Prequantum field theories from Shifted n-plectic structures

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recently again interest (e.g. [Calaque 13])
in transgressing "shifted 2-form" data
to classical field theories
following the original [AKSZ 97]
(which was nicely reviewed in [Roytenberg 06])

but for (non-perturbative) quantization
one needs more than classical data:

1. globally well-defined action functionals on infinite-dimensional stacks of fields;
2. lift of phase space symplectic form to curvature of connection;
3. certain higher codimension data;

i.e. one needs "pre-quantum" field theory.

we give a quick review of

the construction
of prequantum field theories
from shifted n-plectic forms

due to

[Sati-Schreiber-Stasheff 09]
[Fiorenza-Schreiber-Stasheff 10]
[Fiorenza-Rogers-Schreiber 11]
[Fiorenza-Sati-Schreiber 13]

which took its clues from

[Cartan 50], [Brown 73], [Brylinski-McLaughlin 96],
[Carey-Johnson-Murray-Stevenson-Wang 05]
[Ševera 01], [Roytenberg 02], [Henriques 08],

1. The nonabelian derived category
2. L_∞-Algebroids
3. Lie integration to Smooth ∞-stacks
4. Examples: AKSZ models
5. Transgression
6. Example: 3d Chern-Simons theory

1)

The nonabelian derived category

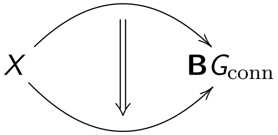
We work in the "nonabelian derived category"
over the site of (derived) smooth manifolds
where sheaves of chain complexes are generalized to
sheaves of Kan complexes [Brown 73] ("smooth ∞-stacks").

This serves to describe nonabelian gauge fields:

For instance there is the sheaf of Kan complexes $\mathbf{B}G_{\mathrm{conn}}$ such that a G-principal connection on any X is a map

$$\nabla : X \longrightarrow \mathbf{B}G_{\mathrm{conn}}$$

and a gauge transformation is a homotopy



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There is also the sheaf of k-forms $\mathbf{\Omega}^k$;
and a k-form on any X is simply a map

$$\omega : X \longrightarrow \mathbf{\Omega}^k \,.$$

Similarly there is the sheaf of $\mathfrak{g}$-valued forms $\mathbf{\Omega}(-, \mathfrak{g})$.

The group G acts on this by gauge transformations
and the homotopy quotient is:

$$\begin{array}{ccc} \mathbf{E} G_{\mathrm{conn}} & := & \mathbf{\Omega}(-, \mathfrak{g}) \\ \downarrow & & \downarrow \\ \mathbf{B} G_{\mathrm{conn}} & \simeq & \mathbf{\Omega}(-, \mathfrak{g})/G \end{array}$$

We want to understand forms (and their prequantizations) on moduli stacks such as this, e.g.

$$\mathbf{B} G_{\mathrm{conn}} \longrightarrow \mathbf{\Omega}^4$$

2)
L_∞-algebroids

Def. An L_∞-algebroid $\mathfrak{a}$ is equivalently a dg-manifold:

- its *Chevalley-Eilenberg algebra* $\mathrm{CE}(\mathfrak{a})$ is the dg-algebra of functions
- its *Weil algebra* $\mathrm{W}(\mathfrak{a})$ is the dg-algebra of differential forms
- a *cocycle* is $\mathfrak{a} \xrightarrow{\mu} \mathbf{B}^{p+2}\mathbb{R}$ hence $\mathrm{CE}(\mathfrak{a}) \longleftarrow \mathrm{CE}(\mathbf{B}^{p+2}\mathbb{R})$;
- an *invariant polynomial* $\langle - \rangle$ on $\mathfrak{a}$ is a closed differential form on the dg-manifold.

If $\langle -, - \rangle$ is binary and non-degenerate, then it's a "graded symplectic form" ([Ševera 01][Roytenberg 02]) or "shifted symplectic structure" ([Pantev-Toën-Vaquie-Vezzosi 11] on the dg-manifold.

For $\mathfrak{g}$ a Lie algebra, then $\mathbf{B}\mathfrak{g}$ is an L_∞-algebroid and all concepts reduce to the traditional ones.

Theorem [Cartan 50][Freed-Hopkins 13]:
For $\mathfrak{g}$ a Lie algebra, then:

$$\begin{array}{ccc}
\mathrm{CE}(\mathbf{B}\mathfrak{g}) & \simeq & \Omega^\bullet_{\underset{\mathrm{cl}}{\mathrm{li}}}(G) \\
\uparrow & & \uparrow \\
\mathrm{W}(\mathbf{B}\mathfrak{g}) & \simeq & \Omega^\bullet(\mathbf{E}G_{\mathrm{conn}}) \\
\uparrow & & \uparrow \\
\mathrm{inv}(\mathbf{B}\mathfrak{g}) & \simeq & \Omega^\bullet(\mathbf{B}G_{\mathrm{conn}})
\end{array}$$

In particular $\underbrace{\langle -, \cdots, - \rangle}_{k\text{-ary}} \;:\; \underset{\nabla \mapsto \langle F_\nabla \wedge \cdots \wedge F_\nabla \rangle}{\mathbf{B}G_{\mathrm{conn}} \longrightarrow \mathbf{\Omega}^{2k}}$.

Consequence: We may use $\mathrm{W}(\mathfrak{a})$ to *force* the universal $\mathfrak{a}$-∞-connection. This is what we do now:

3)

Lie integration to smooth ∞-stacks

Def. [Ševera 01] [Henriques 08]:
universal Lie integration is the sheaf of Kan complexes

$$\exp(\mathfrak{a}) : (\underset{\mathrm{Mfd}}{U}, \underset{\in\mathbb{N}}{k}) \mapsto \{\Omega^\bullet_{\mathrm{vert}}(U \times \Delta^k) \longleftarrow \mathrm{CE}(\mathfrak{a})\}$$

Def. [Fiorenza-Schreiber-Stasheff 10][Fiorenza-Rogers-Schreiber 11]:
differential Lie integration $\exp(\mathfrak{g})_{\mathrm{conn}}$ is

$$\exp(\mathfrak{a})_{\mathrm{conn}} : (U, k) \mapsto \left\{ \begin{array}{ccc} \Omega^\bullet_{\mathrm{vert}}(U \times \Delta^k) & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{a}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(U \times \Delta^k) & \xleftarrow{A} & \mathrm{W}(\mathfrak{a}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(U) & \xleftarrow{\langle F_A \wedge \cdots \wedge F_A \rangle} & \mathrm{inv}(\mathfrak{a}) \end{array} \right\}$$

Def. [Sati-Schreiber-Stasheff 09]: A cocycle μ is in *transgression* with an invariant polynomial $\langle - \rangle$ if there is a *Chern-Simons element* cs making this diagram commutative:

$$\begin{array}{ccc} \mathrm{CE}(\mathfrak{a}) & \xleftarrow{\mu} & \mathrm{CE}(\mathbf{B}^{p+2}\mathbb{R}) \\ \uparrow & & \uparrow \\ \mathrm{W}(\mathfrak{a}) & \xleftarrow{\mathrm{cs}} & \mathrm{W}(\mathbf{B}^{p+2}\mathbb{R}) \\ \uparrow & & \uparrow \\ \mathrm{inv}(\mathfrak{a}) & \xleftarrow{\langle - \rangle} & \mathrm{inv}(\mathbf{B}^{p+2}\mathbb{R}) \end{array}$$

Observation [Fiorenza-Schreiber-Stasheff 10]:
By pasting of diagrams, this data defines a map

$$\exp(\mathrm{cs}) \;:\; \exp(\mathfrak{a})_{\mathrm{conn}} \longrightarrow \exp(\mathbf{B}^{p+2}\mathbb{R})_{\mathrm{conn}}$$

Theorem [Fiorenza-Schreiber-Stasheff 10]:
under $(p+1)$-truncation this descends to

$$
\begin{array}{ccc}
\exp(\mathfrak{a})_{\mathrm{conn}} & \xrightarrow{\exp(\mathrm{cs})} & \exp(\mathbf{B}^{p+2}\mathbb{R})_{\mathrm{conn}} \\
\downarrow{\scriptstyle \mathrm{cosk}_{p+2}} & & \downarrow \\
\mathbf{A}_{\mathrm{conn}} & \xrightarrow{\nabla} & \mathbf{B}^{p+2}(\mathbb{R}/\Gamma)_{\mathrm{conn}}
\end{array}
$$

where

- $\Gamma \hookrightarrow \mathbb{R}$ is the group of periods of μ;
- $\mathbf{B}^{p+2}(\mathbb{R}/\Gamma)_{\mathrm{conn}} := [\Gamma \hookrightarrow \mathbf{\Omega}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathbf{\Omega}^{p+2}]$ "Deligne cplx"

These $\mathbf{A}_{\mathrm{conn}} \xrightarrow{\exp(\mathrm{cs})_{\mathrm{conn}}} \mathbf{B}^{p+2}(\mathbb{R}/\Gamma)_{\mathrm{conn}}$
are our prequantum Lagrangians
for $(p+2)$-dimensional higher gauge field theory.

4)
Examples:
AKSZ models

Theorem
[Brylinski-McLaughlin 96][Fiorenza-Schreiber-Stasheff 10]:

For $\mathfrak{g}$ a semisimple Lie algebra and $\langle -,-\rangle$ the Killing form, then

1. $\exp(\mathrm{cs})_{\mathrm{conn}}$ looks like $(\mathbf{c}_2)_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \longrightarrow \mathbf{B}^3 U(1)_{\mathrm{conn}}$;
2. on globally defined forms $A : \Sigma \to \mathbf{B}G_{\mathrm{conn}}$ it reduces to standard Chern-Simons 3-form $\mathrm{CS}(A)$;
3. the geometric realization of $\mathbf{c}_2 : \mathbf{B}G \to \mathbf{B}^3 U(1)$ is the universal 4-class $c_2 : BG \longrightarrow K(\mathbb{Z}, 4)$.

This turns out to be the prequantum Lagrangian for 3d CS theory.

Remark [Fiorenza-Sati-Schreiber 12]:
The abelian prequantum 3d CS Lagrangian is instead given by the Beilinson-Deligne cup product $\mathbf{B}U(1)_{\mathrm{conn}} \stackrel{\cup_{\mathrm{conn}}}{\longrightarrow} \mathbf{B}^3 U(1)_{\mathrm{conn}}$

Theorem [Fiorenza-Rogers-Schreiber 11]:
For $(\mathfrak{a}, \omega)$ a symplectic Lie $p+1$-algebroid

1. ω is in transgression with the Hamiltonian for $d_{\mathrm{CE}(\mathfrak{a})}$
2. the induced

$$\exp(\mathrm{cs})_{\mathrm{conn}} : \mathbf{A}_{\mathrm{conn}} \longrightarrow \mathbf{B}^{p+2}(\mathbb{R}/\Gamma)_{\mathrm{conn}}$$

   restricts on globally defined $\mathfrak{a}$-valued differential forms to the corresponding AKSZ-model action functional.

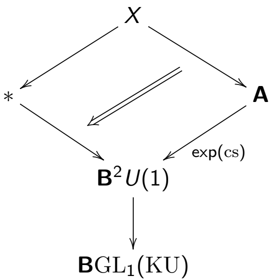
Hence $\exp(\mathrm{cs})_{\mathrm{conn}}$ is globalization of AKSZ-sigma model from Lie n-algebroids to their global smooth n-stacks.

Theorem [Bongers 14] [Nuiten 13]

For (X, π) a Poisson manifold

$\mathfrak{a} := \mathfrak{P}(X, \pi)$ its integrable Poisson Lie algebroid, then

1. $\exp(\mathrm{cs}) : \mathbf{A} \simeq \mathrm{SymplGrp}(X, \pi) \longrightarrow \mathbf{B}^2 U(1)_{\mathrm{conn}}$ is prequantization of the symplectic groupoid;
2. there is a homotopy



3. pull-push through this homotopy-correspondence in twisted equivariant K-theory gives geometric quantization of the symplectic leaves.

Hence a non-perturbative version of [Cattaneo-Felder 99].

5)
Transgression

Let Σ be an oriented closed smooth manifold of dimension d.

Observation: *Fiber integration* of differential forms is morphism of sheaves:

$$[\Sigma, \mathbf{\Omega}^{p+3}] \xrightarrow{\int_\Sigma} \mathbf{\Omega}^{p+3-d}$$

i.e. over each test manifold U this gives the standard

$$\int_{(\Sigma \times U) \to U} \; : \; \Omega^{p+3}(\Sigma \times U) \longrightarrow \Omega^{p+3-d}(U)\,.$$

Observation: *Transgression* of forms $\omega : X \to \mathbf{\Omega}^{p+3}$ to the mapping stack $[\Sigma, X]$ is simply the composite

$$\int_\Sigma [\Sigma, \omega] \; : \; [\Sigma, X] \xrightarrow{[\Sigma,\omega]} [\Sigma, \mathbf{\Omega}^{p+3}] \xrightarrow{\int_\Sigma} \mathbf{\Omega}^{p+3-d}$$

Theorem [Gomi-Terashima 00][Fiorenza-Sati-Schreiber 12]:
This pre-quantizes to integration of higher connection forms:

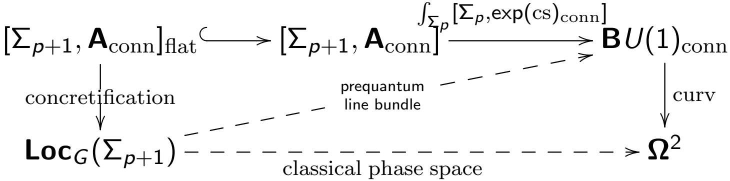
$$
\begin{array}{ccc}
[\Sigma, \mathbf{B}^{p+2} U(1)_{\mathrm{conn}}] & \xrightarrow{\int_\Sigma} & \mathbf{B}^{p+2-d} U(1)_{\mathrm{conn}} \\
\downarrow{\scriptstyle [\Sigma,\mathrm{curv}]} & & \downarrow{\scriptstyle \mathrm{curv}} \\
[\Sigma, \mathbf{\Omega}^{p+3}] & \xrightarrow{\int_\Sigma} & \mathbf{\Omega}^{p+3-d}
\end{array}
$$

Consequence: Transgression of $\nabla : X \to \mathbf{B}^{p+2} U(1)_{\mathrm{conn}}$ to mapping stack is simply the composite

$$
\int_\Sigma [\Sigma, \nabla] \;:\; [\Sigma, X] \xrightarrow{[\Sigma,\nabla]} [\Sigma, \mathbf{B}^{p+2} U(1)_{\mathrm{conn}}] \xrightarrow{\int_\Sigma} \mathbf{B}^{p+2-d} U(1)_{\mathrm{conn}}
$$

Hence a single $\exp(\mathrm{cs})_{\mathrm{conn}}$ induces
hierarchy of transgressions to higher codimension:

- **codimension 0 – global action functional** :
  $\exp(\frac{i}{\hbar}S) : [\Sigma_{p+2}, \mathbf{A}_{\mathrm{conn}}] \longrightarrow U(1)$
- **codimension 1 – prequantized phase space**



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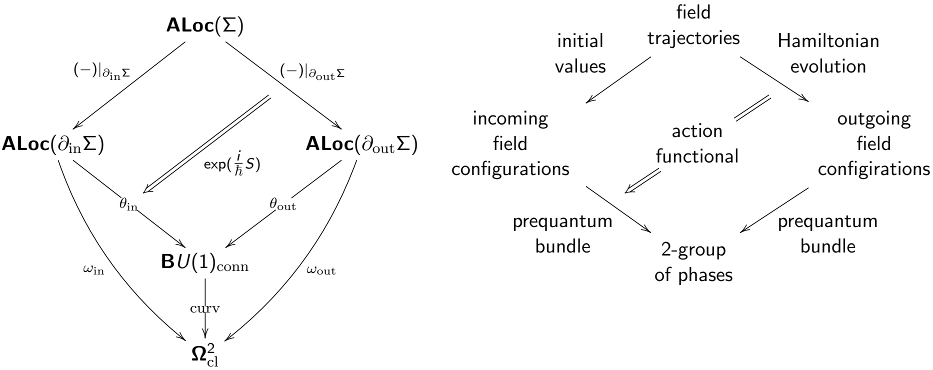
Theorem: The dashed descent exists at least for 3dCS, 2dPSM, CSM with constant 3-form, abelian CS in dim $4k+3$, higher AKSZ models with abelian $p+2$-generators.

This generalizes to Σ with boundary $\partial\Sigma$:

Theorem **[Gomi-Terashima 00]** **[Fiorenza-Sati-Schreiber 13]:**
Transgression to bulk is trivialization-mod-transgressed-curvature of transgression to boundary:

$$
\begin{array}{ccccc}
[\Sigma, X] & \xrightarrow{[\Sigma,\nabla]} & [\Sigma, \mathbf{B}^{p+2}U(1)_{\mathrm{conn}}] & \xrightarrow{\int_\Sigma \mathrm{curv}} & \mathbf{\Omega}^{p+3-d} \\
\Big\downarrow{\scriptstyle (-)|_{\partial\Sigma}} & & \Big\downarrow{\scriptstyle (-)|_{\partial\Sigma}} & \swarrow\!\!\swarrow {\scriptstyle \int_\Sigma} & \Big\downarrow \\
[\partial\Sigma, X] & \xrightarrow[{[\partial\Sigma,\nabla]}]{} & [\partial\Sigma, \mathbf{B}^{p+2}U(1)_{\mathrm{conn}}] & \xrightarrow[\int_{\partial\Sigma}]{} & \mathbf{B}^{p+3-d}U(1)_{\mathrm{conn}}
\end{array}
$$

Corollary: Transgressing $\exp(\mathrm{cs})_{\mathrm{conn}}$ to spacetime with incoming and outgoing boundary yields the prequantized Lagrangian correspondence that exhibits dynamical evolution:



6)
Example:
3d Chern-Simons theory

Consider again the 3dCS prequantum Lagrangian

$$(\mathbf{c}_2)_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \longrightarrow \mathbf{B}^3 U(1)_{\mathrm{conn}}$$

Transgression to **codimension-1** turns the "2-shifted 2-form"

$$\langle -,-\rangle \;:\; \mathbf{B}G_{\mathrm{conn}} \longrightarrow \mathbf{\Omega}^{2+2} \;:\; A \mapsto \langle F_A \wedge F_A\rangle$$

into a 2-form

$$\omega_2 := \int_{\Sigma_2} [\Sigma_2, \langle -,-\rangle] \;:\; G\mathbf{Loc}(\Sigma_2) \longrightarrow \mathbf{\Omega}^2$$

Compute this locally on a plot $A : U \longrightarrow G\mathbf{Loc}(\Sigma_2)$ hence for

$$A \in \Omega^2_{\substack{\mathrm{vert}\\ \mathrm{flat}}}(\Sigma_2 \times U)\,.$$

Then $F_A = d_U A_\Sigma = \delta A$ is the variation of A. Hence we get the traditional symplectic structure on $G\mathbf{Loc}(\Sigma_2)$

$$(\omega_2)|_U = \int_{\Sigma_2} \delta A \wedge \delta A\,.$$

But we also automatically have its prequantum line

$$\int_{\Sigma_2} [\Sigma_2, (\mathbf{c}_2)_{\mathrm{conn}}] \;:\; G\mathbf{Loc}(\Sigma_2) \longrightarrow \mathbf{B}U(1)_{\mathrm{conn}}$$

we may also identify **codim-2** transgression:

Theorem
[Carey-Johnson-Murray-Stevenson-Wang 05]
[Fiorenza-Sati-Schreiber 13]:

1. There is a universal G-connection ∇_{univ} on $G \times S^1$ with $\mathrm{hol}_\nabla(\{g\} \times S^1) = g$.
2. The restriction of the codim-2-transgression of 3dCS along ∇_{univ} to G is the Wess-Zumino-Witten gerbe

$$\nabla_{\mathrm{WZW}} : G \xrightarrow{\nabla_{\mathrm{univ}}} [S^1, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{\int_{S^1}[S^1, (\mathbf{c}_s)_{\mathrm{conn}}]} \mathbf{B}^2 U(1)_{\mathrm{conn}}$$

In summary, prequantum 3d Chern-Simons theory induced from the 2-shifted symplectic form $\langle -, - \rangle$ looks like so in arbitrary codimension:

codim 0	$[\Sigma_3, \mathbf{B}G_{\mathrm{conn}}] \longrightarrow U(1)$	CS invariant
codim 1	$[\Sigma_2, \mathbf{B}G_{\mathrm{conn}}] \longrightarrow \mathbf{B}U(1)_{\mathrm{conn}}$	CS prequantum line
codim 2	$[\Sigma_1, \mathbf{B}G_{\mathrm{conn}}] \longrightarrow \mathbf{B}^2U(1)_{\mathrm{conn}}$	WZW gerbe
codim 3	$[\Sigma_0, \mathbf{B}G_{\mathrm{conn}}] \longrightarrow \mathbf{B}^3U(1)_{\mathrm{conn}}$	Chern-Weil map

Outlook

All these constructions prolong to genuine local field theories defined by prequantum variational data on jet bundles with evolution by higher prequantum correspondences.

See:

Fiorenza, Khavkine, Schreiber:
Local prequantum field theory
ncatlab.org/schreiber/show/Local+prequantum+field+theory

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