2.2.1 Quantum observables on flux-quantized fields.

Recall:

Quantum observables and quantum states.

Given a *star-algebra of quantum observables* consider their	$\big(\mathrm{Obs},\ \cdot\ , (\text{-})^\ast\big)$
expectation values for a given *quantum state* (which is defined thereby)	$\langle - \rangle \,:\, \mathrm{Obs} \xrightarrow{\text{linear}} \mathbb{C}$
respecting involution	$A \in \mathrm{Obs} \;\vdash\; \langle A^\ast \rangle \,=\, \overline{\langle A \rangle}$
and being "positive" on normal observables.	$A \in \mathrm{Obs} \;\vdash\; \langle A^\ast A \rangle \in \mathbb{R}_{\geq 0} \subset \mathbb{C}$
From this a (Hilbert-)space of states is induced	$\mathcal{H} \,:=\, \mathrm{Obs}/\big\{A \in \mathrm{Obs} \,\big\vert\, \langle A^\ast \cdot A \rangle \,=\, 0\big\}$
(the *GNS construction*) with "ground" state	$\vert \psi_0 \rangle := [1]$
supporting an operator-state correspondence	$A \vert \psi_0 \rangle \,=\, [A]$
for the induced inner product, reproducing	$\langle A\psi_0, \vert B\psi_0 \rangle \,=\, \langle \psi_0 \vert A^\ast B \vert \psi_0 \rangle \,:=\, \langle A^\ast B \rangle$
the expectation values in this ground state.	$\langle \psi_0 \vert - \vert \psi_0 \rangle \,=\, \langle - \rangle$

Non-perturbative quantization

of a Poisson-manifold phase space	$\{-,-\} \,:\, C^\infty(P) \otimes C^\infty(P) \to C^\infty(P)$
is a bundle of $C^\ast$-algebras	$\Big\{\mathrm{Obs} \overset{(-)_\hbar}{\twoheadrightarrow} \mathrm{Obs}_\hbar \in C^\ast\mathrm{Alg}\Big\}_{\hbar \in \mathbb{R}}$, $\quad f \in C^0(\mathbb{R}),\ A \in \mathrm{Obs} \;\vdash\; fA \in \mathrm{Obs}\,, (fA)_\hbar = f(\hbar) A_\hbar$
which continuously	$\underset{A \in \widehat{\mathrm{Obs}}}{\forall} \Big(\hbar \mapsto \vert A_\hbar \vert\Big) \in C^0(\mathbb{R})\,, \qquad \vert A \vert = \sup_\hbar \vert A_\hbar \vert$
deforms the classical observables	$Q : C^\infty_{\mathrm{cpt}}(P) \to \mathrm{Obs}\,, \quad Q(-)_0 \,:\, C^\infty_{\mathrm{cpt}}(P) \xhookrightarrow{\text{dense}} A_0$
satisfying Dirac's quantization condition.	$\lim\limits_{\hbar \to 0} \Big\vert \big[Q(f)_\hbar,\, Q(g)_\hbar\big] - \mathrm{i}\hbar\, Q\big(\{f,g\}\big)_\hbar \Big\vert \,=\, 0\,.$

Yang-Mills flux observables.

in $\mathfrak{g}$-Yang-Mills theory on $\mathbb{R}^{0,1} \times X^3$, with an oriented closed surface $\Sigma^2 \hookrightarrow X^3$ measure the weighted integrals of the electric & magnetic flux densities over Σ

$$\Phi^\omega_E \;=\; \textstyle\int_\Sigma \langle \omega, E \rangle \qquad \text{for } \omega \in \Omega^0_{\mathrm{dR}}(X^3; \mathfrak{g})$$
$$\Phi^\omega_B \;=\; \textstyle\int_\Sigma \langle \omega, B \rangle$$

Proposition [SS23-Qnt]: **Non-perturbative quantum observables on quantized YM fluxes.**

A non-perturbative quantization of the Φ_E and Φ_A	for a choice of Lie group G with Lie algebra $\mathfrak{g}$
is the Fréchet-group *convolution algebra* of semidirect product group-valued functions on Σ	$\mathrm{Obs} \;=\; C^\infty\big(\Sigma,\, G \rtimes \mathfrak{g}_0/\Lambda\big)$
Hence the *topological flux observables* form a convolution group algebra of cohomology of Σ:	$\mathrm{Obs}^{\mathrm{top}} \;=\; \mathbb{C}\Big[\pi_0\, C^\infty\big(\Sigma;\, G \ltimes B\Lambda\big)\Big] \;\simeq\; \mathbb{C}\Big[H^0\big(\Sigma; G\big) \ltimes H^1(\Sigma; \Lambda)\Big]$
For Maxwell theory on $\mathbb{R}^{1,1} \times \Sigma$ this coincides with the Pontrjagin-Hopf algebra of loops in the moduli of flux-quantized topological field sectors on Σ	$\simeq H_0\Big(\Omega\, \mathrm{Maps}\big(\Sigma,\, B\mathrm{U}(1) \times B\mathrm{U}(1)\big)\Big)$

This **remarkable re-formulation of quantization**

Non-perturbative quantum observables on topological flux observables in Yang-Mills theory on $\mathbb{R}^{1,1} \times \Sigma$.	$\longleftrightarrow$	Pontrjagin-Hopf algebra of loops in flux-quantized topological fields on Σ.

we next take as the blueprint for the quantization of topological fluxes in M-theory.