Quantum observables of Quantized fluxes

Hisham Sati[*][†]   and   Urs Schreiber[*]

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Abstract

While it has become widely appreciated that defining (higher) gauge theories requires, in addition to ordinary phase space data, also "flux quantization" laws in generalized differential cohomology, there has been little discussion of the general rules, if any, for lifting Poisson-brackets of (flux-)observables and their quantization from traditional phase spaces to the resulting higher moduli stacks of flux-quantized gauge fields.

In this short note, we present a systematic analysis of **(i)** the canonical quantization of flux observables in Yang-Mills theory and **(ii)** of valid flux quantization laws in abelian Yang-Mills, observing **(iii)** that the resulting topological quantum observables form the homology Pontrjagin algebra of the loop space of the moduli space of flux-quantized gauge fields.

This is remarkable because the homology Ponrjagin algebra on loops of moduli makes immediate sense in broad generality for higher and non-abelian (non-linearly coupled) gauge fields, such as for the C-field in 11d supergravity, where it recovers the quantum effects previously discussed in the context of "Hypothesis H".

Contents

[*] Mathematics, Division of Science; and
Center for Quantum and Topological Systems,
NYUAD Research Institute,
New York University Abu Dhabi, UAE.

[†]The Courant Institute for Mathematical Sciences, NYU, NY

Electromagnetic fluxes in (higher) gauge theories are generally subject to "quantization laws" ("flux quantization", "charge quantization", going back to [Di1931][Al85, §2] and others, we follow [FSS23, Intro][SS23c] with exposition in [SS24], see also [Fr00] and §2 below), broadly in the sense of "discretization": In the familiar abelian case without self-sourcing of fluxes, these laws imply that total fluxes through (and hence charges inside) closed hypersurfaces form lattices ("charge lattices"), hence are integer multiples of certain unit fluxes (unit charges). This picture generalizes (with the recent construction of the non-abelian character map in [FSS23]) to theories with non-abelian fluxes (a famous example being the C-field in 11d supergravity, see [SS24, Ex. 2.12][GSS24]), now flux-quantized in non-abelian differential cohomology as discussed in the companion article [SS23c]. In any case, such flux-quantization applies already to classical gauge fields (as soon as they serve as background fields for charged quantum probes).

On top of this, there is the actual quantization of fluxes, whereby quantum observables on fluxes form a non-commutative star-algebra (e.g. [BLOT90, §6]), reflecting quantum uncertainties (see §1).

It should be clear that a deeper understanding of quantum gauge field theory requires an understanding of the combination of these two quantum effects, namely of quantum observables on quantized fluxes (cf. §4). However, existing discussions of the two aspects are mostly disjoint, among the exceptions being [FMS07a][FMS07b][BBSS17] (to which we relate in the following, as we proceed).

In this brief note we mean to clarify some general principles behind topological quantum observables on flux-quantized gauge fields, by making some observations (Thms. 1.1, 1.2 & 3.1) which are not hard to prove but whose import seems not to have been appreciated before, while they arguably touch on the heart of the matter. Background discussion and proofs are relegated to appendix A.

1 Quantum observables on Yang-Mills fluxes

The phase space of fluxes. The following fundamental statement about classical observables on fluxes in Yang-Mills theory is implied by standard facts about the phase space structure (cf. [FP83, §3][BLS84, §2]) but seems not to have been noticed before, in its entirety (key observations are due to [CP17]). Consider $\mathfrak{g}$ a metric Lie algebra with pairing $\langle -, - \rangle$, Σ a closed orientable surface (not necessarily connected) embedded in spacetime, and consider maps $\alpha \in C^\infty(\Sigma, \mathfrak{g})$ as observables on fluxes that send the electric/magnetic flux density to its integral over Σ against $\langle \alpha, - \rangle$.

Theorem 1.1 (Phase space of Yang-Mills fluxes)**.** *The phase space of electromagnetic fluxes in $\mathfrak{g}$-Yang-Mills theory, through a closed orientable surface Σ, is the Lie-Poisson manifold (e.g. [We83, §3]) associated with the Fréchet Lie algebra of smooth maps into the semidirect product, via the adjoint action, of $\mathfrak{g}$ (with Lie bracket rescaled by $\hbar \in \mathbb{R}_{>0}$) on its underlying abelian Lie algebra $\mathfrak{g}_0$:*

$$\underbrace{C^\infty\big(\Sigma, (\mathfrak{g}_\hbar \ltimes_{\mathrm{ad}} \mathfrak{g}_0)\big)}_{\text{Linear observables on fluxes}} \simeq \underbrace{C^\infty\big(\Sigma, \mathfrak{g}_\hbar\big)}_{\text{electric}} \ltimes_{\mathrm{ad}} \underbrace{C^\infty\big(\Sigma, \mathfrak{g}_0\big)}_{\text{magnetic}}. \tag{1}$$

Proof. We discuss this in appendix A.1. $\qquad\square$

Quantum flux observables as group algebras. This is remarkable, because the non-perturbative (aka "strict-" or "C^*-algebraic-"deformation or "Rieffel-") quantizations ([Ri89][Ri94], review in [La99, §2][LR01, §4][Ha08, §2]) of Lie-Poisson phase spaces are well-knonw [Ri90][LR01, Ex. 11.1]

[La99, Ex. 2]. Indeed, upon choosing a Lie group integrating the given Lie algebra, the non-perturbative quantum observables on the Lie-Poisson space form its *group algebra* under the convolution product, formed with due attention to analytic issues. One may think of this (cf. [LR01, Ex. 11.3] and [BHR07]) as a version of the time-honored quantization step from canonical commutation relations in the form of Heisenberg Lie algebras to their exponentiated Weyl form ([vN1931, p. 571]) of quantum observables.

Concretely in the case of Thm. 1.1, choose G a Lie group (not necessarily connected) with Lie algebra $\mathfrak{g}$, and choose a linear representation of G on the underlying vector space of $\mathfrak{g}$ which on the connected component G_e restricts to the adjoint action. Then for $\Lambda \subset \mathfrak{g}$ a lattice (not necessarily of full dimension, in fact possibly zero) which is preserved under this action, we obtain the corresponding semidirect product Lie group of G with the (partial) torus $\mathfrak{g}_0/\Lambda$ and hence a Fréchet Lie group of maps Lie-integrating (1):

$$\underbrace{C^\infty\big(\Sigma,\, G \ltimes (\mathfrak{g}_0/\Lambda)\big)}_{\substack{\text{Exponentiated linear} \\ \text{observables on fluxes}}} \simeq \underbrace{C^\infty\big(\Sigma,\, G\big)}_{\text{electric}} \ltimes \underbrace{C^\infty\big(\Sigma,\, (\mathfrak{g}_0/\Lambda)\big)}_{\text{magnetic}}. \tag{2}$$

A typical example of the choices involved for $\mathfrak{g} = \mathfrak{su}(2)$ is given by $G \equiv \mathrm{SU}(2)$ and $\Lambda = 0 \subset \mathfrak{su}(2)$. However, it is important to notice the freedom of choosing G to be non-connected, which here is part of the usual freedom in choosing quantizations. For instance, already for $\mathfrak{g} = \mathfrak{u}(1) \simeq \mathbb{R}$ we may choose G to be the direct product group $\mathrm{U}(1) \times \mathbb{Z}_2$, $\Lambda = \mathbb{Z} \subset \mathbb{R}$ and the action of $\mathrm{U}(1) \times \mathbb{Z}_2$ on $\mathfrak{u}(1) \simeq \mathbb{R}$ to factor through the $\mathbb{Z}_2$-action by multiplication with -1. With this choice, the coefficient group in (2) is the non-abelian group $\mathrm{U}(1) \times \mathbb{Z}_2 \ltimes \mathrm{U}(1)$, reflecting a non-trivial commutator between electric and magnetic flux observables; see Ex. 1.4 below.

Topological quantum observables on fluxes. Here we are not concerned with the analytical fine-print of the convolution algebra on (2); instead, we focus on just its subsector of *topological flux observables*, namely those that are locally constant as functions on the group manifold, and as such form the subalgebra which is the ordinary group algebra (58) of the group of connected components of (2). Interestingly, this group of connected components of (2) is, by the smooth Oka principle (46), equivalently a (possibly non-abelian) cohomology group of Σ:

$$
\begin{aligned}
\underbrace{\pi_0\, C^\infty\big(\Sigma,\, G \ltimes (\mathfrak{g}_0/\Lambda)\big)}_{\substack{\text{Topological sectors of} \\ \text{exponentiated linear} \\ \text{observables on fluxes}}} &\simeq \pi_0\, \mathrm{Map}\big(\Sigma,\, G \ltimes B\Lambda\big) & \text{by (46)} \\
&\simeq \pi_0\, \mathrm{Map}\big(\Sigma,\, G\big) \ltimes \pi_0\, \mathrm{Map}\big(\Sigma,\, B\Lambda\big) & \text{by (47)} \\
&\simeq \underbrace{H^0\big(\Sigma;\, G\big)}_{\text{electric}} \ltimes \underbrace{H^1\big(\Sigma;\, \Lambda\big)}_{\text{magnetic}} & \text{by (52)}.
\end{aligned}
\tag{3}
$$

In the last line, we retain the topology on the cohomology coefficients G; see Rem. A.1.

In conclusion, combining Thm. 1.1 with Rieffel-quantization of Lie-Poisson structures yields:

Theorem 1.2 (Non-perturbative topological quantum observables on Yang-Mills fluxes). *The convolution group algebra* (58) *on the (possibly non-abelian) cohomology group* (3)

$$\mathbb{C}\big[\pi_0\, \mathrm{Map}(\Sigma;\, G \ltimes B\Lambda)\big] \simeq \mathbb{C}\big[H^0(\Sigma;\, G) \ltimes H^1(\Sigma;\, \Lambda)\big] \tag{4}$$

is a subalgebra of topological observables in a non-perturbative Rieffel-quantization of the phase space from Thm. 3.1 of fluxes in $\mathfrak{g}$-Yang-Mills.

Example 1.3 (Non-perturbative topological quantum observables on Maxwell fluxes). For $\mathfrak{u}(1)$-Yang-Mills theory (vacuum Maxwell theory) an evident choice in (2) of Lie group G and lattice Λ is $G \equiv \mathrm{U}(1)$ and $\Lambda = \mathbb{Z} \hookrightarrow \mathbb{R} \simeq_\mathbb{R} \mathfrak{u}(1)$. In this case $H^0(\Sigma;\, G) \simeq H^1(\Sigma;\, \mathbb{Z})$ (see (53)) and also

$H^1(\Sigma; \Lambda) = H^1(\Sigma; \mathbb{Z})$, so that the algebra of topological flux quantum observables from (4) is this group algebra:

$$\mathbb{C}\big[\mathrm{Map}\big(\Sigma,\ \underbrace{\mathrm{U}(1)}_{\text{electric}} \times \underbrace{\mathrm{U}(1)}_{\text{magnetic}}\big)\big] \ \simeq\ \mathbb{C}\big[\underbrace{H^1(\Sigma; \mathbb{Z})}_{\text{electric}} \times \underbrace{H^1(\Sigma; \mathbb{Z})}_{\text{magnetic}}\big]. \tag{5}$$

Notice that, while this algebra is commutative, it is in general distinct from (and non-isomorphic to) the algebra of classical observables with its pointwise (non-convoluting) product. The pointwise product sees the topological flux sectors as "superselection" sectors, whose Hilbert space decomposes as a direct sum indexed by electric and magnetic flux, such that all observables are block-diagonal with respect to this decomposition.

In contrast, the quantum algebra (5) has operator representations by tuples of unitary operators, mutually commuting with each other but each acting by *shifting by a unit* through the lattice of topological sectors. (This result is different from the proposal in [FMS07b, p. 20], but not unlike in spirit.)

Concretely, consider the simple case that $\Sigma \simeq \mathbb{T}^2$ is a torus. Then $H^1(\mathbb{T}^2; \mathbb{Z}) \simeq \mathbb{Z}_a \times \mathbb{Z}_b$, such that, under the identification with $H^0(\mathbb{T}^2; S^1)$ (see (53)), an element $\vec{n} \equiv (n_a, n_b) \in \mathbb{Z}_a \times \mathbb{Z}_b$ is the homotopy class of a $\mathfrak{u}(1)$-valued smearing function (24) on $\mathbb{T}^2$ exponentiated to a $\mathrm{U}(1)$-valued function $\mathbb{T}^2 \to \mathrm{U}(1)$ which winds n_a and n_b times around $\mathrm{U}(1) \simeq S^1$ as its arguments travel once around one or the other nontrivial cycle of $\mathbb{T}^2$, respectively. Denoting by

$$H^1\big(\mathbb{T}^2; \mathbb{Z}\big) \times H^1\big(\mathbb{T}^2; \mathbb{Z}\big) \longhookrightarrow \mathbb{C}\big[H^1\big(\mathbb{T}^2; \mathbb{Z}\big) \times H^1\big(\mathbb{T}^2; \mathbb{Z}\big)\big]$$
$$\big(\vec{n}^{\,\mathrm{el}}, \vec{n}^{\,\mathrm{mag}}\big) \longmapsto \mathcal{O}\big(\vec{n}^{\,\mathrm{el}}, \vec{n}^{\,\mathrm{mag}}\big)$$

the observable corresponding to these classes of smearing functions, the quantum (operator) product on them is

$$\mathcal{O}\big(\vec{n}^{\,\mathrm{el}}, \vec{n}^{\,\mathrm{mag}}\big) \cdot \mathcal{O}\big(\vec{m}^{\,\mathrm{el}}, \vec{m}^{\,\mathrm{mag}}\big) = \mathcal{O}\big(\vec{n}^{\,\mathrm{el}} + \vec{m}^{\,\mathrm{el}},\ \vec{n}^{\,\mathrm{mag}} + \vec{m}^{\,\mathrm{mag}}\big),$$

as befits observables in Weyl form ([vN1931, p. 571]).

Notice that (the exponentials of) the total flux observables $e^{\frac{i}{\hbar}\int_{\mathbb{T}^2} E}$ and $e^{\frac{i}{\hbar}\int_{\mathbb{T}^2} F_A}$, whose smearing functions are constant, are summands of $\mathcal{O}\big((0,0),(0,0)\big)$. From this, all other topological flux observables are obtained by acting with the generators $\mathcal{O}\big((\pm 1, 0), (0,0)\big)$ and $\mathcal{O}\big((0, \pm 1), (0,0)\big)$ etc., which observe the sectors of the first Fourier modes of fluxes through $\mathbb{T}^2$.

Example 1.4 (Quantum observables on "large fluxes"). In generalizing Ex. 1.3, notice that in quantizing we do have the freedom of choosing in (2) a non-connected Lie group G with Lie algebra $\mathfrak{u}(1)$ – such as $G \equiv \mathrm{U}(1) \times \mathbb{Z}_2$, even while retaining $\mathfrak{u}(1)/\mathbb{Z} \simeq \mathrm{U}(1)$ as the gauge group of the magnetic fluxes, as usual. If we think of the flux observables in (1) as G-gauge transformations – indeed this is how they appear more manifestly below in (19) – then the non-connected components of G correspond to what are known as "large gauge transformations" (e.g. [HT92, p. 31]). In this vein here we may speak of "large fluxes". Now since $\mathbb{Z}_2 \simeq \mathrm{Aut}(\mathbb{Z})$ we may consider a non-trivial action of G on $\mathfrak{g}_0/\Lambda$ in (2). This leads, via (4), to a quantum algebra of topological flux observables:

$$\mathbb{C}\big[\pi_0 \mathrm{Map}\big(\Sigma, (\underbrace{\mathrm{U}(1) \times \mathbb{Z}_2}_{\text{electric}} \ltimes \underbrace{\mathrm{U}(1)}_{\text{magnetic}})\big)\big] \ \simeq\ \mathbb{C}\big[\underbrace{H^1(\Sigma; \mathbb{Z}) \times H^0(\Sigma; \mathbb{Z}_2)}_{\text{electric}} \ltimes \underbrace{H^1(\Sigma; \mathbb{Z})}_{\text{magnetic}}\big], \tag{6}$$

which is non-commutative, due to a non-trivial commutator between magnetic and large electric fluxes. More generally, we could as well choose a semidirect product group $G \equiv \mathrm{U}(1) \rtimes \mathbb{Z}_2$ for the electric fluxes, in which case already the electric topological flux observables among themselves have non-trivial commutators, as is generally the case for non-abelian non-topological flux observables (1). Yet more generally we may replace $\mathbb{Z}_2$ by any discrete group equipped with a pair of homomorphisms $K \to \mathbb{Z}_2$, to obtain the following quantum algebras of topological fluxes:

$$\mathbb{C}\big[\pi_0 \mathrm{Map}\big(\Sigma, (\underbrace{\mathrm{U}(1) \rtimes K}_{\text{electric}} \ltimes \underbrace{\mathrm{U}(1)}_{\text{magnetic}})\big)\big] \ \simeq\ \mathbb{C}\big[\underbrace{H^1(\Sigma; \mathbb{Z}) \rtimes H^0(\Sigma; K)}_{\text{electric}} \ltimes \underbrace{H^1(\Sigma; \mathbb{Z})}_{\text{magnetic}}\big]. \tag{7}$$

4

In these examples, the choice of non-connected integrations of the gauge Lie algebra is, while certainly mathematically admissible, unusual in traditional discussions of gauge theory. We next highlight, in §2, that such global choices of gauge group structure are part of the general phenomenon of flux quantization and as such have a clear relevance that deserves attention.

2 Flux quantization in abelian Yang-Mills

The electromagnetic flux density (the Faraday tensor) $F \in \Omega^2_{\mathrm{dR}}(X^4)$ on a spacetime manifold X^4 (cf. [SS23c, Ex. 3.1]) can be thought of as a map $F : X \to \mathbf{\Omega}^2_{\mathrm{clsd}}$ to the closed-differential form classifier in the topos of smooth sets (cf. [Sc13, §1.2.3.2][GiS23, §2.3][Sc24, p. 4]). A fundamental and now classical insight into quantum gauge theory is the observation that this needs to be accompanied by a map $\chi : X \to \mathbf{B}^2\mathbb{Z}$ to the classifying space for integral 2-cohomology and by a homotopy $\widehat{A}$ in the ∞-topos of smooth ∞-groupoids (cf. [FSS23, Prop. 1.24][1], exposition in [FSS14][Sc24]). This identifies the images of the two in real cohomology (see [FSS23, Ex. 9.4]):

$$
\begin{array}{c}
\xymatrix{
& \mathbf{\Omega}^2_{\mathrm{clsd}} \ar[r]^{de\ Rham\ map} & \\
X^4 \ar[ur]^{F} \ar[dr]_{\chi} & \ \ \Downarrow \widehat{A} & \mathbf{B}^2\mathbb{R} \ . \\
& \underbrace{\mathbf{B}^2\mathbb{Z}}_{B\mathrm{U}(1)} \ar[ur]_{extension\ of\ scalars} &
}
\end{array}
\tag{8}
$$

This extra data exhibits *flux quantization* (often: "Dirac charge quantization", for further discussion and pointers see [Al85, §2][Fr00, §2][FSS23, p. 4][SS23c, §3.1][SS24]) in that it ensures that the integrated magnetic flux through any 2-sphere submanifold $S^2 \hookrightarrow X$ is an integer $\int_{S^2} F \in \mathbb{Z}$,

counting the *number* of elementary magnetic solitons enclosed by S^2. For instance, if

$$
X \equiv \mathbb{R}^{1,1}_+ \wedge \mathbb{R}^2_{\cup\{\infty\}}
$$

is Minkowski spacetime with the "point at infinity" of a spatial hyperplane adjoined – encoding the constraint that fields vanish at infinity along this plane, as is the case for a real laboratory magnetic field through a slab of material in the laboratory – and if

$$
S^2 \simeq \{0,0\} \times \mathbb{R}^2_{\cup\{\infty\}} \longrightarrow X
$$

is the resulting sphere, then flux quantization reflects the experimentally observed phenomenon of integer numbers of Abrikosov vortices in a type-II superconducting material (cf. [SS24, §2.1]).

While this situation in Maxwell theory is commonly felt to be settled, it is not outright clear (and has hardly received consideration) which corresponding quantization condition is to be imposed on the *electric* flux density represented by the Hodge-dual 2-form $\star F$. To even state the question properly, we need to get hold of $\star F$ as an independent flux variable that can be subjected to flux quantization.

Premetric fluxes. This is accomplished by the equivalent "pre-metric" formulation of Maxwell's equations ([Cartan1924, §80], cf. [Fr00, Ex. 3.8][HIO16][BBSS17, Rem. 2.3][LS22, Def. 1.16][LS23, Def. (3)][SS23c, §3.1][SS24, §2.4]), which (in vacuum) subjects not one but two closed flux density variables $(F, E) : X \to \mathbf{\Omega}^2_{\mathrm{clsd}} \times \mathbf{\Omega}^2_{\mathrm{clsd}}$ to a further *constitutive relation* enforcing their Hodge-duality on X^4:

$$
E = \star_4 F \ .
\tag{9}
$$

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[1]When we refer to equation-, definition-, proposition-, page-numbers in [FSS23] we refer to the version published by World Scientific — see `ncatlab.org/schreiber/show/The+Character+Map#PublishedVersion` — which differs from the numbering in the arXiv version (otherwise the content is the same).

Trivially equivalent as this re-formulation is, it makes manifest the maximal "decoupling" of the *cohomological* from the *metric* content of Maxwell's equations, clearly suggesting that a choice of flux quantization of the pair (F, E) needs to be made along the lines of (8), and only afterward the Hodge-duality condition (9) to be reimposed, in some fashion.

Notice that this is just the same approach which in supergravity/string theory is known ([SS24, §2.4]) as the "duality-symmetric" or "democratic" formulation (eg. [MV23, p. 2]) underlying notably the common *Hypothesis K* that RR-fluxes are quantized in topological K-theory (see [GrS22, p.3][SS24, §4.1] for pointers), and likewise for the fields in M-theory (see [Sa10, §4]), here underlying the analogous "Hypothesis H" ([SS24, §4.2], see p. 7 below).

In fact, we may observe (we expand on this in the companion article [SS23c]) that the pre-metric/democratic formulation of vacuum Maxwell theory essentially coincides with its canonical phase space formulation (cf. appendix A.1):

Namely on a globally hyperbolic spacetime $X^4 \simeq \mathbb{R}^{0,1} \times X^3$ in temporal gauge ($A_0 = 0$) the Faraday tensor in $\Omega^2(X^4)_{\mathrm{clsd}}$ decomposes as a magnetic flux density $F \in \Omega^2(X^3)_{\mathrm{clsd}}$ and a temporal component, whose Hodge dual is the electric flux density $E \in \Omega^2(X^3)_{\mathrm{clsd}}$ which takes the role of the field's canonical momentum, and whose closure condition $\mathrm{d}E = 0$ now plays the role of the *Gauß law* constraint (23). This way, F and E are indeed independent field variables on X^3 and the constitutive relation (9) is all absorbed into the prescription by which initial value data (F, E) on X^3 induces temporal evolution in $X^4 = \mathbb{R}^{0,1} \times X^3$ (cf. [SS23c, §3.1]).

Phase space flux quantization. Hence we need to ask: What are the admissible flux quantization laws for flux densities $(F, E) : X^3 \to \Omega^2_{\mathrm{clsd}} \times \Omega^2_{\mathrm{clsd}}$? The general answer is given in [FSS23]: These are given by choices of topological spaces[2] $\mathcal{A}$ whose rationalization is equivalent to $B^2\mathbb{Q} \times B^2\mathbb{Q}$:

$$\mathcal{A} \xrightarrow{\text{rationalization}} B^2\mathbb{Q} \times B^2\mathbb{Q} \xrightarrow[\text{of scalars}]{\text{extension}} B^2\mathbb{R} \times B^2\mathbb{R}. \tag{10}$$
$$\underbrace{\hspace{7cm}}_{\mathbb{R}\text{-rationalization}}$$

For any such choice, the corresponding flux-quantized pre-metric gauge fields are given by homotopies of smooth ∞-groupoids of the following form ([FSS23, Def. 9.3], but for our purpose here the reader need not further be concerned with the details of this construction):

$$
\begin{array}{ccc}
 & \mathbf{\Omega}^2_{\mathrm{clsd}} \times \mathbf{\Omega}^2_{\mathrm{clsd}} & \\
\overset{(F,E)}{\nearrow} & {\scriptstyle \simeq \Downarrow \widehat{A}_{\mathrm{EM}}} \xrightarrow{\text{de Rham map}} & \\
X^3 & & \mathbf{B}^2\mathbb{R} \times \mathbf{B}^2\mathbb{R} \\
\underset{\chi}{\searrow} & \mathcal{A} \xrightarrow[\mathbb{R}\text{-rationalization}]{} & \nearrow
\end{array}
\tag{11}
$$

Notice that there are *many* available choices for such $\mathcal{A}$, and that each choice is a statement (a prediction) about the corresponding physics. For instance, any connected topological space Q all of whose homotopy groups are finite (e.g. classifying spaces of finite groups)

$$K \in \mathrm{Grp}^{\mathrm{fin}} \qquad \vdash \qquad BK \xrightarrow{\text{rationalization}} * \tag{12}$$

has trivial rationalization. This implies that with A also any $A \times Q$ is an admissible flux quantization law (10) for pre-metric electromagnetism.

We make explicit some of the possible choices of electromagnetic flux quantization:
- In view of (8), a suggestive choice may be the one that subjects $\star F$ to same integral flux quantization as F
$$\mathcal{A} \equiv \underbrace{B\mathrm{U}(1)}_{\text{magnetic}} \times \underbrace{B\mathrm{U}(1)}_{\text{electric}}, \tag{13}$$

---

[2] Here we restrict attention to classifying spaces $\mathcal{A}$ which are simply connected with finite-dimensional rational cohomology in each degree. This is not to get sidetracked by technical complications which, while of interest to the issue of flux quantization, are more esoteric and beyond the intended scope of this note, cf. [FSS23, Rem. 5.1].

This corresponds to the choice made in [FMS07a, (1.26)][BBSS17, Rem. 2.3][LS22, Def. 4.1][LS23, Def. 4.3]. In [FMS07b, (3.4)] it says that it "follows immediately" from the Hodge self-duality of Maxwell's equations; but this is to presuppose the answer to the question: Whether flux quantization laws retain all the symmetries of the underlying differential form data is a hypothesis that would ultimately need to be decided by experiment. Mathematically it is consistent, but so would be many other choices.

- Indeed, the mathematical physics literature commonly implies *no* further condition on the electric flux density $\star F$, apart from it being the Hodge-dual of the magnetic flux density. In particular, common discussions of electromagnetism assume that the topological content of an EM-field configuration is all encoded in the class $\chi : X \to B^2 \mathbb{Z}$ of a single U(1)-principal bundle, nothing else. This assumption is reflected in the choice

$$\mathcal{A} \equiv \underbrace{B\mathrm{U}(1)}_{\text{magnetic}} \times \underbrace{B^2 \mathbb{Q}}_{\text{electric}} \; . \tag{14}$$

- In either case, we highlight that the flux quantization laws (10) are subject to choices of pure torsion components in the classifying space. Even if we remain within the traditional assumption that magnetic flux is classified exactly by $B\mathrm{U}(1)$ and that also electric flux should satisfy an integrality constraint as in (13), there is still the freedom to postulate that electric flux is classified by a non-connected extension of U(1), such as $\mathrm{U}(1) \rtimes K$ for any finite group K with any action on U(1):

$$\mathcal{A} \simeq B\big(\underbrace{\mathrm{U}(1)}_{\text{magnetic}} \rtimes \underbrace{K \ltimes \mathrm{U}(1)}_{\text{electric}} \big). \tag{15}$$

(This freedom of choosing "global" non-abelian structure even in abelian Yang-Mills theory has also been observed, from a different angle, in [LS22].)

We will see in §3 that it is not entirely a coincidence that the group in (15) is the one controlling the topological flux observables in (7).

Hypotheses about flux quantization laws. In listing the above examples, our aim is not to dwell on phenomenological questions of experimental quantum electromagnetism (though these are worthwhile, cf. [FMS07b, p. 28][KMW07]), but rather to amplify the previously underappreciated mathematics parameterizing the space of consistent possibilities. Namely, it is important to realize that analogous *choices* of flux quantization laws need to be made in higher gauge theories, and absent further rules any such choice is a *hypothesis* on the fundamental nature of these theories.

Notably, when making the popular statement that RR-flux forms in type I/II supergravity theory are quantized in topological K-theory (for pointers see [GrS22][SS23c, §1.3], we refer to this traditional hypothesis as "Hypothesis K" following [SS23b, Rem. 4.1]), there is secretly a choice that has been made and existing consistency checks of this choice rarely try to differentiate it from other possible choices.

Perhaps more importantly, a similar choice of flux-quantization law needs to be made when considering the C-field fluxes in $D = 11$ supergravity. The similarly canonical-looking choice in this case is "Hypothesis H" ([Sa13, §2.5][FSS19][SS20][FSS21a][FSS21b][SS21a][FSS22][SS23a]). For this choice, we had observed [SS22][CSS23] that the Pontrjagin algebra of the loop space of the moduli space of flux-quantized C-fields looks a lot like an algebra of non-perturbative topological quantum observables on (fluxes sourced by) M-branes (see also §4).

Next, in §3, we observe that the analogous statement is already true for plain electromagnetism.

Notice that another role of non-perturbative Rieffel quantization in relation to C-field fluxes has been considered in [MS15].

3 Pontrjagin algebras of Quantum observables

In unification of §1 and §2, we observe here that algebras of topological flux observables (4) arise as the *homology Pontrjagin algebras* of the loop spaces of moduli spaces of flux-quantized fields.

Topological fields and KK-Reduction. We consider globally hyperbolic spacetimes which, besides the temporal direction, have a line factor singled out, i.e., are product spaces of the form

$$\underbrace{\mathbb{R}^{0,1}}_{\text{time}} \times \underbrace{\left(\mathbb{R}^1 \times \Sigma\right)}_{\text{Cauchy surface}} = \mathbb{R}^{1,1} \times \Sigma. \tag{16}$$

Here we may regard (by (19) below) the $\mathbb{R}^1$-factor as a "decompactified" KK-compactification fiber (i.e., the fully non-perturbative situation, as KK-theory goes). Indeed, a "spontaneous" compactification is automatically implied if we consider topological fields vanishing at spatial infinity (42), as usual (43) for solitonic fields, because:

$$\text{Map}^{*/}\left(\mathbb{R}^{0,1}_{\sqcup\{\infty\}} \wedge \left(\mathbb{R}^1 \times \Sigma\right)_{\cup\{\infty\}}, B(G \ltimes B\Lambda)\right) \qquad \text{moduli space of topological fields vanishing at spatial infinity}$$

$$\underset{\text{whe}}{\simeq} \text{Map}^{*/}\left(\left(\mathbb{R}^1 \times \Sigma\right)_{\cup\{\infty\}}, B(G \ltimes B\Lambda)\right) \qquad \text{by (45)} \tag{17}$$

$$\simeq \text{Map}^{*/}\left(\left(S^1 \wedge \Sigma_{\sqcup\{\infty\}}, B(G \ltimes B\Lambda)\right)\right) \qquad \text{by (37).}$$

(Here we are assuming, just for brevity for exposition, that Σ is already compact itself.)

Elementary as this is mathematically, it is somewhat remarkable as it exhibits the moduli space of topological fields as a loop space:

$$\text{Map}^{*/}\left(\mathbb{R}^{0,1}_{\sqcup\{\infty\}} \wedge \left(\mathbb{R}^1 \times \Sigma\right)_{\cup\{\infty\}}, B(G \ltimes B\Lambda)\right) \qquad \text{moduli space of topological fields vanishing at spatial infinity}$$

$$\underset{\text{whe}}{\simeq} \text{Map}^{*/}\left(\left(S^1 \wedge \Sigma_{\sqcup\{\infty\}}, B(G \ltimes B\Lambda)\right)\right) \qquad \text{by (17)}$$

$$\simeq \text{Map}^{*/}\left(S^1, \text{Map}^{*/}\left(\Sigma_{\sqcup\{\infty\}}, B(G \ltimes B\Lambda)\right)\right) \qquad \text{by (30)} \tag{18}$$

$$\simeq \Omega \text{Map}\left(\Sigma, B(G \ltimes B\Lambda)\right) \qquad \text{by (38) & (31).}$$

But this also means that:

$$\text{Map}^{*/}\left(\mathbb{R}^{0,1}_{\sqcup\{\infty\}} \wedge \left(\mathbb{R}^1 \times \Sigma\right)_{\cup\{\infty\}}, B(G \ltimes B\Lambda)\right) \qquad \text{moduli space of topological fields vanishing at spatial infinity}$$

$$\simeq \Omega \text{Map}\left(\Sigma, B(G \ltimes B\Lambda)\right) \qquad \text{by (18)} \tag{19}$$

$$\simeq \text{Map}\left(\Sigma, (G \ltimes B\Lambda)\right) \qquad \text{by (51).}$$

Hereby the topological fields on $\mathbb{R}^1 \times \Sigma$ are re-expressed as fields on Σ, as befits a KK-reduction. (This is the based version of double dimensional reduction via *free* looping [BMSS19, §2.2]).
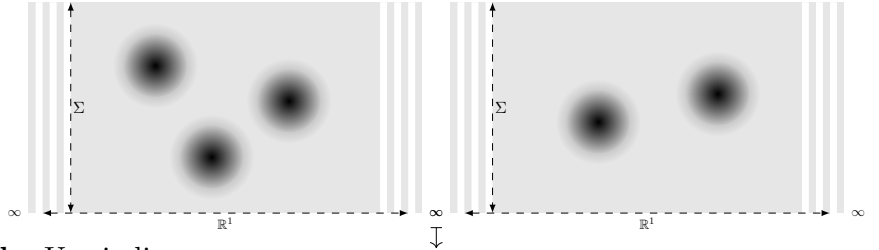
Fusion of topological KK-fields. Unwinding the definitions, one sees that the operation of loop concatenation in (18) corresponds to the "fusion" of field solitons in the KK-direction. For abelian fields, this fusion of solitons is reflected in the addition of their charges.
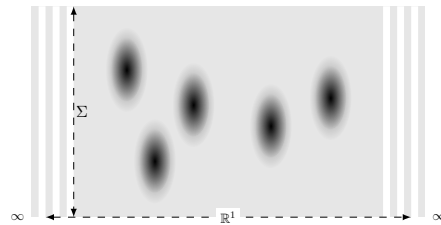
Pontrjagin algebra as Quantum observables. To see how to obtain quantum observables of such moduli, let us step back for a moment and reconsider the notion of quantum observables in the most simplistic non-trivial case, namely on a set $S \in$ Set of disconnected ("superselection") sectors of a physical system. The algebra of such quantum observables is just the linear span $\mathbb{C}[S]$ (whose canonical basis elements are the observations: "system is in sector $s \in S$") equipped with the S-element wise product of complex numbers, and regarded as a star-algebra under S-element wise complex conjugation.

Now regard this simplistic case from a more sophisticated perspective by regarding $S \in$ Set $\hookrightarrow$ kTopSp as a topological space that happens to carry a discrete topology. Then we may equivalently say that the observables span the *homology* $H_\bullet(S; \mathbb{C}) \simeq \mathbb{C}[S]$ of the space of sectors of configurations of the physical system. This is noteworthy, because the notion of homology makes sense, of course, for general topological spaces. The main point is that, while homology groups do not generally form a natural algebra structure – much less a star-algebra structure as required on quantum observables (e.g. [BLOT90, §6]) – they do so on *loop spaces* (38) (generally on "H-spaces"):

To that end, let TopFields$_\Sigma$ be a topological moduli space of topological field configurations of a given physical system, equipped with some basepoint. Then the homology of its loop space becomes a star-algebra (in fact a Hopf algebra) whose product is induced by the concatenation of loops ("Pontrjagin product" [Po39][BoSa53], cf. [Ha02, pp. 287]) and whose star-involution ("dagger"-operation) is induced by reversal of loops:

$$\Omega \, \text{TopFields}_\Sigma \, \times \, \Omega \, \text{TopFields}_\Sigma \xrightarrow[\text{concatenate loops}]{\text{conc}} \Omega \, \text{TopFields}_\Sigma$$

$$H_\bullet\big(\Omega \, \text{TopFields}_\Sigma; \mathbb{C}\big) \otimes H_\bullet\big(\Omega \, \text{TopFields}_\Sigma; \mathbb{C}\big) \xrightarrow[\text{Pontrjagin product}]{(-)\cdot(-) \, := \, \text{conc}_*} H_\bullet\big(\Omega \, \text{TopFields}_\Sigma; \mathbb{C}\big),$$

$$\Omega \, \text{TopFields}_\Sigma \xrightarrow[\text{reverse loops}]{\text{rev}} \Omega \, \text{TopFields}_\Sigma \tag{20}$$

$$H_\bullet\big(\Omega \, \text{TopFields}_\Sigma; \mathbb{C}\big) \xrightarrow[\text{Pontrjagin antipode}]{\text{rev}_*} H_\bullet\big(\Omega \, \text{TopFields}_\Sigma; \mathbb{C}\big)$$

$$\underset{\text{Pontrjagin dagger}}{\diagdown \, (\sim)\dagger} \qquad \underset{\substack{\text{complex} \\ \text{conjugation}}}{\Big\downarrow \overline{(-)}}$$

$$H_\bullet\big(\Omega \, \text{TopFields}_\Sigma; \mathbb{C}\big).$$

In these terms, we now obtain the following main observation of this note:

Theorem 3.1 (Pontrjagin ring of topological quantum observables on Maxwell fluxes)**.** *The Pontrjagin Hopf-algebra* (20) *of the moduli space of topological $\mathfrak{u}(1)$-gauge fields* (18) *subject to the flux-quantization law* (15) *is, in degree=0, isomorphic to the Hopf algebra of topological quantum observables from Ex. 1.4 on fluxes in $\mathfrak{u}(1)$-Yang-Mills:*

$$H_0\Big(\text{Map}^{*/}\Big(\mathbb{R}^{0,1}_{\sqcup\{\infty\}} \wedge \big(\mathbb{R}^1 \times \Sigma\big)_{\cup\{\infty\}}, B\big(\text{U}(1)^2 \rtimes K\big)\Big); \mathbb{C}\Big) \simeq \mathbb{C}\big[H^1(\Sigma; \mathbb{Z})^2 \rtimes H^0(\Sigma; K)\big].$$

Proof. We discuss this in appendix A.2. □

This observation seems noteworthy in that it obtains the topological quantum flux observables directly from the topology of the flux quantization law, short-cutting the analysis of Poisson brackets, and as such immediately generalizes to higher non-abelian gauge theories subject to rather more subtle flux quantization laws, such as the RR-fluxes in string theory and the C-field fluxes in M-theory mentioned on p. 7. We comment on the potential impact in the following outlook §4.

4 Conclusion and Outlook

Above we have focused on Pontrjagin algebras (of loop spaces of moduli spaces of topological gauge fields) in degree $= 0$, showing that these reproduce non-perturbative topological quantum observables on fluxes. It is interesting to notice that also in higher degrees these Pontrjagin algebras look like (higher) algebras of (higher) quantum observables:

Pontrjagin algebras as higher quantum algebras. The rational homotopy type of a simply connected topological space $\mathcal{A}$ (as in footnote 2) is all encoded in its rational Whitehead L_∞-algebra $\mathfrak{l}\mathcal{A} \in L_\infty\mathrm{Alg}$ (essentially its "Quillen model", see [FSS23, Prop. 5.11]), and the Pontrjagin algebra of its loop space is [MM65, p. 262][FHT00, Thm. 16.13] the universal graded enveloping algebra $U(-)$ of the underlying binary Whitehead Lie algebra bracket $[-, -]$ (a graded super-Lie algebra, but this generalizes to the enveloping A_∞-algebra of the full Whitehead L_∞-algebra [M-F22, Thm. 4.1]):
$$\mathcal{A} \text{ simply connected} \quad \Rightarrow \quad H_\bullet(\Omega\,\mathcal{A}; \mathbb{R}) \;\simeq\; U\big(\mathfrak{l}\mathcal{A}, [-,-]\big).$$
Now, of course, the passage from Lie algebras to universal enveloping algebras is again the (now "formal") deformation quantization of the corresponding Lie-Poisson structure [Gu83, (.42)][Gu11, §22][PV00, p. 3], now all in a higher-geometric sense reflected in the grading.

For example, if $\mathcal{A} \equiv S^4$ is the homotopy type of the 4-sphere, then $\mathfrak{l}S^4$ has generators v_3 in degree 3 and v_6 in degree-6, with non-trivial super-Lie bracket being the *M-theory gauge Lie algebra* $[v_3, v_3] = v_6$ ([CJLP98, (2.5)][Sa10, §4][SV23, Ex. 2.2][SS24, (27)]), whose graded universal enveloping algebra is hence
$$H_\bullet(\Omega\,S^4; \mathbb{C}) \;\simeq\; \mathbb{C}\big[v_3, v_6\big]/\big(v_3^2 - v_6\big). \tag{21}$$
This may be thought of as a quantum deformation of the cohomology of the 3-sphere $S^3 \hookrightarrow \Omega S^4$ just as, up to degree shifts, the *quantum cohomology* of $\mathbb{C}P^1$ (cf. [Wi90, p. 275][DGR10]):
$$QH^\bullet(\mathbb{C}P^1; \mathbb{C}) \;\simeq\; \mathbb{C}[v_2, v_4]/\big(v_2^2 - v_4\big)$$
is a deformation of the ordinary cohomology of $\mathbb{C}P^1$ (and, of course, S^3 forms a circle bundle over $\mathbb{C}P^1$, the Hopf fibration, as befits an M-theory lift.)

In view of Thm. 3.1 this suggests that passage to Pontrjagin algebras of loop spaces of moduli spaces may generally be a valid form of quantization, at least for topological observables, applicable in particular also to higher gauge theories. Since this construction shortcuts known forms of quantization, one is led to ask how to think of it as a quantization process in more detail:

Topological Light-cone quantization? A key aspect of Pontrjagin algebras of loops in moduli spaces is (18) that their product operation corresponds to sequencing along a singled-out *spatial* direction in spacetime. But, since the operator product order of quantum observables is well-known to reflect their *temporal* ordering [Fey42, p. 35][Fey48, p. 381] (cf., e.g., [Na99, pp. 33]), it stands to reason that the Pontrjagin product on loop space homology regarded as a quantum operator-product must be reflecting sequences of events which happen by *joint* progression along a "time-axis" *and* along an effectively periodic spatial direction. This is of course the case in ("discretized") light-cone quantization (review in [He01]), such as famously used for non-perturbative quantization of Yang-Mills theory (review in [FS10, §12]) and of sectors of M-theory (review in [Yd18, §3.9/7.9]).
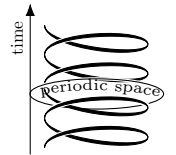
Indeed, the discussion in [SS22, §4.9] with the result of [CSS23] indicate that the method of Pontrjagin algebra quantization applied to topological fluxes sourced by intersecting M-branes reproduces the quantum states of transversal M2/M5-brane bound states in discretized light-cone quantization as previously discussed in the BMS matrix model.

The results presented here suggest that this is not a coincidence, but part of a general role that Pontrjagin algebras play in quantization of fluxes in (higher) gauge theories, cf. [SS23d, §2].

A   Background and Proofs

Here we spell out technical details and prove the claims in the previous sections.

    §A.1 – Phase space of fluxes in Yang-Mills theory

    §A.2 – Homotopy theory of topological field sectors

A.1   Phase space of Yang-Mills fluxes

Phase space of Yang-Mills theory. We start by recalling the canonical phase space structure of Yang-Mills theory in temporal gauge (e.g. [FP83, §3][BLS84, §2]), following the insightful account of [CP17].[3] We slightly generalize these accounts by admitting any metric Lie algebra as gauge algebra, and by admitting non-trivial topological sectors of gauge potentials.

Consider:

- $\mathfrak{g}$ a finite-dimensional real metric Lie algebra, hence equipped with a symmetric ad-invariant non-degenerate (but not necessarily definite) bilinear form $\langle -, - \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$.
- G a Lie group with Lie algebra $\mathfrak{g}$, being the Yang-Mills gauge/structure group.
- X a smooth 3-manifold, thought of as a Cauchy surface in globally hyperbolic spacetime $\mathbb{R}^{0,1} \times X$.
- $\widehat{X} \twoheadrightarrow X$ any differentiably good open cover, i.e., $\widehat{X} \equiv \underset{i \in I}{\sqcup} U_i$ for $\left\{ U_i \overset{\iota_i}{\hookrightarrow} X \right\}_{i \in I}$ a set of open subsets which cover, $\underset{i \in I}{\cup} U_i = X$, and all whose non-empty finite intersections are diffeomorphic to an open ball.
- $g_{\bullet, \bullet} : \widehat{X} \times_X \widehat{X} \to G$ a smooth ("transition"-)function, hence with components $g_{ij} : U_i \cap U_j \to G$, such that on any $U_i \cap U_j \cap U_k$ we have $g_{ij} \cdot g_{jk} = g_{ik}$, encoding a topological sector (a G-principal bundle) P of G-Yang-Mills theory on $\mathbb{R}^{0,1} \times X$.

Write:

- $\Omega^\bullet_{\mathrm{dR}}\big(\widehat{X}; \mathfrak{g}\big) \equiv \Omega^\bullet_{\mathrm{dR}}\big(\widehat{X}\big) \otimes \mathfrak{g}$ for the de Rham complex of smooth differential forms on $\widehat{X}$ with coefficients in $\mathfrak{g}$,
- $[-, -] : \Omega^\bullet_{\mathrm{dR}}\big(\widehat{X}; \mathfrak{g}\big) \otimes \Omega^\bullet_{\mathrm{dR}}\big(\widehat{X}; \mathfrak{g}\big) \to \Omega^\bullet_{\mathrm{dR}}\big(\widehat{X}; \mathfrak{g}\big)$ for the induced super-Lie bracket, given by the wedge product of differential forms in the given order, tensored with the Lie bracket of their coefficients,
- $\langle -, - \rangle : \Omega^\bullet(X; \mathfrak{g}_P) \otimes \Omega^\bullet(X; \mathfrak{g}_P) \to \Omega^\bullet(X)$ for the induced graded pairing, given by the wedge product of differential forms in the given order tensored with the pairing of their coefficients – and then regarded as a plain differential form on X, via the ad-invariance of the pairing,
- $\Gamma_{TX}\big(TP/G\big) = \big\{ A \in \Omega^1_{\mathrm{dR}}(\widehat{X}) \otimes \mathfrak{g} \,\big|\, \forall_{ij} \; A_j = \mathrm{d}g_{ij} + \mathrm{Ad}_{g_{ij}}(A_i) \text{ on } U_i \cap U_j \big\}$ for the set of gauge potentials in the sector P (principal connections on P),
- $\Omega^\bullet_{\mathrm{dR}}(X; \mathfrak{g}_P) = \big\{ \omega \in \Omega^\bullet_{\mathrm{dR}}(\widehat{X}) \otimes \mathfrak{g} \,\big|\, \forall_{i,j} \; \omega_j = \mathrm{Ad}_{g_{ij}}(\omega_i) \text{ on } U_i \cap U_j \big\}$ for the de Rham complex of smooth differential forms on X with values in sections of the P-adjoint bundle,
- $\mathrm{d}_A : \Omega^\bullet_{\mathrm{dR}}(X; \mathfrak{g}_P) \to \Omega^\bullet_{\mathrm{dR}}(X; \mathfrak{g}_P)$ for the covariant de Rham differential (of degree=1) with respect to a given $A \in \Gamma_{TX}(TP/G)$: $\mathrm{d}_A \omega \equiv \mathrm{d}\,\omega + [A, \omega]$,
- $F_A \equiv \mathrm{d}\,A + \frac{1}{2}[A, A] \in \Omega^2(X; \mathfrak{g}_P)$ for the magnetic flux density of A (the curvature form).

Now:

---

[3]The discussion in [CP17] is motivated by Ashtekar's phase space of first-order Einstein-gravity, which famously coincides with that of SU(2)-Yang-Mills theory subject to further constraints. But these further constraints play no role in [CP17] and the restriction to $G \equiv \mathrm{SU}(2)$ is inessential otherwise.

The phase space of G-Yang-Mills theory in temporal gauge on $\mathbb{R}^{0,1} \times X$ with respect to the background field sector P is globally coordinatized by:

- $A \in \Gamma_{TX}(TP/G)$, the gauge potential in temporal gauge, serving as the canonical coordinate,
- $E \in \Omega^2_{\mathrm{dR}}(X; \mathfrak{g}_P)$ the electric flux density, constituting the canonical momentum,

hence with the non-trivial Poisson bracket being

$$\left\{ \int_X \langle \omega, E \rangle, A(\hat{x}) \right\} \;=\; \omega(\hat{x}) \qquad \text{for } \omega \in \Omega^1_{\mathrm{dR}}(X; \mathfrak{g}_P)_{\mathrm{cpt}} \tag{22}$$

subject to a first-order constraint:

- the *Gauß-Law*
$$\mathrm{d}_A E \;=\; 0 \,. \tag{23}$$

Linear flux observables in Yang-Mills theory. Consider in addition:

- $\Sigma \hookrightarrow X$ a closed oriented 2-dimensional submanifold (not necessarily connected), being the surface through which electromagnetic flux is to be observed,
- $\Phi^\alpha_E := \int_\Sigma \langle \alpha, E \rangle$ for $\alpha \in \Omega^0_{\mathrm{dR}}(\Sigma, \mathfrak{g}_P)$, the electric flux through Σ integrated against a weight α,
- $\Phi^\beta_B := \int_\Sigma \langle \beta, F_A \rangle$ for $\beta \in \Omega^0_{\mathrm{dR}}(\Sigma, \mathfrak{g}_P)$, the magnetic flux through Σ integrated against weight β,

where the weights (or: "smearing functions") are smooth Lie-algebra valued functions, precisely:

$$\alpha, \beta \;\in\; \Omega^0_{\mathrm{dR}}(\Sigma; \mathfrak{g}_P) \;\simeq\; \Gamma_\Sigma\big(\mathfrak{g} \rtimes_{\mathrm{ad}} P\big) \,. \tag{24}$$

The subtlety pointed out and resolved in [CP17] is that these Φ^α_E, Φ^β_B are not technically observables on the phase space, since their would-be associated Hamiltonian vector fields are not smooth; but that gauge-equivalent regularized observables are obtained by considering:

- $\widehat{\Sigma} \hookrightarrow X$ the exterior component of a tubular neighborhood of Σ in X, hence a non-compact 3-dimensional submanifold with boundary $\partial \widehat{\Sigma} = \Sigma$;
- $\widehat{\Phi}^\alpha_E := \int_{\widehat{\Sigma}} \langle \mathrm{d}_A \alpha, E \rangle$ for compactly supported $\alpha \in \Omega^0_{\mathrm{dR}}\big(\widehat{\Sigma}; \mathfrak{g}_P\big)_{\mathrm{cpt}}$, the electric flux observable;
- $\widehat{\Phi}^\alpha_E := \int_{\widehat{\Sigma}} \langle \mathrm{d}_A \beta, F_A \rangle$ for compacty supported $\beta \in \Omega^0_{\mathrm{dR}}\big(\widehat{\Sigma}; \mathfrak{g}_P\big)_{\mathrm{cpt}}$, the magnetic flux observable.

The above two forms of the magnetic flux observable are actually equal, due to the Bianchi identity,

$$\begin{aligned}
\widehat{\Phi}^\beta_B &\equiv \int_{\widehat{\Sigma}} \langle \mathrm{d}_A \beta, F_A \rangle \\
&= \int_{\widehat{\Sigma}} \langle \mathrm{d}_A \beta, F_A \rangle + \int_{\widehat{\Sigma}} \langle \beta, \underbrace{\mathrm{d}_A F_A}_{0} \rangle \\
&= \int_{\widehat{\Sigma}} \mathrm{d} \langle \beta, F_A \rangle \\
&= \int_\Sigma \mathrm{d} \langle \beta, F_A \rangle \\
&= \Phi^\beta_B \,,
\end{aligned} \tag{25}$$

and the analogous computation, but now using the Gauß law (23), shows that the two forms of the electric flux observables coincide on the constraint surface (i.e. up to a term proportional to $\mathrm{d}_A E$):

$$\widehat{\Phi}^\alpha_E \;\approx\; \Phi^\alpha_E \,. \tag{26}$$

With the canonical Poisson bracket (22), one finds the Poisson brackets of these regularized electromagnetic linear flux observables, first for electric/electric fluxes (cf. [CP17, (7)])

$$\begin{aligned}
\left\{ \widehat{\Phi}^\alpha_E, \widehat{\Phi}^\beta_E \right\} &\equiv \left\{ \int_{\widehat{\Sigma}} \langle \mathrm{d}_A \alpha, E \rangle, \int_{\widehat{\Sigma}} \langle \mathrm{d}_A \beta, E \rangle \right\} \\
&= \left\{ \int_{\widehat{\Sigma}} \langle \mathrm{d}_A \alpha, E \rangle, \int_{\widehat{\Sigma}} \langle [A, \beta], E \rangle \right\} + \left\{ \int_{\widehat{\Sigma}} \langle [A, \alpha], E \rangle, \int_{\widehat{\Sigma}} \langle \mathrm{d}_A \beta, E \rangle \right\} \\
&= \int_{\widehat{\Sigma}} \Big\langle [\mathrm{d}_A \alpha, \beta], E \Big\rangle - \int_{\widehat{\Sigma}} \Big\langle [\alpha, \mathrm{d}_A \beta], E \Big\rangle \\
&= \int_{\widehat{\Sigma}} \Big\langle \mathrm{d}_A [\alpha, \beta], E \Big\rangle \equiv \widehat{\Phi}^{[\alpha, \beta]}_E \,,
\end{aligned} \tag{27}$$

and for the electric/magnetic fluxes (a statement which seems not to have been recorded before):

$$
\begin{aligned}
\left\{ \widehat{\Phi}_E^\alpha, \ \widehat{\Phi}_B^\alpha \right\} &\equiv \left\{ \int_{\widehat{\Sigma}} \langle \mathrm{d}_A\alpha, \ E \rangle, \ \int_{\widehat{\Sigma}} \langle \mathrm{d}_A\beta, \ F_A \rangle \right\} \\
&\equiv \left\{ \int_{\widehat{\Sigma}} \langle \mathrm{d}_A\alpha, \ E \rangle, \ \int_{\widehat{\Sigma}} \langle \mathrm{d}\,\beta + [A,\beta], \ \mathrm{d}A + \tfrac{1}{2}[A,A] \rangle \right\} \\
&= \int_{\widehat{\Sigma}} \langle [\mathrm{d}_A\alpha, \beta], \ F_A \rangle + \int_{\widehat{\Sigma}} \langle \mathrm{d}_A\beta, \ \mathrm{d}_A \,\mathrm{d}_A \,\alpha \rangle \\
&= \int_{\widehat{\Sigma}} \langle [\mathrm{d}_A\alpha, \beta], \ F_A \rangle + \int_{\widehat{\Sigma}} \langle \mathrm{d}_A\beta, \ [F_A, \,\alpha] \rangle \\
&= \int_{\widehat{\Sigma}} \langle [\mathrm{d}_A\alpha, \beta], \ F_A \rangle + \int_{\widehat{\Sigma}} \langle [\alpha, \,\mathrm{d}_A\beta], \ F_A \rangle \\
&= \int_{\widehat{\Sigma}} \langle \mathrm{d}_A[\alpha, \beta], \ F_A \rangle \ \equiv \ \widehat{\Phi}_B^{[\alpha,\beta]},
\end{aligned}
\tag{28}
$$

where we used that $\langle -, [-,-] \rangle : \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ is invariant under cyclic permutations.

This completes the proof of Thm. 1.1.

A.2 Homotopy theory of topological field sectors

Topology of fields vanishing at infinity. As usual in algebraic topology, we work in the category kTopSp of compactly-generated topological spaces (for pointers see [SS21b, p. 21], and we will just say "topological spaces", for short), where for $X, Y, Z \in$ kTopSp the mapping spaces $\mathrm{Map}(-, -)$ and the product spaces $(-) \times (-)$ are related ("Cartesian closure") by natural homeomorphisms of the form
$$
\mathrm{Map}(X \times Y, \ Z) \ \simeq \ \mathrm{Maps}(X, \ \mathrm{Maps}(Y, \ Z)).
$$
This property is inherited by the category of pointed spaces
$$
\mathrm{kTopSp}^{*/} \ := \ \{ X \in \mathrm{kTop}, \ \infty_X \in X \}
$$
(which here we think of as spaces equipped with a "point at infinity", see around [SS23a, Ntn. 3.3] for more), now with respect to the mapping sub-space $\mathrm{Map}^{*/}(-, -)$ of point-preserving maps and the "smash product" $(-) \wedge (-)$, which identifies everything "at infinity" with a single point at infinity:
$$
X, Y \ \in \ \mathrm{kTopSp}^* \qquad \vdash \qquad X \wedge Y \ := \ \frac{X \times Y}{X \times \{\infty_X\} \ \cup \ \{\infty_Y\} \times Y}, \tag{29}
$$
in that
$$
\mathrm{Map}^{*/}(X \wedge Y, \ Z) \ \simeq \ \mathrm{Map}^{*/}(X, \ \mathrm{Map}^{*/}(Y, \ Z)). \tag{30}
$$

There are several ways to turn a topological space X into a pointed topological space. We write:
- $X_{\sqcup\{\infty\}}$ for X with a disjoint base point adjoined, so that none of the original points of X is "at infinity". Accordingly, preserving a disjoint point at infinity is no extra condition on maps:
$$
\mathrm{Map}^{*/}(X_{\sqcup\{\infty\}}, \ -) \ \simeq \ \mathrm{Map}(X, \ -). \tag{31}
$$
- $X_{\cup\{\infty\}}$ for X with a basepoint adjoined whose open neighborhoods are the complements of closed compact subsets of X (called the "Alexandroff one-point compactification" of X, cf. [Cu20, pp. 5]). This means that continuous paths may reach the "point at infinity".
- X for connected spaces, regarded as pointed by any one of their points (as on the right of (32) and (38) below).

For example:
- Identifying the "ends" (cf. [Pe90]) of Euclidean space with a point at infinity yields a sphere (cf. [SS23a, p. 7]):
$$
(\mathbb{R}^{n \geq 1})_{\cup\{\infty\}} \underset{\mathrm{homeo}}{\simeq} S^n. \tag{32}
$$

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- Adjoining the point at infinity to a product is the smash product of the factor with their separate points-at-infinity [Cu20, Prop. 1.6]:
$$(X \times Y)_{\cup\{\infty\}} \;\simeq\; X_{\cup\{\infty\}} \wedge Y_{\cup\{\infty\}} \,. \tag{33}$$
For example, with (32) this gives
$$S^{n_1} \wedge S^{n_2} \;\simeq\; S^{n_1+n_2}. \tag{34}$$
- If a space Σ is already compact, then the adjoined point at infinity is disjoint:
$$\Sigma \text{ compact} \qquad \Rightarrow \qquad \Sigma_{\cup\{\infty\}} \;\simeq\; \Sigma_{\sqcup\{\infty\}} \,. \tag{35}$$
- The (reduced) suspension
$$S^1 \wedge X_{\sqcup\{\infty\}} \;\equiv\; \frac{\mathbb{R}^1_{\cup\{\infty\}} \times X}{\{\infty\} \times X} \tag{36}$$
is to be thought of as the cylinder $\mathbb{R}^1 \times X$ with both ends regarded as being at infinity.
- If Σ is already compact, then its suspension (36) is equivalently the compactification of its product with the real line:
$$\begin{aligned} \left(\mathbb{R}^1 \times \Sigma\right)_{\cup\{\infty\}} \;&\simeq\; \mathbb{R}^1_{\cup\{\infty\}} \wedge \Sigma_{\cup\{\infty\}} & \text{by (33)} \\ &\simeq\; S^1 \wedge \Sigma_{\sqcup\{\infty\}} & \text{by (32) \& (35).} \end{aligned} \tag{37}$$
- The based loop space of any $X \in \mathrm{kTopSp}^{*/}$ is the pointed mapping space from the circle:
$$\Omega X \;=\; \mathrm{Map}^{*/}(S^1, X)\,, \tag{38}$$
where the circle is equipped with any basepoint.
- The n-th homotopy group of $X \in \mathrm{kTopSp}^{*/}$ is the connected components of the pointed mapping space out of the n-sphere:
$$\pi_n(X, \infty_X) \;=\; \pi_0 \, \mathrm{Map}^{*/}\!\left(S^n, X\right)\,. \tag{39}$$
- With G a topological group, its classifying space BG (pointers in [SS21b, §2.3]) is connected and its based loop space (38) is weakly homotopy equivalent (44) to the underlying space of G:
$$\Omega BG \underset{\text{whe}}{\simeq} G\,. \tag{40}$$
For example,
$$S^1 \underset{\text{whe}}{\simeq} B\mathbb{Z}\,. \tag{41}$$

For classifying spaces of (topological sectors of) physical fields, we are to think of their point at infinity as classifying the *vanishing field*, because the constant pointed classifying map, which factors as
$$X \longrightarrow \{\infty_{BG}\} \hookrightarrow BG\,,$$
classifies the trivial field. It is in this way that general pointed classifying maps literally *vanish at infinity* (cf. [SS22, §2.1][SS23a, Rem. 2.3]):
$$\begin{array}{ccc} X & \longrightarrow & BG \\ \uparrow & & \uparrow \\ \{\infty_X\} & \longrightarrow & \{\infty_{BG}\}\,. \end{array} \tag{42}$$

For example, the familiar classification of Yang-Mills instanton sectors — as $\mathrm{SU}(2)$-valued gauge fields on $\mathbb{R}^4$ which vanish at infinity — is obtained as follows:
$$\begin{aligned} \pi_0 \, \mathrm{Map}^{*/}\!\left(\mathbb{R}^4_{\cup\{\infty\}}, BSU(2)\right) \;&\simeq\; \pi_0 \, \mathrm{Map}^{*/}\!\left(S^4, BSU(2)\right) & \text{by (32)} \\ &\simeq\; \pi_0 \, \mathrm{Map}^{*/}\!\left(S^3 \wedge S^1, BSU(2)\right) & \text{by (33)} \\ &\simeq\; \pi_0 \, \mathrm{Map}^{*/}\!\left(S^3, \mathrm{Map}^{*/}\!\left(S^1, BSU(2)\right)\right) & \text{by (30)} \\ &\simeq\; \pi_0 \, \mathrm{Map}^{*/}\!\left(S^3, \underbrace{\mathrm{SU}(2)}_{\simeq S^3}\right) & \text{by (40)} \\ &\simeq\; \pi_3(S^3) \;=\; \mathbb{Z} & \text{by (39).} \end{aligned} \tag{43}$$

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Homotopy theory of topological field sectors. A *weak homotopy equivalence* is a continuous map that induces isomorphisms on all homotopy groups (39):

$$f \; : \; X \xrightarrow[\text{whe}]{\sim} Y \qquad \Leftrightarrow \qquad \begin{cases} \pi_0(f) \, : \, \pi_0(X) \; \xrightarrow{\sim} \; \pi_0(Y), \\ \underset{\substack{n \in \mathbb{N}_{\geq 1} \\ x \in X}}{\forall} \pi_n(f,x) \, : \, \pi_n(X,x) \; \xrightarrow{\sim} \; \pi_n\big(Y, f(y)\big), \end{cases} \tag{44}$$

making X and Y be "the same for all purposes" of topological homotopy theory (see [FSS23, Ex. 1.1] for review).

For example:

- The smash product (29) of smooth manifolds with contractible smooth manifolds[4] is the identity up to weak homotopy equivalence, and the pointed mapping space construction is insensitive, up to weak homotopy equivalence, to maps out of contractible manifolds:

$$\left. \begin{array}{l} X, Y \; \in \; \mathrm{SmthMfd}, \\ X \underset{\text{whe}}{\simeq} * \end{array} \right\} \qquad \Rightarrow \qquad \begin{cases} X \wedge Y \underset{\text{whe}}{\simeq} Y, \\ \mathrm{Map}^{*/}(X \wedge Y, -) \underset{\text{whe}}{\simeq} \mathrm{Map}^{*/}(Y, -). \end{cases} \tag{45}$$

- If Σ and Y have the structure of smooth manifolds with Σ compact, then there is a weak homotopy equivalence (44) from the Fréchet manifold of smooth functions $\Sigma \xrightarrow{\text{smooth}} Y$ to the mapping space of the underlying topological spaces — an instance of the *smooth Oka principle* [SS21b, Thm. 3.3.63]:

$$C^\infty\big(\Sigma, Y\big) \underset{\text{whe}}{\simeq} \mathrm{Map}(\Sigma, Y). \tag{46}$$

As such, this depends only on the weak homotopy type of Σ and Y themselves. For instance, if $Y \equiv \mathbb{R}^n/\mathbb{Z}^n$ is a torus, whose weak homotopy type is that of the classifying space $B\mathbb{Z}^n$ (41), then

$$C^\infty\big(\Sigma, \mathbb{R}^n/\mathbb{Z}^n\big) \underset{\text{whe}}{\simeq} \mathrm{Map}(\Sigma, B\mathbb{Z}^n). \tag{47}$$

Cohomology classifying topological field sectors. That the space BG (40) is "classifying" refers to the homotopy classes of maps into it corresponding to isomorphism classes of G-principal bundles over smooth manifolds (at least), hence to the degree=1 non-abelian cohomology with coefficients in G (cf. [FSS23, Ex. 2.2]):

$$\pi_0 \, \mathrm{Map}\big(\Sigma, BG\big) \; \simeq \; H^1(\Sigma; G). \tag{48}$$

In the case when $G \equiv A$ is abelian there is (abelian) topological group structure on BA itself so that we iteratively obtain higher classifying space $B^{n+1}A := B^n A$. For A discrete, these are "Eilenberg-MacLane spaces" $K(A, n)$ which classify ordinary cohomology in higher degrees (cf. [FSS23, Ex. 2.1]):

$$\pi_0 \, \mathrm{Map}\big(\Sigma, B^n A\big) \; \simeq \; H^n(\Sigma; A). \tag{49}$$

Notice that, under this equivalence, the usual group structure on ordinary cohomology comes from the pointwise group structure of maps into a topological group.

Hence in general, for any topological group G, we may think of maps into a BG as 1-cocycles of (possibly non-abelian) G-cohomology, and of homotopies between such maps as coboundaries between the corresponding cocycles. In this sense, the mapping space into BG is the *cocycle space* of G-cohomology ([FSS23, Def. 2.1]) and its connected components are the (non-abelian) cohomology classes:

---

[4]The assumption in (45) that X and Y be smooth manifolds is not necessary for this statement, we make it only for brevity of the discussion. A sufficient condition is that X and Y admit the structure of CW-complexes (which is the case for smooth manifolds by the triangulation theorem).

$$\mathrm{Map}(\Sigma,\, BG) \;=\; \left\{ \Sigma \underset{\underset{\text{cocycle}}{c'}}{\overset{\overset{\text{cocycle}}{c}}{\underset{\Downarrow}{\overset{\|}{\text{coboundary}}}}} BG \right\}, \qquad H^1(\Sigma;\, G) \;:=\; \pi_0\,\mathrm{Map}(\Sigma,\, BG)\,. \qquad (50)$$

Similarly, the loop space of the mapping space into BG is the mapping space into G

$$
\begin{aligned}
\Omega\,\mathrm{Map}(\Sigma;\, BG) \;&\simeq\; \mathrm{Map}^{*/}\!\Big(S^1,\, \mathrm{Map}^{*/}\!\big(\Sigma_{\sqcup\{\infty\}};\, BG\big)\Big) && \text{by (38) \& (31)} \\
&\simeq\; \mathrm{Map}^{*/}\!\Big(\Sigma_+,\, \mathrm{Map}^{*/}\!\big(S^1;\, BG\big)\Big) && \text{by (30)} \\
&\simeq\; \mathrm{Map}^{*/}\!\big(\Sigma_{\sqcup\{\infty\}},\, G\big) && \text{by (40)} \\
&\simeq\; \mathrm{Map}\big(\Sigma,\, G\big) && \text{by (31),}
\end{aligned}
\qquad (51)
$$

whose connected components are the 0-cohomology with coefficients in G:

$$H^0(\Sigma;\, G) \;:=\; \pi_0\,\mathrm{Map}(\Sigma,\, G)\,. \qquad (52)$$

Remark A.1 (Ordinary cohomology with topological group coefficients)**.** Notice, with (41), that

$$H^0(\Sigma;\, S^1) \;\simeq\; H^1(\Sigma;\, \mathbb{Z}) \qquad (53)$$

and beware that the usual notation "$H^n(\Sigma;\, \mathrm{U}(1))$" tacitly refers to the circle coefficient understood with its *discrete* topology, hence is quite different. To make this explicit, if we write $\flat A$ for the underlying discrete group of a topological abelian group, then the usual notion of ordinary cohomology with coefficients in A is $H^n(\Sigma;\, \flat A)$ in the above notation (49).

The following derivations are standard for homotopy theorists but may serve as instructive examples of the above notions for other readers :

Lemma A.2. $$\mathrm{Map}(\Sigma, S^1) \underset{\text{whe}}{\simeq} H^1(\Sigma;\, \mathbb{Z}) \times B\big(H^0(\Sigma;\, \mathbb{Z})\big). \qquad (54)$$

Proof. First, observe that the connected components are

$$
\begin{aligned}
\pi_0\,\mathrm{Map}(\Sigma,\, S^1) \;&\simeq\; H^0\big(\Sigma;\, S^1\big) && \text{by (49)} \\
&\simeq\; H^1\big(\Sigma;\, \mathbb{Z}\big) && \text{by (53).}
\end{aligned}
$$

Moreover, the fundamental group at the basepoint is

$$
\begin{aligned}
\pi_1\big(\mathrm{Map}(\Sigma,\, S^1),\, 0\big) \;&\simeq\; \pi_0\big(\Omega\,\mathrm{Map}(\Sigma,\, S^1)\big) && \text{by (39)} \\
&\simeq\; \pi_0\big(\mathrm{Map}(\Sigma,\, \mathbb{Z})\big) && \text{by (51) \& (41)} \\
&\simeq\; H^0(\Sigma;\, \mathbb{Z}) && \text{by (49).}
\end{aligned}
\qquad (55)
$$

Also, all higher homotopy groups at the basepoint vanish:

$$
\begin{aligned}
\pi_{n\geq 2}\big(\mathrm{Map}(\Sigma,\, S^1),\, 0\big) \;&\simeq\; \pi_0\big(\Omega^{n\geq 2}\,\mathrm{Map}(\Sigma,\, S^1)\big) && \text{by (39)} \\
&\simeq\; \pi_0\Big(\mathrm{Map}\big(\Sigma,\, \underbrace{\Omega^{n\geq 2}\,S^1}_{\underset{\text{whe}}{\simeq}\,*}\big)\Big) && \text{by (51) \& (34)} \\
&\simeq\; *\,.
\end{aligned}
\qquad (56)
$$

Hence the connected component of the neutral element of $\mathrm{Map}(\Sigma,\, S^1)$ is weakly homotopy equivalent to $B\big(H^1(\Sigma;\, \mathbb{Z})\big)$.

But the topological group structure on $\mathrm{Map}(\Sigma, S^1)$ implies that the multiplication operation of any of its elements constitutes a homeomorphism from the connected component of that element to that of the neutral element. Therefore, the underlying spaces of all connected components are isomorphic. This implies (54). □

Homology Pontrjagin algebras. We consider homology with complex coefficients, throughout.[5] For $\Gamma \in \mathrm{Grp}(\mathrm{Set})$ a discrete group, regarded as the loop space of its classifying space, $\Gamma \underset{\mathrm{whe}}{\simeq} \Omega BG$, see (40), its homology is just the linear span of the underlying set

$$H_\bullet(\Gamma; \mathbb{C}) = H_0(\Gamma; \mathbb{C}) \simeq \mathbb{C}[\Gamma], \qquad \begin{array}{ccc} \Gamma & \longrightarrow & \mathbb{C}[\Gamma] \\ \gamma & \longmapsto & \mathcal{O}(\gamma) \end{array} \tag{57}$$

so that

$$\mathbb{C}[\Gamma] = \left\{ \sum_{\gamma \in \Gamma} c_\gamma \cdot \mathcal{O}(\gamma) \,\Big|\, c_\gamma \in \mathbb{C}, \ \mathrm{supp}(c_{(-)}) \text{ is finite} \right\}.$$

Furthermore, the Pontrjagin product (20) is the convolution product of the group algebra ([We1931, §III.13], cf. [FH91, §3.4]):

$$\begin{array}{ccc} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] & \xrightarrow{\ (-)\cdot(-)\ } & \mathbb{C}[\Gamma] \\ \mathcal{O}(\gamma) \otimes \mathcal{O}(\gamma') & \longmapsto & \mathcal{O}(\gamma \cdot \gamma') \end{array} \tag{58}$$

$$\left(\sum_{\gamma \in \Gamma} c_\gamma \cdot \mathcal{O}(\gamma) \right) \cdot \left(\sum_{\gamma' \in \Gamma} c'_{\gamma'} \cdot \mathcal{O}(\gamma') \right) = \left(\sum_{\gamma'' \in \Gamma} \left(\sum_{\gamma \in \Gamma} c_\gamma \cdot c'_{\gamma^{-1} \cdot \gamma''} \right) \cdot \mathcal{O}(\gamma'') \right).$$

Now, since homology is invariant under weak homotopy equivalence (44), i.e.,

$$X \underset{\mathrm{whe}}{\simeq} Y \qquad \Rightarrow \qquad H_\bullet(X; \mathbb{C}) \simeq H_\bullet(Y; \mathbb{C}), \tag{59}$$

and by the Künneth Theorem (e.g. [Ha02, Cor. 3B.7])

$$H_n(X \times Y; \mathbb{C}) \simeq \bigoplus_{n_1 + n_2 = n} H_{n_1}(X; \mathbb{C}) \otimes_{\mathbb{C}} H_{n_2}(Y; \mathbb{C}), \tag{60}$$

this remains the case in degree=0 for products with connected CW-complexes (whose homology in degree 0 is $\mathbb{C}$), such as with classifying spaces of abelian groups, $\Gamma \times BA \underset{\mathrm{whe}}{\simeq} \Omega B(\Gamma \times BA)$:

$$H_0(\Gamma \times BA; \mathbb{C}) \simeq H_0(\Gamma; \mathbb{C}). \tag{61}$$

Using all this, we establish Thm. 3.1 as follows:

$$H_0\left(\mathrm{Map}^{*/}\left(\mathbb{R}^{0,1}_{\sqcup\{\infty\}} \wedge (\mathbb{R}^1 \times \Sigma)_{\cup\{\infty\}}, B(\mathrm{U}(1)^2 \rtimes K) \right); \mathbb{C} \right)$$

$$\simeq H_0\left(\Omega\,\mathrm{Map}(\Sigma, B(\mathrm{U}(1)^2 \rtimes K)); \mathbb{C} \right) \qquad\qquad \text{by (18) \& (59)}$$

$$\simeq H_0\left(\mathrm{Map}(\Sigma, \mathrm{U}(1)^2 \rtimes K); \mathbb{C} \right) \qquad\qquad \text{by (19) \& (59)}$$

$$\simeq H_0\left(H^1(\Sigma; \mathbb{Z})^2 \times B(H^0(\Sigma; \mathbb{Z})) \rtimes H^0(\Sigma; K); \mathbb{C} \right) \qquad \text{by (54)}$$

$$\simeq H_0\left(H^1(\Sigma; \mathbb{Z})^2 \rtimes H^0(\Sigma; K); \mathbb{C} \right) \qquad\qquad \text{by (61)}$$

$$\simeq \mathbb{C}\left[H^1(\Sigma; \mathbb{Z})^2 \rtimes H^0(\Sigma; K) \right] \qquad\qquad\qquad \text{by (57).}$$

---

[5]Much further interesting structure appears when considering homology with integer coefficients, but this is beyond the intended scope of the present note.

References

[Al85] O. Alvarez, *Topological quantization and cohomology*, Commun. Math. Phys. **100** 2 (1985), 279-309, [`euclid:cmp/1103943448`].

[BLS84] A. Bassetto, I. Lazzizzera, and R. Soldati, *Yang-Mills theories in space-like axial and planar gauges*, Nucl. Phys. B **236** 2 (1984), 319-335, [`doi:10.1016/0550-3213(84)90538-8`].

[BBSS17] C. Becker, M. Benini, A. Schenkel, R. Szabo, *Abelian duality on globally hyperbolic spacetimes*, Commun. Math. Phys. **349** (2017) 361-392 [`arXiv:1511.00316`] [`doi:10.1007/s00220-016-2669-9`].

[BHR07] E. Binz, R. Honegger, and A. Rieckers, *Infinite dimensional Heisenberg group algebra and field-theoretic strict deformation quantization*, Int. J. Pure Appl. Math. **38** 1 (2007), [`ijpam:2007-38-1/6`].

[BLOT90] N. Bogolyubov, A. A. Logunov, A. I. Oksak, and I. T. Todorov, *General principles of quantum field theory*, Math. Phys. & Appl. Math. **10**, Kluwer (1990), [`doi:10.1007/978-94-009-0491-0`].

[BoSa53] R. Bott and H. Samelson, *On the Pontryagin product in spaces of paths*, Commentarii Math. Helvetici **27** (1953), 320–337, [`doi:10.1007/BF02564566`].

[BMSS19] V. Braunack-Mayer, H. Sati, and U. Schreiber, *Gauge enhancement of Super M-Branes via rational parameterized stable homotopy theory* Commun. Math. Phys. **371** (2019), 197–265, [`doi:10.1007/s00220-019-03441-4`], [`arXiv:1806.01115`].

[Cartan1924] É. Cartan, *Sur les variétés à connexion affine, et la théorie de la relativité généralisée (première partie) (Suite)*, Ann. Scient. l'ÉNS 3e sér. **41** (1924), 1-25, [`numdam:ASENS_1924_3_41__1_0`].

[CP17] A. S. Cattaneo and A. Perez, *A note on the Poisson bracket of 2d smeared fluxes*, Class. Quant. Grav. **34** (2017) 107001, [`doi:10.1088/1361-6382/aa69b4`], [`arXiv:1611.08394`].

[CSS23] D. Corfield, H. Sati, and U. Schreiber *Fundamental weight systems are quantum states*, Lett. Math. Phys. **113** (2023) 112, [`doi:10.1007/s11005-023-01725-4`], [`arXiv:2105.02871`].

[CJLP98] E. Cremmer, B. Julia, H. Lu, and C. Pope, *Dualisation of Dualities, II: Twisted self-duality of doubled fields and superdualities*, Nucl. Phys. B **535** (1998), 242-292, [`doi:10.1016/S0550-3213(98)00552-5`], [`arXiv:hep-th/9806106`].

[Cu20] T. Cutler, *The category of pointed topological spaces* (2020), [`ncatlab.org/nlab/files/CutlerPointedTopologicalSpaces.pdf`]

[Di1931] P. A. M. Dirac, *Quantized Singularities in the Electromagnetic Field*, Proceedings of the Royal Society, A **133** (1931) 60–72 [`doi:10.1098/rspa.1931.0130`]

[DGR10] J. F. Dorfmeister, M. A. Guest, and W. Rossman, *The tt^* Structure of the Quantum Cohomology of $\mathbb{C}P^1$ from the Viewpoint of Differential Geometry*, Asian J. Math. **14** 3 (2010), 417-438, [`doi:10.4310/AJM.2010.v14.n3.a7`].

[FHT00] Y. Félix, S. Halperin, and J.-C. Thomas, *Rational Homotopy Theory*, Graduate Texts in Mathematics **205** Springer, Berlin (2000), [`doi:10.2307/1970615`].

[Fey42] R. P. Feynman, *The Principles of Least Action in Quantum Mechanics*, PhD thesis (1942); reprinted in L. Brown, *Feynman's Thesis – A New Approach to Quantum Theory*, World Scientific, Singapore (2005), [`doi:10.1142/9789812567635_0001`].

[Fey48] R. P. Feynman, *Space-Time Approach to Non-Relativistic Quantum Mechanics*, Rev. Mod. Phys. **20** (1948), 367-387, [`doi:10.1103/RevModPhys.20.367`].

[FSS14] D. Fiorenza, H. Sati, and U. Schreiber, *A higher stacky perspective on Chern-Simons theory*, in: *Mathematical Aspects of Quantum Field Theories*, Mathematical Physics Studies, Springer (2014), 153-211, [`doi:10.1007/978-3-319-09949-1`], [`arXiv:1301.2580`].

[FSS19] D. Fiorenza, H. Sati, and U. Schreiber, *Twisted Cohomotopy implies M-theory anomaly cancellation on 8-manifolds*, Commun. Math. Phys. **377** (2020), 1961-2025, [`doi:10.1007/s00220-020-03707-2`], [`arXiv:1904.10207`].

[FSS21a] D. Fiorenza, H. Sati, and U. Schreiber, *Twisted Cohomotopy implies M5 WZ term level quantization*, Commun. Math. Phys. **384** (2021), 403-432, [`doi:10.1007/s00220-021-03951-0`] [`arXiv:1906.07417`].

[FSS21b] D. Fiorenza, H. Sati, and U. Schreiber: *Twisted cohomotopy implies twisted String structure on M5-branes*, J. Math. Phys. **62** (2021), 042301 , [`doi:10.1063/5.0037786`], [`arXiv:2002.11093`]

[FSS22] D. Fiorenza, H. Sati, and U. Schreiber, *Twistorial Cohomotopy Implies Green-Schwarz anomaly cancellation*, Rev. Math. Phys. **34** 5 (2022) 2250013, [`doi:doi.org/10.1142/S0129055X22500131`], [`arXiv:2008.08544`].

[FSS23] D. Fiorenza, H. Sati, and U. Schreiber: *The Character map in Nonabelian Cohomology — Twisted, Differential and Generalized*, World Scientific (2023), [`doi:10.1142/13422`], [`arXiv:2009.11909`].

[Fr00] D. Freed, *Dirac charge quantization and generalized differential cohomology*, Surv. Differential Geom. **7**, International Press (2000), 129-194, [`doi:10.4310/SDG.2002.v7.n1.a6`], [`arXiv:hep-th/0011220`].

[FMS07a] D. Freed, G. Moore, and G. Segal, *The Uncertainty of Fluxes*, Commun. Math. Phys. **271** (2007), 247-274, [`doi:10.1007/s00220-006-0181-3`], [`arXiv:hep-th/0605198`].

[FMS07b] D. Freed, G. Moore, and G. Segal, *Heisenberg Groups and Noncommutative Fluxes*, Annals Phys. **322** (2007), 236-285, [`doi:10.1016/j.aop.2006.07.014`], [`arXiv:hep-th/0605200`].

[FP83] J. L. Friedman and N. J. Papastamatiou, *On the canonical quantization of Yang-Mills theories*, Nucl. Phys. B **219** 1 (1983), 125-142, [`doi:10.1016/0550-3213(83)90431-5`].

[FS10] Y. Frishman and J. Sonnenschein, *Non-Perturbative Field Theory – From Two Dimensional Conformal Field Theory to QCD in Four Dimensions*, Cambridge University Press (2010), [`doi:10.1017/CBO9780511770838`], open access (2023), [`doi:10.1017/9781009401654`].

[FH91] W. Fulton and J. Harris, *Representation Theory: a First Course*, Springer, Berlin (1991), [`doi:10.1007/978-1-4612-0979-9`].

[GiS23] G. Giotopoulos and H. Sati, *Field Theory via Higher Geometry I: Smooth sets of fields*, [`arXiv:2312.16301`].

[GSS24] G. Giotopoulos, H. Sati and U. Schreiber, *Flux Quantization on 11d Superspace* [`arXiv:2403.16456`].

[GrS22] D. Grady and H. Sati, *Ramond-Ramond fields and twisted differential K-theory*, Adv. Theor. Math. Phys. **26** 5 (2022), 1097–1155, [`doi:10.4310/ATMP.2022.v26.n5.a2`], [`arXiv:1903.08843`].

[Gu83] S. Gutt, *An explicit ∗-product on the cotangent bundle of a Lie group*, Lett. Math. Phys. **7** (1983), 249–258, [`doi:10.1007/BF00400441`].

[Gu11] S. Gutt, *Deformation quantization of Poisson manifolds*, Geom. Topol. Monographs **17** (2011), 171-220, [`gtm:2011/17`], [`arXiv:math/0003107`].

[Ha02] A. Hatcher, *Algebraic Topology*, Cambridge University Press (2002), [`ISBN:9780521795401`], [`https://pi.math.cornell.edu/~hatcher/AT/ATpage.html`].

[Ha08] E. Hawkins, *An Obstruction to Quantization of the Sphere*, Commun. Math. Phys. **283** (2008), 675–699, [`doi:10.1007/s00220-008-0517-2`], [`arXiv:0706.2946`].

[HIO16] F. W. Hehl, Y. Itin, and Y. N. Obukhov, *On Kottler's path: origin and evolution of the premetric program in gravity and in electrodynamics*, Int. J. Mod. Phys. D **25** 11 (2016) 1640016, [`doi:10.1142/S0218271816400162`], [`arXiv:1607.06159`].

[He01] T. Heinzl, *Light-Cone Quantization: Foundations and Applications*, in: *Methods of Quantization*, Lecture Notes in Physics **572**, Springer, New York (2001), [`doi:10.1007/3-540-45114-5_2`], [`arXiv:hep-th/0008096`].

[HT92] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press (1992), [`ISBN:9780691037691`], [`jstor:j.ctv10crg0r`].

[KMW07] A. Kitaev, G. W. Moore, and K. Walker, *Noncommuting Flux Sectors in a Tabletop Experiment*, [`arXiv:0706.3410`].

[La99] N. P. Landsman, *Lie Groupoid C∗-Algebras and Weyl Quantization*, Commun. Math. Phys. **206** (1999), 367–381, [`doi:10.1007/s002200050709`].

[LR01] N. P. Landsman and B. Ramazan, *Quantization of Poisson algebras associated to Lie algebroids*, in: *Groupoids in Analysis, Geometry, and Physics*, Contemporary Math. **282** (2001), 159-192, [`ams:conm/282`], [`arXiv:math-ph/0001005`].

[LS22] C. Lazaroiu and C. S. Shahbazi, *The duality covariant geometry and DSZ quantization of abelian gauge theory*, Adv. Theor. Math. Phys. **26** (2022) 2213–2312 [`arXiv:2101.07236`] [`doi:10.4310/ATMP.2022.v26.n7.a5`].

[LS23] C. Lazaroiu and C. S. Shahbazi, *The geometry and DSZ quantization of four-dimensional supergravity*, Lett Math. Phys. **113** 4 (2023) [`doi:10.1007/s11005-022-01626-y`] [`arXiv:2101.07778`].

[MS15] V. Mathai and H. Sati, *Higher abelian gauge theory associated to gerbes on noncommutative deformed M5-branes and S-duality*, J. Geom. Phys. **92** (22015), 240-251, [`arXiv:1404.2257`], [`doi:10.1016/j.geomphys.2015.02.019`].

[MM65] J. Milnor and J. Moore, *On the structure of Hopf algebras*, Ann. Math. **81** (1965), 211-264, [`doi:10.2307/1970615`].

[MV23] K. Mkrtchyan and F. Valach, *Democratic actions for type II supergravities*, Phys. Rev. D **107** 6 (2023) 066027, [`doi:10.1103/PhysRevD.107.066027`], [`arXiv:2207.00626`].

[M-F22] J. M. Moreno-Fernández, *The Milnor-Moore theorem for L_∞-algebras in rational homotopy theory*, Math. Zeitschrift **300** (2022), 2147–2165, [`doi:10.1007/s00209-021-02838-z`], [`arXiv:1904.12530`].

[Na99] N. Nagaosa, *Quantum Field Theory in Condensed Matter Physics*, Texts and Monographs in Physics, Springer (1999), [`doi:10.1007/978-3-662-03774-4_2`].

[PV00] M. Penkava and P. Vanhaecke, *Deformation Quantization of Polynomial Poisson Algebras*, J. Algebra **227** 1 (2000), 365-393, [`doi:10.1006/jabr.1999.8239`], [`arXiv:math/9804022`].

[Pe90] G. Peschke, *The Theory of Ends*, Nieuw Archief voor Wiskunde, **8** (1990), 1-12, [`ncatlab.org/nlab/files/Peschke-Ends.pdf`]

[Po39] L. Pontrjagin, *Homologies in compact Lie groups*, Rec. Math. (Mat. Sbornik) N. S. **6** (**48**) 3 (1939), 389–422, [`mathnet:5835`].

[Ri89] M. A. Rieffel, *Deformation quantization of Heisenberg manifolds*, Commun. Math. Phys. **122** (1989), 531–562, [`doi:10.1007/BF01256492`].

[Ri90] M. A. Rieffel, *Lie Group Convolution Algebras as Deformation Quantizations of Linear Poisson Structures*, Amer. J. Math. **112** 4 (1990), 657-685, [`doi:10.2307/2374874`].

[Ri94] M. Rieffel, *Quantization and C^*-algebras*, Contemporary Math. **167** (1994), [`https://math.berkeley.edu/~rieffel/papers/quantization.pdf`]

[Sa10] H. Sati, *Geometric and topological structures related to M-branes*, in: R. Doran, G. Friedman and J. Rosenberg (eds.), *Superstrings, Geometry, Topology, and C^*-algebras*, Proc. Symp. Pure Math. **81**, AMS, Providence (2010), 181-236, [`doi:10.1090/pspum/081`], [`arXiv:1001.5020`].

[Sa13] H. Sati, *Framed M-branes, corners, and topological invariants*, J. Math. Phys. **59** (2018) 062304, [`doi:10.1063/1.5007185`], [`arXiv:1310.1060`].

[SS20] H. Sati and U. Schreiber, *Equivariant Cohomotopy implies orientifold tadpole cancellation*, J. Geom. Phys. **156** (2020) 103775, [`doi:10.1016/j.geomphys.2020.103775`], [`arXiv:1909.12277`].

[SS21a] H. Sati, U. Schreiber, *Twisted Cohomotopy implies M5-brane anomaly cancellation*, Lett. Math. Phys. **111** (2021) 120, [`doi:10.1007/s11005-021-01452-8`], [`arXiv:2002.07737`].

[SS21b] H. Sati and U. Schreiber, *Equivariant Principal ∞-Bundles*, [`arXiv:2112.13654`].

[SS22] H. Sati and U. Schreiber, *Differential Cohomotopy implies intersecting brane observables*, Adv. Theor. Math. Phys. **26** 4 (2022), 957-1051, [`doi:10.4310/ATMP.2022.v26.n4.a4`], [`arXiv:1912.10425`].

[SS23a] H. Sati and U. Schreiber, *M/F-Theory as Mf-Theory*, Rev. Math. Phys. **35** 10 (2023), [`doi:10.1142/S0129055X23500289`], [`arXiv:2103.01877`].

[SS23b] H. Sati and U. Schreiber, *Anyonic Defect Branes and Conformal Blocks in Twisted Equivariant Differential (TED) K-Theory*, Rev. Math. Phys. **35** 06 (2023) 2350009, [`arXiv:2203.11838`], [`doi:10.1142/S0129055X23500095`].

[SS23c] H. Sati and U. Schreiber, *Flux quantization on Phase space*, Annales Henri Poincaré (2024, in print) [`arXiv:2312.12517`].

[SS23d] H. Sati and U. Schreiber, *Introduction to Hypothesis H*, lecture notes (2023-4) [`ncatlab.org/schreiber/show/Introduction+to+Hypothesis+H`]

[SS24] H. Sati and U. Schreiber, *Flux Quantization*, Encycl. of Math. Phys. 2nd ed (2024) [`arXiv:2402.18473`]

[SV23] H. Sati and A. Voronov, *Mysterious Triality and Rational Homotopy Theory*, Commun. Math. Phys. **400** (2023), 1915–1960, [`doi:10.1007/s00220-023-04643-7`], [`arXiv:2111.14810`].

[Sc13] U. Schreiber, *Differential cohomology in a cohesive ∞-topos* [`arXiv:1310.7930`].

[Sc24] U. Schreiber, *Higher Topos Theory in Physics*, Encyclopedia of Mathematical Physics 2nd ed., Elsevier (2024), [`arXiv:2311.11026`].

[vN1931] J. von Neumann, *Die Eindeutigkeit der Schrödingerschen Operatoren*, Mathematische Annalen **104** (1931), 570–578, [`doi:10.1007/BF01457956`].

[We83] A. Weinstein, *The local structure of Poisson manifolds*, J. Differential Geom. **18** 3 (1983), 523-557, [`doi:10.4310/jdg/1214437787`].

[We1931] H. Weyl *Gruppentheorie und Quantenmechanik*, S. Hirzel, Leipzig (1931); translated by H. P. Robertson,*The Theory of Groups and Quantum Mechanics*, Dover Publications (1950), [`ISBN:0486602699`], [`ark:/13960/t1kh1w36w`].

[Wi90] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surv. Differential Geom. **1** (1990), 243-310, [`doi:10.4310/SDG.1990.v1.n1.a5`].

[Yd18] B. Ydri, *Matrix Models of String Theory*, Institute of Physics Publishing (2018), [`arXiv:1708.00734`], [`ISBN:978-0-7503-1726-9`].