T-duality in rational homotopy theory

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Abstract

What in string theory is known as topological T-duality between K^0-cocycles in type IIA string theory and K^1-cocycles in type IIB string theory or as Hori's formula can be recognized as a Fourier-Mukai transform between twisted cohomologies when looked through the lenses of rational homotopy theory. Remarkably, the whole construction naturally emerges and is actually derived from noticing that the (super-)Chevalley-Eilenberg algebra of the super-Minkowski space $\mathbb{R}^{8,1|\mathbf{16}+\mathbf{16}}$ carries *two* distinct 2-cocycles, whose product is an exact 4-cochain with an explicit trivializing 3-cochain. The super-form components of the RR-fields in type IIA and IIB string theory are then realized as rational K-theory cocycles twisted by these two 2-cocycles, and the trivializing 3-cochain induces rational topological T-duality between them.

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Contents

1 Twisted de Rham cohomology

LetX be a smooth manifold. Then we can twist the de Rham differential $d\colon \Omega^\bullet(X;\mathbb{R}) \xrightarrow{d} \Omega^\bullet(X;\mathbb{R})$ by a 1-form α, defining the twisted de Rham operator $d_\alpha\colon \Omega^\bullet(X;\mathbb{R}) \xrightarrow{d} \Omega^\bullet(X;\mathbb{R})$ as

$d_\alpha\omega = d\omega + \alpha\wedge\omega$. The operator d_α does not square to zero in general: d_α^2 is the multiplication by the exact 2-form $d\alpha$. This means that precisely when α is a closed 1-form, the operator d_α is a differential, defining an α-twisted de Rham complex $(\Omega^\bullet(X), d_\alpha)$. The cohomology of this complex is called the α-twisted de Rham cohomology of X and it will be denoted by the symbol $H^\bullet_{\mathrm{dR};\alpha}(X)$.

The operator d_α is a connection on the trivial $\mathbb{R}$-bundle over X, which is flat precisely when α is closed. This means that for a closed 1-form α, the α-twisted de Rham cohomology of X is actually a particular instance of flat cohomology or cohomology with local coefficients. Having identified d_α with a connection in the above remark, it is natural to think of gauge transformations as the natural transformations in twisted de Rham cohomology. More precisely, since we are in an abelian setting with a trivial $\mathbb{R}$-bundle, two connections d_{α_1} and d_{α_2} will be gauge equivalent exactly when there exists a smooth function β on X such that $\alpha_1 = \alpha_2 + d\beta$, i.e., when the two closed 1-forms α_1 and α_2 are in the same cohomology class. When this happens, the two twisted de Rham complexes $(\Omega^\bullet(X), d_{\alpha_1})$ and $(\Omega^\bullet(X), d_{\alpha_2})$ are isomorphic, with an explicit isomorphism of complexes given by the multiplication by the smooth function e^β. Namely, if ω is a differential form on X, we have

$$\begin{aligned}
d_{\alpha_2}(e^\beta\wedge\omega) &= d(e^\beta\wedge\omega) + \alpha_2\wedge e^\beta\wedge\omega\\
&= d\beta\wedge e^\beta\wedge\omega + e^\beta\wedge d\omega + \alpha_2\wedge e^\beta\wedge\omega\\
&= e^\beta\wedge((d\beta+\alpha_2)\wedge\omega + d\omega)\\
&= e^\beta\wedge(\alpha_1\wedge\omega + d\omega)\\
&= e^\beta\wedge d_{\alpha_1}\omega.
\end{aligned}$$

In particular, multiplication by e^β induces an isomorphism in twisted cohomology:

$$e^\beta\colon H^\bullet_{\mathrm{dR};\alpha_1}(X) \xrightarrow{\sim} H^\bullet_{\mathrm{dR};\alpha_2}(X).$$

We now investigate the functorial behaviour of twisted cohomology with repsect to a smooth map $\pi\colon Y\to X$. It is immediate to see that, since the pullback morphism $\pi^*\colon\Omega^\bullet(X)\to\Omega^\bullet(Y)$ is a morphism of differential graded commutative algebras, it induces a morphism of complexes

$$\pi^*\colon(\Omega^\bullet(X), d_\alpha)\to(\Omega^\bullet(Y), d_{\pi^*\alpha}).$$

In turn this gives a pullback morphism in twisted cohomology

$$\pi^*\colon H^\bullet_{\mathrm{dR};\alpha}(X)\to H^\bullet_{\mathrm{dR};\pi^*\alpha}(Y).$$

The pushforward morphism is a bit more delicate. To begin with, given a smooth map $\pi\colon Y\to X$ we in general have no pushforward morphism of complexes $\pi_*\colon\Omega^\bullet(Y)\to\Omega^\bullet(Y)$. However we do have such a morphism of complexes, up to a degree shift, if $Y\to X$ is not a general smooth map but it is an oriented fiber bundle with typical fiber F which is a compact closed oriented manifold: in this case π_* is given by integration along the fber and is a morphism of complexes $\pi_*\colon(\Omega^\bullet(Y), d)\to(\Omega^\bullet(X)[-\dim F], d[-\dim F])$. Yet, π_* will not induce a morphism $\pi_*\colon(\Omega^\bullet(Y), d_\alpha)\to(\Omega^\bullet(X)[-\dim F], d_{\pi_*\alpha}[-\dim F])$, and actually a minute reflection reveals that the symbol $d_{\pi_*\alpha}$ just makes no sense. However, when α is not just a generic 1-form on Y but it is a 1-form pulled back from X, then everything works fine. Namely, the projection formula

$$\pi_*(\pi^*\alpha\wedge\omega) = (-1)^{\deg\alpha\dim F}\alpha\wedge\pi_*\omega$$

precisely says that π_* is a morphism of chain complexes

$$\pi_*\colon(\Omega^\bullet(Y), d_{\pi^*\alpha})\to(\Omega^\bullet(X)[-\dim F], d_\alpha[-\dim F])$$

and so it induces a pushforward morphism in twisted cohomology

$$\pi^*\colon H^\bullet_{\mathrm{dR};\pi^*\alpha}(Y)\to H^{\bullet-\dim F}_{\mathrm{dR};\alpha}(X).$$

1.1 Fourier-Mukai transforms in twisted de Rham cohomology

All of the above suggests as to cook up a Fourier-type transform in twisted cohomology. Assume we are given a span of smooth manifolds

$$\begin{array}{ccc} & Y & \\ {\scriptstyle\pi_1}\swarrow & & \searrow{\scriptstyle\pi_2} \\ X_1 & & X_2, \end{array}$$

with $Y \xrightarrow{\pi_2} X_2$ an oriented fiber bundle with compact closed oriented fibers. Let α_i be a closed 1-form on X_i, and assume that the two 1-forms $\pi_1^*\alpha_1$ and $\pi_2^*\alpha_2$ are cohomologous in Y, with $\pi_1^*\alpha_1 - \pi_2^*\alpha_2 = d\beta$. Then we have the sequence of morphisms of chain complexes

$$(\Omega^\bullet(X_1), d_{\alpha_1}) \xrightarrow{\pi_1^*} (\Omega^\bullet(Y), d_{\pi_1^*\alpha_1}) \xrightarrow{e^\beta} (\Omega^\bullet(Y), d_{\pi_2^*\alpha_2}) \xrightarrow{\pi_{2*}} (\Omega^\bullet(X_2)[-\dim F_1], d_{\alpha_2}[-\dim F_2])$$

whose composition defines the Fourier-Mukai transform with kernel β in twisted de Rham cohomology:

$$\Phi_\beta \colon H^\bullet_{\mathrm{dR};\alpha_1}(X_1) \to H^{\bullet-\dim F_2}_{\mathrm{dR};\alpha_2}(X_2).$$

Writing $\int_F$ for π_{2*} and writing $\cdot$ for the right action of $\Omega^\bullet(X)$ on $\Omega^\bullet(Y)$ given by $\eta\cdot\omega = \eta\wedge\pi_1^*\omega$ makes it evident why this is a kind of Fourier transform:

$$\Phi_\beta \colon \omega \mapsto \int_{F_2} e^\beta \cdot \omega.$$

If moreover also $\pi_1 \colon Y \to X_1$ is an oriented fiber bundle with compact closed oriented fibers, then we also have a Fourier-Mukay transform in the inverse direction, with kernel $-\beta$. Notice that by evident degree reasons the trasforms Φ_β and $\Phi_{-\beta}$ are not inverse each other. A particular way of obtaining a span of oriented fiber bundles $X_1 \leftarrow Y \to X_2$ with compact closed oriented fibers is to consider a single oriented fiber bundle $Y \to Z$ with compact closed oriented fiber $F_1 \times F_2$. Then the manifolds X_1 and X_2 are given by the total spaces of the F_2-fiber bundle and F_1-fiber bundles on Z, respectively, associated with the two factors of $F_1 \times F_2$ together with the canonical projections. In particular, an oriented 2-torus bundle $Y \to Z$ produces this way a span $X_1 \leftarrow Y \to X_2$ where both $\pi_i \colon Y \to X_i$ are S^1-bundles. It is precisely a configuration of this kind that we will be interested in.

1.2 From 1-form twists to 3-form twists

Assume now α is a 3-form on X instead of a 1-form. Then we can still define the operator d_α on differential forms as $d_\alpha\omega = d\omega + \alpha\wedge\omega$, but this will no more be a homogeneous degree 1 operator. Yet, as α is odd, this will still be a homogeneous operator if we collapse the grading on differential forms from the usual $\mathbb{Z}$-grading to the associated $\mathbb{Z}/2\mathbb{Z}$-grading. Doing so, and denoting by $\overline{n}$ the $\mod 2$ class of an integer n, the opearor d_α is an operator

$$d_\alpha \colon \Omega^{\overline{\bullet}}(X) \to \Omega^{\overline{\bullet}+\overline{1}}(X),$$

and the above discussion verbatim applies, with the de Rham complex $\Omega^\bullet(X)$ replaced by the 2-periodic de Rham complex $\Omega^{\overline{\bullet}}(X)$. In particular, if we have a span $X_1 \leftarrow Y \to X_2$ of oriented S^1-bundles and if α_i are 3-forms on X_i such that $\pi_1^*\alpha_1 - \pi_2^*\alpha_2 = d\beta$ for some 2-form β on Y, then we have Fourier-Mukai transforms

$$\Phi_\beta \colon H^{\overline{\bullet}}_{\mathrm{dR};\alpha_1}(X_1) \to H^{\overline{\bullet}-\overline{1}}_{\mathrm{dR};\alpha_2}(X_2)$$
$$\Phi_{-\beta} \colon H^{\overline{\bullet}}_{\mathrm{dR};\alpha_2}(X_2) \to H^{\overline{\bullet}-\overline{1}}_{\mathrm{dR};\alpha_1}(X_1)$$

Since $\overline{2} = \overline{0}$, the compositions $\Phi_{-\beta} \circ \Phi_\beta$ and $\Phi_\beta \circ \Phi_{-\beta}$ preserve the $\mathbb{Z}/2\mathbb{Z}$-degree and so there is now no degree obstruction to the possibility that Φ_β and $\Phi_{-\beta}$ are inverse each other. We will come back to this later.

As we have moved from the $\mathbb{Z}$-grading to the $\mathbb{Z}/2\mathbb{Z}$-grading there is apparently no point in considering 3-forms rather than 1-forms or 5-forms. From the the $\mathbb{Z}/2\mathbb{Z}$-graded Rham point of view it is actually pointless even to have differential forms of homogeneous odd degree: the above argument would identically apply to an odd form $\alpha = \alpha_{(1)} + \alpha_{(3)} + \alpha_{(5)} + \cdots$, where $\alpha_{(i)}$ is a differential form of $\mathbb{Z}$-degree i on X. There is however an important geometrical reason to focus on degree 3 forms. namely, when coefficients are taken in a characteristic zero field, even de Rham cohomology is isomorphic (via the Chern character) with even K-theory, and odd de Rham cohomology is isomorphic to odd K-thory. Under these isomorphisms, K-theory twists (which are topologically given by principal $U(1)$-gerbes and so are classified by maps to $B^2U(1) \simeq K(\mathbb{Z}, 3)$) precisely become closed 3-forms. In other words, for α_1 and α_2 closed 3-forms as above, the Fourier-Mukai transform Φ_β is to be thought as a morphism

$$\Phi_\beta \colon K^{\overline{\bullet}}_{\mathcal{G}_1}(X_1) \otimes \mathbb{R} \to K^{\overline{\bullet}-\overline{1}}_{\mathcal{G}_2}(X_2) \otimes \mathbb{R}.$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are the twisting gerbes. This is indeed the rationalization, with real coefficients, of a topological Fourier-Mukai transform

$$\Phi_\beta \colon K^{\overline{\bullet}}_{\mathcal{G}_1}(X_1) \to K^{\overline{\bullet}-\overline{1}}_{\mathcal{G}_2}(X_2).$$

When the span $X_1 \leftarrow Y \rightarrow X_2$ of oriented S^1-bundles is induced by a 2-torus bundle $Y \to Z$, and so by a classifying map $Z \to B(U(1) \times U(1)) \cong BU(1) \times BU(1)$, we say we are given a de Rham T-fold configuration. In the following section we are going to investigate these from the point of view of rational homotopy theory. This will be able to read all of the information contained in the Fourier-Mukai transform associated with a de Rham T-fold configuration, as the rational homotopy type of a space retains all of its cohomology overa characteristic zero field (as well as all of its homotopy up to torsion).

2 Basics of rational homotopy theory

The idea at the heart of rational homotopy theory is that, up to torsion, all of the homotopy type of a simple space[1] with finite rank cohomology groups is encoded in its de Rham algebra with coefficients in a characteristic zero field, as a differential graded commutative algebra, up to homotopy [Qu69, Su77]. Moreover, since one has the freedom to replace the de Rham algebra with any homotopy equivalent DGCA, one sees that up to torsion the homotopy type of a simple space X is encoded into its so called minimal model or Sullivan algebra: a DGCA A_X equipped with a quasi-isomorphism of differential graded commutative algebras $A_X \to \Omega^\bullet(X)$, which is semi-free, i.e., which is a free graded commutative algebra when one forgets the differential, and such that the differential is decomposable, i.e., it has no linear component. In other words, A_X is a DGCA of the form $(\wedge^\bullet \mathfrak{l}X^*, d) = (\mathrm{Sym}^\bullet(\mathfrak{l}X[1]^*), d)$ for a suitable graded vector space $\mathfrak{l}X$ (finitely dimensional in each degree) and a suitable degree 1 differential d with $d(\mathfrak{l}X^*) \subseteq \wedge^{\geq 2}\mathfrak{l}X^*$. Here $\mathfrak{l}X^*$ denotes the graded linear dual of $\mathfrak{l}X$, and the degree shift in the definition of $\wedge^\bullet$ is there in order to match the degree coming from geometry: the de Rham algebra is generated by 1-forms, which are in degree 1. The semifreenes property together with the datum of the quasiisomorphism to the de Rham algebra and the decomposability of the differential uniquely characterize the minimal model up to isomorphism and the quasiisomorphism to the de Rham algebra up to homotopy, so that one can talk of *the* minimal model of a space X. The pair $(\wedge^\bullet \mathfrak{l}X^*, d)$ is what is called a *minimal L_∞-algebra* structure on $\mathfrak{l}X$ in the theory of L_∞-algebras. Equivalently, one says that the DGCA $(\wedge^\bullet \mathfrak{l}X^*, d)$ is the Chevalley-Eilenberg algebra of the L_∞-algebra $\mathfrak{l}X$ (omitting the L_∞ brackets of $\mathfrak{l}X$ from the notation), and writes

[1] i.e., a connected topological space that has a homotopy type of a CW complex and whose fundamental group is abelian and acts trivially on the homotopy and homology of the universal covering space

$$(A_X, d_X) \cong (\mathrm{CE}(\mathfrak{l}X), d_X)$$

as the defining equation of the L_∞-algebra $\mathfrak{l}X$. We say that the L_∞-algebra $\mathfrak{l}X$ is the rational approximation of X. Geometrically, it can be thought of as the tangent L_∞-algebra to the ∞-group given by the based loop space of X (as X is simple, the choice of a basepoint is irrelevant). A smooth map $f\colon Y \to X$ is faithfully encoded into the DGCA morphism $f^*\colon \Omega^\bullet(X) \to \Omega^\bullet(Y)$, so that the rational approximation of f is encoded into a DGCA morphism, which we will continue to denote f^*,

$$f^*\colon A_Y \to A_X.$$

In turn, by definition, this is a morphism of L_∞-algebras $\mathfrak{l}f\colon \mathfrak{l}X \to \mathfrak{l}Y$. Finally, up to homotopy, every L_∞ algebra is equivalent to a minimal one: this is the dual statement of the fact that every (well behaved) DGCA is homotopy equivalent to a minimal DGCA. Therefore we get the fundamental insight of rational homotopy theory: *the category of simple homotopy types over a characteristic zero field $\mathbb{K}$ is (equivalent to) the homotopy category of L_∞-algebras over $\mathbb{K}$.*

Example 2.1. The above description of rational homotopy theory may have erroneously suggested it is a quite abstract construction. So let us make an example to make it concrete. Consider the classifying space $BU(1)$. Its real cohomology is $H^\bullet(BU(1);\mathbb{R}) \cong \mathbb{R}[x_2]$, where x_2 is a degree 2 element, the universal first Chern class. As $H^\bullet(BU(1);\mathbb{R})$ is a free polynomial algebra, we can think of it as a semifree DGCA with trivial differential. Moreover, choosing a de Rham representative for the first chern Class defines a quasi-isomorphism

$$(\mathbb{R}[x_2], 0) \to (\Omega^\bullet(BU(1)), d)$$

exhibiting $(\mathbb{R}[x_2], 0)$ as the Sullivan model of $BU(1)$. The equation

$$(\mathbb{R}[x_2], 0) \cong (\mathrm{CE}(\mathfrak{l}BU(1)), d_{BU(1)})$$

then characterizes $\mathfrak{l}BU(1))$ as the L_∞-algebra consisting of the cochain complex $\mathbb{R}[1]$ consisting of the vector space $\mathbb{R}$ in degree -1 and zero in all other degrees (with zero differential). We will denote this L_∞-algebra by the symbol $\mathfrak{bu}_1$. A principal $U(1)$-bundle $P \to X$ is classified by a map $X \to BU(1)$. The rational approximation of this map is an L_∞-morphism

$$\mathfrak{l}X \to \mathfrak{bu}_1.$$

Equivalently, by definition, this is a DGCA morphism

$$(\mathbb{R}[x_2], 0) \to (A_X, d_X),$$

i.e., it is a degree 2 closed element in A_X. By pushing it forward along the quasiisomorphism $(A_X, d_X) \xrightarrow{\sim} (\Omega^\bullet(X), d)$ we get a closed 2-form ω_2 on X associated to the principal $U(1)$-bundle $P \to X$. Since the quasiisomorphism $(A_X, d_X) \xrightarrow{\sim} (\Omega^\bullet(X), d)$ is only unique up to homotopy, the 2-form ω_2 is only well defined up to an exact term so that it is the cohomology class $[\omega_2]$ to be actually canonically associated with $P \to X$. No surprise, $[\omega_2]$ is the image in de Rham cohomology of the first Chern class of $P \to X$.

Given the identification between simple homotopy types and L_∞-algebras mentioned above, from now on we will mostly work directly with L_∞-algebras, with no reference to the space they can be a rationalization of. Thereore, a span $X_1 \leftarrow Y \to X_2$ as in the discussion of Fourier-Mukai transforms in twisted de Rham cohomology will become a span

$$\begin{array}{ccc} & \mathfrak{h} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathfrak{g}_1 & & \mathfrak{g}_2 \end{array}$$

of L_∞-algebras. As we want that the π_i's represent the (rationalization of) S^1-bundles our next step is the characterization of those L_∞-morphism that correspond to principal $U(1)$-bundles.

3 Central extensions of L_∞-algebras

A principal $U(1)$-bundle over a smooth manifold X is encoded up to homotopy into a map $f\colon X \to BU(1)$ from X to the classifying space $U(1)$. The total space P as well as the projection $P \to X$ are recovered by f by taking its homotopy fiber, i.e., by considering the homotopy pullback

$$\begin{array}{ccc} P & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BU(1) \end{array} .$$

As rationalization commutes with homotopy pullbacks, the rational approximation of the above diagram is

$$\begin{array}{ccc} \mathfrak{l}P & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathfrak{l}X & \xrightarrow{\mathfrak{l}f} & b\mathfrak{u}_1 \end{array} .$$

Dually, this means that we have a homotopy pushout of DGCAs

$$\begin{array}{ccc} (\mathbb{R}[x_2], 0) & \longrightarrow & (\mathbb{R}, 0) \\ {\scriptstyle f^*}\downarrow & & \downarrow \\ (A_X, d_X) & \longrightarrow & (A_P, d_P) \end{array} .$$

This is easily computed. All we have to do is to replace the DCGA morphism $\mathbb{R}[x_2] \to \mathbb{R}$ with an equivalent cofibration. The easiest way of doing this is to factor $\mathbb{R}[x_2] \to \mathbb{R}$ as

$$(\mathbb{R}[x_2], 0) \hookrightarrow (\mathbb{R}[y_1, x_2], dy_1 = x_2) \xrightarrow{\sim} \mathbb{R}.$$

Then A_P is computed as an ordinary pushout

$$\begin{array}{ccc} (\mathbb{R}[x_2], 0) & \longrightarrow & (\mathbb{R}[y_1, x_2], dy_1 = x_2) \\ {\scriptstyle f^*}\downarrow & & \downarrow \\ (A_X, d_X) & \longrightarrow & (A_P, d_P) \end{array} ,$$

i.e.,

$$(A_P, d_P) = (A_X[y_1], d_P\omega = d_X\omega \text{ for } \omega \in A_X,\ d_P y_1 = f^* x_2).$$

This immediately generalizes to the case of an arbitrary morphism $f\colon \mathfrak{g} \to b\mathfrak{u}_1$. The homotopy fiber of f will be the L_∞-algebra $\hat{\mathfrak{g}}$ characterized by

$$\mathrm{CE}(\hat{\mathfrak{g}}) = \mathrm{CE}(\mathfrak{g})[y_1],$$

where y_1 ia a variable in degree 1 and where the differential in $\mathrm{CE}(\hat{\mathfrak{g}})$ extends that in $\mathrm{CE}(\mathfrak{g})$ by the rule $d_{\hat{\mathfrak{g}}} y_1 = f^*(x_2)$.

Example 3.1. If $\mathfrak{g}$ is a Lie algebra (over $\mathbb{R}$), then an L_∞-morphism $f\colon \mathfrak{g} \to b\mathfrak{u}_1$ is precisely a Lie algebra 2-cocycle on $\mathfrak{g}$ with values in $\mathbb{R}$. The L_∞-algebra $\hat{\mathfrak{g}}$ is again a Lie algebra in this case, and it is the central extension of $\mathfrak{g}$ by $\mathbb{R}$ classified by the 2-cocycle f.

The above construction admits an immediate generalization. Instead of $b\mathfrak{u}_1$ we can consider the L_∞-algebra $b^n\mathfrak{u}_1$ given by the cochain complex $\mathbb{R}[n]$ consisting of $\mathbb{R}$ in degree $-n$ and zero in all other degrees. The corresponding Chevalley-Eilenberg algebra is

$$(\mathrm{CE}(b^n\mathfrak{u}_1), d_{b^n\mathfrak{u}_1}) = (\mathbb{R}[x_{n+1}], 0),$$

where x_{n+1} is in degree $n+1$. One sees that $b^n\mathfrak{u}_1$ is a rational model (with coefficients in $\mathbb{R}$) for the classifying space $B^nU(1)$ of principal $U(1)$-n bundles (or principal $U(1)$-$(n-1)$-gerbes), which is a $K(\mathbb{Z}, n+1)$. If $\mathfrak{g}$ is a Lie algebra, then an L_∞-morphism $\mathfrak{g} \to b^n\mathfrak{u}_1$ is precisely a Lie algebra $(n+1)$-cocycle on $\mathfrak{g}$ with coefficients in $\mathbb{R}$. More generally, an L_∞-morphism $\mathfrak{g} \to b^n\mathfrak{u}_1$ with $\mathfrak{g}$ an L_∞-algebra will also be called an $(n+1)$-cocycle. The dual picture makes this terminology transparent: an $(n+1)$-cocycle on $\mathfrak{g}$ is a DGCA morphism

$$(\mathbb{R}[x_{n+1}], 0) \to (\mathrm{CE}(\mathfrak{g}), d_\mathfrak{g})$$

so it is precisely a closed degree $n+1$ element in $\mathrm{CE}(\mathfrak{g})$. The description of homotopy fibers of 2-cocycles immediately generalizes to higher cocycles: the homotopy fiber $\hat{\mathfrak{g}}$ of an $(n+1)$-cocycle $\mathfrak{g} \to b^n\mathfrak{u}_1$ is characterized by

$$\mathrm{CE}(\hat{\mathfrak{g}}) = \mathrm{CE}(\mathfrak{g})[y_n],$$

where y_n ia a variable in degree n and where the differential in $\mathrm{CE}(\hat{\mathfrak{g}})$ extends that in $\mathrm{CE}(\mathfrak{g})$ by the rule $d_{\hat{\mathfrak{g}}}y_1 = f^*(x_{n+1})$. By analogy with the case of 2-cocycles on Lie algebras, one calls $\hat{\mathfrak{g}}$ a higher central extension of $\mathfrak{g}$. Geometrically, $\hat{\mathfrak{g}}$ is to be thought as the total space of a rational $U(1)$-n-bundle over $\mathfrak{g}$

4 Twisted L_∞-algebra cohomology

As we remarked, a (finite dimensional in each degree) L_∞-algebra $\mathfrak{g}$ is encoded into its Chevalley-Eilenberg algebra $(\mathrm{CE}(\mathfrak{g}), d_\mathfrak{g})$. As this is differential graded commutaive algebra, we can consider its cohomology which, by definition, is the L_∞-algebra cohomology of $\mathfrak{g}$:

$$H^\bullet_{L_\infty}(\mathfrak{g};\mathbb{R}) = H^\bullet\left(\mathrm{CE}(\mathfrak{g}), d_\mathfrak{g}\right).$$

When $\mathfrak{g}$ is a Lie algebra this reproduces the Lie algebra cohomology of $\mathfrak{g}$. If $\mathfrak{g}$ is the L_∞-algebra representing the rational omotopy type of a simple space X, then the L_∞-algebra cohomology of $\mathfrak{g}$ computes the de Rham cohomology of X. Namely,

$$H^\bullet_{L_\infty}(\mathfrak{l}X;\mathbb{R}) = H^\bullet\left(\mathrm{CE}(\mathfrak{l}X), d_X\right) = H^\bullet\left(A_X, d_X\right)) \cong H^\bullet\left(\Omega^\bullet(X), d\right)) = H^\bullet_{\mathrm{dR}}(X).$$

Example 4.1. If $\mathfrak{g}$ is the Lie algebra of a compact Lie group G, then one recovers the classical statement that the Lie algebra cohomology of $\mathfrak{g}$ computes the de Rham cohomology of G:

$$H^\bullet_{\mathrm{Lie}}(\mathfrak{g};\mathbb{R}) \cong H^\bullet_{\mathrm{dR}}(G).$$

This has actually been one of the motivating examples in the definition of Lie algebra cohomology.

Exactly as we twisted de Rham cohomology we can twist L_∞-algebra cohomology: if a is a degree 3 cocycle on $\mathfrak{g}$ then we can consider the odd operator $d_{\mathfrak{g};a}\colon x \mapsto d_\mathfrak{g}x + a\,x$ on the ($\mathbb{Z}/2\mathbb{Z}$-graded algebra underlying the) Chevalley-Eilenberg algebra of $\mathfrak{g}$ and define

$$H^{\overline{\bullet}}_{L_\infty;a}(\mathfrak{g};\mathbb{R}) = H^{\overline{\bullet}}\left(\mathrm{CE}(\mathfrak{g}), d_{\mathfrak{g};a}\right).$$

As in the de Rham case, if a_1 and a_2 are cohomologous 3-cocycles with $a_1 - a_2 = db$ then e^b is a cochain complexes isomorphism between $(\mathrm{CE}(\mathfrak{g}), d_{\mathfrak{g};a_1})$ and $(\mathrm{CE}(\mathfrak{g}), d_{\mathfrak{g};a_2})$ and so induces an isomorphism

$$e^b\colon H^{\overline{\bullet}}_{L_\infty;a_1}(\mathfrak{g};\mathbb{R}) \xrightarrow{\sim} H^{\overline{\bullet}}_{L_\infty;a_2}(\mathfrak{g};\mathbb{R}).$$

If $f\colon \mathfrak{h} \to \mathfrak{g}$ is an L_∞ morphism, then by definition f is a DGCA morphism $f^*\colon \mathrm{CE}(\mathfrak{g}) \to \mathrm{CE}(\mathfrak{h})$ so that f^*a is a 3-cocycle on $\mathfrak{h}$ for any 3-cocycle a on $\mathfrak{g}$, and f^* is a morphism of cochain complexes between $(\mathrm{CE}(\mathfrak{g}), d_{\mathfrak{g};a})$ and $(\mathrm{CE}(\mathfrak{h}), d_{\mathfrak{h};f^*a})$, thus inducing a morphism between the twisted cohomologies

$$f^*\colon H^{\overline{\bullet}}_{L_\infty;a}(\mathfrak{g};\mathbb{R}) \xrightarrow{\sim} H^{\overline{\bullet}}_{L_\infty;f^*a}(\mathfrak{h};\mathbb{R}).$$

We therefore see that in order to define Fourier-Mukai transforms at the level of L_∞-algebra cohomology the only ingredient we miss is a pushforward morphism

$$\pi_*\colon (\mathrm{CE}(\hat{\mathfrak{g}}), d_{\hat{\mathfrak{g}}}) \to (\mathrm{CE}(\mathfrak{g})[-1], d_{\mathfrak{g}}[-1])$$

for any central extension $\pi\colon \hat{\mathfrak{g}} \to \mathfrak{g}$ induced by a 2-cocycle $\mathfrak{g} \to b\mathfrak{u}_1$, which is a morphism of cochain complexes and which satisfies the projection formula identity. We are going to exibit such a morphism in the next section.

5 Fiber integration along $U(1)$-bundles in rational homotopy theory

Let $P \to X$ be a principal $U(1)$-bundle. Since $U(1)$ is a copact Lie group, every differential form on P can be averaged so to become invariant under the $U(1)$-action on P. Moreover, taking average is a homotopy inverse to the inclusion of $U(1)$-invariant forms into all forms on P so that

$$\Omega^\bullet(P)^{U(1)} \hookrightarrow \Omega^\bullet(P)$$

is a quasiisomorphism of DGCAs. The DGCA $\Omega^\bullet(P)^{U(1)}$ has a very simple description in terms of the DGCA $\Omega^\bullet(X)$. Namely, identifying $\Omega^\bullet(X)$ with its image in $\Omega^\bullet(P)$ via π^* one sees that $\Omega^\bullet(X)$ is actually a subalgebra of $\Omega^\bullet(P)^{U(1)}$. The subalgebra $\Omega^\bullet(X)$ however does not exhaust all of the $U(1)$-invariant forms on P: those forms that restrict to a scalar multiple of the volume form on the fibers (for some choice of a $U(1)$-invariant metric on P) are left out. Picking one such a form ω_1 is equivalent to the datum of a $U(1)$-connection ∇ on P and

$$(\Omega^\bullet(P)^{U(1)}, d) = (\Omega^\bullet(X)[\omega_1], d\omega_1 = F_\nabla),$$

where F_∇ is the curvature of ∇, so that we have a quasiisomorphism of DGCAs

$$(\Omega^\bullet(X)[\omega_1], d\omega_1 = F_\nabla) \xrightarrow{\sim} (\Omega^\bullet(P), d).$$

This is the geometric counterpart of the isomorphism

$$(\mathrm{CE}(\hat{\mathfrak{g}}), d_{\hat{\mathfrak{g}}}) = (\mathrm{CE}(\mathfrak{g})[y_1], d_{\hat{\mathfrak{g}}} y_1 = f^* x_2)$$

we met in Section 3, so that we see that the degree 1 element y_1 in the Chevalley-Eilenberg of the central extension $\hat{\mathfrak{g}}$ does indeed represent a vertical volume form. The fiber integration $\pi_*\colon (\Omega^\bullet(P), d) \to (\Omega^\bullet(X)[-1], d[-1])$, restricted to $U(1)$-invariant forms reads

$$\begin{gathered}\pi_*\colon (\Omega^\bullet(X)[\omega_1], d\omega_1 = F_\nabla) \to (\Omega^\bullet(X)[-1], d[-1]) \\ \alpha + \omega_1 \wedge \beta \mapsto \beta,\end{gathered}$$

so it is natural to define the fiber integration morphism π_* associated with the central extension $\pi\colon \hat{\mathfrak{g}} \to \mathfrak{g}$ determined by the 2-cocycle $f\colon \mathfrak{g} \to \mathfrak{u}_1$ as

$$\begin{gathered}\pi_*\colon (\mathrm{CE}(\mathfrak{g})[y_1], d_{\hat{\mathfrak{g}}} y_1 = f^* x_2) \to (\mathrm{CE}(\mathfrak{g})[-1], d_{\mathfrak{g}}[-1]) \\ a + y_1\, b \mapsto b,\end{gathered}$$

It is immediate to see that π_* is indeed a morphism of chain complexes:

$$d_{\mathfrak{g}}[-1](\pi_*(a + y_1\, b)) = -d_{\mathfrak{g}} b = \pi_*(d_{\mathfrak{g}} a + (f^* x_2)\, b - y_1\, d_{\mathfrak{g}} b) = \pi_*(d_{\hat{\mathfrak{g}}}(a + y_1\, b)).$$

Next, let us show that the projection formula holds. Since the morphism $\pi^*\colon (\mathrm{CE}(\mathfrak{g}), d_{\mathfrak{g}}) \to (\mathrm{CE}(\hat{\mathfrak{g}}), d_{\hat{\mathfrak{g}}})$ is the inclusion of $\mathrm{CE}(\mathfrak{g})$ into $\mathrm{CE}(\mathfrak{g})[y_1]$, we find:

$$\pi_*((\pi^* a)\,(b + y_1\, c)) = \pi_*(a\, b + (-1)^a y_1\, ac)) = (-1)^a ac = (-1)^a a\, \pi_*(b + y_1\, c).$$

Summing up, we have reproduced at the L_∞-algebra/rational homotopy theory level all of the ingredients we needed to define Fourier-Mukai transforms. That is, give a span $\mathfrak{g}_1 \xleftarrow{\pi_1} \mathfrak{h} \xrightarrow{\pi_2} \mathfrak{g}_2$ of central extensions (by the abelian Lie algebra $\mathbb{R}$) of L_∞-algebras, and given a triple (a_1, a_2, b) consisting of 3-cocycles a_i on $\mathfrak{g}_i$ and of a degree 4 element b in $\mathrm{CE}(\mathfrak{h})$ such that $d_{\mathfrak{h}} b = \pi_1^* a_1 - \pi_2^* a_2$ we have Fourier-Mukai transforms

$$\Phi_b \colon H^{\bar{\bullet}}_{L_\infty; a_1}(\mathfrak{g}_1) \to H^{\bar{\bullet}-\bar{1}}_{L_\infty; a_2}(\mathfrak{g}_2)$$
$$\Phi_{-b} \colon H^{\bar{\bullet}}_{L_\infty; a_2}(\mathfrak{g}_2) \to H^{\bar{\bullet}-\bar{1}}_{L_\infty; a_1}(\mathfrak{g}_1).$$

6 The hofiber/cyclification adjunction

We are going to see how to produce a quintuple $(\pi_1, \pi_2, a_1, a_2, b)$ inducing a Fourier-Mukai transform in Section 7. But first let us spend a few more words on the geometric properties of the pushforward morphism π_*. As $\pi_* \colon (\mathrm{CE}(\hat{\mathfrak{g}}), d_{\hat{\mathfrak{g}}}) \to (\mathrm{CE}(\mathfrak{g})[-1], d_{\mathfrak{g}}[-1])$ is a morphism of cochain complexes, it in particular maps degree $n+1$ cocycles in $\mathrm{CE}(\hat{\mathfrak{g}})$ to degree n cocycles in $\mathrm{CE}(\mathfrak{g})$. But, if $\mathfrak{h}$ is any L_∞-algebra, we have seen that a degree k cocycle in $\mathrm{CE}(\mathfrak{h})$ is precisely an L_∞-morphism $\mathfrak{h} \to b^{k-1}\mathfrak{u}_1$. Therefore we see that π_* induces a morphism of sets

$$\mathrm{Hom}_{L_\infty}(\hat{\mathfrak{g}}, b^n \mathfrak{u}_1) \to \mathrm{Hom}_{L_\infty}(\mathfrak{g}, b^{n-1}\mathfrak{u}_1).$$

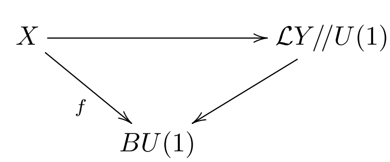
This is actually part of a much larger picture, to see which we need a digression on free loop spaces. So let again X be our smooth manifold and let $\pi \colon P \to X$ be a principal $U(1)$-bundle over X, and let $\varphi \colon P \to Y$ a map from P to another smooth manifold Y. Let $\gamma \colon P \times U \to Y$ be the composition

$$P \times U(1) \to P \xrightarrow{\varphi} Y$$

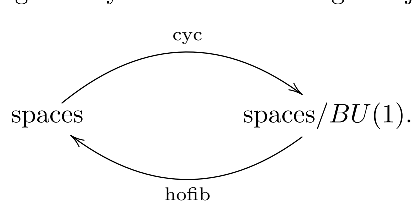
where the first map is the right $U(1)$-avtion on P. By the multiplication by S^1/free loop space adjunction, γ is equivalently a morphism from P to the free loop space $\mathcal{L}Y$ of Y. More explicitly, a point $x \in P$ is mapped to the loop $\varphi_x \colon U(1) \to Y$ defined by $\varphi_x(e^{i\theta}) = \varphi(x \cdot e^{i\theta})$. The map $\varphi \colon P \to \mathcal{L}Y$ is equivariant with respect to the right $U(1)$-action on$P$ and the right $U(1)$-action on $\mathcal{L}Y$ given by loop rotation: $\eta \cdot e^{i\theta} = \rho_\theta^* \eta$, where $\rho_\theta \colon U(1) \to U(1)$ is the rotation by angle θ. Namely, one has

$$((\varphi_x) \cdot e^{i\theta})(e^{i\theta_0}) = (\rho_\theta^* \varphi_x)(e^{i\theta_0}) = \varphi_x(e^{i\theta} e^{i\theta_0}) = \varphi((x \cdot e^{i\theta}) \cdot e^{i\theta_0}) = \varphi_{x \cdot e^{i\theta}}(e^{i\theta_0}).$$

Therefore, equivalently, φ is a morphism between the homotopy quotients $P//U(1)$ and $\mathcal{L}Y//U(1)$ over $BU(1)$. Moreover, as P is a principal $U(1)$-bundle over X, the homotopy quotient $P//U(1)$ is equivalent to the ordinary quotient and so is equivalent to the base X, and the natural map $P//U(1) \to BU(1)$ is identified with the morphism $f \colon X \to BU(1)$ classifying the principal bundle P. In other words, a morphism $\varphi \colon P \to Y$ is equivalently a morphism



from f to the canonical morphism $\mathcal{L}Y//U(1) \to BU(1)$ in the overcategory of spaces over $BU(1)$. Writing $\mathrm{cyc}(Y)$ for the "cyclification" $\mathcal{L}Y//U(1)$ and recalling that the total space P is the homotopy fiber of the morphism $f \colon X \to BU(1)$, we see that the above discussion can be elegantly summarised by saying that cyclification is the right adjoint to homotopy fiber,



6.1 Cyclification of L_∞-algebras

The above topological construction immediately translates to the L_∞-algebra setting, where we find an adjunction

$$L_\infty\text{-algebras} \underset{\text{hofib}}{\overset{\text{cyc}}{\rightleftarrows}} L_\infty\text{-algebras}/b\mathfrak{u}_1.$$

We have already seen that the homotopy fiber functor from L_∞-algebras over $b\mathfrak{u}_1$(i.e., L_∞-algebras equipped with an $\mathbb{R}$-valued 2-cocycles) to L_∞-algebras consists in forming the $\mathbb{R}$-central extension classified by the 2-cocycle. So we have now to complete the picture by describing the cyclification functor. As usual, we start from geometry, and consider an L_∞-algebra $\mathfrak{l}X$ representing the rational homotopy type of a simple space X. If X is 2-connected (so that its free loop space is surely simply connected and therefore simple) an L_∞-algebra representing the rational homotopy type of the free loop space $\mathcal{L}X$ is easily deduced from the multiplication by S^1/free loop space adjunction. As a Sullivan model for $Y \times S^1$ is $A_{Y\times S^1} = A_Y \otimes A_{S^1} = A_Y[t_1]$ with $dt_1 = 0$, one sees that if $A_X = (\wedge^\bullet \mathfrak{l}X^*, d_X)$, then

$$A_{\mathcal{L}X} = (\wedge^\bullet(\mathfrak{l}X^* \oplus s\mathfrak{l}X^*), d_{\mathcal{L}X})$$

where $s\mathfrak{l}X^* = \mathfrak{l}X^*[1]$ is a shifted copy of $\mathfrak{l}X^*$, with $d_{\mathcal{L}X}\big|_{A_X} = d_X$ and $[d_{\mathcal{L}X}, s] = 0$, where $s\colon A_{\mathcal{L}X} \to A_{\mathcal{L}X}$ is the shift operator $s\colon \mathfrak{l}X^* \xrightarrow{\sim} (s\mathfrak{l}X^*)[-1]$ extended as a degree -1 differential. See [VPB85] for details. This immediately suggests the following definition: for an arbitrary L_∞-algebra $\mathfrak{g}$ we write $\mathfrak{L}\mathfrak{g}$ for the L_∞-algebra defined by

$$(\mathrm{CE}(\mathcal{L}\mathfrak{g}), d_{\mathfrak{L}\mathfrak{g}}) = (\wedge^\bullet(\mathfrak{g}^* \oplus s\mathfrak{g}^*), d_{\mathfrak{L}\mathfrak{g}}\big|_{\mathrm{CE}(\mathfrak{g})} = d_\mathfrak{g}, [d_{\mathfrak{L}\mathfrak{g}}, s] = 0]).$$

Deriving an L_∞-algebra model for the cyclification $\mathrm{cyc}(X)$ is a bit more involved, ad has been worked out in [VS76]. One finds

$$A_{\mathrm{cyc}(X)} = (\wedge^\bullet(\mathfrak{l}X^* \oplus s\mathfrak{l}X^* \oplus b\mathfrak{u}_1^*), d_{\mathrm{cyc}(X)}) = (\wedge^\bullet(\mathfrak{l}X^* \oplus s\mathfrak{l}X^*)[x_2], d_{\mathrm{cyc}(X)}),$$

where x_2 is a degree 2 closed variable and $d_{\mathrm{cyc}(X)}$ acts on an element $a \in \mathfrak{l}X^* \oplus s\mathfrak{l}X^*$ as $d_{\mathrm{cyc}X}a = d_{\mathfrak{L}\mathfrak{g}}a + x_2 \wedge sa$. From this one has the natural generalization to an arbitrary L_∞-algbera $\mathfrak{g}$: its cyclification is the L_∞-algebra $\mathrm{cyc}(\mathfrak{g})$ defined by

$$\mathrm{CE}(\mathrm{cyc}(\mathfrak{g})) = (\wedge^\bullet(\mathfrak{g} \oplus s\mathfrak{g} \oplus b\mathfrak{u}_1)^*, d_{\mathrm{cyc}(\mathfrak{g})}) = ((\wedge^\bullet(\mathfrak{g} \oplus s\mathfrak{g})^*)[x_2], d_{\mathrm{cyc}(\mathfrak{g})}),$$

where x_2 is a degree 2 variable with $d_{\mathrm{cyc}(\mathfrak{g})}x_2 = 0$ and $d_{\mathrm{cyc}(\mathfrak{g})}$ acts on an element $a \in \mathfrak{g}^*[-1] \oplus \mathfrak{g}^*$ as

$$d_{\mathrm{cyc}(\mathfrak{g})}a = d_{\mathfrak{L}\mathfrak{g}}a + x_2 \wedge sa.$$

Notice that there is a canonical inclusion of DGCAs $\mathrm{R}[x_2] \hookrightarrow \mathrm{CE}(\mathrm{cyc}(\mathfrak{g}))$, giving a canonical 2-cocycle $\mathrm{cyc}(\mathfrak{g}) \to b\mathfrak{u}_1$. It is then not hard to see that, if $f\colon \mathfrak{g} \to b\mathfrak{u}_1$ is a 2-cocycle classifying a central extenson $\hat{\mathfrak{g}}$, then there is a natural bijection

$$\mathrm{Hom}_{L_\infty}(\mathrm{hofib}(f), \mathfrak{h}) \cong \mathrm{Hom}_{L_\infty/b\mathfrak{u}_1}(\mathfrak{g}, \mathrm{cyc}(\mathfrak{h})),$$

for any L_∞-algebra $\mathfrak{h}$, where on the right hand side with a little abuse of notation we have written the sources in places of the morphisms. Namely, in the dual Chevalley-Eilenberg picture this amounts to a natural bijection

$$\mathrm{Hom}_{DGCA}(\mathrm{CE}(\mathfrak{h}), \mathrm{CE}(\hat{\mathfrak{g}})) \cong \mathrm{Hom}_{R[x_2]/DGCA}(\mathrm{CE}(\mathrm{cyc}(\mathfrak{h})), \mathrm{CE}(\mathfrak{g})).$$

As $\mathrm{CE}(\mathfrak{h})$ is freely generated by $\mathfrak{h}^*[-1]$ as a polynomial algebra, a morphism on the left amounts to a graded linear map $\mathfrak{h}^*[-1] \to \mathrm{CE}(\hat{\mathfrak{g}})$ constrained by the compatibility with the differentials condition. As $\mathrm{CE}(\hat{\mathfrak{g}}) = \mathrm{CE}(\mathfrak{g})[y_1]$, where y_1 ia a variable in degree 1 with $d_{\hat{\mathfrak{g}}} y_1 = f^*(x_2)$, as a graded vector space we have

$$\mathrm{CE}(\hat{\mathfrak{g}}) = \mathrm{CE}(\mathfrak{g}) \oplus y_1\, \mathrm{CE}(\mathfrak{g}) = \mathrm{CE}(\mathfrak{g}) \oplus \mathrm{CE}(\mathfrak{g})[-1],$$

so that a graded linear map $\mathfrak{h}^*[-1] \to \mathrm{CE}(\hat{\mathfrak{g}})$ is equivalent to a pair of graded linear maps from $\mathfrak{h}^*[-1]$ to $\mathrm{CE}(\mathfrak{g})$ and to $\mathrm{CE}(\mathfrak{g})[-1]$, respectively. In turn, this pair is a graded linear map $\mathfrak{h}^*[-1] \oplus \mathfrak{h}^* \to \mathrm{CE}(\mathfrak{g})$. We can extend this to a graded linear map

$$\mathfrak{h}^*[-1] \oplus \mathfrak{h}^* \oplus b\mathfrak{u}_1^*[-1] \to \mathrm{CE}(\mathfrak{g})$$

by mapping the linear generator x_2 of $b\mathfrak{u}_1^*[-1]$ to the element $f^*(x_2)$ of $\mathrm{CE}(\mathfrak{g})$. This way we define a graded commutative algebra map

$$\wedge^\bullet(\mathfrak{h}^*[-1] \oplus \mathfrak{h}^* \oplus b\mathfrak{u}_1^*[-1]) \to \mathrm{CE}(\mathfrak{g})$$

which a direct computation shows to be a morphism of DGCAs making the diagram

$$\begin{array}{ccc} & \mathbb{R}[x_2] & \\ \swarrow & & \searrow^{f^*} \\ \mathrm{CE}(\mathrm{cyc}(\mathfrak{h})) & \longrightarrow & \mathrm{CE}(\mathfrak{g}) \end{array}$$

commute. See [FSS16] for details.

6.2 Fiber integration revisited

The L_∞ algebras $b^n\mathfrak{u}_1$ have a particularly simple cyclicization. Namely, as $\mathrm{CE}(b^n\mathfrak{u}_1) = (\mathbb{R}[x_{n+1}], 0)$, we see from the explicit description of cyclicization given in the previous section that as a polynomial algebra $\mathrm{CE}(\mathrm{cyc}(b^n\mathfrak{u}_1))$ is obtained from $\mathbb{R}[x_{n+1}]$ by adding a generator $y_n = s x_{n+1}$ in degree n and a generator z_2 in degree 2. The differential is given by

$$dx_{n+1} = z_2\, y_n; \quad dy_n = 0; \quad dz_2 = 0.$$

From this one immediatly sees that we have an injection $(\mathbb{R}[y_n], 0) \hookrightarrow (\mathrm{CE}(\mathrm{cyc}(b^n\mathfrak{u}_1)), d)$ and so dually a fibration

$$\mathrm{cyc}(b^n\mathfrak{u}_1) \to b^{n-1}\mathfrak{u}_1$$

of L_∞-algebras. Then given an $\mathbb{R}$ central extension $\pi\colon \hat{\mathfrak{g}} \to \mathfrak{g}$ we can form the composition of morphisms of sets

$$\mathrm{Hom}_{L_\infty}(\hat{\mathfrak{g}}, b^n\mathfrak{u}_1) \cong \mathrm{Hom}_{L_\infty/b\mathfrak{u}_1}(\mathfrak{g}, \mathrm{cyc}(b^n\mathfrak{u}_1)) \to \mathrm{Hom}_{L_\infty}(\mathfrak{g}, \mathrm{cyc}(b^n\mathfrak{u}_1)) \to \mathrm{Hom}_{L_\infty}(\mathfrak{g}, b^{n-1}\mathfrak{u}_1),$$

and a direct inspection easily reveals that this coincides with the fiber integration morhism

$$\pi_*\colon \mathrm{Hom}_{L_\infty}(\hat{\mathfrak{g}}, b^n\mathfrak{u}_1) \to \mathrm{Hom}_{L_\infty}(\mathfrak{g}, b^{n-1}\mathfrak{u}_1)$$

from Section 5.

7 Rational homotopy theory of T-duality configurations

As we already noticed, the same way as the classifying space $BU(1)$ of principal $U(1)$-bundles is a $K(\mathbb{Z}, 2)$, the classifying space $B^3U(1)$ of principal $U(1)$-2-bundles (or principal $U(1)$-2-gerbes) is a $K(\mathbb{Z}; 4)$. This implies that the cup product map

$$\cup\colon K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 4)$$

is equivalently a map

$$\cup\colon BU(1)\times BU(1)\to B^3U(1),$$

i.e., to any pair of principal $U(1)$ bundles P_1 and P_2 on a manifold X is canonically associated a $U(1)$-2-gerbe $P_1\cup P_2$ on X. By definition, a topological T-duality configuration is the datum of two such principal $U(1)$-bundles together with a trivialization of their cup product. In other words, a topological T-duality configuration on a manifold X is a homotopy commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ BU(1)\times BU(1) & \xrightarrow{\cup} & B^3U(1) \end{array} \quad .$$

By the universal property of the homotopy pullback this is in turn equivalent to a map from X to the homotopy fiber of the cup product, which will therefore be the classifying space for topological T-duality configurations. To fix notations, let us call $BTfolds$ this classifying space, so that $BTfold$ is defined by the homotopy pullback

$$\begin{array}{ccc} BTfold & \longrightarrow & * \\ \downarrow & & \downarrow \\ BU(1)\times BU(1) & \xrightarrow{\cup} & B^3U(1) \end{array} \quad .$$

The rationalization of $BTfold$ is obtained as the L_∞-algebra $\mathfrak{btfold}$ given by the homotopy pullback

$$\begin{array}{ccc} \mathfrak{btfold} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathfrak{bu}_1\times\mathfrak{bu}_1 & \xrightarrow{\cup} & \mathfrak{b}^3\mathfrak{u}_1 \end{array} \quad ,$$

and in order to get an explicit description of it we only need to give an explicit description of the 4-cocycle $\mathfrak{bu}_1\times\mathfrak{bu}_1\xrightarrow{\cup}\mathfrak{b}^3\mathfrak{u}_1$. This is easily read in the dual picture: it is the obvious morphism of CGDAs

$$\begin{aligned} (\mathbb{R}[x_4],0) &\to (\mathbb{R}[\check{x}_2,\tilde{x}_2],0)\cong(\mathbb{R}[x_2],0)\otimes(\mathbb{R}[x_2],0) \\ x_4 &\mapsto \check{x}_2\,\tilde{x}_2 \end{aligned}$$

The Chevalley-Eilenberg algebra of $\mathfrak{btfold}$ is then given by the homotopy pushout

$$\begin{array}{ccc} (\mathbb{R}[x_4],0) & \longrightarrow & (\mathbb{R},0) \\ \downarrow{\scriptstyle \cup^*} & & \downarrow \\ (\mathbb{R}[\check{x}_2,\tilde{x}_2],0) & \longrightarrow & (\mathrm{CE}(\mathfrak{btfold}),d) \end{array} \quad ,$$

i.e., by the pushout

$$\begin{array}{ccc} (\mathbb{R}[x_4],0) & \longrightarrow & (\mathbb{R}[y_3,x_4],dy_3=x_4) \\ \downarrow{\scriptstyle \cup^*} & & \downarrow \\ (\mathbb{R}[\check{x}_2,\tilde{x}_2],0) & \longrightarrow & (\mathrm{CE}(\mathfrak{btfold}),d) \end{array} \quad .$$

Explicitly, this means that

$$(\mathrm{CE}(\mathfrak{btfold}),d)=(\mathbb{R}[\check{x}_2,\tilde{x}_2,y_3],d\check{x}_2=0,d\tilde{x}_2=0,dy_3=\check{x}_2\,\tilde{x}_2),$$

and so an L_∞-morphism $\mathfrak{g} \to b\mathfrak{tfold}$ is precisely what we should have expected it to be: a pair of 2-cocycles on $\mathfrak{g}$ together with a trivialization of their product. Moreover, one manifestly has an isomorphism

$$(\mathrm{CE}(b\mathfrak{tfold}), d) \cong (\mathrm{CE}(\mathrm{cyc}(b^2\mathfrak{u}_1), d)$$

so that the $b\mathfrak{tfold}$ L_∞-algebra is isomorphic to the cyclicization of $b^2\mathfrak{u}_1$. This result actually already holds at the topological level, i.e., there is a homotopy equivalence $BTfold \cong \mathrm{cyc}(K(\mathbb{Z},3)) \cong \mathrm{cyc}(B^2U(1))$. Proving this equivalence beyond the rational approximation is however harder; see [BS05] for a proof.

The L_∞-algebra $b\mathfrak{tfold}$ has two independent 2-cocycles $f_1, f_2 \colon b\mathfrak{tfold} \to \mathfrak{u}_1$ given in the dual picture by $f_1^*(x_2) = \check{x}_2$ and by $f_2^*(x_2) = \tilde{x}_2$. Let us denote by $\mathfrak{p}_1$ and $\mathfrak{p}_2$ the central extensions of $b\mathfrak{tfold}$ corresponding to f_1 and f_2, respectively. They are clearly isomorphic as L_∞-algebras; however they are not equivalent as L_∞-algebras over $b\mathfrak{tfold}$ as the two classifying morphisms f_1 and f_2 are not homotopy equivalent.

Let us now write $\mathbb{R}[x_3]$ for $\mathrm{CE}(b^2\mathfrak{u}_1)$ so that in the notation of Section 6.1 we have $\mathrm{CE}(\mathrm{cyc}(b^2\mathfrak{u}_1)) = \mathbb{R}[x_3, y_2, z_2]$ with $dx_3 = z_2 y_2$, $dy_2 = 0$ and $dz_2 = 0$, and with the canonical 2-cocycle $\mathrm{cyc}(b^2\mathfrak{u}_1) \to b\mathfrak{u}_1$ being given dually by

$$\begin{aligned} f_{\mathrm{cyc}}^* \colon \mathbb{R}[x_2] &\to \mathbb{R}[x_3, y_2, z_2] \\ x_2 &\mapsto z_2. \end{aligned}$$

The isomorphism of L_∞-algebras $\varphi_1 \colon b\mathfrak{tfold} \to \mathrm{cyc}(b^2\mathfrak{u}_1)$ dually given by $x_3 \mapsto y_3$, $y_2 \mapsto \tilde{x}_2$ and $z_2 \mapsto \check{x}_2$ is such that the diagram of DGCAs

$$\begin{array}{ccccc} & & \mathrm{CE}(b\mathfrak{u}_1) & & \\ & \swarrow^{f_{\mathrm{cyc}}^*} & & \searrow^{f_1^*} & \\ \mathrm{CE}(\mathrm{cyc}(b^2\mathfrak{u}_1)) & & \xrightarrow{\quad\varphi_1^*\quad} & & \mathrm{CE}(b^2\mathfrak{tfold}) \end{array}$$

commutes, i.e., φ_1 is an isomorphism over $b\mathfrak{u}_1$. Hence, by the hofiber/cyclicization adjunction, it corresponds to an L_∞ morphism from the homotopy fiber of f_1 to $b^2\mathfrak{u}_1$, i.e., to a 3-cocycle $a_{3,1}$ over $\mathfrak{p}_1$. Repeating the same reasoning for f_2 we get a canonical 3-cocycle $a_{3,2}$ over $\mathfrak{p}_2$. Therefore we see how some of the ingredients of a rational T-duality configuration naturally emerge form the T-fold L_∞-algebra. The cocycles $a_{3,1}$ and $a_{3,2}$ can be easily given an explicit description, by unwinding the hofiber/cyclicization adjunction in this case. Let us do this for a_1. The homotopy fiber $\mathfrak{p}_1$ of f_1 is defined by the homotopy of DGCAs

$$\begin{array}{ccc} (\mathbb{R}[x_2], 0) & \longrightarrow & (\mathbb{R}, 0) \\ \Big\downarrow{\scriptstyle f_1^*} & & \Big\downarrow \\ (\mathbb{R}[\check{x}_2, \tilde{x}_2, y_3], d\check{x}_2 = d\tilde{x}_2 = 0, dy_3 = \check{x}_2\tilde{x}_2) & \longrightarrow & (\mathrm{CE}(\mathfrak{p}_1), d_{\mathfrak{p}_1}) \end{array},$$

and so it is given by

$$(\mathrm{CE}(\mathfrak{p}_1), d_{\mathfrak{p}_1}) = (\mathbb{R}[\check{y}_1, \check{x}_2, \tilde{x}_2, y_3], d\check{y}_1 = \check{x}_2, d\check{x}_2 = d\tilde{x}_2 = 0, dy_3 = \check{x}_2\tilde{x}_2).$$

One immediately sees that

$$dy_3 = d(\check{y}_1\tilde{x}_2),$$

i.e., that $y_3 - \check{y}_1\tilde{x}_2$ is a 3-cocycle on $\mathfrak{p}_1$. Under the hofiber/cyclicization adjunction this 3-cocycle corresponds to the morphism of DGCAs $\mathrm{CE}(\mathrm{cyc}(b^2\mathfrak{u}_1)) \to \mathrm{CE}(b^2\mathfrak{tfold})$ mapping x_3 to y_3, y_2 to $\tilde{x}_2$ and z_2 to $\check{x}_2$, i.e., to the morphism φ_1. In other words,

$$a_{3,1} = y_3 - \check{y}_1\tilde{x}_2.$$

In a perfectly similar way $a_{3,2} = y_3 - \check{x}_2\tilde{y}_1$. Finally, let us form the homotopy fiber product $\mathfrak{t} = \mathfrak{p}_1 \times_{b\mathfrak{tfold}} \mathfrak{p}_2$. It is described by the Chevalley-Eilenberg algebra

$$(\mathrm{CE}(\mathfrak{t}), d_{\mathfrak{t}}) = (\mathbb{R}[\check{y}_1, \tilde{y}_1, \check{x}_2, \tilde{x}_2, y_3], d\check{y}_1 = \check{x}_2, d\tilde{y}_1 = \tilde{x}_2, dy_3 = \check{x}_2\tilde{x}_2),$$

with the projections $\pi_i \colon \mathfrak{t} \to \mathfrak{p}_i$ given in the dual picture by the obvious inclusions. By construction, π_1 and π_2 are $\mathbb{R}$-central extensions, classified by the 2-cocycles $\tilde{x}_2$ and $\hat{x}_2$, respectively. One computes

$$\pi_1^* a_{3,1} - \pi_2^* a_{3,2} = (y_3 - \check{y}_1\tilde{x}_2) - (y_3 - \check{x}_2\tilde{y}_1) = -\check{y}_1\tilde{x}_2 + \check{x}_2\tilde{y}_1 = -\check{y}_1(d\tilde{y}_1) + (d\check{y}_1)\tilde{y}_1 = d(\check{y}_1\tilde{y}_1),$$

i.e.,

$$\pi_1^* a_{3,1} - \pi_{3,2}^* a_2 = db_2,$$

where $b_2 \in \mathrm{CE}(\mathfrak{t})$ is the degree 2 element $b = \check{y}_1\tilde{y}_1$. Thus we see that the L_∞-algebra $b\mathfrak{tfold}$ actually contains all the data of a quintuple $(\pi_1, \pi_2, a_{3,1}, a_{3,2}, b_2)$ inducing a Fourier-Mukai transform.

7.1 Maps to $b\mathfrak{tfold}$

All of the construction of the quintuple $(\pi_1, \pi_2, a_1, a_2, b)$ out of the the L_∞-algebra $b\mathfrak{tfold}$ can be pulled back along a morphism of L_∞-algebras $\mathfrak{g} \to b\mathfrak{tfold}$. That is, given such a morphism one has two $\mathbb{R}$-central extensions $\mathfrak{g}_1$ and $\mathfrak{g}_2$ of $\mathfrak{g}$ together with 3-cocycles $a_{3,1}$ and $a_{3,2}$ on $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively, and a degree 2 element b_2 on the (homotopy) fiber product L_∞-algebra $\mathfrak{g}_1 \times_{\mathfrak{g}} \mathfrak{g}_2$ with $\pi_1^* a_{3,1} - \pi_2^* a_{3,2} = db_2$. Let us see in detail how this works. To begin with, the datum of a morphism $\mathfrak{g} \to b\mathfrak{tfold}$ is precisely the datum of two 2-cocycles $\check{c}_2$ and $\tilde{c}_2$ on $\mathfrak{g}$ together with a degree 3 element $h_3 \in \mathrm{CE}(\mathfrak{g})$ such that $dh_3 = \check{c}_2\tilde{c}_2$. The two cocycles $\check{c}_2$ and $\tilde{c}_2$ define the two $\mathbb{R}$-central extensions $\mathfrak{g}_1$ and $\mathfrak{g}_2$ of $\mathfrak{g}$ defined by

$$(\mathrm{CE}(\mathfrak{g}_1), d_{\mathfrak{g}_1}) = (\mathrm{CE}(\mathfrak{g})[\check{e}_1], d\check{e}_1 = \check{c}_2)$$
$$(\mathrm{CE}(\mathfrak{g}_2), d_{\mathfrak{g}_2}) = (\mathrm{CE}(\mathfrak{g})[\tilde{e}_1], d\tilde{e}_1 = \tilde{c}_2),$$

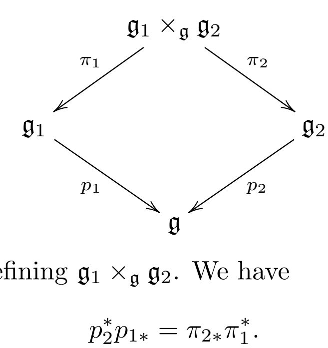
respectively. On the L_∞-algebra $\mathfrak{g}_1$ we have the 3-cocycle $a_{3,1} = h_3 - \check{e}_1\tilde{c}_2$, and on the L_∞-algebra $\mathfrak{g}_2$ we have the 3-cocycle $a_{3,2} = h_3 - \check{c}_2\tilde{e}_1$. Finally, the homotopy fiber product $\mathfrak{g}_1 \times_{\mathfrak{g}} \mathfrak{g}_2$ is given by

$$\mathrm{CE}(\mathfrak{g}_1 \times_{\mathfrak{g}} \mathfrak{g}_2), d_{\mathfrak{g}_1 \times_{\mathfrak{g}} \mathfrak{g}_2}) = (\mathrm{CE}(\mathfrak{g})[\check{e}_1, \tilde{e}_1], d\check{e}_1 = \check{c}_2, d\tilde{e}_1 = \tilde{c}_2),$$

and so in $\mathrm{CE}(\mathfrak{g}_1 \times_{\mathfrak{g}} \mathfrak{g}_2)$ we have $\pi_1^* a_{3,1} - \pi_2^* a_{3,2} = db_2$, where π_1^* and π_2^* are the obvious inclusions and $b_2 = \check{e}_1\tilde{e}_1$. Notice that $\mathrm{CE}(\mathfrak{g}_1 \times_{\mathfrak{g}} \mathfrak{g}_2)$ is built from $\mathrm{CE}(\mathfrak{g}_1)$ by adding the additional generator $\tilde{e}_1$ and from $\mathrm{CE}(\mathfrak{g}_2)$ by adding the additional generator $\check{e}_1$. We can now make completely explicit the Fourier-Mukai transform

$$\Phi_{b_2} \colon H^{\bar{\bullet}}_{L_\infty;a_{3,1}}(\mathfrak{g}_1) \to H^{\bar{\bullet}-\bar{1}}_{L_\infty;a_{3,2}}(\mathfrak{g}_2).$$

To fix notation, let



be the homotpy fiber product defining $\mathfrak{g}_1 \times_{\mathfrak{g}} \mathfrak{g}_2$. We have

$$p_2^* p_{1*} = \pi_{2*}\pi_1^*.$$

Indeed, for any $\omega_k = \alpha_k - \check{e}_1\beta_{k-1}$ in $\mathrm{CE}(\mathfrak{g}_1)$, we have

$$\pi_{2*}\pi_1^*\omega_k = \pi_{2*}\pi_1^*(\alpha_k - \check{e}_1\beta_{k-1}) = \pi_{2*}(\alpha_k - \check{e}_1\beta_{k-1}) = \beta_{k-1} = p_{1*}\omega_k = p_2^* p_{1*}\omega_k.$$

An $a_{3,1}$-twisted even cocycle for $\mathfrak{g}_1$ is an element $\omega \in \mathrm{CE}(\mathfrak{g}_1)$ such that

$$\omega = \sum_{n\in\mathbb{Z}} \omega_{2n},$$

where ω_{2n} is a degree $2n$ element, and such that

$$d_{\mathfrak{g}_1}\omega + a_{3,1}\,\omega = 0.$$

This equation is in turn equivalent to the system of equations

$$d_{\mathfrak{g}_1}\omega_{2n} + a_{3,1}\omega_{2n-2} = 0, \qquad n \in \mathbb{Z}.$$

Writing

$$\omega_{2n} = \alpha_{2n} - \check{e}_1\beta_{2n-1},$$

and recalling that

$$a_{3,1} = h_3 - \check{e}_1\tilde{c}_2$$

this becomes

$$d_{\mathfrak{g}}\alpha_{2n} - \check{c}_2\beta_{2n-1} + \check{e}_1 d_{\mathfrak{g}}\beta_{2n-1} + h_3\alpha_{2n-2} - \check{e}_1\tilde{c}_2\alpha_{2n-2} + \check{e}_1 h_3\beta_{2n-3} = 0,$$

i.e.,

$$\begin{cases} d_{\mathfrak{g}}\alpha_{2n} + h_3\alpha_{2n-2} = \check{c}_2\beta_{2n-1} \\ d_{\mathfrak{g}}\beta_{2n-1} + h_3\beta_{2n-3} = \tilde{c}_2\alpha_{2n-2} \end{cases}$$

The Fourier-Mukai transform Φ_{b_2} maps the twisted cocycle ω to $\pi_{2*}(e^{b_2}\pi_1^*\omega)$. Since π_1^* is just the inclusion and

$$e^{b_2} = e^{\check{e}_1\tilde{e}_1} = 1 + \check{e}_1\tilde{e}_1,$$

we find

$$\begin{aligned}
\Phi_{b_2}(\omega) &= \pi_{2*}(\omega + \check{e}_1\tilde{e}_1\omega) \\
&= \sum_{n\in\mathbb{Z}} \pi_{2*}(\alpha_{2n} - \check{e}_1\beta_{2n-1} + \check{e}_1\tilde{e}_1(\alpha_{2n-2} - \check{e}_1\beta_{2n-3})) \\
&= \sum_{n\in\mathbb{Z}} \pi_{2*}(\alpha_{2n} - \check{e}_1\beta_{2n-1} + \check{e}_1\tilde{e}_1\alpha_{2n-2}) \\
&= \sum_{n\in\mathbb{Z}} (\beta_{2n-1} - \tilde{e}_1\alpha_{2n-2})
\end{aligned}$$

Let $\tilde{\omega}_{2n-1} = \beta_{2n-1} - \tilde{e}_1\alpha_{2n-2}$ and $\tilde{\omega} = \sum_{n\in\mathbb{Z}} \tilde{\omega}_{2n-1}$, so that $\tilde{\omega}$ is an odd element in $\mathrm{CE}(\mathfrak{g}_2)$ and $\tilde{\omega} = \Phi_{b_2}(\omega)$. We know from the general construction of Fourier-Mukai transforms we have been developing that $\tilde{\omega}$ is an $a_{3,2}$-twisted cocycle. We can directly show this as follows:

$$\begin{aligned}
d_{\mathfrak{g}_2}\tilde{\omega}_{2n-1} &= d_{\mathfrak{g}_2}(\beta_{2n-1} - \tilde{e}_1\alpha_{2n-2}) \\
&= d_{\mathfrak{g}}\beta_{2n-1} - \tilde{c}_2\alpha_{2n-2} + \tilde{e}_1 d_{\mathfrak{g}}\alpha_{2n-2} \\
&= (-h_3\beta_{2n-3} + \tilde{c}_2\alpha_{2n-2}) - \tilde{c}_2\alpha_{2n-2} + \tilde{e}_1(-h_3\alpha_{2n-4} + \check{c}_2\beta_{2n-3}) \\
&= -a_{3,2}\beta_{2n-3} - \tilde{e}_1 a_{3;2}\alpha_{2n-4} \\
&= -a_{3,2}\tilde{\omega}_{2n-3}.
\end{aligned}$$

Looking at the explicit formula for Φ_{b_2} we have now determined, we see that Φ_{b_2} acts as

$$\{\alpha_{2n} - \check{e}_1\beta_{2n-1}\}_{n\in\mathbb{Z}} \mapsto \{\beta_{2n-1} - \tilde{e}_1\alpha_{2n-2}\}_{n\in\mathbb{Z}}$$

So it is manifestly a bijection between even $a_{3,1}$-twisted cocycles on $\mathfrak{g}_1$ and odd $a_{3,2}$-twisted cocycles on $\mathfrak{g}_2$. Repeating verbatim the above argument one sees that Φ_{b_2} is also a bijection between odd $a_{3,1}$-twisted cocycles on $\mathfrak{g}_1$ and even $a_{3,2}$-twisted cocycles on $\mathfrak{g}_2$. No surprise, the inverse morphism is, in both cases Φ_{-b_2}. We have therefore proved that the Fourier-Mukai transform associated to an L_∞-morphism $\mathfrak{g} \to \mathfrak{btfold}$ is an isomorphism

$$\Phi_{b_2} \colon H^{\bar{\bullet}}_{L_\infty;a_{3,1}}(\mathfrak{g}_1) \xrightarrow{\sim} H^{\bar{\bullet}-\bar{1}}_{L_\infty;a_{3,2}}(\mathfrak{g}_2)$$

8 An example from string theory: the superMinkowski space $\mathbb{R}^{1,8|\mathbf{16}+\mathbf{16}}$

All of the above immediatley generalizes from L_∞-algebras to super-L_∞-algebras, and it is precisely in this more general setting that we find an interesting example from the string theory literature.

Let $\mathbf{16}$ be the unique irreducible real representation of $\mathrm{Spin}(8,1)$ and let $\{\gamma_a\}_{a=0}^{d-1}$ be the corresponding Dirac representation on $\mathbb{C}^{16}$ of the Lorentzian $d = 9$ Clifford algebra. Write $\mathbf{16} + \mathbf{16}$ for the direct sum of two copies of the representation $\mathbf{16}$, and write

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

with ψ_1 and ψ_2 in $\mathbf{16}$ fon an element ψ in $\mathbf{16} + \mathbf{16}$. Finally, for $a = 0, \cdots, 8$ let

$$\Gamma^a = \begin{pmatrix} 0 & \gamma^a \\ \gamma^a & 0 \end{pmatrix}$$

The super-Minkowski super Lie algebra

$$\mathbb{R}^{8,1|\mathbf{16}+\mathbf{16}}$$

is the super Lie algebra whose dual Chevalley-Eilenberg algebra is the differential $(\mathbb{Z}, \mathbb{Z}/2)$-bigraded commutative algebra generated from elements $\{e^a\}_{a=0}^{8}$ in bidegree $(1, \mathrm{even})$ and from elements $\{\psi^\alpha\}_{\alpha=1}^{32}$ in bidegree $(1, \mathrm{odd})$ with differential given by

$$d\psi^\alpha = 0 \quad , \quad de^a = \overline{\psi}\Gamma^a\psi \, ,$$

where

$$\overline{\psi}\Gamma^a\psi = (C\Gamma^a)_{\alpha\beta}\, \psi^\alpha\psi^\beta \, ,$$

with C the charge conjugation matrix for the real representation $\mathbf{16} + \mathbf{16}$. Let now

$$\Gamma_9^{\mathrm{IIA}} = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix} \qquad \text{and} \qquad \Gamma_9^{\mathrm{IIB}} = \begin{pmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{pmatrix} \, ,$$

where I is the identity matrix. Since $d\psi^\alpha = 0$ for any α, both

$$c_2^{\mathrm{IIA}} = \overline{\psi}\Gamma_9^{\mathrm{IIA}}\psi \qquad \text{and} \qquad c_2^{\mathrm{IIB}} = \overline{\psi}\Gamma_9^{\mathrm{IIB}}$$

are degree (2,even) cocycles on $\mathbb{R}^{8,1|\mathbf{16}+\mathbf{16}}$. The central extensions they classify are obtained by adding a new degree (1,even) generator e^9_A or e^9_B to $\mathrm{CE}(\mathbb{R}^{8,1|\mathbf{16}+\mathbf{16}})$ with differential

$$de^9_A = \overline{\psi}\Gamma_9^{\mathrm{IIA}}\psi \qquad \text{and} \qquad de^9_B = \overline{\psi}\Gamma_9^{\mathrm{IIB}}\psi,$$

respectively. These two central extensions are therefore themselves super-Minkowski super Lie algebras. Namely, the extensions classified by c_2^{IIA} and c_2^{IIB} are

$$\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \qquad \text{and} \qquad \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}},$$

respectively. Finally, set

$$\Gamma_{10} = \begin{pmatrix} i\mathrm{I} & 0 \\ 0 & -i\mathrm{I} \end{pmatrix},$$

and let μ_{F1}^{IIA} be the degree (3,even) element in $\mathrm{CE}(\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}})$ given by

$$\mu_{F1}^{\mathrm{IIA}} = \mu_{F1}^{8,1} - i\overline{\psi}\Gamma_9^{\mathrm{IIA}}\Gamma_{10}\psi e_A^9 = -i\sum_{a=0}^{8}\overline{\psi}\Gamma_a\Gamma_{10}\psi e^a - i\overline{\psi}\Gamma_9^{\mathrm{IIA}}\Gamma_{10}\psi e_A^9.$$

The element μ_{F1}^{IIA} is actually a cocycle [CdAIPB00], so that

$$d\mu_{F1}^{8,1} = (i\overline{\psi}\Gamma_9^{\mathrm{IIA}}\Gamma_{10}\psi)(\overline{\psi}\Gamma_9^{\mathrm{IIA}}\psi).$$

A simple direct computation shows

$$\Gamma_9^{\mathrm{IIB}} = i\,\Gamma_9^{\mathrm{IIA}}\Gamma_{10},$$

so that

$$d\mu_{F1}^{8,1} = (\overline{\psi}\Gamma_9^{\mathrm{IIB}}\psi)(\overline{\psi}\Gamma_9^{\mathrm{IIA}}\psi) = c_2^{\mathrm{IIA}}c_2^{\mathrm{IIB}}.$$

As the element $\mu_{F1}^{8,1}$, as well as the elements c_2^{IIA} and c_2^{IIB} actually belong to the differential bigraded subalgebra $\mathrm{CE}(\mathbb{R}^{8,1|\mathbf{16}+\mathbf{16}})$ of $\mathrm{CE}(\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}})$, the relation

$$d\mu_{F1}^{8,1} = c_2^{\mathrm{IIA}}c_2^{\mathrm{IIB}}$$

actually holds in $\mathrm{CE}(\mathbb{R}^{8,1|\mathbf{16}+\mathbf{16}})$, so that the triple $(c_2^{\mathrm{IIA}}, c_2^{\mathrm{IIB}}, \mu_{F1}^{8,1})$ defines an L_∞-morphism

$$\mathbb{R}^{8,1|\mathbf{16}+\mathbf{16}} \to \mathfrak{btfold}.$$

The 3-cocycles on $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$ and on $\mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}}$ associated with this L_∞-morphism are

$$\mu_{F1}^{8,1} - e_A^9 c_2^{\mathrm{IIB}} \qquad \text{and} \qquad \mu_{F1}^{8,1} - c_2^{\mathrm{IIA}} e_B^9,$$

respectively. As $\Gamma_9^{\mathrm{IIB}} = i\,\Gamma_9^{\mathrm{IIA}}\Gamma_{10}$, we see that

$$\mu_{F1}^{8,1} - e_A^9 c_2^{\mathrm{IIB}} = \mu_{F1}^{8,1} - e_A^9\overline{\psi}\Gamma_9^{\mathrm{IIB}}\psi = \mu_{F1}^{8,1} - i\overline{\psi}\Gamma_9^{\mathrm{IIA}}\Gamma_{10}\psi e_A^9 = \mu_{F1}^{\mathrm{IIA}}.$$

We then set $\mu_{F1}^{\mathrm{IIB}} = \mu_{F1}^{8,1} - c_2^{\mathrm{IIA}} e_B^9$. An explicit expression for the $(3,\mathrm{even})$-cocycle μ_{F1}^{IIB} on $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$ is

$$\mu_{F1}^{\mathrm{IIB}} = \mu_{F1}^{8,1} - \overline{\psi}\Gamma_9^{\mathrm{IIA}}\psi e_B^9 = -i\sum_{a=0}^{8}\overline{\psi}\Gamma_a\Gamma_{10}\psi e^a - i\overline{\psi}\Gamma_9^{\mathrm{IIB}}\psi e_B^9,$$

where we used $\Gamma_9^{\mathrm{IIA}} = i\Gamma_9^{\mathrm{IIB}}\Gamma_{10}$. We have therefore an explicit Fourier-Mukai isomorphism

$$\Phi_{e_A^9 e_B^9} : H^{\overline{\bullet}}_{L_\infty;\mu_{F1}^{\mathrm{IIA}}}(\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) \xrightarrow{\sim} H^{\overline{\bullet}-\overline{1}}_{L_\infty;\mu_{F1}^{\mathrm{IIB}}}(\mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}}).$$

This isomorphism is known as Hori's formula or as the Buscher rules for RR-fields in the string theory literature [Ho99]. A direct computation shows that it maps the Chryssomalakos-de Azcárraga,-Izquierdo-Pérez Bueno μ_{F1}^{IIA}-twisted cocycle on $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$ to the Sakaguchi μ_{F1}^{IIB}-twisted cocycle on $\mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}}$, see [CdAIPB00, Sa00, FSS16].

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