Urs Schreiber

(New York University, Abu Dhabi & Czech Academy of Science, Prague)

Microscopic brane physics from Cohomotopy

talk at

M-Theory and Mathematics

NYU AD 2020

based on joint work with

H. Sati

0) Introduction

1) Microscopic Brane Charge

2) Orientifold Tadpole Cancellation

3) D6 $\perp$ D8 **-Brane Intersections**

4) Hanany-Witten Theory

5) Chan-Paton Data

6) BMN Matrix Model States

7) M2/M5 Brane Bound States

(0)

Introduction

Open Problem M and Hypothesis H

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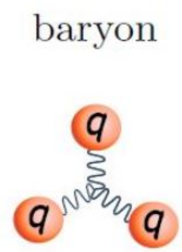
Open problem QCD

Confined QCD

"Millennium problem"

- QCD-cosmology
- nucleosynthesis
- form factors
- ...



Flavored QCD

"Flavor problem"

Higgs sector
- cosm. constant
- EW hierarchy
- vacuum stability
- V_{cb}-puzzle
- flavour anomalies

$\xrightarrow{\text{Leptoquark}}$ GUT

- ...

???

Solution

Open problem QCD $\xrightarrow[\text{'t Hooft doubling}]{}$

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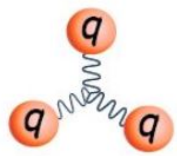
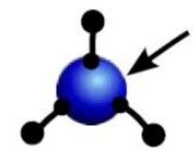
- . . .

baryon



D4 with N_c strings



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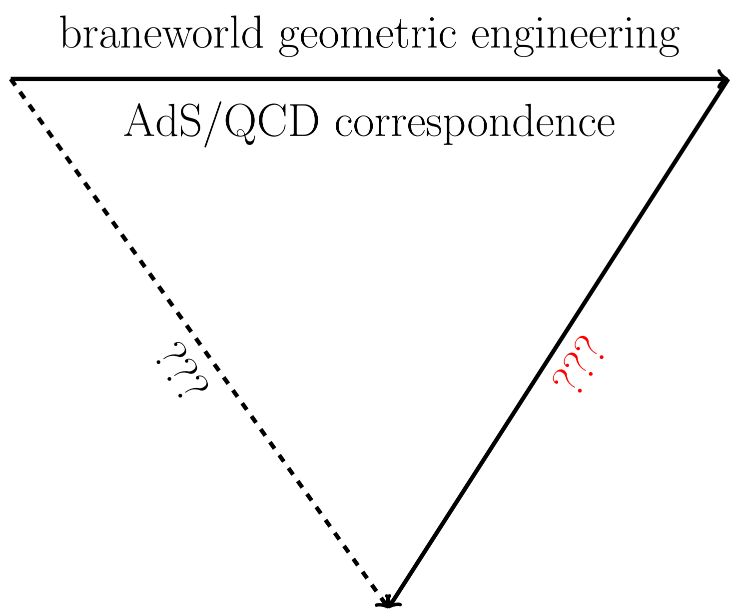
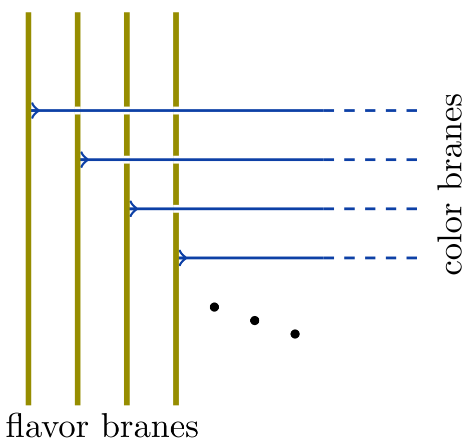
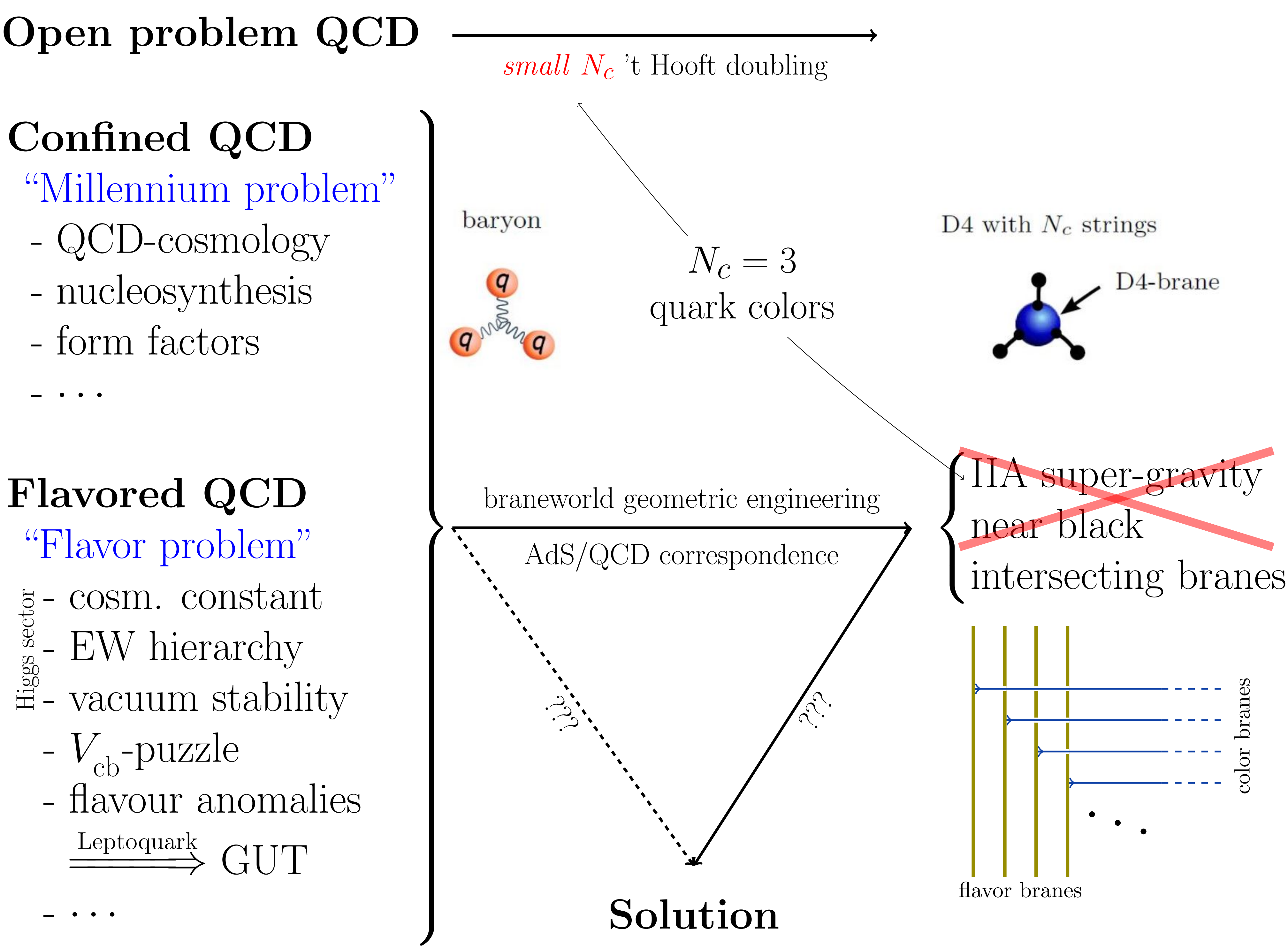


IIA super-gravity
near black
intersecting branes



Solution



Open problem QCD
small N_c 't Hooft doubling
Confined QCD
"Millennium problem"
- QCD-cosmology
- nucleosynthesis
- form factors
- ...
baryon
q
q
q
$N_c = 3$
quark colors
D4 with N_c strings
D4-brane
Flavored QCD
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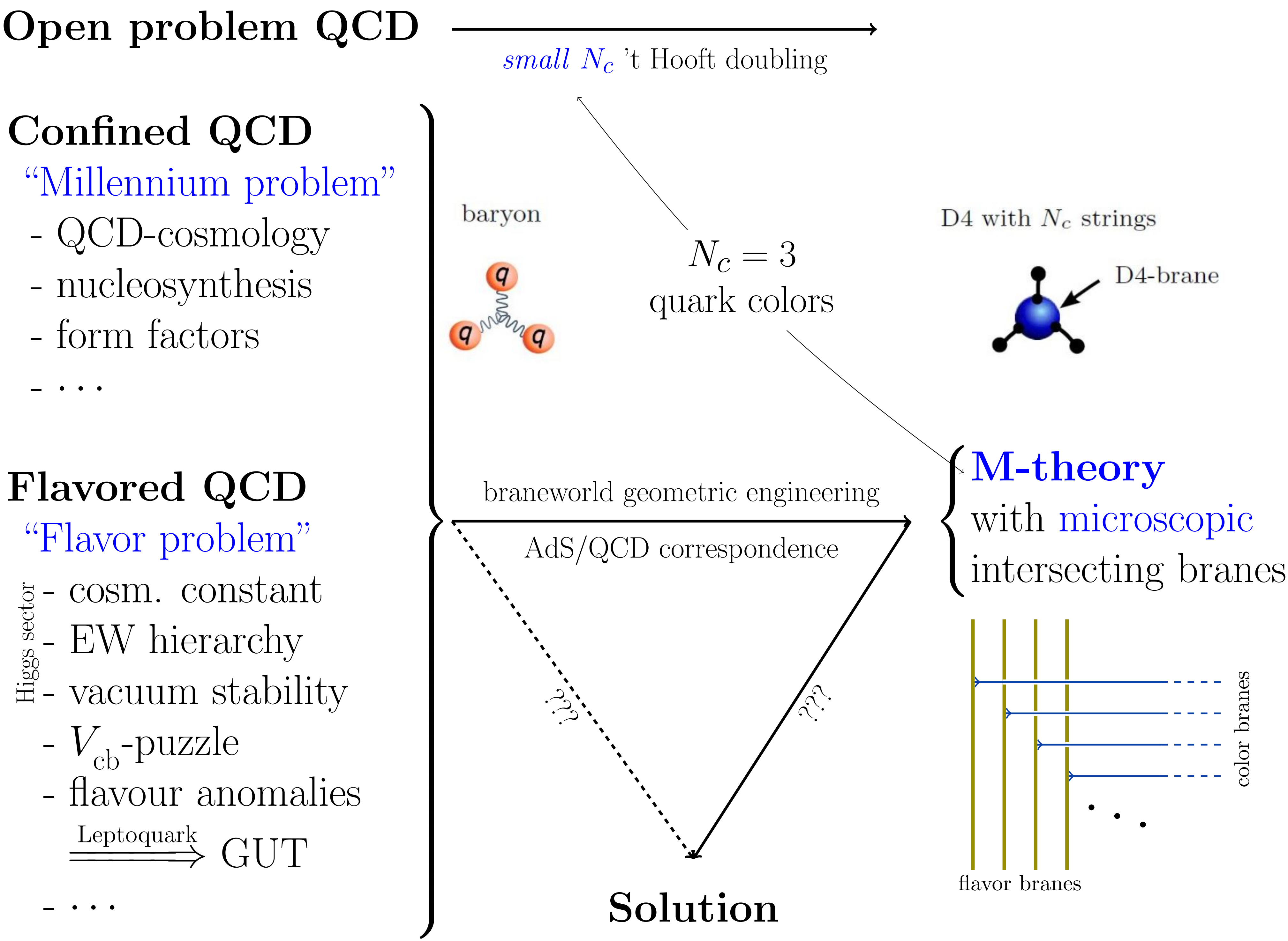
$\xrightarrow{\text{Leptoquark}}$ GUT
- ...
braneworld geometric engineering
AdS/QCD correspondence
~~IIA super-gravity near black~~
intersecting branes
???
???
color branes
flavor branes
Solution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with microscopic
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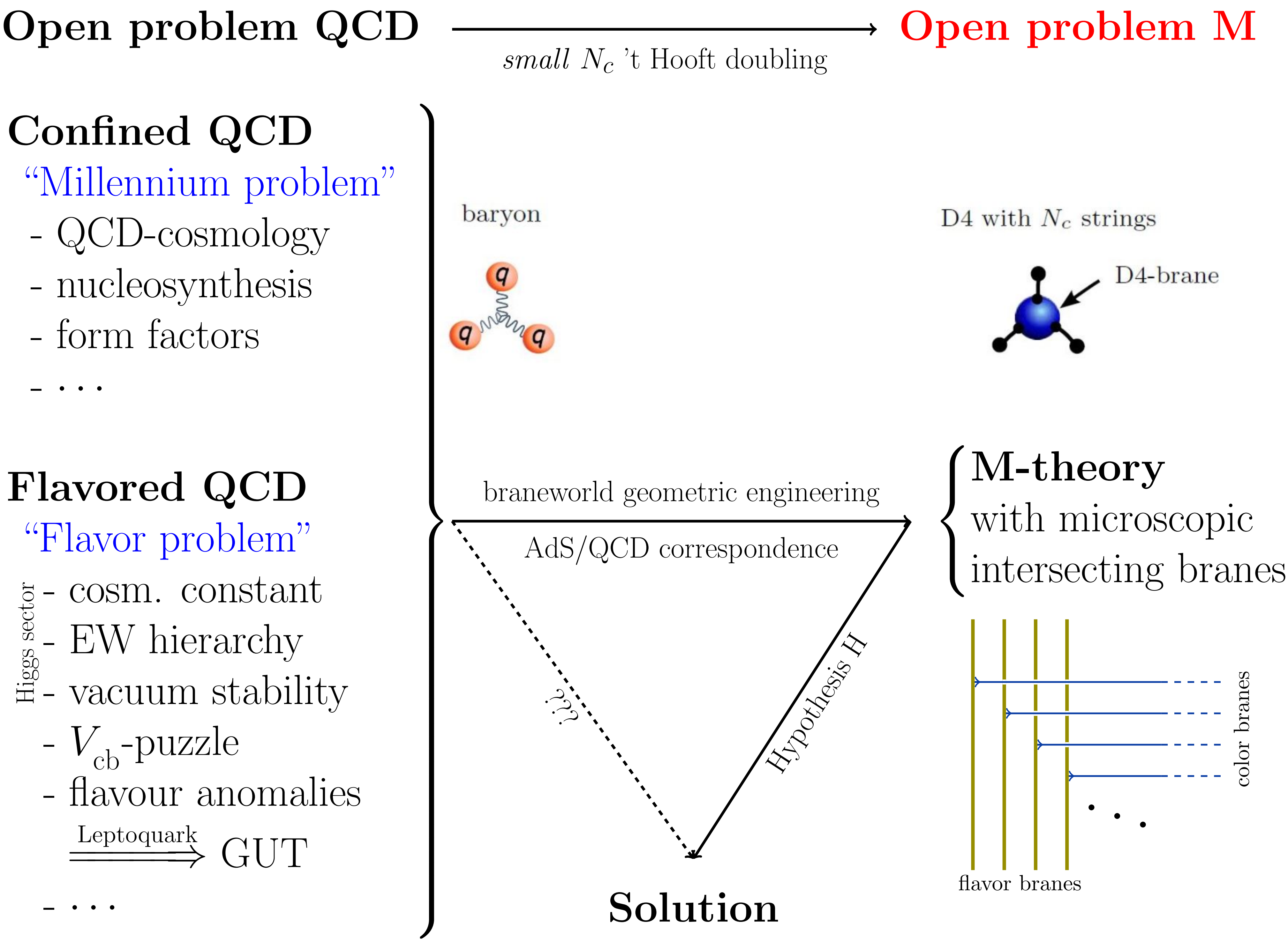

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Hypothesis H
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M-theory
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flavor branes

$$\textbf{Open problem QCD} \xrightarrow[\textit{small } N_c \text{ 't Hooft doubling}]{} \textbf{Open problem M}$$

$$\textbf{Open problem QCD} \xrightarrow[\textit{small } N_c \text{ 't Hooft doubling}]{} \textbf{Open problem M}$$

first consider *large* $g_{\mathrm{YM}}^2 N_c \Longrightarrow$ **11d supergravity**

$$\textbf{Open problem QCD} \xrightarrow[\textit{small } N_c \text{ 't Hooft doubling}]{} \textbf{Open problem M}$$

$$\text{first consider } \textit{large } g_{\mathrm{YM}}^2 N_c \Longrightarrow \textbf{11d supergravity}$$

$$\textbf{CovariantPhaseSpace}_{\text{11d SuGra}} =$$

spacetimes

formalized as:	G_{ADE}–orbi $\mathbb{R}^{10,1\vert\mathbf{32}}$–folds $\overset{\text{super-orbifold}}{(\mathcal{X})}$	
equipped with:	**0) gravity**	**1) C-field**
formalized as:	Pin$^+$-structure $\overset{\text{super-vielbein}}{(E, \Psi)}$	differential forms $\overset{\text{flux densities}}{(G_4, G_7)}$
subject to:	**Einstein equations** $G = T(\Psi, G_4, G_7)$	**Page equation** $dG_7 + \frac{1}{2} G_4 \wedge G_4 = 0$

$$\textbf{Open problem QCD} \xrightarrow[\textit{small } N_c \text{ 't Hooft doubling}]{} \textbf{Open problem M}$$

first consider *large* $g^2_{\mathrm{YM}} N_c \Longrightarrow$ **11d supergravity**

$$\mathbf{CovariantPhaseSpace}_{\text{11d SuGra}} =$$

spacetimes

formalized as:	G_{ADE}–orbi $\mathbb{R}^{10,1\vert\mathbf{32}}$–folds $(\mathcal{X})$ super-orbifold	
equipped with:	**0) gravity**	**1) C-field**
formalized as:	Pin$^+$-structure (E, Ψ) super-vielbein	differential forms (G_4, G_7) flux densities
subject to:	**Einstein equations**	**Page equation**
equivalent to:	super-torsion $= 0$	flux is in rationalized J-twisted Cohomotopy
	Candiello-Lechner 93, Howe 97	Sati 13, Fiorenza-Sati-S. 19a

Open problem QCD $\xrightarrow[\textit{small } N_c \text{ 't Hooft doubling}]{}$ **Open problem M**

& *large* $g^2_{\mathrm{YM}} N_c$ $\Longrightarrow$ **11d supergravity charge-quantized in Cohomotopy**

$$\mathbf{CovariantPhaseSpace}_{\text{11d SuGra}} =$$

spacetimes

formalized as:	G_{ADE}–orbi $\mathbb{R}^{10,1\vert\mathbf{32}}$–folds $(\mathcal{X})$ super-orbifold	
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Hypothesis H

Sati 13, Fiorenza-Sati-S. 19b,19c

CovariantPhaseSpace$_{\text{M-Theory}}$ $=$

spacetimes

formalized as:	G_{ADE}–orbi $\mathbb{R}^{10,1\vert\mathbf{32}}$–folds $\overset{\text{super-orbifold}}{(\mathcal{X})}$	
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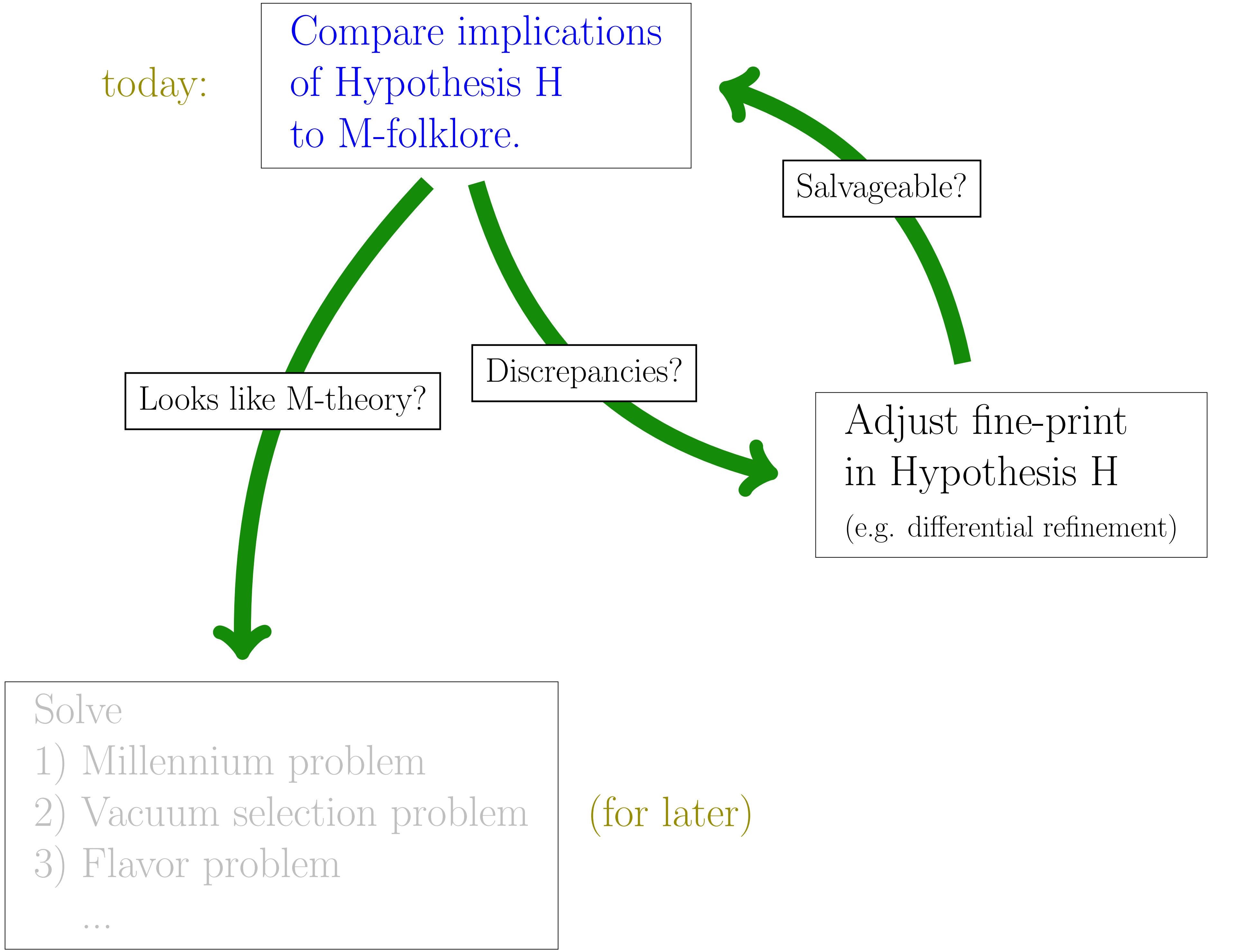
subject to:	**Einstein equations**	**Page equation**
formalized as:	super-torsion $= 0$	flux is in ~~rationalized~~ J-twisted Cohomotopy FSS 19b,19c, SS 19a,19b,19c


today:
Compare implications of Hypothesis H to M-folklore.
Looks like M-theory?
Discrepancies?
Salvageable?
Adjust fine-print in Hypothesis H
(e.g. differential refinement)
Solve
1) Millennium problem
2) Vacuum selection problem
3) Flavor problem
...
(for later)

Implications of Hypothesis H

on **curved but smooth** spacetimes	on **flat but orbi-singular** spacetimes	on spacetimes **with horizons**
FSS 19b, FSS 19c, GS20	BSS 18, SS 19a	SS 19c
topological **anomaly cancellation:**	**equivariant** **anomaly cancellation**	D$p \perp$ D$(p+2)$ **worldvolume QFT**
- shifted C-field flux quantization - C-field tadpole cancellation - M5 Hopf-WZ level quantization - DMW anomaly cancellation - C-field integral eom ...	- M5/MO5 anomaly cancellation - RR-field tadpole cancellation - no irractional D-brane charge	- fuzzy funnels - BLG 3-algebras - BMN matrix model - M2/M5 bound states - AdS3-holography - Coulomb branch indices - Hanany-Witten rules - ...

H. Sati's talk	
D. Fiorenza's talk	my talk

at **M-Theory and Mathematics 2020**

(1)

Microscopic Brane Charge

implied by

Hypothesis H with Pontrjagin-Thom Theorem

Sati-Schreiber 19a [arXiv:1909.12277]

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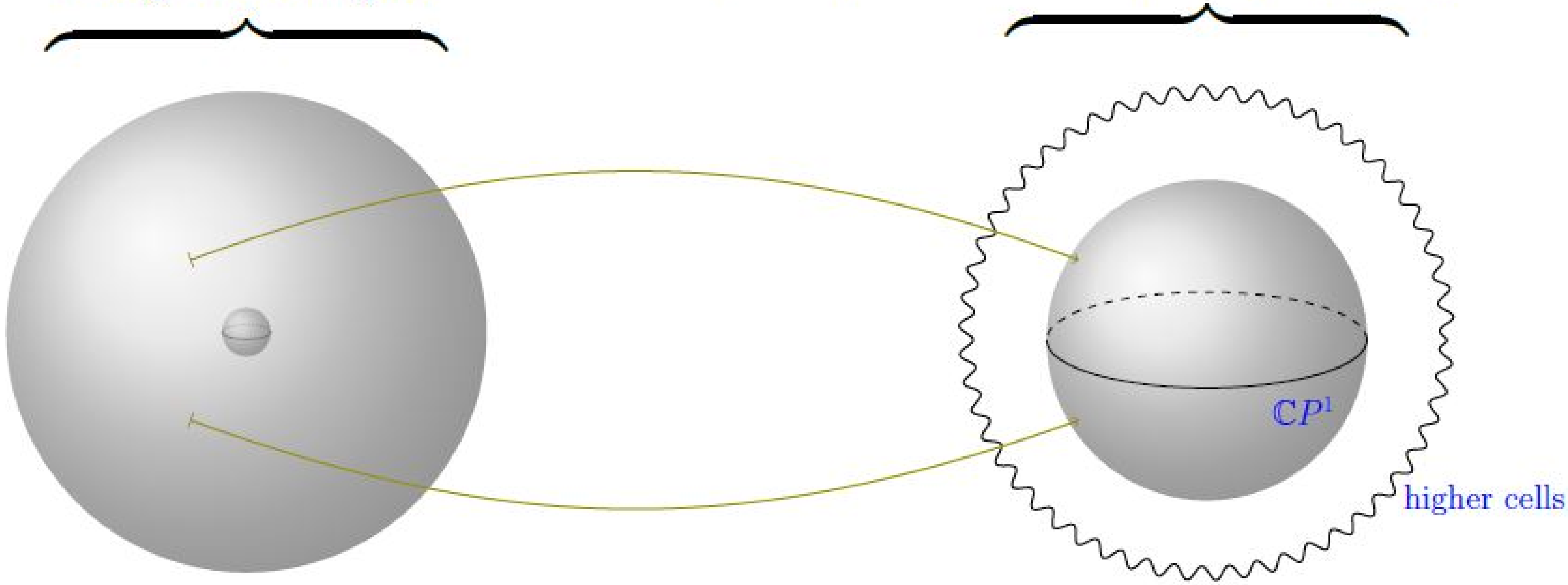
$$X := \mathbb{R}^1 \times \left(\mathbb{R}^3 \setminus \{0\}\right) \simeq S^2 \xrightarrow{\quad c \quad} BU(1) \simeq \mathbb{C}P^\infty$$

spacetime around a magnetic monopole

electromagnetic field sourced by monopole

classifying space of electromagnetic gauge group



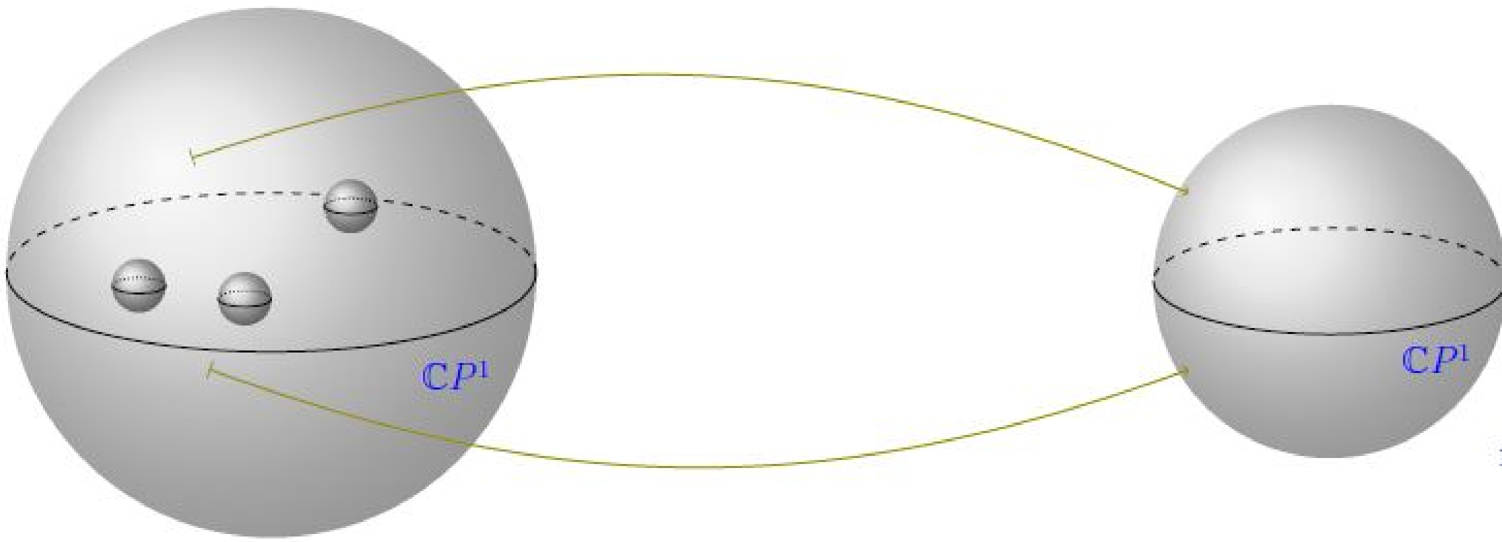
$$[c] \in \left\{ X \longrightarrow BU(1) \right\} \big/_{\sim_{\text{homotopy}}} \simeq \mathbb{Z}$$

charge = homotopy class

charge lattice

Dirac charge quantization – The topological sector of the electromagnetic field is a cocycle in degree-2 ordinary cohomology, with classifying/coefficient space $BU(1)$.

$$\underbrace{\mathbb{R}^1 \times \left(\mathbb{R}^3 \backslash \{\vec{x}_1, \cdots \vec{x}_k\}\right) \supset \mathbb{C}P^1}_{\text{spacetime around Yang-Mills monopoles}} \xrightarrow[\text{nuclear force field sourced by monopole}]{c} \underbrace{\mathbb{C}P^1}_{\text{classifying space of complex Cohomotopy}}$$

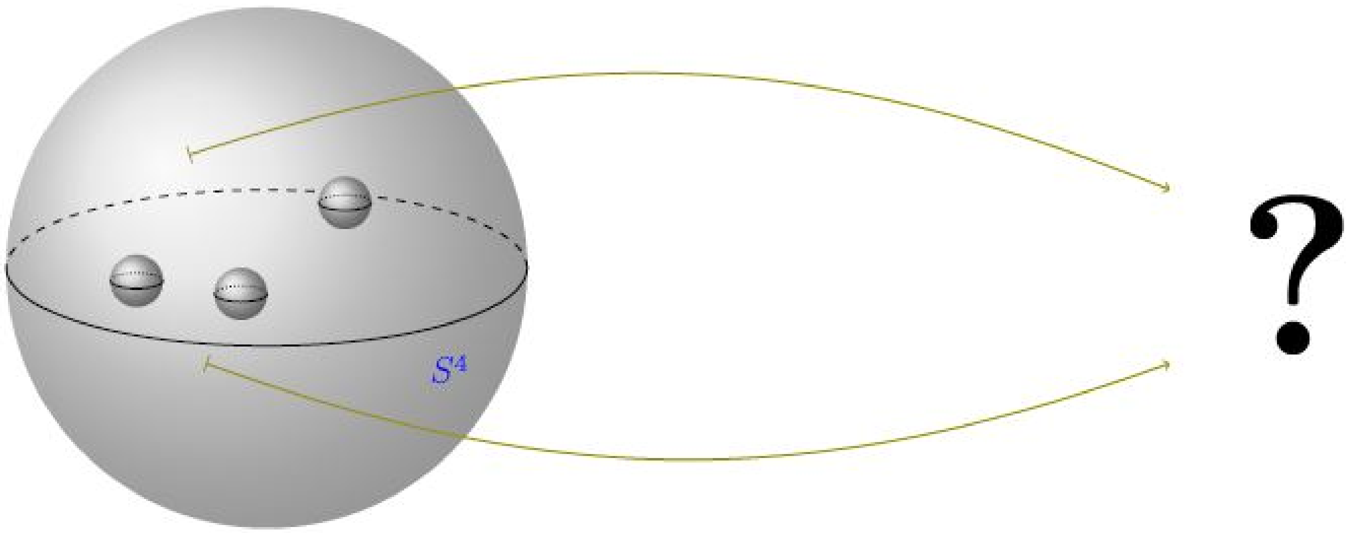


$$[c] \;\in\; \left\{ \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1 \right\} \big/_{\sim_{\text{homotopy}}} \;\simeq\; \underset{\text{charge lattice}}{\mathbb{Z}}$$

charge = homotopy class

Atiyah-Hitchin charge quantization – The moduli space of SU(2) Yang-Mills monopoles is the cocycle space of complex-rational Cohomotopy of any sphere enclosing them.

$$\underbrace{\mathbb{R}^{5,1}\times\left(\mathbb{R}^4\setminus\{\vec{x}_1,\cdots\vec{x}_k\}\right)\times\mathbb{R}^1_{/\mathbb{Z}_2^{\mathrm{HW}}}}_{\substack{\text{spacetime around}\\ \text{M5-branes}}} \xrightarrow[\substack{\text{C-field}\\ \text{sourced by M-branes}}]{\quad c\quad} \underbrace{??}_{\substack{\text{classifying space of}\\ \text{which cohomology theory??}}}$$



$$[c] \in \left\{X \longrightarrow ??\right\}\Big/_{\sim_{\text{homotopy}}} \simeq \underset{\substack{\text{charge}\\ \text{lattice}}}{??}$$

charge = homotopy class

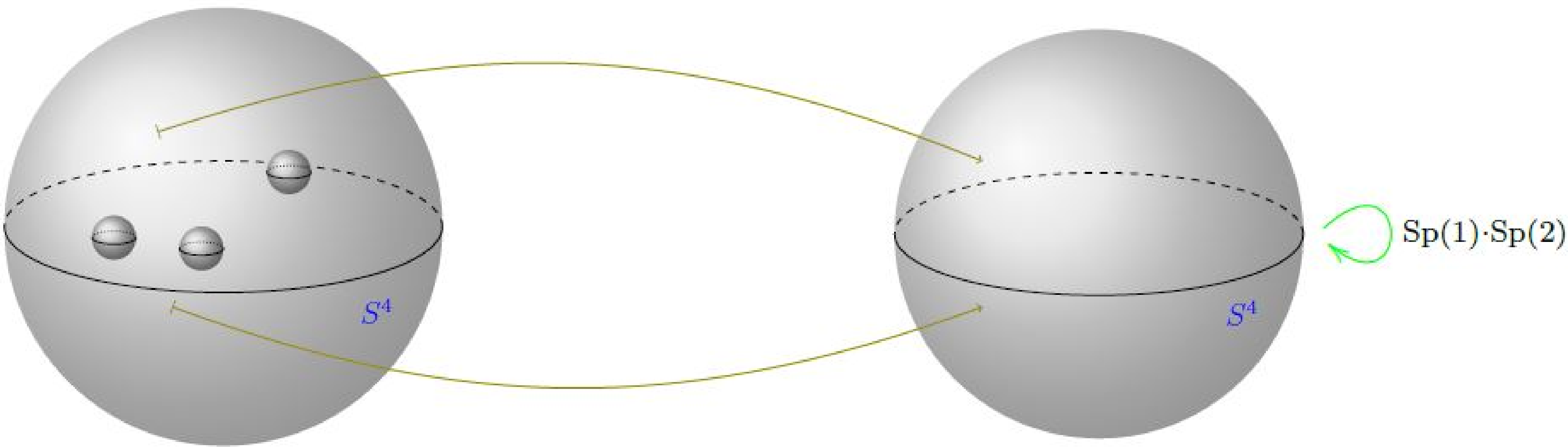
Strominger-Witten: Monopoles are wrapped M5-branes

and the elusive non-perturbative Yang-Mills theory is in M-theory.

$\rightsquigarrow$ Open problem: *Wherein is M5-brane charge quantization?*

$$X \xrightarrow{\quad c \quad} S^4/\!\!/_{\mathrm{Sp}(1)\cdot\mathrm{Sp}(2)}$$

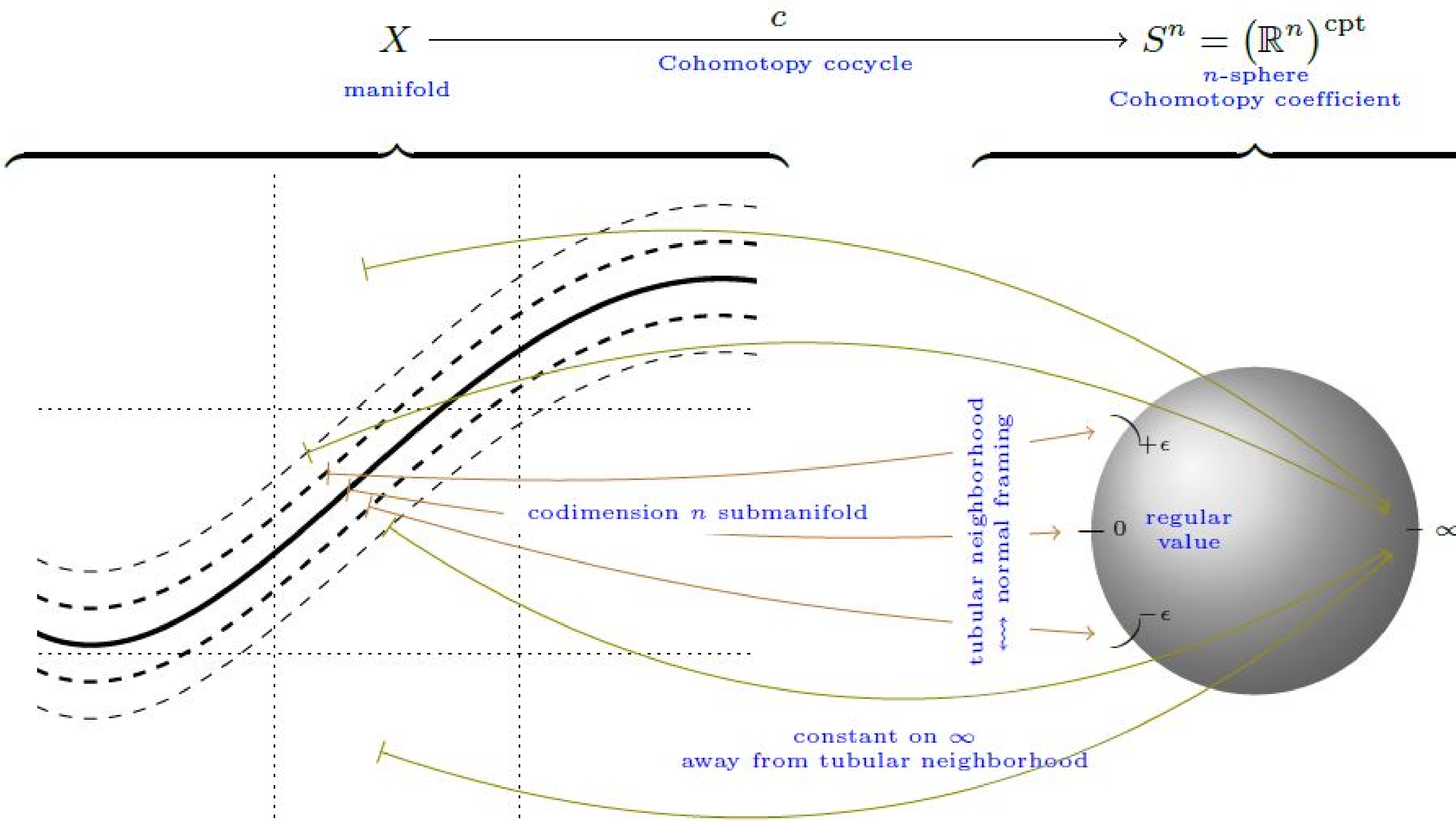
spacetime around M5-branes — C-field sourced by M-branes — classifying space of twisted Cohomotopy theory



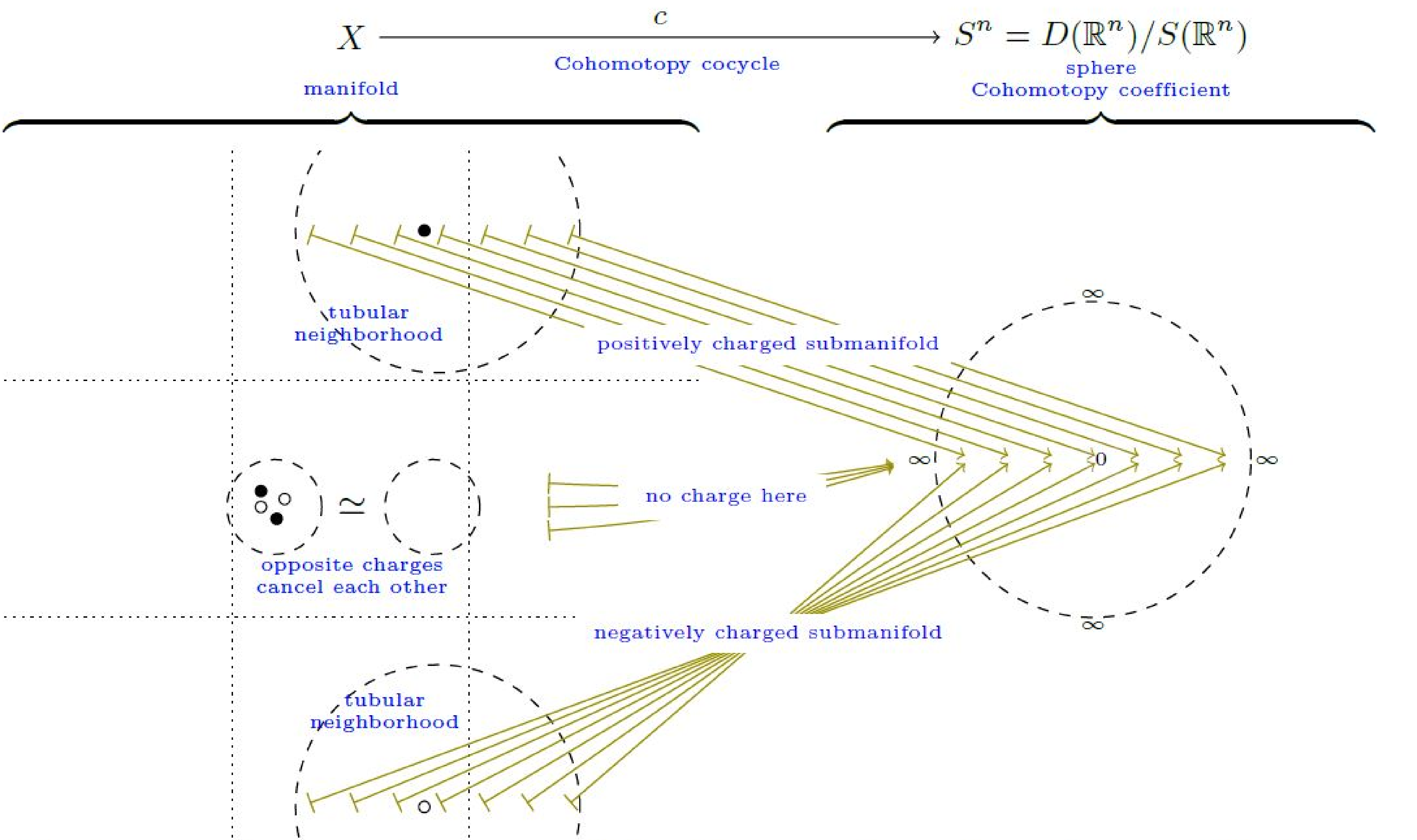
$$[c] \in \left\{ \begin{tikzcd}[column sep=small] X \arrow[rr] \arrow[dr, "TX"'] & & S^4/\!\!/_{\mathrm{Sp}(1)\cdot\mathrm{Sp}(2)} \arrow[dl] \\ & B(\mathrm{Sp}(1)\cdot\mathrm{Sp}(2)) & \end{tikzcd} \right\}\Big/\sim_{\text{homotopy}} \;\simeq\; \underbrace{\mathrm{Cob}^{TX}_{\mathrm{fr}}(X)}_{\text{charge "lattice"}}$$

charge = homotopy class

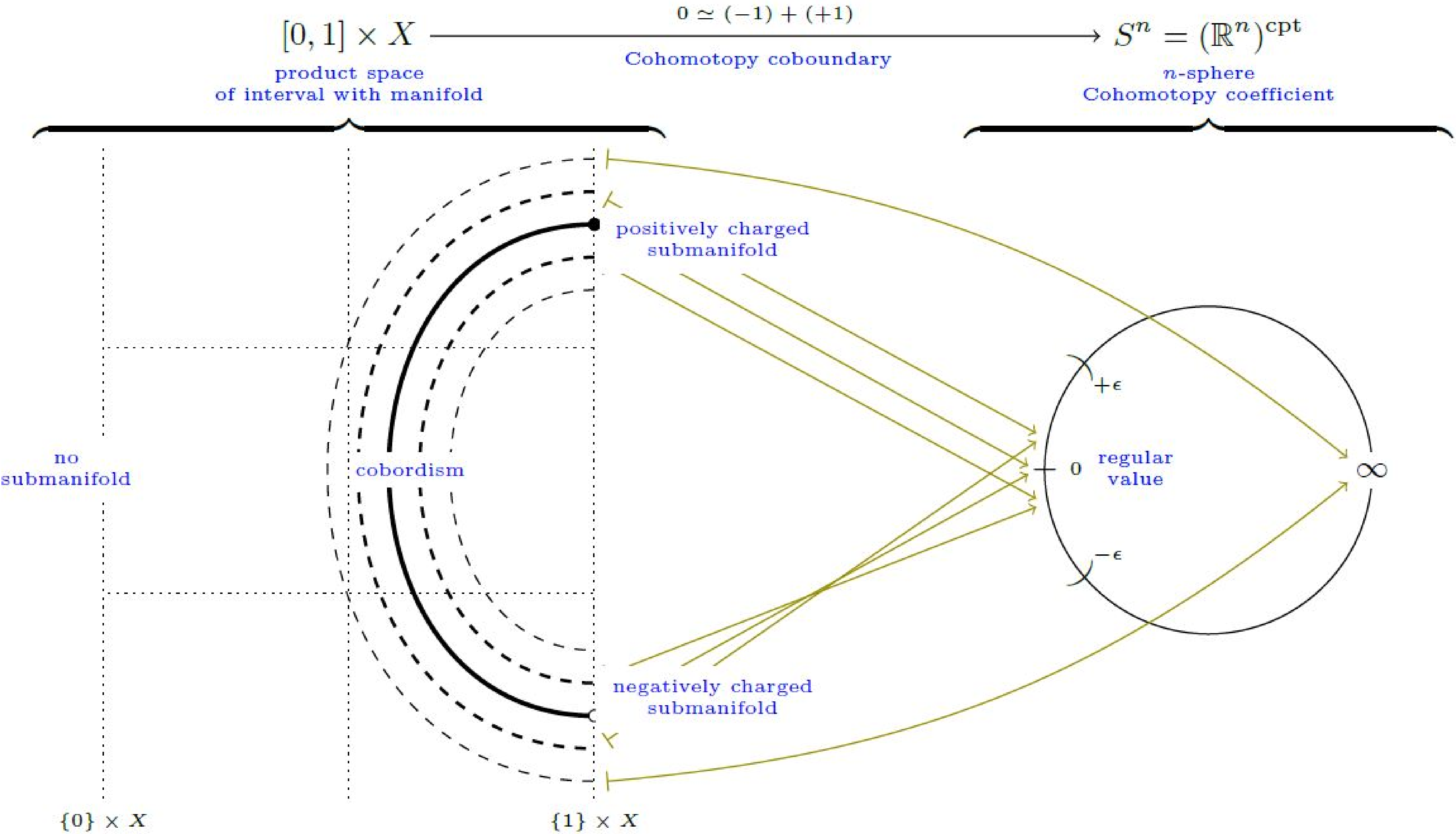
Hypothesis H (Fiorenza-Sati-Schreiber 19):
C-field is charge-quantized in J-twisted Cohomotopy theory.



Cohomotopy charge of normally framed submanifolds
is represented by the submanifold's *asymptotic distance function*,
traditionally known as the *Pontrjagin-Thom collapse*.

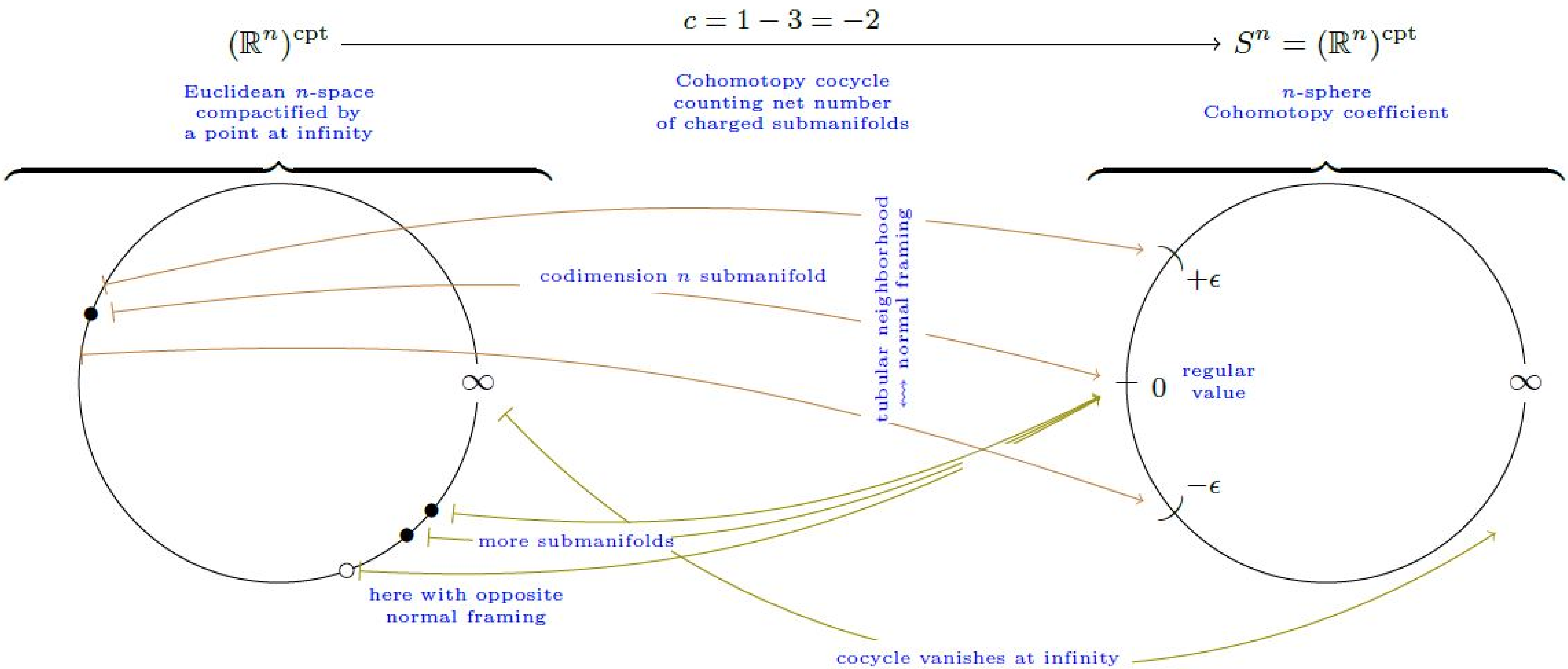


Cohomotopy charge of 0-dimensional submanifolds
(traditionally known as "electric field map" or *scanning map*)
exhibits net brane/anti-brane charge in $\mathbb{Z}$.



Brane/anti-brane pair creation & annihilation

is exhibited, under Hypothesis H, by normally framed cobordism.



Cohomotopy charge vanishing at ∞ on Euclidean n-space is equivalently the Cohomotopy charge of the n-sphere and hence takes values in homotopy groups of spheres.

(2)

Orientifold Tadpole Cancellation

implied by

Hypothesis H with Equivariant Hopf Degree Theorem

Sati-Schreiber 19a [arXiv:1909.12277]

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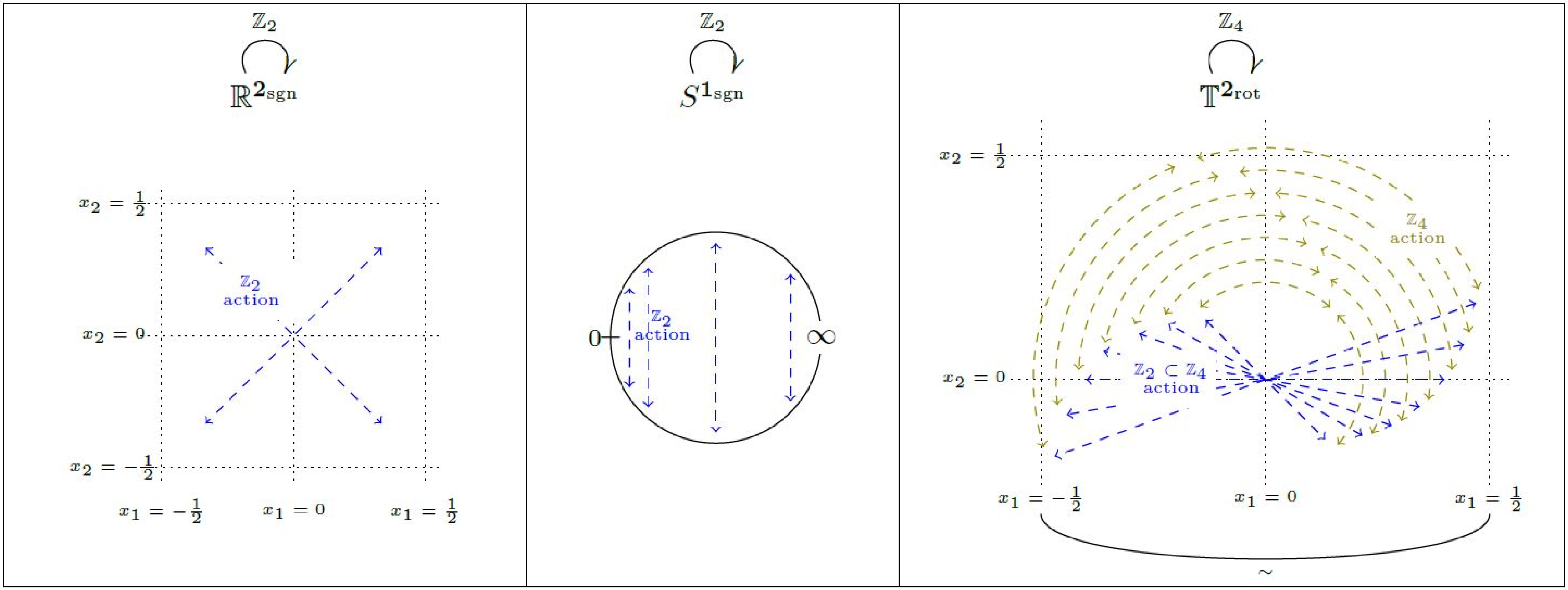
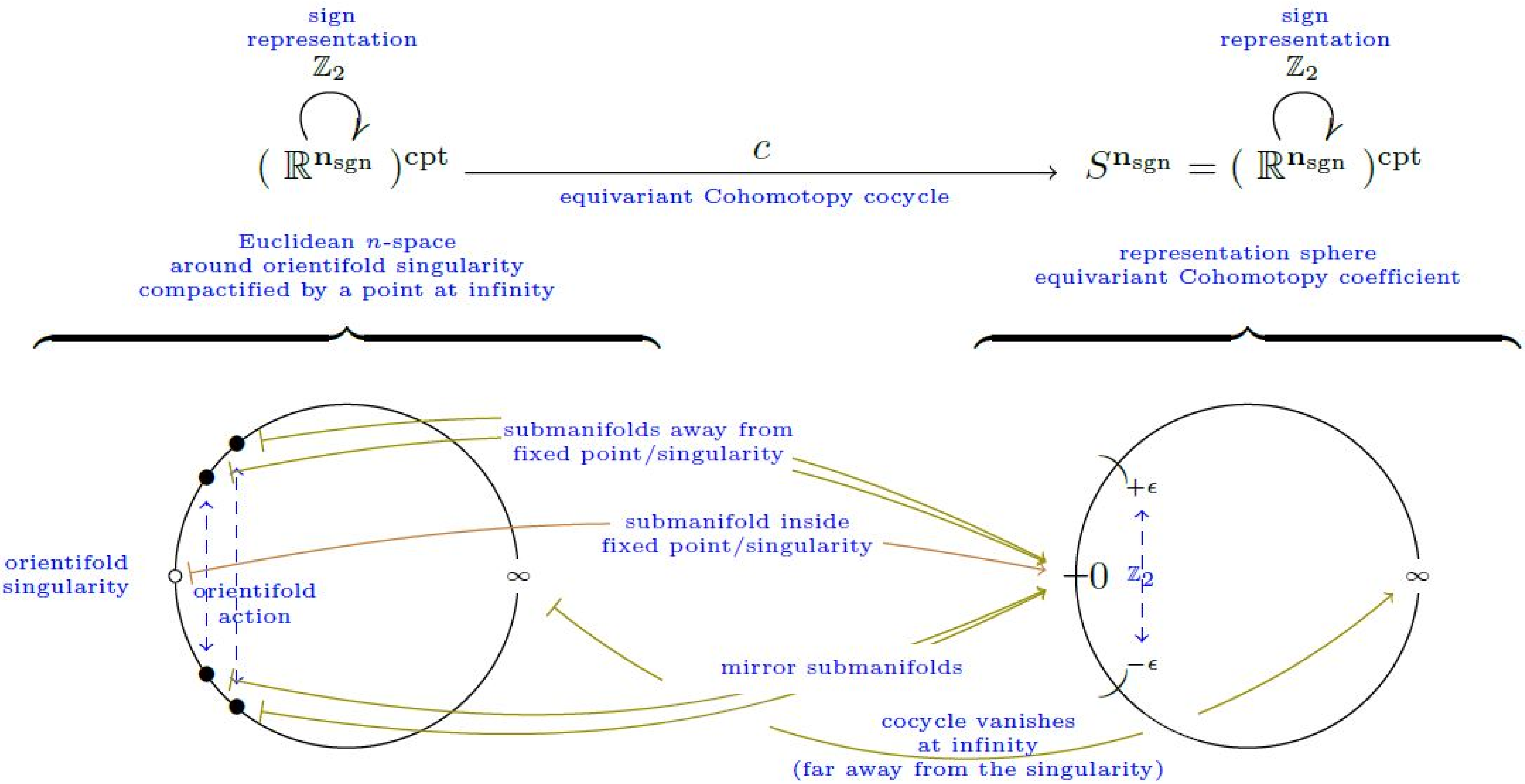
Euclidean linear representation	representation sphere	representation torus



Examples of linear representations and induced G-spaces

Figure H – Hopf degree in the unstable range takes values in the set $\{0,1\}$ (14), corresponding to the binary choice of there being or not being a unit charge at the single point.



The equivariant Hopf degree theorem

says that $\mathbb{Z}_2$-equivariant Cohomotopy charges near singularities are sourced by, possibly, a charge attached to the singularity and any integer number of twice this charge located nearby.

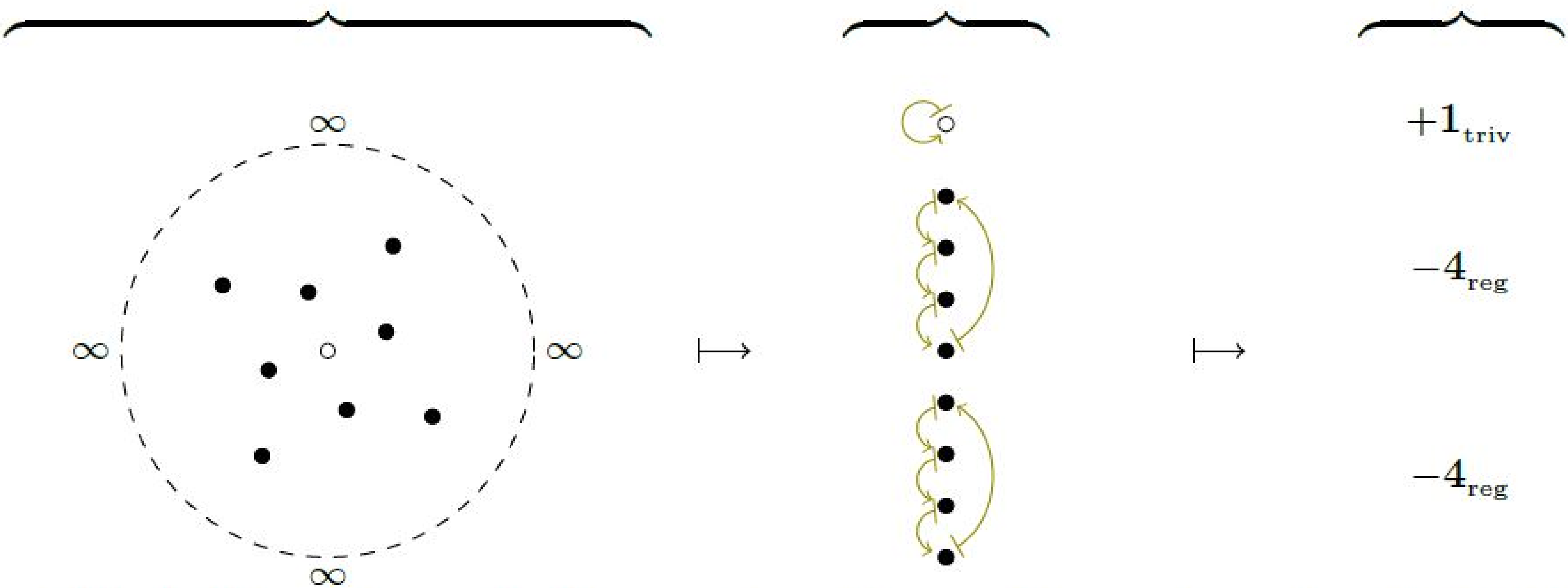
equivariant Cohomotopy vanishing at infinity of Euclidean G-space in compatible RO-degree V — stabilization — stable equivariant Cohomotopy — Boardman homomorphism — equivariant K-theory

$$
\begin{tikzcd}
\pi^V_G\big((\mathbb{R}^V)^{\mathrm{cpt}}\big) \arrow[r, "\Sigma^\infty"] & \mathbb{S}^0_G \arrow[r, "\beta"] \arrow[d, phantom, "\wr"] & \mathrm{KO}^0_G \arrow[d, phantom, "\wr"] \\
 & A_G \arrow[r, "{\mathbb{R}[-]}"'] & \mathrm{RO}(G)
\end{tikzcd}
$$

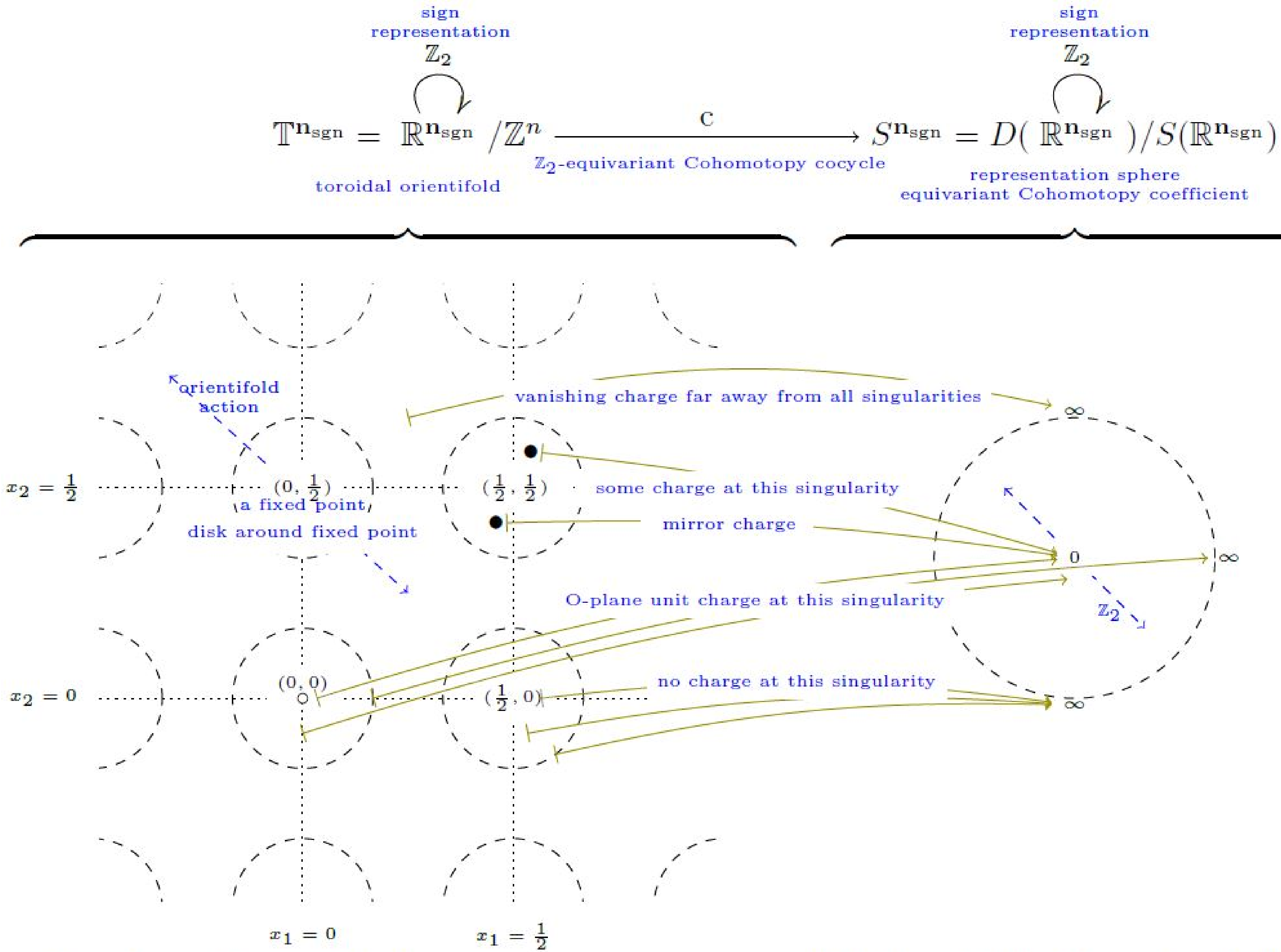
linearization

Burnside ring — representation ring

e.g. one O^--plane and two branes — minus the trivial G-set with two regular G-sets — minus the trivial G-representation plus two times the regular G-representation



Stabilization & linearization of equivariant Cohomotopy lands in equivariant K-theory. In this approximation virtual G-sets of (anti-)branes map to virtual permutation representations.



Equivariant Cohomotopy on toroidal orbifolds glued from local cocycles in the vicinity of singularities. By the equivariant Hopf degree theorem, all global cocucles are obtained this way.

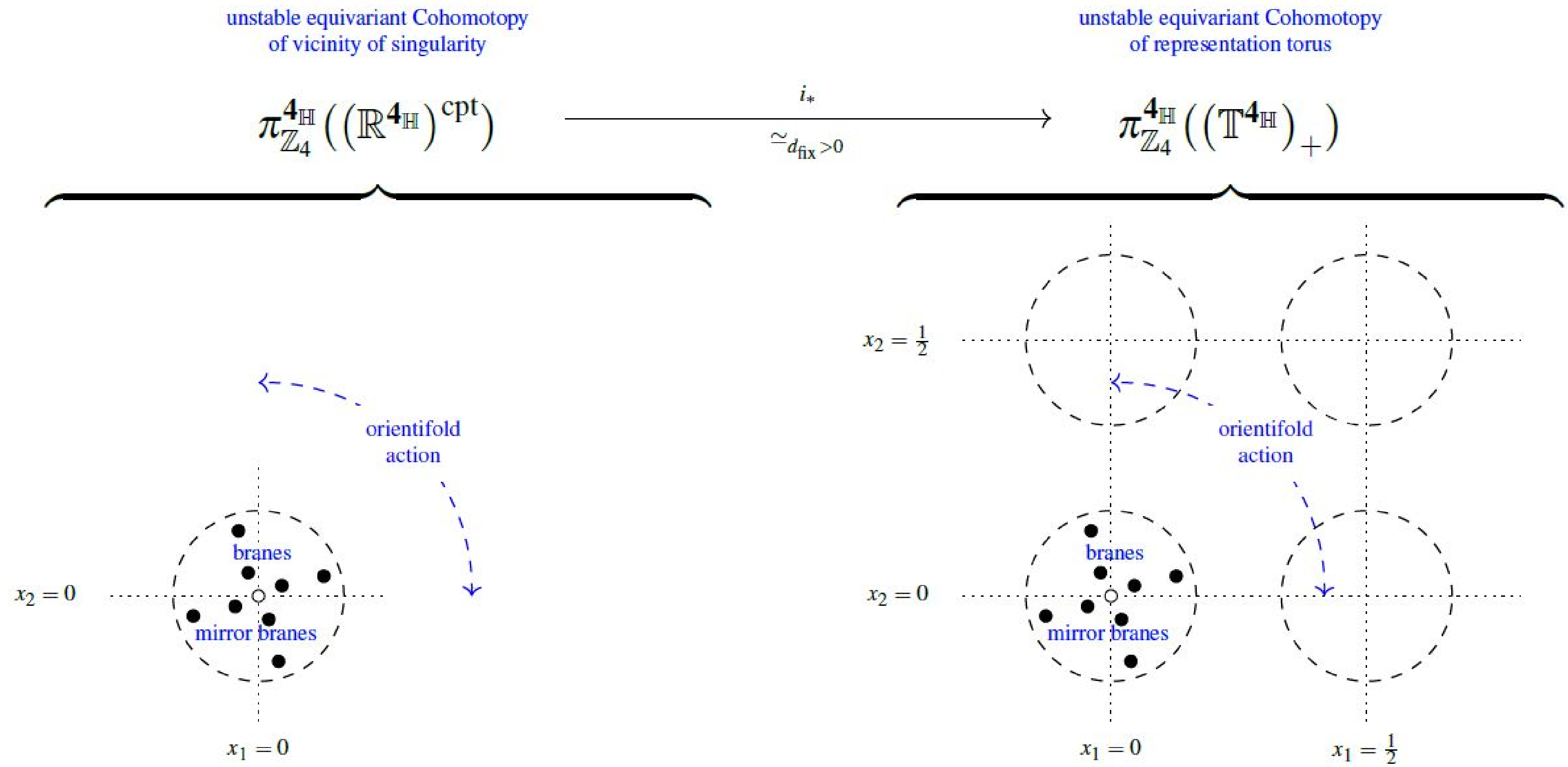


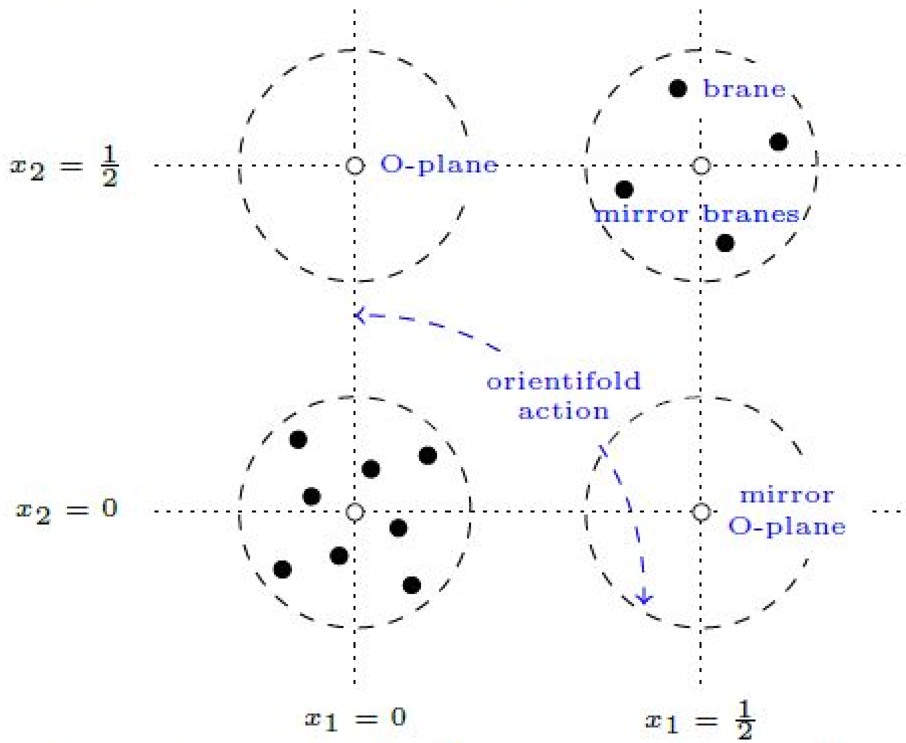
Figure O – Pushforward in equivariant Cohomotopy from the vicinity of a singularity to the full toroidal orientifold is an isomorphism on brane charges and an injection on O-plane charges, by Prop. 3.18. Shown is a case with $G = \mathbb{Z}_4$, as in *Figure M*.

equivariant Cohomotopy
of representation torus
(orientifold Cohomotopy)

equivariant K-theory
of representation torus
= representation ring

$$\pi^{\mathbf{4}_{\mathbb{H}}}_{\mathbb{Z}_4}\Big(\big(\mathbb{T}^{\mathbf{4}_{\mathbb{H}}}\big)_+\Big) \xrightarrow{\text{stabilize and linearize}} \mathrm{KO}^0_{\mathbb{Z}_4} \simeq \mathrm{RO}(\mathbb{Z}_4)$$

$$4\cdot[\mathbb{Z}_4/\mathbb{Z}_4] - 3\cdot[\mathbb{Z}_4/1] \longmapsto 4\cdot\mathbf{1} - 3\cdot\mathbf{4}_{\mathrm{reg}}$$



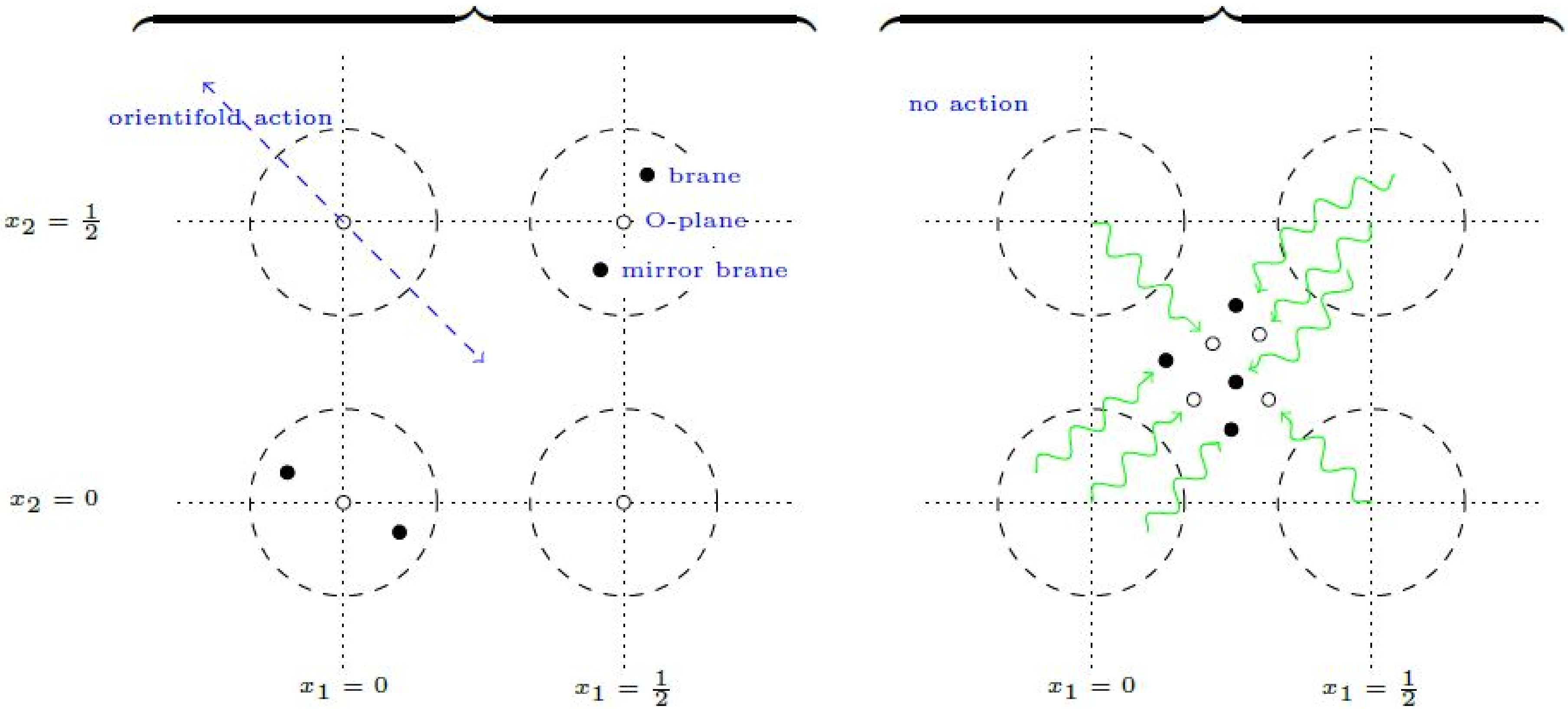
$$\longmapsto \quad \begin{array}{c} 4\cdot\mathbf{1}_{\mathrm{triv}} \\ -3\cdot\mathbf{4}_{\mathrm{reg}} \end{array}$$

Equivariant Cohomotopy implies local tadpole cancellation

by the combined unstable and stable version of the equivariant Hopf degree theorem.

equivariant Cohomotopy of representation torus (orientifold Cohomotopy) | plain Cohomotopy of plain torus

$$\begin{array}{ccc} \pi^{\mathbf{n}_{\mathrm{sgn}}}_{\mathbb{Z}_2}\big((\mathbb{T}^{\mathbf{n}_{\mathrm{sgn}}})_+\big) & \xrightarrow{\text{forget equivariance}} & \pi^{n}\big((\mathbb{T}^{n})_+\big)_{\mathbb{R}} \\ 4\cdot[\mathbb{Z}_2/\mathbb{Z}_2] - 2\cdot[\mathbb{Z}_2/1] & \longmapsto & \begin{array}{l} 4\cdot 1 - 2\cdot 2 \\ = 0 \end{array} \end{array}$$



Super-differential equivariant Cohomotopy implies global tadpole cancellation by forcing the charge to vanish at global Elmendorf stage, and only there.

The following four slides show technical detail of the realization of this mechanism for MO5-planes at ADE-singularities in heterotic M-theory

Skip over technical detail ahead to section (3).

Orientifold		**MO5**	$\frac{1}{2}$**M5**
Global quotient group	$G =$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{\mathrm{HW}} \times G^{\mathrm{ADE}}$
Global quotient group action	$G \curvearrowright \mathbb{T}^V =$	$\mathbb{Z}_2 \curvearrowright \mathbb{T}^{5\mathrm{sgn}}$	$\mathbb{Z}_2^{\mathrm{HW}} \curvearrowright \mathbb{T}^{1\mathrm{sgn}} \times G^{\mathrm{ADE}} \curvearrowright \mathbb{T}^{4\mathrm{sgn}}$
Fixed/singular points	$(T^V)^G =$	$\{0, \frac{1}{2}\}^5 = \overline{32}$	
Far horizon-limit of M5 SuGra solution?		no	yes

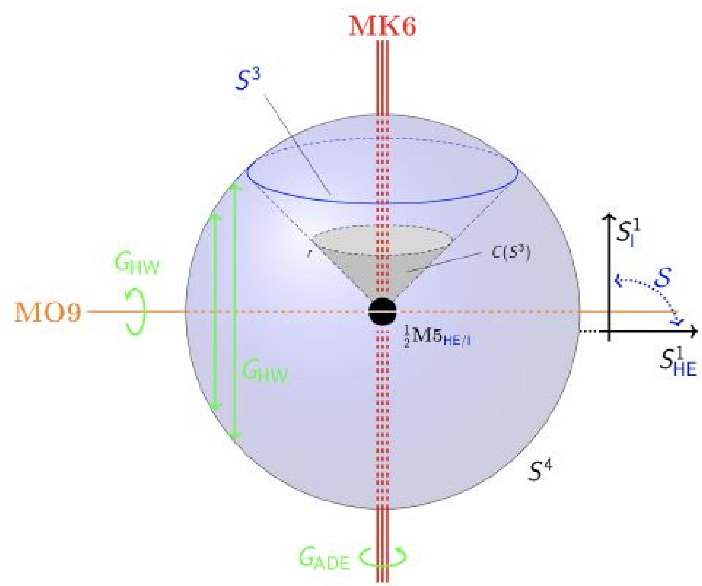


Figure S – Singularity structure of heterotic M-theory on ADE-singularities, as in Figure R, [HSS18, 2.2.2, 2.2.7]. The corresponding toroidal orbifolds (as per *Table 5*) are illustrated in *Figure V* and *Table 8*.

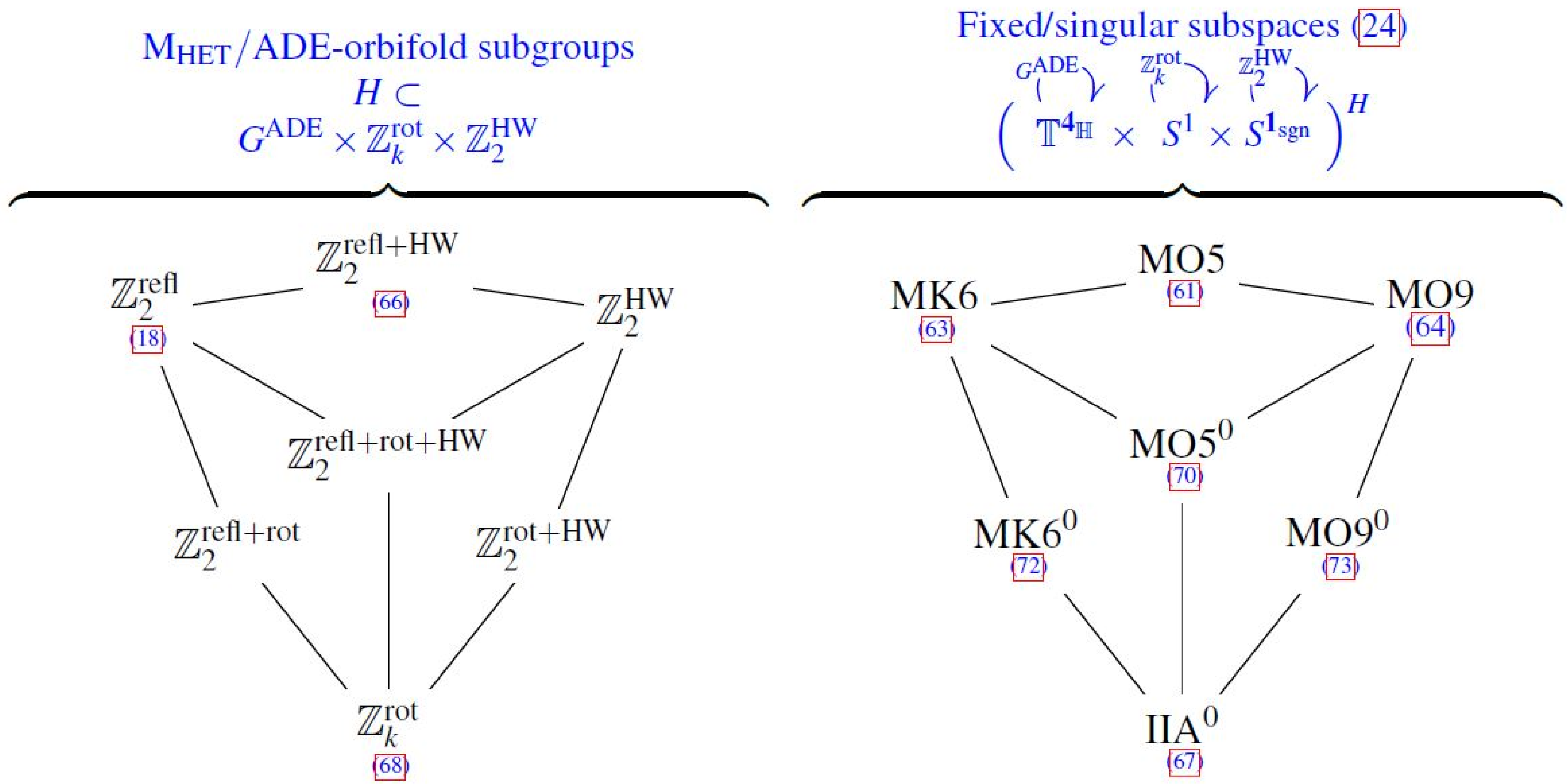


Figure T – Subgroup lattice and fixed/singular subspaces in the singularity structure from *Table 7*. On the left, groups associated to the middle of a sub-simplex are diagonal subgroups inside the direct product of subgroups associated to the vertices, as indicated by the superscripts. On the right, all fixed loci with superscript $(-)^0$ are actually empty, but appear as superficially non-empty (un-charged) singularities after M/IIA KK-reduction (68), e.g. $\mathrm{O4}^0$ (71), $\mathrm{O8}^0$ (74), as on the right of *Figure OP*. The numbered subscripts (xx) indicate the corresponding expression in the text.

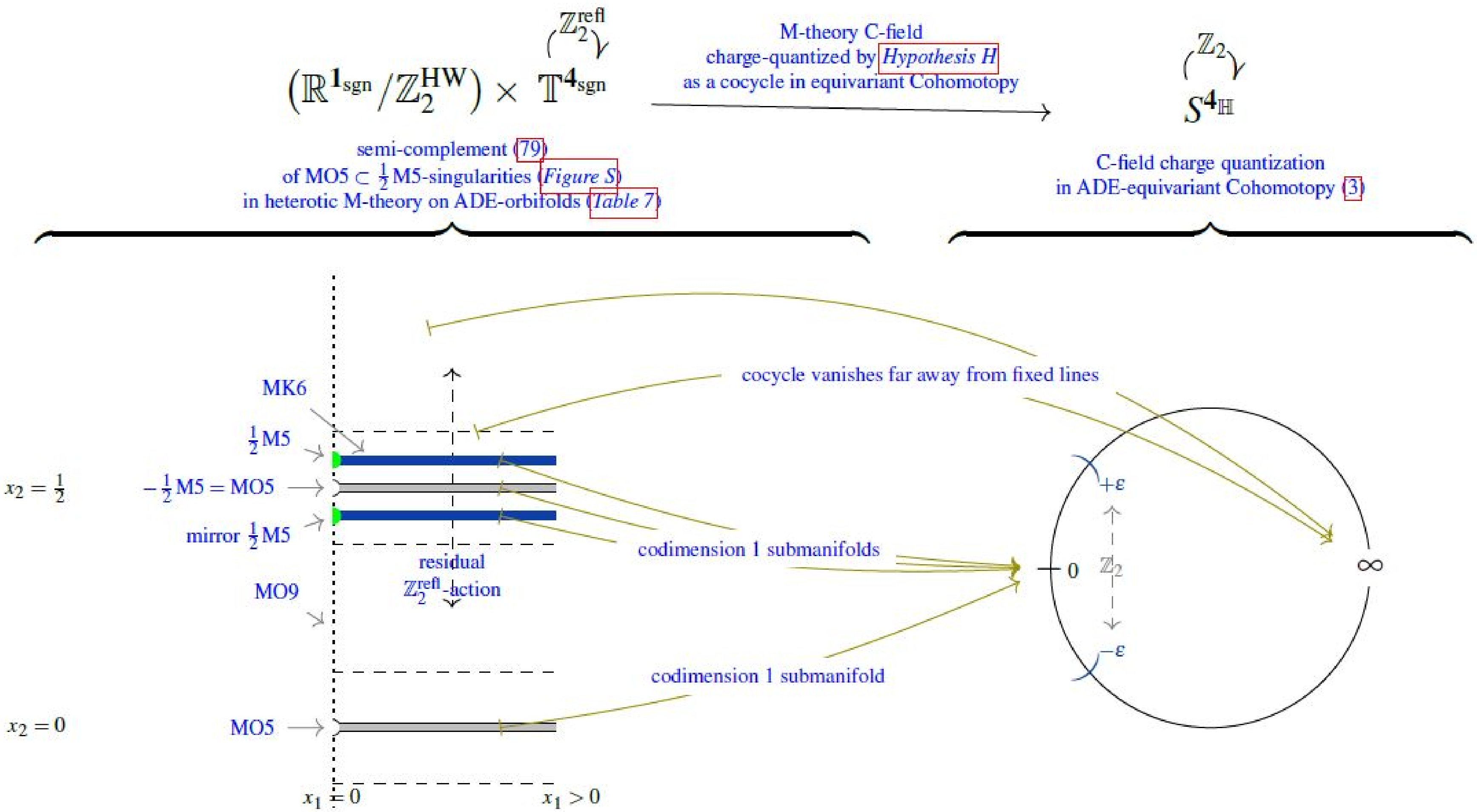
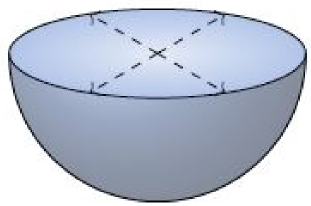
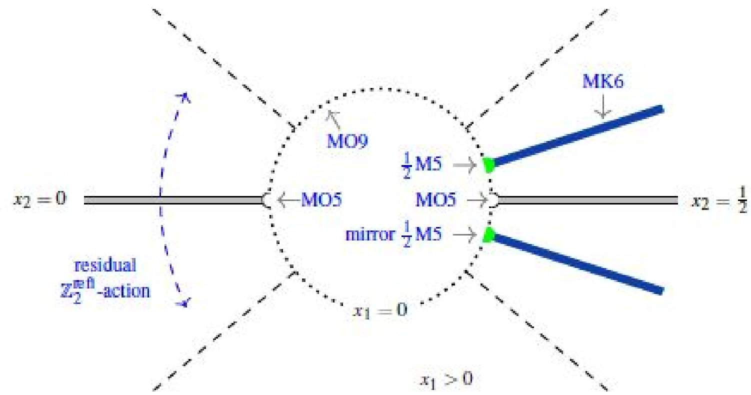
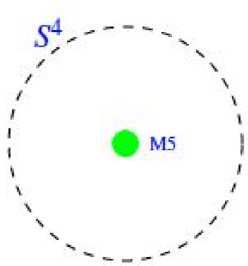
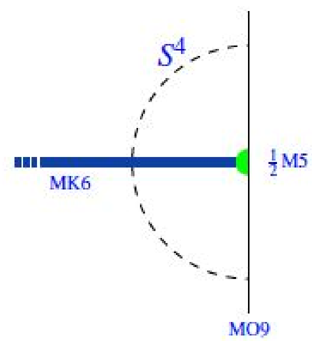


Figure V – Equivariant Cohomotopy of ADE-orbifolds in heterotic M-theory with singularity structure as in *Figure S*. The resulting charge classification (Cor. 4.4) implies, via the unstable PT isomorphism (§2.1), the $\frac{1}{2}$M5 $=$ MO9 $\cap$ MK6-brane configurations (65) similarly shown in [FLO99, Fig. 1][KSTY99, p. 7][FLO00a, Fig. 1][FLO00b, Fig. 2][FLO00c, Fig. 1][GKST01, p. 4, 68, 71]. This is as in *Figure L* but with points (M5s) extended to half-line (MK6s), see Remark 4.7 and *Table 8*.

<table>
<tr><th colspan="4">Spacetimes on which to measure flux sourced by M5/MO5-charge</th></tr>
<tr><td>Definition</td><td>$X_{\mathrm{MO5}} \simeq_{\mathrm{htpy}} S(\mathbb{R}^{\mathbf{1}_{\mathrm{sgn}}+\mathbf{4}_{\mathrm{sgn}}})/\mathbb{Z}_2^{\mathrm{het+HW}}$
(75)</td><td colspan="2">$X_{1/2\mathrm{M5}} \simeq_{\mathrm{htpy}} S(\mathbb{R}^{\mathbf{1}_{\mathrm{sgn}}})/\mathbb{Z}_2^{\mathrm{HW}} \times \mathbb{T}^{\mathbf{4}_{\mathbb{H}}} /\!\!/ \mathbb{Z}_2^{\mathrm{refl}}$
(79)</td></tr>
<tr><td>Illustration</td><td></td><td colspan="2">
</td></tr>
<tr><td>Geometry</td><td>smooth but curved</td><td colspan="2">singular but flat</td></tr>
<tr><th colspan="4">Cohomological charge quantization
by Hypothesis H</th></tr>
<tr><td>Cohomology theory
(by Table 4)</td><td>J-twisted Cohomotopy $\pi^{TX}(X)$
[FSS19b][FSS19c]</td><td colspan="2">equivariant Cohomotopy $\pi^{V}(\mathbb{T}^{V})$
§3</td></tr>
<tr><td>Illustration
(Remark 4.7)</td><td>
</td><td colspan="2">
</td></tr>
<tr><td>Charge classification</td><td>$c_{\mathrm{tot}} = 1 - N \cdot 2$
(77)</td><td>$c_{\mathrm{tot}} = N_{\mathrm{MO5}} \cdot \mathbf{1}_{\mathrm{triv}} - N_{\mathrm{M5}} \cdot \mathbf{2}_{\mathrm{reg}}$
(Cor. 4.4)</td><td>$\Leftrightarrow \begin{aligned} \lvert Q_{\mathrm{tot}} \rvert &= 0 \\ N_{\mathrm{M5}} &= 8 \end{aligned}$
(Cor. 4.6)</td></tr>
</table>

(3)

$\mathrm{D6} \perp \mathrm{D8}$ -brane intersections

implied by

Hypothesis H with May-Segal Theorem

Sati-Schreiber 19c [arXiv:1912.10425]

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Cohomotopy cocycle space

pointed mapping space

$$\boldsymbol{\pi}^4(X) \;:=\; \mathrm{Maps}^{*/}(X, S^4)$$

boldface!

$$\pi_0\big(\boldsymbol{\pi}^4(X)\big) \;=\; \left\{ \begin{array}{c} \text{Cohomotopy} \\ \text{cohomology} \\ \text{classes} \end{array} \right\} \;=\; \pi^4(X)$$

not boldface

Cohomotopy set

$$\pi_1\big(\boldsymbol{\pi}^4(X)\big) \;=\; \left\{ \begin{array}{c} \text{Cohomotopy} \\ \text{gauge} \\ \text{transformations} \end{array} \right\}$$

$$\pi_2\big(\boldsymbol{\pi}^4(X)\big) \;=\; \left\{ \begin{array}{c} \text{Cohomotopy} \\ \text{gauge of gauge} \\ \text{transformations} \end{array} \right\}$$

$\vdots$

Cohomotopy cocycle space

vanishing at ∞ on Euclidean 3-space

May-Segal theorem

$$\boldsymbol{\pi}^4\big((\mathbb{R}^3)^{\mathrm{cpt}}\big) \xleftarrow[\text{assign unit charge in Cohomotopy to each point}]{\text{hmtpy}\,\simeq} \mathrm{Conf}\big(\mathbb{R}^3, \mathbb{D}^1\big)$$

configuration space of points

in $\mathbb{R}^3 \times \mathbb{R}^1$

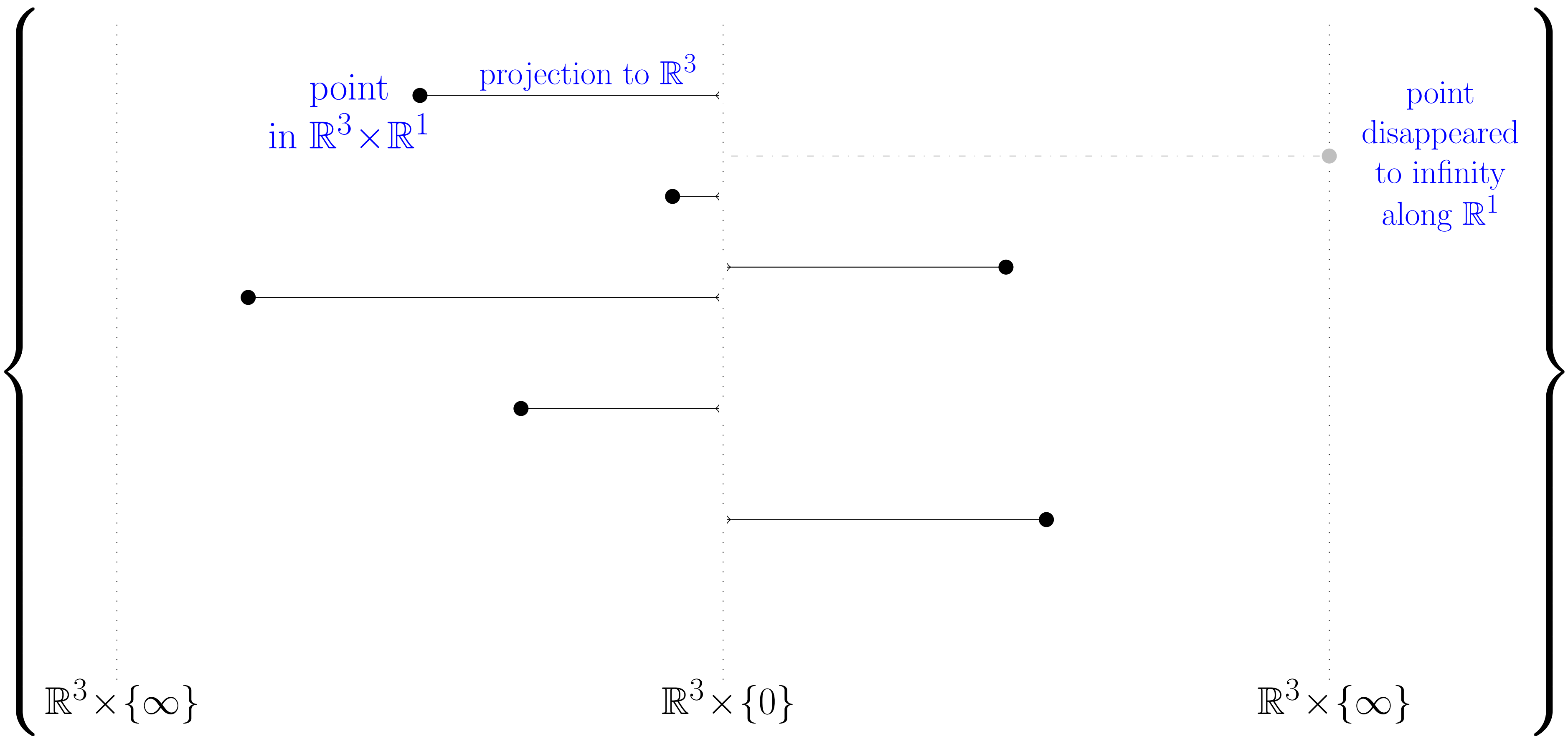
which are:

1) unordered

2) distinct after projection to $\mathbb{R}^3$

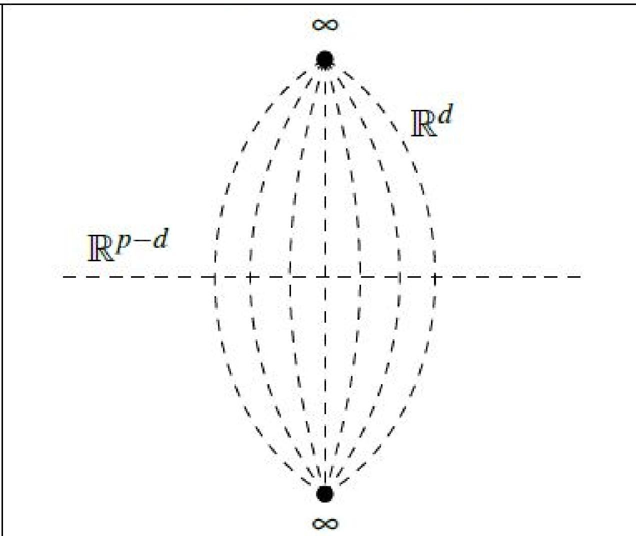
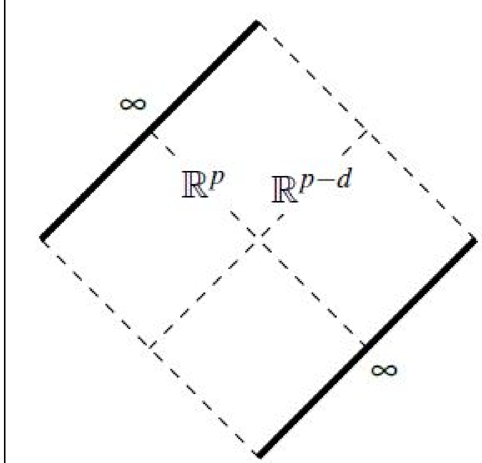
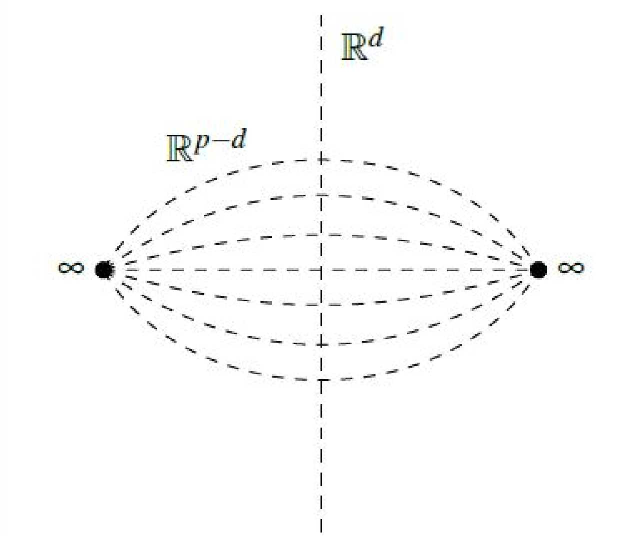
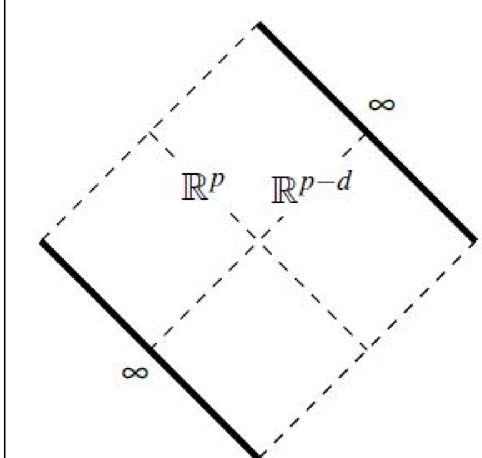
3) allowed to vanish to ∞ along $\mathbb{R}^1$

$=$



hence: a form of differential Cohomotopy assigns configuration spaces:

$$\boldsymbol{\pi}^4\Big((\mathbb{R}^d)^{\mathrm{cpt}} \wedge (\mathbb{R}^{4-d})_+\Big) \xleftarrow[\simeq]{\mathrm{hmtpy}} \boldsymbol{\pi}^4_{\mathrm{diff}}\Big((\mathbb{R}^d)^{\mathrm{cpt}} \wedge (\mathbb{R}^{4-d})_+\Big) := \mathrm{Conf}\big(\mathbb{R}^d, \mathbb{D}^{4-d}\big)$$

<table>
<tr><th rowspan="2">Smash product of pointed topological spaces</th><th colspan="2">Visualization</th></tr>
<tr><th>with point at infinity</th><th>as Penrose diagram</th></tr>
<tr><td>$\overbrace{(\,\mathbb{R}^d\,)}^{\text{cocycles vanish at infinity along these direction}}{}^{\mathrm{cpt}} \wedge \underbrace{(\mathbb{R}^{p-d})}_{\text{...but not necessarily along these}}{}_+$</td><td>
</td><td>
</td></tr>
<tr><td>$\underbrace{(\mathbb{R}^d)}_{\text{...but not necessarily along these}}{}_+ \wedge \overbrace{(\mathbb{R}^{p-d})}^{\text{cocycles vanish at infinity along these direction}}{}^{\mathrm{cpt}}$</td><td>
</td><td>
</td></tr>
</table>

Lemma:

Un-ordered configurations of points in $\mathbb{R}^D$ with labels in $\mathbb{D}^1$

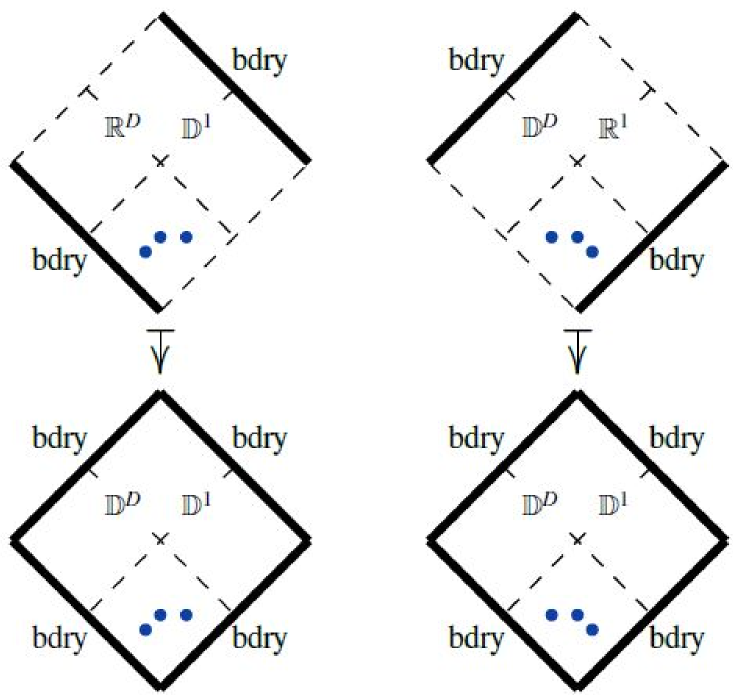
Un-ordered configurations of points in $\mathbb{R}^1$ with labels in $\mathbb{R}^D$

$$\underset{n\in\mathbb{N}}{\bigsqcup} \mathrm{Conf}_{\{1,\cdots,n\}}\big(\mathbb{R}^D\big) \underset{hmtpy}{\simeq} \mathrm{Conf}\big(\mathbb{R}^D,\mathbb{D}^1\big) \underset{\mathrm{Conf}(\mathbb{D}^{D+1})}{\times} \mathrm{Conf}\big(\mathbb{R}^1,\mathbb{D}^D\big)$$

Ordered configurations of points in $\mathbb{R}^D$

Un-ordered configurations of points in $\mathbb{D}^{D+1}$

$$\left(\begin{tikzcd} \mathrm{Conf}\big(\mathbb{R}^D,\mathbb{D}^1\big) \arrow[dr, hook, "((i_L)^*)_*"'] & & \mathrm{Conf}\big(\mathbb{R}^1,\mathbb{D}^D\big) \arrow[dl, hook', "((i_R)^*)_*"] \\ & \mathrm{Conf}\big(\mathbb{D}^{D+1}\big) & \end{tikzcd}\right)$$



Lemma:

$$\underset{n\in\mathbb{N}\{1,\cdots,n\}}{\sqcup} \mathrm{Conf}\big(\mathbb{R}^D\big) \underset{hmtpy}{\simeq} \mathrm{Conf}\big(\mathbb{R}^D,\mathbb{D}^1\big) \underset{\mathrm{Conf}(\mathbb{D}^{D+1})}{\times} \mathrm{Conf}\big(\mathbb{R}^1,\mathbb{D}^D\big)$$

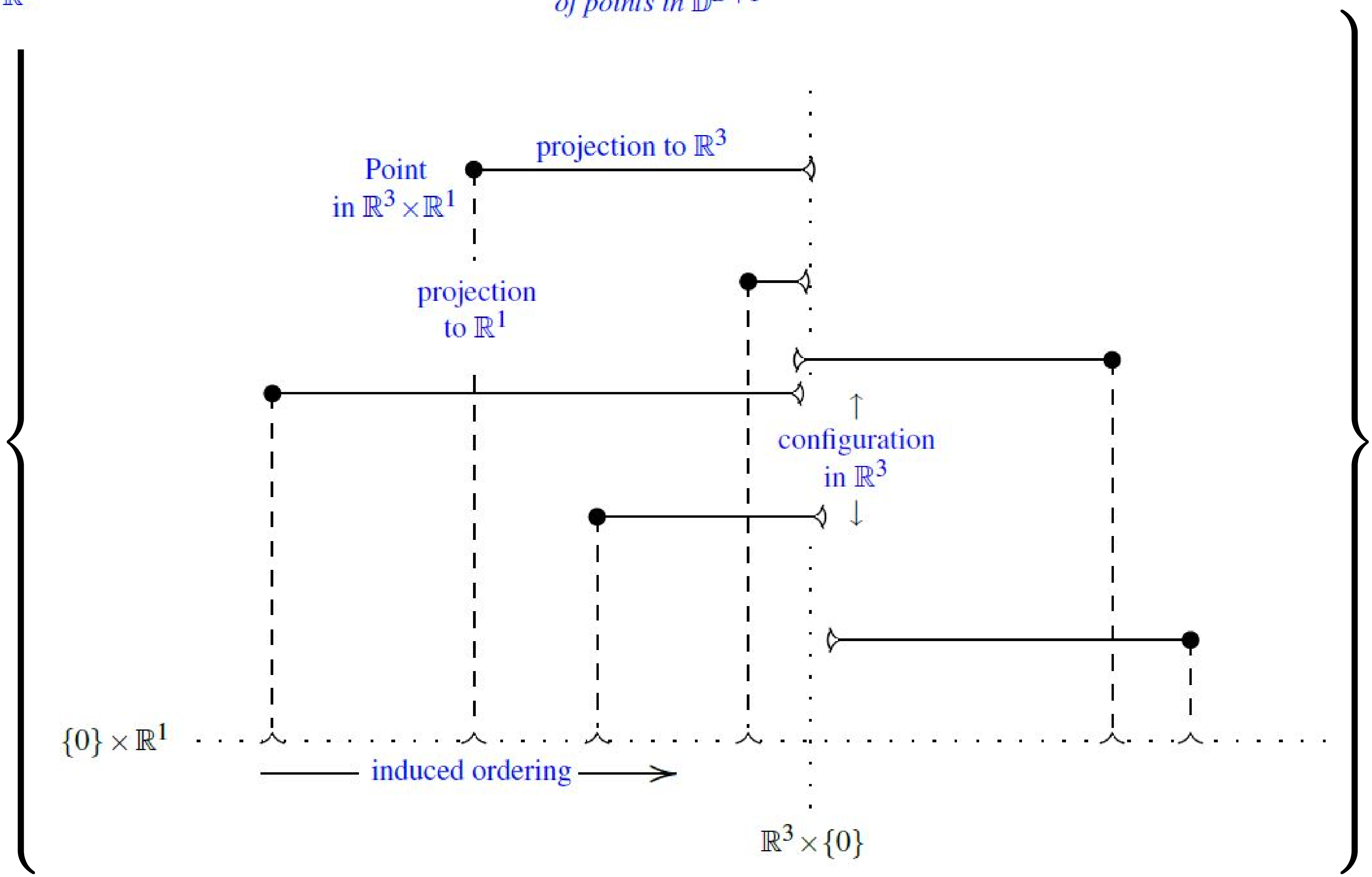
Ordered configurations of points in $\mathbb{R}^D$

Un-ordered configurations of points in $\mathbb{R}^D$ with labels in $\mathbb{D}^1$

Un-ordered configurations of points in $\mathbb{D}^{D+1}$

Un-ordered configurations of points in $\mathbb{R}^1$ with labels in $\mathbb{R}^D$

$=$

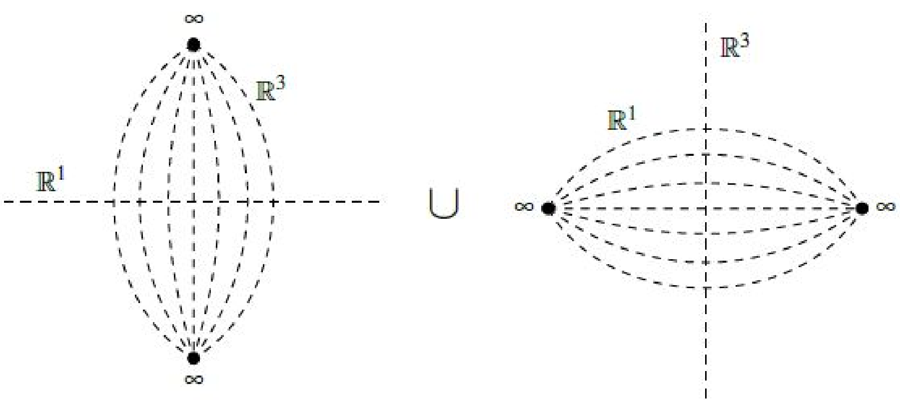


Consequence:

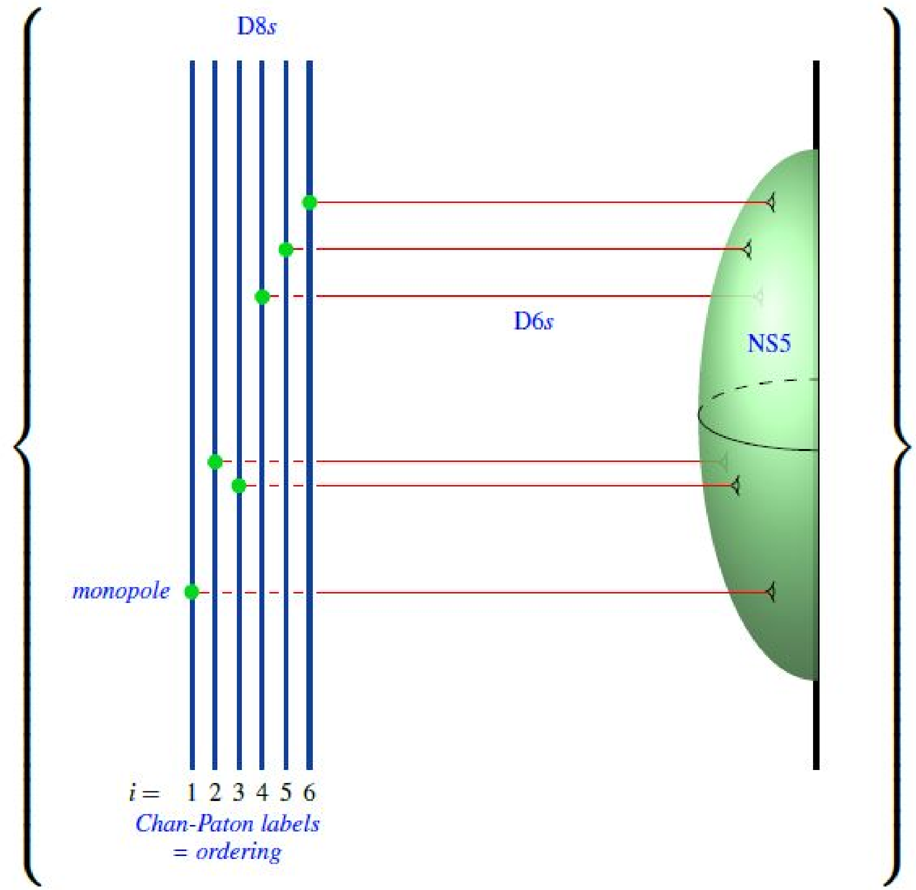
assuming
Hypothesis H:

Transversal space to 3-codim branes hence to D6-branes		*Transversal space to 1-codim branes hence to D8-branes*	*Differential Cohomotopy*	differential Cohomotopy cocycle space reflecting D6 ⊥ D8 -charges

$$(\mathbb{R}^3)^{\mathrm{cpt}} \wedge (\mathbb{R}^1)_+ \;\cup\; (\mathbb{R}^3)_+ \wedge (\mathbb{R}^1)^{\mathrm{cpt}} \xmapsto{\;\boldsymbol{\pi}^4_{\mathrm{diff}}\;} \coprod_{n \in \mathbb{N}} \mathrm{Conf}_{\{1,\cdots,n\}}\big(\mathbb{R}^3\big)$$



$\longmapsto$



(4)

Hanany-Witten Theory

implied by

Hypothesis H with Fadell-Husseini Theorem

Sati-Schreiber 19c [arXiv:1912.10425]

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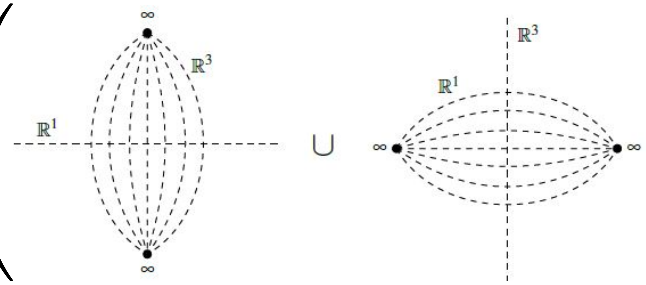
higher co-observables on D6 ⊥ D8 -intersections

$$H_\bullet\left(\underset{[c]}{\bigsqcup}\,\Omega_c\,\boldsymbol{\pi}^4_{\mathrm{diff}}\left(\cdots\right)\right)$$

topological phase space

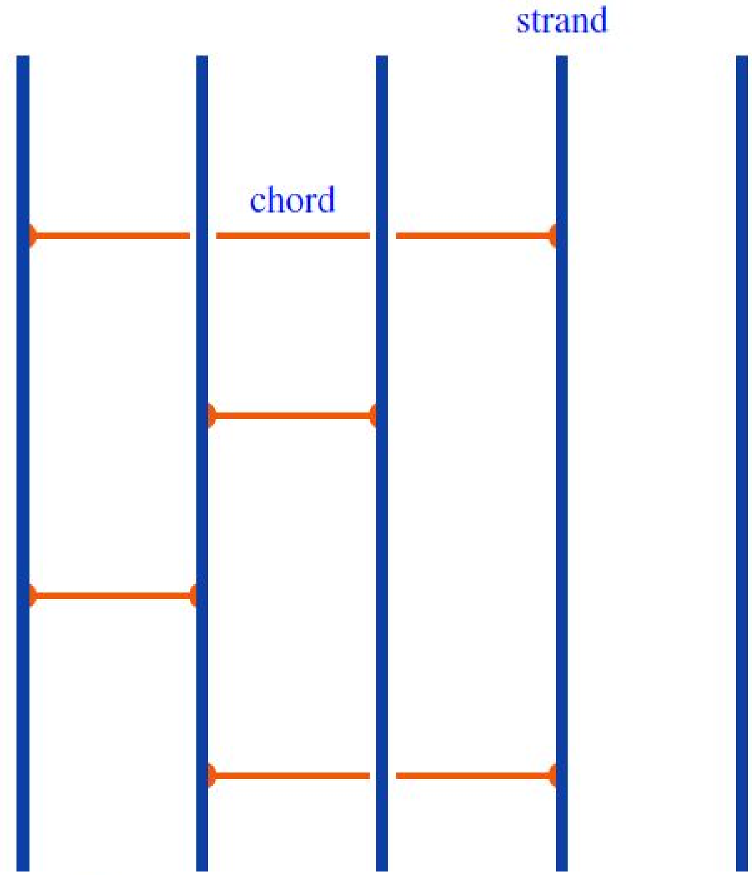


$$\simeq H_\bullet\left(\underset{N_{\mathrm{f}}\in\mathbb{N}}{\bigsqcup}\,\Omega\mathrm{Conf}_{\{1,\cdots,N_{\mathrm{f}}\}}\left(\mathbb{R}^3\right)\right)$$ (by the above)

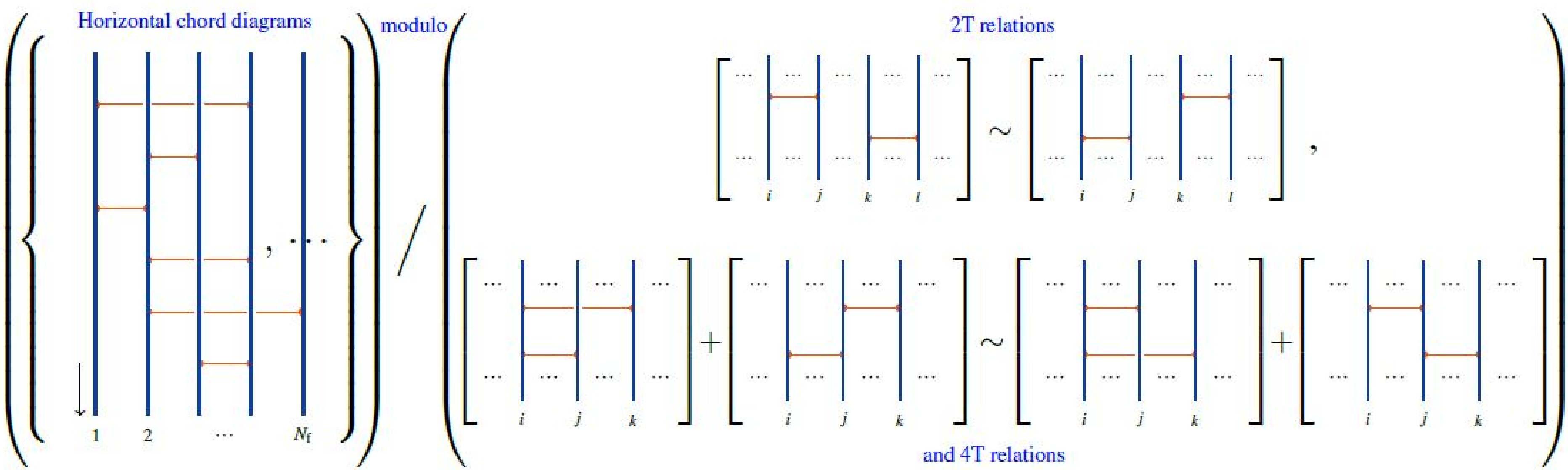
$$\simeq \underset{N_{\mathrm{f}}\in\mathbb{N}}{\bigoplus}\,\mathcal{A}^{\mathrm{pb}}_{N_{\mathrm{f}}}$$

Fadell-Husseini theorem

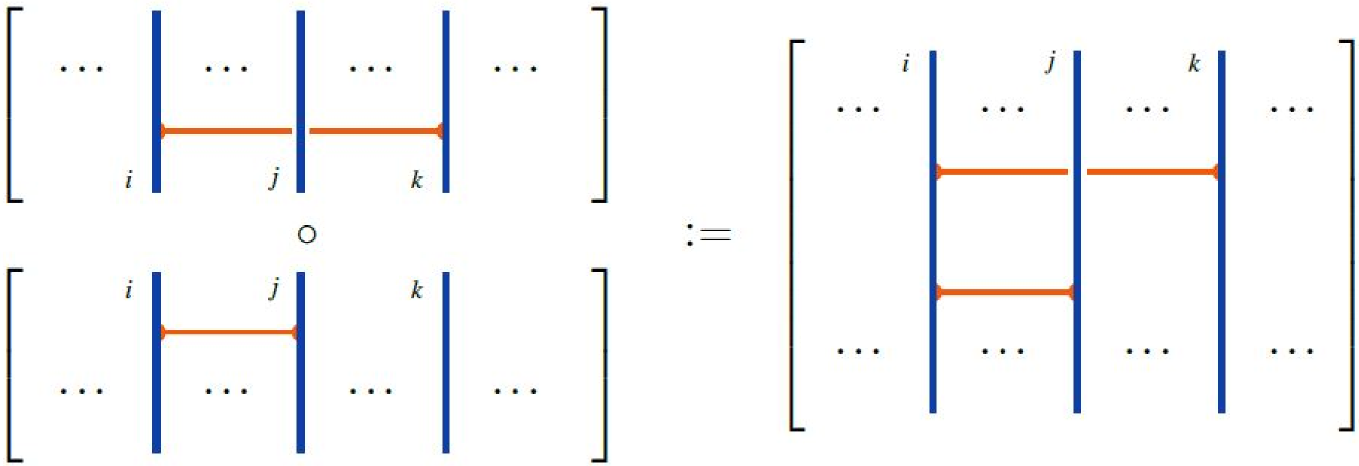
are algebra of **horizontal chord diagrams**:



$$\mathscr{A}^{\mathrm{pb}}_{N_{\mathrm{f}}} := \mathrm{Span}$$



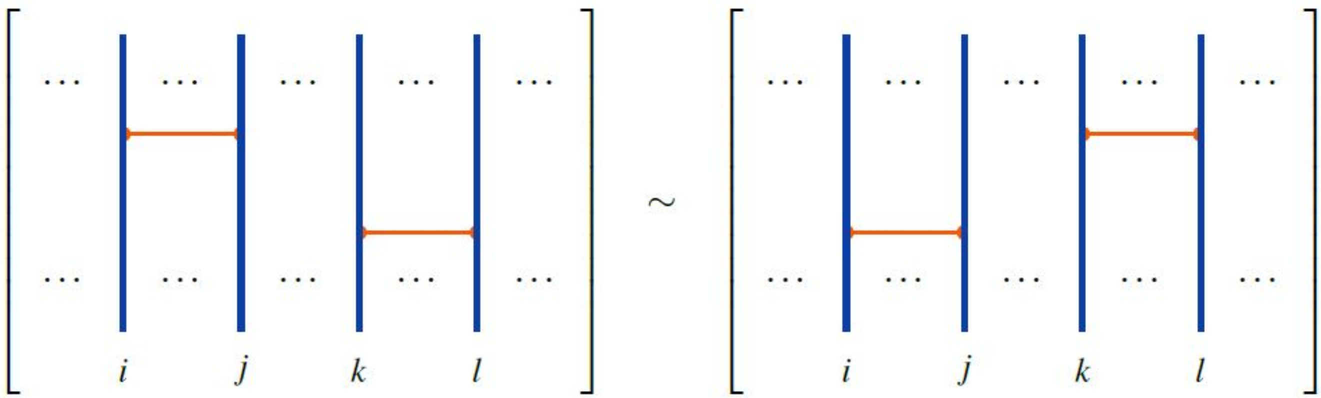
Horizontal chord diagrams form **algebra under concatenation of strands**.



$$t_{ik} \circ t_{ij} \;=\; t_{ik} t_{ij}$$

This is universal enveloping algebra of the infinitesimal braid Lie algebra (Kohno):

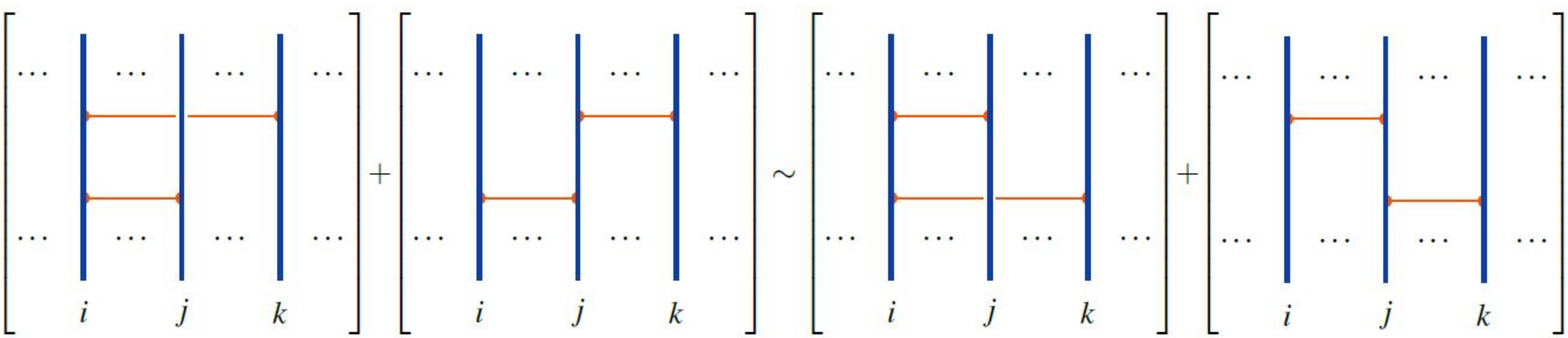
(i) the *2T relations*:



$$[t_{ij}, t_{kl}] = 0$$

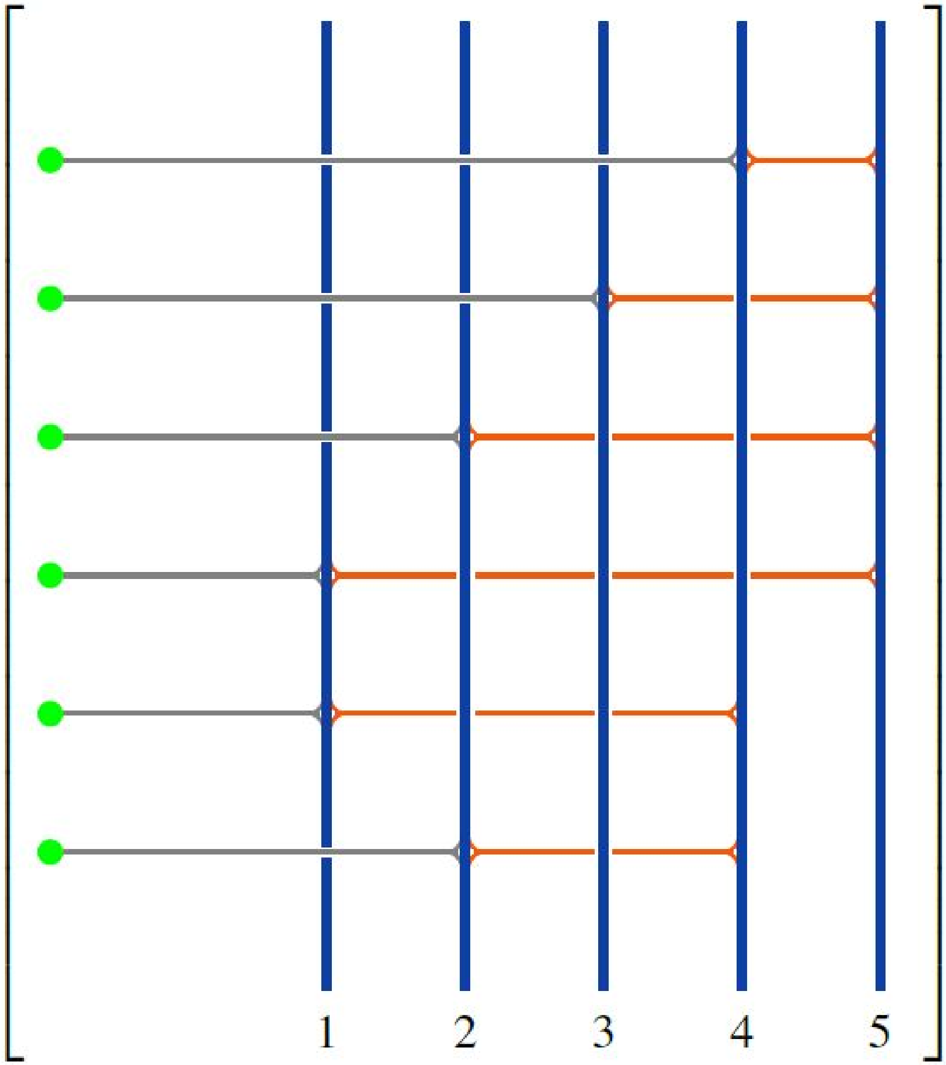
(ii) the *4T relations*

$$[t_{ij}, t_{ik} + t_{jk}] = 0$$



Consider the subspace of skew-symmetric co-observables,

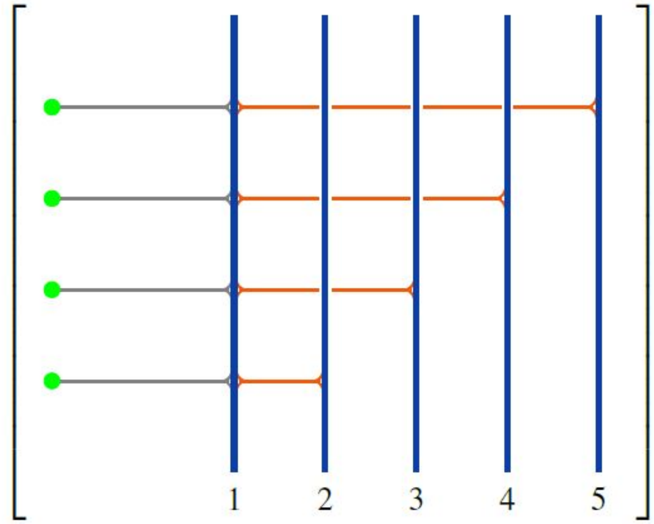
denote elements as follows:



$$= t_{45} \wedge t_{35} \wedge t_{25} \wedge t_{15} \wedge t_{14} \wedge t_{24}$$

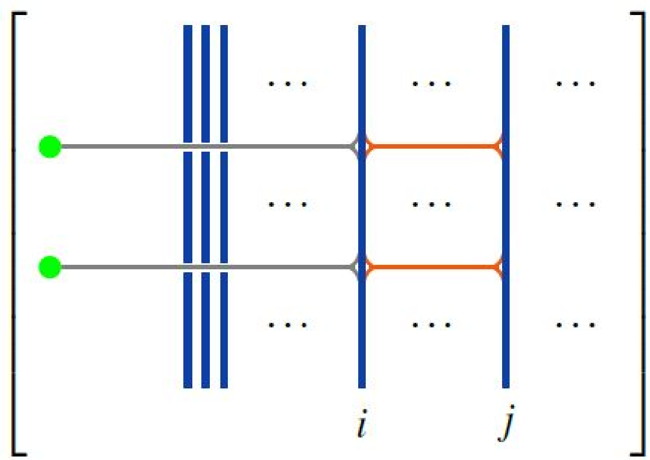
In the subspace of skew-symmetric co-observables we find:

the 2T relations become the *ordering constraint*



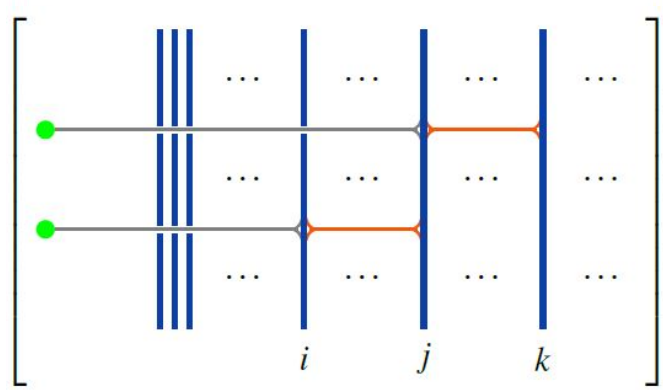
= form of any non-vanishing element

skew-symmetry becomes the *s-rule*

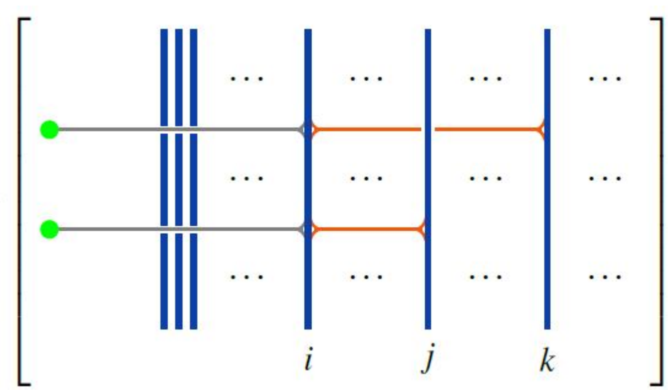


$= 0$

the 4T relations become the *breaking rule*



$= -$



these are the rules of Hanany-Witten theory
for NS5 $\perp$ Dp $\perp$ D$(p+2)$-brane intersections

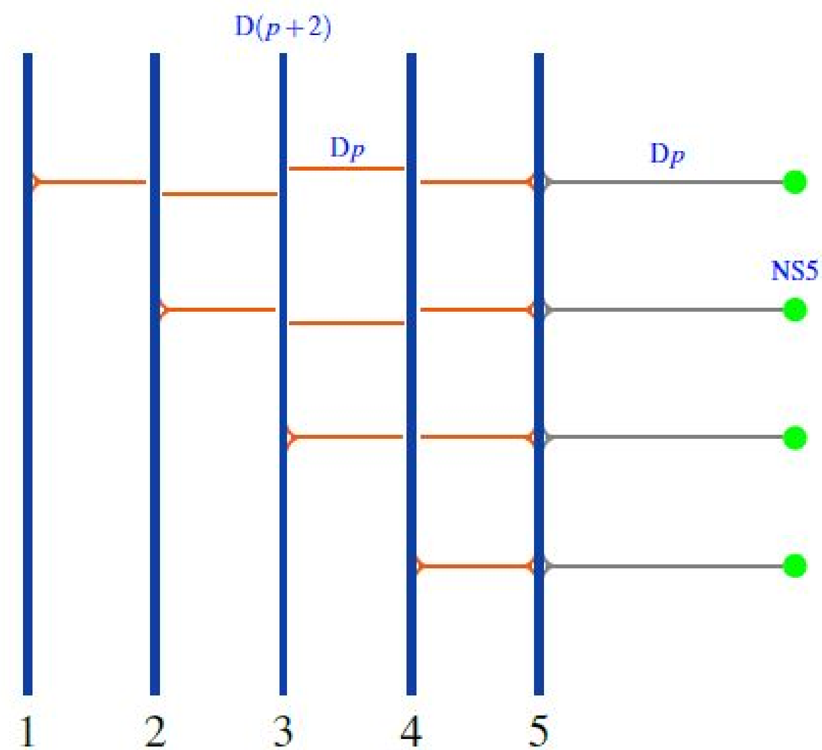
if we identify horizontal chord diagrams as follows:

(i) strands as D$(p+2)$-branes;

(ii) chords as Dp-branes,
stretching between D$(p+2)$s;

(iii) green dots as NS5-branes;

(iv) gray lines as Dp-branes,
stretching from NS5 to D$(p+2)$.



(5)

Chan-Paton data

implied by

Hypothesis H with Bar-Natan theorem

Sati-Schreiber 19c [arXiv:1912.10425]

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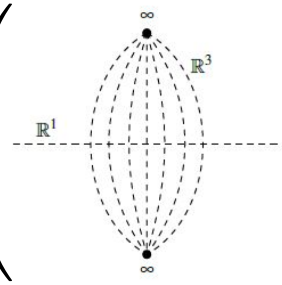
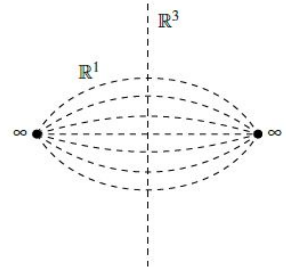
higher co-states on D6 $\perp$ D8-intersections

$$H^\bullet\Bigg(\underset{[c]}{\sqcup}\,\Omega_c\,\boldsymbol{\pi}^4_{\mathrm{diff}}\Big($$



$$\cup$$



$$\Big)\Bigg)$$

(topological phase space)

$$\simeq H^\bullet\Big(\underset{N_{\mathrm{f}}\in\mathbb{N}}{\sqcup}\underset{\{1,\cdots,N_{\mathrm{f}}\}}{\mathrm{Conf}}\big(\mathbb{R}^3\big)\Big) \qquad \text{(by the above)}$$

$$\simeq \bigoplus_{N_{\mathrm{f}}\in\mathbb{N}} \mathcal{W}^{\mathrm{pb}}_{N_{\mathrm{f}}}$$

Kohno & Cohen-Gitler theorem

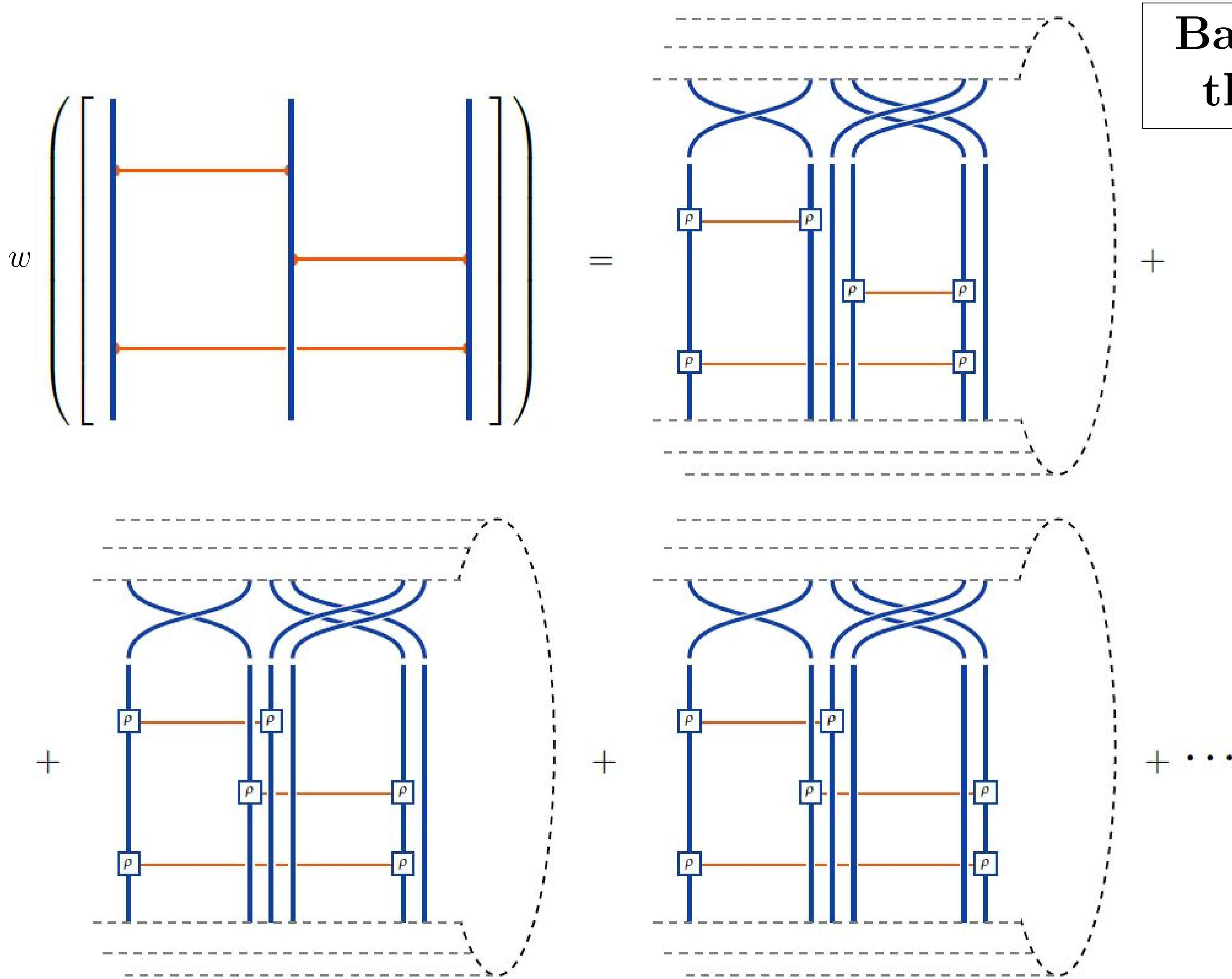
are **horizontal weight systems**:

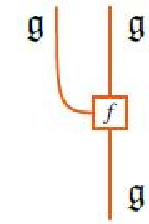
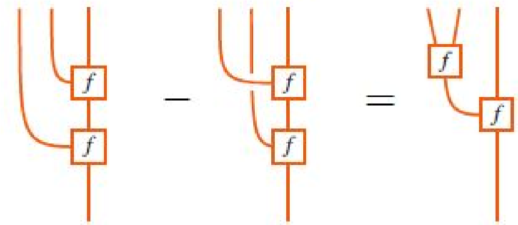
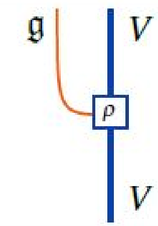
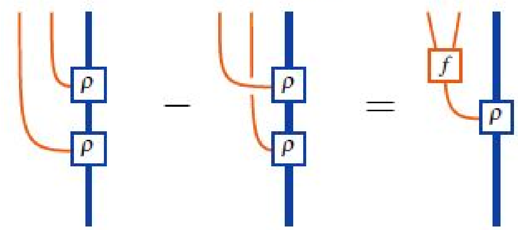
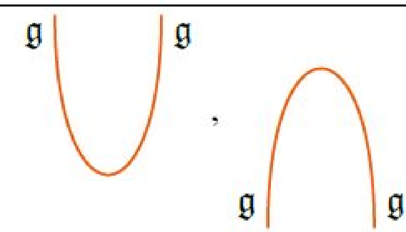
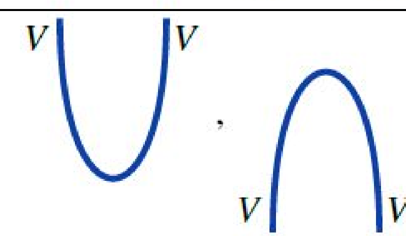
	Cohomology		weight systems			
Higher observables	$H^\bullet\big(\underbrace{\underset{N_{\mathrm{f}}\in\mathbb{N}}{\sqcup}\;\Omega\underset{\{1,\cdots,N_{\mathrm{f}}\}}{\mathrm{Conf}}(\mathbb{R}^3)}_{\text{phase space}}\big)$	$\simeq$	$\mathscr{W}$	$=$	$(\mathscr{A})^*$	Higher co-states
			dualization			
Higher co-observables	$H_\bullet\big(\overbrace{\underset{N_{\mathrm{f}}\in\mathbb{N}}{\sqcup}\;\Omega\underset{\{1,\cdots,N_{\mathrm{f}}\}}{\mathrm{Conf}}(\mathbb{R}^3)}\big)$	$\simeq$	$\mathscr{A}$	$=$	$(\mathscr{W})^*$	Higher states
	Homology		chord diagrams			

All **horizontal weight systems** $w : \mathcal{A}^{\mathrm{pb}} \to \mathbb{C}$ **come from Chan-Paton data**:

1) metric Lie representations ρ	2) stacks of coincident strands	3) winding monodromies:

Bar-Natan theorem



Data of metric Lie representation	Category notation	Penrose notation	Index notation
Lie bracket	$\mathfrak{g}\otimes\mathfrak{g} \xrightarrow{f} \mathfrak{g}$		$f_{ab}{}^{c}$
Jacobi identity	$\begin{tikzcd} \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \arrow[r, "\mathrm{id}\otimes f - f\otimes\mathrm{id}"] \arrow[d, "\sigma_{213}\circ(\mathrm{id}\otimes f)"'] & \mathfrak{g}\otimes\mathfrak{g} \arrow[d, "f"] \\ \mathfrak{g}\otimes\mathfrak{g} \arrow[r, "f"'] & \mathfrak{g} \end{tikzcd}$		$f_{ae}{}^{d} f_{bc}{}^{e} - f_{be}{}^{d} f_{ac}{}^{e} = f_{ec}{}^{d} f_{ab}{}^{e}$
Lie action	$\mathfrak{g}\otimes V \xrightarrow{\rho} V$		$\rho_{a}{}^{i}{}_{j}$
Lie action property	$\begin{tikzcd} \mathfrak{g}\otimes\mathfrak{g}\otimes V \arrow[r, "\mathrm{id}\otimes\rho - f\otimes\mathrm{id}"] \arrow[d, "\sigma_{213}\circ(\mathrm{id}\otimes\rho)"'] & \mathfrak{g}\otimes V \arrow[d, "\rho"] \\ \mathfrak{g}\otimes V \arrow[r, "\rho"'] & V \end{tikzcd}$		$\rho_{a}{}^{j}{}_{l}\,\rho_{b}{}^{l}{}_{i} - \rho_{b}{}^{j}{}_{l}\,\rho_{a}{}^{l}{}_{i} = f_{ab}{}^{c}\,\rho_{c}{}^{j}{}_{i}$
Metric	$\mathfrak{g}\otimes\mathfrak{g} \xrightarrow{g} \mathbf{1}$, $\mathbf{1} \xrightarrow{g^{-1}} \mathfrak{g}\otimes\mathfrak{g}$		g_{ab} , g^{ab}
	$V\otimes V \xrightarrow{k} \mathbf{1}$, $\mathbf{1} \xrightarrow{k^{-1}} V\otimes V$		k_{ij} , k^{ij}

(6)

BMN Matrix Model States

implied by

Hypothesis H

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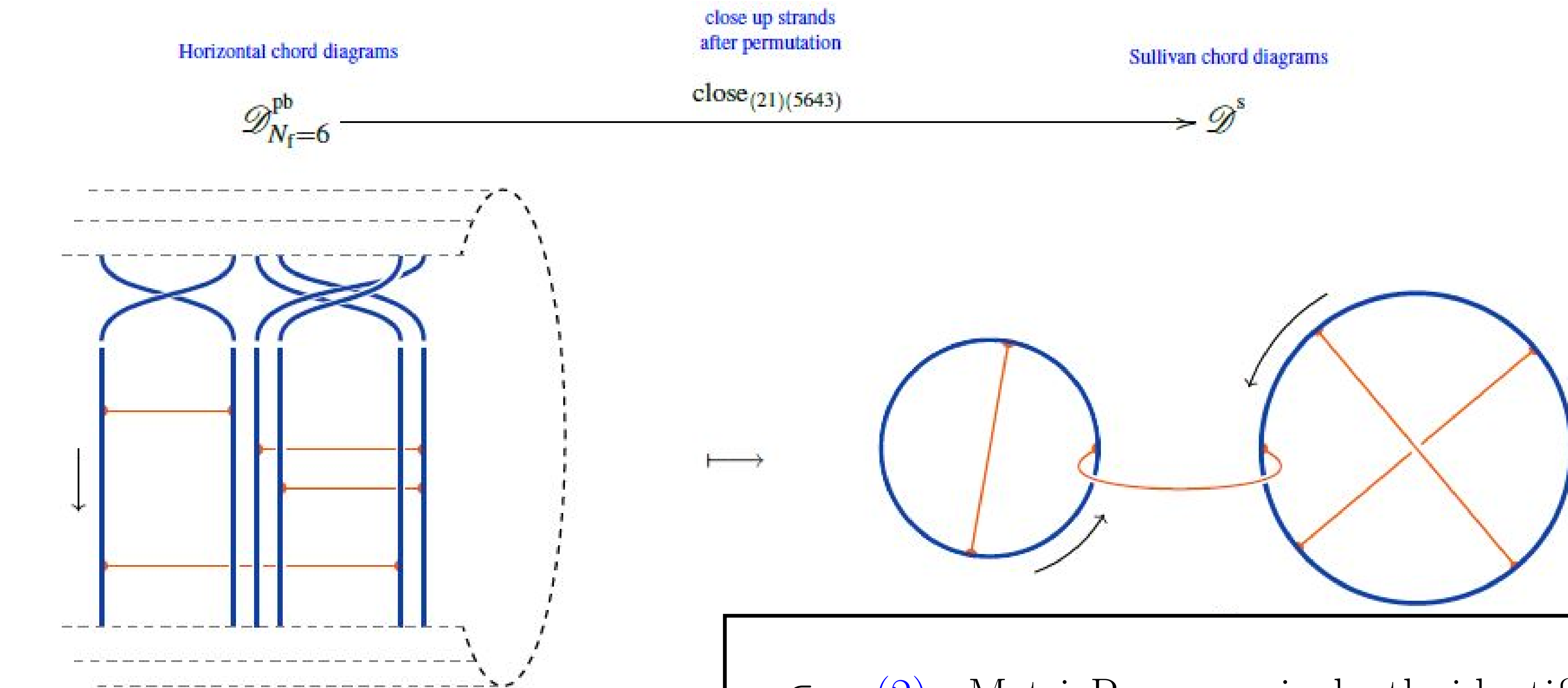
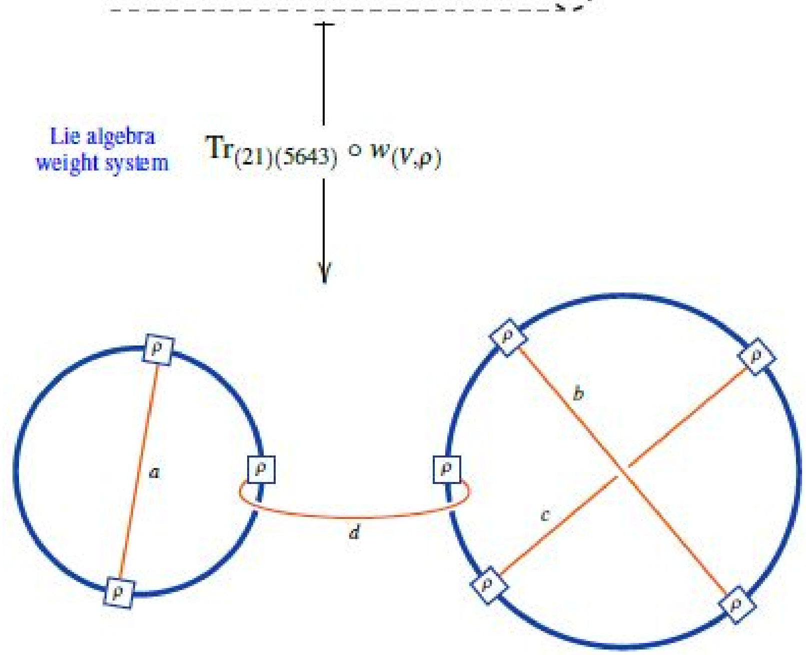




$$= \mathrm{Tr}_V \left(\rho_a \cdot \rho_d \cdot \rho^a \right) \mathrm{Tr}_V \left(\rho_b \cdot \rho_c \cdot \rho^d \cdot \rho^b \cdot \rho^c \right)$$

multi-trace observable

$\rho \in \mathfrak{su}(2)_{\mathbb{C}}$ MetricReps equivalently identified with:

0) configuration of concentric fuzzy 2-spheres
1) fuzzy funnel state in DBI model for $\mathrm{D}p \perp \mathrm{D}(p+2)$
2) susy state in BMN matrix model for M2/M5

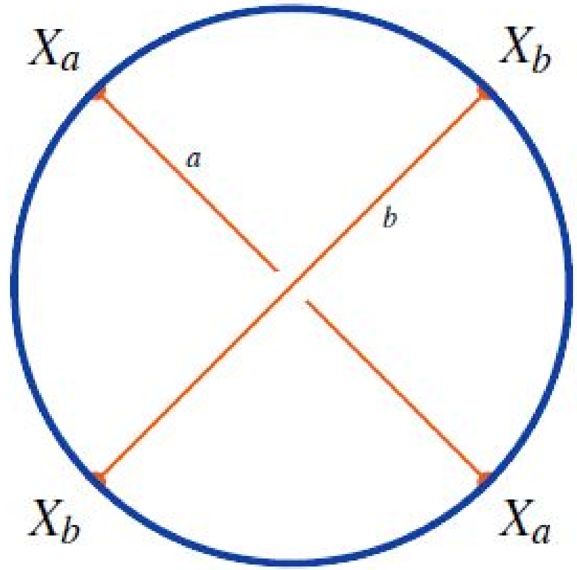
corresponding weight systems $w_{(\rho,\sigma)} : \mathcal{A}^{\mathrm{pb}} \to \mathbb{C}$ are:

0) radius fluctuation amplitudes of fuzzy 2-spheres
1)
2) invariant multi-trace observables in $\left\{ \begin{array}{l} \text{DBI model} \\ \text{BMN model} \end{array} \right.$

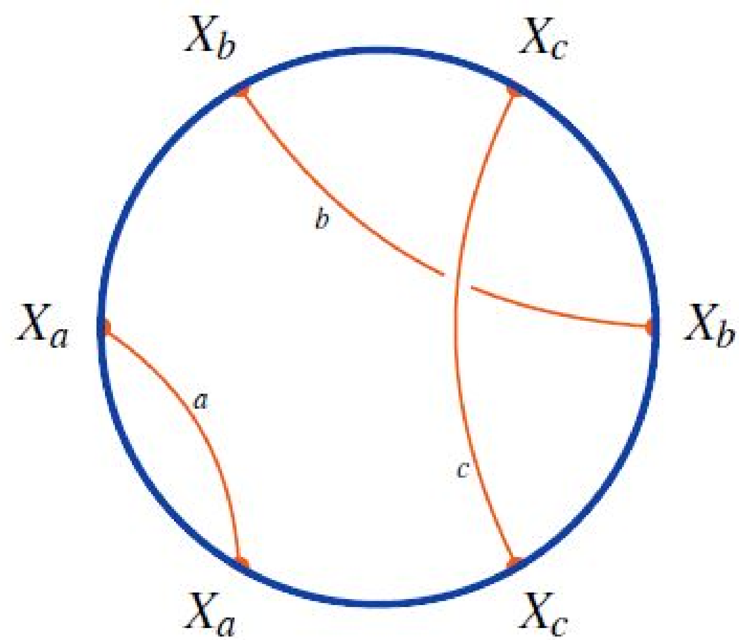
,

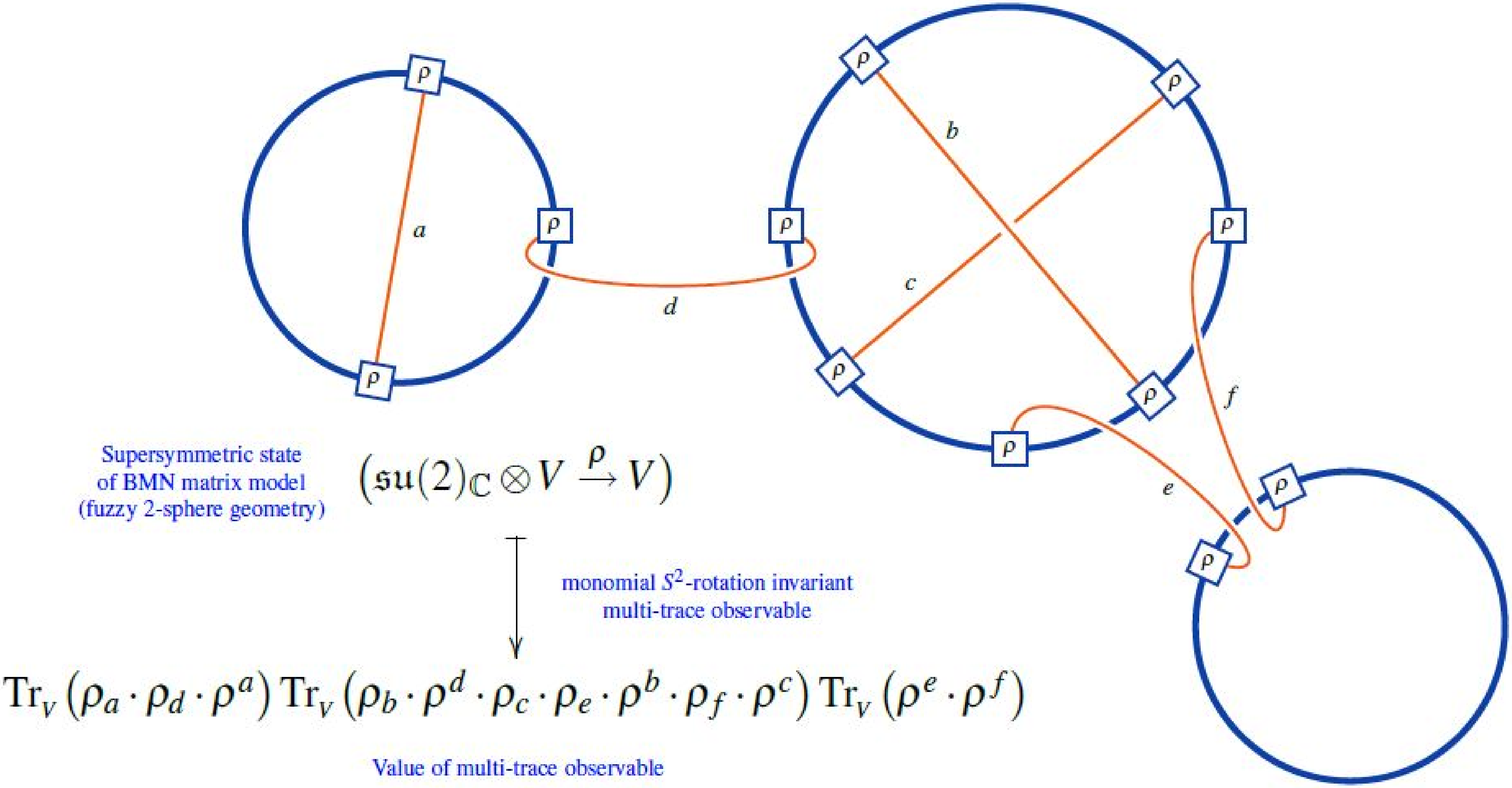
0) Radius fluctuation observables on N-bit fuzzy 2-spheres S^2_N are $\mathbf{N} \in \mathfrak{su}(2)_{\mathbb{C}}$MetricReps weight systems on chord diagrams:

$$\int_{S^2_N} (R^2)^2 = \frac{4\pi}{\sqrt{N^2-1}} \mathrm{Tr}\big(X_a \cdot X_b \cdot X^a \cdot X^b\big)$$



$$\int_{S^2_N} (R^2)^3 = \frac{4\pi}{\sqrt{N^2-1}} \mathrm{Tr}\big(X_a \cdot X^a \cdot X_b \cdot X_c \cdot X^b \cdot X^c\big)$$





1,2) weight system w_ρ is the observable aspect of matrix model state ρ:

linear combinations of
finite-dim $\mathfrak{su}_{\mathbb{C}}$-representations

weight systems on
horizontal chord diagrams

$$\mathrm{Span}\big(\mathfrak{su}(2)_{\mathbb{C}}\mathrm{MetricReps}\big) \xrightarrow{\;\rho \mapsto w_\rho\;} \mathcal{W}^{\mathrm{pb}}$$

naive funnel- / susy-states of
DBI model / BMN matrix model

states of DBI model / BMN matrix mode
as observed by invariant multi-trace observables

(7)

M2/M5 Brane Bound States

implied by

Hypothesis H

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Given a *sequence* of susy states in the BMN matrix model

$$\overbrace{(V,\rho)}^{\substack{\text{M2/M5-brane state}\\ (\text{finite-dim } \mathfrak{su}(2)_{\mathbb{C}}\text{-rep})}} \;:=\; \underbrace{\bigoplus_i}_{\substack{\text{stacks of coincident branes}\\ (\text{direct sum over irreps})}} \overbrace{\Big(N_i^{(\mathrm{M2})} \cdot \mathbf{N}_i^{(\mathrm{M5})}\Big)}^{\substack{\text{M2/M5-brane charge in } i\text{th stack}\\ (i\text{th irrep with multiplicity})}} \;\in\; \mathfrak{su}(2)_{\mathbb{C}}\mathrm{MetricReps}_{/\sim}$$

---

this is argued to converge to macroscopic M2- *or* M5-branes depending on how the sequence behaves in the large N limit:

Stacks of macroscopic...

	M2-branes	*M5-branes*	
If for all i	$N_i^{(\mathrm{M5})} \to \infty$	$N_i^{(\mathrm{M2})} \to \infty$	(the relevant large N limit)
with fixed	$N_i^{(\mathrm{M2})}$	$N_i^{(\mathrm{M5})}$	(the number of coincident branes in the ith stack)
and fixed	$N_i^{(\mathrm{M2})}/N$	$N_i^{(\mathrm{M5})}/N$	(the charge/light-cone momentum carried by the ith stack)

Given a *sequence* of susy states in the BMN matrix model

$$\overbrace{(V,\rho)}^{\substack{\text{M2/M5-brane state} \\ \text{(finite-dim } \mathfrak{su}(2)_{\mathbb{C}}\text{-rep)}}} := \underbrace{\bigoplus_i}_{\substack{\text{stacks of coincident branes} \\ \text{(direct sum over irreps)}}} \overbrace{\left(N_i^{(\mathrm{M2})} \cdot \mathbf{N}_i^{(\mathrm{M5})}\right)}^{\substack{\text{M2/M5-brane charge in } i\text{th stack} \\ (i\text{th irrep with multiplicity})}} \in \mathfrak{su}(2)_{\mathbb{C}}\mathrm{MetricReps}_{/\sim}$$

---

the large N limit does *not* exist here:

but does exist in weight systems

$$\mathrm{Span}\big(\mathfrak{su}(2)_{\mathbb{C}}\mathrm{MetricReps}\big) \xrightarrow{\rho \mapsto w_\rho} \mathcal{W}^{\mathrm{pb}}$$

if we normalize by the scale of the fuzzy 2-sphere geometry:

$$\underbrace{\frac{4\pi\, 2^{2n}}{\left(\left(N^{(\mathrm{M5})}\right)^2 - 1\right)^{1/2+n}} w_{\mathbf{N}^{(\mathrm{M5})}}}_{\substack{\text{Single M2-brane state in BMN model} \\ \text{(multiple of } \mathfrak{su}_{\mathbb{C}}\text{-weight system)}}} \in \underbrace{\mathcal{W}^{\mathrm{pb}}}_{\substack{\text{states as seen by multi-trace observables} \\ \text{(weight systems on chord diagrams)}}}$$

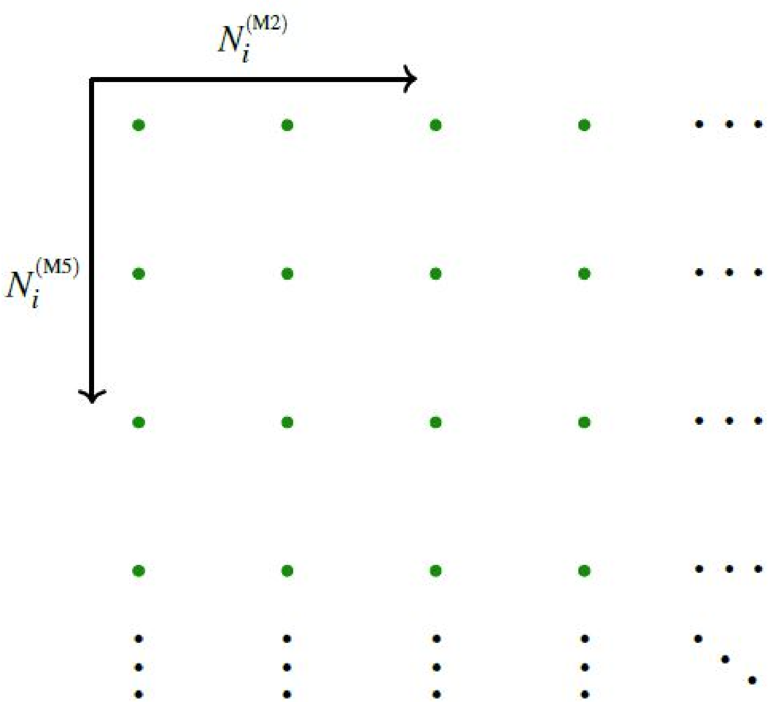
Fuzzy 2-sphere geometries (metric representations of $\mathfrak{su}(2)_{\mathbb{C}}$)

M2-M5-brane bound states (normalized Lie algebra weights)

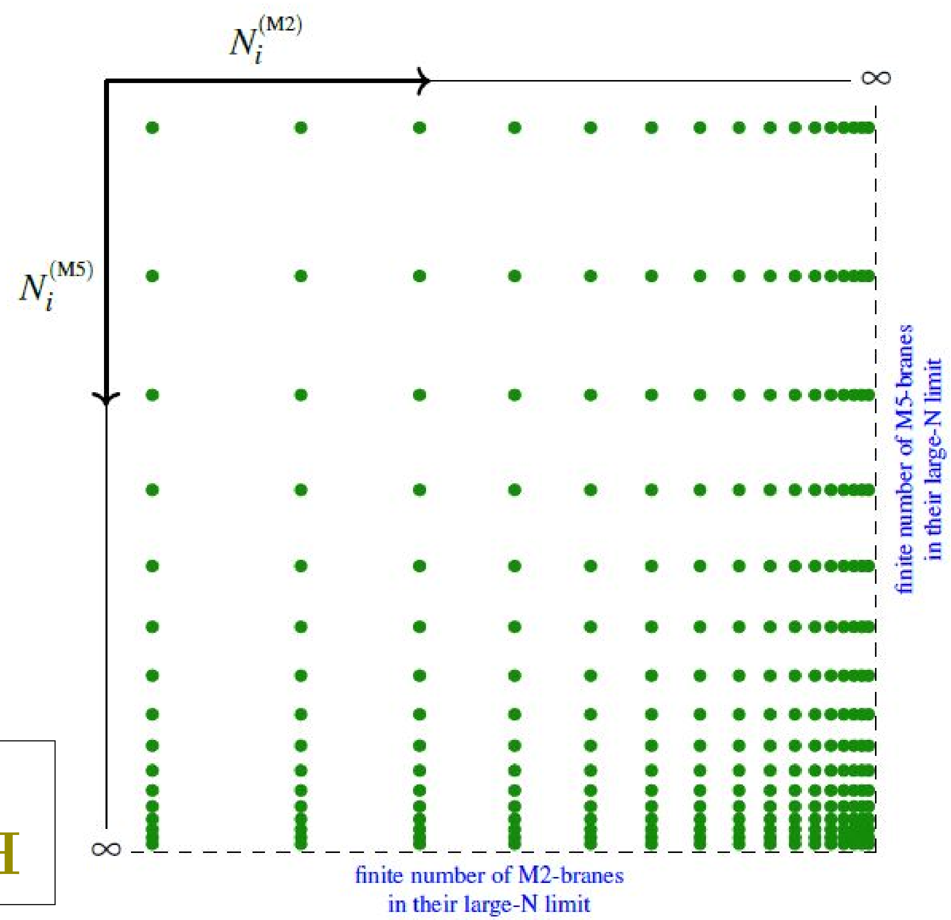
Supersymmetric states of BMN matrix model (weight systems on Sullivan chord diagrams)

$$
\begin{tikzcd}
\mathfrak{su}(2)_{\mathbb{C}}\mathrm{MetMod}_{/\sim} \arrow[r, "\Psi_{(-)}"] \arrow[d, "\wr"] & \prod\limits_{n\in\mathbb{N}} \mathscr{W}^n
\end{tikzcd}
$$

$$
\left\{ \bigoplus_{i\in\mathbb{N}} \Big(\overbrace{\underbrace{N_i^{(\mathrm{M2})}}_{\text{multiplicity}} \cdot \underbrace{\mathbf{N}_i^{(\mathrm{M5})}}_{\substack{\text{irrep of}\\ \dim_{\mathbb{C}} = N_i^{(\mathrm{M5})}}}}^{\substack{\text{Charges carried by}\\ i\text{th stack of branes}}} \Big) \;\middle|\; \begin{array}{l} \left\{\left(N_i^{(\mathrm{M2})}, N_i^{(\mathrm{M5})}\right)\right\}_{i\in\mathbb{N}} \\ \in \bigoplus\limits_{i\in\mathbb{N}} (\mathbb{N}\times\mathbb{N}) \end{array} \right\}
\longrightarrow
\left\{ \overbrace{\frac{1}{\sum\limits_{i\in\mathbb{N}} N_i^{(\mathrm{M2})}} \sum_{i\in\mathbb{N}}}^{\text{Mixture}} \overbrace{\frac{N_i^{(\mathrm{M2})}\, 4\pi\, 2^{2n}}{\left(\left(N_i^{(\mathrm{M5})}\right)^2 - 1\right)^{1/2+n}}}^{\text{Normalized radii}} \overbrace{w_{\mathbf{N}_i^{(\mathrm{M5})}}}^{\text{Lie weights}} \;\middle|\; \begin{array}{l} \left\{\left(N_i^{(\mathrm{M2})}, N_i^{(\mathrm{M5})}\right)\right\}_{i\in\mathbb{N}} \\ \in \bigoplus\limits_{i\in\mathbb{N}} (\mathbb{N}\times\mathbb{N}_{\geq 1}) \end{array} \right\} \Big/ \sim
\;\hookrightarrow\; \prod_{n\in\mathbb{N}} \mathscr{W}^n
$$



$\longmapsto$



M2/M5-brane bound states
as emergent under Hypothesis H

End.

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