Quantum and Reality

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Abstract

Formalizations of quantum information theory in category theory and type theory, for the design of verifiable quantum programming languages, need to express its two fundamental characteristics: (1) parameterized linearity and (2) metricity. The first is naturally addressed by dependent-linearly typed languages such as Proto-`Quipper` or, following our recent observations [SS23d][SS23c]: Linear Homotopy Type Theory (`LHoTT`). The second point has received much attention (only) in the form of semantics in "dagger-categories", where operator adjoints are axiomatized but their specification to Hermitian adjoints still needs to be imposed by hand.

In this brief note, we describe a natural emergence of Hermiticity which is rooted in principles of equivariant homotopy theory, lends itself to homotopically-typed languages and naturally connects to topological quantum states classified by twisted equivariant KR-theory. Namely, we observe that when the complex numbers are considered as a monoid internal to $\mathbb{Z}_2$-equivariant real linear types, via complex conjugation (the "Real numbers"), then (finite-dimensional) Hilbert spaces do become self-dual objects among internally-complex Real modules. This move absorbs the dagger-structure into the type structure; for instance, a complex linear map is unitary iff seen internally to Real modules it is orthogonal.

The point is that this construction of Hermitian forms requires the ambient linear type theory nothing further than a negative unit term of tensor unit type. We observe that just such a term is constructible in plain `LHoTT`, where it interprets as the non-trivial degree 0 element of the ∞-group of units of the sphere spectrum, interestingly tying the foundations of quantum theory to homotopy theory. We close by indicating how this observation allows for encoding (and verifying) the unitarity of quantum gates and of quantum channels in quantum languages embedded into `LHoTT`, as described in [SS23d].

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Quantum theory rests on two principles: (1) parameterized linearity and (2) metricity. The former is embodied by tensor-linear algebra and controls quantum phenomena such as the no-cloning/no-deleting and entanglement. The latter is embodied by quadratic forms and controls (via the Born rule) the probabilistic content of quantum physics and hence its relation to observable reality. (Extensive review and references may be found in [SS23d, §1.1, 1.2].)

The Hermitian core of Quantum physics. Ever since the foundations of quantum physics were laid, [vN30, p. 64][vN32, p. 21], the quadratic form on vector spaces H of quantum states is known to arise from a Hermitian form[1] $\langle \cdot | \cdot \rangle$ (e.g. [La17, §A.1]), hence complex *sesqui*-linear instead of complex bi-linear (e.g. [Bo81, §V.1.1]):

$$\langle c_1 \psi_1 \big| c_2 \psi_2 \rangle \;=\; \bar{c}_1 c_2 \langle \psi_1 | \psi_2 \rangle\,, \qquad \langle \psi_2 | \psi_1 \rangle \;=\; \overline{\langle \psi_1 | \psi_2 \rangle}\,. \tag{1}$$

The whole probabilistic interpretation of quantum physics via the Born rule (see pointers in [SS23d, p. 23]), hence its observable content, relies on this mild *failure*, if you will, of complex-linearity in an otherwise complex-linear theory.

This is not a logical tautology, but a core feature of quantum physics: Complex bi-linear forms do certainly play a role elsewhere in mathematics, for instance as Killing forms on semisimple complex Lie algebras. Here we consider the question at the foundations of quantum theory: How to fundamentally (category-theoretically) think of the phenomenon of Hermitian structure underlying quantum physics?

The subtle abstract nature of Hermitian structure. Abstractly, sesqui-linear forms are more subtle than complex bi-linear forms, since the plain monoidal category $(C, \otimes, \mathbb{1})$ of complex vector spaces $(\mathrm{Mod}_{\mathbb{C}}, \otimes_{\mathbb{C}}, \mathbb{C})$ does *not* support them: While a bi-linear form is equivalently an isomorphism in this category to the dual space, $\mathcal{V} \xrightarrow{\sim} \mathcal{V}^* := \mathrm{Map}_{\mathbb{C}}(\mathcal{V}, \mathbb{C})$ (cf. [Se12]), a sesquilinear form is an anti-linear such isomorphism or equivalently a complex-linear isomorphism but to the anti-dual space $\mathcal{V} \xrightarrow{\sim} \overline{\mathcal{V}^*}$: The first does not exist in $\mathrm{Mor}(\mathrm{Mod}_{\mathbb{C}})$, while the latter does not have a universal construction among $\mathrm{Obj}(\mathrm{Mod}_{\mathbb{C}})$ – not without invoking further structure. [2]

The problem in typed quantum programming languages. This does become an issue when formalizing quantum information theory (eg. [NC00]), such as in formulating typed functional quantum programming languages: Modern quantum type systems such as Proto-`Quipper` [FKS20] or `LHoTT` [Ri22][Ri23] (as explained in [SS23d]), axiomatize *linear types* with semantics in (parameterized versions of) distributive symmetric closed monoidal categories [RS18][RFL21][SS23c], but do not explicitly express anti-linear structure. In these formal languages, one can speak about inner products in terms of self-dual objects in any such monoidal category, but one just cannot speak directly about sesqui-linearity.

Of course, it is possible (see [MR24]) to add inference rules to linear type theories which force an involution $\overline{(\text{-})}$ on the type system such that function types $\mathcal{H}_1 \to \overline{\mathcal{H}_2}$ behave like spaces of antilinear maps. However, part of the beauty at least of `LHoTT` is that it simultaneously serves as a proof system for general abstract parameterized stable homotopy theory (cf. [SS23d, §1.5]); and here we would rather discover sesqui-linear structure than to artificially enforce it.

Motivation: Anyonic quantum states as Real K-modules. Concretely, we are motivated by our recent observation [SS23a] [SS23b][MSS23] that spaces of anyonic quantum states — thought to be needed for topological quantum computation — have a profound description as (twisted equivariant) *cohomology groups* (of configuration spaces of points), receiving the Chern-character map from the same twisted equivariant K-theory which is thought to classify topological ground states of crystalline quantum materials. This makes the quantum state spaces appear via modules over the Eilenberg-MacLane spectrum $H^{\mathrm{ev}}\mathbb{C}$ which is the complex-rationalization of the complex K-theory spectrum KU (the classifying spectrum for "K-theory with complex coefficients").

Or rather — and this is the pivotal point — since generally these topological phases are classified by Real K-theory (with capital "R") whose equivariant spectrum $\mathbb{Z}_2 \circlearrowright \mathrm{KU}$ encapsulates the $\mathbb{Z}_2$-action by complex conjugation (e.g. [SS21, Ex. 4.5.4]), anyonic quantum ground states really arise via modules of the equivariant EM-spectrum $\mathbb{Z}_2 \circlearrowright H^{\mathrm{ev}}\mathbb{C}$, being modules over the monoid $\mathbb{Z}_2 \circlearrowright \mathbb{C}$ of complex numbers equipped with their involution by complex-conjugation.

Real modules and Real vector bundles. This suggests that the category of $\mathbb{Z}_2 \circlearrowright \mathbb{C}$-modules over base $\mathbb{Z}_2$-spaces is a good context for understanding Hermitian forms. This is Atiyah's category of *Real vector bundles* [At66]. The now traditional capitalization "Real" was introduced by later authors who felt the need for disambiguation, while the original [At66] just speaks of "real vector bundles", suggestive of the relevance of this generalization. This hunch is maybe further confirmed by our following observations:

[1]For the purpose of this article, "Hermitian form", "metric", and "inner product" all imply the non-degeneracy and (conjugate-)symmetry of the pairing but not necessarily its signature.

[2]That further structure may and often is taken to be that of a *dagger-category* ([Se07], cf. [HV19, §2.3], which recovers inner products according to [AC04, Def. 7.5], see e.g. [HV19] for general review and [StSt23] for more recent developments). Just to highlight that dagger-structure serves to parameterize the issue more than it illuminates it: Besides the usual dagger structure for Hermitian inner products, $\mathrm{Mod}_{\mathbb{C}}$ also carries a dagger-structure encoding bilinear inner products. Dagger-theory by itself does not select one over the other for the purpose of quantum theory.

The Real modules. First to recall (from [At66, §1]) that Real vector bundles over the point (we will say *Real modules*, for short) are simply $\mathbb{C}$-vector spaces V equipped with anti-linear involutions C, and homomorphisms between them are $\mathbb{C}$-linear maps which intertwine these involutions:

$$
\mathrm{Mod}_{\circlearrowright\mathbb{C}} \;\equiv\; \left\{
\begin{array}{c}
\begin{array}{ccc}
\overset{C_1}{\circlearrowright} V_1 & \xrightarrow[\text{Real homomorphism}]{f} & V_2 \overset{C_1}{\circlearrowright} \\
\text{Real module} & & \text{Real module}
\end{array}
\\[1em]
\begin{array}{ccc}
\mathbb{C}\text{-vector space} & & \mathbb{C}\text{-vector space} \\
W_1 & \xrightarrow[\mathbb{C}\text{-linear map}]{f} & W_2 \\
{\scriptstyle \text{anti-lin involut.}} \updownarrow C_1 & & C_2 \updownarrow {\scriptstyle \text{anti-lin involut.}} \\
W_1 & \xrightarrow{\quad f \quad} & W_2
\end{array}
\end{array}
\right\}. \tag{2}
$$

This becomes a symmectric closed monoidal category $(C, \otimes, \mathbb{1}, \sigma)$ [EK66] under forming $\mathbb{C}$-linear tensor products equipped with the tensor involutions; the tensor unit $\mathbb{1}$ is the "Real numbers" $\circlearrowright\mathbb{C}$ given by $\mathbb{C}$ equipped with its involution by complex conjugation $\overline{(\text{-})}$ and the symmetric braiding σ is that of $\mathrm{Mod}_{\mathbb{C}}$ and swapping the involutions. This monoidal category of Real modules is equivalent to that of $\mathbb{R}$-vector spaces, via complexification (cf. [At66, p. 369]):

$$
\begin{array}{ccc}
\overset{\mathbb{R}\text{-vector spaces}}{(\mathrm{Mod}_{\mathbb{R}}, \otimes_{\mathbb{R}}, \mathbb{R}, \sigma_{\mathbb{R}})} & \xrightarrow[\text{complexification}]{\sim} & \overset{\text{Real modules}}{(\mathrm{Mod}_{\circlearrowright\mathbb{C}}, \otimes_{\circlearrowright\mathbb{C}}, \circlearrowright\mathbb{C}, \sigma_{\mathbb{R}})} \\[1em]
V & & V \otimes_{\mathbb{R}} \mathbb{C} \;\circlearrowright \overline{(\text{-})} \text{ complex conjugation} \\
\phi \big\downarrow & \longmapsto & \big\downarrow \phi \otimes_{\mathbb{R}} \mathbb{C} \\
V' & & V \otimes_{\mathbb{R}} \mathbb{C} \;\circlearrowright \overline{(\text{-})}
\end{array} \tag{3}
$$

We observe that it is interesting to pass inner product spaces through this equivalence:

Hermitian forms are Real inner products. It is classical that $\mathbb{R}$-inner product spaces (V, g) with isometric complex structure J (e.g. [Hu04, Def. 1.2.1, 1.2.11])

$$
V \in \mathrm{Mod}_{\mathbb{R}}\,, \qquad
\begin{array}{ll}
J : V \to V\,, & J \circ J = -\mathrm{id}_V\,, \\
g : V \otimes_{\mathbb{R}} V \to \mathbb{R}\,, & g(J(-), J(-)) = g(-,-)
\end{array} \tag{4}
$$

uniquely *determine* Hermitian forms on the corresponding complex vector space V_J, via the formula [3]

$$
\langle -|-\rangle \;:\; V_{-J} \otimes_{\mathbb{C}} V_{+J} \longrightarrow \mathbb{C}\,, \qquad \langle -|-\rangle \;\equiv\; g(-,-) + \mathrm{i}g(J(-),-)\,, \tag{5}
$$

(e.g. [Hu04, Lem. 1.2.15]). However, we highlight that under the equivalence (3) they actually *become* these Hermitian forms, as follows:

$$
\begin{array}{ccc}
(\mathrm{Mod}_{\mathbb{R}}, \otimes_{\mathbb{R}}, \mathbb{R}) & \xrightarrow[\text{equivalence (3)}]{\sim} & (\mathrm{Mod}_{\circlearrowright\mathbb{C}}, \otimes_{\circlearrowright\mathbb{C}}, \circlearrowright\mathbb{C}) \\[1em]
V \otimes_{\mathbb{R}} V & \mapsto & (V_{-J} \oplus V_{+J}) \otimes (V_{-J} \oplus V_{+J}) \\
\big\downarrow \mathrm{Eig}(J \otimes J, +1) & & \big\downarrow \mathrm{Eig}(\mathrm{I} \otimes \mathrm{I}, +1) \\
 & & V_{-J} \otimes V_{+J} \;\oplus\; V_{+J} \otimes V_{-J} \\
g \big\downarrow \; \mathbb{R}\text{-inner product corresponds to Hermitian form} & & \langle -|-\rangle \searrow \quad \swarrow \langle -|-\rangle \circ \sigma \\
\mathbb{R} & \mapsto & \mathbb{C} \;\circlearrowright \overline{(\text{-})}
\end{array} \tag{6}
$$

(This map (6) of quadratic forms is closely related to the "hyperbolic functor" [Bas65, §5.2] of key relevance in Hermitian K-theory [Ka73, p. 307][Bak77, §3][Ka10, §1.10]. For discussion relating this to KR-theory see also [Cr80, p. 96].)

To see that (6) indeed follows from (3), observe the $\circlearrowright\mathbb{C}$-module isomorphism (cf. [Hu04, Lem. 1.2.5])

[3] In symplectic geometry, the data $(V, J, \mathrm{i}g(J(-),-))$ in (5) is called a *Kähler vector space* (e.g. [Be01, p. 25]) with g its *Kähler metric* and $g(J(-),-)$ the corresponding *symplectic form*.

$$
\begin{array}{ccc}
v + \mathrm{i}v' & \mapsto & \tfrac{1}{\sqrt{2}}\big(v - J(v'),\; v + J(v')\big) \\[4pt]
\overline{(-)} \circlearrowright \mathbb{C} \otimes_{\mathbb{R}} V & \xleftrightarrow{\;\sim\;} & V_{-J} \oplus V_{+J} \\[4pt]
\tfrac{1}{\sqrt{2}}(v_- + v_+) + \tfrac{\mathrm{i}}{\sqrt{2}}\big(J(v_-) - J(v_+)\big) & \mapsfrom & (v_-,\, v_+)
\end{array}
\tag{7}
$$

which serves to diagonalize $J \otimes_{\mathbb{R}} \mathbb{C}$:

$$
\begin{array}{ccc}
(V \otimes_{\mathbb{R}} \mathbb{C}) & \xrightarrow{\;(J \otimes_{\mathbb{R}} \mathbb{C})\;} & (V \otimes_{\mathbb{R}} \mathbb{C}) \\
\wr\uparrow & & \downarrow\wr \\
(V_{-J} \oplus V_{+J}) & \xrightarrow{\;\mathrm{I}\;} & (V_{-J} \oplus V_{+J}) \\
(v_-,\, v_+) & \mapsto & (-\mathrm{i}v,\, +\mathrm{i}v)\,.
\end{array}
\tag{8}
$$

With (7), the isometry condition $g \circ (J \otimes J) = g$ on the left of (6) — which means that g factors through the $(+1)$-eigenspace of $J \otimes_{\mathbb{R}} J$ — implies by functoriality of the equivalence (3) that the pairing on the right of (5) factors through the $(+1)$-eigenspace of $\mathrm{I} \otimes \mathrm{I}$ (8), which already makes it a Hermitian form on V_{+J}, as shown on the right of (6). Explicit computation shows that this is indeed the one given by the traditional formula (5):

$$
\begin{array}{ccccccc}
V_{-J} \otimes_{\mathbb{C}} V_{+J} & \hookrightarrow & (V_{-J} \oplus V_{+J}) \otimes_{\mathbb{C}} (V_{-J} \oplus V_{+J}) & \underset{(7)}{\xrightarrow{\;\sim\;}} & (V \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (V \otimes_{\mathbb{R}} \mathbb{C}) & \xrightarrow{\;(g \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (g \otimes_{\mathbb{R}} \mathbb{C})\;} & \mathbb{C} \\
v_- \otimes_{\mathbb{C}} v_+ & \mapsto & (v_-, 0) \otimes_{\mathbb{C}} (0, v_+) & \mapsto & \tfrac{1}{2}\big(v_- + \mathrm{i}J(v_-)\big) \otimes_{\mathbb{C}} \big(v_+ - \mathrm{i}J(v_+)\big) & \mapsto & g(v_-, v_+) + \mathrm{i}g\big(J(v_-), v_+\big).
\end{array}
$$

Inner products with isometric complex structure internal to Real modules are Hermitian forms. It is instructive to restate this equivalence in the other direction: Given a Real module equipped with a symmetric self-duality structure and an isometric complex structure (all internal to the category of Real modules!, see [SS23d, §1.4] for background):

$$
\begin{array}{lccc}
\text{Real module} & C \circlearrowright \mathcal{H} \;\in\; \mathrm{Mod}_{\circlearrowright\mathbb{C}} & & \\[8pt]
\begin{array}{c}\text{Symmetric}\\ \text{inner product}\end{array} & \overline{(-)} \circlearrowright \mathbb{C} \xrightarrow{\;\Delta\;} \mathcal{H} \otimes_{\mathbb{C}} \mathcal{H} \circlearrowleft C \otimes C \xrightarrow{\;(-,-)\;} \mathbb{C} \circlearrowleft \overline{(-)} & & \begin{array}{rcl} \big((-) \otimes (-,-)\big) \circ \big(\Delta \otimes (-)\big) & = & \mathrm{id}_{\mathcal{H}} \\ \big((-,-) \otimes (-)\big) \circ \big((-) \otimes \Delta\big) & = & \mathrm{id}_{\mathcal{H}} \\ (-,-) \circ \sigma & = & (-,-) \end{array} \\[8pt]
\begin{array}{c}\text{Isometric}\\ \text{complex structure}\end{array} & C \circlearrowright \mathcal{H} \xrightarrow{\;\mathrm{I}\;} \mathcal{H} \circlearrowleft C & & \begin{array}{rcl} \mathrm{I} \circ \mathrm{I} & = & -\mathrm{id}_{\mathcal{H}} \\ \big(\mathrm{I}(-),\, \mathrm{I}(-)\big) & = & (-,-) \end{array}
\end{array}
\tag{9}
$$

we recognize it as encoding a Hermitian form as follows. Observing that I is $\mathbb{C}$-diagonalizable with $\mp\mathrm{i}$-eigenspaces swapped by the anilinear involution (for instance by applying the previous argument (8) to $J \;\equiv\; \mathrm{I}^{\mathbb{Z}_2}$)

$$
C \circlearrowright \mathcal{H} \xleftrightarrow{\;\sim\;} \overline{H} \oplus H\,,
\tag{10}
$$

the isometry property of I implies that $(-,-)$ is non-vanishing only on the $+1$ eigenspace of $\mathrm{I} \otimes \mathrm{I}$, which is the mixed summands $\overline{H} \otimes H \;\oplus\; H \otimes \overline{H}$. Here the inner product $(-,-)$ exhibits $\overline{H}$ as the dual object (cf. [DP84, §1][HV19, §3.1]) of H, inducing a $\mathbb{C}$-linear identification of H with its anti-dual space, this defining (e.g. [Ka10, §1]) a sesqui-linear form $\langle - | - \rangle$:

$$
\begin{array}{ccc}
H & \xrightarrow{\;\sim\;} & \overline{H}^{*} \\
|\psi\rangle & \mapsto & \langle\psi|\,,
\end{array}
$$

which is forced to be Hermitian by symmetry and $\mathbb{Z}_2$-equivariance of $(-,-)$ (on the right, $\{|w\rangle\}_{w \in W}$ is any orthonormal basis):

$$
\begin{array}{cccc}
(H \oplus H^*) \otimes (H \oplus H^*) & \xrightarrow{\;(-,-)\;} & & \mathbb{C} \circlearrowleft \overline{(-)} \\
\langle\psi_1| \otimes |\psi_2\rangle & \longmapsto & & \langle\psi_1|\psi_2\rangle \\
\downarrow & & & \downarrow \\
|\psi_1\rangle \otimes \langle\psi_2| & \mapsto & \langle\psi_2|\psi_1\rangle = & \overline{\langle\psi_1|\psi_2\rangle} \\[6pt]
|\psi_1\rangle \otimes |\psi_2\rangle & \longmapsto & & 0 \\
\downarrow & & & \downarrow \\
\langle\psi_1| \otimes \langle\psi_2| & \longmapsto & & 0
\end{array}
\qquad
\begin{array}{ccc}
\overline{(-)} \circlearrowright \mathbb{C} & \xrightarrow{\;\Delta\;} & (H \oplus H^*) \otimes (H \oplus H^*) \\
1 & \longmapsto & \sum_w |w\rangle \otimes \langle w| \; + \sum_w \langle w| \otimes |w\rangle \\
\downarrow & & \downarrow \\
1 & \longmapsto & \sum_w \langle w| \otimes |w\rangle \; + \sum_w |w\rangle \otimes \langle w|
\end{array}
\tag{11}
$$

The dagger emerges. Observe now that the *internally complex-linear* morphisms $G : \mathcal{H}_1 \to \mathcal{H}_2$ of self-dual Real modules with internal complex structure I (9) — hence the Real homomorphisms G satisfying $G \circ \mathrm{I}_1 = \mathrm{I}_2 \circ G$ — bijectively correspond to ordinary $\mathbb{C}$-linear maps $g : H_1 \to H_2$ between the (+i)-eigenspaces (10) – because $\mathbb{Z}_2$-equivariance forces the other component to be the Hermitian adjoint operator $g^\dagger$, defined by $\langle - | g^\dagger - \rangle = \langle g - | - \rangle$ (e.g. [La17, (A.15)]):

$$
\begin{array}{ccc}
g : H_1 & \xrightarrow{\ \mathbb{C}\text{-linear map}\ } & H_2 \\
|\psi\rangle & \mapsto & g|\psi\rangle
\end{array}
\quad \leftrightarrow \quad
\begin{array}{ccccc}
G : \mathcal{H}_1 & & \xrightarrow{\ \substack{\text{I-complex-linear} \\ \text{Real-module homom.}}\ } & & \mathcal{H}_2 \\
|\psi\rangle & & \longmapsto & & g|\psi\rangle \\
\rotatebox{90}{$\mapsto$} & & & & \rotatebox{90}{$\mapsto$} \\
\langle\psi| & \longmapsto & \langle\psi|\, g^\dagger & = & \langle g\psi| \,.
\end{array}
\tag{12}
$$

In particular, the *internally isometric* internally-complex-linear Real module homomorphisms correspond bijectively to the complex isometries on the underlying Hermitian spaces:

$$
\begin{array}{ccccc}
\mathcal{H}_1 \otimes \mathcal{H}_1 & & \xrightarrow{\quad G \otimes G \quad} & & \mathcal{H}_2 \otimes \mathcal{H}_2 \\
& \langle\phi| \otimes |\psi\rangle & \longmapsto & \langle\phi| g^\dagger \otimes g|\psi\rangle & \\
{\scriptstyle (-,-)_1} \Big\downarrow & \Big\downarrow & & \Big\downarrow & \Big\downarrow {\scriptstyle (-,-)_2} \\
& \langle\phi|\psi\rangle & = & \langle\phi|g^\dagger g|\psi\rangle & \\
\circlearrowleft \mathbb{C} & & = & & \circlearrowleft \mathbb{C}
\end{array}
\tag{13}
$$

hence to unitary operators iff they are also invertible; and the dagger-functor on the category of Hermitian spaces is now just the duality-involution (cf. eg. [HV19, Def. 3.11]) on this category of self-dual Real modules, as shown here:

$$
\begin{array}{ccccccc}
& & \mathcal{H}_1 & = & \mathcal{H}_1 & = & \mathcal{H}_1 \\
& \circlearrowleft \mathbb{C} \xrightarrow{\ \Delta\ } & \otimes & & \otimes & & \\
& & \mathcal{H}_1 & \xrightarrow{\quad G \quad} & \mathcal{H}_2 & & \\
& & \otimes & & \otimes & \xrightarrow{\ (-,-)\ } & \circlearrowleft \mathbb{C} \\
\mathcal{H}_2 & = & \mathcal{H}_2 & = & \mathcal{H}_2 & & \\[6pt]
|\psi\rangle & \mapsto & \begin{array}{l} \sum_w |w\rangle \otimes \langle w| \otimes |\psi\rangle \\ + \sum_w \langle w| \otimes |w\rangle \otimes |\psi\rangle \end{array} & \mapsto & \begin{array}{l} \sum_w |w\rangle \otimes \langle w|\, g^\dagger \otimes |\psi\rangle \\ + \sum_w \langle w| \otimes \; g\,|w\rangle \otimes |\psi\rangle \end{array} & \mapsto & \underbrace{\sum_w |w\rangle\langle w|G^\dagger|\psi\rangle}_{g^\dagger\,|\psi\rangle}
\end{array}
\tag{14}
$$

Hermitian operators as Real matrices. Similarly, we find that "density matrices", hence Hermitian operators on (finite-dimensional, in our case) Hilbert spaces (see [SS23d, §1.2] for background) are just the complex *symmetric* matrices internal to Real modules. The Real space $\mathrm{CSMat}(\mathcal{H})$ of internally-complex (C) and symmetric (S) matrices on an internally complex self-dual Real space $\mathcal{H}$ is the following iterated equalizer:

$$
\begin{array}{ccccc}
\mathrm{CSMat}(\mathcal{H}) & \longrightarrow & \mathrm{CMat}(\mathcal{H}) & \dashrightarrow^{\ \mathrm{braid}^{\otimes}_{\mathrm{CMat}(\mathcal{H})}\ } & \mathrm{CMat}(\mathcal{H}) \\
\Big\downarrow & \text{(pb)} & \Big\downarrow {\scriptstyle \mathrm{eq}} & & \Big\downarrow {\scriptstyle \mathrm{eq}} \\
\mathrm{SMat}(\mathcal{H}) & \xhookrightarrow{\ \mathrm{eq}\ } & \mathcal{H} \otimes \mathcal{H} & \underset{\mathrm{id}}{\overset{\mathrm{braid}^{\otimes}_{\mathcal{H}\otimes\mathcal{H}}}{\rightrightarrows}} & \mathcal{H} \otimes \mathcal{H} \\
\Big\downarrow {\scriptstyle \mathrm{I}\otimes\mathrm{I}_{|\mathrm{SMat}(\mathcal{H})}} & & {\scriptstyle \mathrm{id}} \Big\downarrow \Big\downarrow {\scriptstyle \mathrm{I}\otimes\mathrm{I}} & & {\scriptstyle \mathrm{id}} \Big\downarrow \Big\downarrow {\scriptstyle \mathrm{I}\otimes\mathrm{I}} \\
\mathrm{SMat}(\mathcal{H}) & \xhookrightarrow{\ \mathrm{eq}\ } & \mathcal{H} \otimes \mathcal{H} & \underset{\mathrm{id}}{\overset{\mathrm{braid}^{\otimes}_{\mathcal{H}\otimes\mathcal{H}}}{\rightrightarrows}} & \mathcal{H} \otimes \mathcal{H}
\end{array}
$$

Unwinding the definitions, one has

$$
\begin{array}{cccc}
\mathrm{CMat}(\mathcal{H}) \;\simeq & (H \otimes_{\mathbb{C}} H^* \,\oplus\, H^* \otimes_{\mathbb{C}} H) & \xrightarrow{\ \mathrm{braid}^{\otimes}_{\mathrm{CMat}(\mathcal{H})}\ } & (H \otimes_{\mathbb{C}} H^* \,\oplus\, H^* \otimes_{\mathbb{C}} H) \\
& |\psi\rangle \otimes \langle\phi| \;\leftrightarrow\; \langle\psi| \otimes |\phi\rangle & \longmapsto & |\phi\rangle \otimes \langle\psi| \;\leftrightarrow\; \langle\phi| \otimes |\psi\rangle \,,
\end{array}
$$

which means that the $\mathbb{Z}_2$-fixed locus of $\mathrm{CMat}(\mathcal{H})$ is identified with the $\mathbb{R}$-vector space underlying $H \otimes_{\mathbb{C}} H^*$, under which identification the braiding acts as the Hermitian adjoint operation $(-)^\dagger : |\psi\rangle\langle\phi| \mapsto |\phi\rangle\langle\psi|$:

$$
\begin{array}{ccccc}
\mathrm{CMat}(\mathcal{H})^{\mathbb{Z}_2} & H \otimes_{\mathbb{C}} H^* & & & H \otimes_{\mathbb{C}} H^* \\
 & & |\psi\rangle\langle\phi| & \longmapsto & |\phi\rangle\langle\psi| \\
\big\downarrow & \big\downarrow & \big\downarrow & & \big\downarrow \quad\quad \big\downarrow \\
 & & |\psi\rangle \otimes \langle\phi| \; + \; \langle\psi| \otimes |\phi\rangle & \longmapsto & |\phi\rangle \otimes \langle\psi| \; + \; \langle\phi| \otimes |\psi\rangle \\
\mathrm{CMat}(\mathcal{H}) & \big(H \otimes_{\mathbb{C}} H^* \,\oplus\, H^* \otimes_{\mathbb{C}} H\big) & \xrightarrow[\mathrm{braid}^{\otimes}_{\mathrm{CMat}(\mathcal{H})}]{} & & \big(H \otimes_{\mathbb{C}} H^* \,\oplus\, H^* \otimes_{\mathbb{C}} H\big)
\end{array}
$$

This reveals CSMat($\mathcal{H}$) as the Real space corresponding under (3) to the $\mathbb{R}$-vector space of Hermitian operators:

$$
\mathrm{CSMat}(\mathcal{H}) \;\simeq\; \mathrm{Herm}(H,\, \langle \text{-}|\text{-}\rangle) \otimes_{\mathbb{R}} \mathbb{C} \circlearrowleft \overline{(\text{-})}
$$

The Real incarnation of a unitary quantum channel induced by a unitary operator (12) is hence simply:

$$
\begin{array}{ccc}
|\psi\rangle\langle\phi| & \longmapsto & g\,|\psi\rangle\langle\phi|\,g^\dagger \\
\mathcal{H} \otimes \mathcal{H} & \xrightarrow{\;G \otimes G\;} & \mathcal{H} \otimes \mathcal{H} \\
\big\uparrow & & \big\uparrow \\
\mathrm{CSMat}(\mathcal{H}) & \longrightarrow & \mathrm{CSMat}(\mathcal{H}) \\
\rho & \longmapsto & g \cdot \rho \cdot g^\dagger
\end{array}
$$

In conclusion so far, the above shows that the Hermitian structures in the foundations of quantum information theory are naturally expressed by isometrically complex self-dual objects (9) *internal* to the category of Real modules. Since the latter is monoidally equivalent (3) to that of $\mathbb{R}$-vector spaces, the same is already true there, where the above constructions amount to strictly referring to Hermitian forms only through their real part (5) regarded as a symmetric form on the underlying $\mathbb{R}$-vector space (4) – which is an elementarily equivalent but unusual perspective. It is only when that same construction is transported back through the equivalence with Real modules (3) that all the familiar $\mathbb{C}$-component formulas of quantum theory manifest themselves, such as for Hermitian adjoints (14).

However, abstractly, such as for formal languages, the component expressions are invisible/irrelevant anyway – they are part of the "semantics" (the model) but not of the "syntax" (the theory). One may understand these observations as one way of seeing "why" the (spectral) theory of Hermitian forms on $\mathbb{C}$-vector famously parallels that of symmetric inner products on $\mathbb{R}$-vector spaces: They secretly *are* symmetric inner products when seen internally to Real modules.

In amplification of this point, we now indicate how one may formalize (fin-dim) Hilbert space structure internal to a formal language like LHoTT, without introducing further inference rules for anti-linear maps.

Formalization in linearly homotopy-typed languages. Looking back, one may notice that the only property of the tensor unit $\mathbb{1} \;\equiv\; \mathbb{R}$ in the symmetric closed monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \sigma) \;\equiv\; (\mathrm{Mod}_{\mathbb{R}}, \otimes_{\mathbb{R}}, \mathbb{R}, \sigma_{\mathbb{R}})$ that we used above — namely for saying "internal complex structure" in (4) and (9) — is that it contains a negative unit scalar, in the sense of a non-trivial involutive endomorphism:

$$
-\mathrm{id} \,:\, \mathbb{1} \to \mathbb{1}, \qquad (\,-\,\mathrm{id} = \mathrm{id}) \,\to\, \varnothing, \qquad -\mathrm{id} \circ -\mathrm{id} \;=\; \mathrm{id}\,. \tag{15}
$$

Trivial as this may seem, it is not generally the case for tensor units of monoidal categories and it needs an argument that a term like (15) is constructible in a given formal language for such categories. Interestingly, that this is the case for LHoTT relies on its homotopy-theoretic nature. In indicating now how this works, we will here not go into the formal syntax of LHoTT, but discuss the relevant universal constructions in its expected categorical sematics which interpret this term:

Namely, in refinement of how plain HoTT has categorical semantics in general ∞-toposes (see [MSS23, p. 45] for pointers)[4] and expresses the universal constructions that are generically available in all of them, so LHoTT is meant to have categorical semantics in "tangent ∞-toposes" of parameterized R-module spectra (for R an E_∞-ring, for pointers see [SS23d, §1.5]) and to express the universal constructions generically available in all of these.

Since such tangent ∞-topos is meant to be (categorical semantics for) the type universe of LHoTT we suggestively denote it by "Type", as in [SS23d], with the reflective sub-∞-categories of purely classical types (homotopy types, ∞-groupoids) and of purely linear (quantum) types (module spectra) denoted

$$
\begin{array}{ccccc}
\mathrm{ClaType} & \underset{\hookrightarrow}{\overset{\natural}{\twoheadleftarrow}}{\scriptstyle\perp} & \mathrm{Type} & \underset{\hookleftarrow}{\overset{\triangleright}{\twoheadrightarrow}}{\scriptstyle\perp} & \mathrm{QuType} \\
\left\{\left[\begin{array}{c} 0 \\ \downarrow \\ X \end{array}\right]\right\} & & \left\{\left[\begin{array}{c} V_\bullet \\ \downarrow \\ X \end{array}\right]\right\} & & \left\{\left[\begin{array}{c} V \\ \downarrow \\ * \end{array}\right]\right\}.
\end{array}
\tag{16}
$$

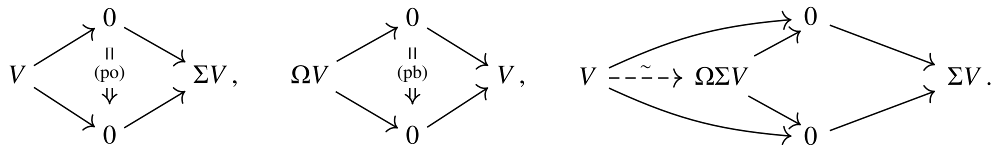
[4] We will notationally suppress the "∞-"-prefix in the following. All constructions are understood to be homotopy-theoretic. Similarly, all diagrams are filled by 2-morphisms, even when these are not explicitly indicated.

The doubly symmetric closed monoidal structure. Here

- ClaType is Cartesian monoidal with Cartesian product denoted $\times$ and tensor unit denoted $*$,
- QuType is symmetric monoidal with tensor product $\otimes$ and tensor unit $\mathbb{1}$, distributing over a Cartesian biproduct (direct sum) $\oplus$ with tensor unit the zero object,
- Type is doubly monoidal with the corresponding "external" Cartesian and tensor products (cf. [SS23c], here to be denoted by the same symbols $\times$ and $\otimes$, respectively),

such that all functors in (16) are strong monoidal.

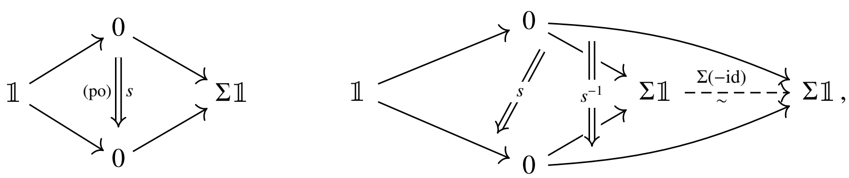
Moreover, QuType is stable (in the sense of [Lu17, §1], namely under suspension), meaning that for $V \in$ QuType the canonical comparison map to the looping $\Omega(-)$ of its suspension $\Sigma(-)$ is an equivalence (cf. [Lu17, p. 24]):



(17)

Of interest for our application to quantum information theory is the case where the ground E_∞-ring is the real Eilenberg-MacLane spectrum $R \equiv H\mathbb{R}$, in which case Type is equivalently the flat ∞-vector bundles or ∞-local systems, see [SS23c], hence QuType is equivalently given by $\mathbb{R}$-chain complexes; see (20) below.

The negative unit scalar in `LHoTT`. If we denote the homotopy exhibiting the suspension type (cf. [MSS23, (117)]) of the tensor unit $\mathbb{1}$ by s, then its inverse s^{-1} induces an endomorphism of $\Sigma\mathbb{1}$ (by the universal property of the pushout) whose looping is the scalar which we may denote by "$-\mathrm{id}$":



$$-\,\mathrm{id} \;=\; \Omega\Sigma(-\mathrm{id}) \;:\; \mathbb{1} \to \mathbb{1}\,. \qquad (18)$$

The desired characteristic properties (15) of this $-\mathrm{id}$ (18) follow immediately from similar uses of the universal property of the defining pushouts. In the language of higher algebra, this is the non-trivial element of degree=0 in $\mathrm{GL}(1, \mathbb{S})$, the "group of units of the sphere spectrum" (cf. [ABGHR14, §1.2][FSS23, (3.18)]): $\pi_0(\mathrm{GL}(1,\mathbb{S})) = (\pi_0\mathbb{S})^\times = \mathbb{Z}^\times = \mathbb{Z}_2$.

The heart of `LHoTT`. With the negative unit scalar (18) in hand, it is immediate to define data structures (see [MSS23, p. 53] for background) in QuType of symmetric inner product spaces with isometric complex structure according to (9). The only further subtlety to take care of is that the definition (9) comes out as intended (only) on linear base types which are in the "heart" of QuType. For completeness, we briefly explain this. In plain `HoTT` the n-truncation modality (for $n \in \mathbb{N}$, see [MSS23, p. 50] for pointers) sends a homotopy type X : Type to a type $[X]_n$: Type which left-universally approximates X while containing no non-trivial $(n+1)$-fold homotopies. In particular, the ordinary definitions of mathematical structures, such as of groups (cf. [MSS23, (140)]), come out as intended (only) on 0-truncated base types $X = [X]_0$. Instead, for higher base types these definitions need to be refined to "higher structures". While higher structures are profoundly interesting — and while we have here at our fingertips a definition of higher "$(\infty, 1)$-Hilbert spaces" (of finite type) which is worth exploring further — for the scope of this note we want to restrict attention to ordinary quantum state spaces.

To that end, the further subtlety in `LHoTT` is that in a stable ∞-category like the QuType, the plain notion of truncation becomes meaningless: For all $n \in \mathbb{N}$, the *only* object of QuType which is n-truncated is the 0-object ([Lu17, Warning 1.2.1.9]). Instead, the proper notion that replaces n-truncation in stable ∞-categories are "t-structures" ([Lu17, §1.2.1]), and the stable (i.e., linear) analog of the 0-truncated sector is the "heart" of the t-structure ([Lu17, Def. 1.2.1.11]).

Concretely, the heart of `LHoTT` may be characterized as follows. Consider the function which sends a quantum type to

$$\Omega^\infty \;\equiv\; \natural(\mathbb{1} \to (-)) \;:\; \mathrm{QuType} \longrightarrow \mathrm{ClaType}\,,$$

the classicalization (16) of its internal hom out of the tensor unit (cf. [Ri22, Def. 2.1.26][SS23d, Prop. 2.7]). Then a linear type belongs to the heart iff

(i) under classicization $\natural$ (16) it is identified with its 0-truncation,

(ii) the 0-truncations of the classicization of all its suspensions are contractible: (cf. [Lu17, Prop. 1.4.3.4]): [5]

$$\mathrm{QuType}^\heartsuit \;\equiv\; \Big(V : \mathrm{QuType}, \quad \Omega^\infty V = [\Omega^\infty V]_0, \quad \underset{n\in\mathbb{N}}{\forall}\, [\Omega^\infty \Sigma^n V]_0 \,=\, * \Big). \qquad (19)$$

[5] In terms of module spectra this may be readily understood from the fact that the stable homotopy groups of the suspensions of a spectrum relate to those of its underlying homotopy type by $\pi_k(\Omega^\infty\Sigma^n V) = \pi_{n-k}(V)$. In view of this, the heart condition (19) says equivalently that the $\mathbb{Z}$-graded stable homotopy groups $\pi_\bullet(V)$ are concentrated in degree 0.

In particular, the heart of $H\mathbb{R}$-module-spectra is the ordinary category of $\mathbb{R}$-vector spaces:

$$(\mathrm{Mod}_{H\mathbb{R}}, \otimes_{H\mathbb{R}}, H\mathbb{R}, \sigma_{H\mathbb{R}})^{\heartsuit} \simeq (\mathrm{Mod}_{\mathbb{R}}, \otimes_{\mathbb{R}}, \mathbb{R}, \sigma_{\mathbb{R}}) . \tag{20}$$

In conclusion, for the purpose of using LHoTT as a universal quantum programming and certification language (as laid out in [SS23d], in certifiable refinement of the Proto-Quipper language), one may code (fin-dim) Hilbert spaces in LHoTT as heart-types (19) among purely linear types (16) equipped with symmetric self-duality and isometric complex structure as in (9), using the negative unit scalar type (18). When interpreted into $H\mathbb{R}$-module spectra (as in [SS23c]), this gives, up to equivalence (6), the ordinary symmetric closed monoidal category of (fin-dim) $\mathbb{C}$-Hilbert spaces with isometries between them, whose dagger-structure is induced by plain dualization (14).

Outlook. The restriction to the heart of LHoTT in (19) serves the purpose (only) of showing that traditional quantum information theory does embed faithfully into LHoTT. If we just drop this constraint then the language construct (9) with suitable higher coherences added speaks about a generalization of (fin-dim) Hilbert spaces to higher "$(\infty, 1)$-Hilbert spaces" (of finite type), such as modeled by (unbounded) $\mathbb{R}$-chain complexes equipped with self-duality under the tensor product of chain complexes (cf. [Lu09, Ex. 2.4.28]). Progenitors of such $(\infty, 1)$-Hilbert spaces appear in [MSS23], where spaces of $\mathfrak{su}(2)$-anyonic quantum ground states arise as the homology of chain complexes for twisted/equivariant cohomology of configuration spaces of points. Generally, topological quantum effects needed for topological quantum computing are described by (linear) higher homotopy theory ("higher structures") of the kind expressed by the LHoTT language.

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