } \mathfrak{su}_{\mathbb{C}}\text{-weight system)}}} \in \underbrace{\mathcal{W}^{\mathrm{pb}}}_{\substack{\text{states as seen by multi-trace observables} \\ \text{(weight systems on chord diagrams)}}}$$



M2/M5-brane bound states

as emergent under Hypothesis H

End.

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