Effective Quantum Certification via Linear Homotopy Types

Urs Schreiber (NYU Abu Dhabi)
on joint work at CQTS with
D. J. Myers, M. Riley,
and Hisham Sati

CENTER FOR
QUANTUM &
TOPOLOGICAL
SYSTEMS

presentation at:

The Topos Institute Colloquium, 13 Apr 2023

slides and further pointers at: ncatlab.org/Quantum+Certification+via+Linear+Homotopy+Types#TI2023

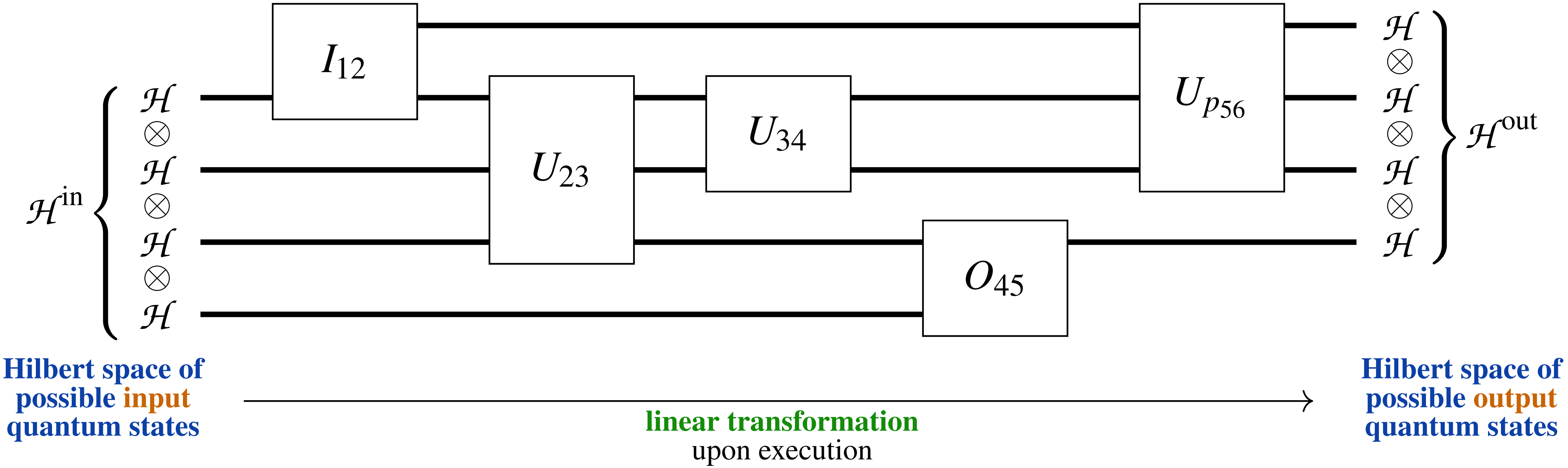
The Problem

in

Quantum Computing

Pure quantum circuits are easy...

Linear operator composed & tensored from given *quantum logic gates*



Hilbert space of possible input quantum states

linear transformation upon execution

Hilbert space of possible output quantum states

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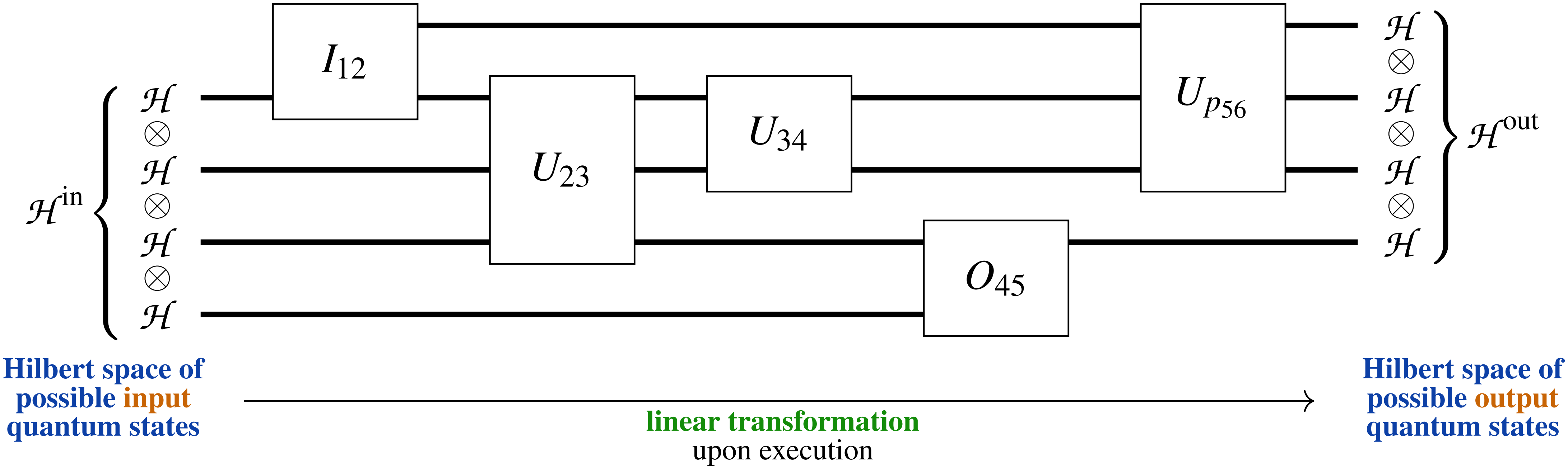


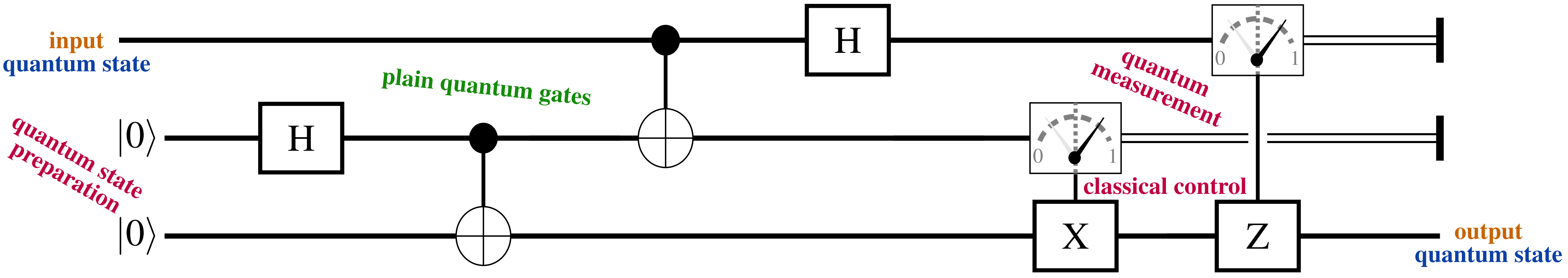
Hilbert space of possible input quantum states

linear transformation upon execution

Hilbert space of possible output quantum states

but real quantum circuits have classical control & effects

(Example: QBit Teleportation protocol)



input quantum state

quantum state preparation

plain quantum gates

quantum measurement

classical control

output quantum state

full reality is a loop: Classical $\underset{\text{prepare}}{\overset{\text{measure}}{\leftrightarrows}}$ Quantum

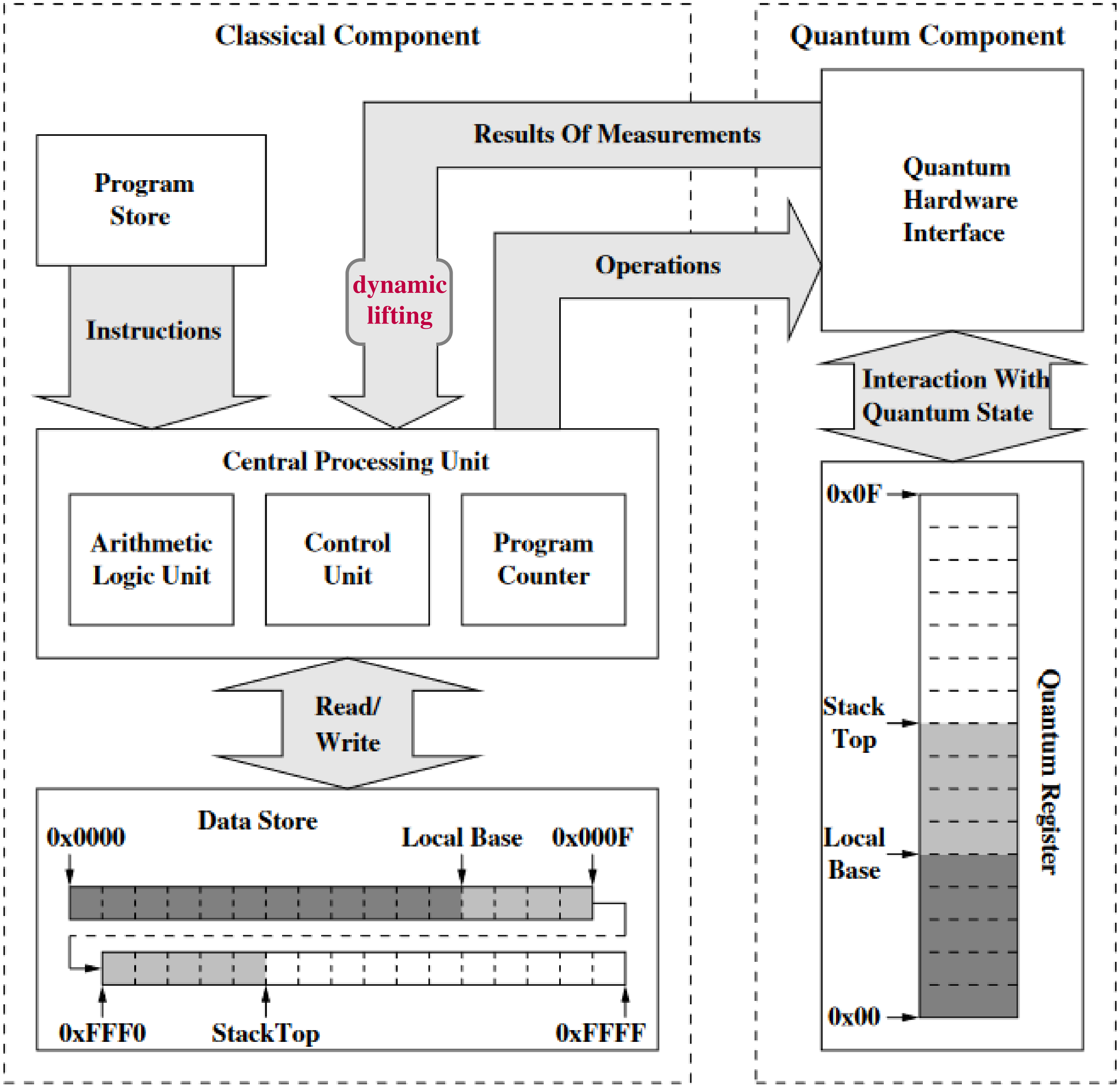


diagram adapted from Nagarajan et al. (2007)

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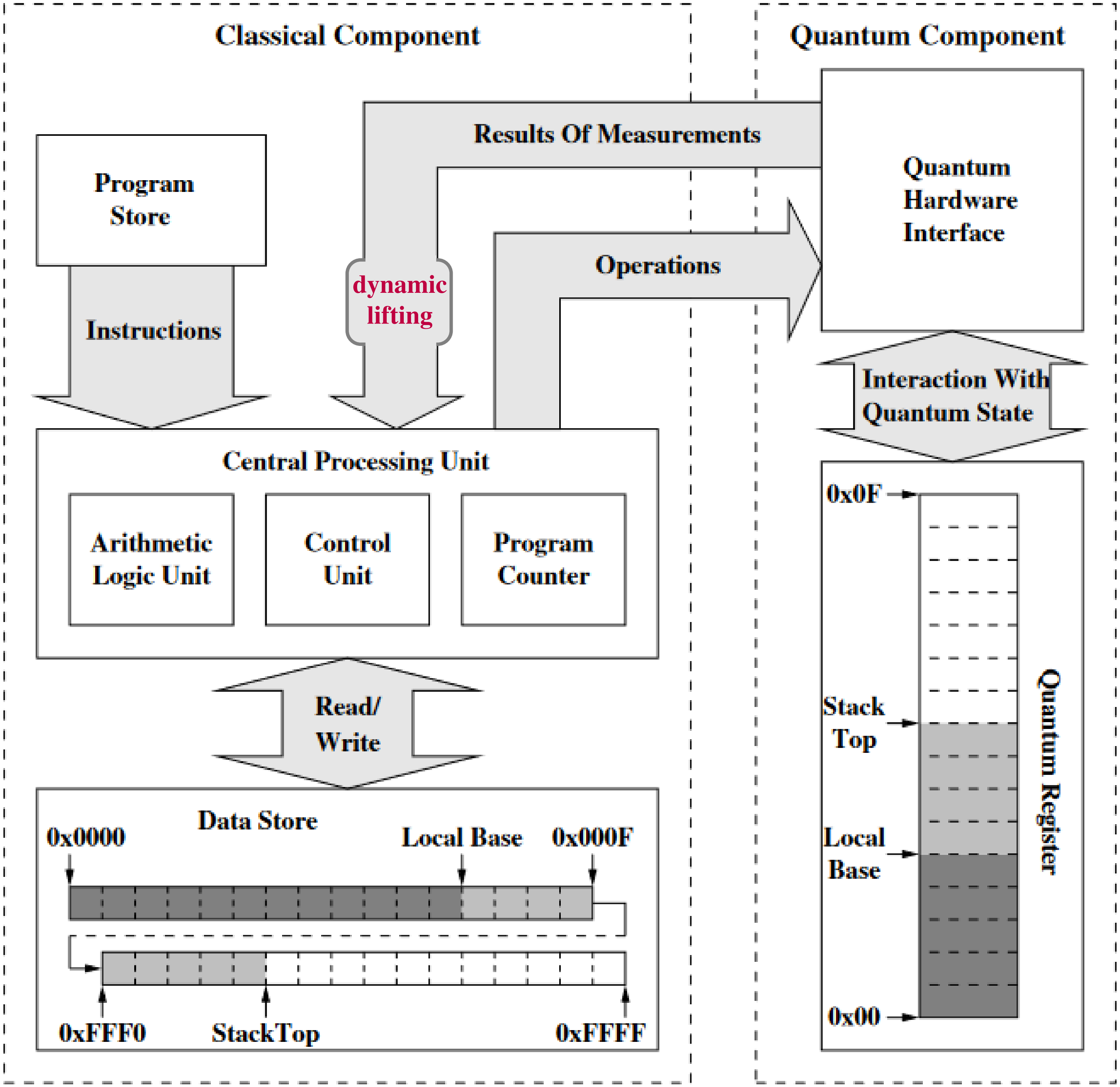
Classical Component

Program Store

Instructions

dynamic lifting

Results Of Measurements

Operations

Central Processing Unit

Arithmetic Logic Unit

Control Unit

Program Counter

Read/ Write

Data Store

0x0000

Local Base

0x000F

0xFFF0

StackTop

0xFFFF

Quantum Component

Quantum Hardware Interface

Interaction With Quantum State

0x0F

Stack Top

Local Base

0x00

Quantum Register

diagram adapted from Nagarajan et al. (2007)

full loop needed e.g. for quantum error correction

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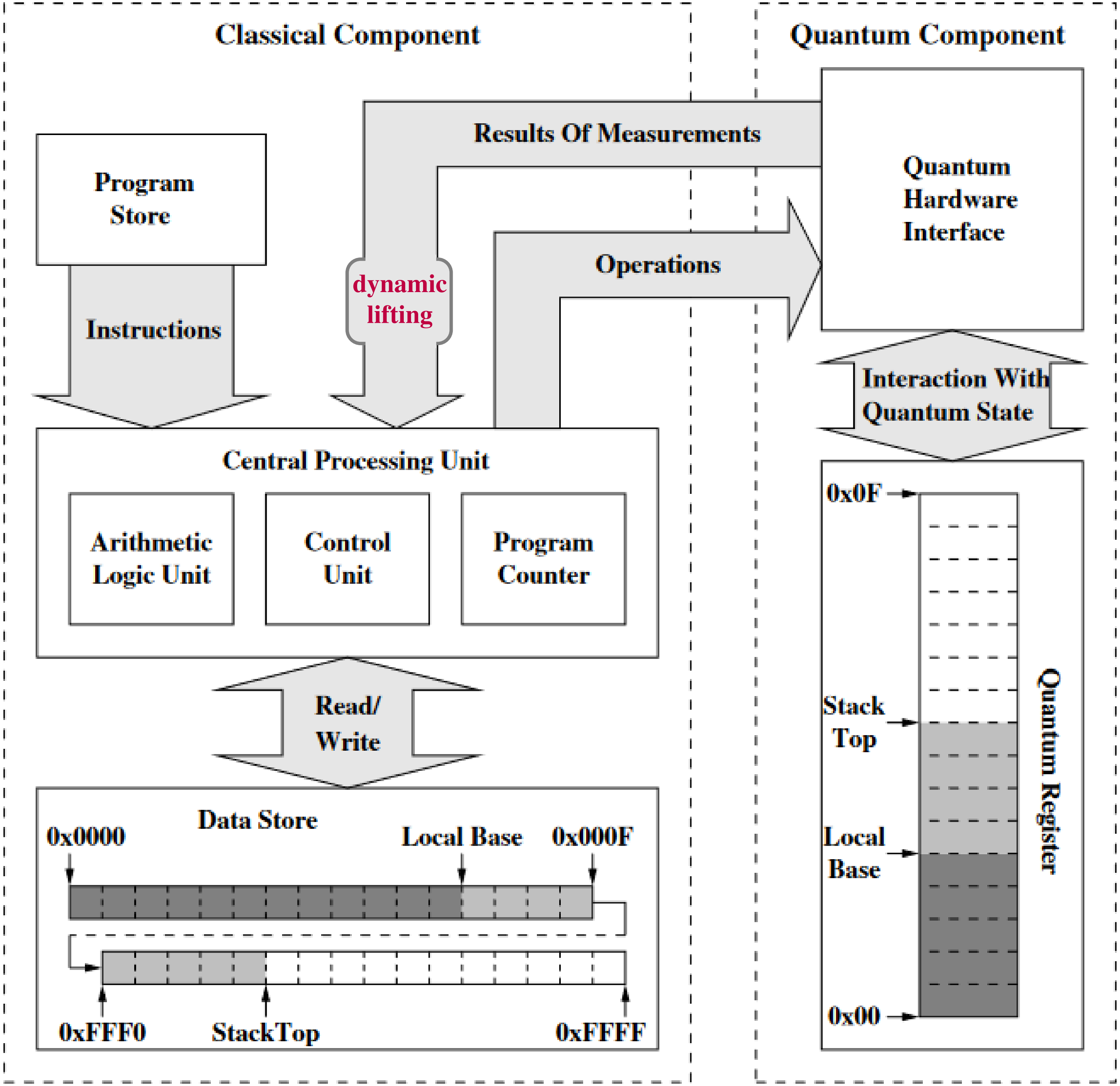

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but its formal language theory had remained thin

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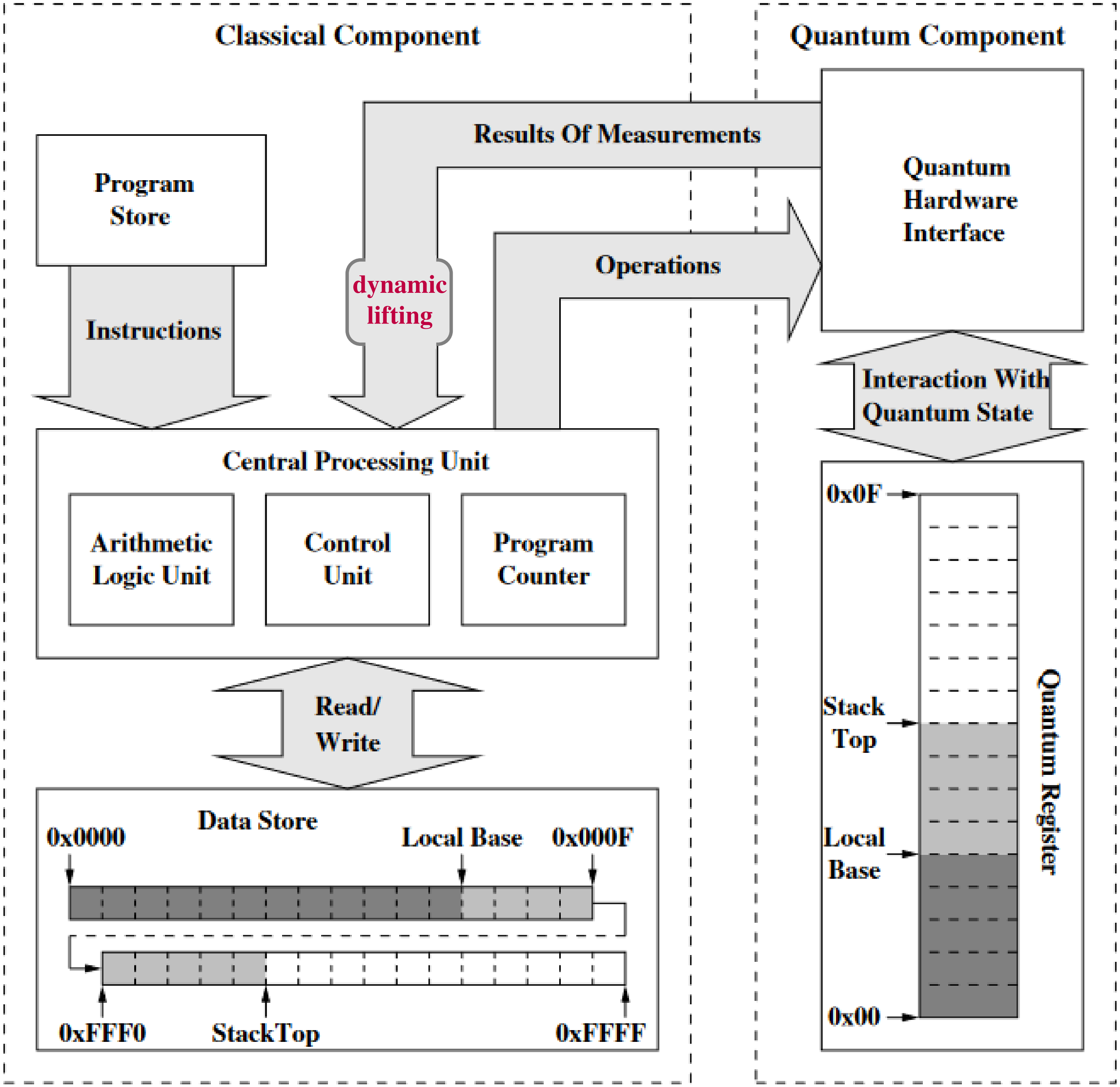

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existing models for dynamic lifting are ad hoc & unverified

Existing quantum typed circuit languages

are embedded inside *classical* type theories:

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Classical Type Theory (e.g. `Haskell`, `Coq`, ...)

for universal classical computation

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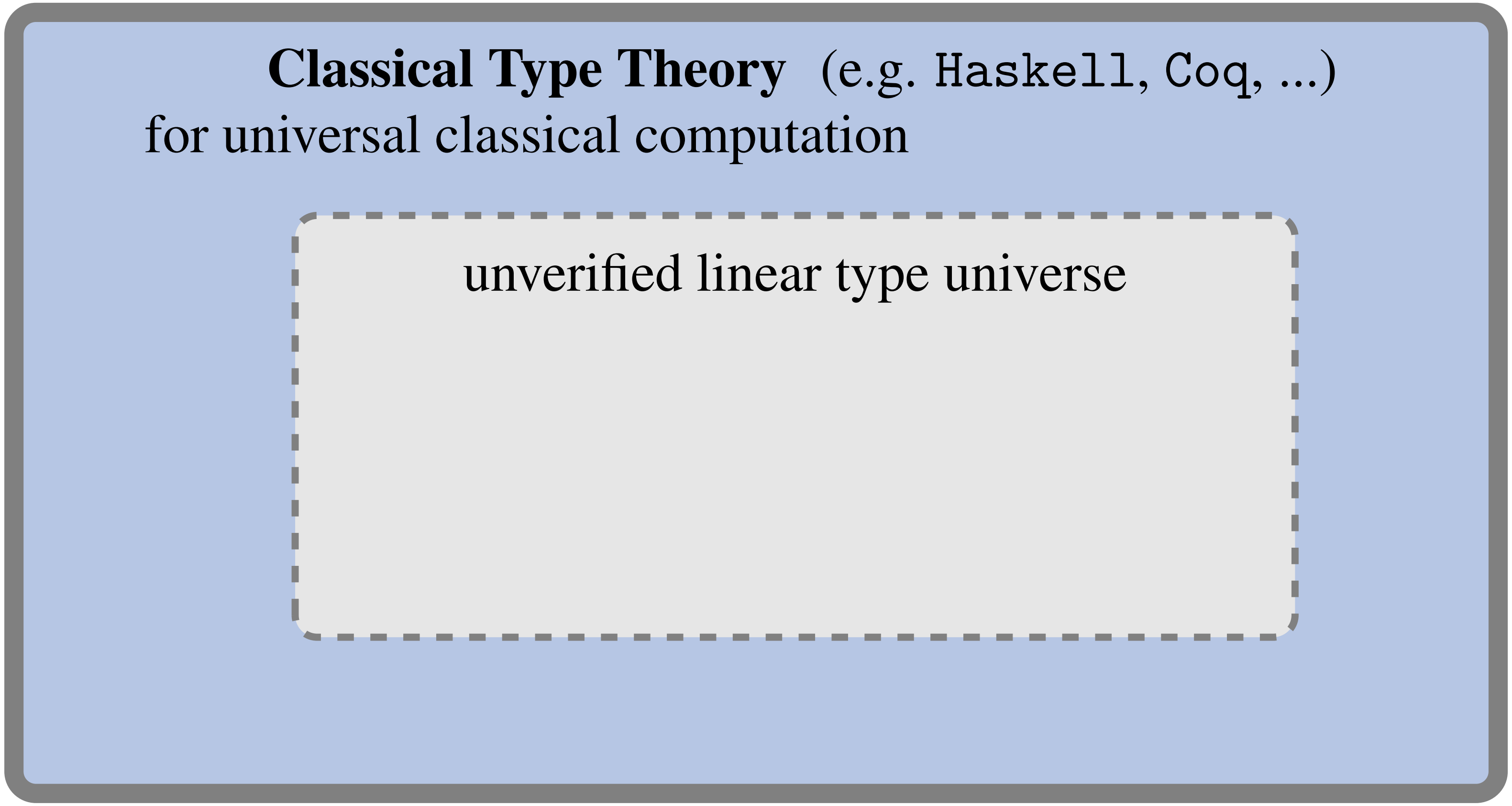

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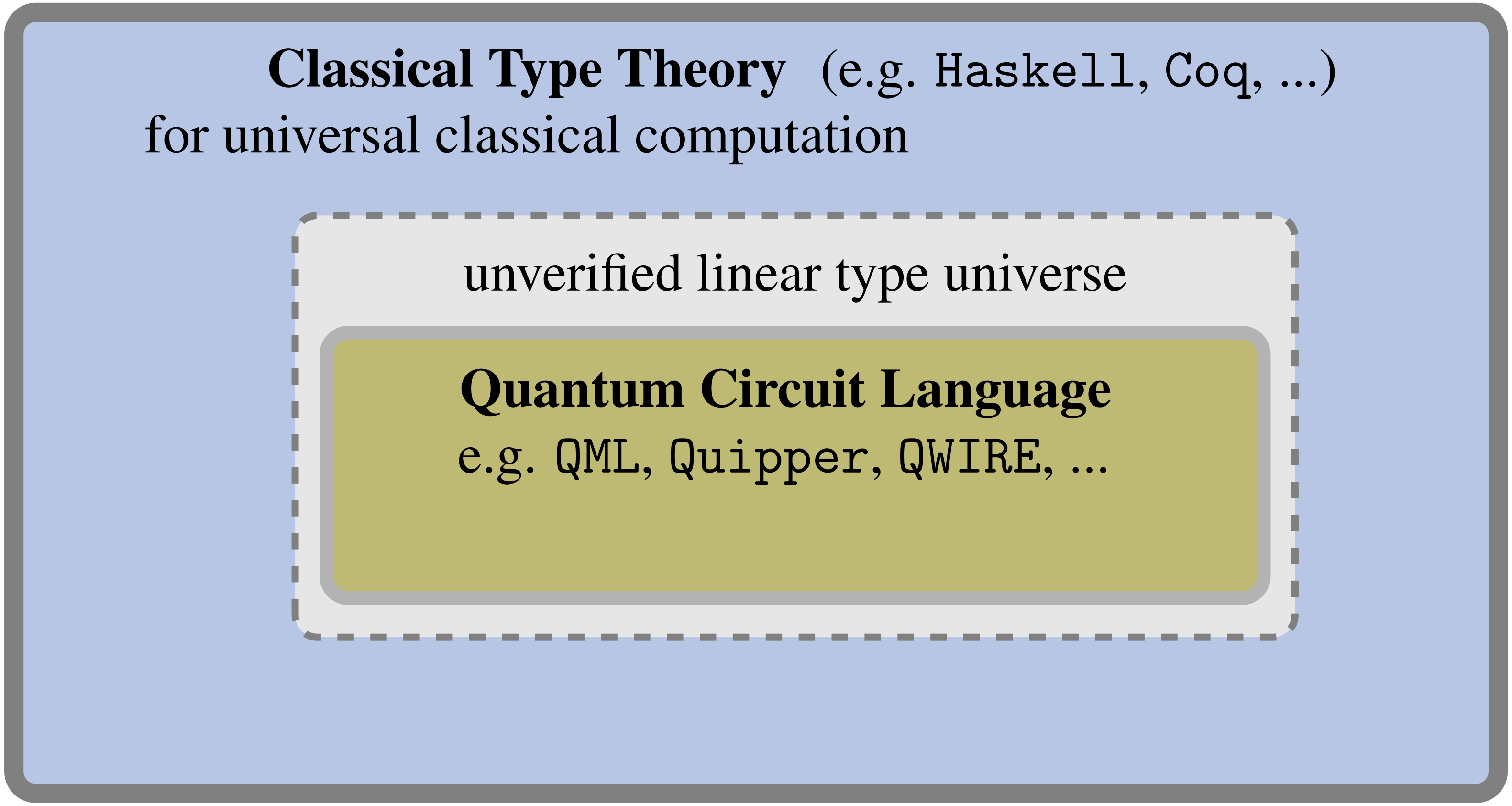

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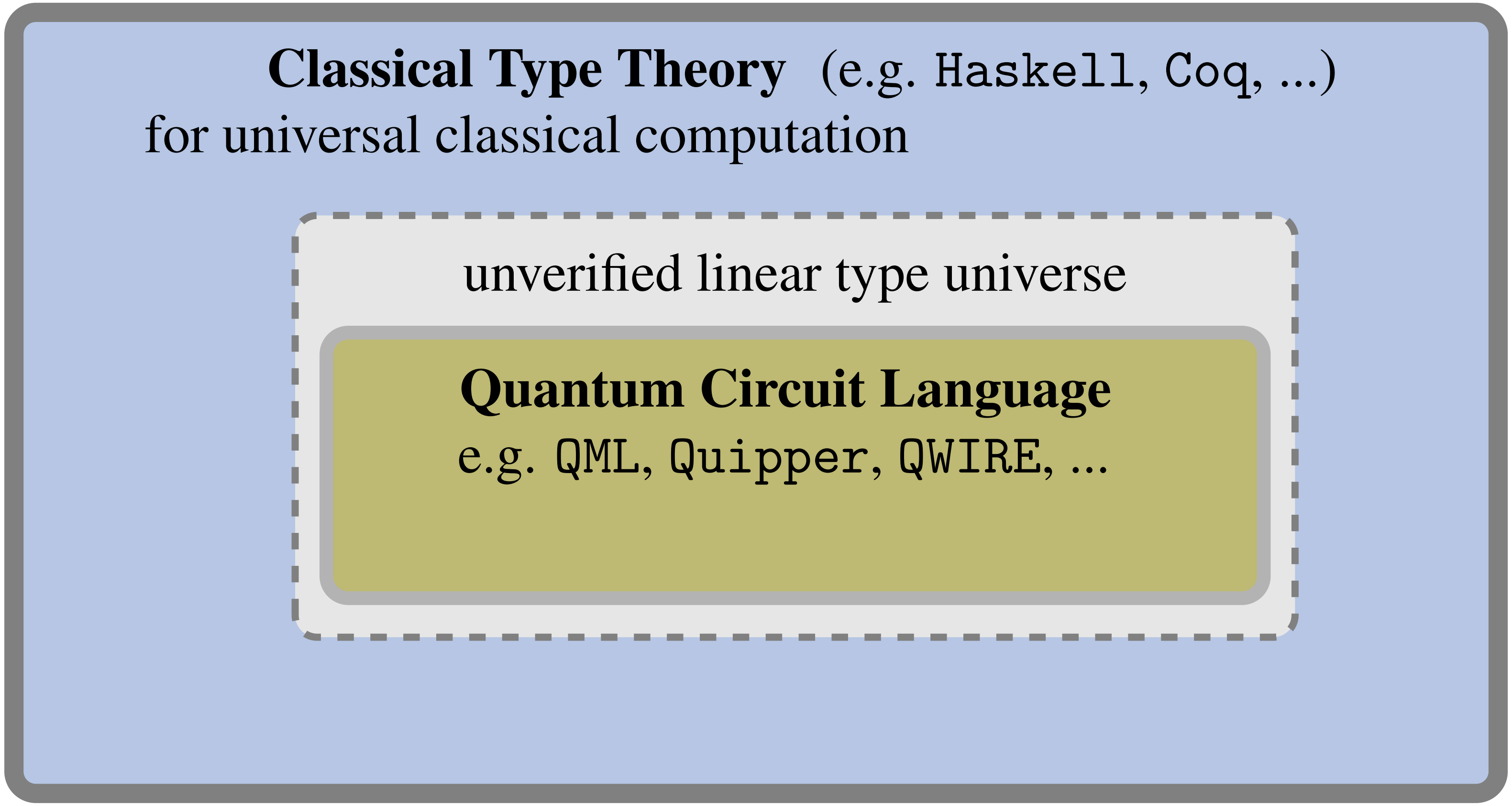

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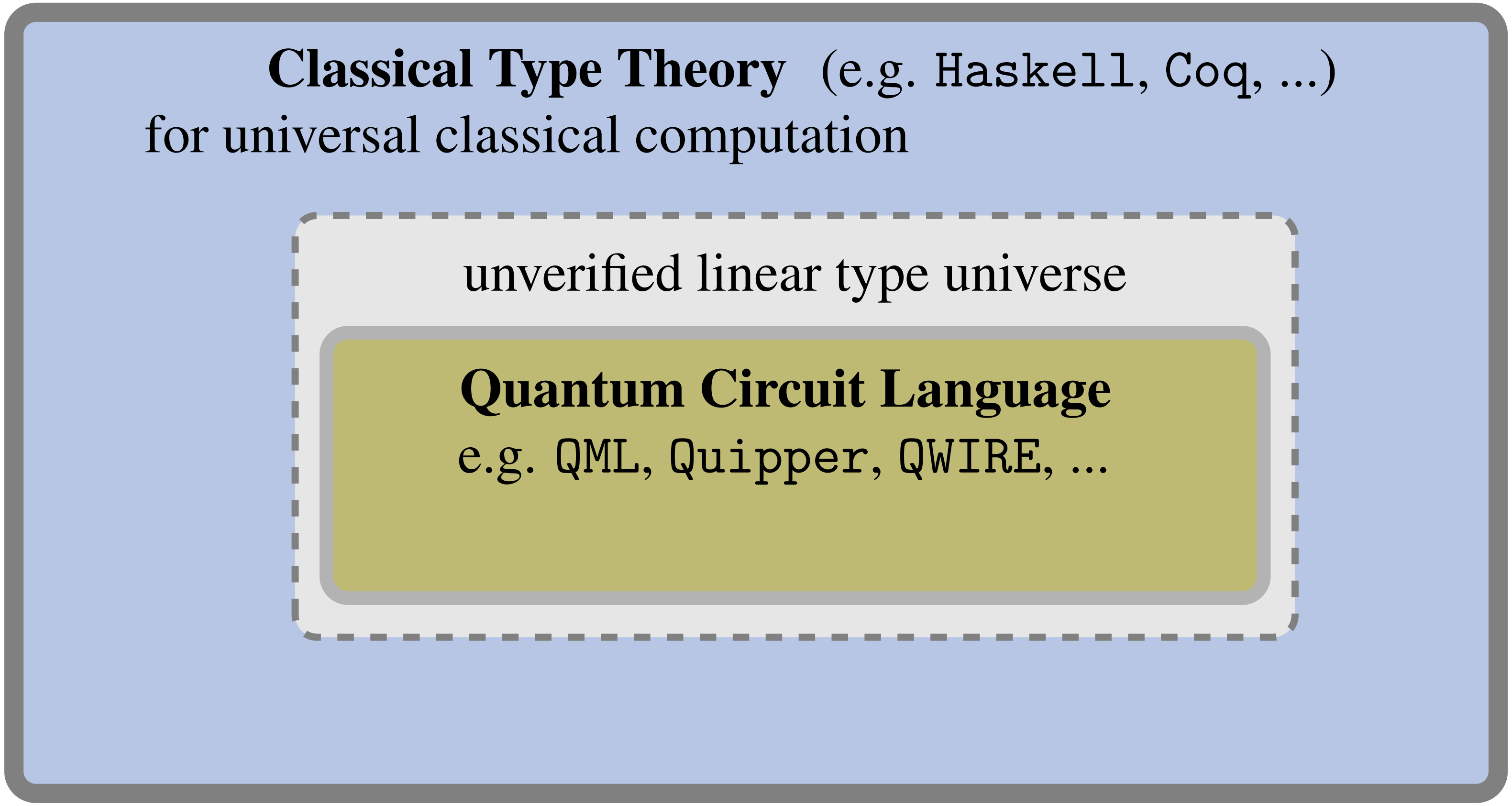

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Why did that not exist?

The Problem

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Type Theory

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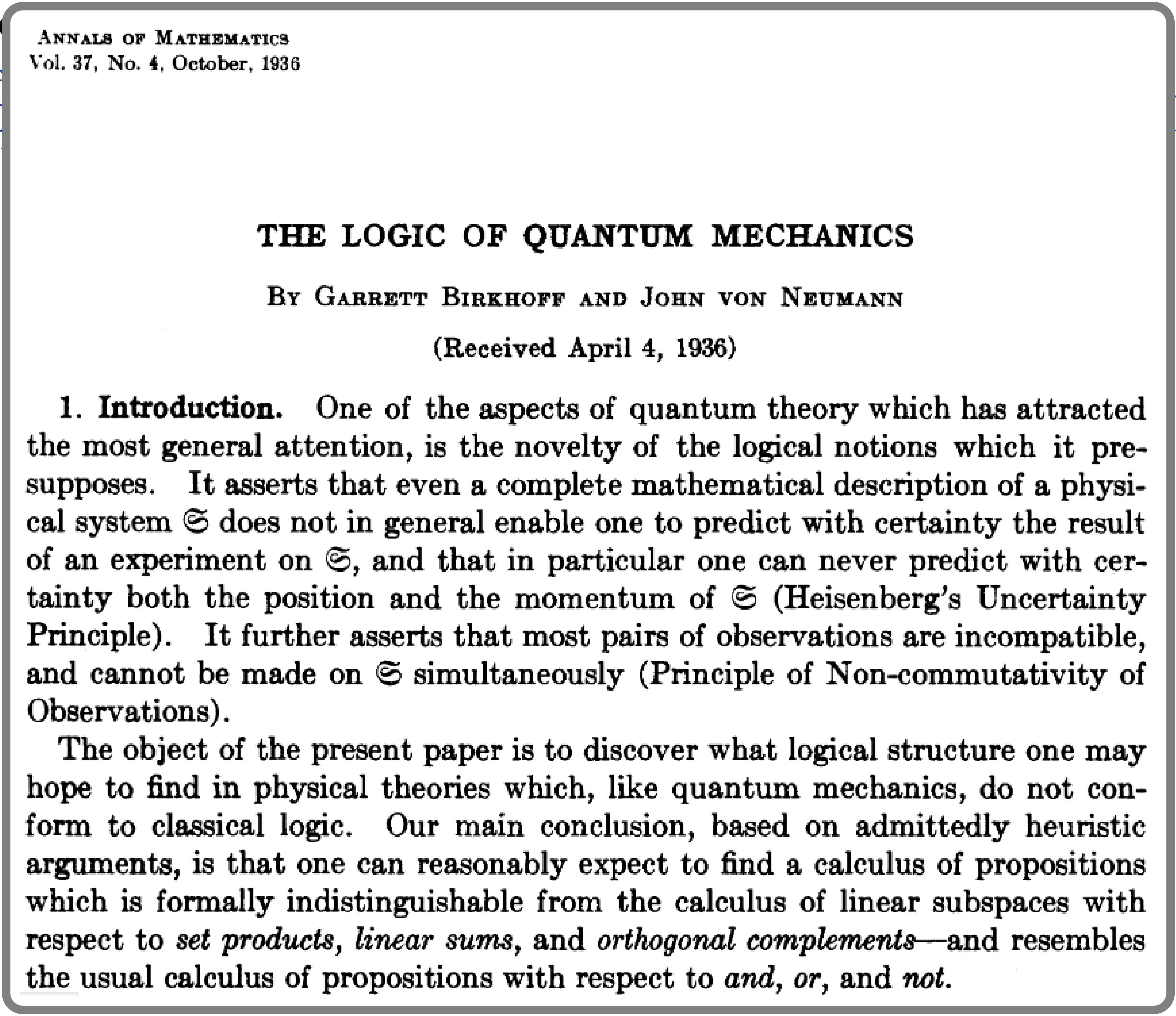

Annals of Mathematics
Vol. 37, No. 4, October, 1936

THE LOGIC OF QUANTUM MECHANICS

By Garrett Birkhoff and John von Neumann

(Received April 4, 1936)

1. Introduction. One of the aspects of quantum theory which has attracted the most general attention, is the novelty of the logical notions which it presupposes. It asserts that even a complete mathematical description of a physical system $\mathfrak{S}$ does not in general enable one to predict with certainty the result of an experiment on $\mathfrak{S}$, and that in particular one can never predict with certainty both the position and the momentum of $\mathfrak{S}$ (Heisenberg's Uncertainty Principle). It further asserts that most pairs of observations are incompatible, and cannot be made on $\mathfrak{S}$ simultaneously (Principle of Non-commutativity of Observations).

The object of the present paper is to discover what logical structure one may hope to find in physical theories which, like quantum mechanics, do not conform to classical logic. Our main conclusion, based on admittedly heuristic arguments, is that one can reasonably expect to find a calculus of propositions which is formally indistinguishable from the calculus of linear subspaces with respect to *set products*, *linear sums*, and *orthogonal complements*—and resembles the usual calculus of propositions with respect to *and*, *or*, and *not*.

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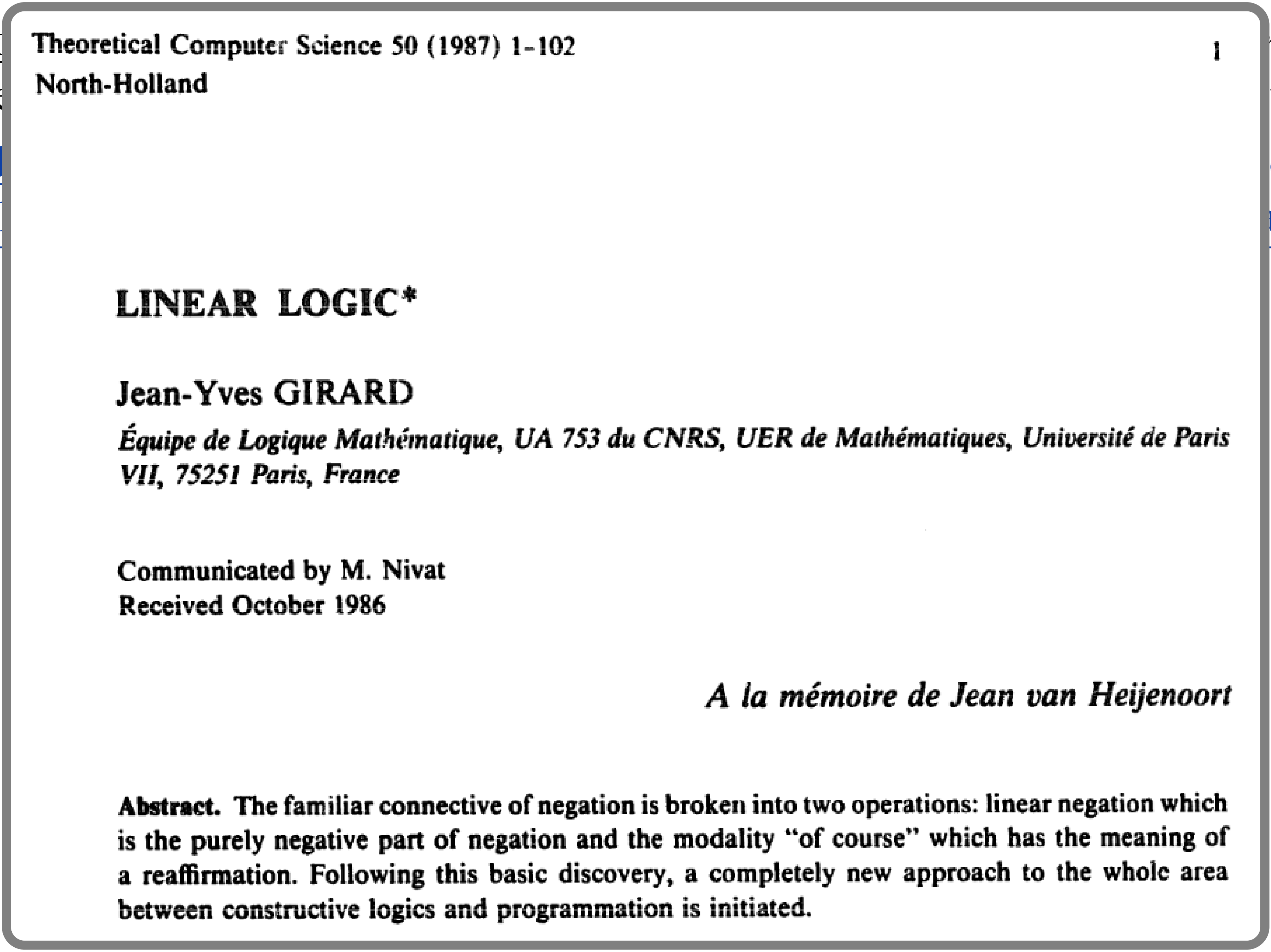

Theoretical Computer Science 50 (1987) 1-102 1
North-Holland

LINEAR LOGIC*

Jean-Yves GIRARD

Équipe de Logique Mathématique, UA 753 du CNRS, UER de Mathématiques, Université de Paris VII, 75251 Paris, France

Communicated by M. Nivat
Received October 1986

A la mémoire de Jean van Heijenoort

Abstract. The familiar connective of negation is broken into two operations: linear negation which is the purely negative part of negation and the modality “of course” which has the meaning of a reaffirmation. Following this basic discovery, a completely new approach to the whole area between constructive logics and programmation is initiated.

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A categorical quantum logic

Published online by Cambridge University Press: **04 July 2006**

SAMSON ABRAMSKY and ROSS DUNCAN

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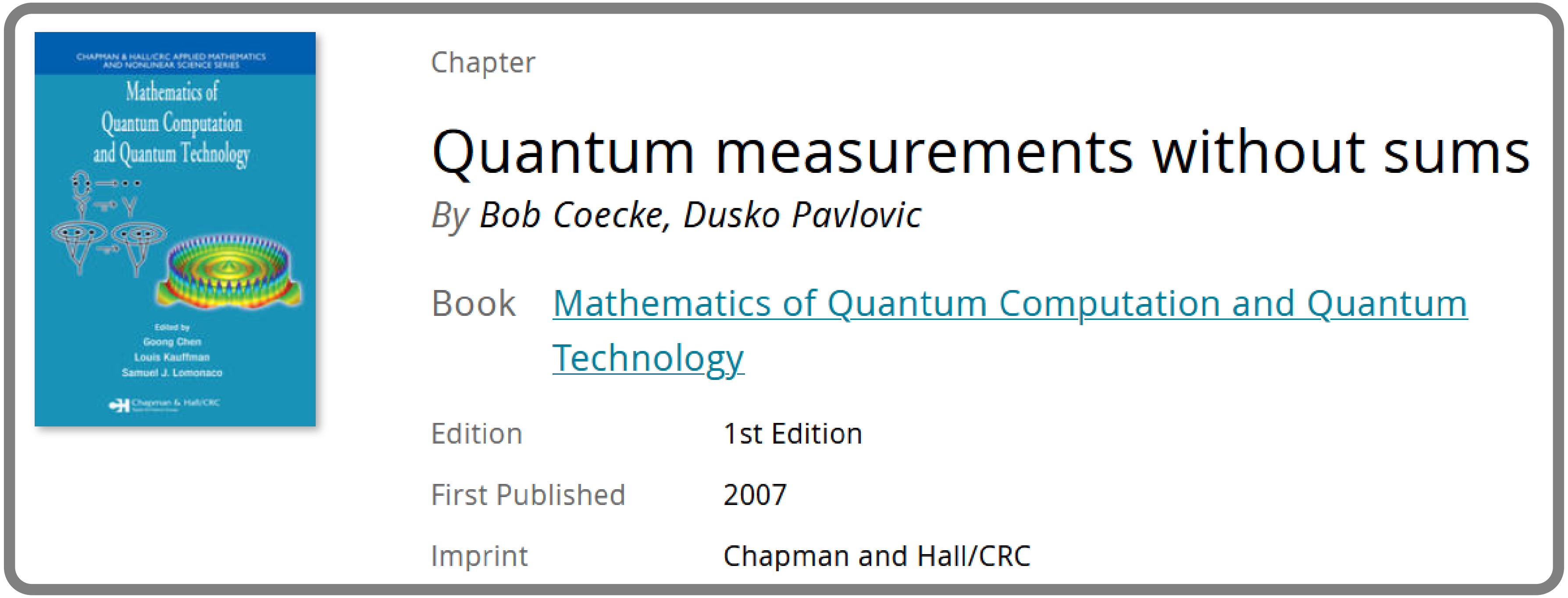

Chapter

Quantum measurements without sums

By Bob Coecke, Dusko Pavlovic

Book Mathematics of Quantum Computation and Quantum Technology

Edition 1st Edition

First Published 2007

Imprint Chapman and Hall/CRC

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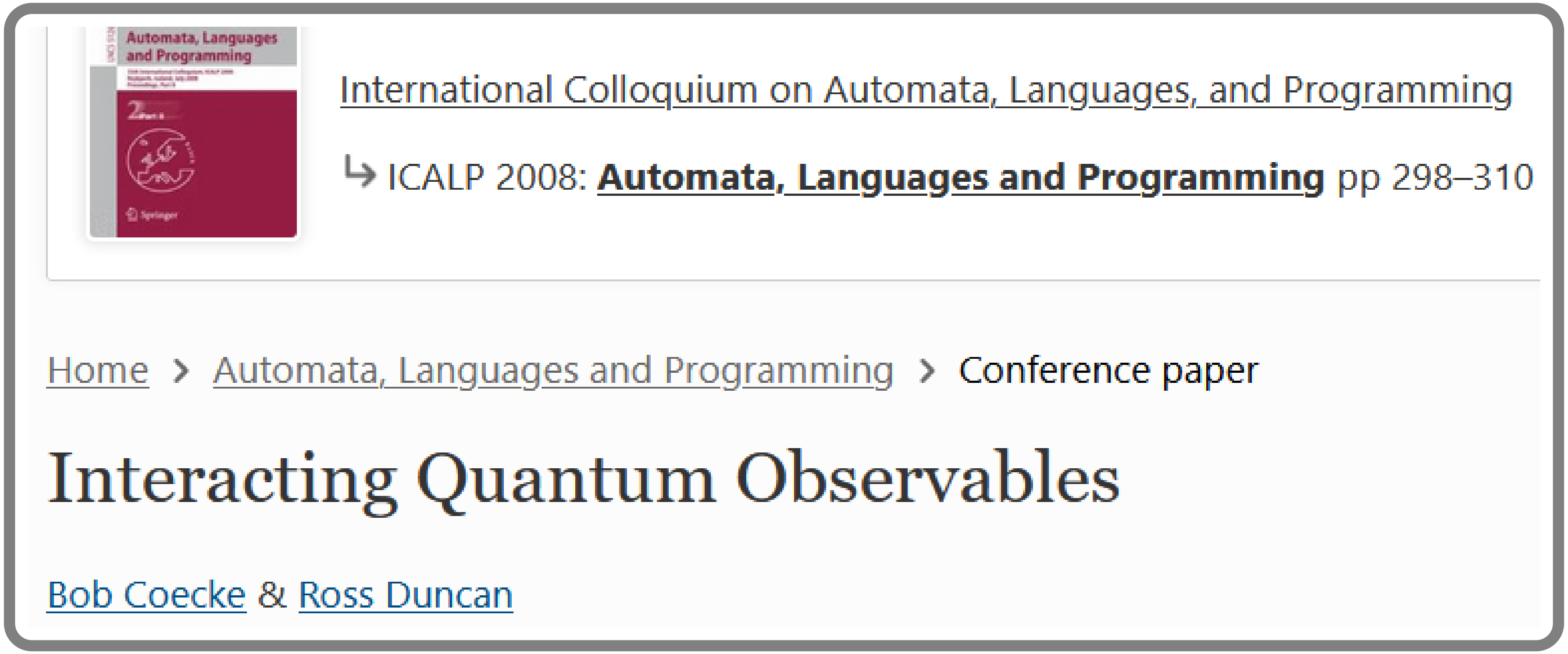

International Colloquium on Automata, Languages, and Programming

↳ ICALP 2008: **Automata, Languages and Programming** pp 298–310

Home › Automata, Languages and Programming › Conference paper

Interacting Quantum Observables

Bob Coecke & Ross Duncan

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$\begin{tikzcd} \mathcal{P} \arrow[d, hook, "p"] \\ \mathcal{H} \end{tikzcd}$	$\begin{tikzcd} \mathcal{P}_1 \arrow[dr, hook, "p_1"] & \mathcal{P}_1 \cap \mathcal{P}_2 \arrow[l] \arrow[r] \arrow[d, hook, "p_1 \wedge p_2"] & \mathcal{P}_2 \arrow[dl, hook', "p_2"] \\ & \mathcal{H} & \end{tikzcd}$	$\begin{tikzcd} & \mathcal{P}_1 \oplus \mathcal{P}_2 \arrow[d, two heads] & \\ \mathcal{P}_1 \arrow[ur] \arrow[dr, hook, "p_1"] & \mathrm{Span}\left(\mathcal{P}_1, \mathcal{P}_2\right) \arrow[d, hook, "p_1 \vee p_2"] & \mathcal{P}_2 \arrow[ul] \arrow[dl, hook', "p_2"] \\ & \mathcal{H} & \end{tikzcd}$

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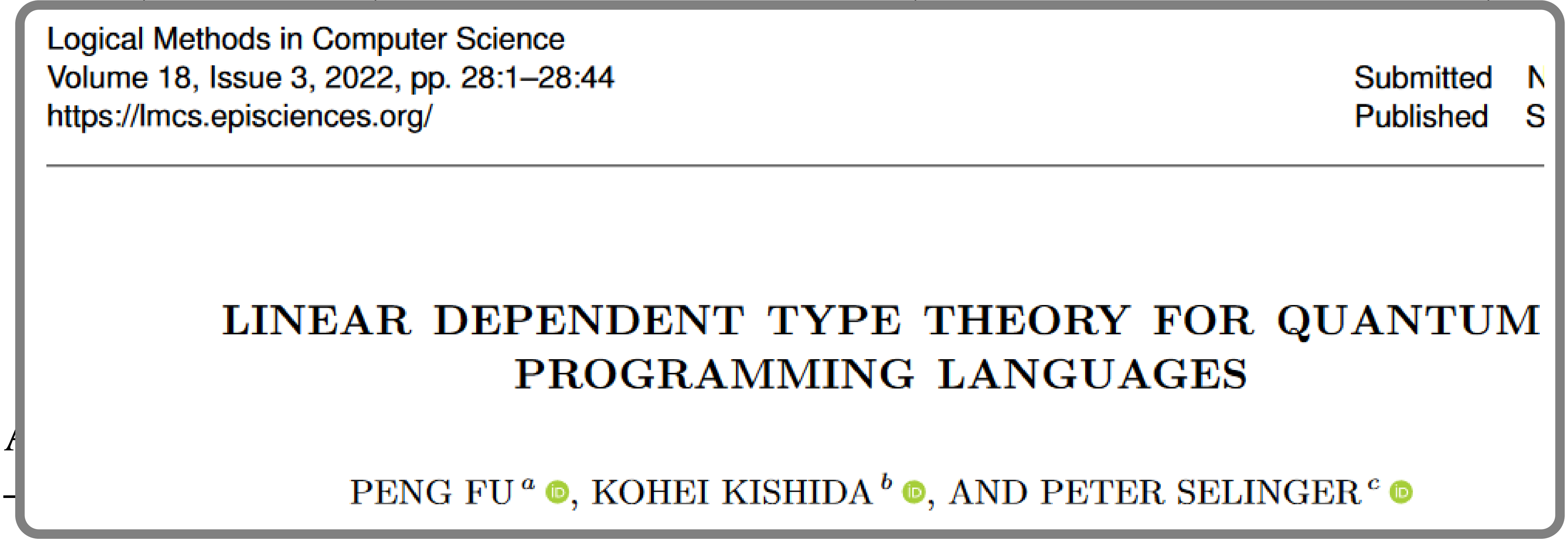

Logical Methods in Computer Science
Volume 18, Issue 3, 2022, pp. 28:1–28:44
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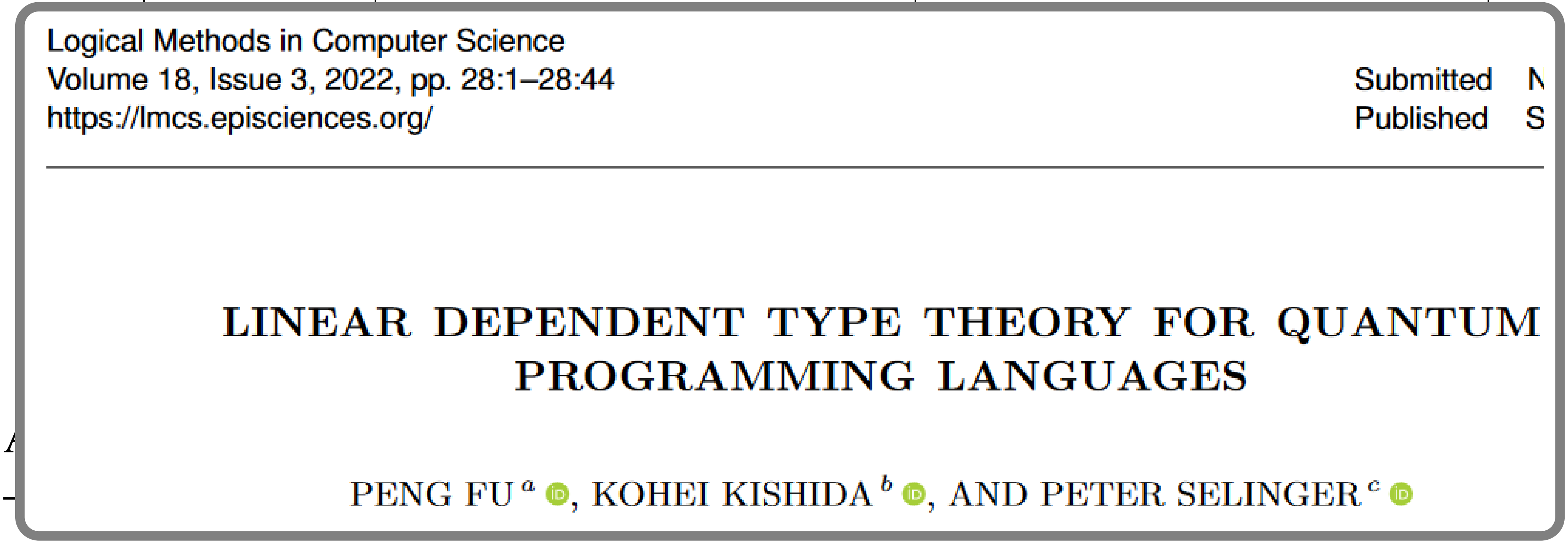

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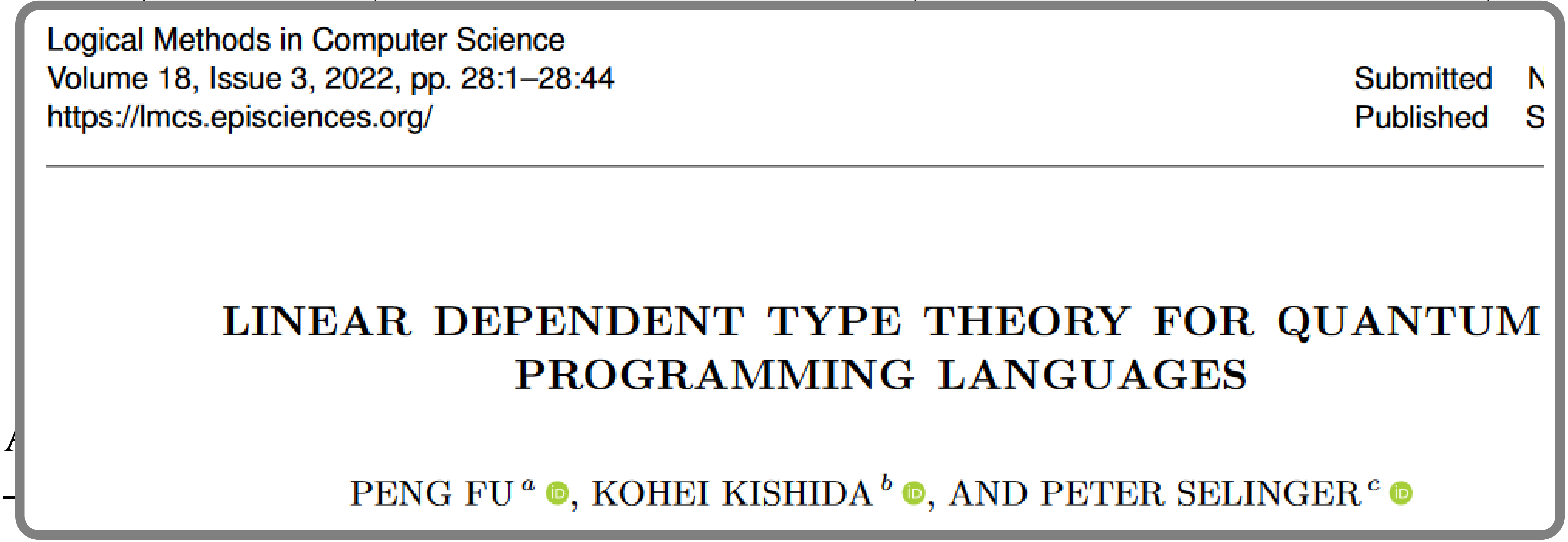

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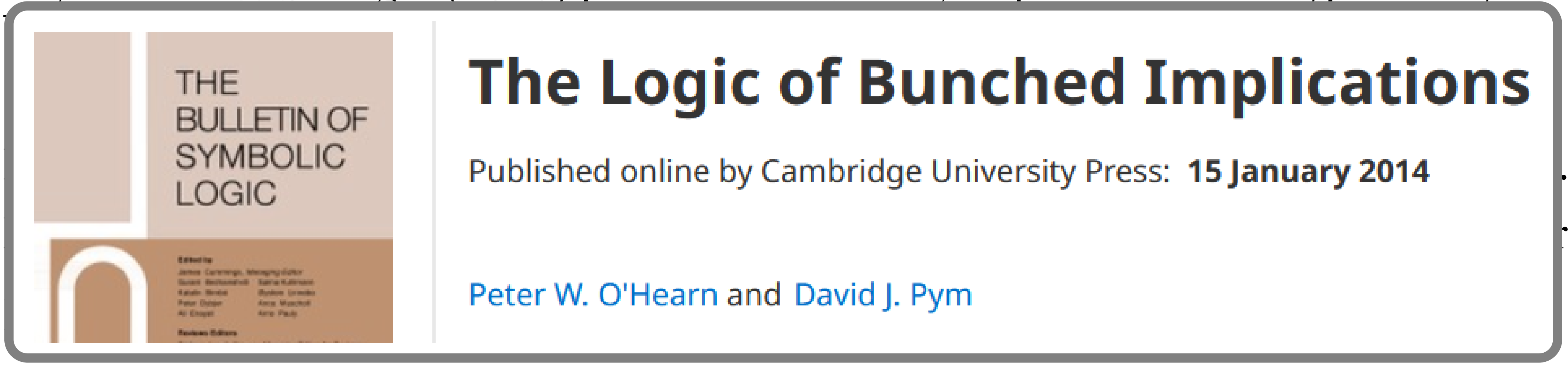

THE BULLETIN OF SYMBOLIC LOGIC

The Logic of Bunched Implications

Peter W. O'Hearn and David J. Pym

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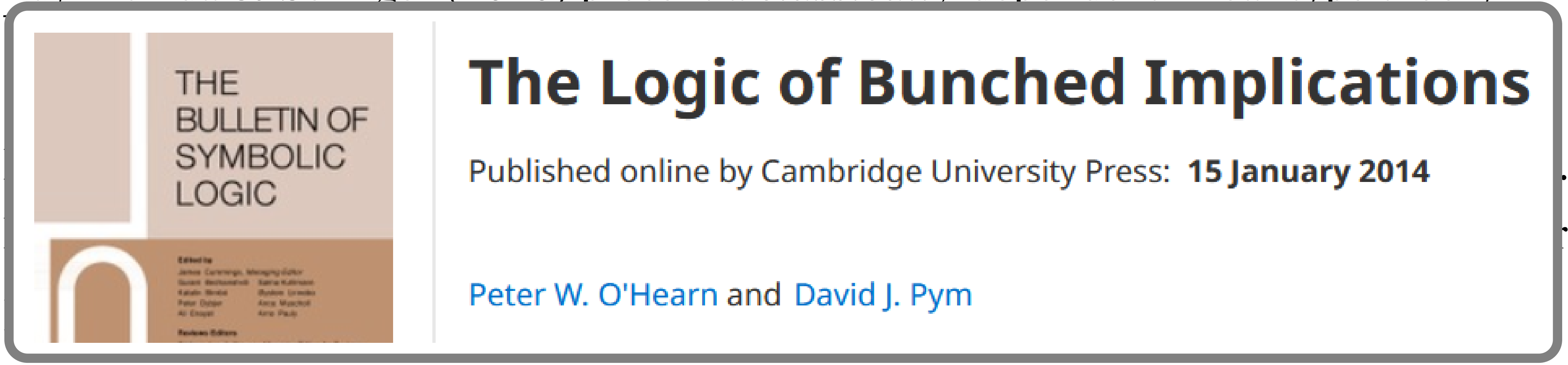

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Published online by Cambridge University Press: **15 January 2014**

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Schöpp & Stark (2004) make progress towards bunched dependent type theory

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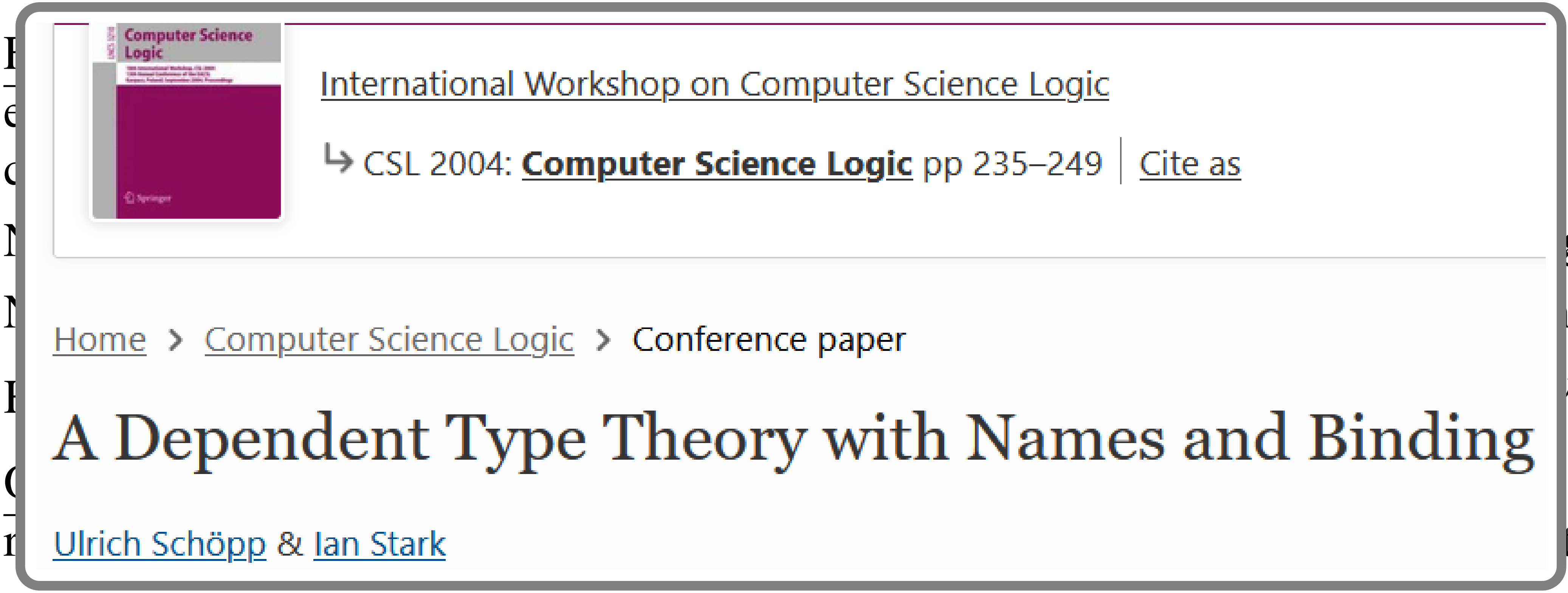

International Workshop on Computer Science Logic

↳ CSL 2004: **Computer Science Logic** pp 235–249 | Cite as

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A Dependent Type Theory with Names and Binding

Ulrich Schöpp & Ian Stark

Schöpp & Stark (2004) make progress towards bunched dependent type theory

$$\begin{tikzcd}
\mathcal{P} \arrow[d, hook, "p"] \\
\mathcal{H}
\end{tikzcd}$$

$$\begin{tikzcd}
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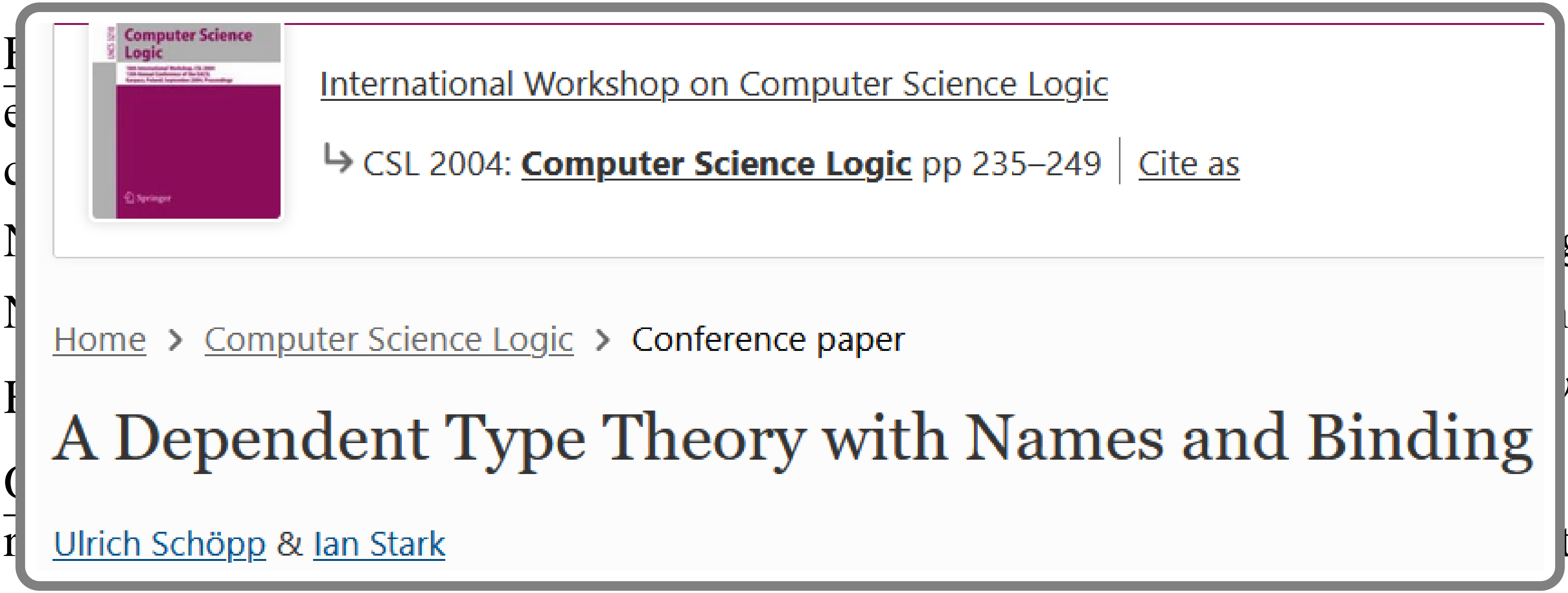

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& \mathcal{H} &
\end{tikzcd}$$

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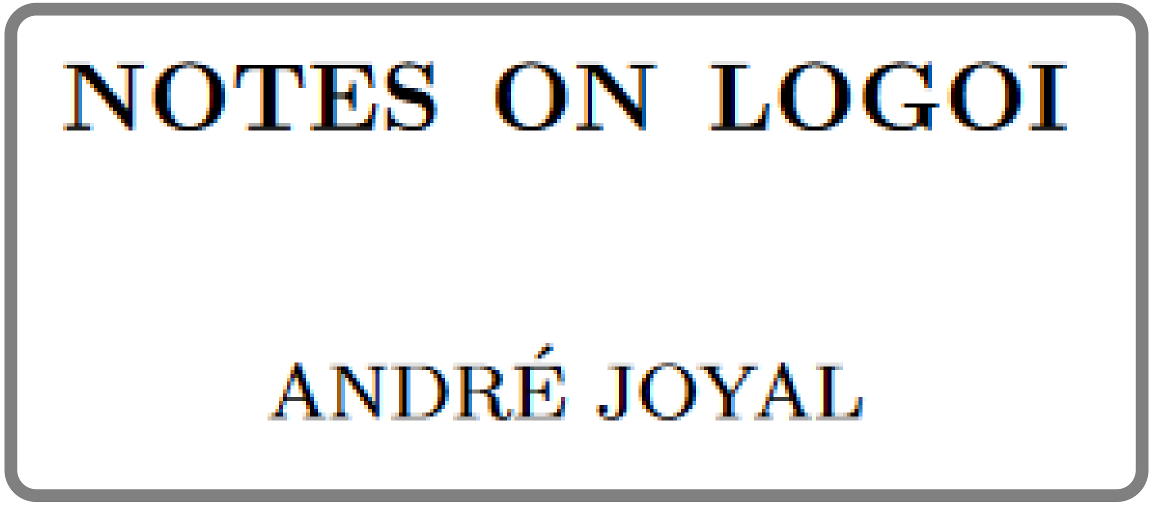

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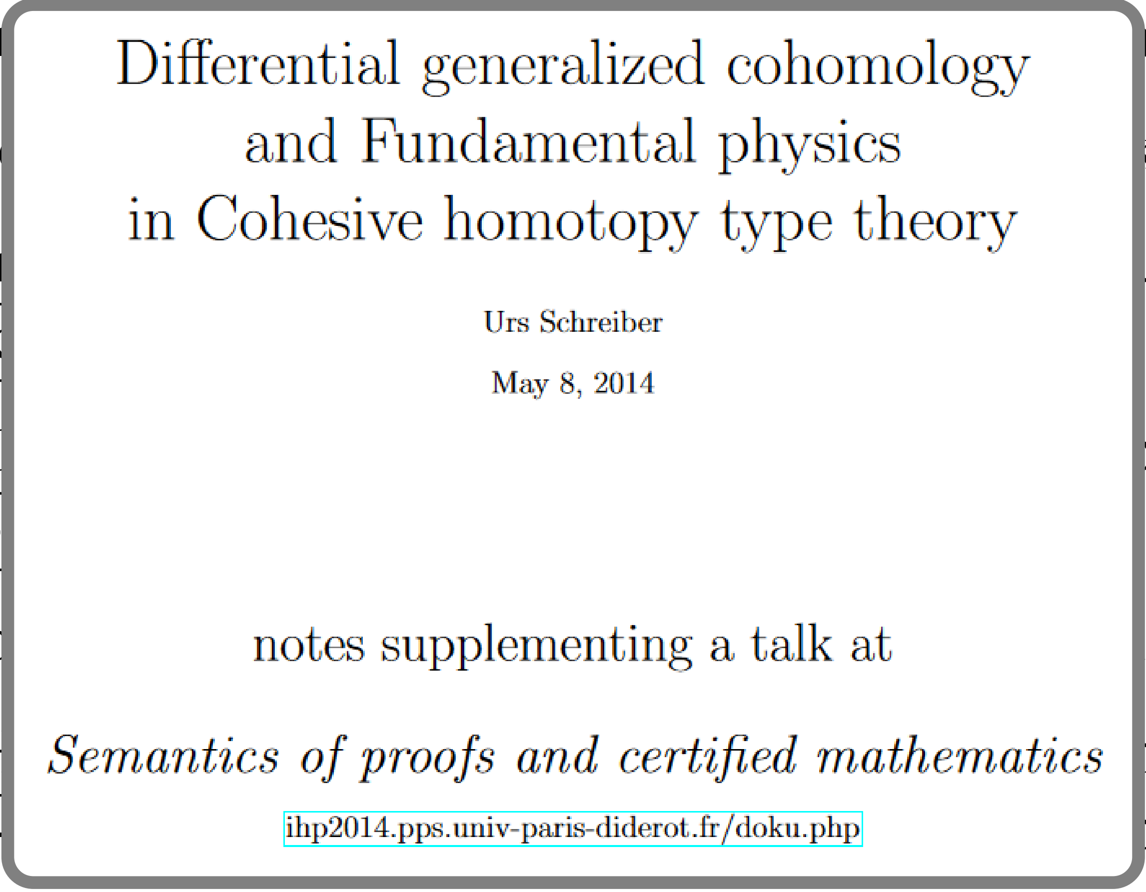

Differential generalized cohomology
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Urs Schreiber

May 8, 2014

notes supplementing a talk at

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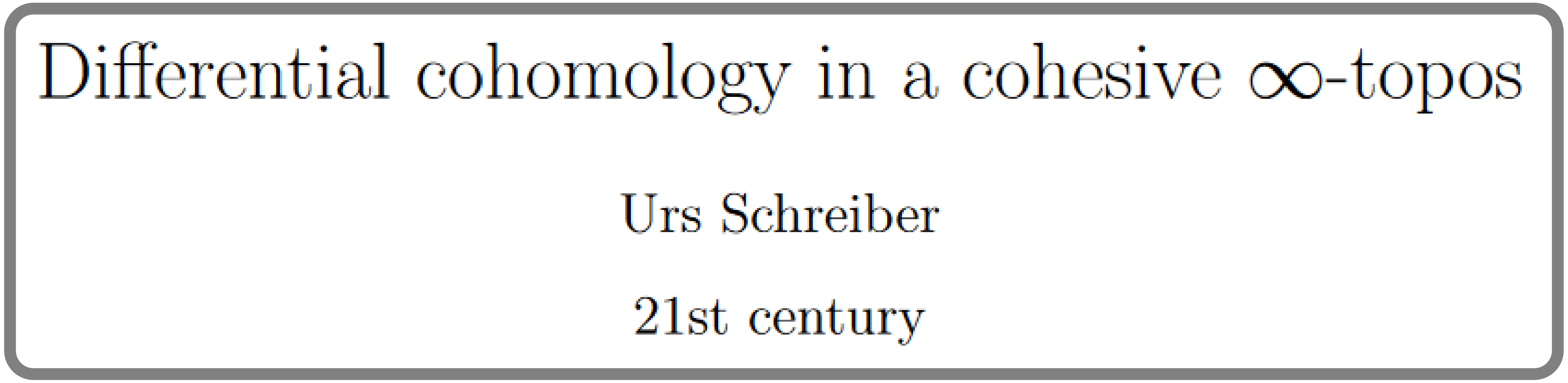

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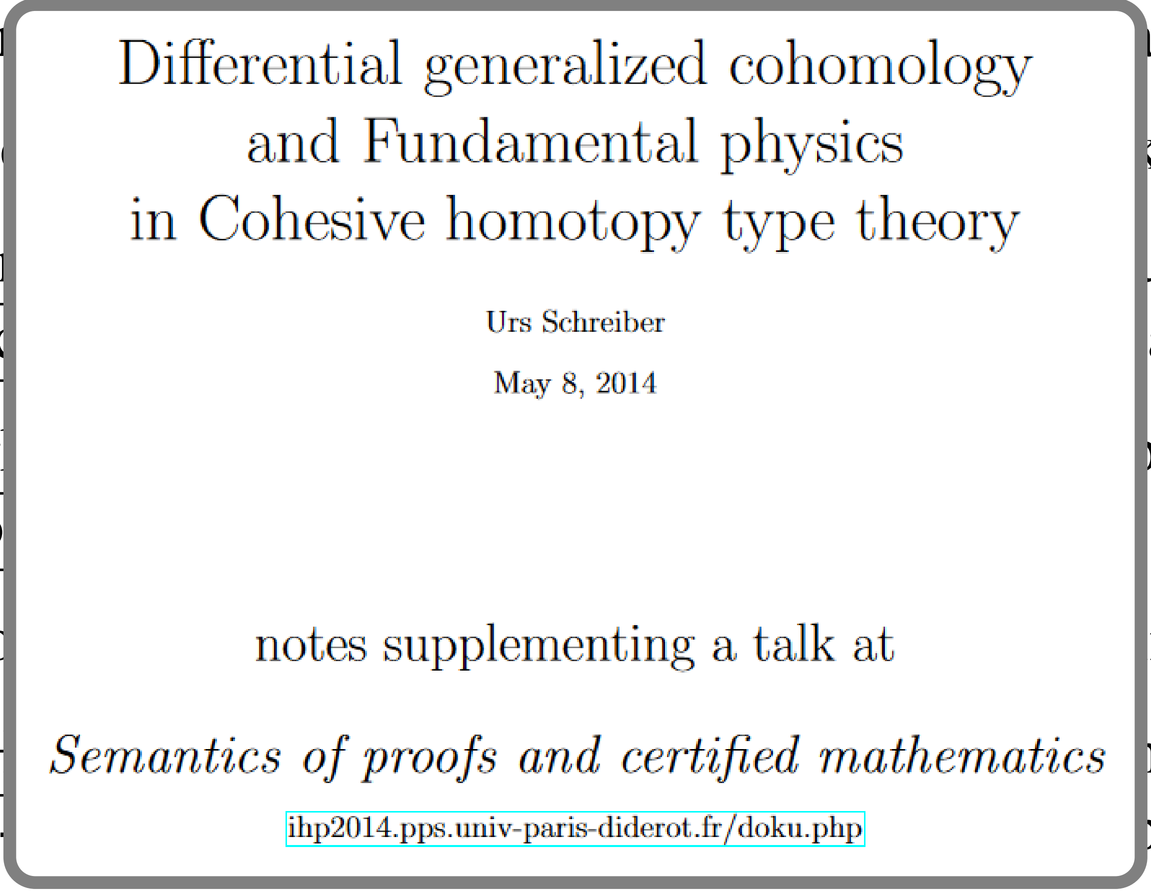

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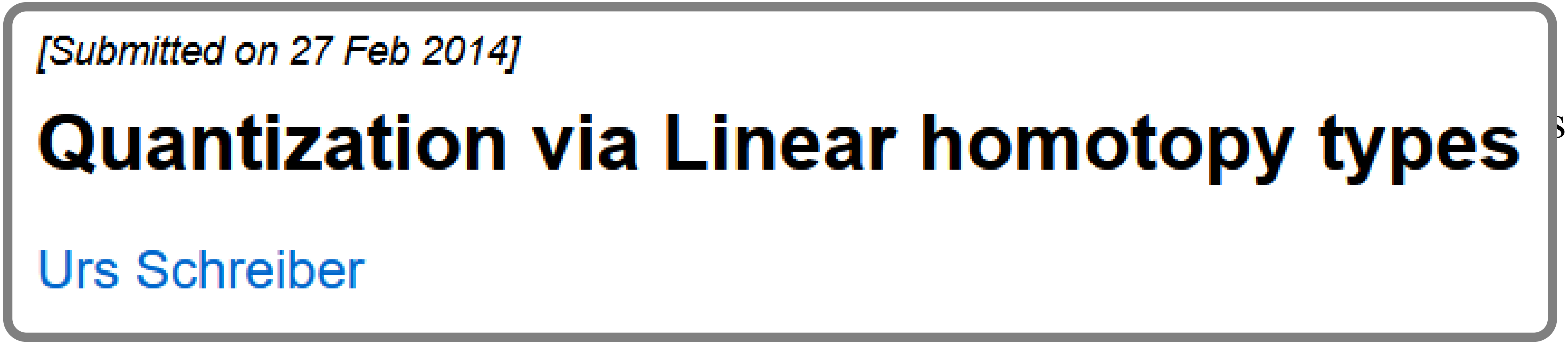

[Submitted on 27 Feb 2014]

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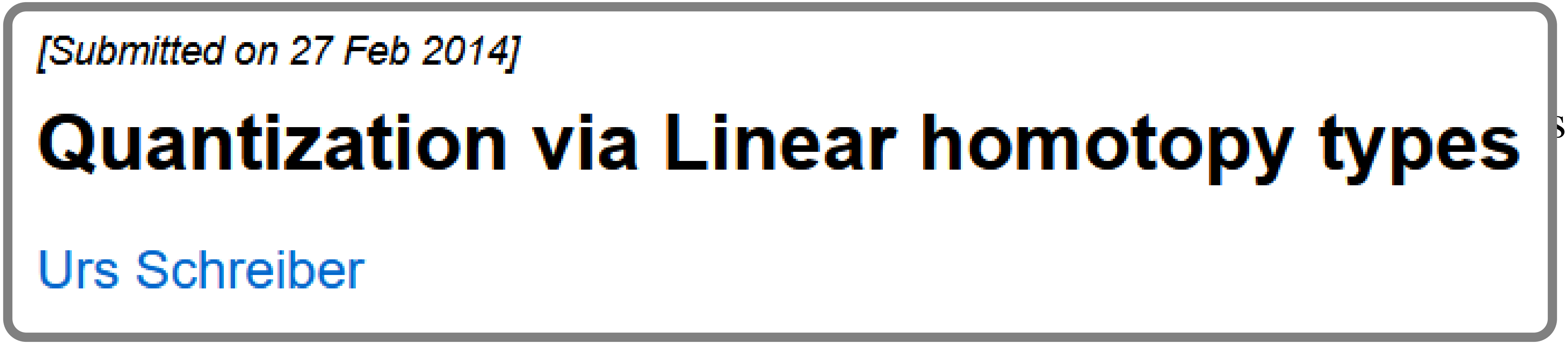

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[Submitted on 8 Feb 2021]

Synthetic Spectra via a Monadic and Comonadic Modality

Mitchell Riley, Eric Finster, Daniel R. Licata

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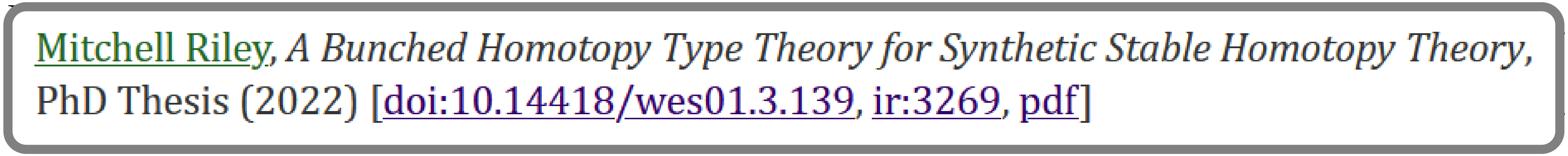

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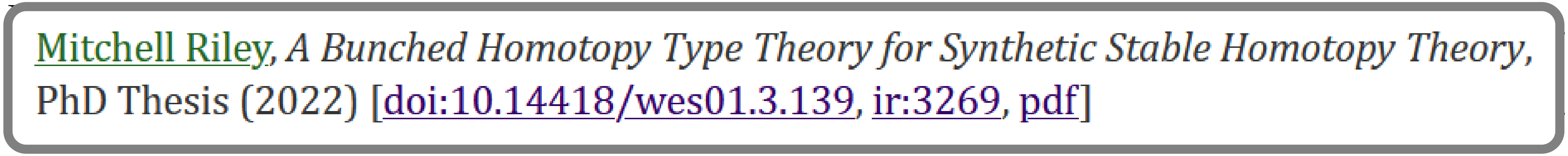

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Effective Quantum Certification via Linear Homotopy Types

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CQTS (2023) understand this *Linear Homotopy Type Theory* (`LHoTT`) as the missing quantum certification language (close to `Quipper`)...

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Our Solution

Linear Homotopy Type Theory (`LHoTT`)

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Idea: Frobenius monad on type system carves out classical types

classical types

bireflective among

all types

$$\begin{tikzcd}
\mathrm{ClType} \arrow[r, hook, "\iota" description] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathrm{Type} \arrow[l, shift right=6, "\beta"'] \arrow[l, shift left=6, "\beta"] \arrow[loop right, "\natural"]
\end{tikzcd}$$

classical modality

such that:

$$\begin{tikzcd}
\natural \arrow[r, "\varepsilon^{\natural}"] \arrow[dr, equal] & \mathrm{id}_{\mathscr{B}} \arrow[d, "\eta^{\natural}"] \\
& \natural
\end{tikzcd}$$

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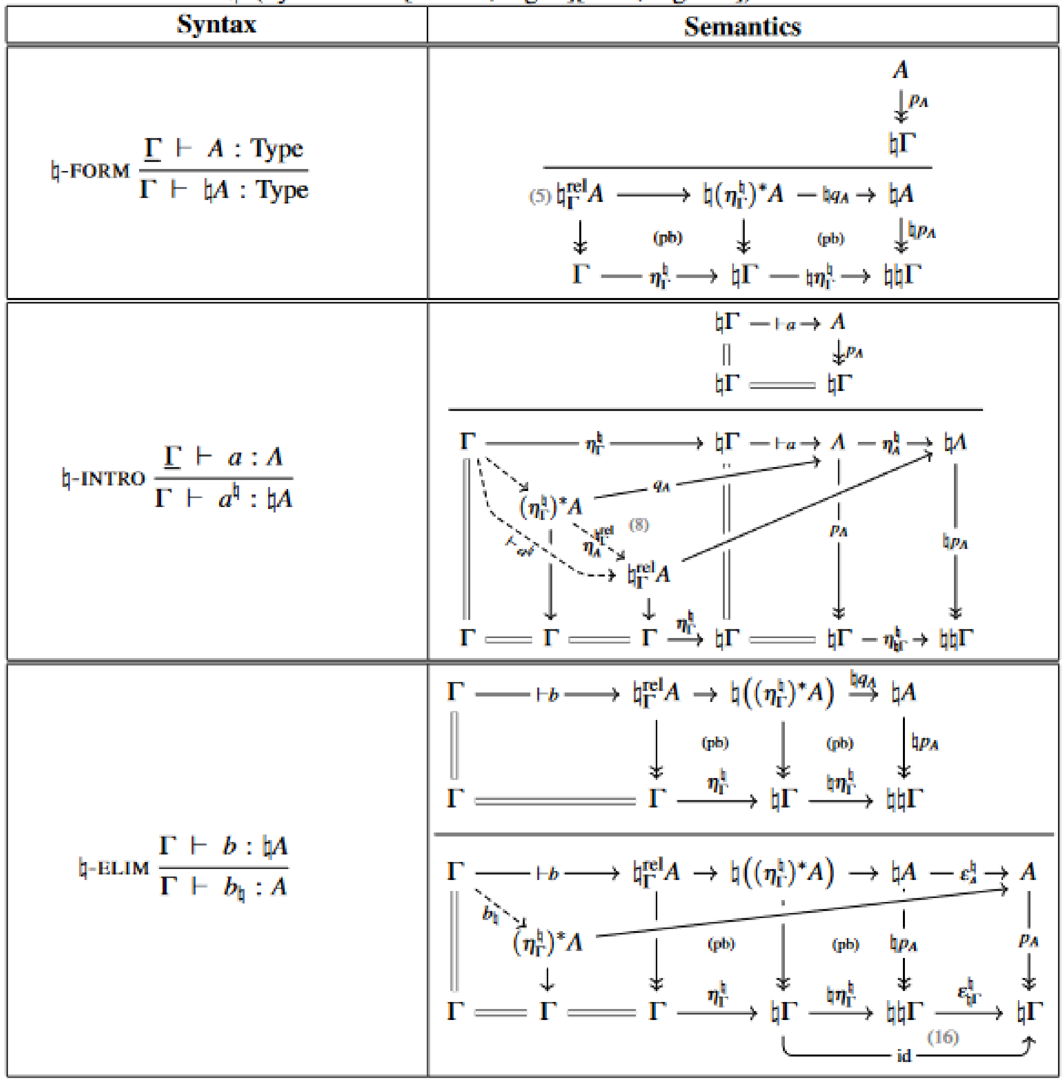
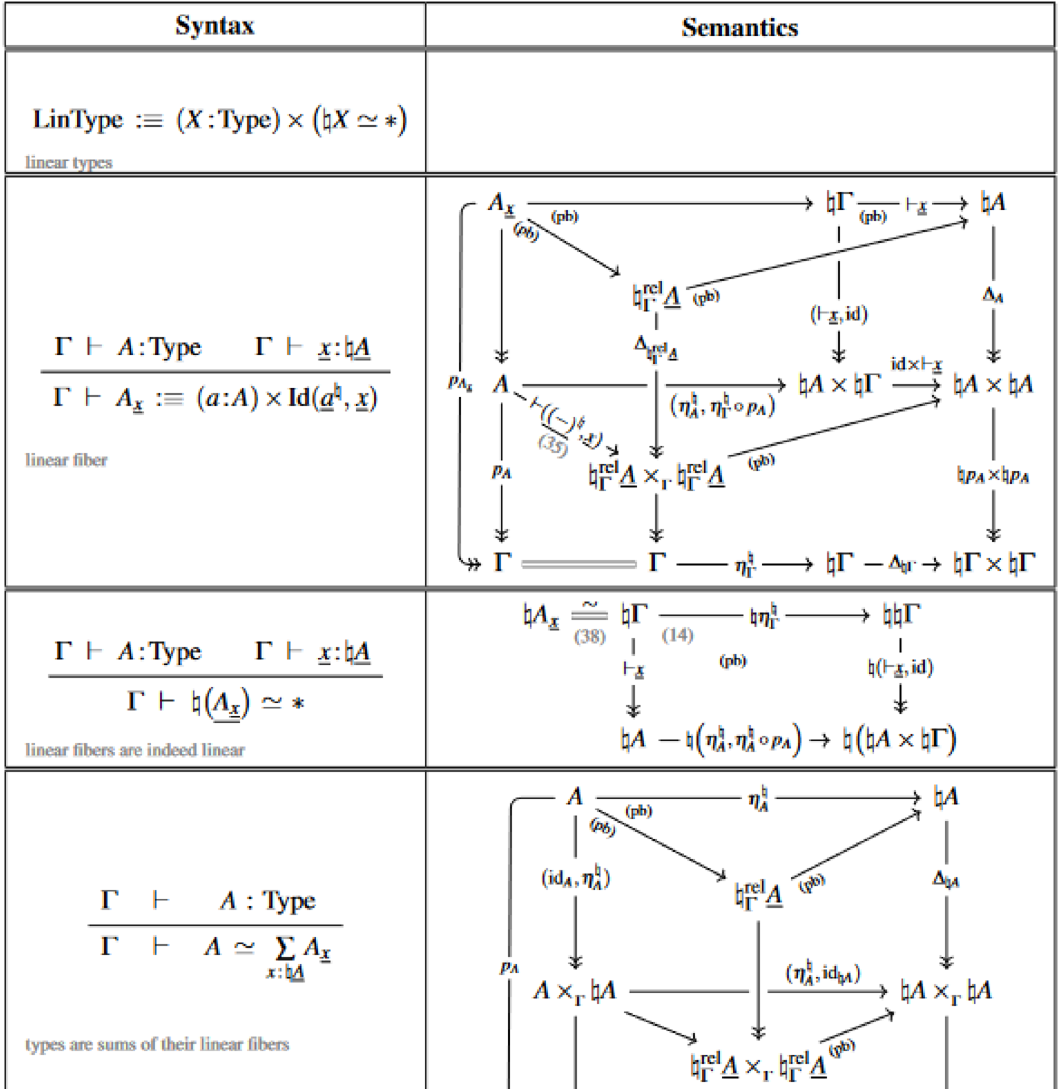
Inference rules for $\natural$. (Syntax from [RFL21, Fig. 2][Ri22, Fig. 1.2]).



Syntax	Semantics
$\natural$-FORM $\dfrac{\Gamma \vdash A : \mathrm{Type}}{\Gamma \vdash \natural A : \mathrm{Type}}$	$\begin{tikzcd} A \arrow[d, two heads, "p_A"] \\ \natural\Gamma \end{tikzcd}$ $\begin{tikzcd} (5)\ \natural^{\mathrm{rel}}_{\Gamma} A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "(\mathrm{pb})"] & \natural(\eta^{\natural}_{\Gamma})^{*} A \arrow[r, "\natural q_A"] \arrow[d, two heads] \arrow[dr, phantom, "(\mathrm{pb})"] & \natural A \arrow[d, two heads, "\natural p_A"] \\ \Gamma \arrow[r, "\eta^{\natural}_{\Gamma}"] & \natural\Gamma \arrow[r, "\natural\eta^{\natural}_{\Gamma}"] & \natural\natural\Gamma \end{tikzcd}$
$\natural$-INTRO $\dfrac{\Gamma \vdash a : A}{\Gamma \vdash a^{\natural} : \natural A}$	$\begin{tikzcd} \natural\Gamma \arrow[r, "\vdash a"] \arrow[d, equal] & A \arrow[d, two heads, "p_A"] \\ \natural\Gamma \arrow[r, equal] & \natural\Gamma \end{tikzcd}$ $\begin{tikzcd} \Gamma \arrow[rrr, "\eta^{\natural}_{\Gamma}"] \arrow[ddd, equal] \arrow[dr, dashed] \arrow[ddrr, dashed, bend right, "\vdash a^{\natural}"'] & & & \natural\Gamma \arrow[r, "\vdash a"] \arrow[ddd, equal] & A \arrow[r, "\eta^{\natural}_{A}"] \arrow[ddd, two heads, "p_A"] & \natural A \arrow[ddd, two heads, "\natural p_A"] \\ & (\eta^{\natural}_{\Gamma})^{*} A \arrow[urrr, "q_A"] \arrow[dr, dashed, "\eta^{\natural\mathrm{rel}}_{A}"] \arrow[dd] & (8) & & & \\ & & \natural^{\mathrm{rel}}_{\Gamma} A \arrow[uurrr] \arrow[d] & & & \\ \Gamma \arrow[r, equal] & \Gamma \arrow[r, equal] & \Gamma \arrow[r, "\eta^{\natural}_{\Gamma}"] & \natural\Gamma \arrow[r, equal] & \natural\Gamma \arrow[r, "\eta^{\natural}_{\natural\Gamma}"] & \natural\natural\Gamma \end{tikzcd}$
$\natural$-ELIM $\dfrac{\Gamma \vdash b : \natural A}{\Gamma \vdash b_{\natural} : A}$	$\begin{tikzcd} \Gamma \arrow[r, "\vdash b"] \arrow[d, equal] & \natural^{\mathrm{rel}}_{\Gamma} A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "(\mathrm{pb})"] & \natural\big((\eta^{\natural}_{\Gamma})^{*} A\big) \arrow[r, "\natural q_A"] \arrow[d, two heads] \arrow[dr, phantom, "(\mathrm{pb})"] & \natural A \arrow[d, two heads, "\natural p_A"] \\ \Gamma \arrow[r, equal] & \Gamma \arrow[r, "\eta^{\natural}_{\Gamma}"] & \natural\Gamma \arrow[r, "\natural\eta^{\natural}_{\Gamma}"] & \natural\natural\Gamma \end{tikzcd}$ $\begin{tikzcd} \Gamma \arrow[rr, "\vdash b"] \arrow[dd, equal] \arrow[dr, dashed, "b_{\natural}"'] & & \natural^{\mathrm{rel}}_{\Gamma} A \arrow[r] \arrow[dd, two heads] \arrow[ddr, phantom, "(\mathrm{pb})"] & \natural\big((\eta^{\natural}_{\Gamma})^{*} A\big) \arrow[r] \arrow[dd, two heads] \arrow[ddr, phantom, "(\mathrm{pb})"] & \natural A \arrow[r, "\varepsilon^{\natural}_{A}"] \arrow[dd, two heads, "\natural p_A"] & A \arrow[dd, two heads, "p_A"] \\ & (\eta^{\natural}_{\Gamma})^{*} A \arrow[urrrr] \arrow[d] & & & & \\ \Gamma \arrow[r, equal] & \Gamma \arrow[r, equal] & \Gamma \arrow[r, "\eta^{\natural}_{\Gamma}"] & \natural\Gamma \arrow[r, "\natural\eta^{\natural}_{\Gamma}"] \arrow[rr, bend right, "\mathrm{id}"'] & \natural\natural\Gamma \arrow[r, "\varepsilon^{\natural}_{\natural\Gamma}"] \arrow[d, phantom, "(16)"] & \natural\Gamma \end{tikzcd}$

Linear types. (Syntax from [RFL21, pp. 24][Ri22, §2.1])



Syntax	Semantics
$\mathrm{LinType} :\equiv (X : \mathrm{Type}) \times (\natural X \simeq *)$ linear types	
$\dfrac{\Gamma \vdash A : \mathrm{Type} \qquad \Gamma \vdash \underline{x} : \natural A}{\Gamma \vdash A_{\underline{x}} :\equiv (a : A) \times \mathrm{Id}(\underline{a}^{\natural}, \underline{x})}$ linear fiber	$\begin{tikzcd} A_{\underline{x}} \arrow[rr, "(\mathrm{pb})"'] \arrow[dr, "(\mathrm{pb})"'] \arrow[dd, two heads] \arrow[dddd, two heads, bend right, "p_{A_{\underline{x}}}"'] & & \natural\Gamma \arrow[r, "\vdash \underline{x}", "(\mathrm{pb})"'] \arrow[dd, two heads, "(\vdash \underline{x}, \mathrm{id})"] & \natural A \arrow[dd, two heads, "\Delta_A"] \\ & \natural^{\mathrm{rel}}_{\Gamma} \underline{A} \arrow[urr, "(\mathrm{pb})"'] \arrow[dd, two heads, "\Delta_{\natural^{\mathrm{rel}}_{\Gamma} \underline{A}}"] & & \\ A \arrow[rr, "(\eta^{\natural}_{A}, \eta^{\natural}_{\Gamma} \circ p_A)"'] \arrow[dr, "\vdash((-)^{\natural}, \underline{x})", "(35)"'] \arrow[dd, two heads, "p_A"] & & \natural A \times \natural\Gamma \arrow[r, "\mathrm{id} \times \vdash \underline{x}"] & \natural A \times \natural A \arrow[dd, two heads, "\natural p_A \times \natural p_A"] \\ & \natural^{\mathrm{rel}}_{\Gamma} \underline{A} \times_{\Gamma} \natural^{\mathrm{rel}}_{\Gamma} \underline{A} \arrow[urr, "(\mathrm{pb})"'] \arrow[d, two heads] & & \\ \Gamma \arrow[r, equal] & \Gamma \arrow[r, "\eta^{\natural}_{\Gamma}"] & \natural\Gamma \arrow[r, "\Delta_{\natural\Gamma}"] & \natural\Gamma \times \natural\Gamma \end{tikzcd}$
$\dfrac{\Gamma \vdash A : \mathrm{Type} \qquad \Gamma \vdash \underline{x} : \natural A}{\Gamma \vdash \natural(A_{\underline{x}}) \simeq *}$ linear fibers are indeed linear	$\begin{tikzcd} \natural A_{\underline{x}} \arrow[r, equal, "\sim", "(38)"'] & \natural\Gamma \arrow[rr, "\natural\eta^{\natural}_{\Gamma}", "(14)"'] \arrow[d, two heads, "\vdash \underline{x}"] \arrow[drr, phantom, "(\mathrm{pb})"] & & \natural\natural\Gamma \arrow[d, two heads, "\natural(\vdash \underline{x}, \mathrm{id})"] \\ & \natural A \arrow[rr, "\natural(\eta^{\natural}_{A}, \eta^{\natural}_{A} \circ p_A)"] & & \natural\big(\natural A \times \natural\Gamma\big) \end{tikzcd}$
$\dfrac{\Gamma \vdash A : \mathrm{Type}}{\Gamma \vdash A \simeq \sum\limits_{\underline{x} : \natural A} A_{\underline{x}}}$ types are sums of their linear fibers	$\begin{tikzcd} A \arrow[rr, "\eta^{\natural}_{A}", "(\mathrm{pb})"'] \arrow[dr, "(\mathrm{pb})"'] \arrow[dd, two heads, "(\mathrm{id}_A, \eta^{\natural}_{A})"] & & \natural A \arrow[dd, two heads, "\Delta_{\natural A}"] \\ & \natural^{\mathrm{rel}}_{\Gamma} \underline{A} \arrow[ur, "(\mathrm{pb})"'] \arrow[d, two heads] & \\ A \times_{\Gamma} \natural A \arrow[rr, "(\eta^{\natural}_{A}, \mathrm{id}_{\natural A})"] \arrow[r] & \natural^{\mathrm{rel}}_{\Gamma} \underline{A} \times_{\Gamma} \natural^{\mathrm{rel}}_{\Gamma} \underline{A} \arrow[r, "(\mathrm{pb})"] & \natural A \times_{\Gamma} \natural A \end{tikzcd}$ p_A

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closed base type | B-dependent linear types

$$B : \mathrm{BType} \quad \vdash \quad \mathrm{LinType}_B \;:\equiv\; B \to \mathrm{LinType}$$

linear base change

closed function

$$f : B \to B' \quad \vdash \quad \begin{tikzcd} \mathrm{LinType}_B \arrow[r, shift left=3, "f_!"] \arrow[r, shift right=3, "f_*"'] & \mathrm{LinType}_{B'} \arrow[l, "f^*"'] \end{tikzcd}$$

$$E_{(-)} : \mathrm{LinType}_{B'} \quad \vdash \quad \big(f^* E_{(-)}\big) \;:\equiv\; b \mapsto E_{f(b)}$$

precomposition

$$E_{(-)} : \mathrm{LinType}_{B} \quad \vdash \quad \big(f_* E_{(-)}\big) \;:\equiv\; b' \mapsto \prod_{(b,p) : \mathrm{fib}_f(b')} E_b$$

dependent product

$$E_{(-)} : \mathrm{LinType}_{B} \quad \vdash \quad \big(f_! E_{(-)}\big) \;:\equiv\; b' \mapsto \bigvee_{(b,p) : \mathrm{fib}_f(b')} E_b$$

HIT cofiber of
zero inclusion in
dependent sum

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`LHoTT` is like a quantum microscope for Classical Data Types B

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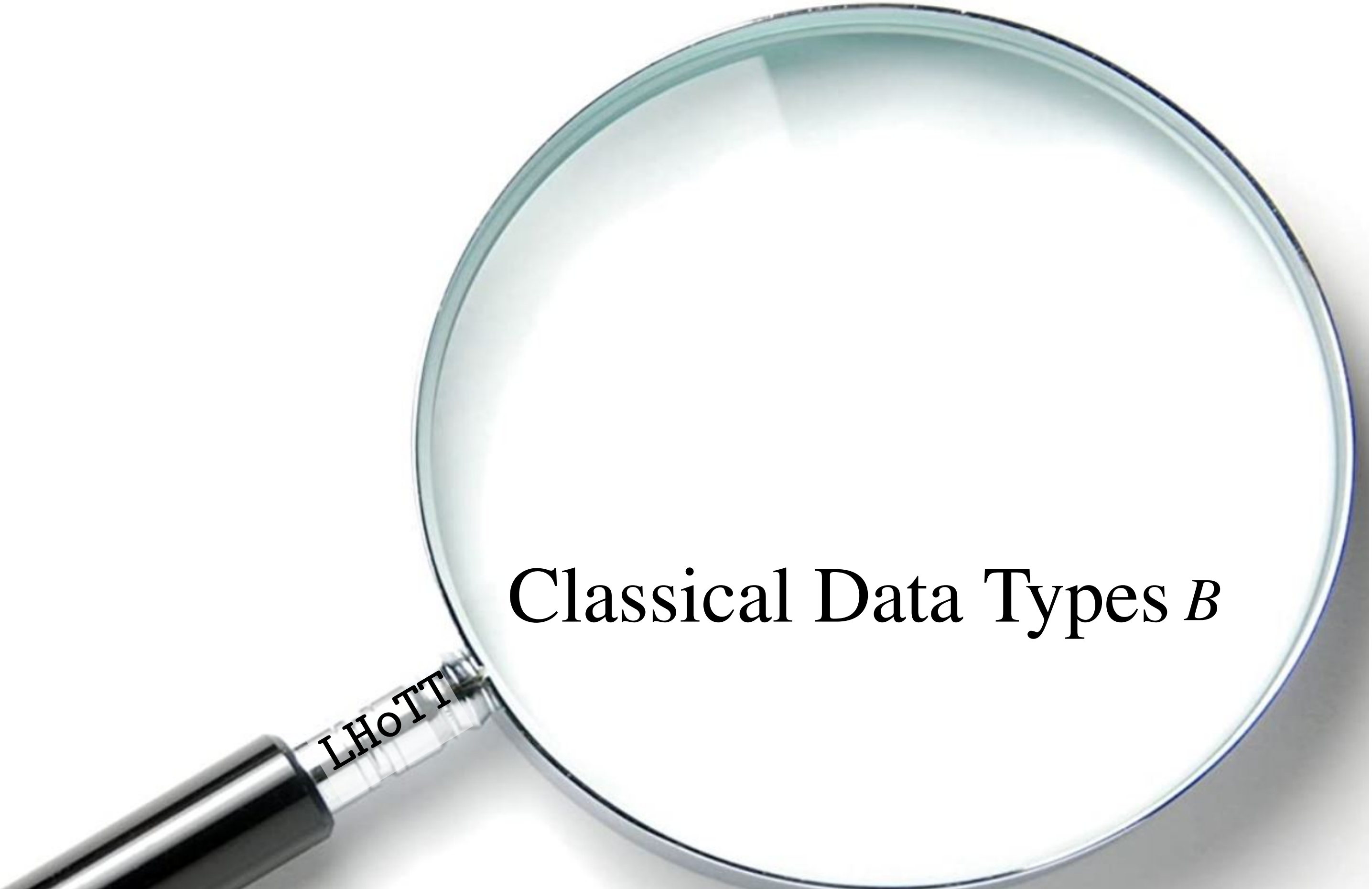

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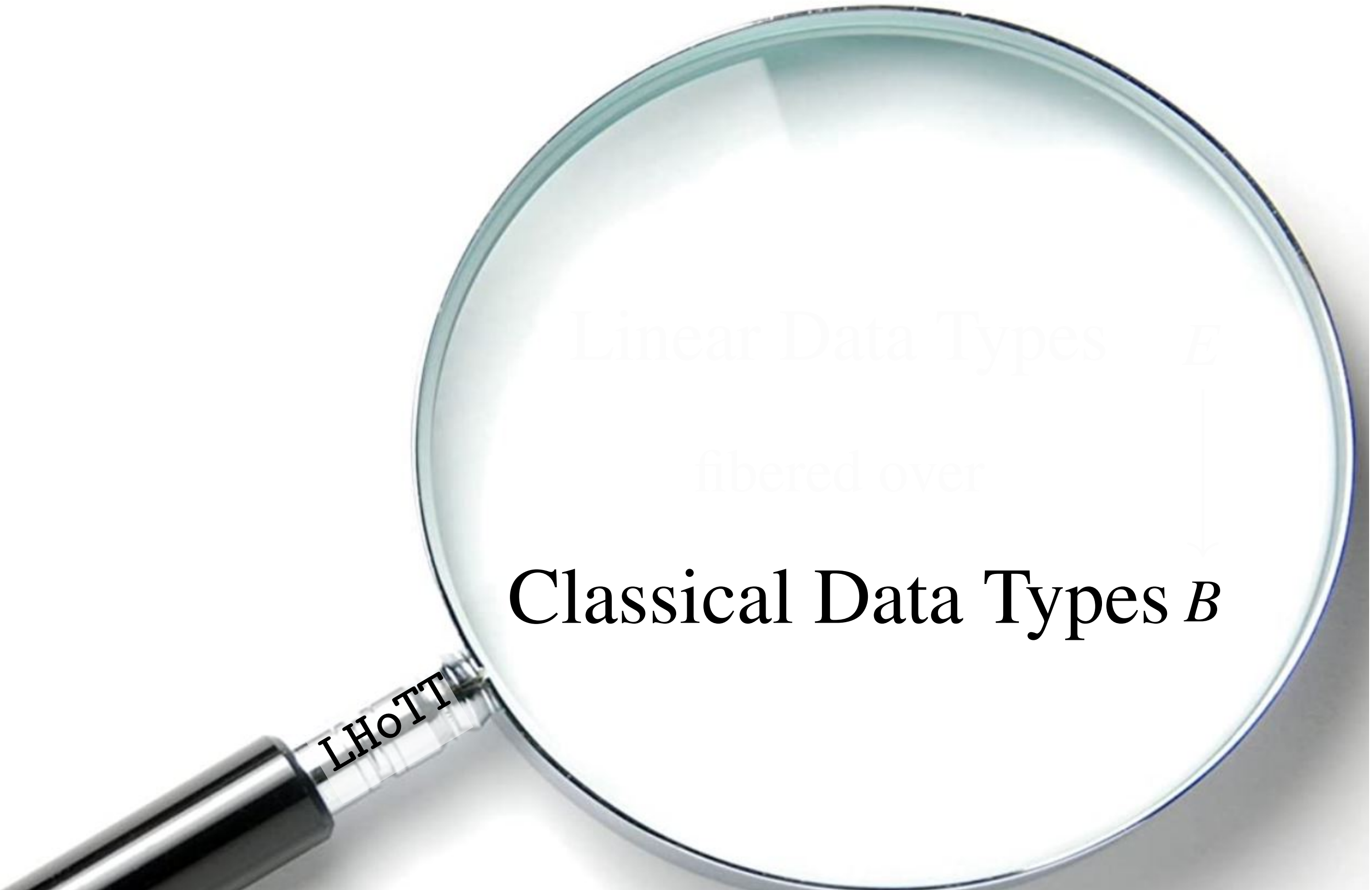

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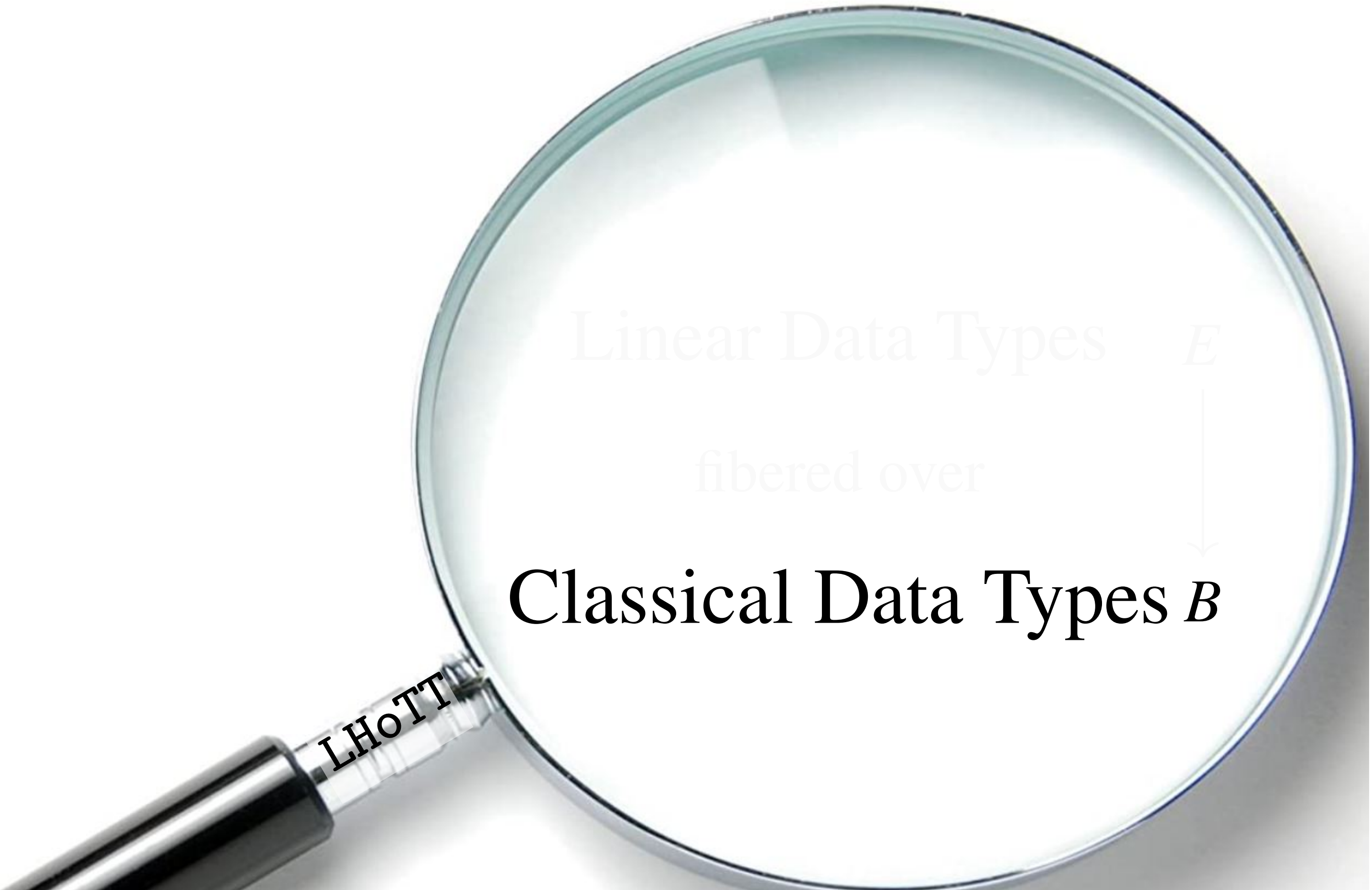

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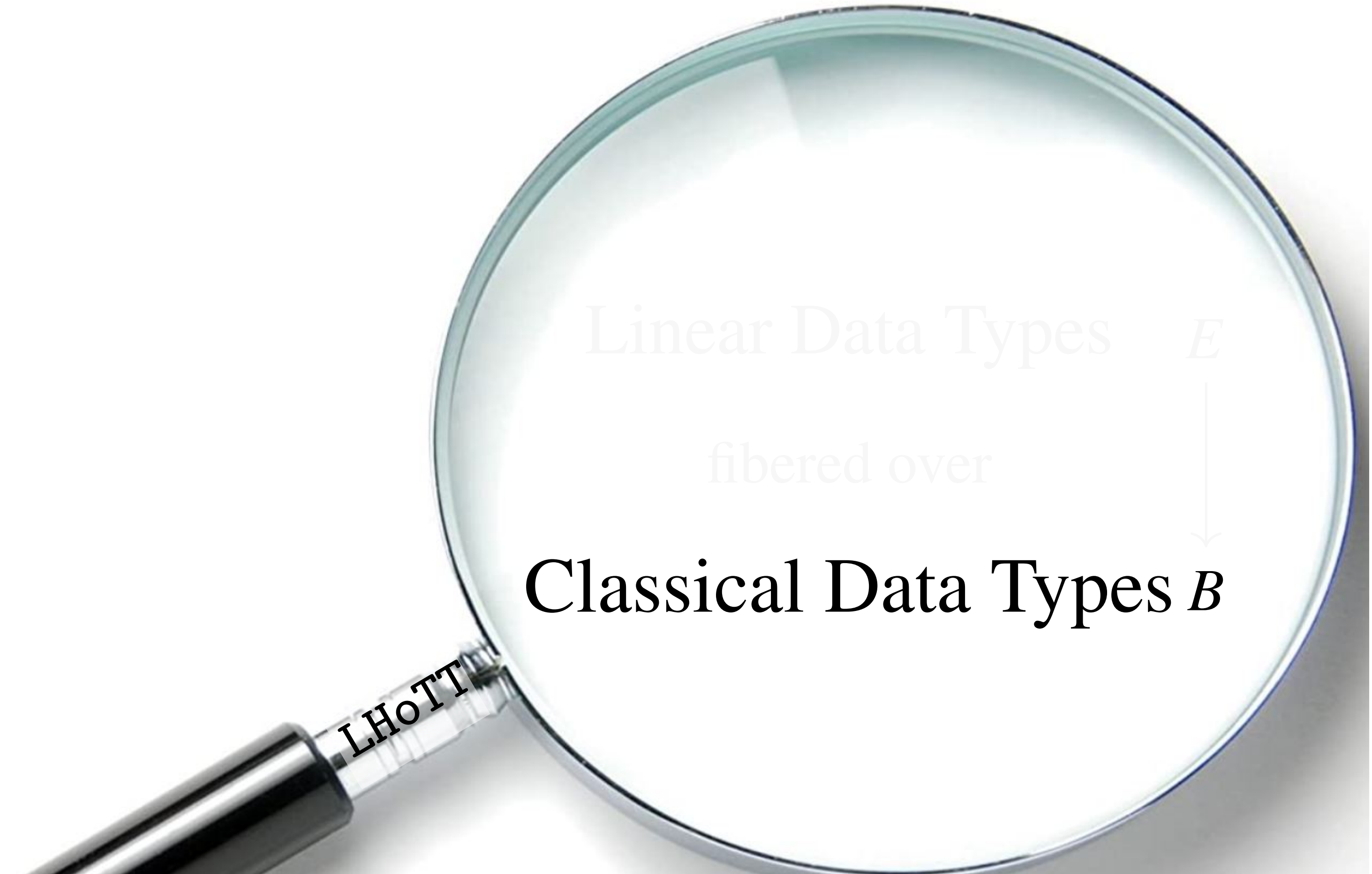

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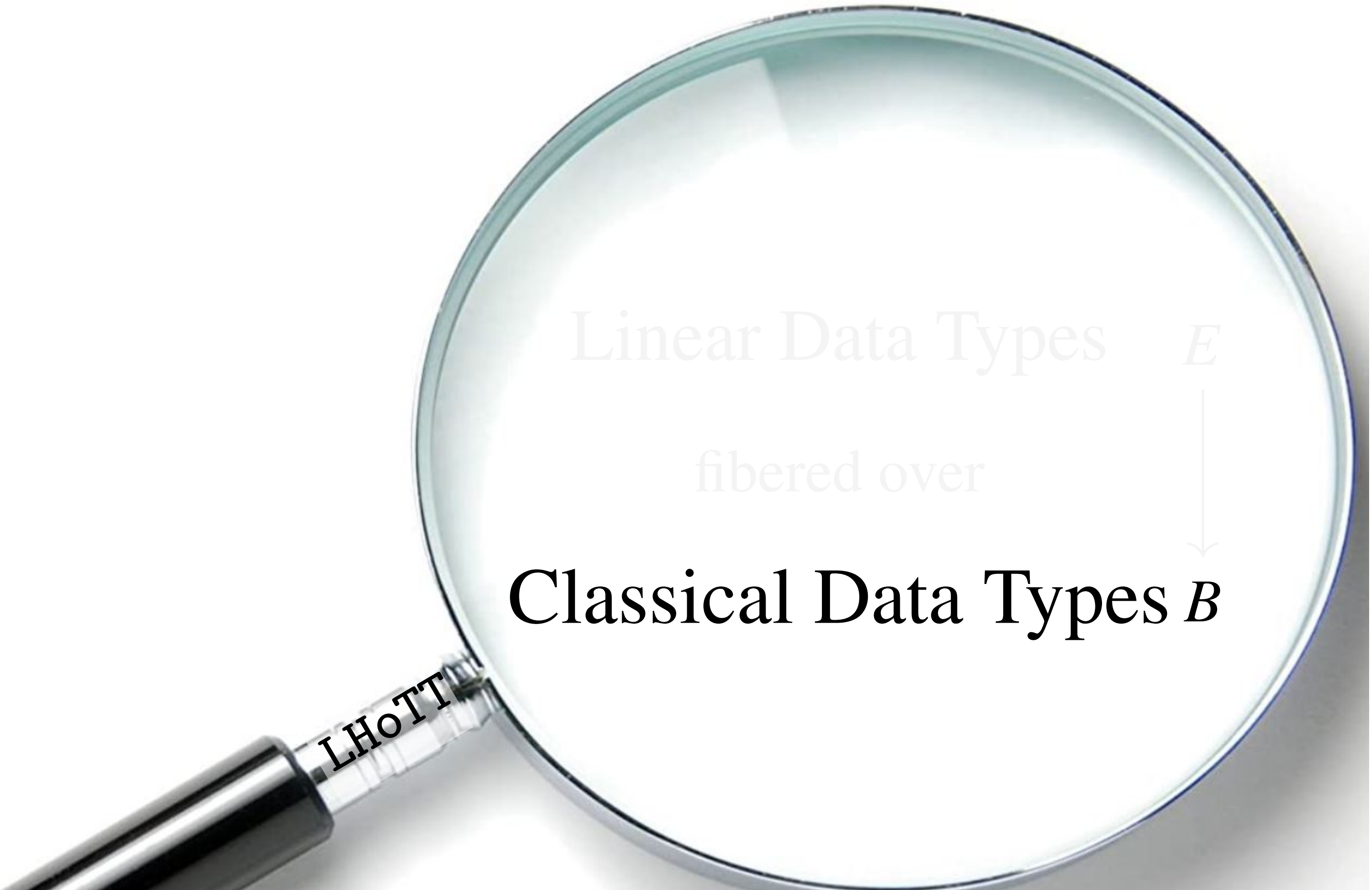

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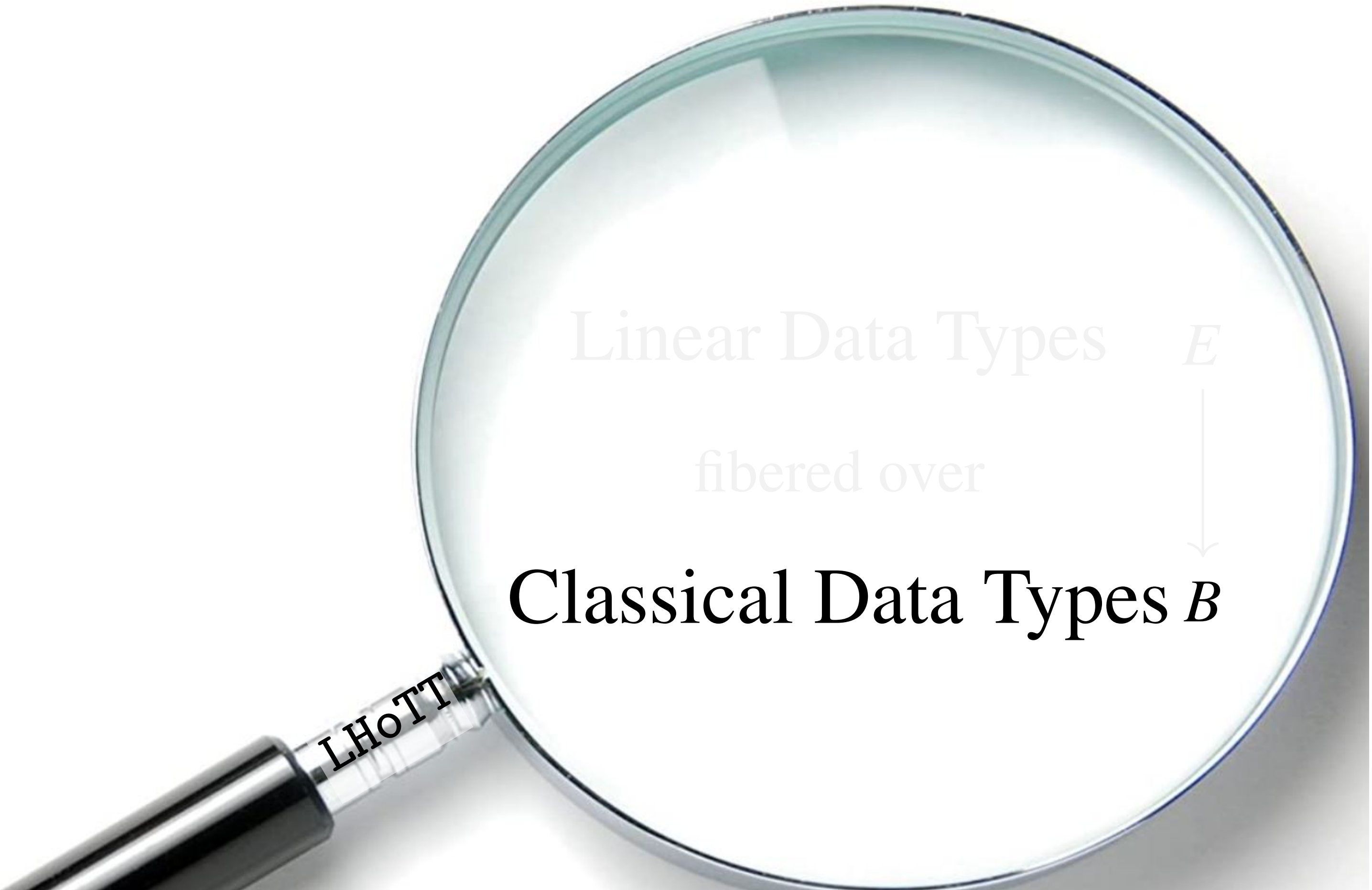

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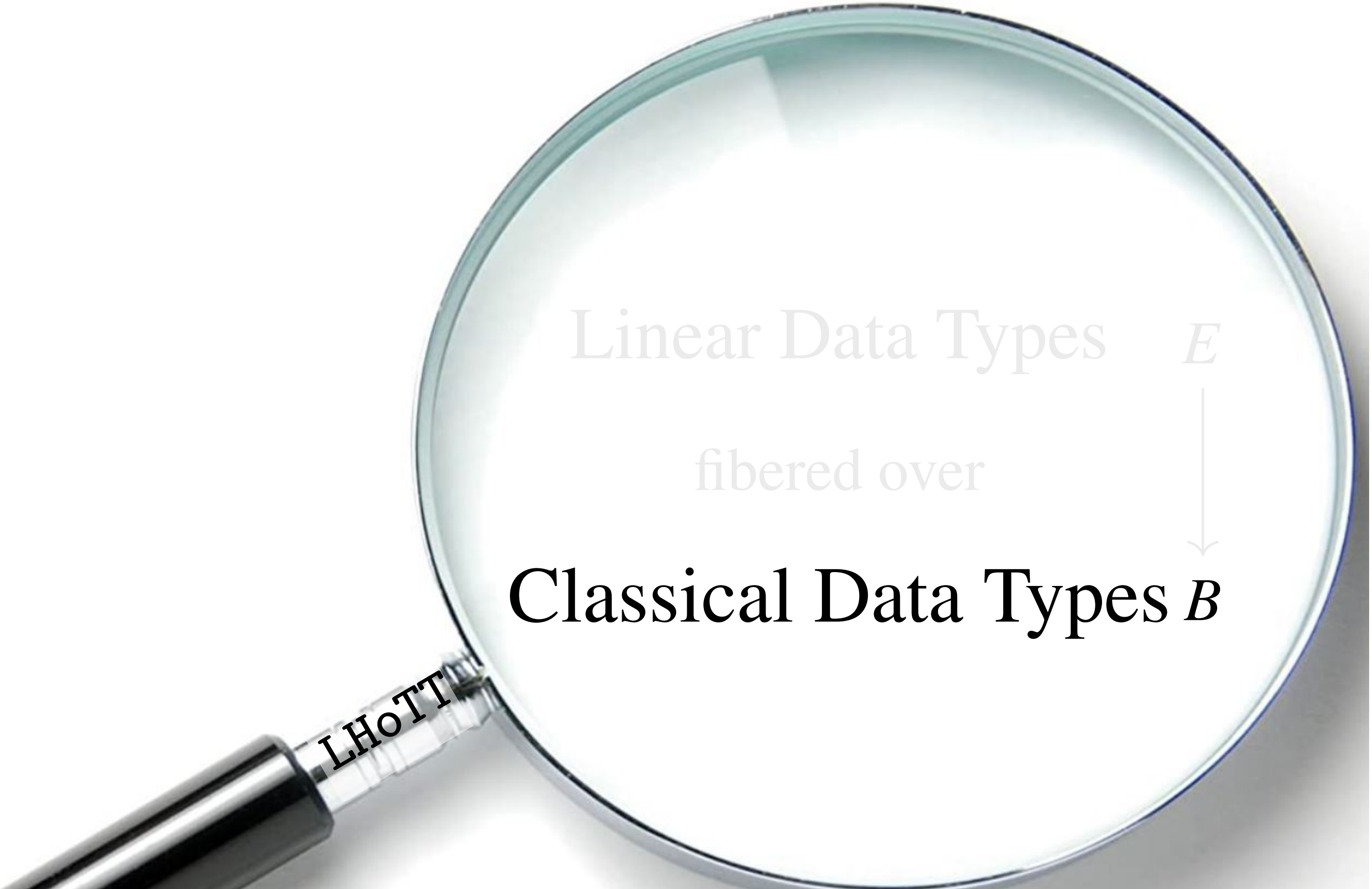

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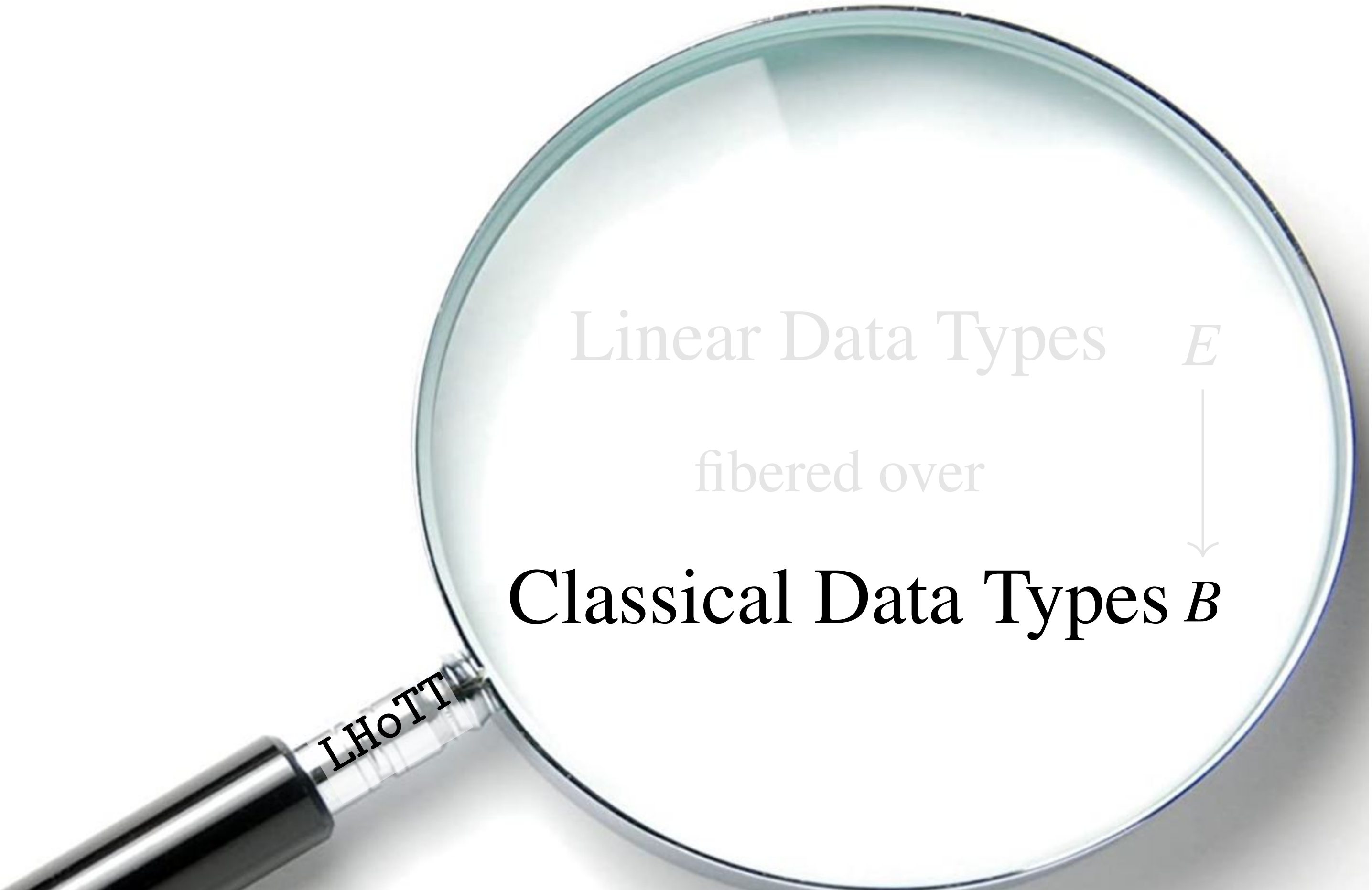

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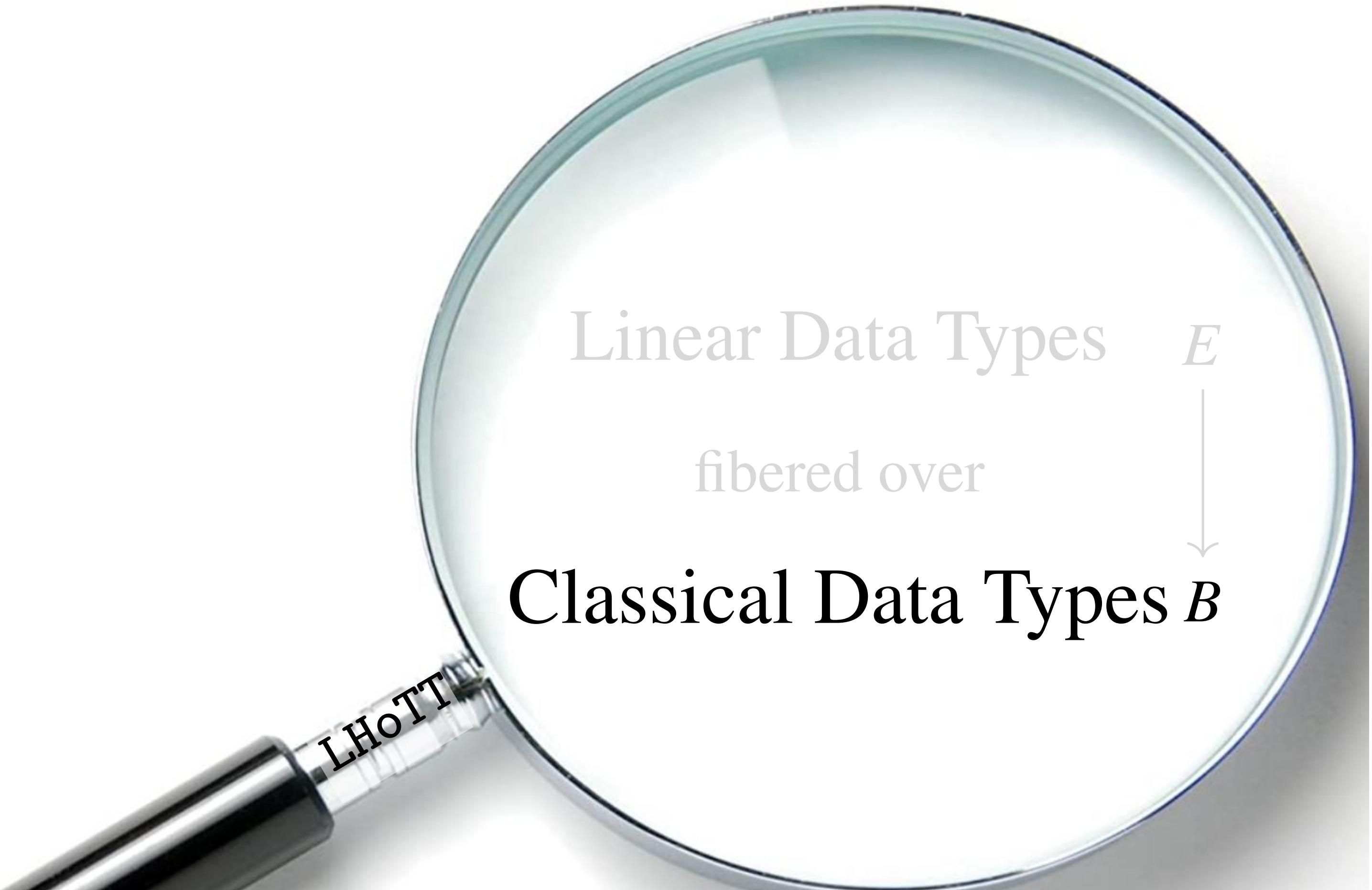

Linear Data Types E

fibered over

Classical Data Types B

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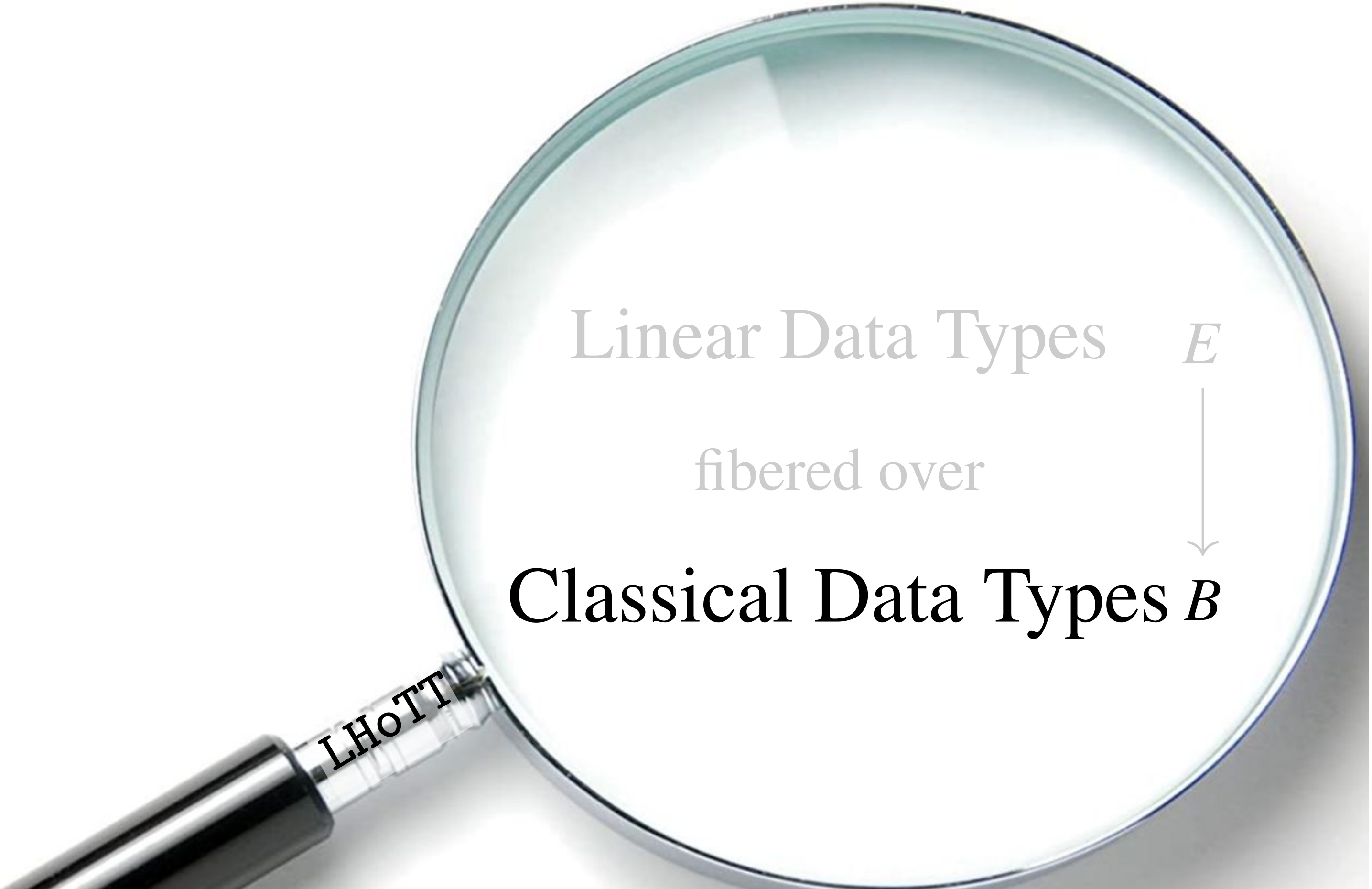

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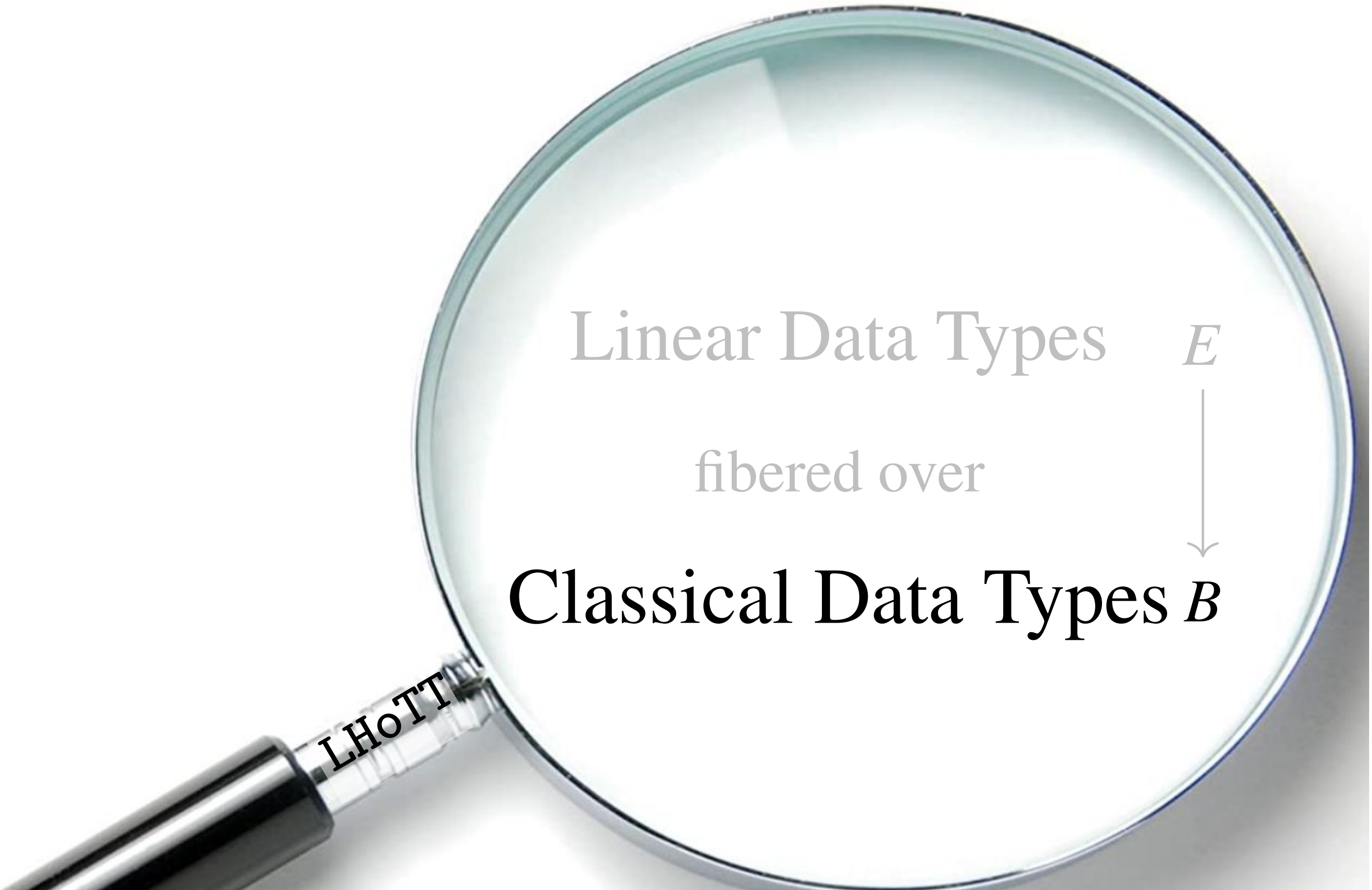

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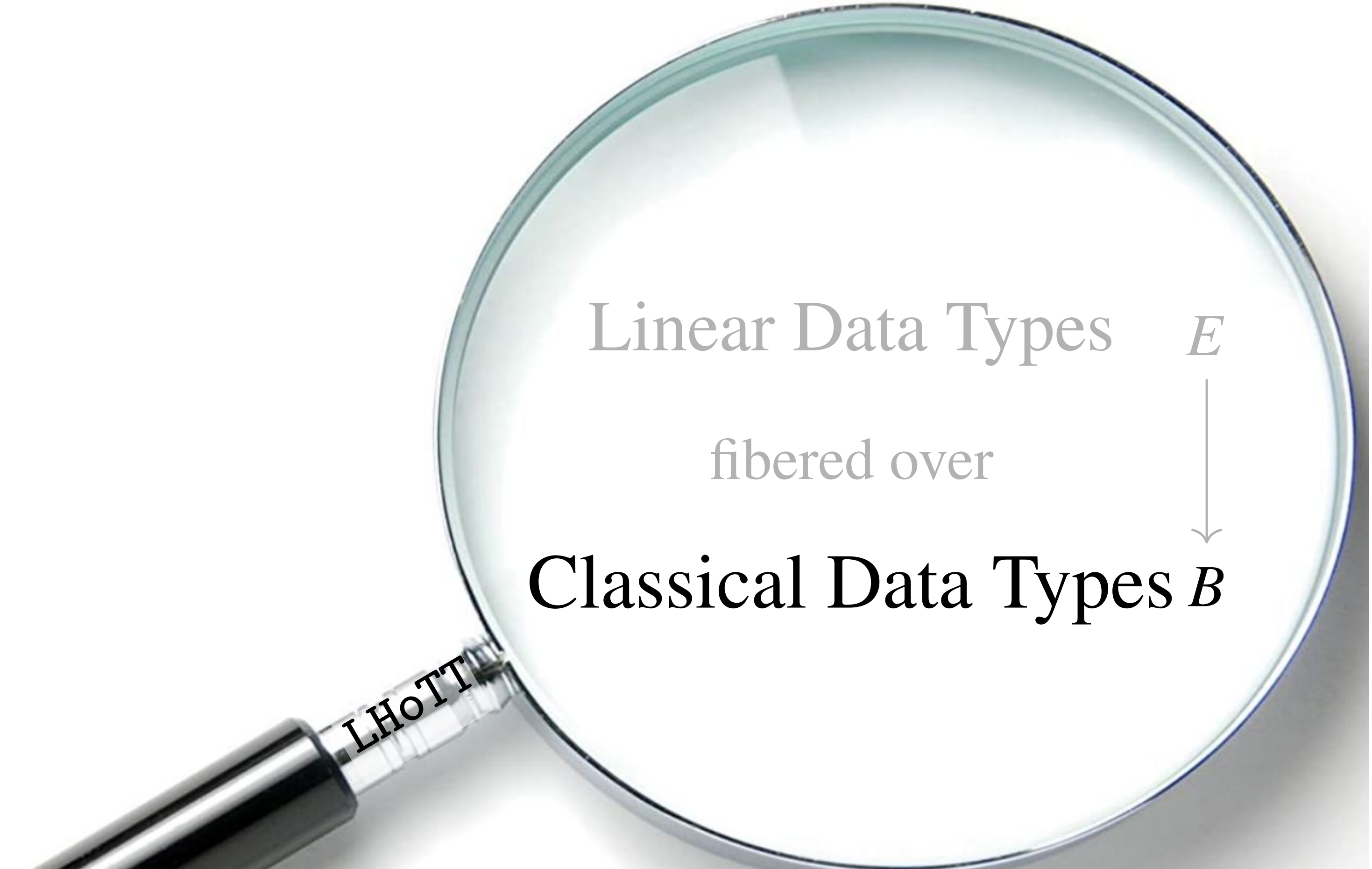

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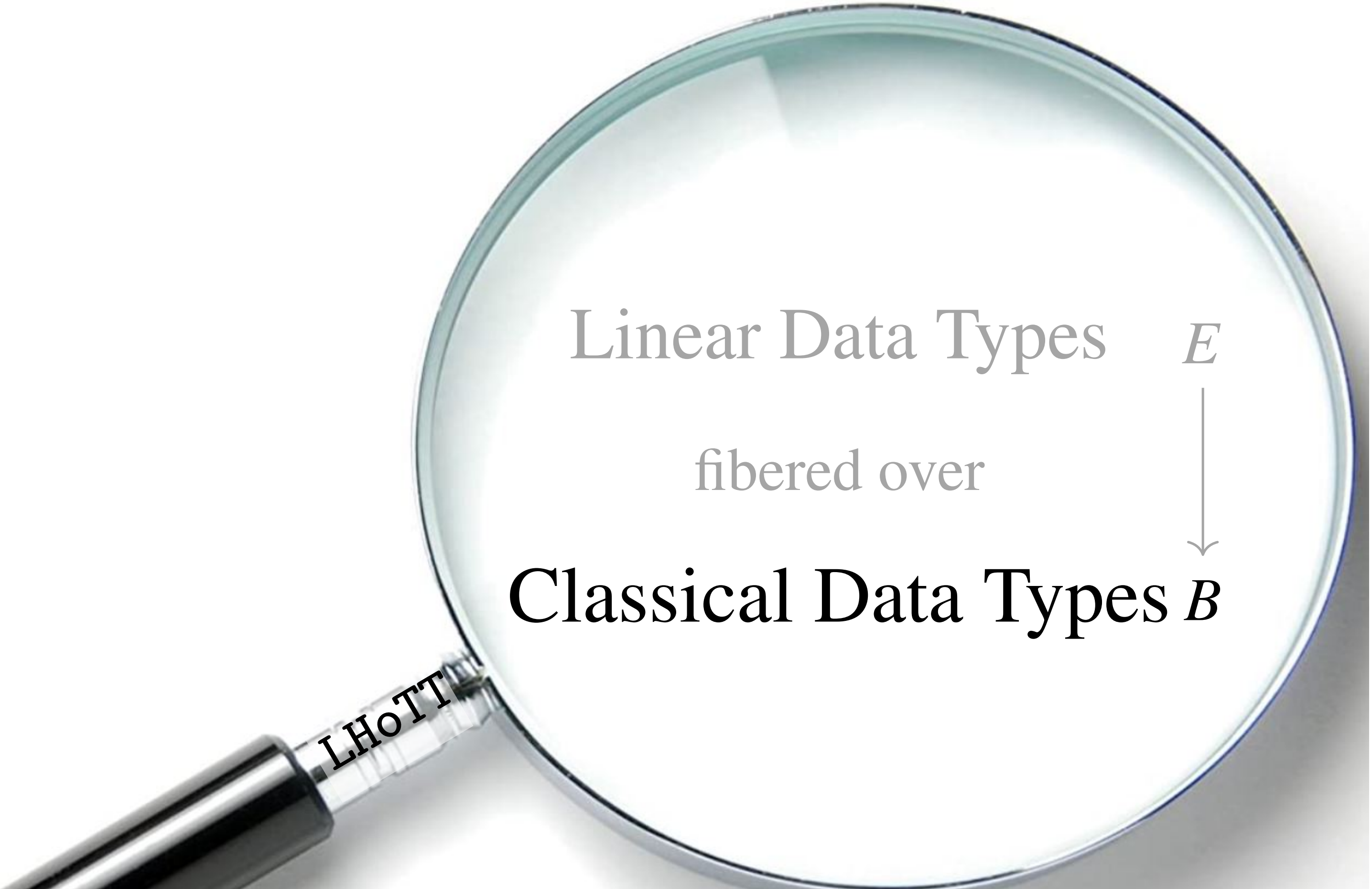

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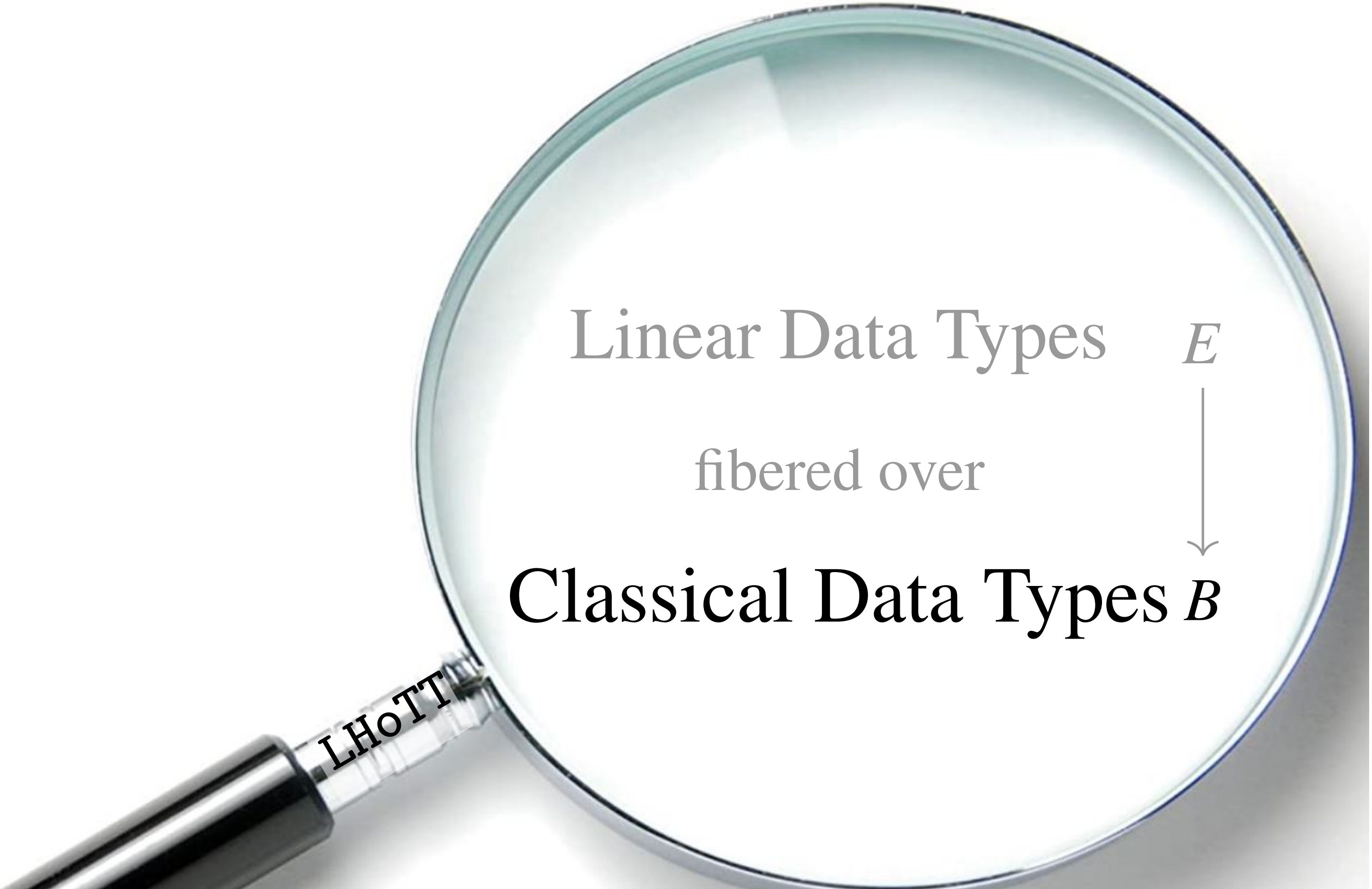

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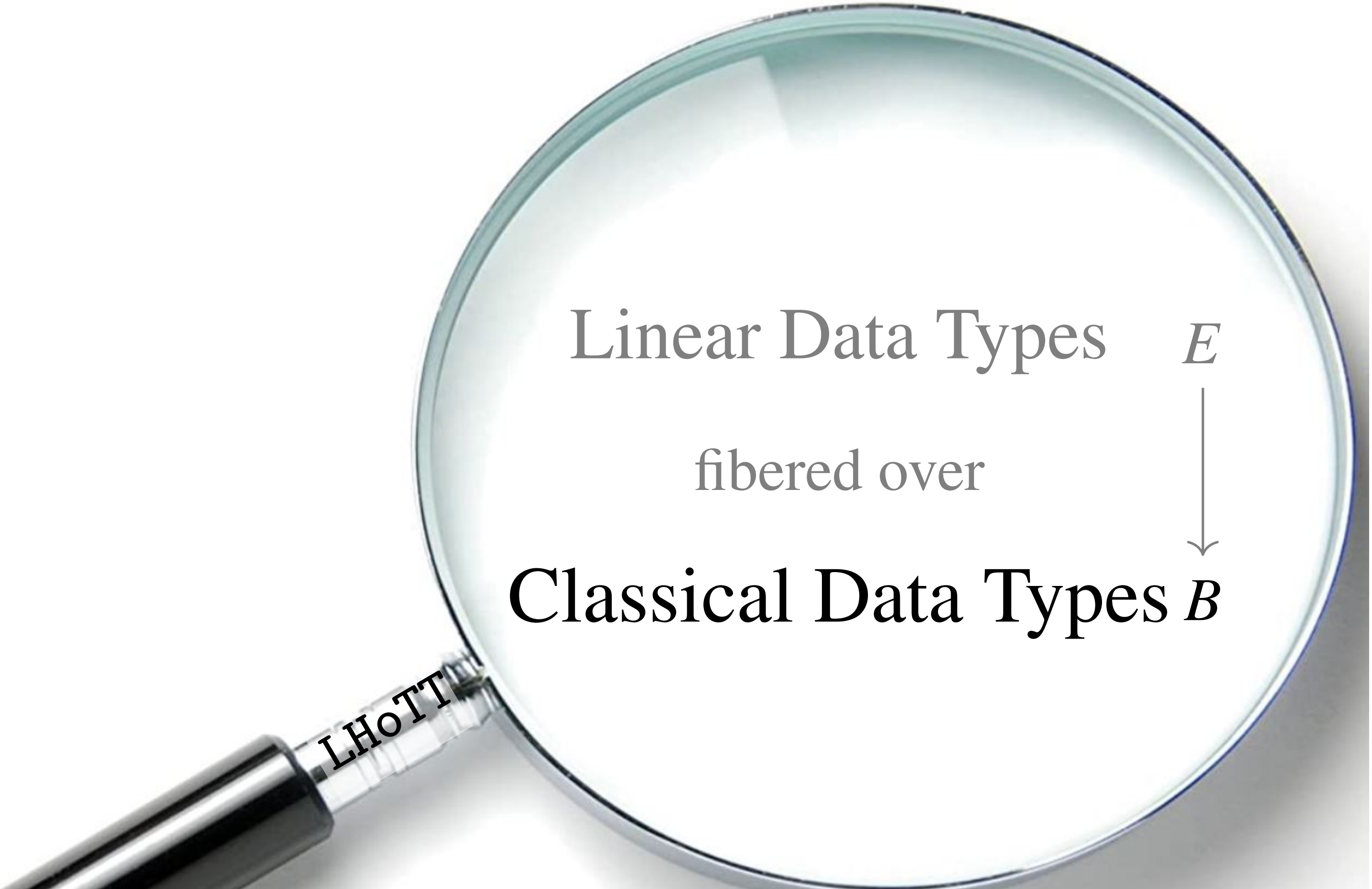

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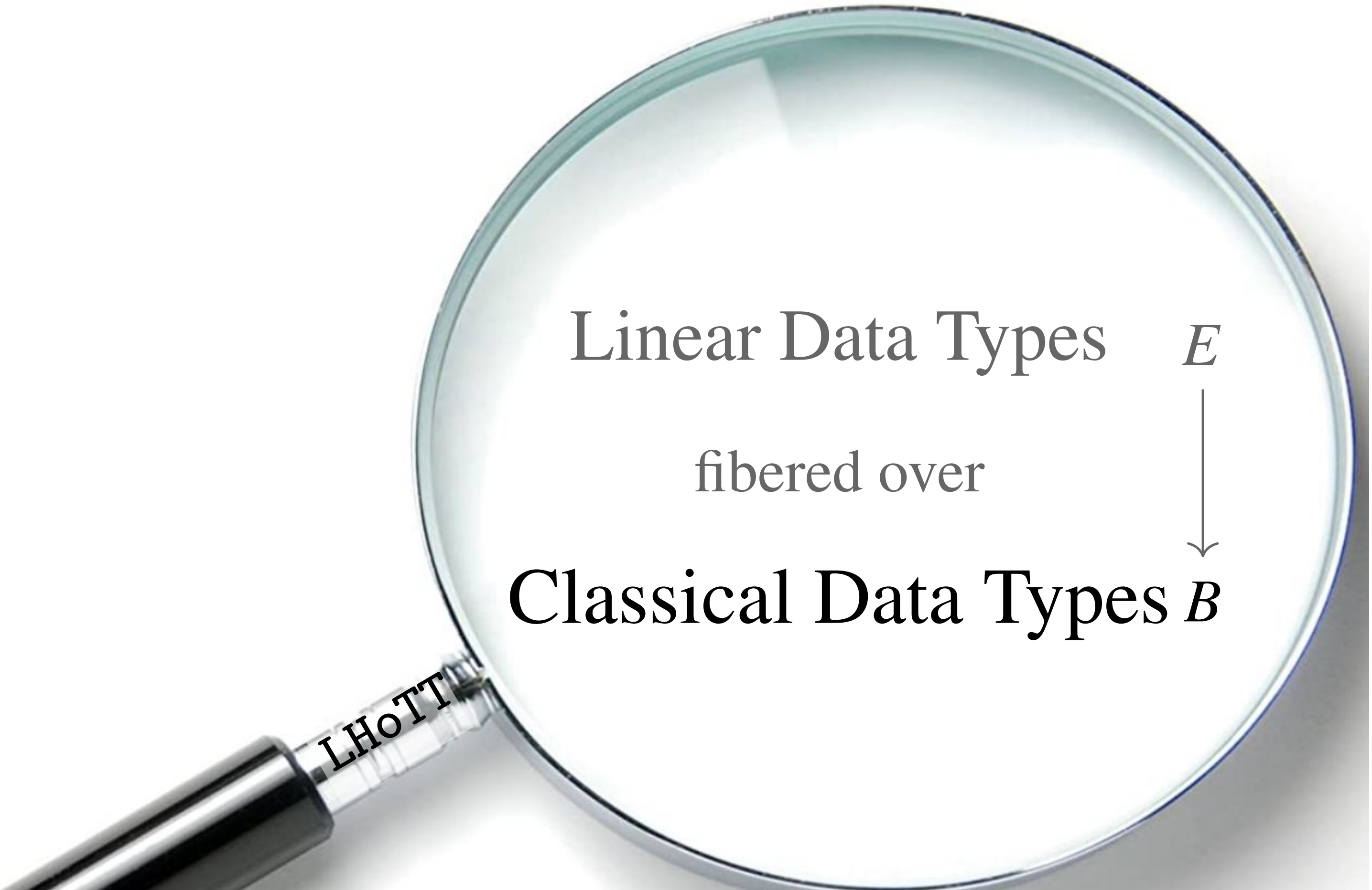

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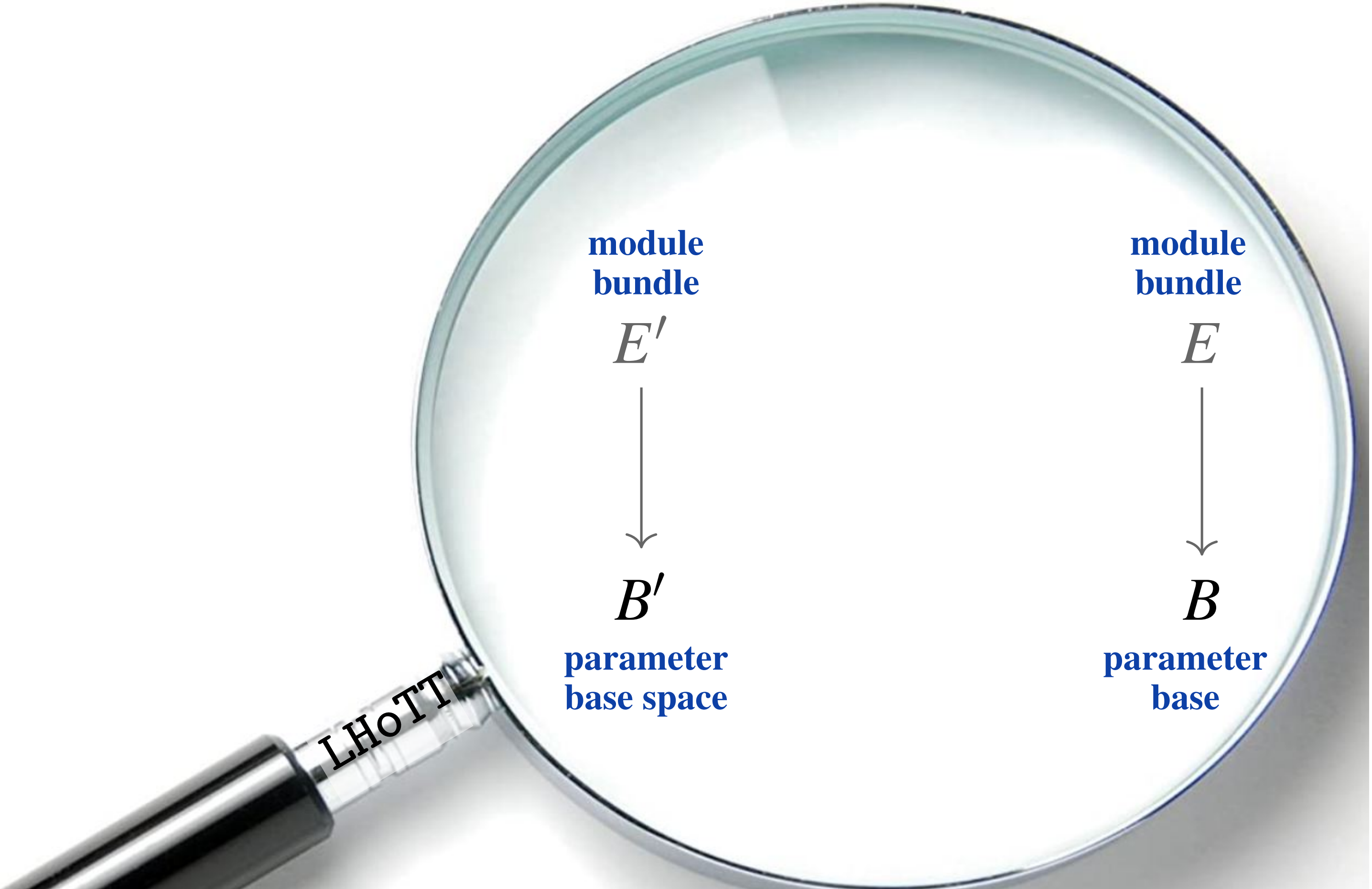

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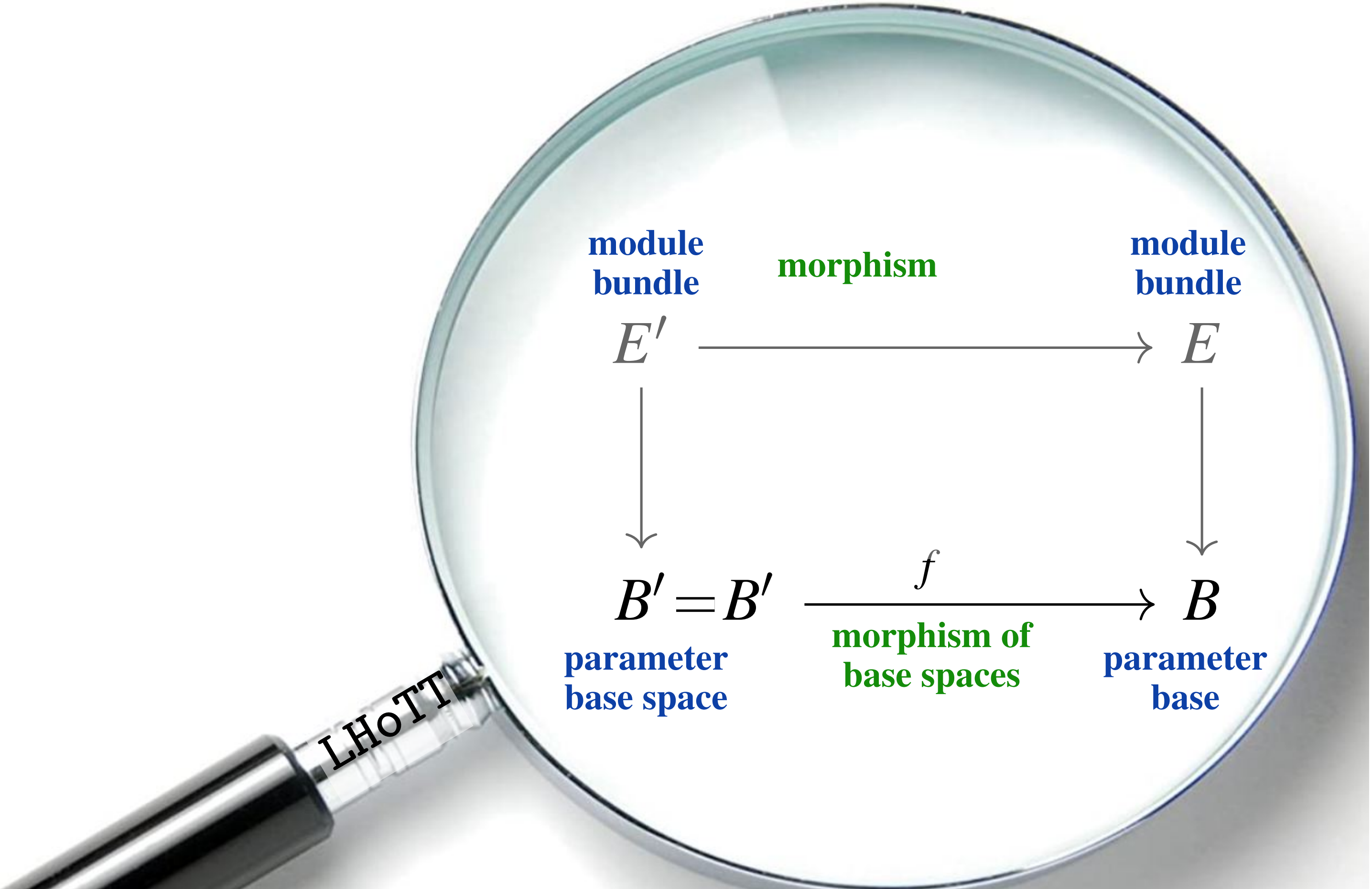

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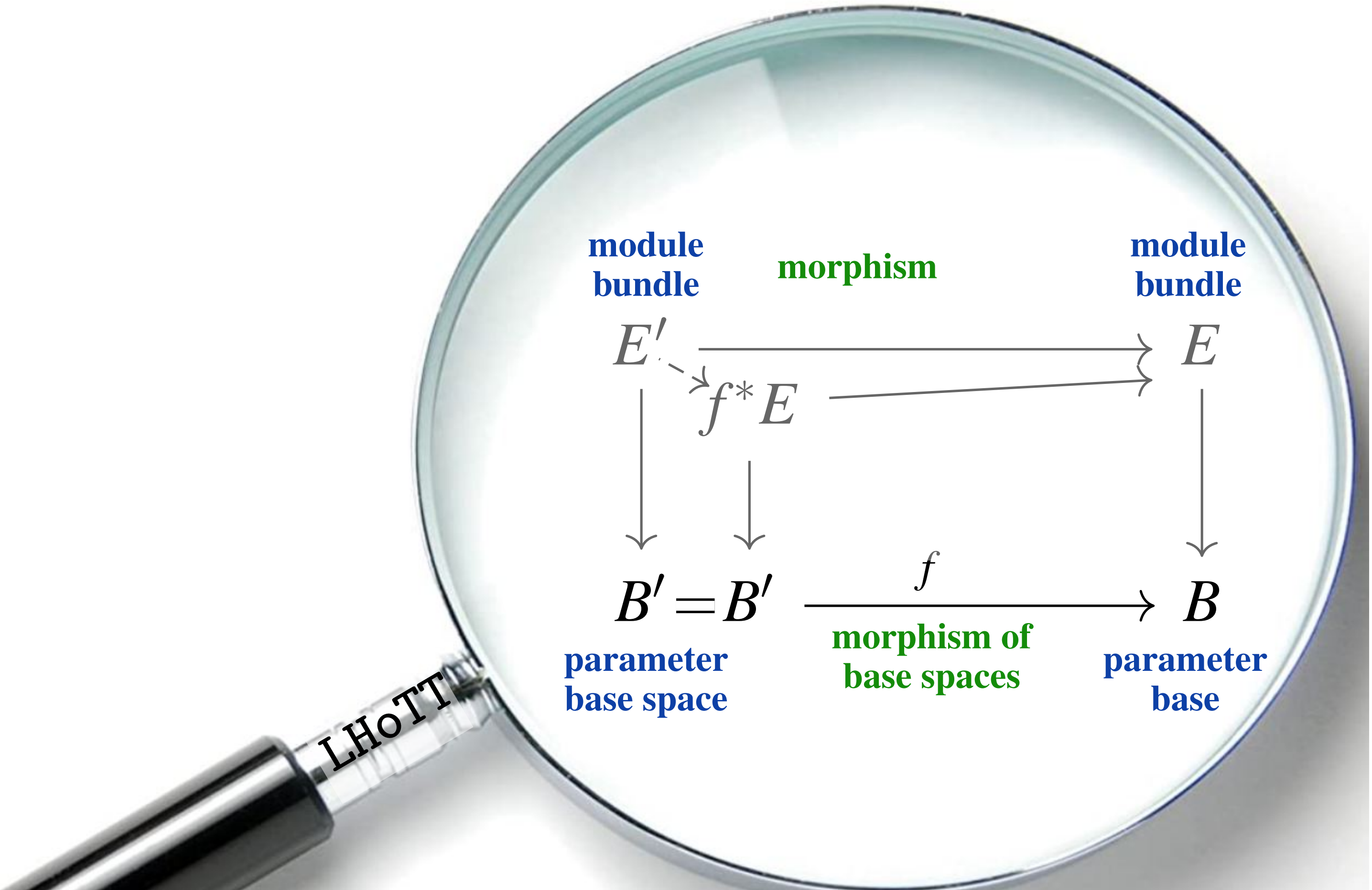

module bundle **morphism** **module bundle**

$$\begin{tikzcd}
E' \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & E \arrow[dd] \\
& f^*E \arrow[ur] \arrow[d] & \\
B' \arrow[r, equal] & B' \arrow[r, "f"] & B
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parameter base space **morphism of base spaces** **parameter base**

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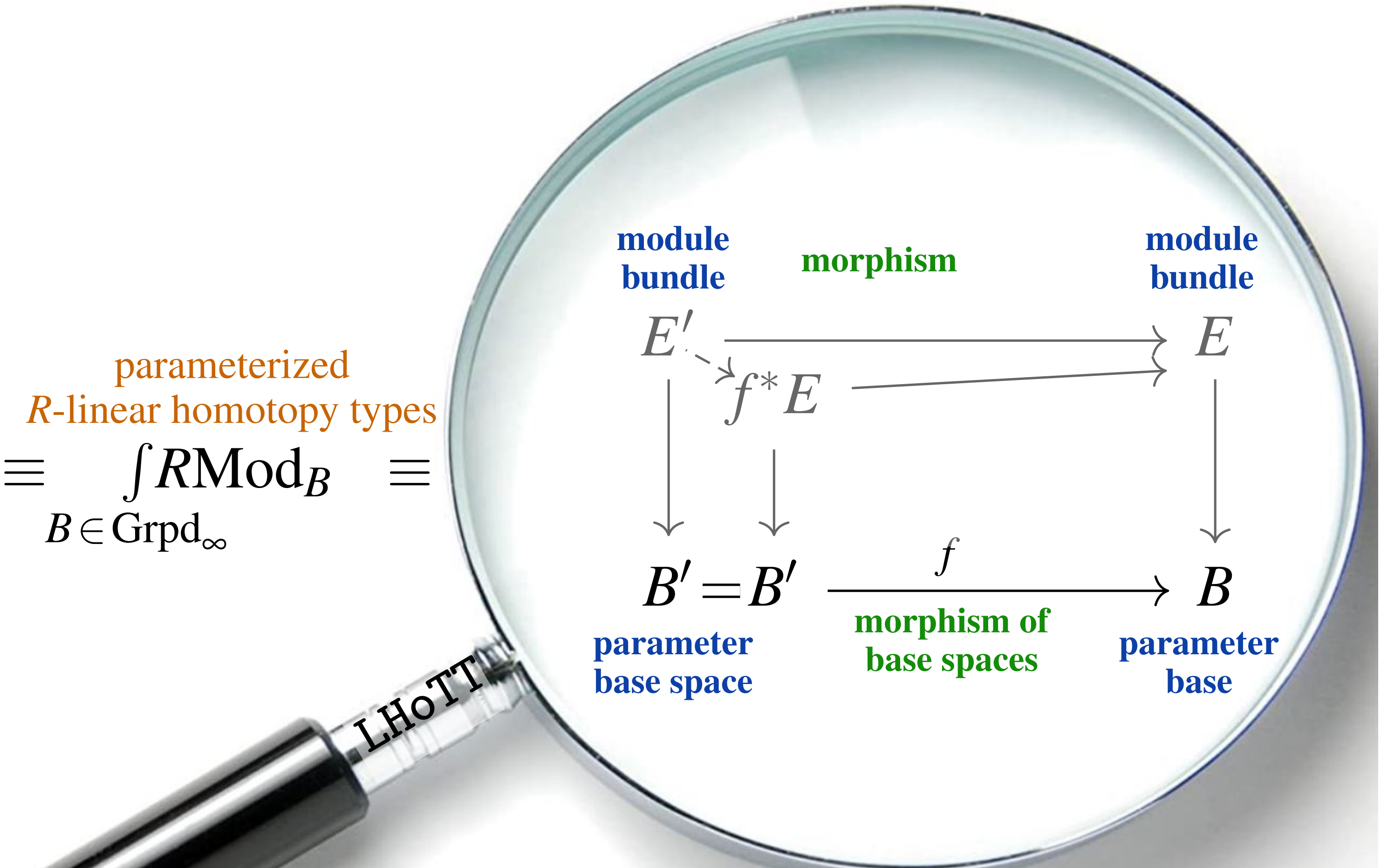
tangent ∞-topos of

parameterized
R-linear homotopy types

$$T\mathrm{Grpd}_\infty \;:\equiv\; \int_{B \in \mathrm{Grpd}_\infty} R\mathrm{Mod}_B \;\equiv$$



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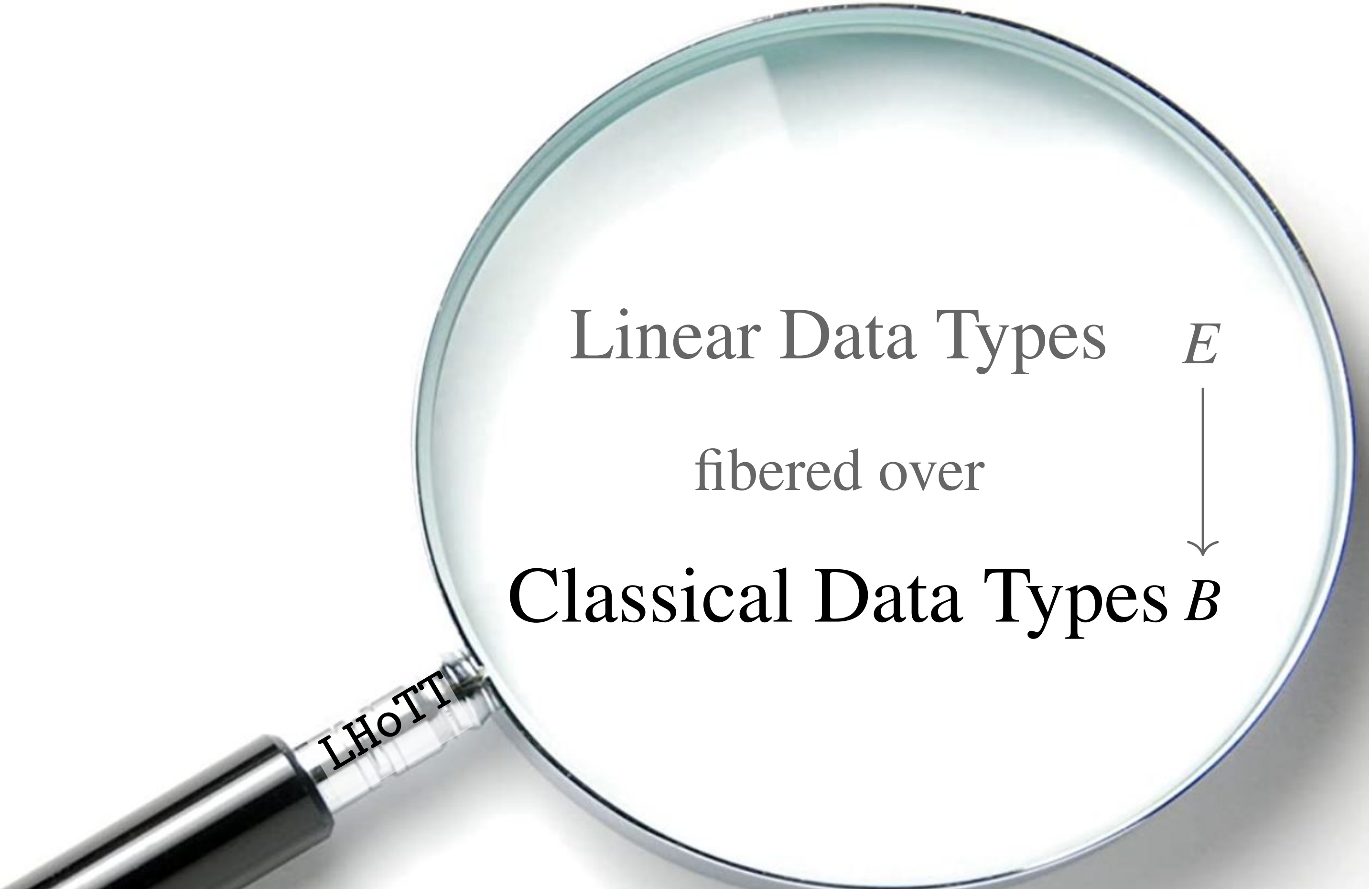

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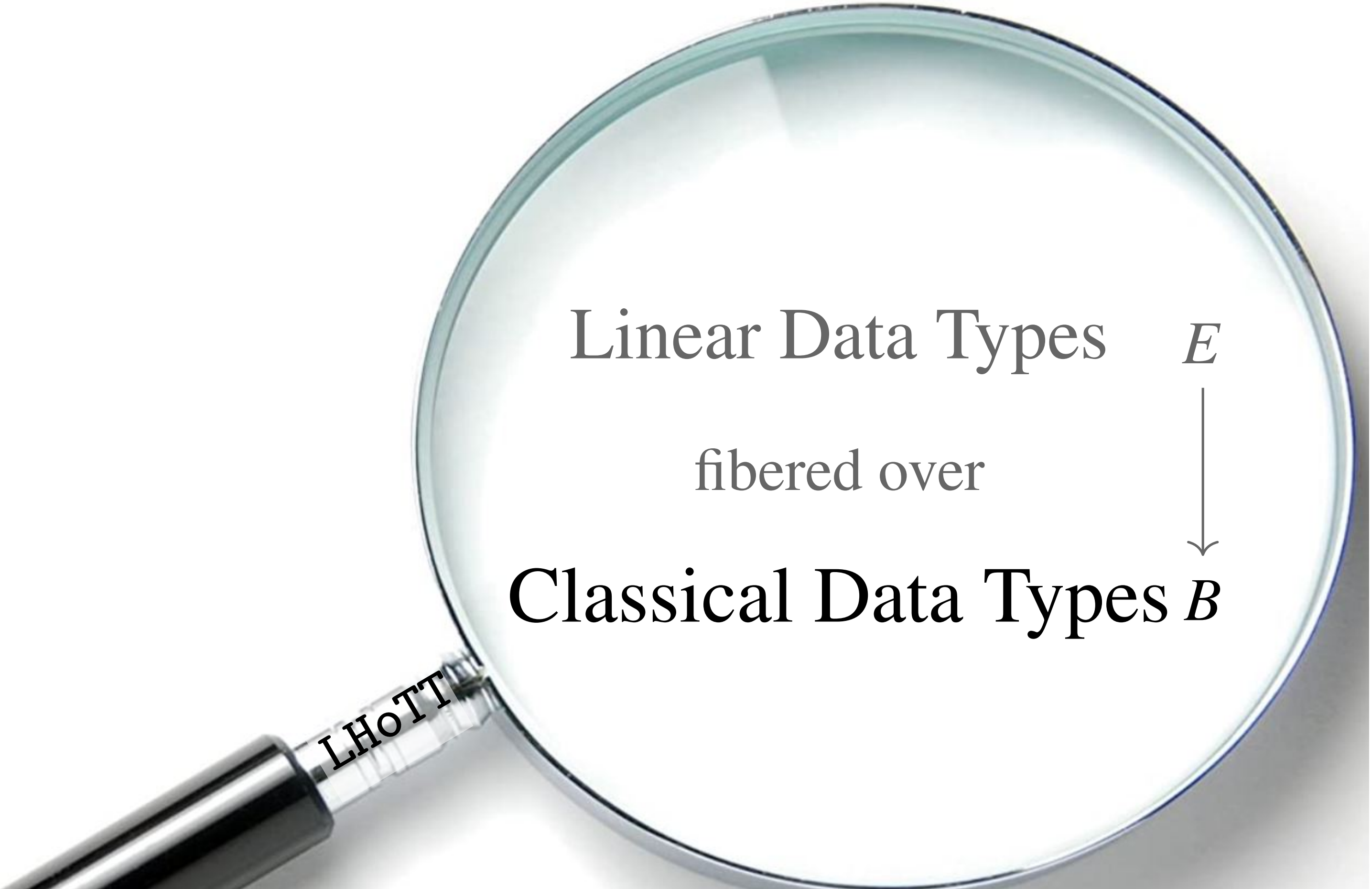

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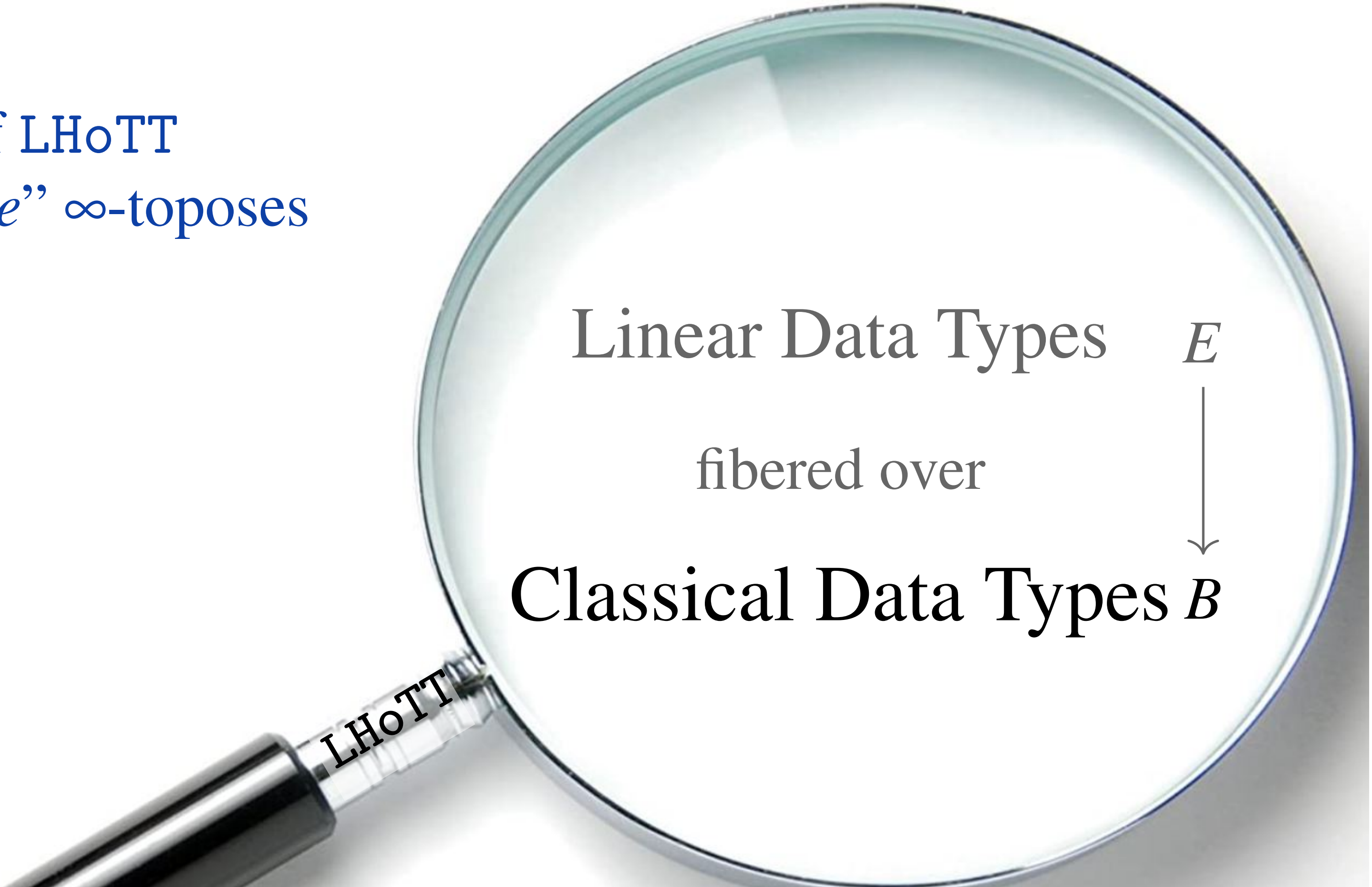

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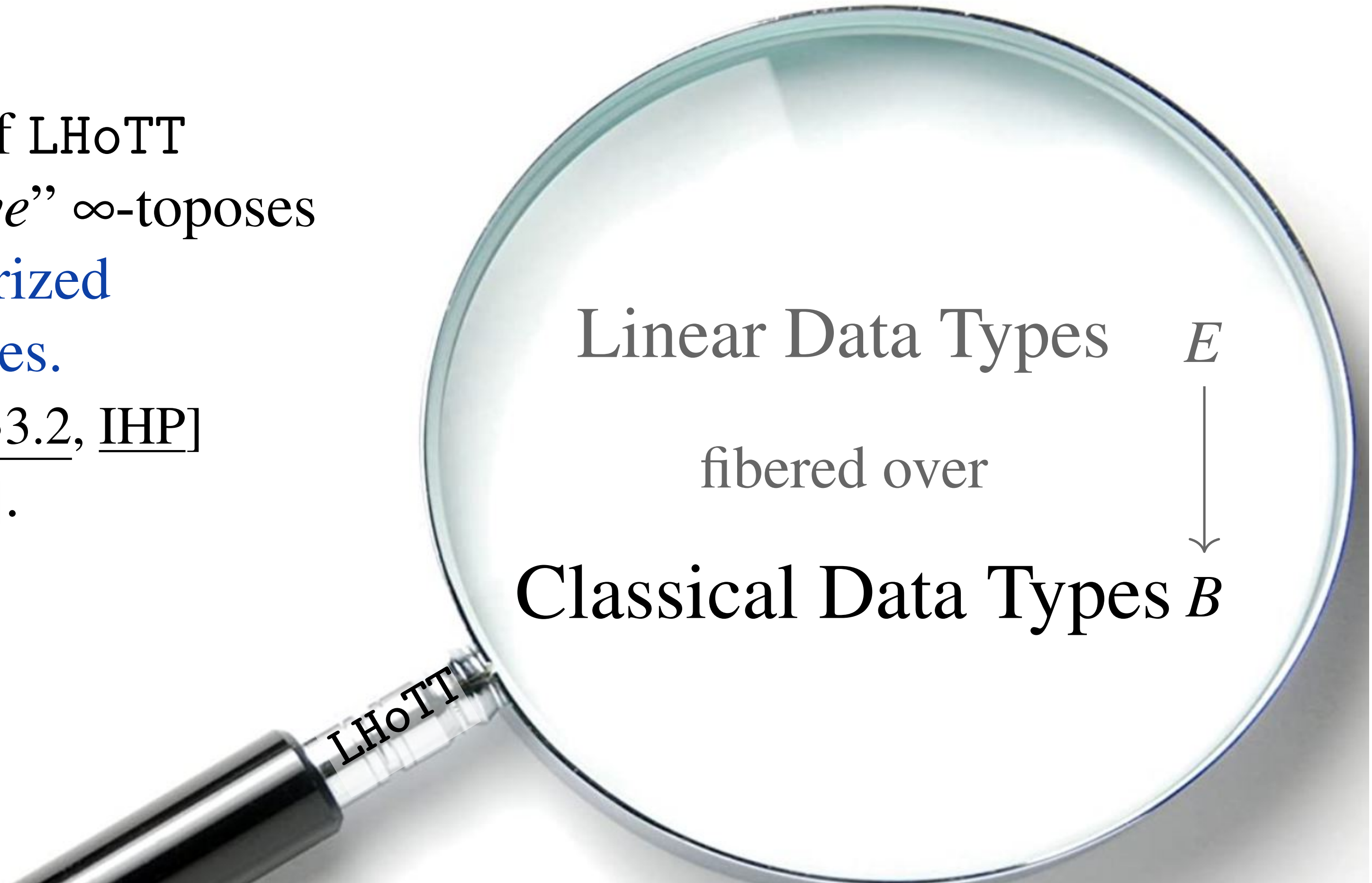

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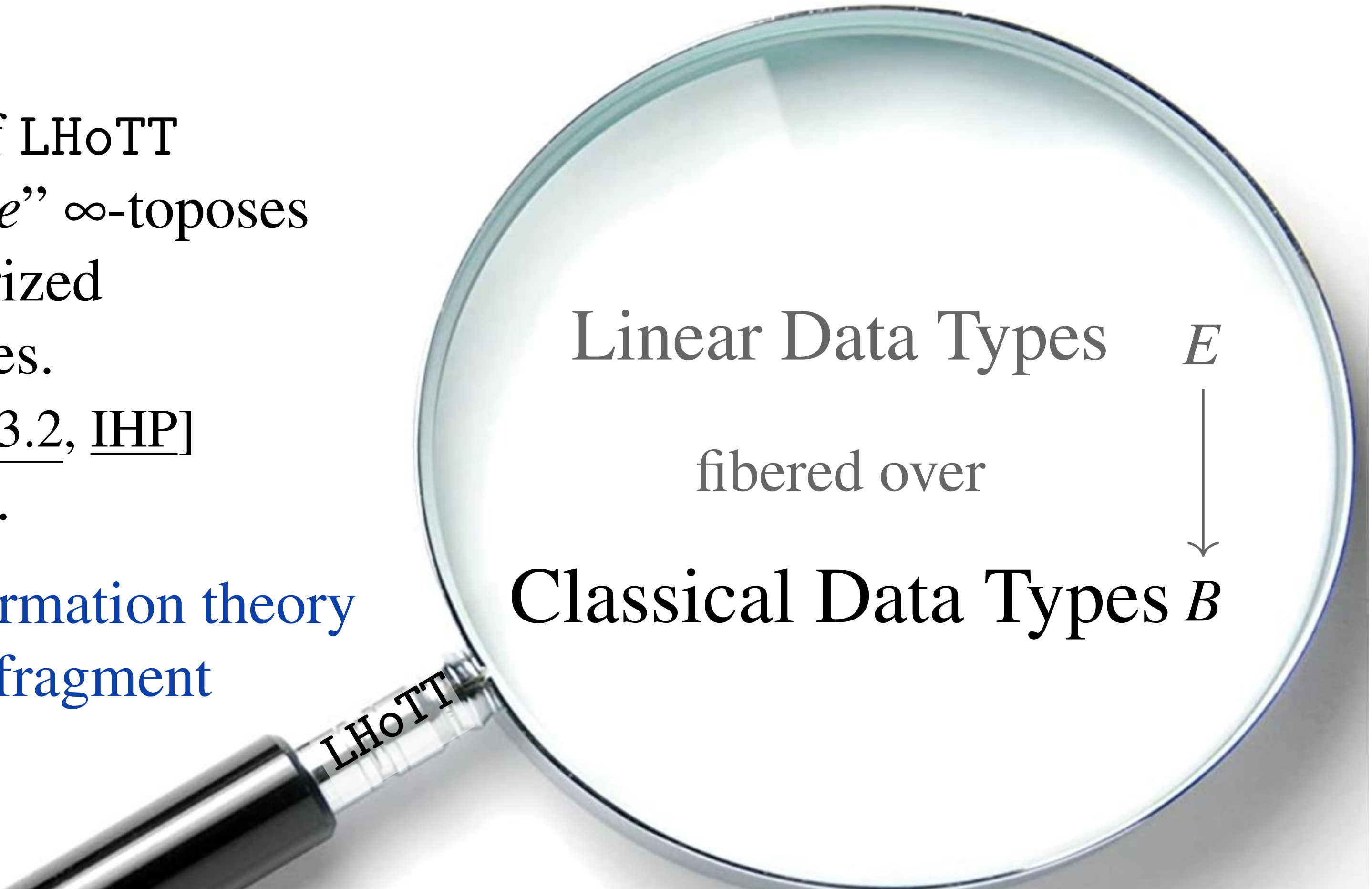

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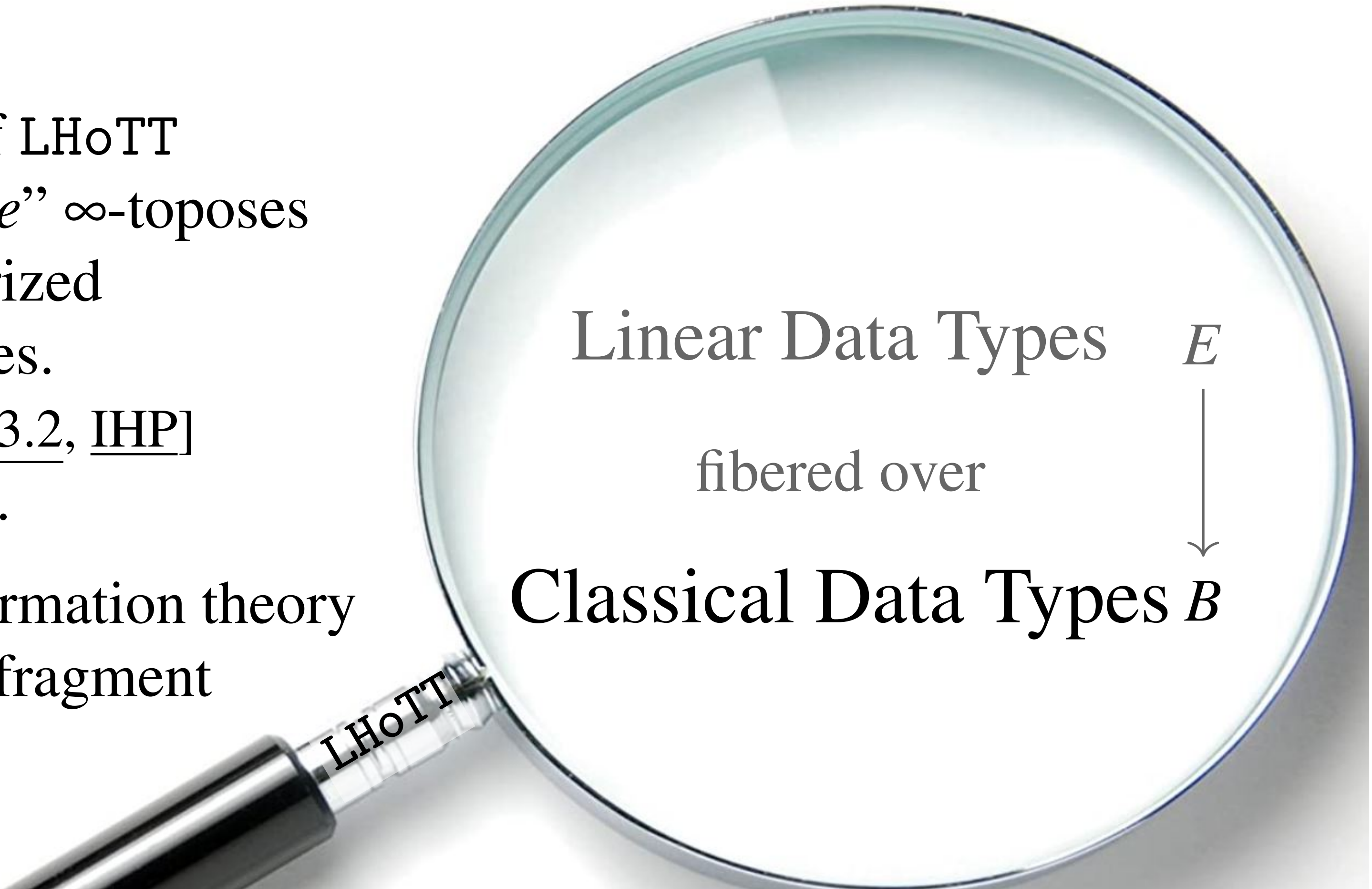

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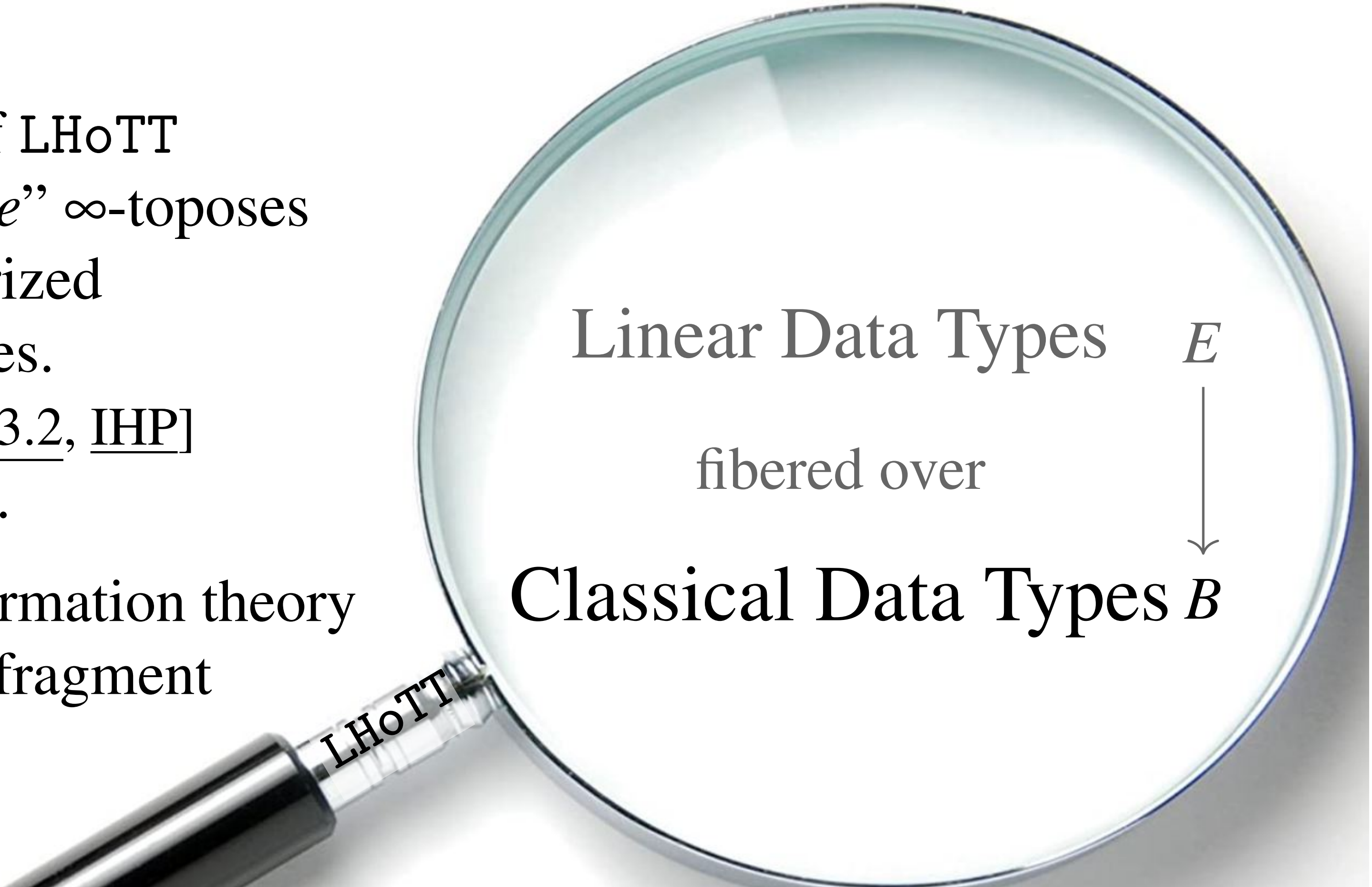

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$$\overset{\text{complex EM-spectrum}}{R :\equiv H\mathbb{C}} \quad \Rightarrow \quad \overset{\text{EM-module spectra}}{(H\mathbb{C})\mathrm{Mod}} \;\simeq\; \overset{\text{chain complexes}}{\mathrm{Ch}_\bullet(\mathbb{C})}$$

$$\begin{tikzcd}
(H\mathbb{C})\mathrm{Mod} \arrow[loop left] \arrow[r, shift right=1.5ex, "\underset{\text{connective cover}}{\tau_{\geq 0}}"'] \arrow[r, phantom, "\perp"] & \overset{\text{connective spectra}}{\underset{\text{t-structure}}{(H\mathbb{C})\mathrm{Mod}_{\geq 0}}} \arrow[l, hook', shift right=1.5ex]
\end{tikzcd}$$

$$\underset{\text{the heart}}{(H\mathbb{C})\mathrm{Mod}^{\heartsuit}} \;:\equiv\; (H\mathbb{C})\mathrm{Mod}_{\geq 0} \cap (H\mathbb{C})\mathrm{Mod}_{\leq 0} \;\simeq\; \underset{\text{complex vector spaces}}{\mathbb{C}\mathrm{Vect}}$$

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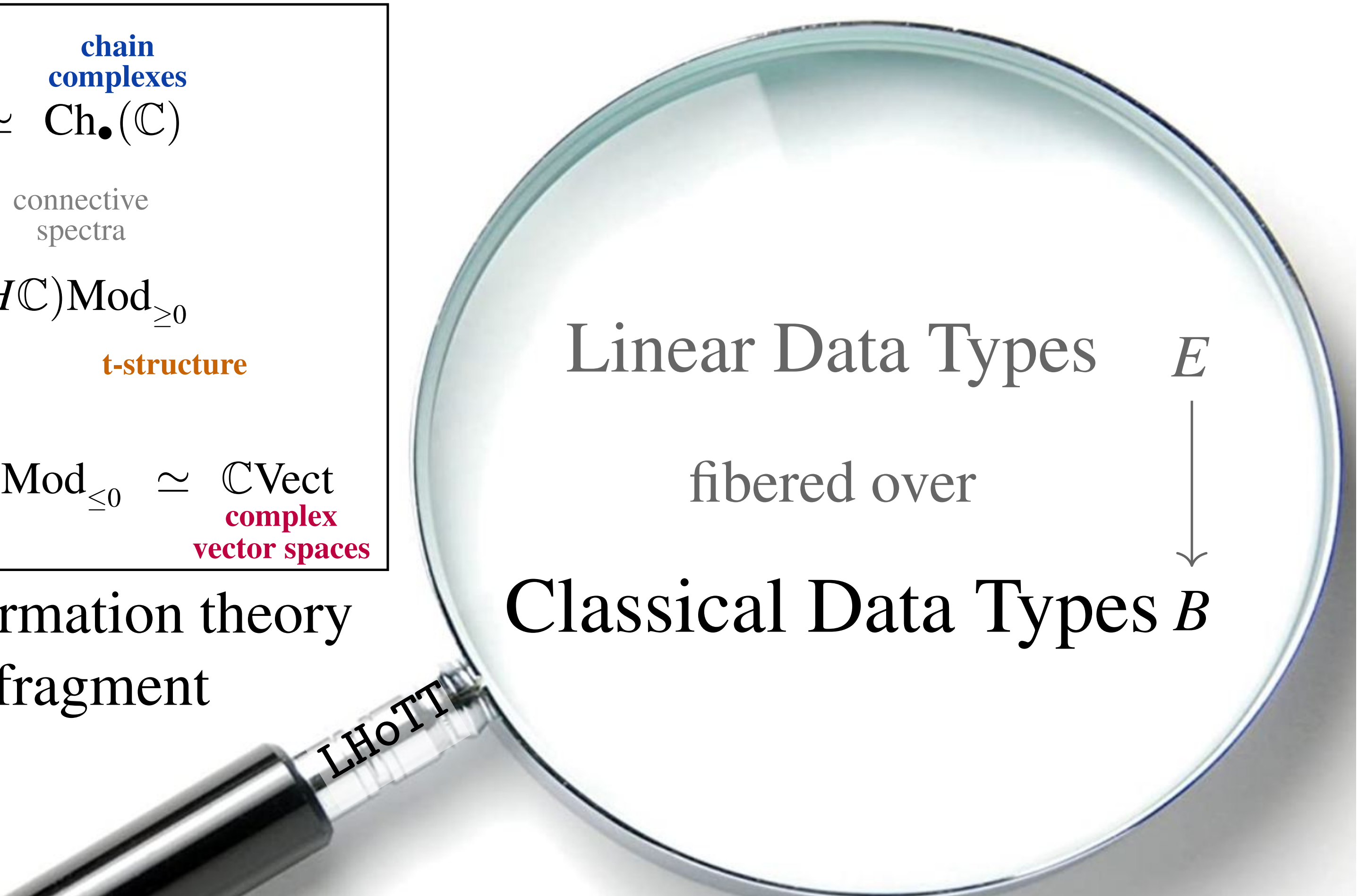

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$\rightsquigarrow$ full-blown Quantum Systems language emerges embedded in `LHoTT`

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Linear Homotopy Type Theory (`LHoTT`)
for universal algorithmic quantum computation

QS – Quantum Systems language @ CQTS

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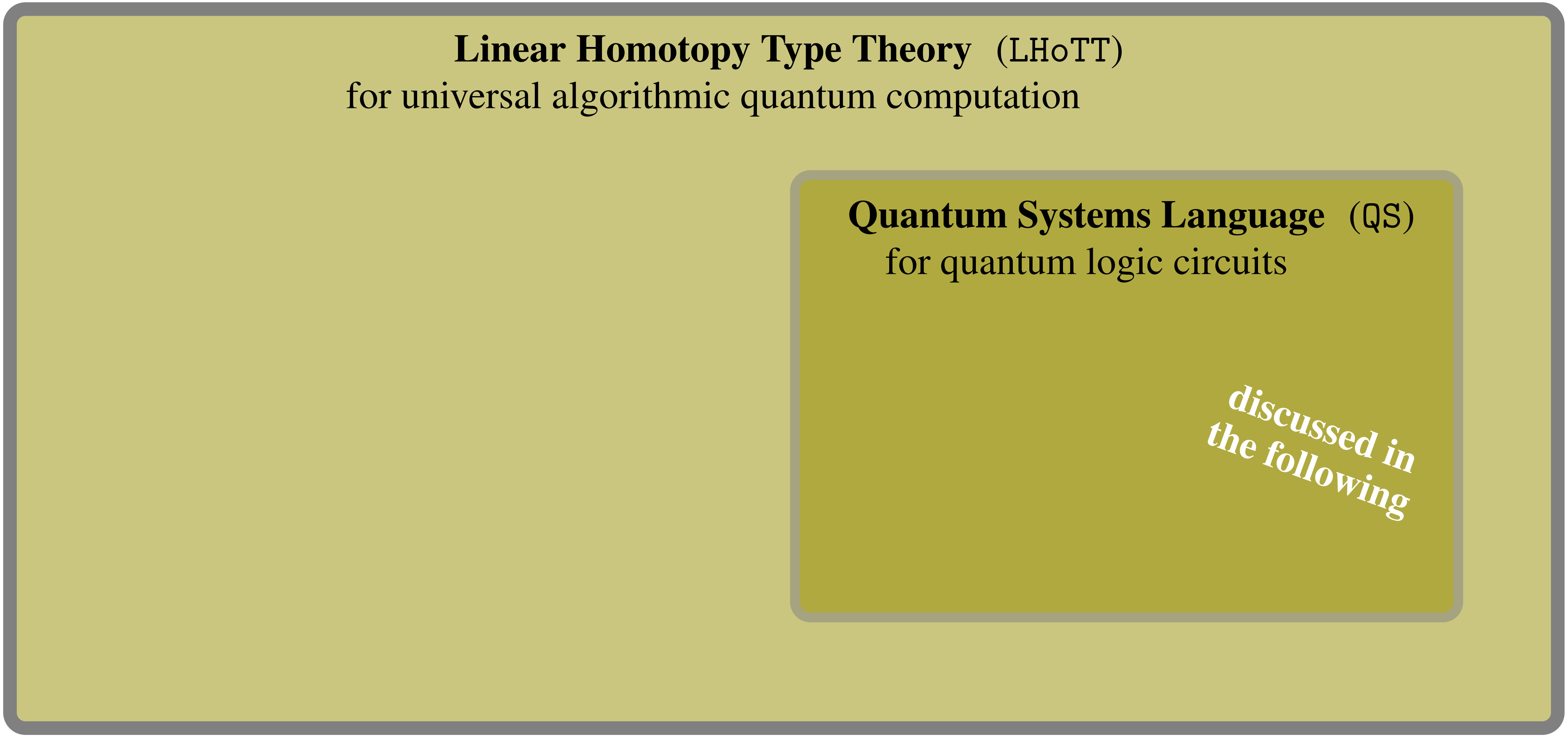

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Quantum Systems Language (QS)
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discussed in the following

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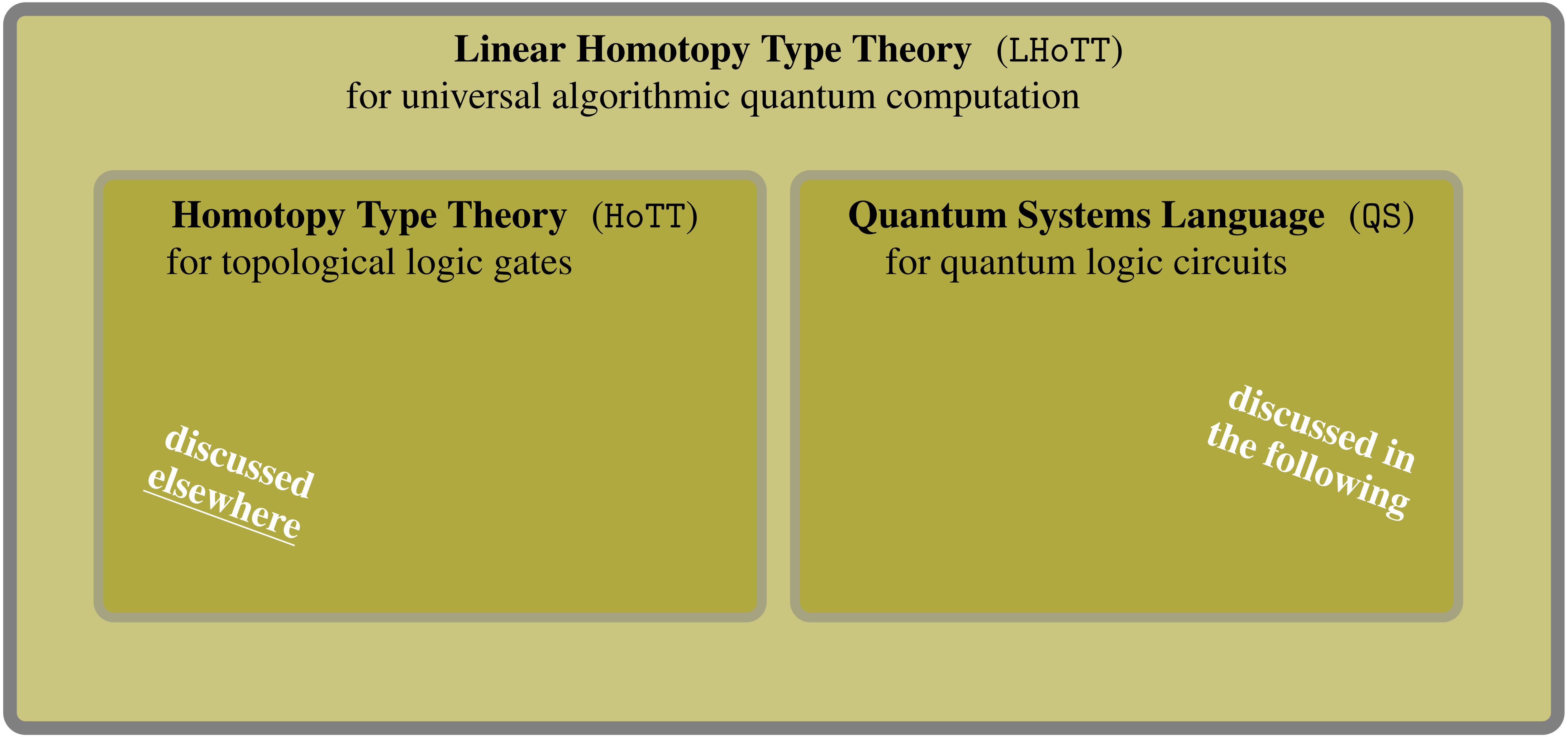

Linear Homotopy Type Theory (LHoTT)
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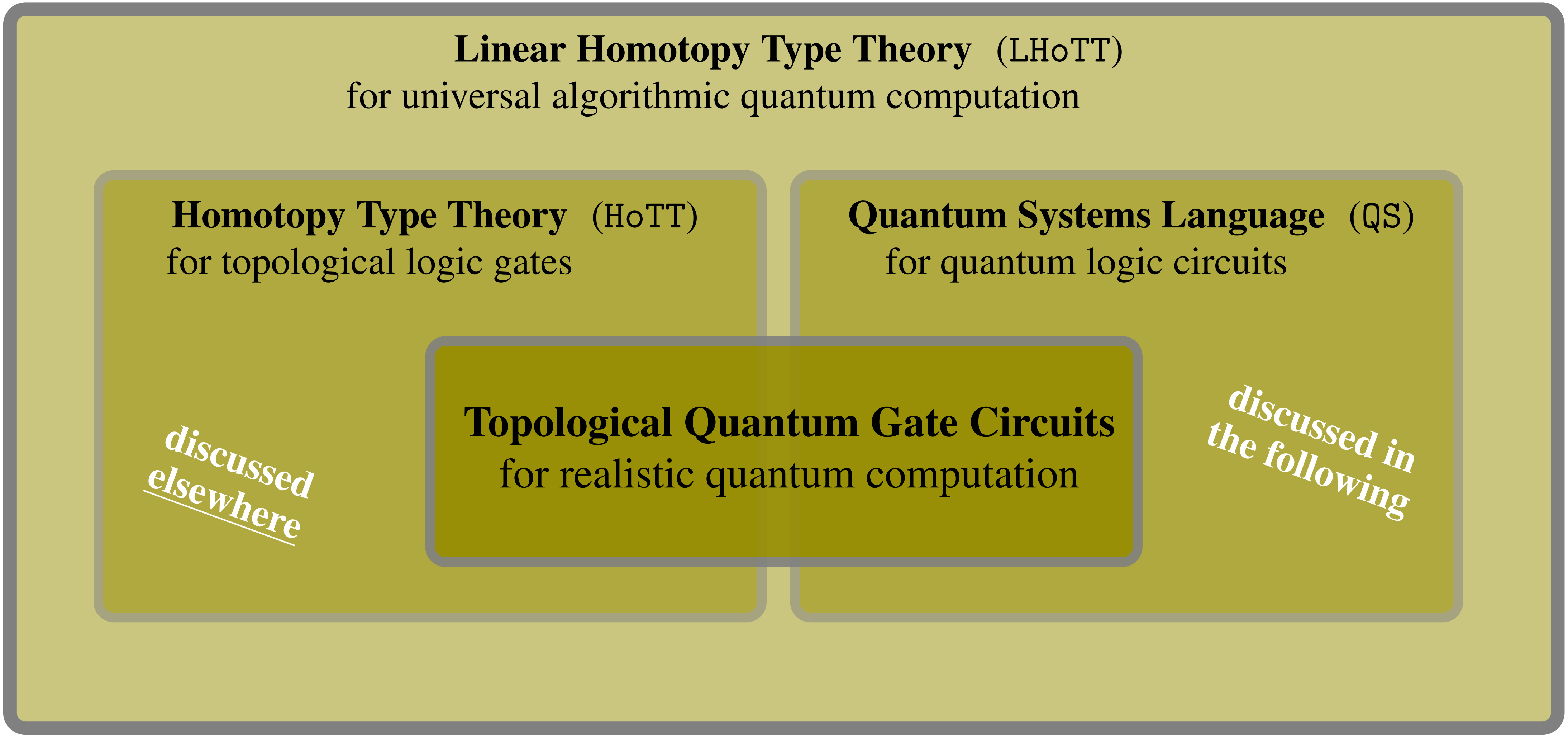

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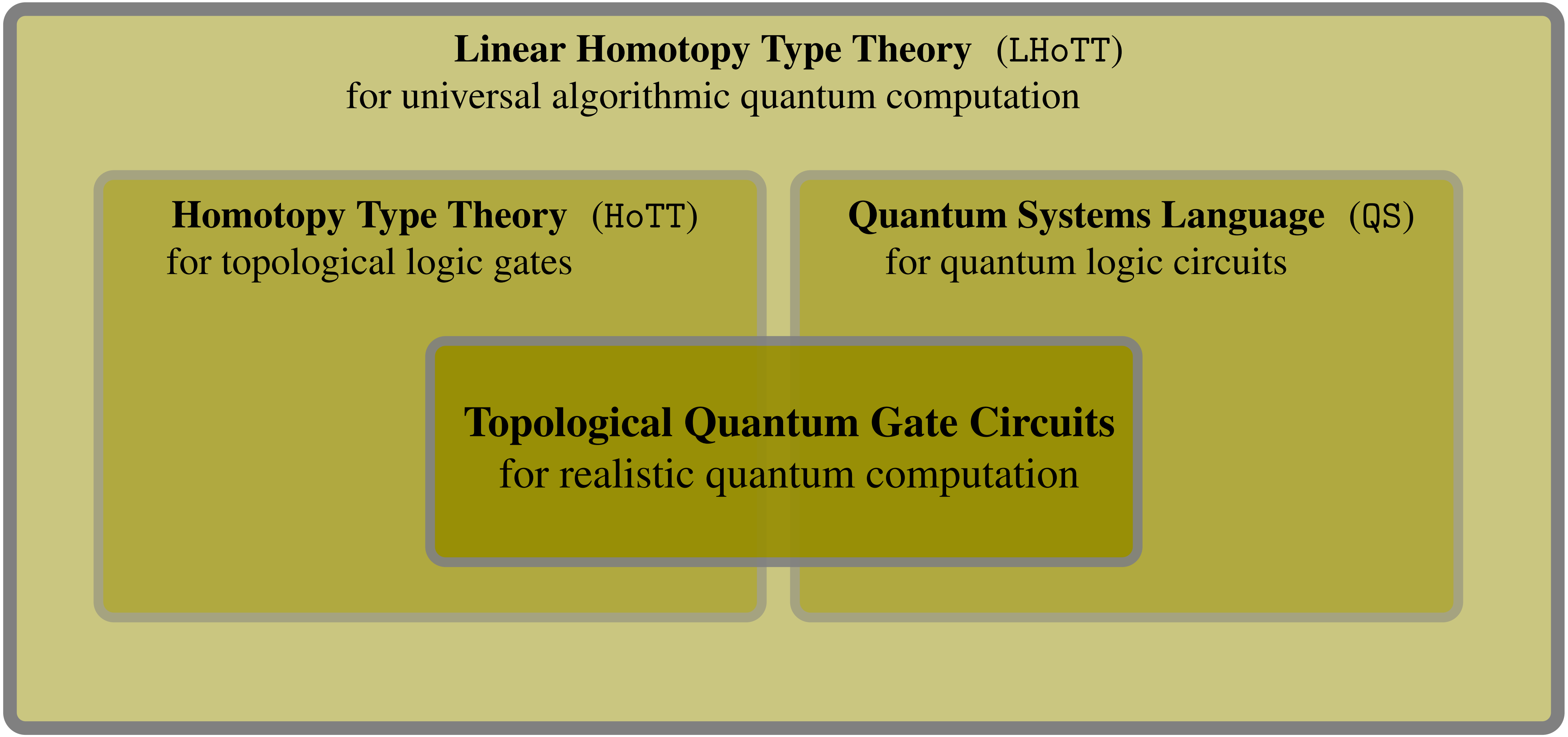

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Topological Quantum Gate Circuits
for realistic quantum computation

ambient LHoTT	verifies	classically dependent quantum linear types
ambient HoTT	provides	specification of topological quantum gates
ambient dTT	provides	full verified classical control

Quantum Data Types

Linear/Quantum Data Types

<table>
<tr><td>Characteristic Property</td><td></td><td></td><td></td></tr>
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<tr><td>Linear Logic</td><td></td><td></td><td></td></tr>
<tr><td>Physics Meaning</td><td></td><td></td><td></td></tr>
</table>

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:		
Symbol			
Formula (for B : FinType)			
AlgTop Jargon			
Linear Logic			
Physics Meaning			

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:		
Symbol	$\oplus$ direct sum		
Formula (for B : FinType)			
AlgTop Jargon			
Linear Logic			
Physics Meaning			

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	
Symbol	$\oplus$ direct sum		
Formula (for B : FinType)			
AlgTop Jargon			
Linear Logic			
Physics Meaning			

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	
Symbol	$\bigoplus$ direct sum	$\bigotimes$ tensor product	
Formula (for B : FinType)			
AlgTop Jargon			
Linear Logic			
Physics Meaning			

Linear/Quantum Data Types

<table>
<tr><th>Characteristic Property</th><th>1. their cartesian product blends into the co-product:</th><th>2. a tensor product appears & distributes over direct sum</th><th>3. a linear function type appears adjoint to tensor</th></tr>
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Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
Symbol	$\oplus$ direct sum	$\otimes$ tensor product	$\multimap$ linear function type
Formula (for B : FinType)			
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Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
Symbol	$\oplus$ direct sum	$\otimes$ tensor product	$\multimap$ linear function type
Formula (for B : FinType)	$\overset{\text{cart. product}}{\prod_B \mathcal{H}_b} \simeq \underset{\text{direct sum}}{\bigoplus_B \mathcal{H}_b} \simeq \overset{\text{co-product}}{\coprod_B \mathcal{H}_b}$		
AlgTop Jargon			
Linear Logic			
Physics Meaning			

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
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AlgTop Jargon			
Linear Logic			
Physics Meaning			

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
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AlgTop Jargon			
Linear Logic			
Physics Meaning			

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
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AlgTop Jargon	biproduct, stability, ambidexterity	Frobenius reciprocity	mapping spectrum
Linear Logic			
Physics Meaning			

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
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Linear Logic			
Physics Meaning			

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
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Linear Logic	additive disjunction	multiplicative conjunction	linear implication
Physics Meaning			

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
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Physics Meaning	superselection sectors / quantum parallelism	compound quantum systems / quantum entanglement	QRAM systems

Linear/Quantum Data Types

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Dependent linear Type Formers

finite classical context
(variables, parameters, ...)

reference context

$$\begin{tikzcd}
B \arrow[rr, "p_B"] & & \ast \\
\mathrm{ClType}_B & & \mathrm{ClType}
\end{tikzcd}$$

classical type system
dependent on context

classical
type system

Linear/Quantum Data Types

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Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd} B \arrow[r, "p_B"] & * \end{tikzcd}$$

classical context extension

classical type system dependent on context

classical type system

$$\begin{tikzcd} \mathrm{ClType}_B & \mathrm{ClType} \arrow[l, "*_B \times"'] \end{tikzcd}$$

Linear/Quantum Data Types

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Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd} B \arrow[rr, "p_B"] & & * \end{tikzcd}$$

co-product

$$\begin{tikzcd} \mathrm{ClType}_B \arrow[rr, bend left, "\coprod_{b:B}"] \arrow[rr, phantom, "\perp"] & & \mathrm{ClType} \arrow[ll, bend left, "*_B \times"] \end{tikzcd}$$

classical type system dependent on context

classical type system

Linear/Quantum Data Types

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Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd} B \arrow[r, "p_B"] & * \end{tikzcd}$$

co-product

classical type system dependent on context

$$\begin{tikzcd}[column sep=huge]
\mathrm{ClType}_B \arrow[r, shift left=5, "\coprod_{b:B}"] \arrow[r, shift right=5, "\prod_{b:B}"'] \arrow[r, phantom, shift left=2.5, "\perp"] \arrow[r, phantom, shift right=2.5, "\perp"] & \mathrm{ClType} \arrow[l, "*_B \times" description]
\end{tikzcd}$$

classical type system

product

Linear/Quantum Data Types

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Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd} B \arrow[rr, "p_B"] & & * \end{tikzcd}$$

co-product

$$\begin{tikzcd} \mathrm{ClType}_B \arrow[rr, shift left=8, "\coprod_{b:B}"] \arrow[rr, phantom, shift left=4, "\perp"] & & \mathrm{ClType} \arrow[ll, "*_B \times" description] \arrow[ll, phantom, shift left=4, "\perp"] \\ \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{ClType}_B \arrow[rr, "\prod_{b:B}"'] & & \mathrm{ClType} \end{tikzcd}$$

product

classical type system dependent on context

classical type system

classical base change / classical quantification

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
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Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd} B \arrow[rr, "p_B"] & & \ast \end{tikzcd}$$

classical type system dependent on context

co-product

$$\begin{tikzcd}[column sep=huge] \mathrm{ClType}_B \arrow[r, shift left=14pt, "\coprod_{b:B}"] \arrow[r, phantom, shift left=7pt, "\perp"] \arrow[r, phantom, shift right=7pt, "\perp"] \arrow[r, shift right=14pt, "\prod_{b:B}"'] & \mathrm{ClType} \arrow[l, "\ast_B \times"'] \end{tikzcd}$$

product

classical type system

classical base change /
classical quantification

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
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Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd}[column sep=huge] B \arrow[r, "p_B"] & * \end{tikzcd}$$

classical type system dependent on context

co-product

$$\begin{tikzcd}[column sep=huge, row sep=large] \mathrm{ClType}_B \arrow[r, shift left=5, "\coprod_{b:B}"] \arrow[r, phantom, shift left=2.5, "\perp"] \arrow[r, phantom, shift right=2.5, "\perp"] \arrow[r, shift right=5, "\prod_{b:B}"'] & \mathrm{ClType} \arrow[l, "*_B \times" description] \end{tikzcd}$$

product

classical type system

linear type system in classical context

$$\Big(\mathrm{LType}_B,\ \overset{\text{tensor}}{\otimes_B}\Big) \qquad\qquad \Big(\mathrm{LType},\ \overset{\text{tensor}}{\otimes}\Big)$$

linear type system

classical base change / classical quantification

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
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Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd}
B \arrow[r, "p_B"] & *
\end{tikzcd}$$

classical type system dependent on context

classical type system

$$\begin{tikzcd}[column sep=huge]
\mathrm{ClType}_B \arrow[r, shift left=10, "\overset{\text{co-product}}{\coprod_{b:B}}"] \arrow[r, phantom, shift left=5, "\perp"] \arrow[r, phantom, shift right=5, "\perp"] \arrow[r, shift right=10, "\underset{\text{product}}{\prod_{b:B}}"'] & \mathrm{ClType} \arrow[l, "{*_B \times}"']
\end{tikzcd}$$

classical base change / classical quantification

linear type system in classical context

linear type system

$$\begin{tikzcd}[column sep=huge]
\Big(\mathrm{LType}_B, \overset{\text{tensor}}{\otimes_B}\Big) & \Big(\mathrm{LType}, \overset{\text{tensor}}{\otimes}\Big) \arrow[l, "\overset{\textbf{linear context extension}}{\mathbb{1}_B \otimes}"']
\end{tikzcd}$$

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
Symbol	$\bigoplus$ direct sum	$\otimes$ tensor product	$\multimap$ linear function type
Formula (for B : FinType)	**cart. product** **co-product** $\prod_B \mathcal{H}_b \simeq \bigoplus_B \mathcal{H}_b \simeq \coprod_B \mathcal{H}_b$ **direct sum**	$\mathcal{V} \otimes \Big(\bigoplus\limits_{b:B} \mathcal{H}_b \Big) \simeq \bigoplus\limits_{b:B} \big(\mathcal{V} \otimes \mathcal{H}_b \big)$	$(\mathcal{V} \otimes \mathcal{H}) \multimap \mathcal{K}$ $\simeq \;\; \mathcal{V} \multimap (\mathcal{H} \multimap \mathcal{K})$

Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd}
B \arrow[r, "p_B"] & *
\end{tikzcd}$$

classical type system dependent on context

$$\begin{tikzcd}[column sep=huge]
\mathrm{ClType}_B \arrow[r, shift left=4, "\overset{\text{co-product}}{\coprod_{b:B}}"] \arrow[r, shift right=4, "\underset{\text{product}}{\prod_{b:B}}"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathrm{ClType} \arrow[l, "{*_B} \times" description]
\end{tikzcd}$$

classical type system

linear type system in classical context

$$\begin{tikzcd}[column sep=huge]
\Big(\mathrm{LType}_B, \overset{\text{tensor}}{\otimes_B} \Big) \arrow[r, shift left=4, "\overset{\text{direct sum}}{\bigoplus_{b:B}}"] \arrow[r, shift right=4, "\bigoplus_{b:B}"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \Big(\mathrm{LType}, \overset{\text{tensor}}{\otimes} \Big) \arrow[l, "\mathbb{1}_B \otimes" description]
\end{tikzcd}$$

linear type system

classical base change / classical quantification

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
Symbol	$\oplus$ direct sum	$\otimes$ tensor product	$\multimap$ linear function type
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Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd} B \arrow[r, "p_B"] & \ast \end{tikzcd}$$

classical type system dependent on context

$$\begin{tikzcd}[column sep=huge] \mathrm{ClType}_B \arrow[r, shift left=4, "\overset{\text{co-product}}{\coprod_{b:B}}"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, shift right=4, "\underset{\text{product}}{\prod_{b:B}}"'] & \mathrm{ClType} \arrow[l, "\ast_B \times" description] \end{tikzcd}$$

classical type system

classical base change / classical quantification

linear type system in classical context

$$\begin{tikzcd}[column sep=huge] \big(\mathrm{LType}_B, \overset{\text{tensor}}{\otimes_B}\big) \arrow[r, shift left=4, "\overset{\text{direct sum}}{\bigoplus_{b:B}}"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, shift right=4, "\bigoplus_{b:B}"'] & \big(\mathrm{LType}, \overset{\text{tensor}}{\otimes}\big) \arrow[l, "\mathbb{1}_B \otimes" description] \end{tikzcd}$$

linear type system

quantum base change / Motivic Yoga

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
Symbol	$\oplus$ direct sum	$\otimes$ tensor product	$\multimap$ linear function type
Formula (for B : FinType)	$\overset{\text{cart. product}}{\prod_B \mathcal{H}_b} \simeq \underset{\text{direct sum}}{\bigoplus_B \mathcal{H}_b} \simeq \overset{\text{co-product}}{\coprod_B \mathcal{H}_b}$	$\mathcal{V} \otimes \Big(\bigoplus_{b:B} \mathcal{H}_b \Big) \simeq \bigoplus_{b:B} (\mathcal{V} \otimes \mathcal{H}_b)$	$(\mathcal{V} \otimes \mathcal{H}) \multimap \mathcal{K}$ $\simeq \mathcal{V} \multimap (\mathcal{H} \multimap \mathcal{K})$

Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd}[column sep=huge] B \arrow[r, "p_B"] & * \end{tikzcd}$$

classical type system dependent on context

$$\begin{tikzcd}[column sep=huge] \mathrm{ClType}_B \arrow[r, shift left=8, "\overset{\text{co-product}}{\coprod_{b:B}}"] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] \arrow[r, shift right=8, "\underset{\text{product}}{\prod_{b:B}}"'] & \mathrm{ClType} \arrow[l, "*_B \times"'] \end{tikzcd}$$

classical type system

classical base change / classical quantification

linear type system in classical context

$$\begin{tikzcd}[column sep=huge] \big(\mathrm{LType}_B, \overset{\text{tensor}}{\otimes_B}\big) \arrow[r, shift left=8, "\overset{\text{direct sum}}{\bigoplus_{b:B}}"] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] \arrow[r, shift right=8, "\bigoplus_{b:B}"'] & \big(\mathrm{LType}, \overset{\text{tensor}}{\otimes}\big) \arrow[l, "\mathbb{1}_B \otimes"'] \end{tikzcd}$$

linear type system

quantum base change / Motivic Yoga

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
Symbol	$\oplus$ direct sum	$\otimes$ tensor product	$\multimap$ linear function type
Formula (for B : FinType)	$\overset{\text{cart. product}}{\prod_B \mathcal{H}_b} \simeq \underset{\text{direct sum}}{\bigoplus_B \mathcal{H}_b} \simeq \overset{\text{co-product}}{\coprod_B \mathcal{H}_b}$	$\mathcal{V} \otimes \Big(\bigoplus_{b:B} \mathcal{H}_b \Big) \simeq \bigoplus_{b:B} (\mathcal{V} \otimes \mathcal{H}_b)$	$(\mathcal{V} \otimes \mathcal{H}) \multimap \mathcal{K}$ $\simeq \mathcal{V} \multimap (\mathcal{H} \multimap \mathcal{K})$

all available in LHoTT

Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd}[column sep=huge]
B \arrow[r, "p_B"] & *
\end{tikzcd}$$

classical type system dependent on context

$$\begin{tikzcd}[column sep=huge]
\mathrm{ClType}_B \arrow[r, shift left=18pt, "\overset{\text{co-product}}{\coprod_{b:B}}"] \arrow[r, shift right=18pt, "\underset{\text{product}}{\prod_{b:B}}"'] \arrow[r, phantom, shift left=9pt, "\perp"] \arrow[r, phantom, shift right=9pt, "\perp"] & \mathrm{ClType} \arrow[l, "{*_B} \times"']
\end{tikzcd}$$

classical type system

classical base change / classical quantification

linear type system in classical context

$$\begin{tikzcd}[column sep=huge]
\big(\mathrm{LType}_B, \overset{\text{tensor}}{\otimes_B}\big) \arrow[r, shift left=18pt, "\overset{\text{direct sum}}{\bigoplus_{b:B}}"] \arrow[r, shift right=18pt, "\bigoplus_{b:B}"'] \arrow[r, phantom, shift left=9pt, "\perp"] \arrow[r, phantom, shift right=9pt, "\perp"] & \big(\mathrm{LType}, \overset{\text{tensor}}{\otimes}\big) \arrow[l, "\mathbb{1}_B \otimes"']
\end{tikzcd}$$

linear type system

quantum base change / Motivic Yoga

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
Symbol	$\bigoplus$ direct sum	$\otimes$ tensor product	$\multimap$ linear function type
Formula (for B : FinType)	$\overset{\text{cart. product}}{\prod_B \mathcal{H}_b} \simeq \underset{\text{direct sum}}{\bigoplus_B \mathcal{H}_b} \simeq \overset{\text{co-product}}{\coprod_B \mathcal{H}_b}$	$\mathcal{V} \otimes \Big(\bigoplus_{b:B} \mathcal{H}_b \Big) \simeq \bigoplus_{b:B} \big(\mathcal{V} \otimes \mathcal{H}_b \big)$	$(\mathcal{V} \otimes \mathcal{H}) \multimap \mathcal{K}$ $\simeq \;\; \mathcal{V} \multimap (\mathcal{H} \multimap \mathcal{K})$

Dependent linear Type Formers

all available in LHoTT

finite classical context (variables, parameters, ...)

reference context

$$B \xrightarrow{\;\;p_B\;\;} *$$

co-product

classical type syst
dependent on con

Observation: Induces quantum effects on linear types. ⟶

Type

classical type system

classical base change / classical quantification

linear type system in classical context

$$\begin{tikzcd}
\Big(\mathrm{LType}_B, \overset{\text{tensor}}{\otimes_B}\Big) \arrow[r, shift left=12, "\overset{\text{direct sum}}{\bigoplus_{b:B}}"] \arrow[r, phantom, shift left=6, "\perp"] \arrow[r, leftarrow, "\mathbb{1}_B \otimes" description] \arrow[r, phantom, shift right=6, "\perp"] \arrow[r, shift right=12, "\bigoplus_{b:B}" description] & \Big(\mathrm{LType}, \overset{\text{tensor}}{\otimes}\Big)
\end{tikzcd}$$

linear type system

quantum base change / Motivic Yoga

Quantum Effects

Recall: **Monadic computational effects.**

A *monad* $\mathscr{E}(-)$ on a data type system encodes *computational effects*:

effectful program

$$D_1 \xrightarrow{\mathrm{prog}_{12}} \mathscr{E}(D_2)$$

output data of nominal type D_2
causing effects of type $\mathscr{E}(-)$

Recall: **Monadic computational effects.**

A *monad* $\mathscr{E}(-)$ on a data type system encodes *computational effects*:

first program

$$D_1 \xrightarrow{\mathrm{prog}_{12}} \mathscr{E}(D_2)$$

output data of nominal type D_2
causing effects of type $\mathscr{E}(-)$

second program

$$D_2 \xrightarrow{\mathrm{prog}_{23}} \mathscr{E}(D_3)$$

input data of type D_2
causing effects of type $\mathscr{E}(-)$

Recall: **Monadic computational effects.**

A *monad* $\mathscr{E}(-)$ on a data type system encodes *computational effects*:

first program

second program

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}"] & \mathscr{E}(D_2) & D_2 \arrow[r, "\mathrm{prog}_{23}"] & \mathscr{E}(D_3) \arrow[d, mapsto, bend left, "\text{bind previous effects into second program}"] \\
D_1 \arrow[r, "\mathrm{prog}_{12}"] & \mathscr{E}(D_2) & \mathscr{E}(D_2) \arrow[r, "\mathrm{bind}^{\mathscr{E}}\,\mathrm{prog}_{23}"] & \mathscr{E}(D_3)
\end{tikzcd}$$

output data of nominal type D_2
causing effects of type $\mathscr{E}(-)$

input data of type D_2
causing effects of type $\mathscr{E}(-)$

bind previous effects
into second program

carry any previous $\mathscr{E}(-)$-effects along

Recall: **Monadic computational effects.**

A *monad* $\mathscr{E}(-)$ on a data type system encodes *computational effects*:

first program

second program

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}"] & \mathscr{E}(D_2) & D_2 \arrow[r, "\mathrm{prog}_{23}"] & \mathscr{E}(D_3) \arrow[d, mapsto, bend left, "\text{bind previous effects into second program}"] \\
D_1 \arrow[r, "\mathrm{prog}_{12}"] & \mathscr{E}(D_2) & \mathscr{E}(D_2) \arrow[r, "\mathrm{bind}^{\mathscr{E}}\,\mathrm{prog}_{23}"] & \mathscr{E}(D_3) \arrow[d, mapsto, bend left, "\text{compose}"] \\
D_1 \arrow[rrr, "\mathrm{bind}^{\mathscr{E}}\,\mathrm{prog}_{23}\circ\mathrm{prog}_{12}"] & & & \mathscr{E}(D_3)
\end{tikzcd}$$

output data of nominal type D_2
causing effects of type $\mathscr{E}(-)$

input data of type D_2
causing effects of type $\mathscr{E}(-)$

bind previous effects
into second program

carry any previous $\mathscr{E}(-)$-effects along

compose

$\mathscr{E}$-composite program

causing cumulative $\mathscr{E}(-)$-effects

Recall: **Monadic effect handlers.**

$$D_1 \xrightarrow[\textbf{in-effectful program}]{\text{prog}_{12}} D_2 \quad \textbf{data type to absorb } \mathscr{E}\textbf{-effects}$$

Recall: **Monadic effect handlers.**

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}", "\textbf{in-effectful program}"'] & D_2 & {} \arrow[d, mapsto, bend left, "\substack{\textbf{incorporate handling}\\ \textbf{of } \mathscr{E}(-)\textbf{-effects}}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\,\mathrm{prog}_{12}", "\substack{\textbf{in-effectful program}\\ \textbf{handling effects of type } \mathscr{E}(-)}"'] & D_2 & {}
\end{tikzcd}$$

Recall: **Monadic effect handlers.**

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}", "\textbf{in-effectful program}"'] & D_2 \arrow[d, mapsto, bend left, "\substack{\textbf{incorporate handling}\\ \textbf{of } \mathscr{E}(-)\textbf{-effects}}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}", "\substack{\textbf{in-effectful program}\\ \textbf{handling effects of type } \mathscr{E}(-)}"'] & D_2
\end{tikzcd}$$

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{ret}^{\mathscr{E}}_{D_1}", "\substack{\textbf{produce}\\ \textbf{trivial effect}}"'] \arrow[rr, bend right=40, "\substack{\mathrm{prog}_{12}\\ \textbf{no effect}}"'] & \mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}", "\substack{\textbf{handle effects}\\ \textbf{running program}}"'] & D_2
\end{tikzcd}$$

consistency conditions

Recall: **Monadic effect handlers.**

$$\begin{tikzcd}[column sep=huge] D_1 \arrow[r, "\mathrm{prog}_{12}", "\text{in-effectful program}"'] & D_2 \end{tikzcd}$$

$\mapsto$ incorporate handling of $\mathscr{E}(-)$-effects

$$\begin{tikzcd}[column sep=huge] \mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}", "\text{in-effectful program handling effects of type } \mathscr{E}(-)"'] & D_2 \end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge] D_1 \arrow[r, "\mathrm{ret}^{\mathscr{E}}_{D_1}", "\text{produce trivial effect}"'] \arrow[rr, bend right, "\mathrm{prog}_{12}"', "\text{no effect}"] & \mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}", "\text{handle effects running program}"'] & D_2 \end{tikzcd}$$

consistency conditions

$$\begin{tikzcd}[column sep=huge] \mathscr{E}(D_0) \arrow[r, "\mathrm{bind}^{\mathscr{E}}\mathrm{prog}_{01}", "\text{carry effects along}"'] \arrow[rr, bend right] & \mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}", "\text{handle cumulative effects}"'] & D_2 \end{tikzcd}$$

$$\mathrm{hndl}^{\mathscr{E}}_{D_2}\Big(D_0 \xrightarrow{\mathrm{prog}_{01}} \mathscr{E}(D_1) \xrightarrow{\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}} D_2\Big)$$

handle effects... consecutively

Recall: **Data type system of Monadic effect handlers.**

$$\begin{tikzcd}[column sep=huge, row sep=huge]
D_1 \arrow[r, "\mathrm{prog}_{12}", "\textbf{in-effectful program}"'] & D_2 \arrow[d, mapsto, bend left, "\textbf{incorporate handling of } \mathscr{E}(-)\textbf{-effects}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}", "\substack{\textbf{in-effectful program} \\ \textbf{handling effects of type } \mathscr{E}(-)}"'] & D_2 \quad \text{“}\mathscr{E}\textbf{-modal data type}\text{”}
\end{tikzcd}$$

Monadicity:

$\mathscr{E}$-modales in Type
(“EM-category”)

$\mathrm{Type}^{\mathscr{E}}$

Recall: **Data type system of Monadic effect handlers.**

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}"] & D_2 \arrow[d, mapsto, bend left, "\text{incorporate handling of } \mathscr{E}(-)\text{-effects}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}"] & D_2
\end{tikzcd}$$

in-effectful program

incorporate handling of $\mathscr{E}(-)$-effects

“$\mathscr{E}$-modal data type”

in-effectful program handling effects of type $\mathscr{E}(-)$

Monadicity:

$$\begin{tikzcd}
\mathrm{Type} \arrow[loop right, "\mathscr{E}"]
\end{tikzcd}$$

monad

$\mathscr{E}$-modales in Type (“EM-category”) $\mathrm{Type}^{\mathscr{E}}$

Recall: **Data type system of Monadic effect handlers.**

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}", "\text{in-effectful program}"'] & D_2 \arrow[d, mapsto, bend left, "\text{incorporate handling of } \mathscr{E}(-)\text{-effects}"] & \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\,\mathrm{prog}_{12}", "\text{in-effectful program handling effects of type } \mathscr{E}(-)"'] & D_2 & \text{``}\mathscr{E}\text{-modal data type''}
\end{tikzcd}$$

Monadicity:

$$\begin{tikzcd}
 & \mathrm{Type} \arrow[loop right, "\mathscr{E}"] \arrow[dl, shift right=2, "F^{\mathscr{E}}"'] \arrow[dl, phantom, "\perp"] & \text{induced monad} \\
\text{$\mathscr{E}$-modales in Type (``EM-category'')} \quad \mathrm{Type}^{\mathscr{E}} \arrow[ur, shift right=2, "U^{\mathscr{E}}"', "\text{monadic adjunction}"'] & &
\end{tikzcd}$$

Recall: **Data type system of Monadic effect handlers.**

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}", "\text{in-effectful program}"'] & D_2 \arrow[d, maps to, bend left, "\text{incorporate handling of } \mathscr{E}(-)\text{-effects}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}", "\text{in-effectful program handling effects of type } \mathscr{E}(-)"'] & D_2 \;\; \text{“}\mathscr{E}\text{-modal data type”}
\end{tikzcd}$$

Monadicity:

$$\begin{tikzcd}
\text{any adjunction on type system} & \mathrm{Type}' \arrow[rr, shift right, "U"'] \arrow[rr, phantom, "\perp"] \arrow[d, "K^{UF}", "\text{terminal comparison functor}"'] & & \mathrm{Type} \arrow[ll, shift right, "F"'] \arrow[dll, shift right, "F^{\mathscr{E}}"'] \arrow[loop right, "\mathscr{E}"] & \text{induced monad} \\
\mathscr{E}\text{-modales in } \mathrm{Type} \; (\text{“EM-category”}) & \mathrm{Type}^{\mathscr{E}} \arrow[urr, shift right, "U^{\mathscr{E}}"'] \arrow[urr, phantom, "\perp"] \arrow[urr, phantom, "\text{monadic adjunction}"', shift right=3ex] & & &
\end{tikzcd}$$

Recall: **Data type system of Monadic effect handlers.**

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}", "\text{in-effectful program}"'] & D_2 \arrow[d, mapsto, bend left, "\begin{array}{c}\text{incorporate handling}\\ \text{of } \mathscr{E}(-)\text{-effects}\end{array}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}", "\begin{array}{c}\text{in-effectful program}\\ \text{handling effects of type } \mathscr{E}(-)\end{array}"'] & D_2 \quad \text{“}\mathscr{E}\text{-modal data type”}
\end{tikzcd}$$

Monadicity:

$$\begin{tikzcd}
\begin{array}{r}\text{free } \mathscr{E}\text{-modales in Type}\\ \text{(“Kleisli category”)}\end{array} & \mathrm{Type}_{\mathscr{E}} \arrow[d, hook, "K_{UF}", "\begin{array}{c}\text{initial}\\ \text{comparison}\\ \text{functor}\end{array}"'] \arrow[dr, shift right, "U_{\mathscr{E}}"'] \arrow[dr, phantom, "\perp"] & \\
\begin{array}{r}\text{any adjunction}\\ \text{on type system}\end{array} & \mathrm{Type}' \arrow[d, "K^{UF}", "\begin{array}{c}\text{terminal}\\ \text{comparison}\\ \text{functor}\end{array}"'] \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{Type} \arrow[l, shift right, "F"'] \arrow[ul, two heads, shift right, "F^{\mathscr{E}}"'] \arrow[dl, shift right, "F^{\mathscr{E}}"'] \arrow[loop right, "\mathscr{E}"] \quad \text{induced monad} \\
\begin{array}{r}\mathscr{E}\text{-modales in Type}\\ \text{(“EM-category”)}\end{array} & \mathrm{Type}^{\mathscr{E}} \arrow[ur, shift right, "U^{\mathscr{E}}"', "\text{monadic adjunction}"] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}$$

Recall: **Data type system of Monadic effect handlers.**

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}"] & D_2 \arrow[d, mapsto, bend left] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}"] & D_2
\end{tikzcd}$$

in-effectful program

incorporate handling of $\mathscr{E}(-)$-effects

"$\mathscr{E}$-modal data type"

in-effectful program handling effects of type $\mathscr{E}(-)$

Monadicity:

$$\begin{tikzcd}
\mathrm{Type}_{\mathscr{E}} \arrow[d, hook, "K_{UF}"] \arrow[drr, shift left, "U_{\mathscr{E}}"] & & \\
\mathrm{Type}' \arrow[d, "K^{UF}"] \arrow[rr, shift right, "U"'] & & \mathrm{Type} \arrow[ll, shift right, "F"'] \arrow[ull, two heads, shift left, "F_{\mathscr{E}}"] \arrow[ull, phantom, "\perp"] \arrow[ll, phantom, "\perp"] \arrow[dll, shift right, "F^{\mathscr{E}}"'] \arrow[dll, phantom, "\perp"] \arrow[loop right, "\mathscr{E}"] \\
\mathrm{Type}^{\mathscr{E}} \arrow[urr, shift right, "U^{\mathscr{E}}"'] & &
\end{tikzcd}$$

free $\mathscr{E}$-modales in Type
("Kleisli category")

initial comparison functor

any adjunction
on type system

terminal comparison functor

$\mathscr{E}$-modales in Type
("EM-category")

monadic adjunction

induced monad

Now just to work this out
for the **effects induced by
dependent data type formers**
in LHoTT →

Given B : ClType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent data type formers** constitute modalities of **actual and potential B-measurements**:

classical data types

possibly **randomly**

$$\begin{tikzcd}
\textbf{actual data} & \perp & \mathrm{Type}_B \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] \arrow[rr, shift left=4, "\coprod_B"] \arrow[rr, phantom, shift left=2, "\perp"] \arrow[rr, phantom, shift right=2, "\perp"] \arrow[rr, shift right=4, "\prod_B"'] & & \mathrm{Type} \arrow[ll, "\times B"] \arrow[loop above, "\star_B"'] \arrow[loop below, "\bigcirc_B"] & \perp & \textbf{potential data}
\end{tikzcd}$$

necessarily **indefinitely**

Given B : ClType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent data type formers** constitute modalities of **actual and potential B-measurements**:

classical data types

possibly **randomly**

actual data **potential data**

$$\begin{tikzcd}
\mathrm{Type}_B \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] \arrow[r, shift left=6, "\coprod_B"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, shift right=6, "\prod_B"'] & \mathrm{Type} \arrow[l, "\times B"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"']
\end{tikzcd}$$

$\Diamond_B \dashv \Box_B$ $\quad$ $\star_B \dashv \bigcirc_B$

necessarily **indefinitely**

---

necessarily $P_\bullet$

$$\Box_B P_\bullet$$

$$b : B \;\vdash\; \prod_{b':B} P_{b'}$$

Given $B : \mathrm{ClType}$ of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent data type formers** constitute modalities of **actual and potential B-measurements**:

classical data types

possibly **randomly**

actual data **potential data**

$$\begin{tikzcd}
\mathrm{Type}_B \arrow[loop above, "\lozenge_B"] \arrow[loop below, "\square_B"] \arrow[r, shift left=4, "\coprod_B"] \arrow[r, shift right=4, "\prod_B"'] & \mathrm{Type} \arrow[l, "\times B" description] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"]
\end{tikzcd}$$

$\lozenge_B \dashv \square_B$, $\coprod_B \dashv \times B \dashv \prod_B$, $\star_B \dashv \bigcirc_B$

necessarily **indefinitely**

---

necessarily $P_\bullet$ **entails** **actually $P_\bullet$**

$$\square_B P_\bullet \xrightarrow{\;\epsilon^{\square_B}_{P_\bullet}\;} P_\bullet$$

$$b : B \;\vdash\; \prod_{b':B} P_{b'} \xrightarrow{(p_{b'})_{b':B} \,\mapsto\, p_b} P_b$$

Given $B : \mathrm{ClType}$ of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent data type formers** constitute modalities of **actual and potential B-measurements**:

classical data types

possibly **randomly**

actual data **potential data**

$$\begin{tikzcd}
\mathrm{Type}_B \arrow[loop above, "\lozenge_B"] \arrow[loop below, "\Box_B"'] \arrow[r, shift left=3, "\coprod_B"] \arrow[r, phantom, shift left=1.5, "\perp"] \arrow[r, phantom, shift right=1.5, "\perp"] \arrow[r, shift right=3, "\prod_B"'] & \mathrm{Type} \arrow[l, "\times B" description] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"']
\end{tikzcd}$$

$\lozenge_B \perp \Box_B$ $\qquad$ $\star_B \perp \bigcirc_B$

necessarily **indefinitely**

necessarily $P_\bullet$ **entails** **actually $P_\bullet$** **entails** **possibly $P_\bullet$**

$$\begin{tikzcd}
\Box_B P_\bullet \arrow[r, "\epsilon^{\Box_B}_{P_\bullet}" description] & P_\bullet \arrow[r, "\eta^{\lozenge_B}_{P_\bullet}" description] & \lozenge_B P_\bullet
\end{tikzcd}$$

$$b : B \;\vdash\; \begin{tikzcd}
\prod\limits_{b':B} P_{b'} \arrow[r, "(p_{b'})_{b':B} \mapsto p_b"] & P_b \arrow[r, "p_b \mapsto (b, p_b)"] & \coprod\limits_{b':B} P_{b'}
\end{tikzcd}$$

Given B : ClType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent data type formers** constitute modalities of **actual and potential B-measurements**:

classical data types

possibly **randomly**

$$\begin{tikzcd}
\textbf{actual data} & \perp & \mathrm{Type}_B \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] \arrow[rr, shift left=3, "\coprod_B"] \arrow[rr, shift right=3, "\prod_B"'] & \begin{array}{c}\perp\\ \\ \perp\end{array} & \mathrm{Type} \arrow[ll, "\times B" description] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"'] & \perp & \textbf{potential data}
\end{tikzcd}$$

necessarily **indefinitely**

---

necessarily $P_\bullet$ **entails** **actually $P_\bullet$** **entails** **possibly $P_\bullet$**

$$\begin{tikzcd}
\Box_B P_\bullet \arrow[r, "\epsilon^{\Box_B}_{P_\bullet}"] & P_\bullet \arrow[r, "\eta^{\Diamond_B}_{P_\bullet}"] & \Diamond_B P_\bullet
\end{tikzcd}$$

$$\begin{tikzcd}
b : B \;\vdash\; \prod\limits_{b':B} P_{b'} \arrow[r, "(p_{b'})_{b':B} \mapsto p_b"] & P_b \arrow[r, "p_b \mapsto (b, p_b)"] & \coprod\limits_{b':B} P_{b'}
\end{tikzcd}$$

randomly P

$$\star_B P$$

$$\coprod_{b:B} P$$

Given B : ClType of possible measurement outcomes ("possible worlds") **the monadic effects of B-dependent data type formers** constitute modalities of **actual and potential B-measurements**:

classical data types

possibly **randomly**

actual data $\perp$ **potential data** $\perp$

necessarily **indefinitely**

$$\begin{tikzcd}
\mathrm{Type}_B \arrow[loop, out=150, in=90, "\Diamond_B"] \arrow[loop, out=270, in=210, "\Box_B"'] \arrow[r, shift left=5, "\coprod_B"] \arrow[r, phantom, shift left=2.5, "\perp"] \arrow[r, phantom, shift right=2.5, "\perp"] \arrow[r, shift right=5, "\prod_B"'] & \mathrm{Type} \arrow[l, "\times B"'] \arrow[loop, out=90, in=30, "\star_B"] \arrow[loop, out=330, in=270, "\bigcirc_B"']
\end{tikzcd}$$

necessarily $P_\bullet$ **entails** **actually $P_\bullet$** **entails** **possibly $P_\bullet$**

$$\Box_B P_\bullet \xrightarrow{\epsilon^{\Box_B}_{P_\bullet}} P_\bullet \xrightarrow{\eta^{\Diamond_B}_{P_\bullet}} \Diamond_B P_\bullet$$

$$b : B \;\vdash\; \prod_{b':B} P_{b'} \xrightarrow{(p_{b'})_{b':B} \mapsto p_b} P_b \xrightarrow{p_b \mapsto (b, p_b)} \coprod_{b':B} P_{b'}$$

randomly P **entails** **potentially P**

$$\star_B P \xrightarrow{\epsilon^{\star_B}_{P}} P$$

$$\coprod_{b:B} P \xrightarrow{(b,p) \mapsto p} P$$

Given $B : \mathrm{ClType}$ of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent data type formers** constitute modalities of **actual and potential B-measurements**:

classical data types

possibly **randomly**

actual data **potential data**

$$\begin{tikzcd}[column sep=huge]
\mathrm{Type}_B \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] \arrow[r, shift left=6, "\coprod_B"] \arrow[r, shift right=6, "\prod_B"'] & \mathrm{Type} \arrow[l, "\times B" description] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"']
\end{tikzcd}$$

$$\Diamond_B \dashv \Box_B \qquad \coprod_B \dashv \times B \dashv \prod_B \qquad \star_B \dashv \bigcirc_B$$

necessarily **indefinitely**

necessarily $P_\bullet$ **entails** **actually $P_\bullet$** **entails** **possibly $P_\bullet$**

$$\Box_B P_\bullet \xrightarrow{\;\epsilon^{\Box_B}_{P_\bullet}\;} P_\bullet \xrightarrow{\;\eta^{\Diamond_B}_{P_\bullet}\;} \Diamond_B P_\bullet$$

$$b : B \;\vdash\; \prod_{b':B} P_{b'} \xrightarrow{(p_{b'})_{b':B} \,\mapsto\, p_b} P_b \xrightarrow{p_b \,\mapsto\, (b, p_b)} \coprod_{b':B} P_{b'}$$

randomly P **entails** **potentially P** **entails** **indefinitely P**

$$\star_B P \xrightarrow{\;\epsilon^{\star_B}_{P}\;} P \xrightarrow{\;\eta^{\bigcirc_B}_{P}\;} \bigcirc_B P$$

$$\coprod_{b:B} P \xrightarrow{(b,p) \,\mapsto\, p} P \xrightarrow{p \,\mapsto\, (p)_{b:B}} \prod_{b:B} P$$

Given B : ClType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements**.

quantum data types

possibly **randomly**

actual data $\perp$ **potential data** $\perp$

$$\begin{tikzcd}
 & (\mathrm{LType}_B)_{\Diamond_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\Diamond_B} \arrow[dr, dash, "\sim"] & \\
\mathrm{LType}_B \simeq \mathrm{LType}^{\bigcirc_B} \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] \arrow[ur, two heads, "\oplus_B"] \arrow[dr, two heads, "\oplus_B"'] & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[d, phantom, "\perp"] & & \mathrm{LType} \arrow[ll, two heads, "\mathbb{1}_B \otimes"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"'] \\
 & (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B} \arrow[ur, dash, "\sim"'] &
\end{tikzcd}$$

necessarily **indefinitely**

Given B : ClType of possible measurement outcomes ("possible worlds") **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements**.

quantum data types

$$\begin{tikzcd}
 & & & (\mathrm{LType}_B)_{\lozenge_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\lozenge_B} \arrow[dr, phantom, "\simeq" sloped] & & \\
\textbf{actual data} \perp & \mathrm{LType}_B \simeq \mathrm{LType}^{\bigcirc_B} \arrow[loop above, "\overset{\textbf{possibly}}{\lozenge_B}"] \arrow[loop below, "\underset{\textbf{necessarily}}{\Box_B}"'] \arrow[urr, two heads, "\oplus_B"] \arrow[drr, two heads, "\oplus_B"'] & & \mathrm{LType}_{\bigcirc_B} \arrow[ll, hook'] \arrow[d, phantom, "\perp"] & & \mathrm{LType} \arrow[ll, two heads, "\mathbb{1}_B \otimes"'] \arrow[loop above, "\overset{\textbf{randomly}}{\star_B}"] \arrow[loop below, "\underset{\textbf{indefinitely}}{\bigcirc_B}"'] & \perp \textbf{potential data} \\
 & & & (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B} \arrow[ur, phantom, "\simeq" sloped] & &
\end{tikzcd}$$

necessarily $\mathcal{H}_\bullet$

$\Box_B \mathcal{H}_\bullet$

Given... **obtain...**

$b : B \quad \vdash \quad \mathcal{H}$

measurement result

where $\mathcal{H} := \bigoplus_{b':B} \mathcal{H}_{b'}$

Given B : ClType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements**.

quantum data types

possibly **randomly**

actual data $\perp$ $\quad\quad$ $\perp$ **potential data**

$$\begin{tikzcd}
 & & (\mathrm{LType}_B)_{\Diamond_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\Diamond_B} \arrow[dr, phantom, "\simeq"] & \\
\mathrm{LType}_B \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] \arrow[urr, ->>, "\oplus_B"] \arrow[drr, ->>, "\oplus_B"'] & \simeq \mathrm{LType}^{\bigcirc_B} & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[d, phantom, "\perp"] & & \mathrm{LType} \arrow[ll, ->>, "\mathbb{1}_B\otimes"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"'] \\
 & & (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B} \arrow[ur, phantom, "\simeq"] &
\end{tikzcd}$$

necessarily **indefinitely**

necessarily $\mathcal{H}_\bullet$ $\quad$ **entails** $\quad$ **actually $\mathcal{H}_\bullet$**

$$\Box_B\,\mathcal{H}_\bullet \xrightarrow{\quad \varepsilon^{\Box_B}_{\mathcal{H}_\bullet} \quad} \mathcal{H}_\bullet$$

Given... $\quad$ **obtain...**

$$b : B \;\vdash\; \mathcal{H} \xrightarrow[\text{measurement collapse}]{\sum_{b'} |\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle} \mathcal{H}_b \qquad \text{where } \mathcal{H} := \bigoplus_{b':B} \mathcal{H}_{b'}$$

measurement result

Given B : ClType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements**.

quantum data types

$$\begin{tikzcd}
 & & (\mathrm{LType}_B)_{\Diamond_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\Diamond_B} \arrow[dr, phantom, "\wr"] & & \\
\textbf{actual data} \perp & \mathrm{LType}_B \simeq \mathrm{LType}^{\bigcirc_B} \arrow[loop above, "\Diamond_B", "\textbf{possibly}"'] \arrow[loop below, "\Box_B"', "\textbf{necessarily}"] \arrow[ur, two heads, "\oplus_B"] \arrow[dr, two heads, "\oplus_B"'] & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[d, phantom, "\perp"] & & \mathrm{LType} \arrow[ll, two heads, "\mathbb{1}_B \otimes"'] \arrow[loop above, "\star_B", "\textbf{randomly}"'] \arrow[loop below, "\bigcirc_B"', "\textbf{indefinitely}"] & \perp \textbf{ potential data} \\
 & & (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B} \arrow[ur, phantom, "\wr"] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\textbf{necessarily } \mathcal{H}_\bullet & \textbf{entails} & \textbf{actually } \mathcal{H}_\bullet & \textbf{entails} & \textbf{possibly } \mathcal{H}_\bullet \\
\Box_B \mathcal{H}_\bullet \arrow[rr, "\epsilon^{\Box_B}_{\mathcal{H}_\bullet}"] & & \mathcal{H}_\bullet \arrow[rr, "\eta^{\Diamond_B}_{\mathcal{H}_\bullet}"] & & \Diamond_B \mathcal{H}_\bullet
\end{tikzcd}$$

$$\begin{tikzcd}
\textbf{Given...} & \textbf{obtain...} & & & \\
b : B & \vdash & \mathcal{H} \arrow[r, two heads, "\Sigma_{b'} \lvert\psi_{b'}\rangle \;\mapsto\; \lvert\psi_b\rangle", "\textbf{measurement collapse}"'] \arrow[rr, bend right, "\textbf{linear projector onto sub-Hilbert space } \mathcal{H}_b"'] & \mathcal{H}_b \arrow[r, hook, "\lvert\psi_b\rangle \mapsto \oplus_{b'} \begin{cases} \lvert\psi_b\rangle \text{ if } b'{=}b \\ 0 \quad \text{else} \end{cases}", "\textbf{state preparation}"'] & \mathcal{H}, \\
\textbf{measurement result} & & & &
\end{tikzcd}$$

where $\mathcal{H} := \bigoplus\limits_{b':B} \mathcal{H}_{b'}$

Given B : ClType of possible measurement outcomes ("possible worlds") **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements**.

quantum
data types

possibly **randomly**

actual data $\perp$ **potential data**

necessarily **indefinitely**

$$\begin{tikzcd}
 & (\mathrm{LType}_B)_{\lozenge_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\lozenge_B} \arrow[dr, phantom, "\simeq"] & \\
\mathrm{LType}_B \simeq \mathrm{LType}^{\bigcirc_B} \arrow[loop above, "\lozenge_B"] \arrow[loop below, "\Box_B"'] \arrow[ur, two heads, "\oplus_B"] \arrow[dr, two heads, "\oplus_B"'] & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[d, phantom, "\perp"] & & \mathrm{LType} \arrow[ll, two heads, "\mathbb{1}_B \otimes"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"'] \\
 & (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B} \arrow[ur, phantom, "\simeq"] &
\end{tikzcd}$$

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principle of quantum compulsion:

necessarily $\mathcal{H}_\bullet$ **entails** **actually $\mathcal{H}_\bullet$** **entails** **possibly $\mathcal{H}_\bullet$** **is** **necessarily $\mathcal{H}_\bullet$**

$$\begin{tikzcd}
\Box_B \mathcal{H}_\bullet \arrow[r, "\epsilon^{\Box_B}_{\mathcal{H}_\bullet}"] & \mathcal{H}_\bullet \arrow[r, "\eta^{\lozenge_B}_{\mathcal{H}_\bullet}"] & \lozenge_B \mathcal{H}_\bullet \arrow[r, phantom, "\simeq"] & \Box_B \mathcal{H}_\bullet
\end{tikzcd}$$

ambidexterity

Given... **obtain...**

$$b : B \quad \vdash \quad \begin{tikzcd}
\mathcal{H} \arrow[r, two heads, "\sum_{b'} |\psi_{b'}\rangle \mapsto |\psi_b\rangle", "\text{measurement collapse}"'] \arrow[rr, bend right, "\text{linear projector onto sub-Hilbert space } \mathcal{H}_b"'] & \mathcal{H}_b \arrow[r, hook, "|\psi_b\rangle \mapsto \oplus_{b'} \begin{cases} |\psi_b\rangle & \text{if } b'=b \\ 0 & \text{else} \end{cases}", "\text{state preparation}"'] & \mathcal{H},
\end{tikzcd} \qquad \text{where } \mathcal{H} := \bigoplus_{b':B} \mathcal{H}_{b'}$$

measurement result

Given B : ClType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements**.

quantum data types

possibly **randomly**

actual data $\perp$ $\perp$ **potential data**

$$\begin{tikzcd}
 & (\mathrm{LType}_B)_{\lozenge_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\lozenge_B} \arrow[dr, phantom, "\simeq"] & \\
\mathrm{LType}_B \simeq \mathrm{LType}^{\bigcirc_B} \arrow[loop above, "\lozenge_B"] \arrow[loop below, "\square_B"] \arrow[ur, two heads, "\oplus_B"] \arrow[dr, two heads, "\oplus_B"'] & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[d, phantom, "\perp"] & & \mathrm{LType} \arrow[ll, two heads, "\mathbb{1}_B \otimes"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"] \\
 & (\mathrm{LType}_B)_{\square_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\square_B} \arrow[ur, phantom, "\simeq"] &
\end{tikzcd}$$

necessarily **indefinitely**

---

principle of quantum compulsion:

necessarily $\mathcal{H}_\bullet$ **entails** **actually $\mathcal{H}_\bullet$** **entails** **possibly $\mathcal{H}_\bullet$** **is** **necessarily $\mathcal{H}_\bullet$**

$$\begin{tikzcd}
\square_B\, \mathcal{H}_\bullet \arrow[r, "\varepsilon^{\square_B}_{\mathcal{H}_\bullet}"] & \mathcal{H}_\bullet \arrow[r, "\eta^{\lozenge_B}_{\mathcal{H}_\bullet}"] & \lozenge_B\, \mathcal{H}_\bullet \arrow[r, phantom, "\simeq"] & \square_B\, \mathcal{H}_\bullet
\end{tikzcd}$$

ambidexterity

Given... **obtain...**

$b : B \;\vdash$

measurement result

$$\begin{tikzcd}
\mathcal{H} \arrow[r, two heads, "\sum_{b'} |\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle", "\text{measurement collapse}"'] \arrow[rr, bend right, "\text{linear projector onto sub-Hilbert space } \mathcal{H}_b"'] & \mathcal{H}_b \arrow[r, hook, "|\psi_b\rangle \mapsto \oplus_{b'} \begin{cases} |\psi_b\rangle & \text{if } b'=b \\ 0 & \text{else} \end{cases}", "\text{state preparation}"'] & \mathcal{H},
\end{tikzcd}$$

where $\mathcal{H} := \bigoplus\limits_{b':B} \mathcal{H}_{b'}$

---

randomly $\mathcal{H}$

$\star_B \mathcal{H}$

$\bigoplus\limits_{b:B} \mathcal{H}$

Given $B : \mathrm{ClType}$ of possible measurement outcomes (“possible worlds”)
the monadic effects of B-dependent linear data type formers constitute
modalities of **actual and potential quantum B-measurements**.

quantum
data types

possibly **randomly**

actual data $\perp$ $\perp$ **potential data**

$$\begin{tikzcd}
& (\mathrm{LType}_B)_{\Diamond_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\Diamond_B} \\
\mathrm{LType}_B \simeq \mathrm{LType}^{\bigcirc_B} \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] \arrow[ur, two heads, bend left, "\oplus_B"] \arrow[dr, two heads, bend right, "\oplus_B"'] & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[d, phantom, "\perp"] & \mathrm{LType} \arrow[l, two heads, "\mathbb{1}_B \otimes"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"'] \arrow[u, phantom, "\simeq" sloped] \arrow[d, phantom, "\simeq" sloped] \\
& (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B}
\end{tikzcd}$$

necessarily **indefinitely**

principle of quantum compulsion:

necessarily $\mathcal{H}_\bullet$ **entails** **actually $\mathcal{H}_\bullet$** **entails** **possibly $\mathcal{H}_\bullet$** **is** **necessarily $\mathcal{H}_\bullet$**

$$\Box_B \mathcal{H}_\bullet \xrightarrow{\;\epsilon^{\Box_B}_{\mathcal{H}_\bullet}\;} \mathcal{H}_\bullet \xrightarrow{\;\eta^{\Diamond_B}_{\mathcal{H}_\bullet}\;} \Diamond_B \mathcal{H}_\bullet \underset{\text{ambidexterity}}{\simeq} \Box_B \mathcal{H}_\bullet$$

Given... **obtain...**

$b : B \;\vdash$

measurement result

$$\begin{tikzcd}
\mathcal{H} \arrow[r, two heads, "\Sigma_{b'} |\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle", "\text{measurement collapse}"'] \arrow[rr, bend right, "\text{linear projector onto sub-Hilbert space } \mathcal{H}_b"'] & \mathcal{H}_b \arrow[r, hook, "|\psi_b\rangle \mapsto \oplus_{b'} \begin{cases} |\psi_b\rangle & \text{if } b' = b \\ 0 & \text{else} \end{cases}", "\text{state preparation}"'] & \mathcal{H},
\end{tikzcd}$$

where $\mathcal{H} := \bigoplus\limits_{b' : B} \mathcal{H}_{b'}$

randomly $\mathcal{H}$ **entails** **potentially $\mathcal{H}$**

$$\star_B \mathcal{H} \xrightarrow{\;\epsilon^{\star_B}_{\mathcal{H}}\;} \mathcal{H}$$

$$\bigoplus_{b:B} \mathcal{H} \xrightarrow[\text{quantum superposition}]{\oplus_b |\psi_b\rangle \mapsto \Sigma_b |\psi_b\rangle} \mathcal{H}$$

Given B : ClType of possible measurement outcomes ("possible worlds") **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements**.

quantum
data types

possibly **randomly**

actual data $\perp$ $\perp$ **potential data**

$$\begin{tikzcd}
 & (\mathrm{LType}_B)_{\Diamond_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\Diamond_B} \arrow[dr, phantom, "\sim" sloped] & \\
\mathrm{LType}_B \simeq \mathrm{LType}^{\bigcirc_B} \arrow[ur, two heads, "\oplus_B"] \arrow[dr, two heads, "\oplus_B"'] \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[d, phantom, "\perp"] & & \mathrm{LType} \arrow[ll, two heads, "\mathbb{1}_B \otimes"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"'] \\
 & (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B} \arrow[ur, phantom, "\sim" sloped] &
\end{tikzcd}$$

necessarily **indefinitely**

---

principle of quantum compulsion:

necessarily $\mathcal{H}_\bullet$ **entails** **actually $\mathcal{H}_\bullet$** **entails** **possibly $\mathcal{H}_\bullet$** **is** **necessarily $\mathcal{H}_\bullet$**

$$\begin{tikzcd}
\Box_B \mathcal{H}_\bullet \arrow[r, "\epsilon^{\Box_B}_{\mathcal{H}_\bullet}"] & \mathcal{H}_\bullet \arrow[r, "\eta^{\Diamond_B}_{\mathcal{H}_\bullet}"] & \Diamond_B \mathcal{H}_\bullet \arrow[r, phantom, "\simeq"] & \Box_B \mathcal{H}_\bullet
\end{tikzcd}$$

ambidexterity

Given... **obtain...**

$$\begin{tikzcd}
b : B \vdash & \mathcal{H} \arrow[r, two heads, "\Sigma_{b'} |\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle", "\text{measurement collapse}"'] \arrow[rr, bend right, "\text{linear projector onto sub-Hilbert space } \mathcal{H}_b"'] & \mathcal{H}_b \arrow[r, hook, "|\psi_b\rangle \mapsto \oplus_{b'} \begin{cases} |\psi_b\rangle & \text{if } b'=b \\ 0 & \text{else} \end{cases}", "\text{state preparation}"'] & \mathcal{H},
\end{tikzcd}$$

measurement result

where $\mathcal{H} := \bigoplus\limits_{b':B} \mathcal{H}_{b'}$

- - -

randomly $\mathcal{H}$ **entails** **potentially $\mathcal{H}$** **entails** **indefinitely $\mathcal{H}$**

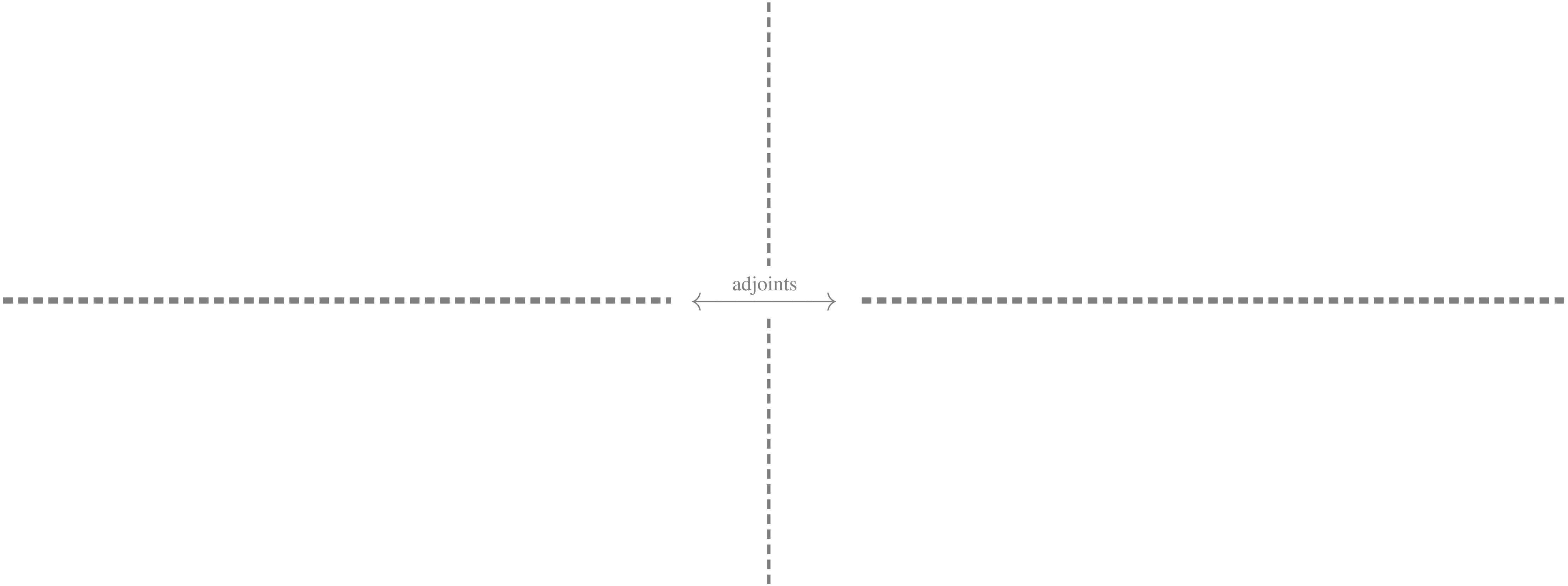
$$\begin{tikzcd}
\star_B \mathcal{H} \arrow[r, "\epsilon^{\star_B}_{\mathcal{H}}"] & \mathcal{H} \arrow[r, "\eta^{\bigcirc_B}_{\mathcal{H}}"] & \bigcirc_B \mathcal{H}
\end{tikzcd}$$

$$\begin{tikzcd}
\bigoplus\limits_{b:B} \mathcal{H} \arrow[r, "\oplus_b |\psi_b\rangle \mapsto \Sigma_b |\psi_b\rangle", "\text{quantum superposition}"'] & \mathcal{H} \arrow[r, "|\psi\rangle \mapsto \oplus_b |\psi\rangle_b", "\text{quantum parallelization}"'] & \bigoplus\limits_{b:B} \mathcal{H}
\end{tikzcd}$$

The pure effects of these modalities of dependent linear data type formation

are remarkable in their sheer quantum information-theoretic content.

To repeat:



The pure effects of these modalities of dependent linear data type formation

are remarkable in their sheer quantum information-theoretic content.

To repeat:

$$\overbrace{(p_B)^*(p_B)_*}^{\Box_B}\mathcal{H}_\bullet \xrightarrow[\textbf{necessity counit}]{\varepsilon^{\Box_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet$$

$$b : B \vdash \bigoplus_{b' : B} \mathcal{H}_{b'} \xrightarrow[\text{quantum } \textbf{measurement}]{\oplus_{b'}|\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle} \mathcal{H}_b$$

“ the necessary becomes actual ”

adjoints

$\longleftrightarrow$

The pure effects of these modalities of dependent linear data type formation

are remarkable in their sheer quantum information-theoretic content.

To repeat:

$$\overbrace{(p_B)^*(p_B)_*}^{\Box_B}\mathcal{H}_\bullet \xrightarrow[\textbf{necessity counit}]{\epsilon^{\Box_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet$$

$$b : B \vdash \bigoplus_{b':B} \mathcal{H}_{b'} \xrightarrow[\text{quantum }\textbf{measurement}]{\oplus_{b'}|\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle} \mathcal{H}_b$$

“ the necessary becomes actual ”

adjoints

$$\longleftrightarrow$$

$$\mathcal{H}_\bullet \xrightarrow[\textbf{possibility unit}]{\eta^{\Diamond_B}_{\mathcal{H}_\bullet}} \overbrace{(p_B)^*(p_B)_!}^{\Diamond_B}\mathcal{H}_\bullet$$

$$b : B \vdash \mathcal{H}_b \xrightarrow[\text{quantum }\textbf{state preparation}]{|\psi_b\rangle \;\mapsto\; \oplus_{b'}\left\{\begin{array}{ll}|\psi_b\rangle & \text{if } b'{=}b \\ 0 & \text{else}\end{array}\right.} \bigoplus_{b':B} \mathcal{H}_{b'}$$

“ the actual is possible ”

The pure effects of these modalities of dependent linear data type formation

are remarkable in their sheer quantum information-theoretic content.

To repeat:

$$\overbrace{(p_B)^*(p_B)_*}^{\Box_B}\mathcal{H}_\bullet \xrightarrow[\textbf{necessity counit}]{\epsilon^{\Box_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet$$

$$b : B \vdash \bigoplus_{b':B}\mathcal{H}_{b'} \xrightarrow[\text{quantum } \textbf{measurement}]{\oplus_{b'}|\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle} \mathcal{H}_b$$

“ the necessary becomes actual ”

$$\mathcal{H}_\bullet \xrightarrow[\textbf{possibility unit}]{\eta^{\Diamond_B}_{\mathcal{H}_\bullet}} \overbrace{(p_B)^*(p_B)_!}^{\Diamond_B}\mathcal{H}_\bullet$$

$$b : B \vdash \mathcal{H}_b \xrightarrow[\text{quantum } \textbf{state preparation}]{|\psi_b\rangle \;\mapsto\; \oplus_{b'}\begin{cases}|\psi_b\rangle & \text{if } b'=b\\ 0 & \text{else}\end{cases}} \bigoplus_{b':B}\mathcal{H}_{b'}$$

“ the actual is possible ”

$$\xleftrightarrow{\text{adjoints}}$$

“ the random becomes potential ”

$$\overbrace{(p_B)_!(p_B)^*}^{\star_B}\mathcal{H} \xhookrightarrow[\textbf{randomness counit}]{\epsilon^{\star_B}_{\mathcal{H}}} \mathcal{H}$$

$$\bigoplus_{b:B}\mathcal{H} \xrightarrow[\text{quantum } \textbf{superposition}]{\oplus_b|\psi_b\rangle \;\mapsto\; \Sigma_b\,|\psi_b\rangle} \mathcal{H}$$

The pure effects of these modalities of dependent linear data type formation

are remarkable in their sheer quantum information-theoretic content.

To repeat:

$$\overbrace{(p_B)^*(p_B)_*}^{\Box_B}\mathcal{H}_\bullet \xrightarrow[\textbf{necessity counit}]{\varepsilon^{\Box_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet$$

$$b : B \vdash \bigoplus_{b':B} \mathcal{H}_{b'} \xrightarrow[\text{quantum }\textbf{measurement}]{\oplus_{b'}|\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle} \mathcal{H}_b$$

“ the necessary becomes actual ”

$$\mathcal{H}_\bullet \xrightarrow[\textbf{possibility unit}]{\eta^{\Diamond_B}_{\mathcal{H}_\bullet}} \overbrace{(p_B)^*(p_B)_!}^{\Diamond_B}\mathcal{H}_\bullet$$

$$b : B \vdash \mathcal{H}_b \xrightarrow[\text{quantum }\textbf{state preparation}]{|\psi_b\rangle \;\mapsto\; \oplus_{b'}\begin{cases}|\psi_b\rangle & \text{if } b'=b\\ 0 & \text{else}\end{cases}} \bigoplus_{b':B}\mathcal{H}_{b'}$$

“ the actual is possible ”

$$\xleftrightarrow{\text{adjoints}}$$

“ the random becomes potential ”

$$\overbrace{(p_B)_!(p_B)^*}^{\star_B}\mathcal{H} \xhookrightarrow[\textbf{randomness counit}]{\varepsilon^{\star_B}_{\mathcal{H}}} \mathcal{H}$$

$$\bigoplus_{b:B}\mathcal{H} \xrightarrow[\text{quantum }\textbf{superposition}]{\oplus_b|\psi_b\rangle \;\mapsto\; \Sigma_b|\psi_b\rangle} \mathcal{H}$$

“ the potential is indeterminate ”

$$\mathcal{H} \xrightarrow[\textbf{indefiniteness unit}]{\eta^{\bigcirc_B}_{\mathcal{H}}} \overbrace{(p_B)_*(p_B)^*}^{\bigcirc_B}\mathcal{H}$$

$$\mathcal{H} \xrightarrow[\text{quantum }\textbf{parallelism}]{|\psi\rangle \;\mapsto\; \oplus_b|\psi\rangle_b} \bigoplus_{b:B}\mathcal{H}$$

Q-bits are the free linear indefiniteness-effect handlers over $\mathrm{Bit} = \{0, 1\}$

Coherent q-bits:

$$\begin{tikzcd} \mathrm{QBit} \arrow[d, phantom, "\rotatebox{90}{$=:$}"] & : & \mathrm{LType} \arrow[r, hook, "\mathbb{1}_{\mathrm{Bit}}\otimes"] & \mathrm{LType}_{\mathrm{Bit}} \arrow[r, "\bigoplus_{\mathrm{Bit}}", "\sim"'] & \mathrm{LType}^{\bigcirc_B} \\ \bigcirc_{\mathrm{Bit}}\mathbb{1} \end{tikzcd}$$

Quantum gate with q-bit output:

De-cohered (measured) q-bits:

Q-bits are the free linear indefiniteness-effect handlers over $\mathrm{Bit} = \{0, 1\}$

Coherent q-bits:

$$\mathrm{QBit} \;:\; \mathrm{LType} \xhookrightarrow{\mathbb{1}_{\mathrm{Bit}}\otimes} \mathrm{LType}_{\mathrm{Bit}} \xrightarrow[\sim]{\bigoplus_{\mathrm{Bit}}} \mathrm{LType}^{\bigcirc_B}$$

$$\overset{\cdot\,\cdot}{\Vert}$$

$$\bigcirc_{\mathrm{Bit}}\mathbb{1} \;=\; \bigoplus_{\{0,1\}}\mathbb{C} \;=\; \mathbb{C}\cdot|0\rangle \oplus \mathbb{C}\cdot|1\rangle$$

De-cohered (measured) q-bits:

Quantum gate with q-bit output:

Q-bits are the free linear indefiniteness-effect handlers over $\mathrm{Bit} = \{0, 1\}$

Coherent q-bits:

$$\mathrm{QBit} : \mathrm{LType} \xhookrightarrow{\mathbb{1}_{\mathrm{Bit}}{}^{\otimes}} \mathrm{LType}_{\mathrm{Bit}} \xrightarrow[\sim]{\bigoplus_{\mathrm{Bit}}} \mathrm{LType}^{\bigcirc_B}$$

$$\overset{\cdot\cdot}{\|}$$

$$\bigcirc_{\mathrm{Bit}} \mathbb{1} = \bigoplus_{\{0,1\}} \mathbb{C} = \mathbb{C} \cdot |0\rangle \oplus \mathbb{C} \cdot |1\rangle$$

De-cohered (measured) q-bits:

Quantum gate with q-bit output:

Q-bits are the free linear indefiniteness-effect handlers over $\mathrm{Bit} = \{0, 1\}$

Coherent q-bits:

$$\begin{array}{l}
\mathrm{QBit} \;:\; \mathrm{LType} \xhookrightarrow{\mathbb{1}_{\mathrm{Bit}}{}^{\otimes}} \mathrm{LType}_{\mathrm{Bit}} \xrightarrow[\sim]{\bigoplus_{\mathrm{Bit}}} \mathrm{LType}^{\bigcirc_B} \\
\overset{\cdot\cdot}{\|} \\
\bigcirc_{\mathrm{Bit}}\mathbb{1} \;=\; \bigoplus_{\{0,1\}}\mathbb{C} \;=\; \mathbb{C}\cdot|0\rangle \oplus \mathbb{C}\cdot|1\rangle
\end{array}$$

$$\begin{array}{l}
\mathrm{QBit} \\
\otimes \\
\mathcal{H} \\
\| \\
\bigcirc_{\mathrm{Bit}}\mathcal{H} \;=\; \bigoplus_{\{0,1\}}\mathcal{H} \;=\; \mathcal{H}\otimes|0\rangle \;\oplus\; \mathcal{H}\otimes|0\rangle
\end{array}$$

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━━━ $\mathrm{QBit} \;:\; \mathrm{LType} \xhookrightarrow{\mathbb{1}_{\mathrm{Bit}}\otimes} \mathrm{LType}_{\mathrm{Bit}} \xrightarrow[\sim]{\bigoplus_{\mathrm{Bit}}} \mathrm{LType}^{\bigcirc_B}$

$\overset{\cdot\cdot}{\|}$

$\bigcirc_{\mathrm{Bit}}\mathbb{1} \;=\; \bigoplus_{\{0,1\}}\mathbb{C} \;=\; \mathbb{C}\cdot|0\rangle \oplus \mathbb{C}\cdot|1\rangle$

━━━ QBit

$\otimes$

━━━ $\mathcal{H}$

$\|$

$\bigcirc_{\mathrm{Bit}}\mathcal{H} \;=\; \bigoplus_{\{0,1\}}\mathcal{H} \;=\; \mathcal{H}\otimes|0\rangle \;\oplus\; \mathcal{H}\otimes|0\rangle$

De-cohered (measured) q-bits:

═══ $\mathbb{1}_{\mathrm{Bit}} \;:\; \mathrm{LType}_{\mathrm{Bit}} \xrightarrow[\sim]{\bigoplus_{\mathrm{Bit}}} \mathrm{LType}^{\bigcirc_{\mathrm{Bit}}}$

$b : \mathrm{Bit} \quad\vdash\quad \mathbb{C}\cdot|b\rangle \;:\; \mathrm{LType}$

Quantum gate with q-bit output:

Q-bits are the free linear indefiniteness-effect handlers over $\mathrm{Bit} = \{0,1\}$

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$$\mathrm{QBit} \;:\; \mathrm{LType} \xhookrightarrow{\mathbb{1}_{\mathrm{Bit}}\otimes} \mathrm{LType}_{\mathrm{Bit}} \underset{\sim}{\xrightarrow{\bigoplus_{\mathrm{Bit}}}} \mathrm{LType}^{\bigcirc_B}$$

$$\overset{..}{\parallel}$$

$$\bigcirc_{\mathrm{Bit}} \mathbb{1} \;=\; \bigoplus_{\{0,1\}} \mathbb{C} \;=\; \mathbb{C}\cdot|0\rangle \oplus \mathbb{C}\cdot|1\rangle$$

$$\mathrm{QBit}$$

$$\otimes$$

$$\mathcal{H}$$

$$\parallel$$

$$\bigcirc_{\mathrm{Bit}} \mathcal{H} \;=\; \bigoplus_{\{0,1\}} \mathcal{H} \;=\; \mathcal{H}\otimes|0\rangle \;\oplus\; \mathcal{H}\otimes|0\rangle$$

De-cohered (measured) q-bits:

$$\mathbb{1}_{\mathrm{Bit}} \;:\; \mathrm{LType}_{\mathrm{Bit}} \underset{\sim}{\xrightarrow{\bigoplus_{\mathrm{Bit}}}} \mathrm{LType}^{\bigcirc_{\mathrm{Bit}}}$$

$$b : \mathrm{Bit} \quad\vdash\quad \mathbb{C}\cdot|b\rangle \;:\; \mathrm{LType}$$

$$\mathbb{1}_{\mathrm{Bit}}$$

$$\otimes \qquad b : \mathrm{Bit} \quad\vdash\quad \mathcal{H}\otimes|b\rangle \;:\; \mathrm{LType}$$

$$\mathcal{H}$$

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$$\overset{\cdot\cdot}{\|}$$

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$$\mathrm{QBit}$$

$$\otimes$$

$$\mathcal{H}$$

$$\|$$

$$\bigcirc_{\mathrm{Bit}}\mathcal{H} \;=\; \bigoplus_{\{0,1\}}\mathcal{H} \;=\; \mathcal{H}\otimes|0\rangle \;\oplus\; \mathcal{H}\otimes|0\rangle$$

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$$\mathbb{1}_{\mathrm{Bit}} \;:\; \mathrm{LType}_{\mathrm{Bit}} \underset{\sim}{\xrightarrow{\bigoplus_{\mathrm{Bit}}}} \mathrm{LType}^{\bigcirc_{\mathrm{Bit}}}$$

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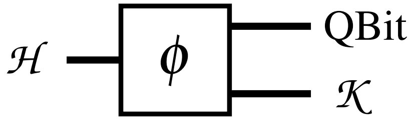
$$\mathbb{1}_{\mathrm{Bit}}$$

$$\otimes \qquad b : \mathrm{Bit} \quad\vdash\quad \mathcal{H}\otimes|b\rangle \;:\; \mathrm{LType}$$

$$\mathcal{H}$$

Quantum gate with q-bit output:

A quantum gate which
may handle $\bigcirc_{\mathrm{Bit}}$-effects
is one with a QBit-output:



$$\mathcal{H} \xrightarrow{\phi} \mathrm{QBit}\otimes\mathcal{K} \simeq \bigcirc_{\mathrm{Bit}}\mathcal{K}$$

Q-bits are the free linear indefiniteness-effect handlers over $\mathrm{Bit} = \{0,1\}$

Coherent q-bits:

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$$\overset{\cdot\cdot}{\|}$$

$$\bigcirc_{\mathrm{Bit}}\mathbb{1} \;=\; \bigoplus_{\{0,1\}}\mathbb{C} \;=\; \mathbb{C}\cdot|0\rangle \oplus \mathbb{C}\cdot|1\rangle$$

$$\mathrm{QBit}$$

$$\otimes$$

$$\mathcal{H}$$

$$\|$$

$$\bigcirc_{\mathrm{Bit}}\mathcal{H} \;=\; \bigoplus_{\{0,1\}}\mathcal{H} \;=\; \mathcal{H}\otimes|0\rangle \;\oplus\; \mathcal{H}\otimes|0\rangle$$

De-cohered (measured) q-bits:

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$$b : \mathrm{Bit} \quad\vdash\quad \mathbb{C}\cdot|b\rangle \;:\; \mathrm{LType}$$

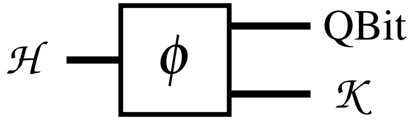
$$\mathbb{1}_{\mathrm{Bit}}$$

$$\otimes \qquad b : \mathrm{Bit} \quad\vdash\quad \mathcal{H}\otimes|b\rangle \;:\; \mathrm{LType}$$

$$\mathcal{H}$$

Quantum gate with q-bit output:

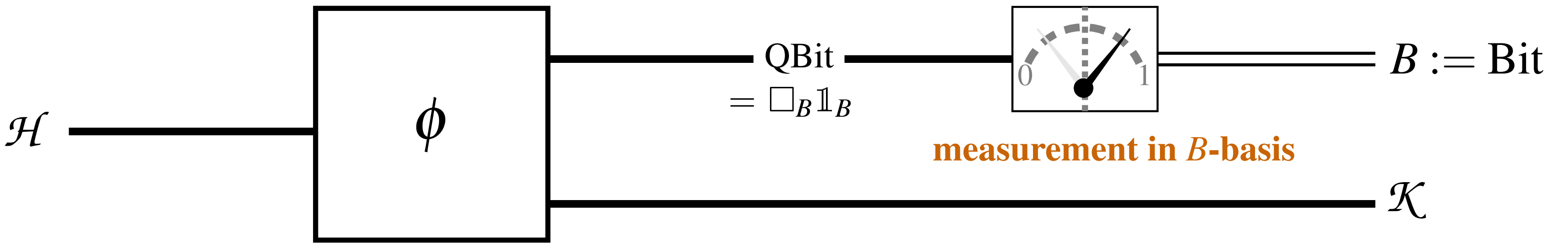
A quantum gate which may handle $\bigcirc_{\mathrm{Bit}}$-effects is one with a QBit-output:



$$\mathcal{H} \xrightarrow{\phi} \mathrm{QBit}\otimes\mathcal{K} \simeq \bigcirc_{\mathrm{Bit}}\mathcal{K}$$

Quantum measurement is Linear indefiniteness-effect handling.

quantum circuit



quantum gate

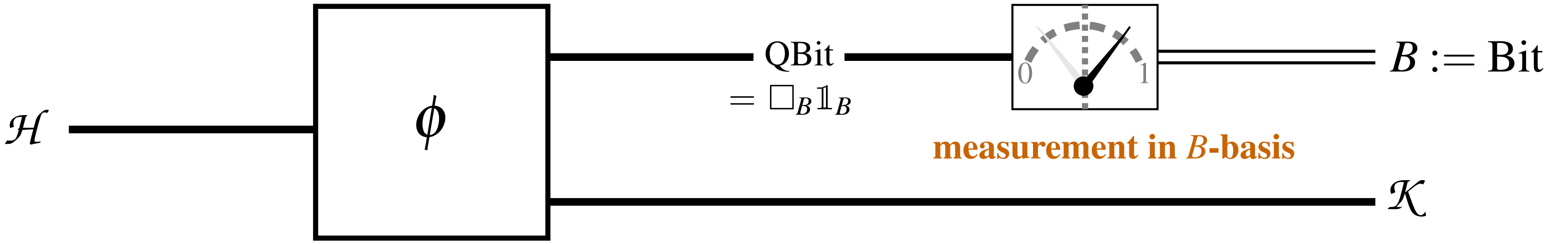
$$\mathcal{H} \xrightarrow{\phi} \mathrm{QBit} \otimes \mathcal{K} \simeq \bigcirc_B \mathcal{K}$$

$\bigcirc_B$-effect handling

Quantum measurement is Linear indefiniteness-effect handling.

$$\begin{tikzcd}
\text{quantum circuit} \arrow[d, squiggly, "\text{formalization}"] \\
\bigcirc_B\text{-modal linear types}
\end{tikzcd}$$

$$\mathrm{LType}_{\bigcirc_B}$$



measurement in B-basis

quantum gate

$\bigcirc_B$-effect handling

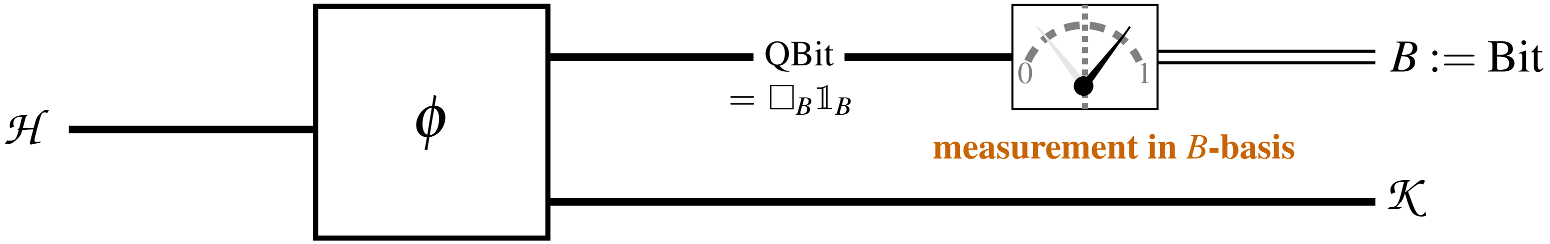
$$\mathcal{H} \xrightarrow{\phi} \mathrm{QBit} \otimes \mathcal{K} \simeq \bigcirc_B \mathcal{K}$$

Quantum measurement is Linear indefiniteness-effect handling.

$$\begin{tikzcd}
\textbf{quantum circuit} \arrow[d, squiggly, "\text{formalization}"] \\
\mathrm{LType}_{\bigcirc_B} \arrow[d, hook, "K_{(p_B)^*(p_B)_*}"', "\text{comparison functor}"] \\
\mathrm{LType}_B
\end{tikzcd}$$

$\bigcirc_B$-modal linear types $\mathrm{LType}_{\bigcirc_B}$

LType_B **B-dependent linear types**



subjection to $\bigcirc_B$-effects

quantum gate

$$\begin{tikzcd}
\bigcirc_B \mathcal{H} \arrow[loop above, "\bigcirc_B"] \arrow[r, "\bigcirc_B\left(\mathcal{H} \xrightarrow{\phi} \mathrm{QBit} \otimes \mathcal{K} \simeq \bigcirc_B \mathcal{K}\right)"] & \bigcirc_B \bigcirc_B \mathcal{K} \arrow[loop above, "\bigcirc_B"] \arrow[r, "\mathrm{hndl}^{\bigcirc_B}_{\bigcirc_B \mathcal{K}}", "\bigcirc_B\text{-effect handling}"'] & \bigcirc_B \mathcal{K} \arrow[loop above, "\bigcirc_B"]
\end{tikzcd}$$

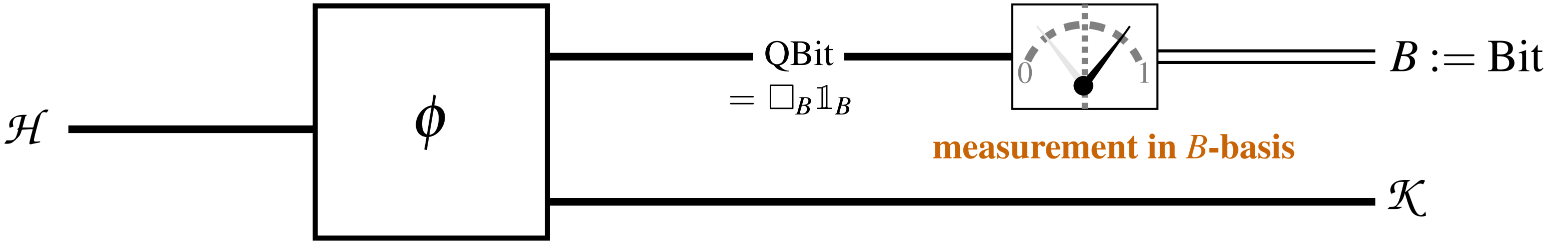
Quantum measurement is Linear indefiniteness-effect handling.

quantum circuit

$$\begin{tikzcd}
\text{quantum circuit} \arrow[d, squiggly, "\text{formalization}"] \\
\mathrm{LType}_{\bigcirc_B} \arrow[d, "K_{(p_B)^*(p_B)_*}"', "\text{comparison functor}"] \\
\mathrm{LType}_B
\end{tikzcd}$$

$\bigcirc_B$-modal linear types

B-dependent linear types



$$\begin{tikzcd}
\bigcirc_B \mathcal{H} \arrow[loop above, "\bigcirc_B"] \arrow[rr, "{\substack{\text{quantum gate} \\ \bigcirc_B\left(\mathcal{H} \xrightarrow{\phi} \mathrm{QBit} \otimes \mathcal{K} \simeq \bigcirc_B \mathcal{K}\right)}}"] & & \bigcirc_B \bigcirc_B \mathcal{K} \arrow[loop above, "\bigcirc_B"] \arrow[rr, "\mathrm{hndl}^{\bigcirc_B}_{\bigcirc_B \mathcal{K}}", "\bigcirc_B\text{-effect handling}"'] & & \bigcirc_B \mathcal{K} \arrow[loop above, "\bigcirc_B"]
\end{tikzcd}$$

subjection to $\bigcirc_B$-effects

$$\begin{tikzcd}
\mathbb{1}_B \otimes \mathcal{H} \arrow[rr, "\mathbb{1}_B \otimes \phi"] & & \Box_B \mathbb{1}_B \otimes \mathcal{K} \arrow[rr, "\epsilon^{\Box_B}_{\mathbb{1}_B} \otimes \mathcal{K}", "\Box_B\text{-counit}"'] & & \mathbb{1}_B \otimes \mathcal{K}
\end{tikzcd}$$

$$\begin{tikzcd}
b : B \vdash & \mathcal{H} \arrow[rr, "\phi"] & & \mathrm{QBit} \otimes \mathcal{K} \arrow[rr, "\text{quantum state collapse}"] & & \mathcal{K}
\end{tikzcd}$$

measurement result

$$\begin{array}{l} |0\rangle \otimes |\psi_0\rangle \\ + |1\rangle \otimes |\psi_1\rangle \end{array} \mapsto |\psi_b\rangle$$

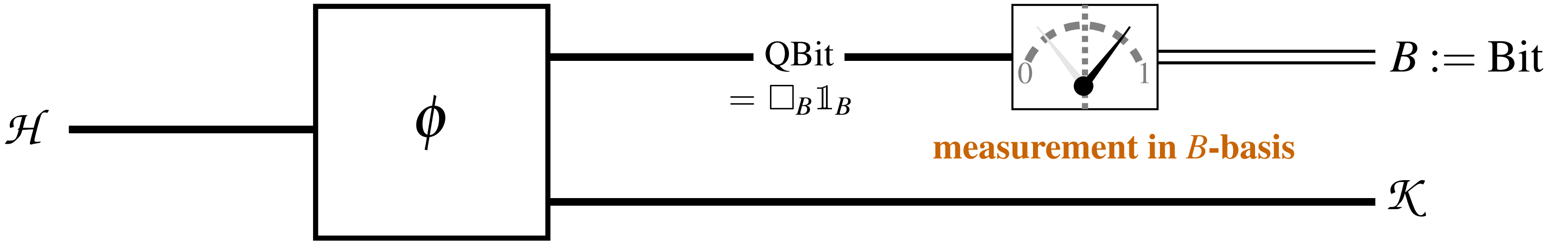
Quantum measurement is Linear indefiniteness-effect handling.

quantum circuit

$$\begin{tikzcd} \text{quantum circuit} \arrow[d, squiggly, "\text{formalization}"] \\ \mathrm{LType}_{\bigcirc_B} \arrow[d, "\text{comparison functor}", "K_{(p_B)^*(p_B)_*}"'] \\ \mathrm{LType}_B \end{tikzcd}$$

$\bigcirc_B$-modal linear types

B-dependent linear types



subjection to $\bigcirc_B$-effects **quantum gate**

$$\begin{tikzcd}[column sep=huge] \bigcirc_B \mathcal{H} \arrow[loop above, "\bigcirc_B"] \arrow[r, "\bigcirc_B\left(\mathcal{H} \xrightarrow{\phi} \mathrm{QBit} \otimes \mathcal{K} \simeq \bigcirc_B \mathcal{K}\right)"] & \bigcirc_B \bigcirc_B \mathcal{K} \arrow[loop above, "\bigcirc_B"] \arrow[r, "\mathrm{hndl}^{\bigcirc_B}_{\bigcirc_B \mathcal{K}}", "\bigcirc_B\text{-effect handling}"'] & \bigcirc_B \mathcal{K} \arrow[loop above, "\bigcirc_B"] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge] \mathbb{1}_B \otimes \mathcal{H} \arrow[r, "\mathbb{1}_B \otimes \phi"] & \square_B \mathbb{1}_B \otimes \mathcal{K} \arrow[r, "\varepsilon^{\square_B}_{\mathbb{1}_B} \otimes \mathcal{K}", "\square_B\text{-counit}"'] & \mathbb{1}_B \otimes \mathcal{K} \end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge] b : B \vdash \mathcal{H} \arrow[r, "\phi"] & \mathrm{QBit} \otimes \mathcal{K} \arrow[r, "\text{quantum state collapse}"] & \mathcal{K} \end{tikzcd}$$

measurement result

$$\begin{array}{l} |0\rangle \otimes |\psi_0\rangle \\ + |1\rangle \otimes |\psi_1\rangle \end{array} \mapsto |\psi_b\rangle$$

full linearly-typed detail of quantum measurement logic is emergent effect in LHoTT

Quantum measurement is Linear indefiniteness-effect handling.

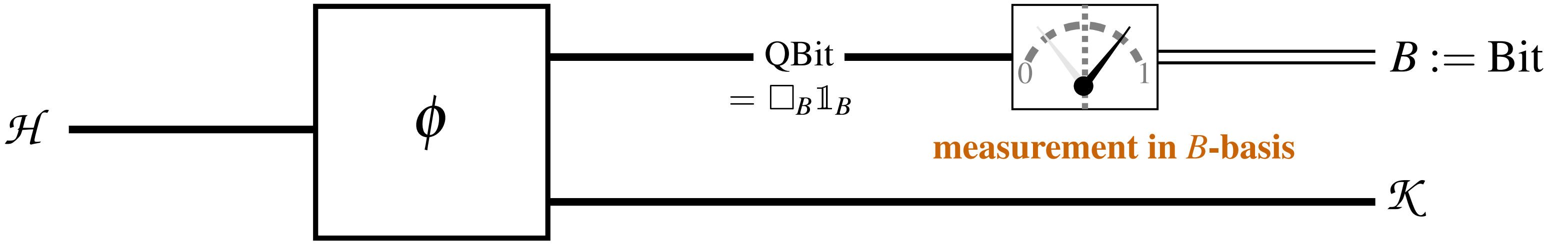
quantum circuit

$$\begin{tikzcd} \text{quantum circuit} \arrow[d, squiggly, "\text{formalization}"] \\ \bigcirc_B\text{-modal linear types} \end{tikzcd}$$

$\bigcirc_B$-modal linear types

$$\begin{tikzcd} \mathrm{LType}_{\bigcirc_B} \arrow[d, "K_{(p_B)^*(p_B)_*}"', "\text{comparison functor}"] \\ \mathrm{LType}_B \end{tikzcd}$$

B-dependent linear types



subjection to $\bigcirc_B$-effects

quantum gate

$$\begin{tikzcd} \bigcirc_B \mathcal{H} \arrow[loop above, "\bigcirc_B"] \arrow[r, "\bigcirc_B\left(\mathcal{H}\xrightarrow{\phi}\mathrm{QBit}\otimes\mathcal{K}\simeq\bigcirc_B\mathcal{K}\right)"] & \bigcirc_B\bigcirc_B\mathcal{K} \arrow[loop above, "\bigcirc_B"] \arrow[r, "\mathrm{hndl}^{\bigcirc_B}_{\bigcirc_B\mathcal{K}}", "\bigcirc_B\text{-effect handling}"'] & \bigcirc_B\mathcal{K} \arrow[loop above, "\bigcirc_B"] \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{1}_B \otimes \mathcal{H} \arrow[r, "\mathbb{1}_B\otimes\phi"] & \Box_B\mathbb{1}_B \otimes \mathcal{K} \arrow[r, "\varepsilon^{\Box_B}_{\mathbb{1}_B}\otimes\mathcal{K}", "\Box_B\text{-counit}"'] & \mathbb{1}_B \otimes \mathcal{K} \end{tikzcd}$$

$$\begin{tikzcd} b : B \vdash \mathcal{H} \arrow[r, "\phi"] & \mathrm{QBit}\otimes\mathcal{K} \arrow[r, "\text{quantum state collapse}"] & \mathcal{K} \end{tikzcd}$$

measurement result

$$\begin{array}{l} |0\rangle\otimes|\psi_0\rangle \\ +|1\rangle\otimes|\psi_1\rangle \end{array} \mapsto |\psi_b\rangle$$

full linearly-typed detail of
quantum measurement logic
is emergent effect in LHoTT

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$$
\begin{array}{c}
\bigcirc_B \\
\rotatebox[origin=c]{90}{$\in$} \\
\mathrm{Monad}\big(\mathrm{LType}\big)
\end{array}
$$

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$$
\begin{array}{c}
\bigcirc_B \\
\rotatebox{-90}{$=:$} \\
B\text{-Reader} \\
\\
\\
\rotatebox{-90}{$\in$} \\
\mathrm{Monad}\big(\mathrm{LType}\big)
\end{array}
$$

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$$B : \mathrm{FinType} \;\vdash\; \begin{tikzcd} \bigcirc_B \arrow[d, phantom, sloped, ":="] \\ B\text{-Reader} \arrow[d, phantom, sloped, "\simeq"] \\ \mathbb{1}^B\text{-Writer} \arrow[d, phantom, sloped, "\in"] \\ \mathrm{Monad}\big(\mathrm{LType}\big) \end{tikzcd}$$

$$\mathbb{1}^B\text{-Writer}(D) := \mathbb{1}^B \oplus D$$

$$\mathrm{bind}^{\mathbb{1}^B\mathrm{Writer}}\Big(D_1 \xrightarrow{\mathrm{prog}} \mathbb{1}^B \oplus D_2\Big) := \mathbb{1}^B \oplus D_1 \xrightarrow{\mathbb{1}^B \oplus \mathrm{prog}} \mathbb{1}^B \oplus \mathbb{1}^B \oplus D_2 \xrightarrow{\mu \oplus \mathrm{id}_{D_2}} \mathbb{1}^B \oplus D_2$$

Where $\mathbb{1}^B = \bigoplus\limits_{b:B} \mathbb{C} \cdot P_b \;\in\; \mathrm{CMon}\big(\mathrm{LType}\big)$ is algebra of B-projection operators :

unit

$$\mathbb{1} \xrightarrow[\;1 \mapsto \sum\limits_{b:B} P_b\;]{\eta} \mathbb{1}^B$$

product

$$\mathbb{1}^B \otimes \mathbb{1}^B \xrightarrow[\;P_b \otimes P_{b'} \mapsto \begin{cases} P_b & \text{if } b=b' \\ 0 & \text{else} \end{cases}\;]{\mu} \mathbb{1}^B$$

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$$\begin{tikzcd}
 & & \bigcirc_B \arrow[d, phantom, sloped, "=:"] \\
 & & B\text{-Reader} \arrow[d, phantom, sloped, "\simeq"] \\
 & & \mathbb{1}^B\text{-Writer} \arrow[dl, phantom, sloped, "\simeq"] \\
B : \text{FinType} \;\; \vdash & \mathbb{1}^B\text{-DualWriter} \arrow[dr, phantom, sloped, "\subset"] & \\
 & & \text{Monad}(\text{LType}) \\
 & \text{FrobMonad}(\text{LType}) \arrow[ur] &
\end{tikzcd}$$

$$\mathbb{1}^B\text{-Writer}(D) :\equiv \mathbb{1}^B \oplus D$$

$$\text{bind}^{\mathbb{1}^B\text{Writer}}\Big(D_1 \xrightarrow{\text{prog}} \mathbb{1}^B \oplus D_2\Big) :\equiv$$

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Where $\mathbb{1}^B = \bigoplus_{b:B} \mathbb{C} \cdot P_b \;\in\; \text{CMon}(\text{LType})$ is Frobenius algebra of B-projection operators :

$$\begin{tikzcd}[column sep=huge]
\mathbb{1} \arrow[r, "\overset{\textbf{unit}}{\eta}", "1 \mapsto \sum_{b:B} P_b"'] & \mathbb{1}^B \arrow[r, "\overset{\textbf{co-product}}{\delta}", "P_b \mapsto P_b \otimes P_b"'] & \mathbb{1}^B \otimes \mathbb{1}^B \arrow[r, "\overset{\textbf{product}}{\mu}", "P_b \otimes P_{b'} \mapsto \begin{cases} P_b & \text{if } b=b' \\ 0 & \text{else} \end{cases}"'] & \mathbb{1}^B \arrow[r, "\overset{\textbf{co-unit}}{\varepsilon}", "P_b \mapsto 1"'] & \mathbb{1}
\end{tikzcd}$$

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$$\begin{tikzcd}
 & & \bigcirc_B \arrow[d, phantom, "\shortparallel"] \\
 & & B\text{-Reader} \arrow[d, phantom, "\wr"] \\
 & \mathbb{1}^B\text{-CoWriter} & & \mathbb{1}^B\text{-Writer} \\
B : \text{FinType} \;\; \vdash & & \mathbb{1}^B\text{-DualWriter} \arrow[ul, phantom, "\simeq"] \arrow[ur, phantom, "\simeq"] \arrow[d, phantom, "\Cap"] & \\
 & \text{CoMonad}\big(\text{LType}\big) & & \text{Monad}\big(\text{LType}\big) \\
 & & \text{FrobMonad}\big(\text{LType}\big) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$$\mathbb{1}^B\text{-Writer}(D) :\equiv \mathbb{1}^B \oplus D$$

$$\text{bind}^{\mathbb{1}^B\text{Writer}}\Big(D_1 \xrightarrow{\text{prog}} \mathbb{1}^B \oplus D_2\Big) :\equiv$$

$$\mathbb{1}^B \oplus D_1 \xrightarrow{\mathbb{1}^B \oplus \text{prog}} \mathbb{1}^B \oplus \mathbb{1}^B \oplus D_2 \xrightarrow{\mu \oplus \text{id}_{D_2}} \mathbb{1}^B \oplus D_2$$

Where $\mathbb{1}^B = \bigoplus_{b:B} \mathbb{C} \cdot P_b \;\in\; \text{CMon}(\text{LType})$ is Frobenius algebra of B-projection operators :

$$\mathbb{1} \xrightarrow[1 \mapsto \sum_{b:B} P_b]{\overset{\textbf{unit}}{\eta}} \mathbb{1}^B \xrightarrow[P_b \mapsto P_b \otimes P_b]{\overset{\textbf{co-product}}{\delta}} \mathbb{1}^B \otimes \mathbb{1}^B \xrightarrow[P_b \otimes P_{b'} \mapsto \begin{cases} P_b \text{ if } b=b' \\ 0 \text{ else} \end{cases}]{\overset{\textbf{product}}{\mu}} \mathbb{1}^B \xrightarrow[P_b \mapsto 1]{\overset{\textbf{co-unit}}{\varepsilon}} \mathbb{1}$$

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$$\begin{tikzcd}
 & & & \bigcirc_B \arrow[d, phantom, "\rotatebox{90}{$=:$}"] \\
 & B\text{-CoReader} \arrow[d, phantom, "\rotatebox{90}{$\simeq$}"] & & B\text{-Reader} \arrow[d, phantom, "\rotatebox{90}{$\simeq$}"] \\
 & \mathbb{1}^B\text{-CoWriter} & & \mathbb{1}^B\text{-Writer} \\
B : \mathrm{FinType} \;\vdash & & \mathbb{1}^B\text{-DualWriter} \arrow[ul, phantom, sloped, "\simeq"] \arrow[ur, phantom, sloped, "\simeq"] & \\
 & \mathrm{CoMonad}\big(\mathrm{LType}\big) & \rotatebox{-90}{$\in$} & \mathrm{Monad}\big(\mathrm{LType}\big) \\
 & & \mathrm{FrobMonad}\big(\mathrm{LType}\big) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$$\mathbb{1}^B\text{-Writer}(D) := \mathbb{1}^B \oplus D$$

$$\mathrm{bind}^{\mathbb{1}^B\mathrm{Writer}}\Big(D_1 \xrightarrow{\mathrm{prog}} \mathbb{1}^B \oplus D_2\Big) := \mathbb{1}^B \oplus D_1 \xrightarrow{\mathbb{1}^B \oplus \mathrm{prog}} \mathbb{1}^B \oplus \mathbb{1}^B \oplus D_2 \xrightarrow{\mu \oplus \mathrm{id}_{D_2}} \mathbb{1}^B \oplus D_2$$

Where $\mathbb{1}^B = \bigoplus_{b:B} \mathbb{C} \cdot P_b \in \mathrm{CMon}(\mathrm{LType})$ is Frobenius algebra of B-projection operators :

$$\mathbb{1} \xrightarrow[1 \mapsto \sum_{b:B} P_b]{\overset{\textbf{unit}}{\eta}} \mathbb{1}^B \xrightarrow[P_b \mapsto P_b \otimes P_b]{\overset{\textbf{co-product}}{\delta}} \mathbb{1}^B \otimes \mathbb{1}^B \xrightarrow[P_b \otimes P_{b'} \mapsto \begin{cases} P_b \text{ if } b = b' \\ 0 \text{ else} \end{cases}]{\overset{\textbf{product}}{\mu}} \mathbb{1}^B \xrightarrow[P_b \mapsto 1]{\overset{\textbf{co-unit}}{\varepsilon}} \mathbb{1}$$

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$B : \mathrm{FinType} \;\; \vdash$

$$\begin{tikzcd}
\star_B \arrow[d, phantom, sloped, "\eqqcolon"] & & \bigcirc_B \arrow[d, phantom, sloped, "\eqqcolon"] \\
B\text{-CoReader} \arrow[d, phantom, sloped, "\simeq"] & & B\text{-Reader} \arrow[d, phantom, sloped, "\simeq"] \\
\mathbb{1}^B\text{-CoWriter} \arrow[dr, phantom, sloped, "\simeq"] & & \mathbb{1}^B\text{-Writer} \arrow[dl, phantom, sloped, "\simeq"] \\
& \mathbb{1}^B\text{-DualWriter} \arrow[d, phantom, sloped, "\in"] & \\
\mathrm{CoMonad}\big(\mathrm{LType}\big) & & \mathrm{Monad}\big(\mathrm{LType}\big) \\
& \mathrm{FrobMonad}\big(\mathrm{LType}\big) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$\mathbb{1}^B\text{-Writer}(D) := \mathbb{1}^B \oplus D$

$\mathrm{bind}^{\mathbb{1}^B\mathrm{Writer}}\Big(D_1 \xrightarrow{\mathrm{prog}} \mathbb{1}^B \oplus D_2\Big) :=$

$\mathbb{1}^B \oplus D_1 \xrightarrow{\mathbb{1}^B \oplus \mathrm{prog}} \mathbb{1}^B \oplus \mathbb{1}^B \oplus D_2 \xrightarrow{\mu \oplus \mathrm{id}_{D_2}} \mathbb{1}^B \oplus D_2$

Where $\mathbb{1}^B = \bigoplus_{b:B} \mathbb{C} \cdot P_b \;\in\; \mathrm{CMon}\big(\mathrm{LType}\big)$ is Frobenius algebra of B-projection operators :

$$\mathbb{1} \xrightarrow[1 \,\mapsto\, \sum\limits_{b:B} P_b]{\overset{\textbf{unit}}{\eta}} \mathbb{1}^B \xrightarrow[P_b \,\mapsto\, P_b \otimes P_b]{\overset{\textbf{co-product}}{\delta}} \mathbb{1}^B \otimes \mathbb{1}^B \xrightarrow[P_b \otimes P_{b'} \,\mapsto\, \begin{cases} P_b & \text{if } b = b' \\ 0 & \text{else} \end{cases}]{\overset{\textbf{product}}{\mu}} \mathbb{1}^B \xrightarrow[P_b \,\mapsto\, 1]{\overset{\textbf{co-unit}}{\varepsilon}} \mathbb{1}$$

Aside: **Linear indefiniteness monad recovers Coecke’s “classical structures”.**

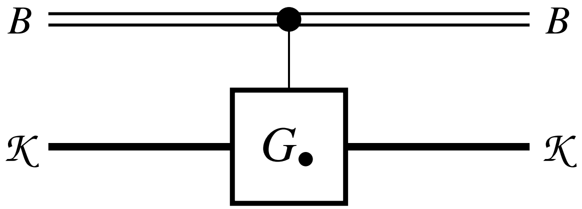
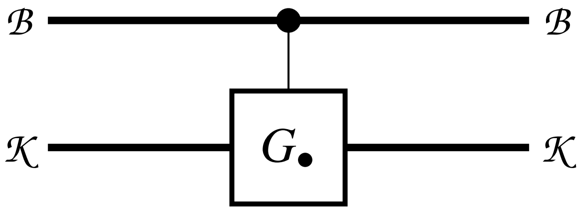
(see nLab:quantum+reader+monad)

$$\begin{tikzcd}
 & & \star_B \arrow[rr, equal] \arrow[dr, equal] & & \bigcirc_B \arrow[dl, equal] \\
B : \mathrm{FinType} \;\vdash & & & \mathbb{1}^B\text{-DualWriter} & \\
 & & \mathrm{CoMonad}\big(\mathrm{LType}\big) & \rotatebox{90}{$\in$} & \mathrm{Monad}\big(\mathrm{LType}\big) \\
 & & & \mathrm{FrobMonad}\big(\mathrm{LType}\big) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Where $\mathbb{1}^B = \bigoplus_{b:B} \mathbb{C} \cdot P_b \;\in\; \mathrm{CMon}(\mathrm{LType})$ is Frobenius algebra of B-projection operators :

$$\begin{tikzcd}[column sep=huge]
\mathbb{1} \arrow[r, "\overset{\textbf{unit}}{\eta}", "1 \mapsto \sum\limits_{b:B} P_b"'] & \mathbb{1}^B \arrow[r, "\overset{\textbf{co-product}}{\delta}", "P_b \mapsto P_b \otimes P_b"'] & \mathbb{1}^B \otimes \mathbb{1}^B \arrow[r, "\overset{\textbf{product}}{\mu}", "P_b \otimes P_{b'} \mapsto \begin{cases} P_b \text{ if } b = b' \\ 0 \text{ else} \end{cases}"'] & \mathbb{1}^B \arrow[r, "\overset{\textbf{co-unit}}{\varepsilon}", "P_b \mapsto 1"'] & \mathbb{1}
\end{tikzcd}$$

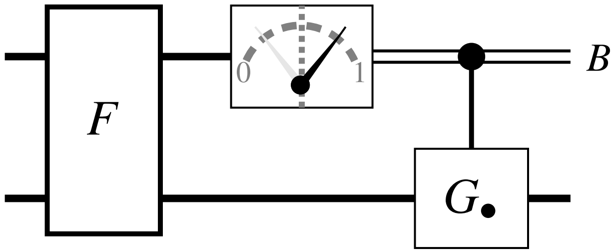
Exmp: Deferred measurement principle – Proven by monadic effect logic.

classically controlled gate	**quantumly controlled gate**
 $\mathcal{B}_\bullet \boxtimes \mathcal{K} \xrightarrow{G_\bullet} \mathcal{B}_\bullet \boxtimes \mathcal{K}$ $b : B \;\vdash\; \mathcal{K} \xrightarrow{G_b} \mathcal{K}$	 $\Box_B \mathcal{B}_\bullet \boxtimes \mathcal{K} \xrightarrow{\Box_B G_\bullet} \Box_B \mathcal{B}_\bullet \boxtimes \mathcal{K}$ $b : B \;\vdash\; \bigoplus_{b' B} \mathcal{K} \xrightarrow{\bigoplus_{b' : B} G_{b'}} \bigoplus_{b' B} \mathcal{K}$

Exmp: Deferred measurement principle – Proven by monadic effect logic.

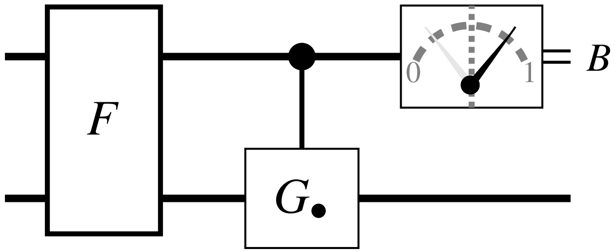
$$\Box_B \mathcal{H}_\bullet \xrightarrow{F} \Box_B \mathcal{H}_\bullet \xrightarrow{\varepsilon^{\Box_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet \xrightarrow{G_\bullet} \mathcal{H}_\bullet \quad \mapsto \quad \Box_B \mathcal{H}_\bullet \xrightarrow{\mathrm{diag}_B(G_\bullet)\circ F} \Box_B \mathcal{H}_\bullet \quad \mapsto \quad \Box_B \mathcal{H}_\bullet \xrightarrow{\mathrm{diag}_B(G_\bullet)\circ F} \Box_B \mathcal{H}_\bullet \xrightarrow{\varepsilon^{\Box_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet$$

measurement-controlled quantum gate **quantum-controlled quantum gate...** **...followed by measurement**

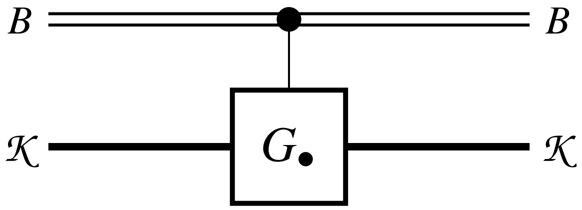


Deferred Measurement Principle

$\longleftrightarrow$



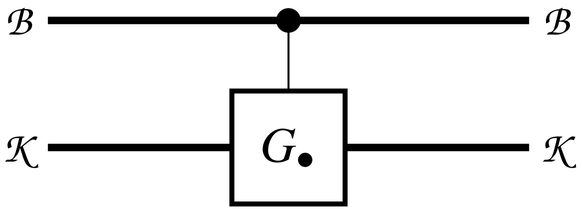
classically controlled gate



$$\mathcal{B}_\bullet \boxtimes \mathcal{K} \xrightarrow{\quad G_\bullet \quad} \mathcal{B}_\bullet \boxtimes \mathcal{K}$$

$$b : B \;\vdash\; \mathcal{K} \xrightarrow{\quad G_b \quad} \mathcal{K}$$

quantumly controlled gate



$$\Box_B \mathcal{B}_\bullet \boxtimes \mathcal{K} \xrightarrow{\quad \Box_B G_\bullet \quad} \Box_B \mathcal{B}_\bullet \boxtimes \mathcal{K}$$

$$b : B \;\vdash\; \bigoplus_{b' B} \mathcal{K} \xrightarrow{\quad \bigoplus_{b' : B} G_{b'} \quad} \bigoplus_{b' B} \mathcal{K}$$

Exmp: Deferred measurement principle – Proven by monadic effect logic.

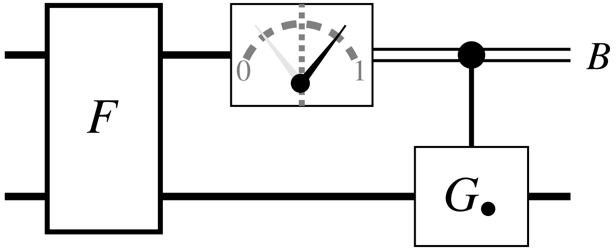
$$\begin{tikzcd} \mathrm{Kl}(\square_B) \arrow[r, "\sim", "\delta^B \circ \square_B(-)"'] \arrow[rr, bend left, "\mathrm{id}"] & \mathrm{LType}_{B\square_B} \arrow[r, "\sim", "\varepsilon^{\square_B} \circ (-)"'] & \mathrm{Kl}(\square_B) \end{tikzcd}$$

Kleisli equivalence

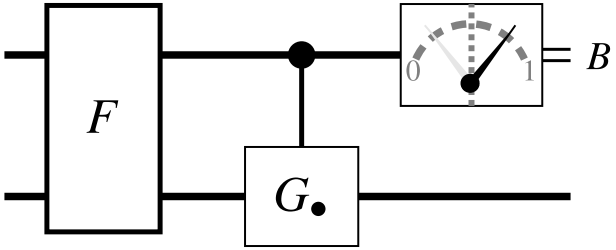
$\square_B$-Kleisli morphisms **$\square_B$-coalgebra homomorphisms** **$\square_B$-Kleisli morphisms**

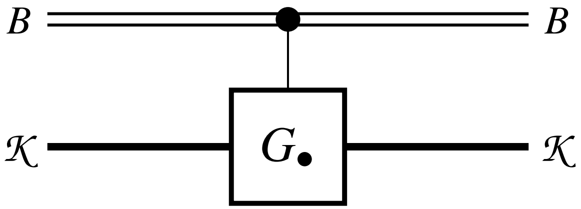
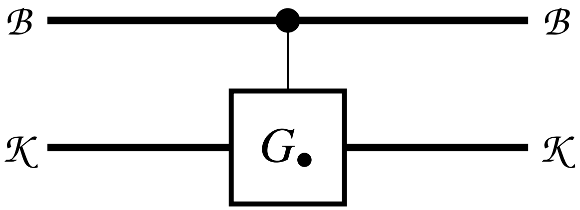
$$\square_B \mathcal{H}_\bullet \xrightarrow{F} \square_B \mathcal{H}_\bullet \xrightarrow{\varepsilon^{\square_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet \xrightarrow{G_\bullet} \mathcal{H}_\bullet \quad \mapsto \quad \square_B \mathcal{H}_\bullet \xrightarrow{\mathrm{diag}_B(G_\bullet) \circ F} \square_B \mathcal{H}_\bullet \quad \mapsto \quad \square_B \mathcal{H}_\bullet \xrightarrow{\mathrm{diag}_B(G_\bullet) \circ F} \square_B \mathcal{H}_\bullet \xrightarrow{\varepsilon^{\square_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet$$

measurement-controlled quantum gate **quantum-controlled quantum gate...** **...followed by measurement**



Deferred Measurement Principle ⟷



classically controlled gate	**quantumly controlled gate**
 $\mathcal{B}_\bullet \boxtimes \mathcal{K} \xrightarrow{G_\bullet} \mathcal{B}_\bullet \boxtimes \mathcal{K}$ $b : B \vdash \quad \mathcal{K} \xrightarrow{G_b} \mathcal{K}$	 $\square_B \mathcal{B}_\bullet \boxtimes \mathcal{K} \xrightarrow{\square_B G_\bullet} \square_B \mathcal{B}_\bullet \boxtimes \mathcal{K}$ $b : B \vdash \quad \bigoplus_{b' B} \mathcal{K} \xrightarrow{\bigoplus_{b' : B} G_{b'}} \bigoplus_{b' B} \mathcal{K}$

The Quantum modality.

Also the *exponential modality* traditionally postulated in linear logic is an emergent effect in `LHoTT`,

linear randomization
aka: stabilization/motivization

$$\begin{tikzcd} \mathrm{Type} \arrow[r, shift left=2, "\Sigma^\infty_+ \,:\, B \,\mapsto\, \star_B \mathbb{1}"] \arrow[r, phantom, "\perp"] & \mathrm{LType} \arrow[l, shift left=2, "\Omega^\infty"] \arrow[loop right, "!"] \end{tikzcd}$$

exponential modality

classical types

linear types

The Quantum modality.

Also the *exponential modality* traditionally postulated in linear logic
is an emergent effect in `LHoTT`,
as is the crucial *Quantum Modality*, not considered before:

linear randomization

aka: stabilization/motivization

quantum modality

$$\begin{tikzcd} \mathrm{Type} \arrow[loop left, "Q"] \arrow[r, shift left=2, "\Sigma^\infty_+ \,:\, B \,\mapsto\, \star_B \mathbb{1}"] \arrow[r, phantom, "\perp"] & \mathrm{LType} \arrow[l, shift left=2, "\Omega^\infty"] \arrow[loop right, "!"] \end{tikzcd}$$

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exponential modality

classical types

linear types

The Q-monad plays a crucial role in the full formulation of the `QS`-language.
It is the secret actor behind QBit $=$ Q(Bit)...

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exponential modality

classical types

linear types

The Q-monad plays a crucial role in the full formulation of the `QS`-language.
It is the secret actor behind QBit $=$ Q(Bit)...

Quantum Circuits

Quantum effects are compatible with tensor product.

Linear Randomness and Indefiniteness are “very strong” effects, in that:

$$\bigcirc_B\big(D \otimes D'\big) \;\simeq\; \big(\bigcirc_B D\big) \otimes D' \,, \qquad \star_B\big(D \otimes D'\big) \;\simeq\; \big(\star_B D\big) \otimes D'$$

There is a whole system of them:

$$\bigcirc_B \bigcirc_{B'} \;\simeq\; \bigcirc_{B\times B'} , \quad \text{NB: } \bigcirc_B \bigcirc'_B \;\simeq\; \bigcirc_B \mathbb{1} \otimes \bigcirc'_B$$

which under dynamic lifting (monadicity comparison functor)
gives the external tensor product of dependent linear types:

free $\bigcirc_B$-effect handlers
in linear data types

comparison
functor

B-dependent
linear data types

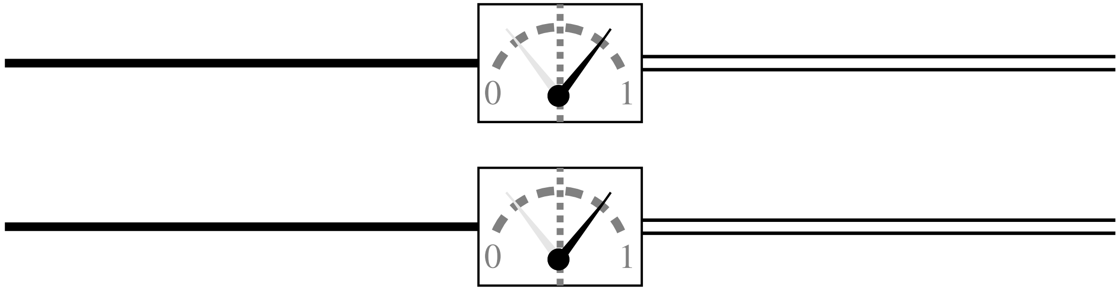
$$\begin{tikzcd}
\mathrm{LType}_{\bigcirc_B} \arrow[d, "K_{\oplus_B \mathbb{1}_B \otimes}"] & \overset{\curvearrowright\;\bigcirc_{B\times B'}}{\bigcirc_{B\times B'} \bigcirc_{B\times B'} \mathcal{H}} \arrow[d, mapsto] \\
\mathrm{LType}_B & \big(\Box_B \mathbb{1}_B\big) \boxtimes \big(\Box_{B'} \mathbb{1}_{B'}\big) \otimes \mathcal{H}
\end{tikzcd}$$

external
tensor product

Quantum circuits with classical control & effects

are the *effectful* string diagrams in the linear type system

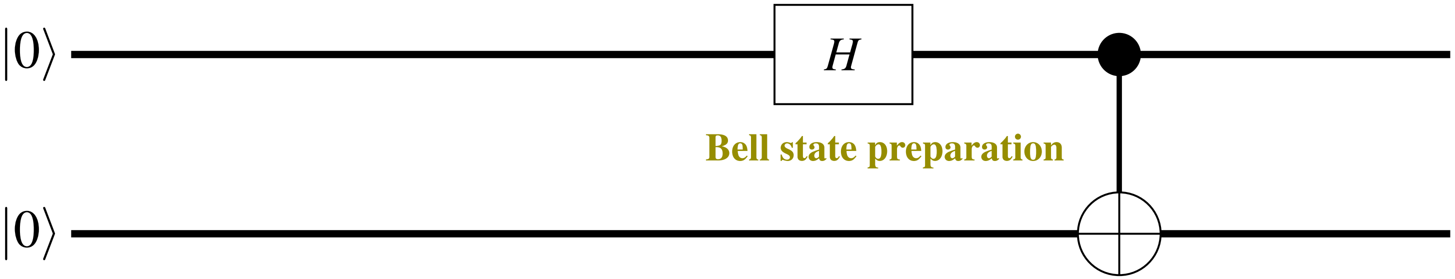
E.g.
The dependent linear type of a measurement on a pair of qbits:



$$\overset{\text{type of a pair of coherent qbits}}{\Box_{\mathrm{Bit}^2}\big(\mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet\big)} \xrightarrow{\overset{\text{pair of measurements}}{\varepsilon_{\mathrm{Bit}^2}(\mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet)}} \overset{\text{type of collapsed qbits dependent on measured bits } b, b'}{\mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet}$$

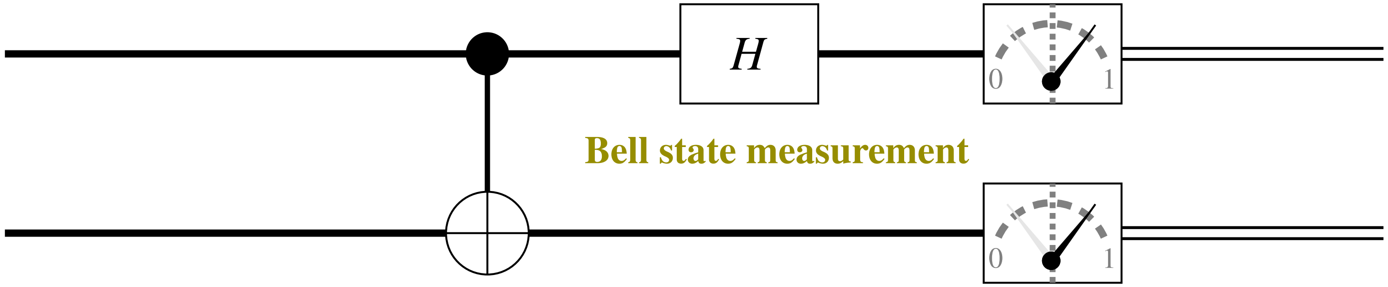
$$\overset{\text{measured bits}}{(b,b') : \mathrm{Bit}^2} \;\;\vdash\;\; \Box_{\mathrm{Bit}^2}\big(\mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet\big)_{(b,b')} \simeq \mathbb{C}^2 \otimes \mathbb{C}^2 \xrightarrow[\text{collapse of the quantum state}]{\sum_{d,d'} q_{dd'}\;|d\rangle\otimes|d'\rangle \;\mapsto\; q_{bb'}\;|b\rangle\otimes|b'\rangle} \mathbb{C}\,.$$

Example: Bell states of q-bits are typed as follows (regarded in $\mathrm{LType}_{\mathrm{Bit}\times\mathrm{Bit}}$):



$$\mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet \longrightarrow \big(\lozenge_{\mathrm{Bit}}\mathrm{QBit}_\bullet\big) \boxtimes \big(\lozenge_{\mathrm{Bit}}\mathrm{QBit}_\bullet\big) \simeq \square_{\mathrm{Bit}^2}\big(\mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet\big) \longrightarrow \square_{\mathrm{Bit}^2}\big(\mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet\big)$$

$$b, b' : \mathrm{Bit} \vdash \;\; \mathbb{C} \xrightarrow{\;1 \;\mapsto\; |0\rangle\otimes|0\rangle \;\mapsto\; \frac{1}{\sqrt{2}}\big(|0\rangle+|1\rangle\big)\otimes|0\rangle \;\mapsto\; \frac{1}{\sqrt{2}}\big(|0\rangle\otimes|0\rangle+|1\rangle\otimes|1\rangle\big)\;} \mathbb{C}^2\otimes\mathbb{C}^2$$

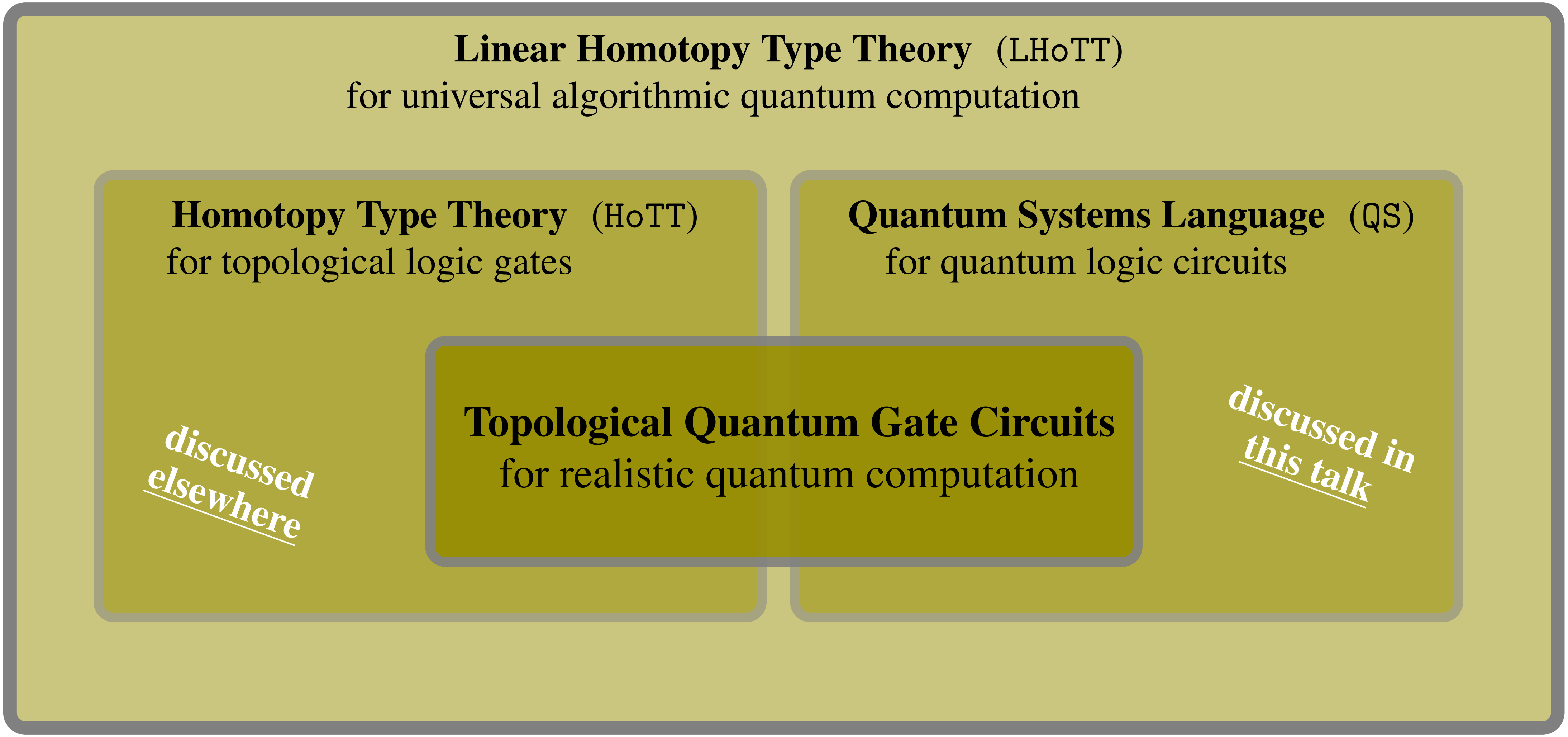


$$\square_{\mathrm{Bit}^2}\big(\mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet\big) \longrightarrow \mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet$$

$$b_1, b_2 : \mathrm{Bit} \quad \vdash \quad \mathbb{C}^2\otimes\mathbb{C}^2 \xrightarrow{\;\Sigma_{b'_1 b'_2} q_{b'_1 b'_2}\cdot|b'_1\rangle\otimes|b'_2\rangle \;\mapsto\; \big(q_{0,b_2}+(-1)^{b_1}\cdot q_{1,(1-b_2)}\big)\cdot|b_1\rangle\otimes|b_2\rangle\;} \mathbb{C}$$

`QS` – Quantum Systems language @ CQTS

⇝ full-blown Quantum Systems language emerges embedded in `LHoTT`



Linear Homotopy Type Theory (`LHoTT`)
for universal algorithmic quantum computation

Homotopy Type Theory (`HoTT`)
for topological logic gates

Quantum Systems Language (`QS`)
for quantum logic circuits

Topological Quantum Gate Circuits
for realistic quantum computation

Effective Quantum Certification via Linear Homotopy Types

Urs Schreiber (NYU Abu Dhabi)
on joint work at CQTS with
D. J. Myers, M. Riley,
and Hisham Sati

Thanks!

CENTER FOR
QUANTUM &
TOPOLOGICAL
SYSTEMS

presentation at:

The Topos Institute Colloquium, 13 Apr 2023

slides and further pointers at: ncatlab.org/Quantum+Certification+via+Linear+Homotopy+Types#TI2023