M/F-Theory as Mf-Theory

Hisham Sati*† and Urs Schreiber*

July 1, 2023

Abstract

In the quest for mathematical foundations of M-theory, the *Hypothesis H* that fluxes are quantized in Cohomotopy theory, implies, on flat but possibly singular spacetimes, that M-brane charges locally organize into equivariant homotopy groups of spheres. Here we show how this leads to a correspondence between phenomena conjectured in M-theory and fundamental mathematical concepts/results in stable homotopy, generalized cohomology and Cobordism theory Mf:

- stems of homotopy groups correspond to charges of probe p-branes near black b-branes;
- stabilization within a stem is the boundary-bulk transition;
- the Adams d-invariant measures G_4-flux;
- trivialization of the d-invariant corresponds to H_3-flux;
- refined Toda brackets measure H_3-flux;
- the refined Adams e-invariant sees the H_3-charge lattice;
- vanishing Adams e-invariant implies consistent global C_3-fields;
- Conner-Floyd's e-invariant is the H_3-flux seen in the Green-Schwarz mechanism;
- the Hopf invariant is the M2-brane Page charge ($\widetilde{G}_7$-flux);
- the Pontrjagin-Thom theorem associates the polarized brane worldvolumes sourcing all these charges.

In particular, spontaneous K3-reductions with 24 branes are singled out from first principles:

- Cobordism in the third stable stem witnesses spontaneous KK-compactification on K3-surfaces;
- the order of the third stable stem implies the 24 NS5/D7-branes in M/F-theory on K3.

Finally, complex-oriented cohomology emerges from Hypothesis H, connecting it to all previous proposals for brane charge quantization in the chromatic tower: K-theory, elliptic cohomology, etc.:

- quaternionic orientations correspond to unit H_3-fluxes near M2-branes;
- complex orientations lift these unit H_3-fluxes to heterotic M-theory with heterotic line bundles.

In fact, we find quaternionic/complex Ravenel-orientations bounded in dimension; and we find the bound to be 10, as befits spacetime dimension 10+1.

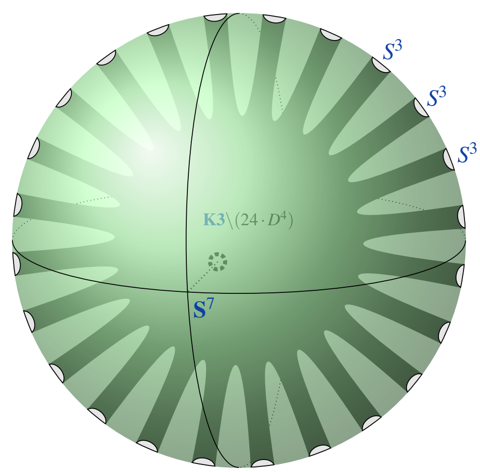
---

* Mathematics, Division of Science; and
Center for Quantum and Topological Systems,
NYUAD Research Institute,
New York University Abu Dhabi, UAE.

†The Courant Institute for Mathematical Sciences, NYU, NY

The authors acknowledge the support by *Tamkeen* under the *NYU Abu Dhabi Research Institute grant* `CG008`.

Contents



1 Introduction

Charges and Generalized cohomology. There is a close link between *non-perturbative physics* (e.g. [BaS10]) and *algebraic topology* (e.g. [Ma99]): Pure algebraic topology may be understood as the study of homotopy types of spaces through functorial assignment of their algebraic invariants; but seen through the lens of mathematical physics, this is all about identifying *charges* carried by solitonic charged objects in spacetimes (black holes, black p-branes) as *cohomology classes* in some generalized cohomology theory evaluated on the ambient spacetime:

Non-perturbative physics	$\longleftrightarrow$	**Algebraic Topology**
Charges carried by microscopic p-branes	$\Big[\underset{\text{ambient spacetime}}{X} \xrightarrow{\text{fields}} \underset{\text{classifying space}}{A}\Big]$	Generalized cohomology of external spacetime

Field strengths or *flux densities* in physics are collections of differential forms F_i on spacetime, satisfying differential relations: *Bianchi identities*. The corresponding rational charges are reflected in the periods of these differential forms, expressing the total flux through any closed hypersurfaces. On these fields, a *charge quantization law* (see [Fr00][Sa10][FSS20c]) is a non-abelian cohomology theory $\widetilde{A}(-)$ [FSS20c, §2][SS20b, p. 6] with the requirement that true fields are cocycles in (differential) A-cohomology whose flux densities are just their images under the *character map* [FSS20c, §4][SS20c, §3.4].

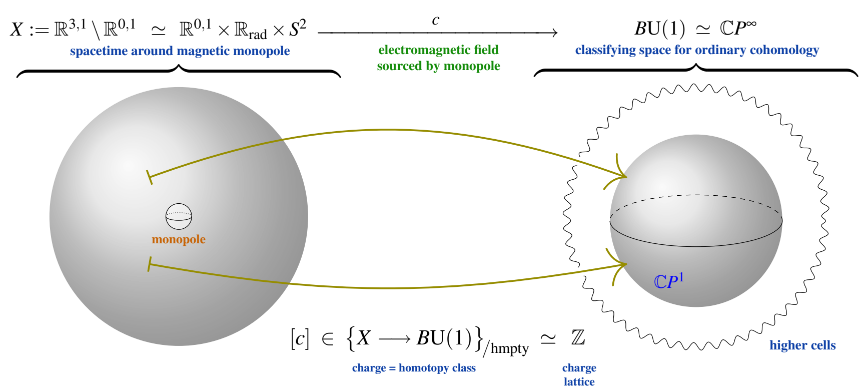
$$
\underset{\substack{A\text{-cohomology of}\\ \text{spacetime } X}}{\widetilde{A}(X)} \xrightarrow[\text{character map}]{\mathrm{ch}_A} \underset{\substack{\text{de Rham cohomology with coefficients}\\ \text{in Whitehead } L_\infty\text{-algebra}}}{H_{\mathrm{dR}}\big(X; \mathfrak{l}A\big)}
$$

charge quantization law

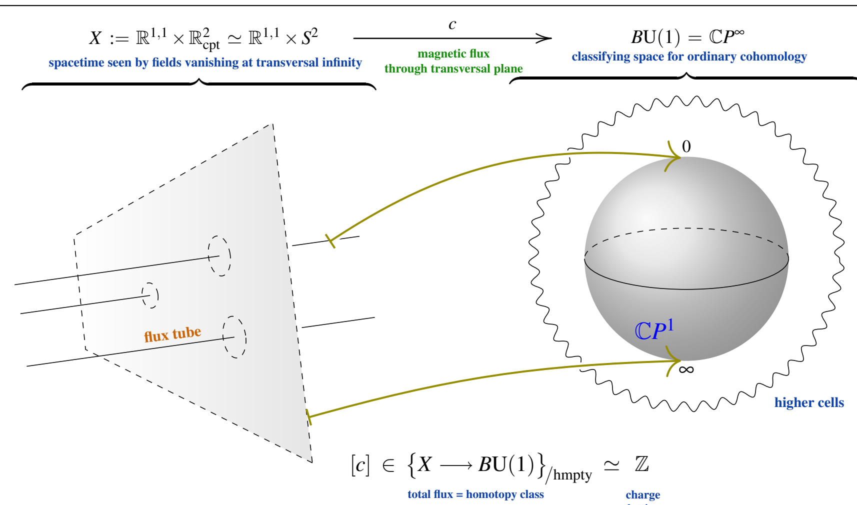
$$
\underset{A\text{-cocycle}}{\big[X \xrightarrow{c} A\big]} \;\;\overset{\substack{\text{classical}\\ \text{approximation}}}{\longmapsto}\;\; \Big\{ \underset{\substack{\text{field strenghts/}\\ \text{flux densities}}}{\big(F_i \in \Omega^\bullet_{\mathrm{dR}}(X)\big)_i} \,\Big|\, \underset{\substack{\text{Bianchi identities/}\\ \text{charge conservation laws}}}{\big(d\,F_i = P_i\big(\{F_j\}\big)_j\big)_i} \Big\}
$$

Since charge quantization thus identifies generalized cohomology theories with microscopic properties of physical objects, definitions/theorems in algebraic topology translate to formulations of putative physical theories, non-perturbatively.

Example: Magnetic charge in ordinary cohomology. The archetypical example is *Dirac's charge quantization* [Di31] (review in [Al85][Fr00, §2]), which is the observation that the quantum nature of electrons requires the magnetic charge carried by a magnetic monopole (say, a charged black hole) to be identified with a class in ordinary integral degree-2 cohomology of the spacetime surrounding the monopole.



While magnetic monopoles remain hypothetical, the same mechanism governs magnetic flux quantization in superconductors (e.g. [Cha00, §2]), which is experimentally observed. The difference here is that, instead of removing the worldline of a singular point source from spacetime, fields are required to *vanish at infinity* along some directions – here: along a plane perpendicular to the superconductor, e.g. [AGZ98, §IV.B]. The result is integer numbers of unit flux tubes: vortex strings [NO73] (review in [To09]).

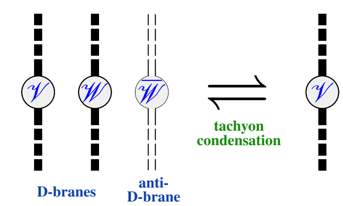


Notice that this *charge/flux quantization reveals the proper nature of the electromagnetic field*, as it reflects the physical reality of the "vector potential", hence of the photon's gauge field, not seen in the old Maxwell theory.

Example: Nuclear and gravitational charge in 1st non-abelian cohomology. Analogous comments apply to the nuclear force fields, and the Atiyah-Hitchin quantization law of their monopole solutions [AH88]. In fact, analogous comments also apply to the field of gravity, as reflected by gravitational instantons (review in [EGH80, §10.2]). In these cases, the cohomology theory in question is *non-abelian* (see [FSS20c, §2, §4.2] for pointers), represented by the classifying spaces BG of the respective structure/gauge groups G, namely $S\mathrm{U}(n)$ and $\mathrm{Spin}(n)$.

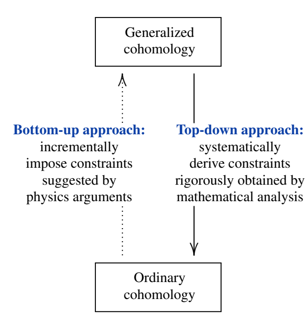
Open problem: Charge quantization in a unified fundamental theory. While charge quantization in gauge theories and in gravity is thus well understood separately, their expected unification in a "theory of everything" such as string theory is as famous as it remains incomplete, certainly in the putative non-perturbative completion (envisioned in the "second superstring revolution" [Sch96]) to M/F-theory (e.g. [Du96][Du99][BBS06]) whose actual (namely: mathematical) formulation remains a wide open problem (e.g., [Du96, §6][Du98, p. 6][Du99, p. 330][Mo14, §12][Wi19, @21:15][Du19, @17:14]). This is the problem with which we are concerned here.

Example: D-Brane charge in K-theory? A prominent example of identifying a physical theory by determining its charge quantization law in generalized cohomology is the conjecture that the charges of D-branes seen in string theory are quantized in topological K-cohomology [MM00] (see also [Wi01][Ev06][ES06][Fred08][GS19]). This conjecture is largely based on the suggestion [Wi98, §3] that, in turn, *Sen's conjecture* about tachyon condensation of open superstring states stretched between D-branes/anti-branes [Sen98] (review in [Sen05]) should imply the defining equivalence relation of topological K-theory on the Chan-Paton vector bundles carried by D-branes.



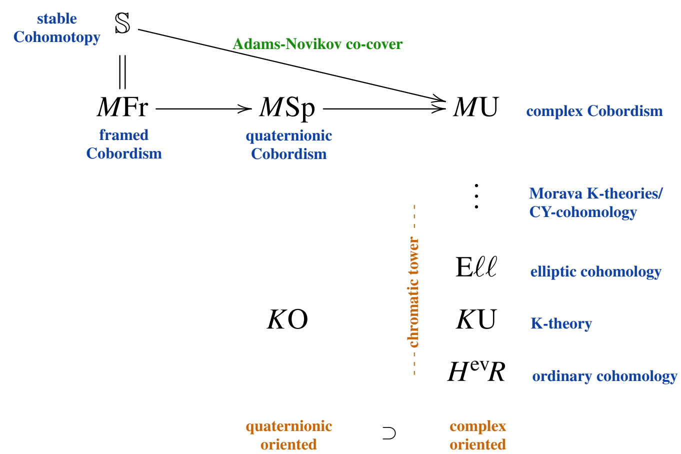
This suggestion still remains to be checked (see [Er13, p. 35][Er19, p. 112]). Even assuming this conjecture, open problems remain within the K-theory hypothesis itself:
(a) In type IIB, it is incompatible with S-duality (see [KS05a][Ev06, §8.3]).
(b) In type IIA, it requires with the D8-brane also a D(-2)-brane, which remain mysterious (e.g. [AJTZ10]).
(c) K-theory seems to predict physically spurious charges (e.g. [dBD$^+$02, §4.5.2][FQ05, p. 1]), such as irrational charges ([BDS00, (2.8)]; for more discussions and recent progress see [BSS19]).
(d) The lift of K-theoretic charge quantization to M-theory has remained elusive (e.g. [dBD$^+$02, §4.6.5]).

Example: M-Brane charge in modified ordinary cohomology? It has been tradition to describe the expected M-brane charge quantization via ordinary cohomology equipped with an incremental sequence of constraints (e.g. [Wi97a][Wi97b][DMW03][DFM03]). From the obstruction-theoretic perspective of [KS04][Sa08][Sa10] this looks just like the first steps of lifting through a character map (p. 3), via an Atiyah-Hirzebruch spectral sequence of a generalized cohomology for M-theory, without that cohomology theory having been determined yet. As a result, the list of conditions to be imposed on ordinary cohomology seems open ended (see [FSS19b, Table 1][GS20]), with every new phenomenon argued for in the string physics literature being another potential candidate to add to the list; such as non-abelian DBI effects (review in [My03]) including M-brane polarizations, seen only through more recent developments in M2-brane theory (review in [BLMP13, §6]). This tradition is hence a perpetual *bottom-up approach* to formulating M-theory.



In contrast, following [Sa10][Sa14][FSS19b][FSS19c][SS19a][SS19b], we consider here a *top-down approach*: *postulating* a generalized cohomology theory for charge quantization in M-theory as advocated early on in [Sa05a][Sa05b][Sa06a], and then rigorously *deriving* its implications for physics.

M-brane charge in complex-oriented cohomology? Several arguments suggest [KS04][KS05a][KS05b][Sa06b][Sa10] that charge quantization in M/F-theory involves *elliptic cohomology*. Interestingly, all these candidates – (0) ordinary cohomology, (1) complex K-theory, (2) elliptic cohomology – are *complex-orientable* cohomology theories (see §3.8). As such, they are the first three stages in the *chromatic tower* of cohomology theories (see [Ra86][Lu10]), whose culmination is complex *Cobordism cohomology* $M\mathrm{U}$ (e.g. [Koc96]). Moreover, complex Cobordism stands out in that the initial map $\mathbb{S} \simeq M\mathrm{Fr} \to M\mathrm{U}$ (36) that it receives from *framed Cobordism cohomology* (see Example 2.4 for pointers) is a co-cover, meaning that computations in $M\mathrm{U}$-theory co-descend to reveal $M\mathrm{Fr}$-theory; this is the statement of the *Adams-Novikov spectral sequence* ([Nov67, Thm. 3.1, 3.3][Ra78, Thm. 3.1], see [Wil13] for modern exposition). One may wonder:



Is M/F-theory charge quantized in $M\mathrm{Fr}$-theory?

Hypothesis H: M-brane charge in Cohomotopy theory. Given that M-theory is supposedly the pinnacle of fundamental physics, we suspect that M-brane charge quantization corresponds not to random but rather to the deep phenomena seen in algebraic topology. This idea has been the guiding light for investigations into M-theory in [Sa10][Sa11a][Sa11b][Sa14]. But the most fundamental of all cohomology theories is Borsuk-Spanier *Cohomotopy theory* $\pi^\bullet$ (12) whose stabilization is the Whitehead-generalized cohomology theory *stable Cohomotopy* $\mathbb{S}^\bullet$ (15), which the Pontrjagin-Thom construction identifies (27) with framed Cobordism cohomology $M\mathrm{Fr}^\bullet$.

Our *Hypothesis H* posits that *M-brane charge is quantized in 4-Cohomotopy* π^4 [Sa13, §2.5][FSS15][FSS17] (exposition in [FSS19a, §7, 12.2]); specifically in tangentially twisted 4-Cohomotopy [FSS19b][FSS19c], which on flat orbifolds means [SS20b, Thm. 5.16] tangentially *RO*-graded equivariant Cohomotopy [HSS19][SS19a][BSS19]. Evidence for Hypothesis H comes from analyzing the image of twisted Cohomotopy...

- **(a)** ... in ordinary rational cohomology [FSS19a, §7][FSS19b][FSS19c][HSS19][BMSS19][SS20a][FSS20b]; under the cohomotopical character map [FSS20c, §5.3][SS20c];
- **(b)** ... in ordinary integral cohomology and differential cohomology [GS20], via the Postnikov tower;
- **(c)** ... in K-theory [SS19a][BSS19], under the Hurewicz-Boardman homomorphism;
- **(d)** ... in equivariant Cobordism [SS19a], under the Pontrjagin construction;
- **(e)** ... in configuration spaces [SS19b], under the May-Segal theorem.

M/F-Brane charge in MFr-Theory. Here we explain how Hypothesis H, specialized (14) to homotopically flat spacetimes (Remark 2.2, hence to the situation where the tangential twisting of Cohomolotopy disappears), connects expected physics of M/F-theory with the conceptual heart of the mathematical field of *stable homotopy theory and generalized cohomology* [Ad74]. Since this field may be advertized (e.g. [Ra86, p. xv][MR87, §1][Wil13, p. 1][IWX20, §1]) as revolving all around the stable homotopy of spheres $\mathbb{S}_{-\bullet}$, hence the stable Cohomotopy $\mathbb{S}^\bullet$, and so equivalently the framed Cobordism cohomology $M\mathrm{Fr}^\bullet$, it makes sense to refer to it as MFr-*theory*, for short. We find that the correspondence that we lay out in §2 (summarized in §4)

$$\underset{\text{fundamental physics}}{\text{M/F-Theory}} \xleftrightarrow{\text{Hypothesis H}} \underset{\text{algebraic topology}}{M\text{Fr-Theory}}$$

not only supports the Hypothesis that M-brane charge is quantized in Cohomotopy theory, by showing that theorems in MFr-theory rigorously imply effects expected in the physics of M/F-theory, but also illuminates MFr-theory by providing coherent physics intuition for several of its fundamental definitions and theorems (discussed in §3).

2 The correspondence

We develop a correspondence (summarized in §4) that translates, under Hypothesis H, fundamental concepts expected in M-theory on flat (Rem. 2.2) and fluxless (Rem. 2.1) backgrounds to fundamental concepts of generalized cohomology and of stable homotopy theory (MFr-theory). This section is expository, focusing on conceptual identifications. All definitions and proofs that we appeal to are treated rigorously in §3.

Before entering the discussion below in §2.1, here are some comments on our running assumption of flat and fluxless backgrounds:

Remark 2.1 (Focus on cohomologically fluxless backgrounds)**.** From §2.4 on we focus on backgrounds for which the 4-flux G_4 is trivialized in cohomology (in *generalized* cohomology, that is, see (155)), its trivialization being the twisted H_3-flux, or equivalently the sum of the C_3-field with a closed H_3-flux (54). Seen in ordinary rational cohomology $H\mathbb{R}$, the G_4-flux is represented as a closed differential form, the *flux density*, whose vanishing in cohomology means that the background is *fluxless* in that the integrated 4-flux through any closed 4-manifold inside spacetime vanishes. Beware that the literature sometimes conflates "vanishing flux" (i.e. flux vanishing in cohomology) with "vanishing flux density". The distinction is fully resolved in enhancement of the present considerations to *differential* cohomology (as in [FSS20c, §4.3]), which we hope to discuss elsewhere (but see Remark 3.55).

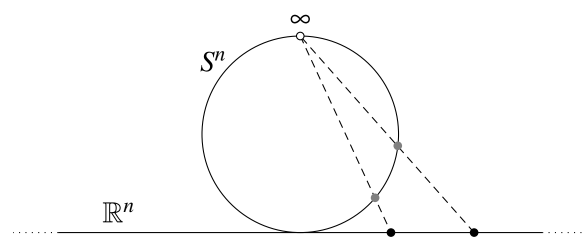
Remark 2.2 (Focus on homotopically flat spacetimes)**.** We focus on spacetime topologies with trivializable tangent bundles, hence admitting a framing, as in [Sa13][Sa14]. For such "homotopically-flat" spacetimes, Hypothesis H says (14) that M-brane charge is measured in plain Cohomotopy (in contrast to tangentially twisted Cohomotopy, since the tangential twist is given by the class of the tangent bundle) on which we wish to concentrate for focus of presentation (details in §2.2).
(a) Typical examples of homotopically flat spacetimes of interest here are product manifolds of Minkowski spacetimes $\mathbb{R}^{d,1}$ with unit spheres in a normed real division algebra, i.e. with 0-spheres $S^0 = * \sqcup *$, circles S^1, 3-spheres S^3 and 7-spheres S^7 ([Ad60, p. 21]).
(b) Restriction to homotopically flat spacetimes is of course a strong constraint, which is however alleviated by understanding it in the generality of *equivariant* homotopy theory (specifically: equivariant Cohomotopy [SS19a][SS20b, Def. 5.28]) where flat spacetimes include flat orbifolds (see pointers in [SS20b, §1]): Such "flat" (often: "Euclidean") orbifolds are really curved spaces for which all curvature is concentrated in singularities. While we do not discuss this here, the point is that all the algebraic topology that we do discuss has fairly immediate equivariant generalization applicable to this case.
(c) While restrictive, the special case is of central interest: Much of the particle physics model building in string theory (review in [IU12]) utilizes flat orbifold spacetimes (review in [Re06]). Moreover, flat and locally fluxless orbifold spacetimes are, as "universal spacetimes" [CGHP08], among the few known classes of exact M-theory backgrounds, i.e. those satisfying supergravity equations of motion subject to the full tower of M-theoretic higher curvature corrections (whatever these may be, see e.g. [CGNT05]).
(d) Since we consider M-brane charge *vanishing at infinity* (see §2.2), we may think of non-compact flat orbifolds as local charts inside a global curved spacetime orbifold with M-brane charge localized inside them. This way our considerations here provide the local building blocks for a fully general discussion of E-cohomological M-brane charge on globally curved spacetimes (to be given elsewhere).
(e) Finally, while flat Minkowski spacetimes themselves are homotopically trivial, it is this constraint that charges vanish at infinity which makes them appear to their charge cohomology theory as *effective* spheres with non-trivial topology.

We close these introductory remarks by further amplification of this last point:

Remark 2.3 (Generalized charge cohomology vanishing at infinity)**.** It is a familiar fact in (higher) gauge theory, recalled in a moment, that the (non-abelian and generalized) cohomology of flat Euclidean n-space, subject to the constraint that *fluxes vanish at infinity*, is equivalent to the unconstrained cohomology of the n-sphere (2): This fact governs the usual classification of gauge instantons in 4d (4) (and of Skyrmions in 3d) as well as the traditionally conjectured classification of D-brane charges in K-theory (5). While this fact is classical and widely used, it may be worthwhile to recall some of its aspects, applications and pertinent references. The discussion in the main text (notably in §2.2) is a direct application of this classical fact; indeed it is the direct analog of (5), with K-theory replaced by Cohomotopy.

Despite the prominent role that n-spheres play in the discussion of finite-flux configurations on flat Euclidean space, it is important to beware that the topological n-sphere in question is just a mathematical tool for reasoning about fields on flat space subject to asymptotic constraints, and not to be conflated with a curved spacetime. In particular, there is no non-trivial contribution of the field of gravity in this discussion, which means that the charge cohomology theories in question are not twisted by non-trivial (generalized) tangent bundles. One way to bring out the actual mechanism at work is:
(1) to understand the n-sphere as being just the homeomorphism type of the "one-point compactification" of Euclidean n-space ([Al24][Ke55, p. 150] review in [Cu20]), hence of Euclidean n-space with a formal "point at infinity" adjoined (17):

$$\overset{\substack{\text{flat Euclidean } n\text{-space} \\ \text{with formal "point-at-infinity}}}{\mathbb{R}^n_{\mathrm{cpt}}} \underset{\text{homeomorphism}}{\simeq} \overset{\text{topological } n\text{-sphere}}{S^n} \tag{1}$$



This "point at infinity" is a mathematical tool for conceptualizing asymptotic boundary conditions, *not* a point in spacetime. It may transparently be understood as the pole $\infty \in S^n$ from which a stereographic projection conformally identifies $\mathbb{R}^n$ with the complement $S^n \setminus \{\infty\} \simeq \mathbb{R}^n$.
(2) to notice that the *cohomology with compact support* of flat $\mathbb{R}^n$ is equivalent to the reduced cohomology of its one-point compactification, hence to pointed-homotopy classes $\pi_0 \mathrm{Map}^{*/}(-,-)$ of continuous maps from this formal S^n to the given classifying space (6):

$$\overset{\substack{A\text{-cohomology} \\ \text{with compact support} \\ \text{of flat Euclidean space}}}{H^0_{\mathrm{cpt}}\big(\mathbb{R}^n; A\big)} \simeq \overset{\substack{\text{pointed homotopy classes} \\ \text{from one-point compactification} \\ \text{to the classifyiong space}}}{\pi_0 \mathrm{Map}^{*/}\big(\mathbb{R}^n_{\mathrm{cpt}}, A\big)} \simeq \overset{\substack{\text{pointed homotopy classes} \\ \text{from } n\text{-sphere} \\ \text{to the classifying space}}}{\pi_0 \mathrm{Map}^{*/}\big(S^n, A\big)} \simeq \overset{\substack{\text{reduced } A\text{-cohomology} \\ \text{of the } n\text{-sphere}}}{\widetilde{A}(S^n)} \simeq \overset{\substack{n\text{th homotopy group} \\ \text{of classifying space}}}{\pi_n(A)}\,. \tag{2}$$

This follows since every continuous function $c : S^n \to A$ which vanishes at ∞ (in that it takes ∞ to the given base-point of the classifying space A) is naturally homotopic to a function $c \circ \phi$ which vanishes in an open neighbourhood $D^n_\infty \subset S^n$ of ∞, hence whose support is in the compact subset $S^n \setminus D^4$. Here $\phi : S^n \to S^n$ may be taken to homeomorphically identify $S^n \setminus \overline{D^n_\infty}$ with $S^n \setminus \{\infty\}$ while sending $D^n_\infty \to \{\infty\}$. Such ϕ is homotopic to the identity on S^n, as may readily be checked by explicit coordinate expressions, but also follows on the general grounds of the Hopf degree theorem (e.g. [Kos93, §IX, Cor. 5.8]).

For instance, K-theory with compact support on flat Euclidean spaces is *defined* this way (2) as

$$\overset{\substack{\text{compactly-supported K-theory} \\ \text{of flat Euclidean space}}}{K_{\mathrm{cpt}}(\mathbb{R}^n)} := \overset{\substack{\text{reduced K-theory of} \\ \text{one-point compactification}}}{\widetilde{K}(\mathbb{R}^n_{\mathrm{cpt}})} = \overset{\substack{\text{reduced K-theory} \\ \text{of } n\text{-sphere}}}{\widetilde{K}(S^n)} \tag{3}$$

(e.g. [Ka78, §II, Ex. 4.4][OS99, (2.20) and below (2.58)]) and the analogous statement holds for any generalized cohomology theory (e.g. [MV03, p. 28]). The left hand side of (3) makes manifest that we are dealing with flat spacetime, even though the equivalent re-expression on the right involves an n-sphere domain.

Classical Example: BPST instanton charge in non-abelian cohomology. When the coefficients in (2) are $A = B\mathrm{SU}(2)$, the classifying space for $\mathrm{SU}(2)$-gauge bundles, then (2) recovers the traditional classification (e.g. [EGH80, §8.2]) of multi-center BPST-instantons on flat 4-space:

$$\overset{\substack{\text{non-abelian cohomology}\\ \text{with compact support}\\ \text{of flat Euclidean 4-space}}}{H^1_{\mathrm{cpt}}\big(\mathbb{R}^4;\mathrm{SU}(2)\big)} \;=\; H^0_{\mathrm{cpt}}\big(\mathbb{R}^4; B\mathrm{SU}(2)\big) \;\simeq\; \overset{\substack{\text{pointed-homotopy classes}\\ \text{from the } n\text{-sphere}\\ \text{to the classifying space}}}{\pi_0 \mathrm{Map}^{*/}\big(S^4, B\mathrm{SU}(2)\big)} \;\simeq\; \pi_4\big(B\mathrm{SU}(2)\big) \;\simeq\; \pi_3\big(\mathrm{SU}(2)\big) \;\simeq\; \overset{\substack{\text{set of multi-center BPST}\\ \text{instanton numbers}}}{\mathbb{Z}}\,. \tag{4}$$

Notice that this concerns instantons in Yang-Mills theory on flat space(-time) without any coupling to gravity, hence without any spacetime curvature.

In contrast, if we would consider a curved spacetime manifold of the form $\mathbb{R}^{d,1} \times S^n$ (say) with non-trivial metric on S^n (either round or squashed), then we would have to consider charges of Einstein-Yang-Mills theory instead of plain Yang-Mills theory, and the plain classifying space $B\mathrm{SU}(2)$ would be generalized to some suitable twisted differential classifying stack. This situation of twisted differential cohomology of curved spacetimes can of course be discussed, too (see [FSS20c]) but is disregarded in the present article for sake of focus.

Traditional Example: D-Brane charge in non-twisted K-theory. When the coefficients in (2) are $A = \mathrm{KU}_0 = \mathbb{Z} \times B\mathrm{U}$, the classifying space for complex K-theory, then (2) reproduces the traditionally conjectured classification of D-brane charge in type IIB string theory for flat transversal space, seen in compactly supported K-theory of the transversal $\mathbb{R}^{9-p}$, understood as the reduced K-theory of its one-point compactification (e.g. [Ka78, §II, Ex. 4.4][OS99, (2.20) and below (2.58)]):

$$\overset{\substack{\text{compactly supported K-theory}\\ \text{of flat transversal space}}}{\mathrm{K}_{\mathrm{cpt}}\big(\mathbb{R}^{9-p}\big)} \;=\; \overset{\substack{\text{reduced K-theory of}\\ \text{transversal compactification}}}{\widetilde{\mathrm{K}}\big(\mathbb{R}^{9-p}_{\mathrm{cpt}}\big)} \;\simeq\; \overset{\substack{\text{reduced K-theory}\\ \text{of } 9-p\text{-sphere}}}{\widetilde{\mathrm{K}}\big(S^{9-p}\big)} \;\simeq\; \pi_0 \mathrm{Map}^{*/}\big(S^{9-p}, B\mathrm{U}\big) \;\simeq\; \pi_{9-p}\big(B\mathrm{U}\big) \;\simeq\; \overset{\substack{\text{set of}\\ \mathrm{D}p\text{-brane charges}}}{\begin{cases} \mathbb{Z} & \text{for } p \text{ odd} \\ 0 & \text{otherwise.} \end{cases}} \tag{5}$$

This is made explicit, for instance, in [Gu00, §2.1][BGH99, §2.1][OS99, §3][Schw01, p. 25], following [Wi98, §4.1][Ho99, §2]. The same kind of expressions hold, more generally, for D-brane charge on orbi-orientifolds, seen in equivariant KR-theory; this is made explicit, for instance, in [Gu00, §3][GS00, §3][GCHS+08, p. 3].

The formulation below in §2.1 of M-brane charge for flat transversal space seen in Cohomotopy vanishing at transversal infinity is the direct and evident analogue of this traditional situation (5), for K-theory replaced by Cohomotopy.

In contrast: Brane charge in twisted generalized cohomology. Again, notice that the traditional classification of D-branes on spacetimes that are not flat but already have themselves the homotopy type of an n-sphere is in general different: The non-trivial background metric (field of gravity) on the sphere will generically come with a non-trivial B-field, and that changes the charge quantization law from plain K-theory to twisted K-theory. (The archetypical and well-studied example of this case is the classification of D-branes for the WZW-model on $\mathrm{SU}(2) \simeq S^3$ with its B-field proportional to the volume class of the 3-sphere, e.g. [FS01][Br04][GG04, §1], review in [Ev06, §3.1].) This, too, is closely analogous to the situation for Cohomotopy instead of K-theory, under *Hypothesis H*: If spacetime itself is a spherical fibration, then Hypothesis H predicts charge quantization in tangentially twisted Cohomotopy, see [SS20a].

While, in either case, brane charges in twisted generalized cohomology (either twisted K-theory or twisted Cohomotopy) on spherical spacetimes is an important subject, we disregard it in the present article for sake of focus. The point to amplify here is that, nevertheless, the compactly-supported cohomology of flat transversal space which we do consider happens toth an expression in terms of non-twisted cohomology of spheres – which is a well-known mathematical phenomenon, familiar from traditional discussion of D-brane charge in K-theory (5).

2.1 M-Brane charge and Borsuk-Spanier Cohomotopy

Higher gauge fields and homotopy & cohomology.

A *non-abelian higher gauge field* species [SSS12][Sch13] is a reduced[1] *non-abelian cohomology theory*, namely [FSS20c, §2] [SS20b, p. 6] (see §3.1) a contravariant functor $\widetilde{A}(-)$ sending pointed topological spaces X to the sets of homotopy-classes of their pointed continuous functions into a given pointed space A, the *classifying space* for A-cohomology:[2]

Physical field	A	**Cohomology theory**
Electromagnetic	$B\mathrm{U}(1)$	Ordinary abelian cohomology
Nuclear	$B\mathrm{SU}(n)$	Ordinary non-abelian cohomology
Gravity	$B\mathrm{Spin}(n)$	Ordinary non-abelian cohomology
Neveu-Schwarz	$B^2\mathrm{U}(1)$	Ordinary abelian cohomology
Ramond-Ramond	$K\mathrm{U}$	Topological K-theory (?)
C-field	$S^4 \simeq B(\Omega S^4)$	Cohomotopy (?)

$$\underset{\text{= pointed topological space}}{\overset{\substack{\text{spacetime with}\\ \text{“point at infinity”}}}{X}} \;\longmapsto\; \underset{\text{= non-abelian reduced } A\text{-cohomology of } X}{\overset{\substack{\text{gauge-equivalence classes of } A\text{-fields on } X\\ \text{vanishing at infinity}}}{\widetilde{A}(X) \;:=\; \pi_0 \mathrm{Maps}^{*/}\big(X,\, A\big)}} \;=\; \left\{ \begin{tikzcd} \underset{\substack{\text{spacetimes =}\\ \text{domain space}}}{X} \arrow[rr, bend left, "\overset{\substack{\text{field configuration =}\\ \text{pointed map / cocycle}}}{c}"] \arrow[rr, bend right, "\underset{\substack{\text{field configuration =}\\ \text{pointed map/cocycle}}}{c'}"'] \arrow[rr, phantom, "\substack{\parallel\\ \text{gauge transf.}\\ \text{= homotopy/}\\ \text{coboundary}\\ \Downarrow}"] & & \underset{\substack{\text{classifying space}\\ \text{for } A\text{-cohomology}}}{A} \end{tikzcd} \right\} \Big/ \substack{\text{gauge equiv.}\\ \text{= homotopy}} \tag{6}$$

Familiar examples include:

- For $A = K(R,n)$ an Eilenberg-MacLane space, this construction (6) is ordinary abelian cohomology as computed by singular- or Čech-cochains (see [FSS20c, Ex. 2.2] for pointers); specifically as computed by (PL-)differential forms in the case that $R = \mathbb{R}$ (see [FSS20c, §3, Ex. 4.9] for pointers). Notice that $K(\mathbb{Z}, n+1) \simeq B^n\mathrm{U}(1)$ is the classifying space for circle $(n-1)$-gerbes (see [FSS20c, Ex. 2.12] for pointers).

  For $n = 1$ this models the electromagnetic field (the “vector potential A-field”, e.g. [Na03, §10.5]), while for $n = 2$ this models the Kalb-Ramond B-field [Ga86][FW99][CJM04]. The case for $n = 3$ is often taken as a first approximation to a model for the supergravity C-field (see p. 4). The point of our discussion here is to improve on this approximation for the C-field.

- For $A = BG$ the classifying space of a compact topological group G (e.g. the infinite Grassmannian $\mathrm{Gr}_n \simeq B\mathrm{O}(n)$), the definition (6) reduces to ordinary non-abelian cohomology in the sense of Chern-Weil theory (see [FSS20c, Ex. 2.2, 2.3] for pointers).

  For G in the Whitehead tower of $\mathrm{SU}(n)$ this models the generic non-abelian/nuclear force gauge field (e.g. [Na03, §10.5]); while for G in the Whitehead tower of $\mathrm{SO}(n)$ this models gravity (see [SSS12, Fig. 1,2]):

$$\overset{\substack{\text{ordinary cohomology}\\ \text{(in particular: abelian)}}}{\widetilde{H}^n\big(X;R\big)} \;\simeq\; \pi_0 \mathrm{Maps}^{*/}\big(X\,,\, \overset{\substack{\text{Eilenberg-MacLane}\\ \text{space}}}{K(R,n)}\big)\,, \qquad \overset{\substack{\text{non-abelian cohomology}\\ \text{(as in Chern-Weil theory)}}}{\widetilde{H}^1(X;G)} \;\simeq\; \pi_0 \mathrm{Maps}^{*/}\big(X\,,\, \overset{\substack{\text{classifying space}\\ \text{of compact group}}}{BG}\big)\,. \tag{7}$$

In generalization of the ordinary abelian case on the left of (7), if $A = E^n$ is a space in a *spectrum* of pointed spaces (“Ω-spectrum”)[3]

$$\overset{\substack{\text{spectrum}\\ \text{(of spaces)}}}{E} \;=\; \Big\{ \overset{\text{graded component spaces (pointed)}}{E^n \;:=\; \Omega^{\infty-n}E \;:=\; \Omega^\infty \Sigma^n E}\,,\;\; E^n \xrightarrow[\simeq]{\substack{\text{classifying maps for}\\ \text{suspension isomorphism}}} \overset{\text{loop spaces}}{\Omega E^{n+1}} \Big\}_{n \in \mathbb{N}} \tag{8}$$

---

[1] We consider reduced cohomology throughout, since this captures the crucial physics concept of fields *vanishing at infinity* (§2.2). Notice that reduced cohomology $\widetilde{A}(-)$ *subsumes* non-reduced cohomology $A(-)$, as $A(X) = \widetilde{A}(X_+)$.

[2] Strictly speaking, bare A-cohomology gives only the “topological sector” or “instanton sector” of the higher gauge field; while the full field content is in *differential* A-cohomology [FSS20c, §4.3]. Here we disregard the differential refinement just for focus of the exposition; we come back to this elsewhere. But see also Remark 3.55 below.

[3] Beware that we write the indices on the component spaces of a spectrum (8) as *super*scripts, instead of the more conventional subscripts. This is to harmonize the index placement in common formulas such as the suspended unit maps $\Sigma^n(1^E) : S^n \to E^n$ in (32) below. Our *sub*script is reserved for the coefficient groups (9), matching the expected index placement for stable homotopy groups of spheres $\pi^s_n = \mathbb{S}_n$ and cobordism rings $\Omega^f_n = (Mf)_n$.

then definition (6) reduces ([FSS20c, Ex. 2.13]) to that of *Whitehead's generalized cohomology*[4] ([Wh62][Ad74])

$$
\overset{\text{Whitehead-generalized}\atop\text{reduced cohomology}}{\widetilde{E}^n(X)} \;\simeq\; \pi_0 \mathrm{Maps}^{*/}\big(X\,,\overset{n\text{th-space in}\atop\text{a spectrum}}{E^n}\big)\,, \quad \text{in particular (Ex. 3.32)}:\; \underset{\text{graded coefficient group}}{E_\bullet \;:=\; \pi_\bullet(E)} \;:=\; \underset{\text{reduced }E\text{-cohomology of spheres}}{\widetilde{E}^k(S^{k+\bullet}) \;\simeq\; \pi_{\bullet+k}\big(E^k\big)}\,. \tag{9}
$$

such as complex K-cohomology (conjectured to model the RR-field, see p. 4) or complex cobordism cohomology (see Examples 2.4):

$$
\overset{\text{complex top.}\atop\text{K-theory}}{\widetilde{K\mathrm{U}}^0(X)} \;\simeq\; \pi_0 \mathrm{Maps}^{*/}\big(X\,,\overset{\text{stable unitary}\atop\text{classifying space}}{B\mathrm{U}}\big)\,, \qquad \overset{\text{complex}\atop\text{cobordism}}{\widetilde{\mathrm{MU}}^0(X)} \;\simeq\; \pi_0 \mathrm{Maps}^{*/}\Big(X\,,\overset{\text{Thom space of cplx. vector bdl.}\atop\text{over classifying space}}{\mathrm{Th}\big(\mathscr{V}_{B\mathrm{U}}\big)}\Big)\,. \tag{10}
$$

But in full non-abelian generalization of the second case in (7), every ∞-group $\mathscr{G}$ – namely every based loop group (see [FSS20c, Prop. 2.8] for pointers) with its homotopy-coherent group operations by concatenation of loops – defines a generalized non-abelian cohomology theory, such as that classifying non-abelian G-gerbes (see [FSS20c, Ex. 2.6] for pointers):

$$
\left.\begin{array}{r} A \simeq B\mathscr{G} \\ \Leftrightarrow \mathscr{G} \simeq \Omega A \end{array}\right\} \quad \overset{\text{general}\atop\text{non-abelian cohomology}}{\widetilde{H}^1\big(X;\mathscr{G}\big)} \;\simeq\; \pi_0 \mathrm{Maps}^{*/}\big(X\,,\overset{\text{any}\atop\text{connected space}}{B\mathscr{G}}\big)\,, \quad \text{e.g.:}\; \overset{\text{non-abelian cohomology}\atop\text{(as in Giraud-Breen theory)}}{\widetilde{H}^1\big(X;\mathrm{Aut}(BG)\big)} \;\simeq\; \pi_0 \mathrm{Maps}^{*/}\big(X\,,\overset{\text{automorphism 2-group}\atop\text{of }G\text{-classifying space}}{\mathrm{Aut}(BG)}\big)\,. \tag{11}
$$

The M-Theory C-field and Borsuk-Spanier Cohomotopy. The most fundamental example of general non-abelian cohomology (6) is unstable *Cohomotopy* [Bo36][Po38][Sp49][Pe56], whose classifying spaces are the n-spheres, $S^n \simeq B\big(\Omega S^n\big)$, hence whose higher gauge group (11) is the loop ∞-group $\mathscr{G} = \Omega S^n$ of n-spheres:

$$
\overset{\text{Cohomotopy}}{\widetilde{\pi}^n(X)} \;:=\; \pi_0 \mathrm{Maps}^{*/}\big(X\,,\overset{n\text{-sphere}}{S^n}\big)\,, \qquad \overset{\text{Cohomotopy}}{\widetilde{\pi}^n(-)} \xrightarrow[\text{non-abelian cohomology operation}\atop\text{induced by any }[g]\in\pi_n(A)]{(S^n \xrightarrow{g} A)_*} \overset{\text{any }A\text{-cohomology}}{\widetilde{A}(-)} \tag{12}
$$

Over a smooth manifold, such as $X = (M^d)^{\mathrm{cpt}} \wedge \mathbb{R}^{p,1}_+$ (17), the *rationalization* of Cohomotopy in even degrees (12)

$$
\begin{tikzcd}
\overset{\text{Cohomotopy}}{\widetilde{\pi}^{2k}\big(M^d_+\big)} \arrow[r, equal] \arrow[drrrr, "\text{cohomotopical character map}", "\mathrm{ch}_{\pi^{2k}}"'] & \mathrm{Maps}^{*/}\big(X,S^n\big) \arrow[r, "\text{rationalization}"] & \mathrm{Maps}^{*/}\big(X,L_{\mathbb{R}}S^n\big) \arrow[r, equal] & \overset{\text{rational}\atop\text{Cohomotopy}}{\widetilde{\pi}^{2k}_{\mathbb{R}}\big(M^d_+\big)} \arrow[r, equal, "\text{non-abelian}\atop\text{de Rham theorem}"] & \overset{\text{non-abelian}\atop\text{de Rham cohomology}}{H_{\mathrm{dR}}\big(M^d, \mathfrak{l}S^n\big)} \arrow[d, equal] \\
& & & & \left\{ \begin{array}{c} G_{2k}, \\ 2G_{4k-1} \end{array} \in \Omega^\bullet_{\mathrm{dR}}\big(M^d\big) \;\middle|\; \begin{array}{rl} d\,G_{2k} & = 0 \\ d\,2G_{4k-1} & = -G_{2k}\wedge G_{2k} \end{array} \right\}
\end{tikzcd} \tag{13}
$$

is equivalent [FSS19b, Prop. 2.5], via a cohomotopical character map [FSS20c, Ex. 5.23], to concordance classes of pairs of differential forms (G_{2k}, G_{4k-1}) that satisfy the differential relations known from the C-field 4-flux form and its Hodge dual [D'AF82, (5.11b)][CDF91, (III.8.53)] (review in [MiSc06, (3.23)]) in 11-dimensional supergravity [CJS78], with their characteristic quadratic dependence.

Noticing the direct analogy with how the image of the twisted K-theoretic Chern character produces differential forms $(H_3, \{F_{2k}\}_k)$ satisfying the differential relations $dF_{2k+2} = H_3 \wedge F_{2k}$ (see [FSS20c, §5.1]) known from the RR-field flux forms $F_{2k\leq 5}$ and their Hodge duals $F_{2k\geq 5}$, the form of the cohomotopical character (13) is the first indication [Sa13, §2.5][FSS17][FSS19a, §7] that the C-field wants to be charge-quantized in Cohomotopy theory.

In the special case of interest here, this *Hypothesis H* reads, in more detail, as follows:

[4]Whitehead's generalized cohomology is traditionally just called *generalized cohomology*, for short. Since this becomes ambiguous as we consider yet more general non-abelian cohomology (6), we re-instantiate Whitehead's name for definiteness.

Hypothesis H on homotopically flat spacetimes: *If X is (the pointed topological space underlying) a homotopically flat 11-dimensional spacetime (Remark 2.2) equipped with a point at infinity* (17)*, then the C-field on X is charge-quantized in reduced Borsuk-Spanier 4-Cohomotopy* (12)*, in that (the topological sector of) a full C-field configuration is a $[c] \in \widetilde{\pi}^4(X)$ whose corresponding 4- and dual 7-flux density are the image $\big[G_4(c), G_7(c)\big] = \mathrm{ch}_{\pi^4}(c)$ under the cohomotopical character map* (13)*:*

$$
\begin{array}{ccc}
\overset{\text{Borsuk-Spanier}}{\overset{\text{4-Cohomotopy}}{\pi^4(X)}} & \xrightarrow[\mathrm{ch}_\pi]{\text{cohomotopical character map}} & \overset{\text{non-abelian}}{\overset{\text{de Rham cohomology}}{H_{\mathrm{dR}}\big(X; \mathfrak{l}S^4\big)}} \\
\underset{\text{in Cohomotopy theory}}{\underset{\text{C-field charge, quantized}}{[c]}} & \longmapsto & \underset{\text{and its 7-form dual}}{\underset{\text{C-field 4-flux density}}{\big[G_4(c), G_7(c)\big]}}
\end{array}
\tag{14}
$$

The abelianized C-field and Stable Cohomotopy. Non-abelian cohomology (6) satisfies, in general, fewer conditions than abelian (Whitehead-generalized) cohomology (9), notably it may lack gradings and suspension isomorphisms (8). This means that non-abelian cohomology is *richer* than abelian Whitehead-generalized cohomology, just as non-abelian groups are richer than abelian groups).

But there are non-abelian cohomology operations ([FSS20c, Def. 2.17]) which approximate non-abelian cohomology by abelian cohomology; a famous example is the Chern-Weil homomorphism (see [FSS20c, §4.2] for pointers). Indeed, every non-abelian cohomology theory A (6) has a universal approximation by a Whitehead-generalized abelian cohomology theory (9), namely that represented by the suspension spectrum $\Sigma^\infty A$ of A. ([FSS20c, Ex. 2.24]); a famous example appears in Snaith's theorem [Sn79][Mat] (and its variants [GeSn08]) which says that complex K-theory theory is close to being the stabilization of the non-abelian cohomology theory classified by the pointed space $\mathbb{C}P^\infty_+$.

The abelianization of Borsuk-Spanier Cohomotopy (12) is *stable Cohomotopy* [Ad74, p. 204][Se74][Ro04], the Whitehead-generalized cohomology theory (9) whose classifying spectrum (8) is the sphere spectrum $\mathbb{S}$ with $\mathbb{S}^n = \Omega^\infty S^{\infty+n}$. Stable Cohomotopy, in turn, has canonical images in any multiplicative abelian cohomology theory, via the Hurewicz-Boardman homomorphism [Ad74, §II.6][Hun95][Arl04], coming from the fact that $\mathbb{S}$ is the initial ring homotopy-commutative spectrum in analogy to how the integers $\mathbb{Z}$ are the initial commutative ring, see (33) and (36) below.

Therefore, the universal cohomology operation from unstable/non-abelian to stable/abelian Cohomotopy initiates, under Hypothesis H, a web of abelian approximations to cohomotopical C-field charge, as measured by a web of Whitehead-generalized cohomology theories, a small part of which looks as follows:

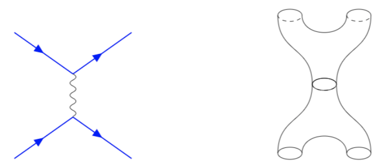
$$
\begin{tikzcd}
 & & & \widetilde{M\mathrm{Sp}}^n(X) \arrow[r] & \widetilde{K\mathrm{O}}^n(X) \arrow[r] & H^{4(n+\bullet)}(X;\mathbb{Z}) \\
\underset{}{\overset{\text{Cohomotopy}}{\widetilde{\pi}^n(X)}} = \pi_0 \mathrm{Maps}^{*/}\big(X, S^n\big) \arrow[r, "\text{stabilization/}"', "\text{abelianization}"'] & \pi_0 \mathrm{Maps}^{*/}\big(X, \mathbb{S}^n\big) = \overset{\text{stable}}{\overset{\text{Cohomotopy}}{\widetilde{\mathbb{S}}^n(X)}} = \overset{\text{framed}}{\overset{\text{Cobordism}}{\widetilde{M\mathrm{Fr}}^n(X)}} \arrow[urr] \arrow[drr] \arrow[drr, bend right, "\text{Hurewicz-Boardman}"', "\text{homomorphism}"'] & & \text{Cobordism} & \text{K-theory} & \overset{\text{ordinary}}{\text{cohomology}} \\
 & & & \widetilde{M\mathrm{U}}^n(X) \arrow[r, "\text{orientation}"'] & \widetilde{K\mathrm{U}}^n(X) \arrow[r, "\text{Chern}"', "\text{character}"'] & H^{2(n+\bullet)}(X;\mathbb{Z})
\end{tikzcd}
\tag{15}
$$

Previously we had discussed *Hypothesis H*
(a) in the (twisted and equivariant) rational approximation (13) in much detail [FSS19b][FSS19c][FSS20b][SS20a][SS20c], and
(b) in the (equivariant) stable approximation (15) in codimension-4 [SS19a][BSS19].

We now set out to discuss the abelian/stable approximation (15) to cohomotopical M-brane charge more systematically.

2.2 M-Brane worldvolumes and Pontrjagin-Thom collapse

Brane interactions and bordisms. At the foundations of perturbative string theory is, famously, the postulate that the multitude of *interactions* of fundamental particles – traditionally encoded by Feynman graphs with labeled interaction vertices – are secretly just (conformal) *bordisms* between strings [Se04][MoSe06][ST11].



While an analogous perturbation theory for fundamental higher dimensional p-branes is not expected to exist, it is expected that the interactions of *probe p-branes* (solitonic branes so light/decoupled as not to disrupt the topology of spacetime) is similarly given by $(p+1)$-dimensional bordisms between them, thought of as the time-evolution of p-dimensional brane volumes merging and splitting, much like soap bubbles do. This expectation is reflected in attempts [Wi10][Fr12][Fr13, p. 32][FT14][BZ14][Mü20] to conceive of at least part of the 5-brane's worldvolume field theory as a functor on a bordism category. Field theories regarded as such functorial bordism representations are well-understood in the "topological sector" [At89][Lu08][Fr13] and are plausible in more generality, where, however, precise definitions are tricky and remain largely elusive.

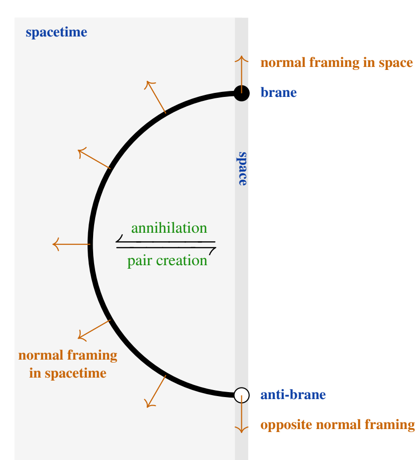
Conserved brane charge and Cobordism cohomology. But as we turn attention away from attempts to formulate a naïve worldvolume dynamics of p-branes, and instead look to the *generalized cohomology theory of their charges in spacetime*, the assumption that brane interactions is encoded by bordisms leads to a robust conclusion:

If total brane charge is to be preserved by brane interactions, then *brane charge is a bordism invariant* of the brane's p-dimensional shape within spacetime. The finest such invariant, hence the universal brane charge, is the *cobordism class* itself, which is an element in (unstable) degree-$(d-p)$ *Cobordism cohomology* of spacetime:

Expected aspect of brane physics	**Formulation in differential topology**
Codimension-$(n=d-p)$ probe p-branes in space M^d	Codimension-n submanifolds of manifold M^d
p-brane interactions in spacetime $X^{d,1}$	Codimension-n bordisms in $M^d \times [0,1]$
(Anti-)brane charge carried by single p-brane	f-structure on normal bundle
Total brane charge preserved by brane interactions	Function of cobordism classes of submanifolds
Finest conserved charge of codimension-n branes in M^d	Cobordism class in unstable Cobordism cohomology
Brane charge seen in bulk spacetime	Class in stable Cobordism cohomology $Mf^n(M^d)$

Table I. If brane interactions are given by bordisms, then **(a)** individual brane/anti-brane charge is reflected in f-structure on the brane's normal bundle and **(b)** conserved total brane charges are f-bordism invariants. The finest such is the f-cobordism class itself, which reflects brane charge quantized in f-Cobordism cohomology. This is unstable (non-abelian) cohomology for branes constrained to asymptotic spacetime boundaries, but approaches stable (abelian) Cobordism cohomology Mf as interaction processes are allowed to probe into higher bulk dimensions.

Anti-Branes and normal f-structure. Here an *anti-brane* is supposed to be a brane with opposite orientation relative to the ambient spacetime, and such that there are interaction processes of brane/anti-brane pair creation/annihilation. With brane interactions perceived as bordisms, this is naturally reflected by bordisms with *f-structure* on their normal bundle, named after fibrations $BG \xrightarrow{f} BO$ factoring the bundle's classifying map. Initial among all f-structures is framing structure $B\mathrm{Fr} \simeq EO \to BO$; where the two possible choices of normal structure on branes of the simple Cartesian shape $\mathbb{R}^p \subset \mathbb{R}^d$ reflect the unit brane charge and its opposite anti-brane charge, while the normally framed cup- or cap-shaped bordisms witness their pairwise (dis-)appearance into/out of the vacuum.
Hence where the refinement of point particles to p-branes serves to *geometrize* both their interactions and their charges, both of these aspects are naturally unified in Cobordism cohomology theory.



Brane charge in K-theory and multiplicative genera. Regarded through the prism of stable homotopy theory, framed Cobordism splits up into a chromatic spectrum of spectra, a small part of which looks as follows:

$$
\begin{tikzcd}
 & & \text{framed} & \begin{array}{c}\text{quaternionic /}\\ \text{symplectic}\end{array} & \begin{array}{c}\text{complex /}\\ \text{unitary}\end{array} & \begin{array}{c}\text{real /}\\ \text{orthogonal}\end{array} & \\
 & & & & M\mathrm{SU} \arrow[r] \arrow[d] & M\mathrm{SO} \arrow[d] & \\
\text{Cohomotopy} & \mathbb{S} \arrow[r, equal] & M\mathrm{Fr} \arrow[r] & M\mathrm{Sp} \arrow[ur] \arrow[dr] \arrow[dd] & & & \text{Cobordism} \\
 & & & & M\mathrm{U} \arrow[r] \arrow[d] & M\mathrm{O} & \\
 & & & K\mathrm{O} \arrow[r] & K\mathrm{U} & & \text{K-theory}
\end{tikzcd}
$$

This reveals that brane charge quantization in Mf-theory is finer than but closely related to charge quantization in K-theory: The Conner-Floyd isomorphism ([CF66, Thm. 10.1][TK06, Thm. 6.35]) says that complex K-theory $K\mathrm{U}$ is the quotient of complex Cobordism cohomology $M\mathrm{U}$, obtained by identifying all submanifolds that do not "wrap compact dimensions" (see (19) below) with the index of their Dirac operator, i.e. with their Todd class. An analogous statement holds for $K\mathrm{O}$ and quaternionic Cobordism.

Therefore, and in view of unresolved issues (e.g. [dBD+02, §4.5.2, 4.6.5][Ev06, §8][FQ05, §1]) surrounding the popular but still conjectural statement that D-brane charge is quantized in K-theory, it is worthwhile to examine M-brane charge quantization in Mf-Cobordism theory.

Brane charge and Pontrjagin-Thom collapse. The above motivation of Cobordism cohomology as the natural home for p-brane charge, summarized in Table I, is in itself only a plausibility argument, just as is the traditional motivation ([Wi98, §3]) of K-theory as the natural home for D-brane charge. However, this physically plausible conclusion is rigorously *implied by Hypothesis H*, and hence is supported by and adds to the other evidence for that Hypothesis: This implication is the statement of *Pontrjagin's isomorphism*, which says that the operation of assigning to a normally framed closed submanifold its *asymptotic directed distance* function (traditionally known as the *Pontrjagin-Thom collapse construction*)

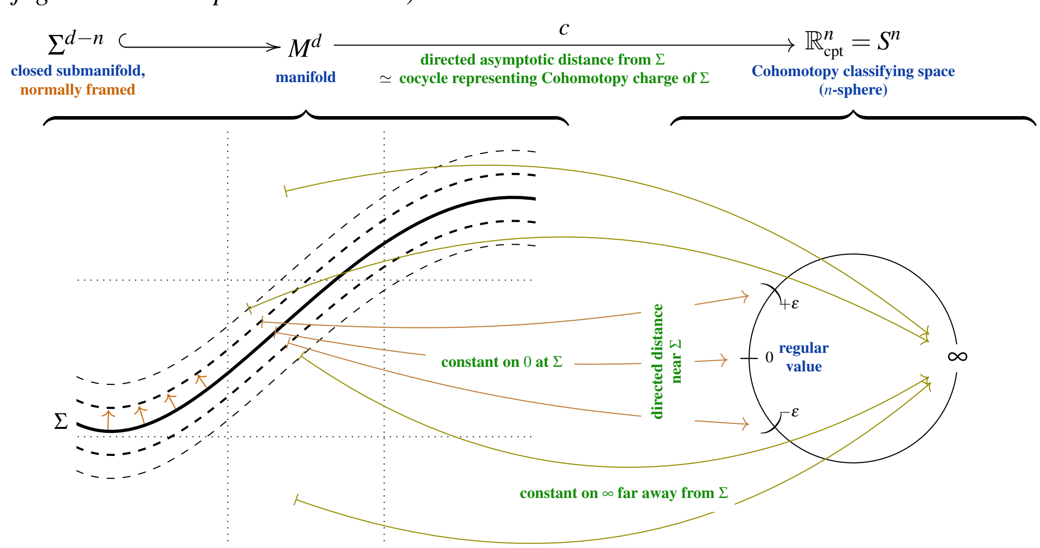


Figure D – The Pontrjagin construction. The *charge in Cohomotopy* of a manifold M^d, sourced by a normally framed closed submanifold Σ^{d-n}, is the homotopy class of the function that assigns directed asymptotic distance from Σ, measured along its normal framing.

identifies framed Cobordism with Cohomotopy, as non-abelian cohomology theories, over *closed manifolds* M^d:

$$
\begin{tikzcd}
\overset{\text{framed unstable } n\text{-Cobordism of } M^d}{\mathrm{Cob}^n_{\mathrm{Fr}}\big(M^d\big)} \arrow[rrr, bend left, "\text{assign Cohomotopy charge}"] & := \mathrm{NFramedSubmflds}_{d-n}\big(M^d\big)_{/\mathrm{brdsm}} \arrow[r, shift left, "\text{directed asymptotic distance}"] \arrow[r, phantom, "\simeq"] & \mathrm{Maps}\big(M^d, S^n\big)_{/\mathrm{hmtpy}} \arrow[l, shift left, "\text{pre-image of regular value } 0"] & =: \overset{\text{unstable } n\text{-Cohomotopy of } M^d}{\pi^n\big(\underset{\substack{\text{closed}\\ \text{manifold}}}{M^d}\big)} . \arrow[lll, bend left, "\text{reconstruct submanifold from its charge}"]
\end{tikzcd}
\tag{16}
$$

Charges vanishing at infinity and reduced cohomology. In the case that $M^d = \mathbb{R}^d$ is contractible (and hence, in particular, not closed), the homotopy-invariance of Cohomotopy theory immediately implies that *all Cohomotopy charges vanish* on such M^d, even though there are non-trivial cobordism classes of normally framed submanifolds in $\mathbb{R}^d$. But translating the trivialization of their Cohomotopy charge back through the reconstruction map (16) shows that this is a result of these submanifolds being allowed to "escape to infinity", carrying their charges away with them. Hence the sensible notion of brane charge quantized in any pointed cohomology theory A is that which is constrained not to escape to infinity, hence to *vanish at infinity*:

A *charge vanishes at infinity* in a non-abelian cohomology theory $A(-)$ (6) with a zero-element $0_A \in A$ in its classifying space, if it is represented by a cocycle map $X \xrightarrow{c} A$ that extends with value 0_A to the *one-point compactification* of X (1):

$$\underset{\substack{\text{one-point-}\\\text{compactification}}}{M_{\mathrm{cpt}}} := \left(\begin{array}{l} M \sqcup \{\infty_X\}, \text{ topologized such that :} \\ M \hookrightarrow M \sqcup \{\infty_X\} \text{ is open embedding} \\ O_\infty \text{ is open nbrhd} \Leftrightarrow M \setminus O_\infty \text{ is cmpct} \end{array} \right) \quad \text{i.e.:} \quad \begin{array}{lcl} \text{map } X \xrightarrow{c} A\text{...} & \leftrightarrow & \text{extension } M_{\mathrm{cpt}} \xrightarrow{\widetilde{c}} A \text{ ...} \\ \text{...vanishes at infinity} & \Leftrightarrow & \text{...equals } 0_A \text{ at } \infty_M \\ \text{...compactly supported} & \Leftrightarrow & \text{...equals } 0_A \text{ on nbrhd of } \infty_M \end{array} \tag{17}$$

The *reduced cohomology* $\widetilde{A}(-)$ of $X := M_{\mathrm{cpt}} \wedge \mathbb{R}^{p,1}_{+}$ is that given by cocycles and coboundaries which vanish at infinity:

$$\overset{\substack{\text{reduced}\\ A\text{-cohomology}}}{\widetilde{A}^n(X)} := \overset{\substack{\text{maps that take}\\ \infty \in X \text{ to } 0 \in A}}{\pi_0 \mathrm{Maps}^{*/}(X, A)}, \qquad \overset{\substack{\text{reduced}\\ \text{Cohomotopy}}}{\widetilde{\pi}^n(X)} := \overset{\substack{\text{maps that take}\\ \infty \in X \text{ to } 0 \in S^n \;\; \text{defined as } \infty \in \mathbb{R}^n_{\mathrm{cpt}}}}{\pi_0 \mathrm{Maps}^{*/}\big(X, \mathbb{R}^n_{\mathrm{cpt}}\big)}. \tag{18}$$

Wrapped branes and the Pontrjagin theorem. In terms of reduced Cohomotopy (18), the Pontrjagin isomorphism (16) takes the following form, now valid for possibly non-compact manifolds M^d (see Prop. 3.24, Rem. 3.25):

$$\overset{\substack{\text{framed unstable}\\ n\text{-Cobordism of } M^d}}{\mathrm{Cob}^n_{\mathrm{Fr}}\big(M^d\big)} \underset{\text{find worldvolume of given charge}}{\overset{\text{assign Cohomotopy charge}}{\rightleftarrows}} \simeq \overset{\substack{\text{reduced}\\ n\text{-Cohomotopy}}}{\widetilde{\pi}^n\Big(M^d_{\mathrm{cpt}} \wedge \mathbb{R}^{p,1}_{+}\Big)}. \tag{19}$$

$$\begin{array}{rccc} \text{branes wrapped on:} & \Sigma^{d-n} & & M^d \\ & \times & \subset & \times \\ \text{extended along:} & \mathbb{R}^{p,1} & & \mathbb{R}^{p,1} \end{array} \qquad \begin{array}{l} \text{charge vanishes at infinity} \\ \text{along these... ...but not along these dimensions} \end{array}$$

On the right of (19) we have included a contractible factor, for conceptual completeness: By the homotopy-invariance of Cohomotopy this does not affect the nature of the charges, which we may interpret as saying that: *The Cohomotopy charge of a brane filling a flat space(-time) factor* $\mathbb{R}^{p,1}$ *and wrapped on a closed submanifold* Σ *inside the remaining spatial dimensions* M^d *depends only on this compact factor* (with its normal framing).

We turn to the archetypical class of examples of this situation:

Probe branes near black brane horizons and homotopy groups of spheres. With the above formulation we may formally grasp the spacetime topologies supporting charges of *probe p-branes* near horizons of *solitonic b-branes* – generalizing the classical case (from p. 3) of magnetic monopoles (solitonic 0-branes) and magnetic flux lines (probe 1-branes):

(a) The *near horizon topology* of (infinitely extended, coincident) *solitonic b-branes* in an 11-dimensional spacetime is the complement $\mathbb{R}^{10,1} \setminus \mathbb{R}^{b,1}$ of their (geometrically singular) worldvolume inside an ambient spacetime chart; and the corresponding *asymptotic boundary of spacetime* is, topologically, the subspace $\simeq \mathbb{R}^{b,1} \times S^{9-b}$ of any fixed radius r (geometrically a limit $r \to 0$) from the singular locus.

(b) The *effective spacetime topology* for (infinitely extended, not necessarily coincident) *probe p-branes* localized (i.e. with charges vanishing at infinity) near the horizon of such solitonic p-branes is the one-point compactification (17) of their transverse space $\mathbb{R}^{p-b} \times S^{9-p}$ inside this near-horizon domain:

$$\overset{\substack{\text{ambient}\\ \text{spacetime}}}{\mathbb{R}^{10,1}} \setminus \underset{\substack{\text{singular } b\text{-brane}\\ \text{worldvolume}}}{\mathbb{R}^{b,1}} \simeq \overset{\substack{\text{asymptotic}\\ \text{worldvolume}}}{\mathbb{R}^{b,1}} \times \underset{\substack{\text{sphere around}\\ b\text{-brane}}}{S^{9-b}} \times \overset{\substack{\text{radial}\\ \text{distance}}}{\mathbb{R}_{\mathrm{rad}}} \underset{\substack{\text{pass to}\\ \text{asymptotic}\\ \text{boundary}}}{\rightsquigarrow} \overset{\substack{\text{asymptotic boundary}\\ \text{of } b\text{-brane spacetime}}}{\mathbb{R}^{b,1} \times S^{9-b}} \simeq \overset{\substack{\text{transversal space to}\\ \text{probe } p\text{-branes near black } b\text{-brane}}}{\underset{\substack{\text{probe brane}\\ \text{worldvolume}}}{\mathbb{R}^{p,1}} \times \mathbb{R}^{b-p} \times S^{9-b}} \underset{\substack{\text{localize}\\ \text{probe } p\text{-branes}\\ \text{near black } b\text{-brane}}}{\rightsquigarrow} \overset{\substack{\text{asymptotic boundary of } b\text{-brane spacetime}\\ \text{seen by localized } p\text{-brane probes}}}{\mathbb{R}^{p,1} \wedge \underset{\substack{\text{space transversal to } p\text{-branes}\\ \text{including transversal } \infty}}{\big(\mathbb{R}^{b-p} \times S^{9-b}\big)_{\mathrm{cpt}}}}. \tag{20}$$

Now, since the reduced 4-Cohomotopy charge (18) of the transversal space is identified (19) under the Pontrjagin isomorphism (16) with $(9-4=5)$-brane worldvolumes extended along $\mathbb{R}^{p,1}$ and wrapped on (cobordism classes of normally framed) submanifolds Σ of the transversal asymptotic boundary (20):

$$
\begin{array}{ccc}
\underbrace{\big(\mathrm{M}p@\mathrm{M}b\big)\ \mathrm{BndrCharges}}_{\substack{\text{charge lattice (under } \textit{Hypothesis H}\text{) of} \\ \text{probe } p\text{-branes at } b\text{-brane horizons}}} := \widetilde{\pi}^4\Big(\mathbb{R}^{p,1}_{+} \wedge \big(\mathbb{R}^{b-p}\times S^{9-b}\big)_{\mathrm{cpt}}\Big) & \xrightarrow[\substack{\text{associate with Cohomotopy charge} \\ \text{the wrapped brane sourcing it}}]{\simeq} & \big\{\mathbb{R}^{p,1}\big\}\times \mathrm{Cob}^4_{\mathrm{Fr}}\big(\mathbb{R}^{b-p}\times S^{9-b}\big) \\
[c]\left(\substack{\text{representative } c \\ \text{factoring through } \mathrm{pr}_2 \\ \text{and regular at } 0\in S^4}\right) & \longmapsto & \big[\mathbb{R}^{p,1}\times\Sigma\big] := \big[c^{-1}(\{0\})\big]
\end{array}
\qquad (21)
$$

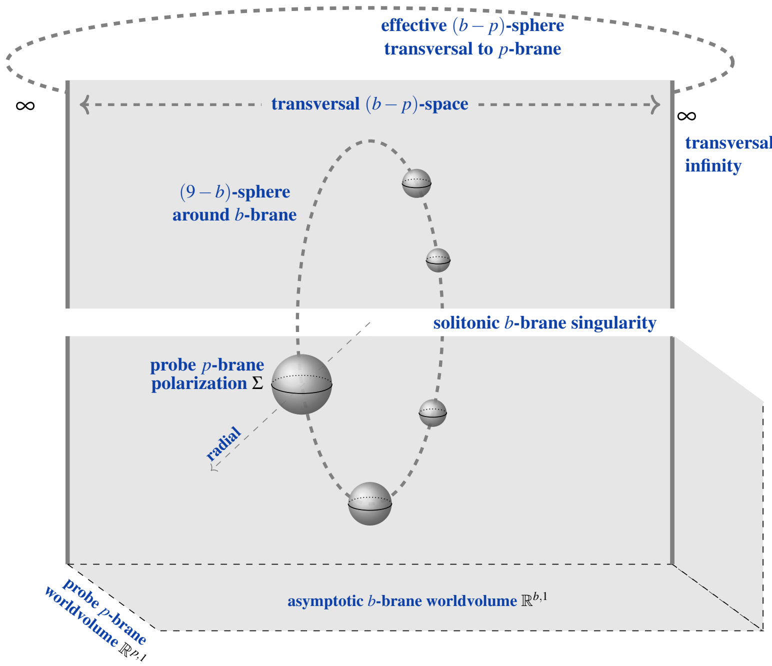
we interpret, under *Hypothesis H*, the product manifold

$$\mathbb{R}^{p,1}\times\Sigma \subset \mathbb{R}^{p,1}\times\big(\mathbb{R}^{b-p}\times S^{9-b}\big)$$

of dimension

$$
\begin{aligned}
\dim\big(\mathbb{R}^{p,1}\times\Sigma\big) &= (p+1)+\big((9-p)-4\big) \\
&= 5+1
\end{aligned}
$$

as the probe 5-brane worldvolume which carries the given Cohomotopy charge $[c]$, and hence interpret Σ as the "brane polarization" (in the sense of [My03, §4]) of the probe p-brane worldvolume, puffing it up to a 5-brane worldvolume, due to the background 5-brane charge $[c]$.



Observe that these Cohomotopy charges (21) subsume low unstable *homotopy groups of spheres*:

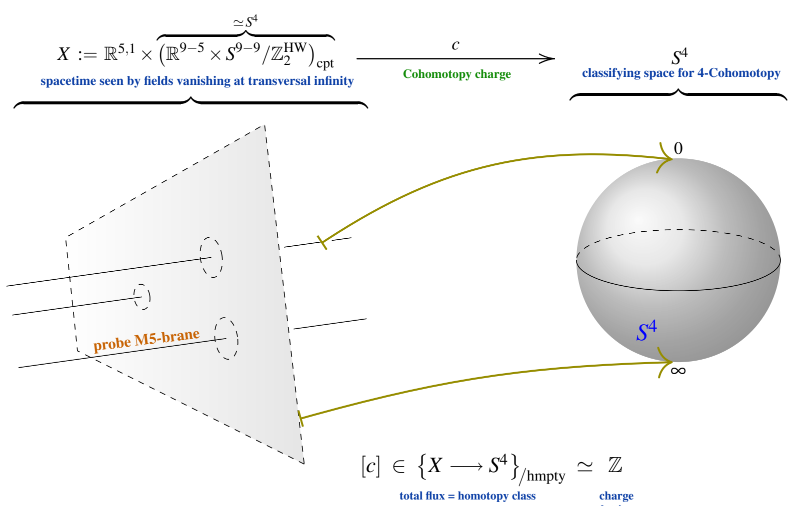
(i) $b=p$ i.e.: probe p-branes near their own horizon $\Rightarrow$ $\big(\mathrm{M}p@\mathrm{M}p\big)$ BndrCharges $\simeq \pi_{9-p}(S^4)$

(ii) $b=9$ i.e.: probe p-branes near an MO9-plane $\Rightarrow$ $\big(\mathrm{M}p@\mathrm{M}9\big)$ BndrCharges $\simeq \pi_{9-p}(S^4)\times\mathbb{Z}_2^{\mathrm{HW}}$

$p=$	0	1	2	3	4	5	6	7	8	9
$\pi_{9-p}(S^4)=$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}\oplus\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	0

(22)

M5-Brane probes at MO9-planes and the Hopf degree theorem is the special case $\boxed{b=9 \text{ and } p=5}$ of (21). This was discussed, under Hypothesis H, in [SS19a][BSS19][SS20c] in the generality of orbifold geometry and equivariant Cohomotopy ([SS20b]): Here the free $\mathbb{Z}_2^{\mathrm{HW}}$-action identifies $S^{9-b}=S^0$ with a point and leaves a transversal space which is an ADE-orbifold of $\mathbb{R}^{9-p}=\mathbb{R}^4$. In this transversal space the probe M5-branes appear, under the Pontrjagin isomorphism (eq. (19)), as points (i.e. without polarization, as expected).



Regarding here n-Cohomotopy of n-manifolds as ordinary n-cohomology, via the Hopf degree theorem (Ex. 2.5 below), this situation is a higher-dimensional analogue of the vortex strings seen in superconductors (p. 3).

M2-Brane polarization and the quaternionic-Hopf fibration. The special case $\boxed{b = 2 \text{ and } p = 2}$ of (20) with $M^7 = S^7$ is that of the near-horizon geometry of black M2-branes ([DS91][DGT95][Pa83], i.e. Freund-Rubin compactifications [FR80] of 11-dimensional supergravity on S^7) with M2-brane probes. Similarly, the case $\boxed{b = 9 \text{ and } p = 2}$ with $X \simeq S^7 \vee S^7$ (by Prop. 3.8) is that of M2-branes probing the vicinity of an MO9-plane in heterotic M-theory. In both cases the set (22) of cohomotopical brane charges contains an integer summand, which is generated by the *quaternionic Hopf fibration* $[h_{\mathbb{H}}] = 1 \in \mathbb{Z} \subset \pi^4(S^7)$. Since this $h_{\mathbb{H}}$ is an $\mathrm{SU}(2) \simeq S^3$-fiber bundle over S^4, the brane worldvolume that corresponds to this cohomotopical unit charge under the Pontrjagin isomorphism (19) is a 6-manifold wrapped on a 3-sphere:

$$
\begin{tikzcd}
\underset{\text{M5-brane wrapped on 3-sphere}}{\overset{\text{polarized M2-brane worldvolume /}}{}} \overset{\substack{\text{unit probe} \\ \text{brane worldvolume} \\ \text{under Hypothesis H}}}{\mathbb{R}^{2,1} \times S^3} \arrow[r, hook, "\substack{\text{Pontrjagin} \\ \text{construction (19)}}", "\mathrm{fib}(h_{\mathbb{H}})"'] \arrow[d, "\mathrm{id} \times h_{\mathbb{C}}"] & \mathbb{R}^{2,1} \times S^7 \arrow[r, "\simeq", "\mathrm{hmtpy}"'] \arrow[d, "\substack{\text{M/IIA circle} \\ \text{fibration}}"'] \arrow[rr, bend left, "\substack{\text{unit M2-brane Page charge} \\ \text{under Hypothesis H}}"] & S^7 \arrow[r, "h_{\mathbb{H}}", "\text{quaternionic Hopf fibration}"'] \arrow[d, "h_{\mathbb{C}}", "\substack{\text{complex} \\ \text{Hopf fibration}}"'] & S^4 \arrow[d, equal] \\
\underset{\text{D4-brane wrapped on 2-sphere}}{\overset{\text{polarized D2-brane worldvolume /}}{}} \mathbb{R}^{2,1} \times S^2 \arrow[r, hook, "\mathrm{fib}(t_{\mathbb{H}})"] & \mathbb{R}^{2,1} \times \mathbb{C}P^3 \arrow[r, "\mathrm{hmtpy}", "\simeq"'] \arrow[rr, bend right, "\substack{\text{unit D2-brane charge} \\ \text{under Hypothesis H}}"'] & \mathbb{C}P^3 \arrow[r, "t_{\mathbb{H}}"', "\text{Atiyah-Penrose fibration}"] & S^4
\end{tikzcd}
\tag{23}
$$

Noticing that unit flux through the S^7 around a black M2 must be M2-brane Page charge (emphasized in [PTW15, (1.2)], we discuss this below in §2.7) induced from non-trivial but cohomologically trivialized 4-flux (discussed below in §2.4) this cohomotopical analysis neatly matches the M$(2 \mapsto 5)$-brane polarization phenomenon expected in the string theory literature [Be00][BN00][BW04][AMS08][GRRV08][BGKM14] (review in [BLMP13, §6]), notably including the expectation [NPR09][BLMP13, §6.4.2] that the S^3-polarization of the M2-branes is, under M/IIA-duality, fibered over the S^2-polarization of D2-branes via the Hopf map $h_{\mathbb{C}}$: under Hypothesis H and via Pontrjagin's isomorphism (19), this is implied by Rem. 3.28 below. In particular, this means that the S^3s seen here in $\mathbb{S}_3 \simeq (M\mathrm{Fr})_3$ do correspond to the fuzzy funnel/spheres seen via Hypothesis H in [SS19b].

We discuss further aspects of this situation in §2.7, §2.8 and §2.9 below, see also the conclusion in Remark 4.1.

D6/D8-Branes in Type I' and the May-Segal theorem. The special case $\boxed{b = 9 \text{ and } p = 6, 8}$ of (21) is that of probe 6-branes and probe 8-branes in the vicinity of an MO9-plane, hence of D6/D8-branes in the vicinity of an O8-plane in strongly coupled Type I' string theory, as discussed in [SS19b]: By (22) the Cohomotopy charges of these D6/D8-branes as such vanish identically; which is in line with the fact that only the $D(\leq 5)$-branes are meant to have M-theoretic lifts to fundamental M-branes (to M2/M5-branes). The nature of D6/D8-branes in M-theory must be more subtle: Indeed, beyond the mere Cohomotopy set lies the full *Cohomotopy cocycle space*:

$$
\overset{\substack{n\text{-Cohomotopy} \\ \text{cocycle space}}}{\widetilde{\boldsymbol{\pi}}^n(X)} := \mathrm{Maps}^{*/}\big(X, S^n\big)\,, \qquad \overset{n\text{-Cohomotopy set}}{\widetilde{\pi}^n(X)} = \pi_0 \widetilde{\boldsymbol{\pi}}^n(X)\,,
$$

a higher homotopy type of which the Cohomotopy classes are only the connected components; hence which, under *Hypothesis H*, is a *moduli stack of brane configurations* with their gauge- and higher gauge-of-gauge transformations, of which the plain Cohomotopy charge (21) only captures the lowest gauge-equivalence classes.

The analog of the Pontrjagin theorem (19) in this situation is the *May-Segal theorem*, which identifies these Cohomotopy cocycle spaces (not just with single brane configurations but) with the *moduli spaces of configurations* of D6/D8-branes as per their positions in their transversal space $\mathbb{R}^{9-p}$

$$
\begin{tikzcd}
\overset{\substack{\text{configuration space of} \\ \text{points in } \mathbb{R}^{9-p} \text{ carrying labels in } \mathbb{R}^{p-5}_{\mathrm{cpt}}}}{\mathrm{Conf}\big(\mathbb{R}^{9-p}; \mathbb{R}^{p-5}_{\mathrm{cpt}}\big)} \arrow[r, bend left, "\text{assign Cohomotopy charge}"] \arrow[r, phantom, "\simeq_{\mathrm{hmtpy}}"] & \overset{\text{4-Cohomotopy cocycle space}}{\widetilde{\boldsymbol{\pi}}^4\big(\mathbb{R}^{9-p}_{\mathrm{cpt}}\big)} \arrow[l, bend left, "\text{find worldvolume of given charge}"]
\end{tikzcd}.
\tag{24}
$$

In a negative-dimensional parallel to the $(p \mapsto 5)$-brane polarization process via Pontrjagin's theorem in (21), the May-Segal theorem (24) asserts that the points/branes in these configurations carry labels in $\mathbb{R}^{p-5}_{\mathrm{cpt}}$, which is just the modulus for an M5-brane positioned *inside* the p-brane (possibly at infinity).

For detailed discussion of Hypothesis H in this low-codimension sector of M-brane physics we refer to [SS19b].

Branes probing bulk spacetime and the suspension homomorphism. The unstable framed Cobordism sets (16) naturally appear in sequences where more and more ambient space dimensions are made available for the bordisms to propagate through (see *Figure S*):

$$
\begin{tikzcd}
 & \Sigma^{d-n} \subset M^d \arrow[r, hook, "{(\mathrm{id},0)}"] & M^d \times \mathbb{R}^1 \arrow[r, hook, "{(\mathrm{id},0)}"] & M^d \times \mathbb{R}^2 \arrow[r, hook, dotted] & M^d \times \mathbb{R}^\infty & \\
\substack{\text{unstable}\\ \text{framed}\\ \text{Cobordism}} & \mathrm{Cob}^n_{\mathrm{Fr}}\big(M^d\big) \arrow[r, "\sigma"] \arrow[d, dashed, "\text{Cohomotopy charge map}"'] & \mathrm{Cob}^{n+1}_{\mathrm{Fr}}\big(M^d \times \mathbb{R}^1\big) \arrow[r, "\sigma"] \arrow[d, "\simeq"] & \mathrm{Cob}^{n+2}_{\mathrm{Fr}}\big(M^d \times \mathbb{R}^2\big) \arrow[r, dotted] \arrow[d, "\simeq"] & \widetilde{M\mathrm{Fr}}^n\big(M^d_{\mathrm{cpt}}\big) \arrow[d, equal, "\text{stable Pontrjagin-Thom isom.}"'] & \substack{\text{stable}\\ \text{framed}\\ \text{Cobordism}} \\
\substack{\text{unstable}\\ \text{Cohomotopy}} & \widetilde{\pi}^n\big(M^d_{\mathrm{cpt}}\big) \arrow[r, "(-)\wedge S^1"] & \widetilde{\pi}^{n+1}\big((M^d \times \mathbb{R}^1)_{\mathrm{cpt}}\big) \arrow[r, "(-)\wedge S^1"] & \widetilde{\pi}^{n+1}\big((M^d \times \mathbb{R}^2)_{\mathrm{cpt}}\big) \arrow[r, dotted] & \widetilde{\mathbb{S}}^n\big(M^d_{\mathrm{cpt}}\big) & \substack{\text{stable}\\ \text{Cohomotopy}}
\end{tikzcd}
\tag{25}
$$

make more ambient spatial dimensions available to bordisms $\longrightarrow$

suspension homomorphism $\longrightarrow$

Seen under the Cohomotopy charge map (19), this is equivalently the sequence of *suspension homomorphisms* on reduced Cohomotopy sets, given by forming the smash product $c \wedge S^1$ of Cohomotopy cocycle maps $X \xrightarrow{c} S^n$ with $\mathbb{R}^1_{\mathrm{cpt}} = S^1$:

$$
\substack{\text{suspension}\\ \text{homomorphism}} \;:\; \widetilde{\pi}^n\big(X^d\big) = \pi_0 \mathrm{Maps}^{*/}\big(X^d, S^n\big) \xrightarrow{(-)\wedge S^1} \pi_0 \mathrm{Maps}^{*/}\big(\Sigma X^d, S^{n+1}\big) = \widetilde{\pi}^{n+1}\big(\Sigma X^d\big)\,. \tag{26}
$$

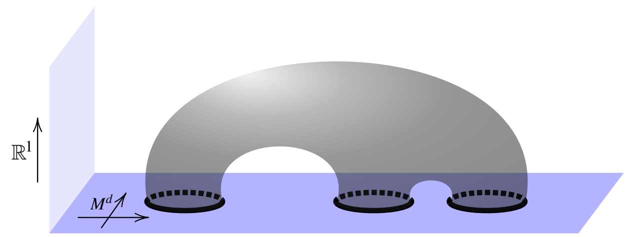


Figure S – The suspension homomorphism on Cobordism $\mathrm{Cob}^n_{\mathrm{Fr}}\big(M^d\big) \xrightarrow{\sigma} \mathrm{Cob}^{n+1}_{\mathrm{Fr}}\big(M^d \times \mathbb{R}^1\big)$ takes (normally framed) submanifolds $\Sigma^{d-n} \subset M^d$ to their equivalence classes under (normally framed) bordisms that may explore an extra bulk dimension.

Holography and Freudenthal suspension theorem. Here Figure S reflects the qualitative picture of
(a) Polyakov's holographic principle, where QCD quarks are boundaries of flux strings that probe into a hidden bulk dimension; which,
(b) in M-theoretic holography translates to bulk $p = 2$-branes with boundaries constrained to asymptotic $b = 5, 9$-brane boundaries. Indeed, the *Freudenthal suspension theorem* says that the suspension homomorphisms (26) become *iso*morphisms after a finite number of k steps:

$$
\underbrace{n+k}_{\text{codimension}} \;>\; \underbrace{(d+k)/2+1}_{\text{ambient dimension}} \qquad \Rightarrow \qquad \mathrm{Cob}^{n+k}_{\mathrm{Fr}}\big(X^d \times \mathbb{R}^k\big) \overset{\sigma}{=\!=\!=} \mathrm{Cob}^{n+k+1}_{\mathrm{Fr}}\big(X^d \times \mathbb{R}^{k+1}\big)
$$

from which stage on the sequence (25) of unstable framed Cobordism/Cohomotopy sets *stabilize* to the *stable* framed Cobordism cohomology group, equivalently the stable Cohomotopy group of M^d_{cpt}. This implies for the probe $p \geq 2$-brane charges in black b-brane backgrounds, from (21) above, that their charge stabilizes after revealing $k = 1$ extra bulk dimensions, hence that; under *Hypothesis H*:

Allowing $(p \geq 2)$-brane interactions to probe the radial bulk direction $\mathbb{R}^1_{\mathrm{rad}}$ of the ambient black b-brane spacetime abelianizes their Cohomotopy charges from unstable Cohomotopy to stable Cohomotopy.

$$
\begin{tikzcd}
 & \substack{\text{charges (21) of probe } p\text{-branes at}\\ \text{black } b\text{-brane horizons}} \\[-2em]
 & \big(\mathrm{M}p@\mathrm{M}b\big)\mathrm{BndrCharges} \arrow[rr, "\substack{\text{allow } p\text{-brane interactions}\\ \text{to probe into the bulk}}"] \arrow[d, equal, "\text{Hypothesis H}"] & & \big(\mathrm{M}p@\mathrm{M}b\big)\mathrm{BulkCharges} \arrow[d, equal] & \\
\substack{\text{unstable}\\ \text{Cohomotopy}} & \widetilde{\pi}^4\Big(\big(\mathbb{R}^{b-p} \times S^{9-b}\big)_{\mathrm{cpt}}\Big) \arrow[rr, dashed, bend left, "\text{stabilization (abelianization)}"] \arrow[d, equal, "\text{Pontrjagin Thm.}"] & \widetilde{\pi}^{4+1}\Big(\big(\mathbb{R}^{b-p} \times S^{9-b} \times \mathbb{R}^1_{\mathrm{rad}}\big)_{\mathrm{cpt}}\Big) \arrow[r, equal, "p\geq 2", "\text{Freudenthal Thm.}"'] \arrow[d, equal, "\text{Pontrjagin Thm.}"] & \widetilde{\mathbb{S}}^4\Big(\big(\mathbb{R}^{b-p} \times S^{9-b}\big)_{\mathrm{cpt}}\Big) \arrow[d, equal, "\text{Pontrjagin-Thom Isom.}"] & \substack{\text{stable}\\ \text{Cohomotopy}} \\
\substack{\text{unstable}\\ \text{framed}\\ \text{Cobordism}} & \mathrm{Cob}^4_{\mathrm{Fr}}\big(\mathbb{R}^{b-p} \times S^{9-b}\big) \arrow[r, "\sigma", "\substack{\text{allow cobordisms}\\ \text{to probe into the bulk}}"'] & \mathrm{Cob}^{4+1}_{\mathrm{Fr}}\big(\mathbb{R}^{b-p} \times S^{9-b} \times \mathbb{R}_{\mathrm{rad}}\big) \arrow[r, equal, dashed] & \widetilde{M\mathrm{Fr}}^4\Big(\big(\mathbb{R}^{b-p} \times S^{9-b}\big)_{\mathrm{cpt}}\Big) & \substack{\text{stable}\\ \text{framed}\\ \text{Cobordism}}
\end{tikzcd}
\tag{27}
$$

Combining all this, we arrive at: **M-Theory compactified on K3 and the third stable stem.** We saw in (23) that, under *Hypothesis H*, M2-branes probing their own asymptotic horizon geometry $\mathbb{R}^{2,1}\times S^7$ appear polarized to M5-brane worldvolumes $\mathbb{R}^{2,1}\times S^3$ wrapped on 3-spheres, whose cohomotopical brane charges (19) are the non-torsion elements $n\in\mathbb{Z}\subset\pi_7(S^4)$. But moreover, only a quotient of these asymptotic brane charges is visible inside the bulk spacetime, namely only their image $n \bmod 24$ in the third stable homotopy group of spheres $\mathbb{Z}_{24}\simeq\mathbb{S}_3$ (52). Under the Pontrjagin isomorphism (19) this means that in the vicinity of 24 probe branes the bulk spacetime $\mathbb{R}^{2,1}\times(\mathbb{R}^8\setminus\{0\})$ admits a compactification

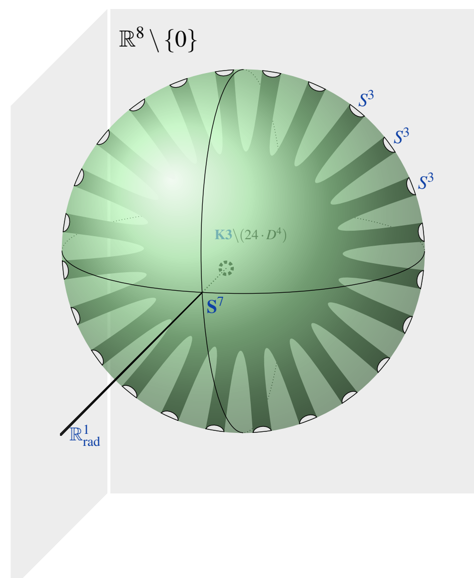
$$
\left.\begin{array}{l} Y^4 = \\ \mathrm{K3}\setminus\sqcup_{24}D^4 \end{array}\right\}
\Rightarrow
\begin{tikzcd}
\underset{\text{spacetime near probe M-branes}}{\overset{Y^4\text{-compactification of}}{\mathbb{R}^{5,1}\times\mathbb{R}^1\times Y^4}} \arrow[r, hook, "\text{Pontrjagin construction}"] & \underset{}{\overset{\text{bulk spacetime near black M2-branes}}{\mathbb{R}^{2,1}\times\mathbb{R}^8\setminus\{0\}}} \arrow[r, hook] & \mathbb{R}^{2,1}\times S^7\times(\mathbb{R}^1_{\mathrm{rad}})_{\mathrm{cpt}} \arrow[r, bend left, "0"] \arrow[r, bend right, "\Sigma(24\cdot[h_{\mathbb{H}}])"'] \arrow[r, phantom, "\vdash Y^4\ \Downarrow"] & \Sigma S^4 \\
\underset{\{1,\cdots,24\}}{\sqcup}\underset{\text{probe M2-branes (polarized to M5s)}}{\left(\mathbb{R}^{2,1}\times S^3\right)}\times\underset{\text{their vicinity (tubular neighborhood)}}{\mathbb{R}^4} \arrow[u, hook, "\text{bdry}"] \arrow[r, hook, "\text{Pontrjagin construction}"] & \underset{\text{asymptotic boundary near black M2-branes}}{\mathbb{R}^{2,1}\times S^7} \arrow[u, hook] \arrow[rr, "\text{24 units of cohomotopical brane charge}", "24\cdot[h_{\mathbb{H}}]"'] & & S^4
\end{tikzcd}
\quad (28)
$$

24-punctured K3-surface

trivialized in bulk

on a 4-dimensional compact manifold Y^4 with 24 punctures – namely given by the tubular neighborhood $\simeq Y^4\times\mathbb{R}^4$ of a normally framed 4d cobordism between the 24 S^3s. But that Y^4 is given, up to higher cobordism, by the 24-punctured K3-manifold (e.g. [WX10, §2.6][SP17]):

$$
\begin{tikzcd}
 & \pi^4(S^7)=\pi_7(S^4) \ni (n,0) \arrow[dl, "\text{bulk interactions / stabilization}"'] \arrow[d, equal] \\
 & \mathrm{Cob}^4_{\mathrm{Fr}}(S^7) \ni n\cdot[S^3=\mathrm{SU}(2)] \arrow[dl, mapsto] \\
\pi^{4+1}\left(S^7\times(\mathbb{R}^1_{\mathrm{rad}})_{\mathrm{cpt}}\right)=\mathbb{S}_3 \ni n \bmod 24 \arrow[d, equal] & \\
\mathrm{Cob}^{4+1}_{\mathrm{Fr}}\left(S^7\times\mathbb{R}^1_{\mathrm{rad}}\right)=M\mathrm{Fr}_3 \ni n\cdot[S^3] \bmod \left(\mathrm{K3}\setminus 24\cdot D^4\right) &
\end{tikzcd}
$$

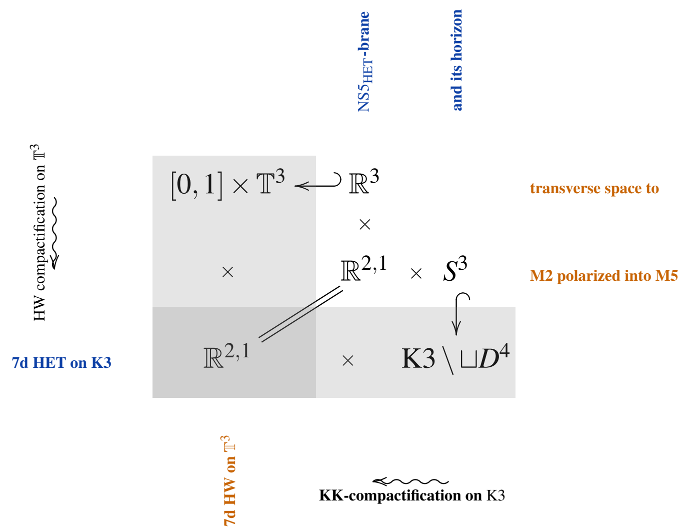


In conclusion:

> *In the bulk spacetime of M-theory near black M2-branes, Hypothesis H implies that the total charge of 24 probe M-branes cancels out, their flux through the 24 encircling 3-spheres being absorbed along the compact K3-fiber of a spontaneous KK-compactification of spacetime near the probe branes.*

Of course, K3-compactifications of M-theory have received much attention, due to a famous argument that they should be dual to heterotic string theory compactified on $\mathbb{T}^2$ [HT95, §6][Wi95, §4]. However, there seems to have been no argument why spontaneous K3-compactifications of M-theory should be dynamically preferred in the first place, hence why the theory would share the theorists' interest in these compactifications. The above conclusion suggests that *Hypothesis H* provides this missing aspect of M-theory: The 24-punctured K3 fiber arises in M-theory near black M2-branes as the framed cobordism which is, under Pontrjagin's theorem, the trivialization of 24 units of cohomotopical M-brane that is implied in the bulk spacetime by Freudenthal's theorem.

Tadpole cancellation of 24 $\mathrm{M5}_{\mathrm{HET}}$-branes via GS mechanism on K3 and the Poincaré-Hopf theorem. While we arrived at the above picture from consideration of M2-branes polarized into M5-branes wrapped on S^3s, it is, in the end, only the measurement of 3-flux through the S^3s that signifies brane charges, while the reconstruction of the brane worldvolumes sourcing these charges depends on the spacetime perspective: If we think of the above configuration from a different perspective, where the $\mathbb{R}^1$-factor in the top left of (28) is interpreted as going along a Hořava-Witten fiber, then the same S^3s appear as encircling $\mathrm{NS5}_{\mathrm{HET}}$-branes in heterotic string theory on K3.



Indeed, heterotic string theory on K3 has been argued to require (in the absence of gauge flux) precisely 24 NS5-branes transverse the the K3, in order for their tadpoles along the compact fiber to cancel: In the traditional derivation ([Sch97, p. 50][Jo98, p. 30][CK19, §1.1]) this follows from the differential relation encoding the Green-Schwarz mechanism, which requires that the H_3-flux density of the $\mathrm{NS5}_{\mathrm{HET}}$-branes trivializes the fractional Pontrjagin 4-form $\frac{1}{2}p_1(\nabla)$ of the compact space, which for a complex surface like K3 is its Euler form $\chi_4(\nabla)$:

$$\underbrace{dH_3 = \tfrac{1}{2}p_1(\nabla)}_{\substack{\text{Green-Schwarz mechanism} \\ \text{(for vanishing gauge flux } F_2\text{)}}} \underbrace{= \chi_4(\nabla)}_{\substack{\text{on complex} \\ \text{surface}}} \in \Omega^4_{\mathrm{dR}}\big(\mathrm{K3} \setminus n \cdot D^4\big) \quad \xRightarrow{\substack{\text{Poincaré} \\ \text{lemma}}} \quad \underbrace{n \cdot \int_{S^3} H^3}_{\substack{\text{charge of} \\ \text{enclosed } \mathrm{NS5}_{\mathrm{HET}}}} = \underbrace{\int_{\mathrm{K3}} \chi_4(\nabla) = 24}_{\text{Euler number of K3}}. \qquad (29)$$

(We highlight in passing that making rigorous even this traditional derivation in itself already requires appeal to Cohomotopy: Namely one needs to prove that an H_3-form on $\mathrm{K3} \setminus 24 \cdot D^4$ may actually be constructed with the required properties (the differential relation $dH_3 = \chi_4(\nabla)$ subject to the normalization $\int_{S^3} H^3 = 1$): This follows by appeal to the Poincaré-Hopf theorem to obtain a cocycle in the *J-twisted* 3-Cohomotopy (as in [FSS19b, §2.5]) on $\mathrm{K3} \setminus 24 \cdot D^4$, whose de Rham image under the cohomotopical character map [FSS20c] is then guaranteed to satisfy (29) (by [FSS19b, Prop. 2.5]).

The constraint that branes transverse to a K3 must appear in multiples of 24, is also known for D7-branes in F-theory compactified on elliptically fibered K3 [Sen96, p. 5][Va96, p. 5][Le99, p. 6][DPS14] in order for their total charges in the compact dimensions to cancel out [De08, p. 34]. Under T-duality this is expected to imply the analogous statement for D6-branes in IIA-theory [Va96, Footn. 2]. While the mechanism for D7-branes is superficially rather different from the above argument (29) for NS5-branes, it has recently been argued that both correspond to each other under suitable stringy dualities ([BBLR18, §III], following analysis in [AM97]).

H_3-Flux and Toda brackets in Cohomotopy. Hence it remains to see how a 3-flux H_3 satisfying a Green-Schwarz condition (29) arises from first principles under *Hypothesis H*. This discussion occupies most of the following sections:

In §2.4 we see the 3-flux appear, charge-quantized via the Adams e-invariant on the stable homotopy groups of spheres.

In §2.6 we see its geometric incarnation in the form of a Green-Schwarz mechanism on cobordisms, namely as Conner-Floyd's formulation of the e-invariant.

In §2.9 we see the more general Green-Schwarz mechanism with gauge flux in the form of heterotic line bundles.

In fact, in its usual incarnation in ordinary cohomology we had derived the H_3-flux and its Green-Schwarz mechanism from Hypothesis H already previously in [FSS20b][SS20c] (see also [FSS20a]). Here we see how the H_3-flux is refined to generalized cohomology, in line with the refinement of G_4-flux to Cohomotopy theory.

2.3 M5-brane charge and the Adams d-Invariant

Unit M5-brane charge and Unital cohomology. Given a Whitehead-generalized cohomology theory E (9), we ask that there is a notion of *unit M5-brane charge* as measured in E-cohomology. Since, under *Hypothesis H*, the actual M5-brane charge of a homotopically-flat spacetime X (Remark 2.2) is measured by classes in Cohomotopy (12), hence by homotopy classes of maps to the 4-sphere, this means that there must be a certain unit element in the E-cohomology of the 4-sphere: $[G^E_{4,\mathrm{unit}}] \in \widetilde{E}^4(S^4)$.

A sufficient condition for this to exist is that the cohomology theory is *multiplicative* (Def. 3.35), so that its coefficient ring $E_\bullet$ (9) has the structure of a $\mathbb{Z}$-graded-commutative ring, and in particular has a multiplicative unit element $1^E \in E_0$. The image under the 4-fold suspension isomorphism (137) of this unit 1^E serves the purpose:

$$\underset{E\text{-cohomology ring...}}{\text{multiplicative unit in}} \quad 1^E \in \pi_0(E) = \widetilde{E}^0(S^0) \underset{\Sigma^4}{\xrightarrow{\simeq}} \widetilde{E}^4(S^4) \quad \underset{\text{in } E\text{-cohomology of 4-sphere.}}{\text{... suspended to canonical unit}} \tag{30}$$

In order to bring out its interpretation as unit M5-brane charge seen in E-cohomology, we denote this E-unit in degree 4 as follows:

$$\underset{\text{seen in } E\text{-cohomology}}{\text{unit M5-brane charge}} \quad \big[G^E_{4,\mathrm{unit}} := \Sigma^4(1^E) \,:\, S^4 \longrightarrow E^4\big] \in \widetilde{E}^4(X). \tag{31}$$

In fact, for any multiplicative cohomology theory E, the suspended unit elements $\Sigma^n(1^E)$ (30) are jointly the components of a *unique* homomorphism of ring spectra (Def. 3.35) out of the sphere spectrum $\mathbb{S}$ (Example 2.4):

$$\begin{array}{ccccc}
 & & \underset{\text{(unique multiplicative map from } \mathbb{S})}{\text{unit of } E\text{-ring spectrum}} & & \\
 & \text{sphere spectrum} \;\; \mathbb{S} & \xrightarrow{\quad e^E \quad} & E \;\; \text{any ring spectrum} & \\
\Sigma^\infty S^n \;\; \simeq & \Sigma^n \mathbb{S} & \xrightarrow{\quad \Sigma^n(e^E) \quad} & \Sigma^n E & \\
 & S^n & \xrightarrow{\quad \Sigma^n(1^E) \quad} & \Omega^{\infty - n} E & =: \;\; E^n \\
 & & \text{suspensions of } 1^E := [e^E] \in \pi_0(E) \;\; \text{(canonical units in } \widetilde{E}(S^n)) & &
\end{array} \tag{32}$$

The corresponding cohomology operations are the *Hurewicz-Boardman homomorphisms* in E-cohomology [Ad74, §II.6][Hun95][Arl04]:

$$\begin{array}{ccc}
 & \text{Hurewicz-Boardman homomorphism} & \\
\underset{\text{Cohomotopy}}{\text{stable}} \;\; \widetilde{\mathbb{S}}^n(X) & \xrightarrow{\quad \beta^n_X \quad} & \widetilde{E}^n(X) \;\; E\text{-cohomology} \\
\big[X \xrightarrow{c} \mathbb{S}^n\big] & \longmapsto & \big[X \xrightarrow{c} \mathbb{S}^n \xrightarrow{\Sigma^n(1^E)} E^n\big].
\end{array} \tag{33}$$

General M5-brane charge and the d-Invariant. General M5-brane charge on a homotopically flat spacetime X (Remark 2.2) is, by *Hypothesis H*, given by classes in the 4-Cohomotopy of X, hence homotopy classes of maps $X \xrightarrow{c} S^4$. For E a multiplicative cohomology theory, the corresponding charge as measured in the E-cohomology of X is the base change along this map of the E-unit M5-brane charge (31):

$$\underset{\text{seen in } E\text{-cohomology}}{\underset{\text{general M5-brane charge}}{\big[G^E_4(c)\big]}} := \underset{\text{along classifying map of its charge in Cohomotopy}}{\underset{\text{pullback of unit M5-brane charge in } E\text{-cohomology}}{\big[c^*\big(G^E_{4,\mathrm{unit}}\big)\big] = \big[X \xrightarrow{c} S^4 \xrightarrow{\Sigma^4(1^E)} E^4\big]}} \in \widetilde{E}^4(X). \tag{34}$$

This construction, mapping Cohomotopy classes to E-cohomology classes, is the *Adams d-invariant* $d_E(c)$ (Def. 3.36) of the Cohomotopy class c seen in E-cohomology:

$$\begin{array}{ccc}
\text{Cohomotopy} & \text{d-invariant} & E\text{-Cohomology} \\
\widetilde{\pi}^4(X) & \xrightarrow{\quad d^4_E \quad} & \widetilde{E}^4(X) \\
\underset{\text{(under Hypothesis H)}}{\underset{\text{full M-brane charge}}{\big[X \xrightarrow{c} S^4\big]}} & \longmapsto & \big[G^E_4(c)\big] := \underset{\text{M-brane charge in } E\text{-cohomology}}{\big[X \xrightarrow{c} S^4 \xrightarrow{\Sigma^4(1^E)} E^4\big]}.
\end{array} \tag{35}$$

Hypothesis H and Initiality of the sphere spectrum. Any homomorphism of multiplicative cohomology theories $E \xrightarrow{\phi} F$ (Def. 3.35) (a multiplicative cohomology operation) induces a natural comparison map between the corresponding M-brane charge quantization laws (35). In this system of multiplicative charge quantization laws, one is universal: The sphere spectrum $\mathbb{S}$ is initial among homotopy-commutative ring spectra, via the unit morphisms (32), and so *stable Cohomotopy is initial* among multiplicative cohomology theories – we may say: among multiplicative charge quantization laws. It seems suggestive that among all imaginable charge quantization laws in physics, M-theory would correspond to this exceptional one. This is what *Hypothesis H* asserts.

Concretely, initiality of $\mathbb{S}$ means that any choice E of M5-brane charge quantization (34) factors through stable Cohomotopy $\mathbb{S}$ via the cohomotopical d-invariant $d_{\mathbb{S}}$ (35) followed by the E-Boardman homomorphism β_E (33):

$$
\begin{tikzcd}
\widetilde{\pi}^4(X) \arrow[r, "d^4_{\mathbb{S}}"] \arrow[rr, bend left, "d^4_E"] & \widetilde{\mathbb{S}}^4(X) \arrow[r, "\beta^4_E"] & \widetilde{E}^4(X) \\
{[c]} \arrow[r, mapsto] & \big[G^{\mathbb{S}}_4(c)\big] \arrow[r, mapsto] & \big[G^E_4(c)\big] \\
\big[X \xrightarrow{c} S^4\big] \arrow[r, mapsto] & \big[X \xrightarrow{c} S^4 \xrightarrow{\Sigma^4(1^{\mathbb{S}})} \mathbb{S}^4\big] \arrow[r, mapsto] & \big[X \xrightarrow{c} S^4 \xrightarrow{\Sigma^4(1^E)} E^4\big] .
\end{tikzcd}
\tag{36}
$$

Since the G_4-flux as seen in any E is, thereby, a function of the G_4-flux seen in stable Cohomotopy $E = \mathbb{S}$, we shall also write

$$
G^{\mathbb{S}}_4\mathrm{Fluxes}\big(X\big) \;:=\; \widetilde{\mathbb{S}}^4(X)
\tag{37}
$$

for the stable 4-Cohomotopy group of a spacetime X, hence for the stable image of the unstable 4-Cohomotopy of X (Remark 3.39). This notation serves to bring out the beginning of a pattern that continues with the group $H^E_{n-1}\mathrm{Fluxes}(X)$ in §2.4 below.

While measurement of the 4-flux G_4 itself requires only the unital structure of a multiplicative cohomology theory, by (34), the product structure enters in describing the *dual* flux G_7 (whose rational image appeared around (13)):

Dual M5-brane flux and Multiplicative cohomology. Using the product structure (cup product) in a given multiplicative cohomology theory E, we may form, in particular, the square of the M5-brane charge (34). But the square of the *unit* M5-brane flux $G^E_{4,\mathrm{unit}}$ (31) trivializes canonically (by Prop. 3.81 below) via a homotopy/gauge transformation (6), which we denote[5] $-2G^E_{7,\mathrm{unit}}$, as shown on the left here:

$$
\begin{tikzcd}
S^4 \arrow[r, "G^E_{4,\mathrm{unit}} = \Sigma^4(1^E)"] \arrow[d] & E^4 \arrow[d, "(-)^2"] \arrow[dl, phantom, "\mathrm{d}\left(2G^E_{7,\mathrm{unit}}\right) = -G^E_{4,\mathrm{unit}} \cup G^E_{4,\mathrm{unit}}"] \\
\ast \arrow[r, "0"'] & E^8
\end{tikzcd}
\quad\Rightarrow\quad
\big(X \xrightarrow{c} S^4\big) \;\mapsto\;
\begin{tikzcd}
X \arrow[r, "G^E_4(c) = c^\ast G^E_{4,\mathrm{unit}}"] \arrow[d] & E^4 \arrow[d, "(-)^2"] \arrow[dl, phantom, "\mathrm{d}\left(2G^E_7(c)\right) = -G^E_4(c) \cup G^E_4(c)"] \\
\ast \arrow[r, "0"'] & E^8
\end{tikzcd}
\tag{38}
$$

By naturality of the cup square, this implies that for given Cohomotopy charge $X \xrightarrow{c} S^4$, the homotopy

$$
\underbrace{G^E_7(c)}_{\substack{\text{dual M5-brane flux} \\ \text{seen in } E\text{-cohomology}}} \;:=\; c^\ast\big(G^E_{7,\mathrm{unit}}\big)
\tag{39}
$$

trivializes the cup product square of the M5-brane charge measured in E-cohomology (34), as shown on the right of (38), and thus is the dual[6] flux analog of (34).

[5] We include a "2" in the notation because, after rationalization (see Example 2.6), this is the conversion factor to the conventional normalization of the 7-flux form G_7 [FSS19c, (3)].

[6] Once the geometric aspect of (super-)gravity is taken into account, its equations of motion force a differential form representative of the rational image of G_7 (see Example 2.6) to be *Hodge dual* to that of G_4, whence the terminology. Our consideration of the cohomological charge quantization of the pair (G_4, G_7) without – or rather before – considering its geometric self-duality constraint is directly analogous to how the K-theoretic charge quantization of the set of RR-flux forms is considered before imposing the respective self-duality constraint.

Examples 2.4 (Examples of multiplicative cohomology theories)**.** Some examples of multiplicative (Def. 3.35, except for the last items) Whitehead-generalized cohomology theories (9), with the notation we will use for them:

E	**Multiplicative generalized cohomology**	[CF66][Ad74][Koc96][TK06]
HR	Ordinary cohomology with R coefficients	[Ei40, p. 243] [EML54b, p. 520-521] [St72, §19][Ma99, §22]
$H^{\mathrm{ev}}R = \bigoplus_k \Sigma^{2k} HR$	Even-periodic ordinary cohomology	
$K\mathrm{U}$	Complex topological K-theory	[AH61][At64] [Ka78]
$K\mathrm{O}$	Orthogonal topological K-theory	
$\mathbb{S} \simeq M\mathrm{Fr}$	Stable Cohomotopy / framed Cobordism	[Ad74, p. 204][Ro04, [p. 1]/ [CF66, p. 27][Sto68, §IV, p. 40]
Mf	Cobordism cohomology	[At61][Qu71] [Sto68] [CF66, §12] [Koc96, §4.4][TK06, §6] [LLP17][CLP19]
$M\mathrm{O}$	Cobordism	
$M\mathrm{U}$	Complex Cobordism	
$M\mathrm{SU}$	Special complex Cobordism	
$Mf/\mathbb{S} = M(f,\mathrm{Fr})$	Cobordism with framed boundaries	
$M(\mathrm{U},\mathrm{Fr})$	Complex Cobordism with framed boundaries	[CF66, §16][CS69, §6][Sm71] [La00]
$M(S\mathrm{U},\mathrm{Fr})$	Special complex Cobordism with framed boundaries	

Example 2.5 (**Ordinary M5-brane charge and Hopf winding degree).** If $E = HR$ is ordinary cohomology, then, under the de Rham homomorphism, the unit G_4-flux (31) is given by the volume form on the 4-sphere:

$$G_{4,\mathrm{unit}}^{HR} \simeq \left[\mathrm{vol}_{S^4}\right] \in H^4(S^4;\mathbb{Z}) \longrightarrow H^4(S^4;R)\,.$$

Thus, if spacetime $X \simeq X^4$ is homotopy equivalent to a connected closed manifold of dimension 4, the *Hopf degree theorem* ([Po55, §9], review in [Kob16, §7.5]) says that the assignment (35) is an isomorphism, where the d-invariant

$$\begin{array}{ccc}
\text{Cohomotopy} & \text{Hopf winding degree map} & \text{ordinary cohomology} \\
\widetilde{\pi}^4\left(X^4\right) & \xrightarrow[\simeq]{\ \deg\ } & \widetilde{HZ}^4(X^4) \simeq \mathbb{Z} \\
\left[X \xrightarrow{c} S^4\right] & \longmapsto & \left[G_4^{HR}(c)\right] := c^*\left[\mathrm{vol}_{S^4}\right]
\end{array} \tag{40}$$

is the *Hopf winding degree* of the given map $X^4 \xrightarrow{c} S^4$ (see also Example 3.26). In this sense, the d-invariant (35) generalizes integer mapping degree to generalized cohomology, whence the name.

In the physics literature, this situation is familiar from Freund-Rubin compactifications of 11-dimensional supergravity on spacetimes locally of the topology $\mathbb{R}^{5,1} \times \mathbb{R}_{\mathrm{rad}} \times S^4$ ([FR80]), which are the near-horizon geometries of n coincident black M5-branes ([Gü92][AFCS99, §2.1.2]). Here the 4-flux G_4 is, on the S^4-factor, n times the volume form. Under Hypothesis H, this is identified as the $d_{H\mathbb{R}}$-invariant (35) of the Cohomotopy cocycle $\left[\mathbb{R}^{5,1} \times \mathbb{R}_{\mathrm{rad}} \times S^4 \simeq S^4 \xrightarrow{c} S^4\right]$ with Hopf degree/winding number (40) equal to n.

(In the more general global case, as per Remark 2.2, on spacetimes that are S^4-fibrations and measuring charge in tangentially twisted Cohomotopy, this statement finds a parametrized generalization reflecting M5-brane anomaly cancellation; this is discussed in [SS20a].)

Example 2.6 (**Ordinary dual M5-brane flux and the Sullivan model of the 4-sphere).** For $E = H\mathbb{R}$ BEING ordinary rational cohomology with real coefficients, as in Example 31 and under the fundamental theorem of rational homotopy theory, the topological equation of motion (38) is the differential relation

$$d\big(2G^{\mathbb{R}}_{7,\mathrm{unit}}\big) = -G^{\mathbb{R}}_{4,\mathrm{unit}} \wedge G^{\mathbb{R}}_{4,\mathrm{unit}}$$

in the Sullivan model for the 4-sphere (recalled as [FSS20c, Ex. 3.68]), and its image under the cohomotopical character map [FSS20c, §5.3] in the rational/de Rham cohomology of spacetime:

$$d\big(2G^{\mathbb{R}}_{7}(c)\big) = -G^{\mathbb{R}}_{4}(c) \wedge G^{\mathbb{R}}_{4}(c)\,.$$

But this is the defining relation of the dual 7-flux form in 11-dimensional supergravity. The fact that this appears as the image of Cohomotopy theory as seen in ordinary rational cohomology is a key motivation for Hypothesis H [Sa13, §2.5][FSS15][FSS17][FSS19a, (57)][FSS19b, Prop. 2.5].

Flux degree and Adams operations. While flux densities are thought of as differential forms (or their classes in De Rham cohomology and hence in ordinary rational cohomology) of a given degree, such may not manifestly be provided by a class in a Whitehead-generalized cohomology E measuring M-brane charge (35). Extra structure on E which does allow to extract form degree under its character map are *Adams operations*. The Adams operations on topological K-theory ([Ad62, §5][Ka78, §IV.7], review in [Wir12, §11]) are a system of cohomology operations indexed by $k \in \mathbb{N}$

$$X \quad \longmapsto \quad \widetilde{KU}^0(X) \xrightarrow{\ \psi^k\ } \widetilde{KU}^0(X) \tag{41}$$

$$X \quad \longmapsto \quad \widetilde{KO}^0(X) \xrightarrow{\ \psi^k\ } \widetilde{KO}^0(X)\,. \tag{42}$$

These being cohomology operations means that they are natural transformations of cohomology groups, hence commute with pullback along any map $X \xrightarrow{f} Y$, in that the following squares commute:

$$\begin{tikzcd} \widetilde{KU}^0(X) \arrow[r, "\psi^k"] & \widetilde{KU}^0(X) \\ \widetilde{KU}^0(Y) \arrow[u, "f^*"] \arrow[r, "\psi^k"] & \widetilde{KU}^0(Y) \arrow[u, "f^*"'] \end{tikzcd} \tag{43}$$

Less subtle but otherwise analogous cohomology operations ψ^k_H exist on ordinary even cohomology $H^{\mathrm{ev}}R$ (Example 2.4) defined as multiplication by k^r in any degree $2r$:

$$\begin{tikzcd} \widetilde{H^{\mathrm{ev}}R}^0(X) \arrow[r, "\psi^k_H"] & \widetilde{H^{\mathrm{ev}}R}^0(X) \\ \widetilde{HR}^{2r}(X) \arrow[u, hook] \arrow[r] & \widetilde{HR}^{2r}(X) \arrow[u, hook] \\ {[\alpha_{2r}]} \arrow[r, mapsto] & k^r \cdot [\alpha_{2r}] \end{tikzcd}$$

We have ([Ad62, Thm. 5.1 (vi)] [Ka78, Thm. V.3.27]) that the Chern character map is compatible with the Adams operations, in that the following diagrams all commute

$$\begin{tikzcd} \widetilde{KU}^0(X) \arrow[r, "\mathrm{ch}"] \arrow[d, "\psi^k"'] & \widetilde{H^{\mathrm{ev}}\mathbb{Q}}^0(X) \arrow[d, "\psi^k_H"] \\ \widetilde{KU}^0(X) \arrow[r, "\mathrm{ch}"] & \widetilde{H^{\mathrm{ev}}\mathbb{Q}}^0(X)\,, \end{tikzcd} \qquad \begin{tikzcd} \widetilde{KO}^0(X) \arrow[r, "\mathrm{ch}"] \arrow[d, "\psi^k"'] & \widetilde{H^{\mathrm{ev}}\mathbb{Q}}^0(X) \arrow[d, "\psi^k_H"] \\ \widetilde{KO}^0(X) \arrow[r, "\mathrm{ch}"] & \widetilde{H^{\mathrm{ev}}\mathbb{Q}}^0(X) \end{tikzcd} \tag{44}$$

This means that a class $\big[V\big] \in \widetilde{KU}^0(X)$ represents a single flux form in degree $2r$ if it is an eigenvector of eigenvalue k^r for the Adams operations ψ^k:

$$\psi^k[V] = k^r \cdot [V] \quad \Rightarrow \quad \mathrm{ch}([V]) \in H^{2r}(X;\mathbb{Q}) \hookrightarrow \widetilde{H^{\mathrm{ev}}\mathbb{Q}}^0(X) \tag{45}$$

For instance, on the $2r$-dimensional sphere, where the Chern character of any reduced class is necessary concentrated on classes in degree $2r$, all reduced K-theory classes are eigenvectors to eigenvalue k^r of the Adams operations ψ^k (e.g. [Wir12, p. 45 (47 of 67)]):

$$\widetilde{KU}^0(S^{2r}) \xrightarrow{\ \psi^k(-) = k^r \cdot (-)\ } \widetilde{KU}^0(S^{2r})\,. \tag{46}$$

2.4 M5 brane H_3-Flux and the Toda brackets

Fluxless backgrounds and Vanishing d-invariant. A *fluxless* background in M-theory is meant to be one for which the G_4-flux vanishes, by which we shall mean that it vanishes in cohomology (Remark 2.1). Under *Hypothesis H* and seen in E-cohomology via (34), this means that the d_E-invariant (35) of the Cohomotopy charge vanishes:

$$[c] \in \widetilde{\pi}^4(X) \text{ is } E\text{-fluxless} \quad \Leftrightarrow \quad \big[G_4^E(c)\big] = 0 \in \widetilde{E}^4(X) \quad \Leftrightarrow \quad d_E^E(c) = 0\,. \tag{47}$$

Notice that the Cohomotopy class $[c]$ itself may be non-trivial even if its E-flux vanishes, which is ([FSS19b, §3.8]) a key source of subtle effects implied by Hypothesis H (we expand on this in Example 2.7 and §2.6 below).

H_3-Flux and Trivializations of the d-invariant. When the $G_4^E(c)$-flux/$d_E(c)$-invariant (35) vanishes, there are E-gauge transformations (6), to be denoted $H_3^E(c)$, which witness this vanishing:

$$\big[G_4^E(c)\big] = 0 \quad \Leftrightarrow \quad \exists \left(\begin{tikzcd} X \arrow[rr] \arrow[d, "c"'] & & \ast \arrow[d, "0"] \\ S^4 \arrow[rr, "G^E_{4,\mathrm{unit}}"'] \arrow[urr, phantom, "d\,H_3^E(c) = G_4^E(c)"] & & E^4 \end{tikzcd} \right). \tag{48}$$

These trivializing homotopies are in general not unique, even up to homotopy. The difference of homotopy classes of any two such homotopies is a class in the 3rd E-cohomology of X:

$$\big[H_3^E(c)_a - H_3^E(c)_b\big] = \left[\begin{tikzcd} & \ast \arrow[dr, bend left, "0"] \arrow[d, phantom, "\Downarrow H_3^E(c)_a"] & \\ X \arrow[ur, bend left] \arrow[rr, "G_4^E(c)"] \arrow[dr, bend right] & {} \arrow[d, phantom, "\Downarrow -H_3^E(c)_b"] & E^4 \\ & \ast \arrow[ur, bend right, "0"'] & \end{tikzcd} \right] \in \pi_1 \mathrm{Maps}^{*/}\big(X\,, E^4\big) \simeq \widetilde{E}^3(X)\,, \tag{49}$$

and every such class arises this way. This means that for each Cohomotopy charge c with vanishing G_4^E-flux (47), hence with trivial d_E-invariant, the set $H_3^E\mathrm{Fluxes}(X,c)$ of possible *choices of trivializations* $\big[H_3^E(c)\big]$ (48) carries the structure of a *torsor* (a principal bundle over the point) for the third E-cohomology group of X (Lemma 3.43):

$$[c] \in \pi^4(X),\ \big[G_4^E(c)\big] = 0 \quad \Rightarrow \quad H_3^E\mathrm{Fluxes}(X,c) \in \big(\widetilde{E}^3(X)\big)\mathrm{Torsors}\,. \tag{50}$$

Noticing that the set of $H_3^E\mathrm{Fluxes}(X,c)$ depends only on the stabilized Cohomotopy charge (15) $[c] \in \widetilde{\pi}^4(X) \xrightarrow{\Sigma^\infty} \widetilde{\mathbb{S}}^4(X)$ we write (see Def. 3.40):

$$\begin{array}{ll} \substack{H_3\text{-fluxes in } E \\ \text{for } G_4\text{-fluxes in } \mathbb{S}} & H_3^E\mathrm{Fluxes}(X) := \displaystyle\coprod_{[c] \in \widetilde{\mathbb{S}}^4(X)} H_3^E\mathrm{Fluxes}(X,c) \\ & \qquad\qquad \Big\downarrow \\ \substack{G_4\text{-fluxes in} \\ \text{stable Cohomotopy}} & G_4^E\mathrm{Fluxes}(X) := \widetilde{\mathbb{S}}^4(X) \end{array} \tag{51}$$

for the set of all such 3-fluxes as the stable Cohomotopy class of the Cohomotopy charge c varies, hence fibered over the set of G_4^EFluxes (37), with fiber over $[c]$ empty if $\big[G_4^E(c)\big] \neq 0$ and otherwise an $\widetilde{E}^3(X)$-torsor (50). This is the (discrete) *moduli space* of (homotopy/gauge equivalence classes of) choices of pairs of 4-flux and 3-flux.

Example 2.7 ($H_3^{\mathbb{S}}$-Flux near M2-branes and the Order of the third stable stem)**.** In the initial case that $E = \mathbb{S}$ is taken to be stable Cohomotopy itself (15) and $X \simeq S^7$ is spacetime in the vicinity of M2-branes (23), the unstable Cohomotopy charges lie in $\mathbb{Z} \times \mathbb{Z}_{12}$ (22), while their underlying 4-flux seen in $\mathbb{S}$-cohomology/stable Cohomotopy vanishes precisely on those such charges that in the first factor are multiples of 24 (as in [FSS19b, (8)]):

$$\begin{array}{ccccccccc} & & \substack{\text{non-abelian/unstable} \\ \text{Cohomotopy group}} & \substack{\text{Boardman} \\ \text{homomorphism}} & \substack{\text{abelian/stable} \\ \text{Cohomotopy group}} & & & \\ \substack{\text{quaternionic} \\ \text{Hopf fibration}} & [S^7 \xrightarrow{h_{\mathbb{H}}} S^4] & \in \ \widetilde{\pi}^4(S^7) & \xrightarrow{\quad \beta^4 \quad} & \widetilde{\mathbb{S}}^4(S^7) & \ni & \Sigma^\infty[S^7 \xrightarrow{h_{\mathbb{H}}} S^4] & \substack{\text{stabilized} \\ \text{quaternionic} \\ \text{Hopf fibration}} \\ & \big\updownarrow & \wr\!\wr & & \wr\!\wr & & \big\updownarrow & \\ \substack{\text{non-torsion} \\ \text{generator}} & (1,0) & \in \ \mathbb{Z} \times \mathbb{Z}_{12} & \xrightarrow[(n,a) \mapsto (n \bmod 24)]{} & \mathbb{Z}_{24} & \ni & 1 & \substack{\text{torsion} \\ \text{generator}} \end{array} \tag{52}$$

Hence:

> *$H_3^{\mathbb{S}}$-Flux near M2-branes for Cohomotopy charge $c = n \cdot [h_{\mathbb{H}}]$ measured in $\mathbb{S}$-Cohomology exists precisely if n is a multiple of 24.* Moreover, comparison with (28) shows that: *This cohomotopical $H_3^{\mathbb{S}}$-flux witnesses the spontaneous compactification of M-theory on K3 near probe M2-branes.*

Example 2.8 (Ordinary and K-theoretic 3-flux)**.** Let spacetime (20) be homotopy-equivalent to any odd-dimensional sphere (such as $X \simeq S^7$ for an M2-brane background (28)) and consider measuring M-brane charge (35) in either even integral cohomology or in complex K-theory (Notation 2.4):

$$X \simeq S^{2n+1} \quad \text{and} \quad E \in \big\{ H^{\mathrm{ev}}\mathbb{Z},\, K\mathrm{U} \big\}\,.$$

Then, by the suspension isomorphism and by (Bott-)periodicity, we have:

- **(a)** $\widetilde{E}^4(X) \simeq 0$, and hence for *all* Cohomotopy charges $\big[X \xrightarrow{c} S^4\big]$ the $d_{K\mathrm{U}}$-invariant (35) vanishes, $\big[G_4^{K\mathrm{U}}(c)\big] = 0$, (47) and an $H_3^{K\mathrm{U}}$-flux exists (48);
- **(b)** $\widetilde{E}^3(X) \simeq \mathbb{Z}$, and hence the possible H_3-fluxes (50) form a $\mathbb{Z}$-torsor.

If we suggestively denote the $\mathbb{Z}$-action on this $\mathbb{Z}$-torsor as addition of multiples of a *unit of closed 3-flux*

$$\begin{array}{ccc} E\mathrm{Fluxes}(X,c) \times \mathbb{Z} & \longrightarrow & E\mathrm{Fluxes}(X,c) \\ \Big(\underbrace{\big[H_3^E(c)\big]}_{\text{any 3-flux}}\,,\, n \Big) & \longmapsto & \big[H_3^E(c)\big] + n \cdot \underbrace{\big[H^E_{3,\mathrm{unit}}\big]}_{\text{unit of closed 3-flux}} \end{array} \tag{53}$$

then this means that for all c, a choice of *reference 3-flux* – to be denoted $C_3^{K\mathrm{U}}(c)$ – is equivalently an isomorphism of $\mathbb{Z}$-torsors, specifically the one of the following form (a trivialization of $\mathbb{Z}$-torsors):

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\quad \simeq \quad} & H_3^E\mathrm{Fluxes}(X,c) \\ n & \longmapsto & \underbrace{\big[C_3^E(c)\big]}_{\text{C-field}} \;+\; n \cdot \underbrace{\big[H^E_{3,\mathrm{unit}}\big]}_{\text{closed 3-flux}} \end{array} \tag{54}$$

Observables on H_3-Flux and Toda brackets. A choice of H_3^E-flux (48) by itself does not give a cohomology class, and is hence not an invariant observable. By forming differences (49) we do get cohomology classes, and hence observables, of pairs of H_3-fluxes, measuring one relative to another. In order to make H_3-flux itself be observable we need such a relative construction with the reference point fixed by other means. Since H_3^E-flux witnesses the vanishing of G_4^E-flux, whose reference point is, in a sense, the unit 4-flux $G^E_{4,\mathrm{unit}}$ (31), we obtain such a reference point in any (non-abelian) cohomology theory A in which the $G_{4,\mathrm{unit}}$-flux has been trivialized, regarded there through a given cohomology operation $E^4 \xrightarrow{\phi} BA$:

$$\begin{tikzcd} S^4 \arrow[r, "G^E_{4,\mathrm{unit}}"] \arrow[d] & E^4 \arrow[d, "\phi"] \arrow[dl, phantom, "\swarrow\, \mathrm{d}\, O^A \,=\, -G^{\phi}_{4,\mathrm{unit}}"] \\ \ast \arrow[r, "0"'] & BA \end{tikzcd} \tag{55}$$

Given this, we may measure H_3^E-flux (48) in A-cohomology, by forming the pasting composite with (55):

$$\underbrace{\begin{tikzcd} X \arrow[rr, bend left, "0"] \arrow[rr, bend right, "0"'] \arrow[rr, phantom, "\swarrow\, O^A(H_3^E(c))"] & & BA \end{tikzcd}}_{\text{Observable on } H_3^E\text{-flux relative to } O^A} \;:=\; \underbrace{\begin{tikzcd} X \arrow[r] \arrow[d, "c"'] & \ast \arrow[d, "0"] \arrow[dl, phantom, "\swarrow\, \mathrm{d}\, H_3^E(c) \,=\, G_4^E(c)"] \\ S^4 \arrow[r, "G^E_{4,\mathrm{unit}}"] \arrow[d] & E^4 \arrow[d, "\phi"] \arrow[dl, phantom, "\swarrow\, \mathrm{d}\, O^A \,=\, -G^{\phi}_{4,\mathrm{unit}}"] \\ \ast \arrow[r, "0"'] & BA \end{tikzcd}}_{\text{Refined Toda bracket}} \;\; \begin{array}{l} \in\;\; \pi_1 \mathrm{Maps}^{*/}(X, BA) \\ \;\;= \pi_0 \mathrm{Maps}^{*/}(X, A) \\ \;\;= \widetilde{A}(X) \end{array} \tag{56}$$

This construction of *observables on trivializing fluxes* hence this *secondary invariant* is a refined *Toda bracket* (Def. 3.44) of (a) the Cohomotopy charge c with (b) its unit $G^{E}_{4,\mathrm{unit}}$ in E-cohomology and (c) its observation through ϕ, as discussed in detail below in §3.4.

In the following we consider a sequence of **examples of Toda-bracket observables on H_3^E-flux**:

1. The stably *universal* H_3^E-observable $O^{E/\mathbb{S}}$ takes values in the *Adams cofiber cohomology theory* $E/\mathbb{S}$ (Def. 3.48) – discussed as Prop. 2.9 below (a special case of Prop. 3.53 further below).
2. The *refined Adams e-invariant* sees a charge lattice of rational values of H_3^{KU}-fluxes – discussed in §2.5.
3. The *refined Conner-Floyd e-invariant* measures the flux of H_3 through 3-spheres around branes transversal to local CY2-compactifications – discussed in (§2.6
4. The *Hopf invariant* is the $H_3^{H\mathbb{Z}}$-observable that measures the *Page charge* of M2-branes corresponding to the given H_3/C_3-field – discussed in §2.7.

First, to better understand observables on the moduli spaces (51) of E-fluxes, we turn attention to the *classifying maps* of the 3-flux:

Classifying maps for 3-flux and Homotopy cofiber sequences. Given a pair of maps out of one domain space X, we denote their *homotopy cofiber-product* or *homotopy pushout* (of one map along the other) by the label "(po)"[7] (e.g. [FSS20c, Def. A.23]), as shown on the left in the following. In particular, when $X \xrightarrow{c} B$ is any map, and $A = *$ is the point, this is the *homotopy cofiber* C_c, shown on the right:

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\text{universal homotopy}}{\Swarrow}\ (\mathrm{po})"] & A \arrow[d] \\
B \arrow[r] & A \underset{X}{\sqcup^h} B \ \ \substack{\textbf{homotopy}\\ \textbf{pushout}}
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r] \arrow[d, "c"'] \arrow[dr, phantom, "\overset{\text{universal homotopy}}{\Swarrow}\ (\mathrm{po})"] & * \ \ \textbf{point space} \arrow[d] \\
B \arrow[r] & C_c \ \ \substack{\textbf{homotopy}\\ \textbf{cofiber space}}
\end{tikzcd}
\tag{57}
$$

These homotopies being universal means that homotopies φ under the original pair of maps are *classified* by maps $\vdash \varphi$ out of the homotopy pushout, via factorization through the universal homotopy:

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\varphi}{\Swarrow}"] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow\ (\mathrm{po})"] & A \arrow[d] \arrow[ddr, bend left, "f"] \\
B \arrow[r] \arrow[drr, bend right, "g"'] & A \underset{X}{\sqcup^h} B \arrow[dr, dashed, "\vdash\varphi"] \\
& & C
\end{tikzcd}
\tag{58}
$$

Moreover, such homotopy pushout squares (57) satisfy the *pasting law*, which says that if in a "homotopy pasting diagram", as shown in the diagram

$$
\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow\ (\mathrm{po})"] & \cdot \arrow[d] \\
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}
\tag{59}
$$

the top square is a homotopy pushout, then the bottom square is so if and only if the total rectangle is. In particular, this means

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "c"'] \arrow[dr, phantom, "\Swarrow\ (\mathrm{po})"] & * \arrow[d] \\
Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow\ (\mathrm{po})"] & C_c \arrow[d] \\
* \arrow[r] & \Sigma X
\end{tikzcd}
\tag{60}
$$

that the homotopy cofiber of a cohomotopy cofiber is the suspension of the original domain. The resulting long *homotopy cofiber sequences*

[7] Often we notationally suppress the homotopy (the double arrow) filling this square. Conversely, *all* cells of all diagrams we display in this article are homotopy-commutative, with coherent homotopies filling them, even if these are not made notationally explicit.

$$X \longrightarrow Y \longrightarrow \Sigma X \longrightarrow \Sigma Y \longrightarrow \Sigma Z \longrightarrow \cdots \tag{61}$$

are sent by any Whitehead-generalized cohomology theory $\widetilde{E}(-)$ to the corresponding long exact sequence (62) in E-cohomology:

$$\cdots \longleftarrow \widetilde{E}^{n+1}(Z) \longleftarrow \widetilde{E}^{n}(X) \longleftarrow \widetilde{E}^{n}(Y) \longleftarrow \widetilde{E}^{n}(Z) \longleftarrow \widetilde{E}^{n-1}(X) \longleftarrow \cdots \tag{62}$$

Applied to the 3-flux homotopies $H_3^E(c)$ from (48), the equivalence (58) says that these are classified by E-cohomology classes $\big[\vdash H_3^E(c)\big]$ of the cofiber space C_c of the Cohomotopy charge c:

$$\big[\vdash H_3^E(c)\big] \;\in\; \widetilde{E}^4\big(C_c\big)\,, \qquad \text{s.t.} \quad q_c^*\big[\vdash H_3^E(c)\big] \;=\; \big[G^E_{4,\mathrm{unit}}\big]\,. \tag{63}$$

$$\begin{tikzcd}
X \arrow[rr] \arrow[d, "c"'] & & \ast \arrow[d, "0"] \arrow[dll, Rightarrow, "d\,H_3^E(c) = G_4^E(c)"'] \\
S^4 \arrow[rr, "G^E_{4,\mathrm{unit}}"'] & & E^4
\end{tikzcd}
\;\simeq\;
\begin{tikzcd}
X \arrow[rr] \arrow[d, "c"'] & & \ast \arrow[d] \arrow[dr, bend left, "0"] \arrow[dll, Rightarrow, "(\mathrm{po})"] & \\
S^4 \arrow[rr, "q_c"] \arrow[rrr, bend right, "G^E_{4,\mathrm{unit}}"'] & & C_c \arrow[r, dashed, "\vdash H_3^E(c)"] & E^4
\end{tikzcd} \tag{64}$$

Furthermore, (59) says that this is controlled by long homotopy cofiber sequence of this form:

$$X \xrightarrow{\;c\;} S^4 \xrightarrow{\;q_c\;} C_c \xrightarrow{\;p_c\;} \Sigma X \longrightarrow \cdots\,, \qquad \text{e.g.} \qquad S^n \xrightarrow{\;c\;} S^4 \xrightarrow{\;q_c\;} C_c \xrightarrow{\;p_c\;} S^{n+1} \longrightarrow \cdots\,. \tag{65}$$

H_3^E-Flux charge quantization and the Adams cofiber $E/\mathbb{S}$-theory. The *stably universal* way to fill the bottom square in (56), and hence the stably universal Toda-bracket observable on H_3^E-flux, is given by the defining homotopy square of the cohomology theory $E/\mathbb{S}$ which is the homotopy cofiber of the unit map (32) – the *Adams cofiber cohomology theory* (Def. 3.48):

Proposition 2.9 (H_3^E-Flux measured in unit cofiber cohomology)**.** *Let spacetime $X \simeq S^{3+d}$ be homotopy equivalent to a sphere*[8] *(§2.2) and measure M-brane charge in a multiplicative cohomology theory E (§2.3). Then, the following stably universal Toda bracket observable* (56) *on H_3^E*

$$H^E_{n-1}\mathrm{Fluxes}\big(S^{n+d-1}\big) \ni
\begin{tikzcd}
S^{n+d-1} \arrow[rr] \arrow[d, "c"'] & & \ast \arrow[d, "0"] \arrow[dll, Rightarrow, "H^E_{n-1}(c)"] \\
S^n \arrow[rr, "\Sigma^n(1^E)"'] & & E^n
\end{tikzcd}
\longmapsto
\begin{tikzcd}
S^{n+d-1} \arrow[rr] \arrow[dd, "c"'] & & \ast \arrow[dr] \arrow[dd] \arrow[ddll, phantom, "\Swarrow (\mathrm{po})"] & \\
& & & \ast \arrow[dd, "0"] \\
S^n \arrow[rr, "q_c"] \arrow[rrrd, "\Sigma^n(1^E)"'] \arrow[dd] & & C_c \arrow[dr, dashed, "\vdash H^E_{n-1}(c)"] \arrow[dd, "p_c"] \arrow[ddll, phantom, "\Swarrow (\mathrm{po})"] & \\
& & & E^n \arrow[dd, "i^E"] \\
\ast \arrow[rr] \arrow[rrrd, bend right, "0"'] & & S^{n+d} \arrow[dr, dashed, "\vdash(G_n^{\mathbb{S}}(c)\,,H^E_{n-1}(c))"] & \\
& & & (E/\mathbb{S})^n
\end{tikzcd} \tag{66}$$

constitutes a lift from the set (51) *of classes of possible H_3^E-fluxes on X to the degree d-cohomology ring* (9) *of the cofiber cohomology theory $E/\mathbb{S}$* (146) *through its boundary map:*

$$\begin{tikzcd}
\big([c],[H_3^E(c)]\big) \arrow[d, mapsto] & H_3^E\mathrm{Fluxes}\big(S^{3+d}\big) \arrow[r, dashed] \arrow[d] & \big(E/\mathbb{S}\big)_d \arrow[d, "\partial"] \\
{[G_4^{\mathbb{S}}(c)]} & G_4^E\mathrm{Fluxes}\big(S^{3+d}\big) \arrow[r, equal] & \mathbb{S}_{d-1}
\end{tikzcd}
\qquad
\big([c],[H_3^E(c)]\big) \longmapsto \big[\vdash\big(G_4^E(c),H_3^E(c)\big)\big] \tag{67}$$

[8] This assumption is not necessary; we make it just for focus of exposition.

We prove this as Prop. 3.53 below.

Remark 2.10 (Observables on H_3^E from multiplicative cohomology operations)**.** One way to extract more specific information from the universal cofiber Toda-bracket observable (Def. 2.9) is by postcomposition with a multiplicative cohomology operation $E \xrightarrow{\phi} F$: Namely, any such canonically sits in a homotopy-commutative square of the form

$$
\begin{tikzcd}
E \arrow[r, "\phi"] \arrow[d, "i^E"'] \arrow[dr, phantom, "\swarrow i^{\phi}"] & F \arrow[d, "i^F"] \\
E/\mathbb{S} \arrow[r, "\phi/\mathbb{S}"'] & F/\mathbb{S}
\end{tikzcd}
\tag{68}
$$

whose pasting composite with the diagram (66) for the universal observable hence yields the following new Toda-bracket observable (56):

$$
\underset{\substack{\textbf{Observable on } H_3^E\textbf{-flux} \\ \textbf{relative to } O^{E/\mathbb{S}} \Box i^{\phi}}}{
\begin{tikzcd}
X \arrow[r, bend left, "0"] \arrow[r, bend right, "0"'] \arrow[r, phantom, "\Downarrow O^{F/\mathbb{S}}(H_3^E(c))"] & F/\mathbb{S}
\end{tikzcd}}
\quad := \quad
\underset{\textbf{composite refined Toda bracket}}{
\begin{tikzcd}
X \arrow[rr] \arrow[d, "c"'] \arrow[drr, phantom, "\swarrow\, d\, H_3^E(c) \,=\, G_4^E(c)"] & & \ast \arrow[d, "0"] \\
S^4 \arrow[rr, "G^E_{4,\mathrm{unit}}"] \arrow[d] \arrow[drr, phantom, "\swarrow\, d\, O^{E/\mathbb{S}} \,=\, -G_{4,\mathrm{unit}}^{(i^E)}"] & & E^4 \arrow[d, "\phi^4"] \arrow[dl, "i^E"'] \arrow[ddl, phantom, "\swarrow (i^{\phi})^4"] \\
\ast \arrow[r, "0"] \arrow[d, equal] & E/\mathbb{S} \arrow[dr, bend right, "(\phi/\mathbb{S})^4"'] & F^4 \arrow[d, "(i^F)^4"] \\
\ast \arrow[rr, "0"] & & F/\mathbb{S}
\end{tikzcd}}
\tag{69}
$$

2.5 M5 brane C_3-Field and the Adams e-Invariant

H_3^{KU}-Flux and the Adams e-Invariant. Since the Chern character on complex K-theory KU and the Pontrjagin character on KO

$$
\begin{tikzcd}
KO \arrow[d, "\mathrm{cplx}"'] \arrow[dr, "\mathrm{ph}", "\text{Pontrjagin character}"'] & \\
KU \arrow[r, "\mathrm{ch}", "\text{Chern character}"'] & H^{\mathrm{ev}}\mathbb{Q}
\end{tikzcd}
\tag{70}
$$

are multiplicative cohomology operations, they induce, via Remark 2.10, Toda-bracket observables $O^{(H^{\mathrm{ev}}\mathbb{Q}/\mathbb{S})}$ (56) on H_3^{KU}- and H_3^{KO}-fluxes. We call these observables the $\widehat{e}_{KU}$*-invariant* and $\widehat{e}_{KO}$*-invariant* (Def. 3.56 below) since, in the case that spacetime (21) is homotopy equivalent to an (odd-dimensional) sphere, this observable

$$
\begin{tikzcd}
H^{KU}_{2n-1}\mathrm{Fluxes}\big(S^{2(n+d)-1}\big) \arrow[r, "O^{KU/\mathbb{S}}"'] \arrow[rrr, bend left, "\widehat{e}_{KU}"] & \big((KU)/\mathbb{S}\big)_{2d-1} \arrow[r, "\mathrm{ch}/\mathbb{S}"'] & \big((H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}\big)_{2d-1} \arrow[r, "\mathrm{spl}_0"'] & \mathbb{Q}
\end{tikzcd}
$$

$$
\begin{tikzcd}
S^{2(n+d)-1} \arrow[r] \arrow[d, "c"'] & * \arrow[d, "0"] \arrow[dl, phantom, "\Swarrow\, H^{KU}_{2n-1}(c)"] \\
S^{2n} \arrow[r, "\Sigma^{2n}(1^{KU})"'] & KU^{2n}
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
S^{2(n+d)-1} \arrow[rr] \arrow[dd, "c"'] & & * \arrow[dr] \arrow[dd] \arrow[dl, phantom, "\Swarrow\,(\mathrm{po})"] & & \\
& & & * \arrow[dd, "0"] & \\
S^{2n} \arrow[rr, "q_c"] \arrow[drrr, "\Sigma^{2n}(1^{KU})"] \arrow[dd] & & C_c \arrow[dr, dashed, "\vdash H^{KU}_{2n-1}(c)"] \arrow[dd, "p_c"] \arrow[dl, phantom, "\Swarrow\,(\mathrm{po})"] & & \\
& & & KU^{2n} \arrow[dr, "\mathrm{ch}"] \arrow[dd, "i^{KU}"] & \\
* \arrow[rr] \arrow[drrr, "0"'] & & S^{2(n+d)} \arrow[dr, dashed, "\vdash (G^{\mathbb{S}}_{2n}(c)\,,\, H^{KU}_{2n-1}(c))"] & & (H^{\mathrm{ev}}\mathbb{Q})^{2n} \arrow[dd, "i^{H^{\mathrm{ev}}\mathbb{Q}}"] \\
& & & (KU/\mathbb{S})^{2n} \arrow[dr, "\mathrm{ch}/\mathbb{S}"'] & \\
& & & & \big((H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}\big)^{2n}
\end{tikzcd}
\tag{71}
$$

coincides with the lifts through $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ of the classical Adams e-invariant (Def. 3.58 below) on Cohomotopy charges (21) in the $(2n-1)$st stem:

Proposition 2.11 (The $\widehat{e}_{KU}$-invariant defines a charge lattice of H_3^{KU}-flux). *For $X \simeq S^{2d+3}$ (2.8), the $\widehat{e}_{KU}$-invariant (71) embeds the $\mathbb{Z}$-torsors of H_3^{KU}-fluxes (from Example 2.8), as $\mathbb{Z}$-sets, into the rational numbers $\mathbb{Q}$, such that this covers the classical Adams* e*-invariant (Def. 3.58) on the underlying Cohomotopy classes, in that we have a commuting diagram as follows:*

$$
\begin{tikzcd}
& \underset{\substack{\mathbb{Z}\text{-torsor of trivializations} \\ \text{of d-invariant } G_4^{KU}(c)}}{H_3^{KU}\mathrm{Fluxes}(X,c)} \arrow[r, hook] \arrow[d, "\substack{\text{forget} \\ \text{3-flux}}"] \arrow[rr, bend right, hook] \arrow[dr, phantom, "(\mathrm{pb})"] & H_3^{KU}\mathrm{Fluxes}(X) \arrow[r, "\widehat{e}_{KU}", "\substack{\text{observable on } H_3^{KU}\text{-fluxes} \\ \text{induced by Pontrjagin character}}"'] \arrow[d] & \mathbb{Q} \arrow[d] & \in & \mathbb{Z}\mathrm{Sets} \arrow[d, "\substack{\text{quotient by} \\ \mathbb{Z}\text{-action}}"] \\
\underset{\substack{\text{a Cohomotopy class} \\ \text{with d-invariant trivialization}}}{\Big\{\big([c]\,,\,H_3^{KU}(c)\big)\Big\}} \arrow[ur, hook, "\substack{\text{M5 charge in Cohomotopy} \\ \text{3-flux in K-theory}}"] \arrow[r] & \underset{\text{a Cohomotopy class}}{\big\{[c]\big\}} \arrow[r, hook] & \underset{\substack{\text{Cohomotopy in} \\ (4n-1)\text{st stem}}}{\pi^4\big(S^{4n+3}\big)} \arrow[r, "e_{\mathbb{R}}", "\text{Adams' } e_{\mathrm{Ad}}\text{-invariant}"'] & \mathbb{Q}/\mathbb{Z} & \in & \mathrm{Sets}\,.
\end{tikzcd}
\tag{72}
$$

We prove this as Theorem 3.62 below.

The C_3-field and vanishing of the Adams e-invariant. Prop. 2.11 says that the $\widehat{e}_{KU}$-invariant gives integer-spaced rational number-values to H_3-fluxes measured in topological K-theory. Hence *when* these rational numbers are all integers, *then* the $\widehat{e}_{KU}$-invariant gives a trivialization of the $\mathbb{Z}$-torsor of 3-fluxes (53), and hence a *choice of C_3-field* (54) seen in K-theory. But the diagrams (72) say that this is the case precisely when the classical Adams e-invariant vanishes:

$$\underbrace{\mathrm{e}_{\mathrm{Ad}}(c) = 0 \;\in\; \mathbb{Q}/\mathbb{Z}}_{\text{Adams e-invariant vanishes}} \qquad \Longleftrightarrow \qquad \overbrace{\begin{array}{ccc} \mathbb{Z} & \xleftarrow[\simeq]{\widehat{e}_{KU}} & H_3^{KU}\mathrm{Fluxes}(X,c) \\ n & \longleftarrow & \underbrace{[C_3^{KU}(c)]}_{\text{C-field}} + n \cdot \underbrace{[H^{KU}_{3,\mathrm{unit}}]}_{\text{closed 3-flux}} \end{array}}^{\widehat{e}_{KU}\text{-observable realizes C-field in } KU\text{-theory}} \tag{73}$$

Concretely, near M2-branes $X \simeq S^7$ (23) with Cohomotopy charge in the 3rd stem being an integral multiple $[c] = n \cdot [h_{\mathbb{H}}]$ of the quaternionic Hopf fibration (Example 2.7), the classical Adams invariant is (by Example 3.59):

$$\mathrm{e}_{\mathrm{Ad}}(n \cdot h_{\mathbb{H}}) = \left[\tfrac{n}{12}\right] \in \mathbb{Q}/\mathbb{Z}. \tag{74}$$

With (73) this means, in conclusion:

When measured in KU-theory, integral H_3-flux relative to a C_3-field exists in the vicinity of M2-branes precisely when the background Cohomotopy charge (52) *is an integer multiple of* 12.

Remark 2.12 (Integrality conditions in various theories)**. (i)** The above condition for integral H_3-flux and a C_3-field near M2-branes to be visible in KU-theory is similar to but a little weaker than the condition that H_3-flux is seen at all in stable Cohomotopy $\mathbb{S}$. The latter requires background Cohomotopy charge in multiples of 24 (Example 2.7) instead of just 12, and is witnessed by a spontaneous KK-compactification (28) on K3 (p. 16), with K3 regarded as a bare 4-manifold.
(ii) We next see in §2.6 below that when the observation of H_3 in KU-theory is made through $MS\mathrm{U} \to M\mathrm{U} \to K\mathrm{U}$, then the K3-compactification emerges again, now with its SU-structure.

2.6 $\mathrm{NS5}_{\mathrm{HET}}$-brane charge and Conner-Floyd's e-invariant

We had found in §2.5 that the refined $\widehat{e}_{KU}$-invariant of a 4-Cohomotopy charge near M2-branes, is a $\mathbb{Q}$-valued measure of the possible H_3-fluxes on such a background ($\mathbb{Z}$-valued when the classical Adams invariant vanishes). But since this Cohomotopy charge corresponds, under the Pontrjagin-construction, to the bordism class of a 3-manifold, interpreted in §2.2 as the polarization of the worldvolume of probe M2-branes into M5-branes on 3-manifolds, it ought to be the case that the refined Adams e-invariant has an equivalent interpretation as a Cobordism invariant measuring actual H_3-flux through these 3-folds.

In the form of a cobordism invariant, an M-theoretic interpretation of the Adams invariant has been discussed in [Sa14]. Here we expand on this interpretation, discussing how it connects to *Hypothesis H* and how, in this context, it encodes H_3-flux through 3-spheres and the Green-Schwarz anomaly cancellation of 24 5-branes transverse to a K3-surface.

Extended brane worldvolumes and $M\mathrm{U}/\mathbb{S}$-Cobordism. In the case that $E = Mf$ is Cobordism cohomology theory (Example 2.4) the Adams cofiber cohomology theory $Mf/\mathbb{S}$ (146) is Cobordism cohomology for f-structured manifolds with *framed boundaries* ([CF66, p. 97][Sto68, p. 102]):

$$\underset{f\text{-structure with framed boundaries}}{\text{Cobordism ring for}} \quad \Omega_\bullet^{f,\mathrm{Fr}} := \big(Mf/\mathbb{S}\big)_\bullet, \qquad \text{e.g.:} \quad \Omega_\bullet^{\mathrm{U},\mathrm{Fr}} := \big(M\mathrm{U}/\mathbb{S}\big)_\bullet, \;\; \Omega_\bullet^{S\mathrm{U},\mathrm{Fr}} := \big(MS\mathrm{U}/\mathbb{S}\big)_\bullet. \tag{75}$$

Therefore, Prop. 2.9 here says that H_3-fluxes measured in $M\mathrm{U}$-theory (§2.4) on spacetimes $X \simeq S^{3+2d}$ are seen via the universal Toda-bracket observable $O^{M\mathrm{U}/\mathbb{S}}$ (66) as bordism classes of $2d$-dimensional U-manifolds whose framed *boundaries* are the (polarized) brane worldvolumes from §2.2:

$$\begin{tikzcd}
\underset{\text{U-Cobordism cohomology}}{H_3\text{-fluxes seen in}} & H_3^{M\mathrm{U}}\mathrm{Fluxes}(X) \arrow[rr, "\text{universal Toda-bracket observable}", "O^{M\mathrm{U}/\mathbb{S}}"'] \arrow[d] & & (M\mathrm{U}/\mathbb{S})_{2d} \arrow[d, "\partial"] & \underset{\text{brane worldvolumes}}{\text{extended complex}} \\
\underset{\text{stable Cohomotopy}}{\text{underlying } G_4\text{-flux seen in}} & G_4^{\mathbb{S}}\mathrm{Fluxes}(X) \arrow[rr, "\simeq", "\text{Pontrjagin-Thom isomorphism}"'] & & (M\mathrm{Fr})_{2d-1} & \underset{\text{(compact part)}}{\text{brane worldvolumes}}
\end{tikzcd} \tag{76}$$

$$\Big(\big[c\big], \big[H_3^{M\mathrm{U}}(c)\big]\Big) \quad \overset{O^{MS\mathrm{U}/\mathbb{S}}}{\longmapsto} \quad
\begin{tikzcd}
S^{3+2d} \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Swarrow M^{2d}_{\mathrm{U,Fr}}"] & & (M\mathrm{U}/\mathbb{S})^4
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
S^{3+2d} \arrow[rr] \arrow[dd, "c"'] \arrow[ddrr, phantom, "\swarrow\,(\mathrm{po})"] & & * \arrow[dr] \arrow[dd] & \\
& & & * \arrow[dd, "0"] \\
S^4 \arrow[rr, "q_c"] \arrow[drrr, bend right=10, "G^{M\mathrm{U}}_{4,\mathrm{unit}}"'] \arrow[dd] \arrow[ddrr, phantom, "\swarrow\,(\mathrm{po})"] & & C_c \arrow[dr, dashed, "\vdash H_3^{M\mathrm{U}}(c)"] \arrow[dd] & \\
& & & M\mathrm{U}^4 \arrow[dd] \\
* \arrow[rr] \arrow[drrr, bend right=10, "0"'] & & S^{4+2d} \arrow[dr, dashed, "\vdash M^{2d}_{\mathrm{U,Fr}}"] & \\
& & & (\mathrm{MU}/\mathbb{S})^4 .
\end{tikzcd}$$

KK-Compactification on K3 and SU-Cobordism. The analogous statement (76) holds for measurement of M-brane charge in $E = MS\mathrm{U}$. Specifically, near M2-brane horizons $X \simeq S^7$ (23) the $O^{MS\mathrm{U}/\mathbb{S}}$-observable sees the $H^{MS\mathrm{U}}$-flux embodied as (the cobordism class of) a 4d SU-manifold with 3d Fr-boundaries, which by the discussion on p. 19 we may dually understand as Kaluza-Klein-compactification fibers punctured by transversal heterotic 5-branes:

$$\begin{tikzcd}
\underset{\text{near black M2-branes}}{\underset{S\text{U-Cobordism cohomology}}{H_3\text{-fluxes seen in}}} & H_3^{MS\mathrm{U}}\mathrm{Fluxes}(S^7) \arrow[rr, "\text{universal Toda-bracket observable}", "O^{MS\mathrm{U}}"'] \arrow[d] & & (MS\mathrm{U}/\mathbb{S})_{2d} \arrow[d, "\partial"] & \underset{\text{transversal 5-branes}}{\underset{\text{punctured by}}{\text{KK-compactification space}}} \\
\underset{\text{stable Cohomotopy}}{\text{underlying } G_4\text{-flux see in}} & G_4^{\mathbb{S}}\mathrm{Fluxes}(S^7) \arrow[rr, "\simeq", "\text{Pontrjagin-Thom isomorphism}"'] & & (M\mathrm{Fr})_3 & \underset{\text{transversal 5-branes}}{\text{3-spheres around}}
\end{tikzcd} \tag{77}$$

But now the abelian group $(MSU/\mathbb{S})_4$ is an extension of $(M\mathrm{Fr})_3$ by $(MSU)_4$ (Prop. 3.49) and the non-torsion elements in $(MSU)_\bullet$ are generated by Calabi-Yau manifolds (Prop. 3.64). Specifically, $(MSU)_4$ is spanned by (the cobordism class of) the K3-surface, regarded as an SU-manifold (Prop. 3.65). This then means that the observable $O^{MSU/\mathbb{S}}$ (66) sees the cancellation of the stable Cohomotopy charge of 24 probe M-branes near black M2 (28) witnessed by the appearance of a transversal K3-surface, as before in plain Cohomotopy (p. 18) but now seen with its SU-structure:

$$
\begin{tikzcd}[column sep=large]
 & & \underset{\text{measured in } MSU\text{-cohomology}}{\overset{H_3\text{-fluxes near black M2s}}{H_3^{MSU}\mathrm{Fluxes}(S^7)}} \arrow[r, "\substack{\text{underlying } G_4\text{-flux}\\ \text{in stable Cohomotopy}}"] \arrow[d, "O^{MSU/\mathbb{S}}"', "\substack{\text{Toda observable}\\ (66)}"] & G_4^{\mathbb{S}}\mathrm{Fluxes}(S^7) \arrow[d, equal, "\text{PT-isomorphism}"] & \\
0 \arrow[r] & \overset{\substack{\text{cobordism classes of}\\ \text{closed 4d } SU\text{-manifolds}}}{(MSU)_4} \arrow[r, hook, "i"] \arrow[d, equal, "\substack{(MSU)_\bullet \text{ is generated}\\ \text{by Calabi-Yau manifolds}\\ \text{Prop. 3.64, 3.65}}"'] & \underset{\substack{\text{cobordism classes of}\\ \text{compact 4d } SU\text{-manifolds}\\ \text{with Fr-boundaries}}}{(MSU/\mathbb{S})_4} \arrow[r, two heads, "\partial", "\text{boundary map}"'] & (M\mathrm{Fr})_3 \arrow[r] \arrow[d, equal] & 0 \\
 & \underset{\substack{\text{copies of}\\ \text{the K3-surface}}}{\mathbb{Z}\big\langle [\mathrm{K3}] \big\rangle} \arrow[r, hook] & \underset{\substack{\text{the cancellation of 24 units of M-brane charge}\\ \text{is seen by } O^{MSU/\mathbb{S}} \text{ as the appearance}\\ \text{of the K3-surface, with its } SU\text{-structure}}}{\mathbb{Z}\big\langle [\mathrm{K3}] \big\rangle \rtimes \mathbb{Z}_{24}} \arrow[r, two heads] & \underset{\substack{\text{in bulk around black M2-branes}\\ \text{charge of 24 probe branes cancels out}}}{\mathbb{Z}_{24}} &
\end{tikzcd}
\tag{78}
$$

D-brane charge and the Conner-Floyd e-invariant. Given a Kaluza-Klein-compactification on a complex curve M^4_{SU} with vanishing first Chern class, such as a K3-surface (78), the traditional expression for its intrinsic D-brane charge, given ([MM00, (1.1)][BMRS08, Cor. 8.5]) as evaluation of the *square root of the Todd class* (170) on the fundamental class $[M^4]$, reduces (173) to half the *Todd number* (Example 3.69):

$$
\underset{\text{of spacetime on } M^4_{SU}}{\text{intrinsic D-brane charge}} \quad \sqrt{\hat{A}}\,\big[M^4_{SU}\big] \overset{SU\text{-structure}}{=} \sqrt{\mathrm{Td}}\,\big[M^4_{SU}\big] \overset{\dim_{\mathbb{R}}=4}{=} \tfrac{1}{2}\mathrm{Td}\,\big[M^4_{SU}\big] \;\; \text{half the Todd number}
\tag{79}
$$

But the Todd character on SU-Cobordism is (Prop. 3.75) a multiplicative cohomology operation, equal to the composite of the Conner-Floyd KO-orientation (Prop. 3.66) with the Pontrjagin character. This implies (Prop. 3.76) that the intrinsic D-brane charge (79) is a Toda-bracket observable (56) on H_3^{MSU}-fluxes, that comes (via Remark 2.10) from the universal MSU-observable (77):

$$
\begin{tikzcd}
\underset{\text{measured in } MSU\text{-theory}}{\overset{H_3\text{-fluxes near black M2s}}{H_3^{MSU}\mathrm{Fluxes}(S^7)}} \arrow[r, "O^{MSU/\mathbb{S}}", "\substack{\text{universally observed as an}\\ SU\text{-manifold with Fr-bdry}}"'] \arrow[rr, bend right, "\text{observed as cofiber K-theory class}"'] \arrow[rrrr, bend right=30, "\substack{\widehat{e}_{KU} = \widehat{e}_{KO}\\ \text{refined e-invariant}}"'] & (MSU/\mathbb{S})_4 \arrow[r, "\sigma_{SU}/\mathbb{S}"] \arrow[rr, bend left, "\mathrm{Td}/\mathbb{S}"] \arrow[rrr, bend left=30, "\mathrm{Td}/\mathbb{S}", "\text{Todd class of cplx mfds with frmd bdry}"'] & (KO/\mathbb{S})_4 \arrow[r, "\mathrm{ph}/\mathbb{S}"] & \big((H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}\big)_4 \arrow[r, "\mathrm{spl}_0"] & \mathbb{Q}
\end{tikzcd}
$$

$$
\big(G_4^{\mathbb{S}}(c), H_3^{MSU}(c)\big) \mapsto
\begin{tikzcd}
S^7 \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\swarrow \mathrm{Td}[M^4_{SU,\mathrm{Fr}}]"] \\
\big((H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}\big)^8
\end{tikzcd}
:=
\begin{tikzcd}
S^7 \arrow[r] \arrow[d, "c"] \arrow[dr, phantom, "\swarrow (\mathrm{po})"] & \ast \arrow[dr] \arrow[dd] & & & \\
 & & \ast \arrow[dd, "0"] & & \\
S^4 \arrow[r] \arrow[drr, "\Sigma^4(1^{MU})"] \arrow[dd] \arrow[dr, phantom, "\swarrow (\mathrm{po})"] & C_c \arrow[dr, dashed, "\vdash H_3^{MSU}(c)"] \arrow[dd] & & & \\
 & & (MSU)^4 \arrow[dd] \arrow[dr, "\sigma_{SU}"] \arrow[ddrr, bend left, "\mathrm{Td}"] & & \\
\ast \arrow[r] \arrow[dr, "0"] & S^{2(n+d)} \arrow[dr, dashed, "M^4_{SU,\mathrm{Fr}}"] & & KO^4 \arrow[dr, "\mathrm{ph}"] \arrow[dd] & \\
 & & (MSU/\mathbb{S})^4 \arrow[dr, "\sigma_{U}/\mathbb{S}"] \arrow[ddrr, bend left, "\mathrm{Td}/\mathbb{S}"] & & (H^{\mathrm{ev}}\mathbb{Q})^4 \arrow[dd] \\
 & & & (KO/\mathbb{S})^4 \arrow[dr, "\mathrm{ph}/\mathbb{S}"] & \\
 & & & & \big((H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}\big)^4
\end{tikzcd}
\tag{80}
$$

Here the top composite operation in (80)

$$
\begin{tikzcd}
{[H_3^{MSU}(c)]} \arrow[r, mapsto] \arrow[d, mapsto] & {[M^4_{SU,\mathrm{Fr}}]} \arrow[r, mapsto] & \mathrm{Td}[M^4_{SU,\mathrm{Fr}}] \arrow[d, mapsto] & \in & \mathbb{Q} \arrow[d] \\
{[c]} \arrow[rr, mapsto, "\mathrm{e}_{\mathrm{CF}}", "\text{Conner-Floyd e-invariant}"'] & & \mathrm{Td}[M^4_{SU,\mathrm{Fr}}] & & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}
\tag{81}
$$

is the construction of Conner-Floyd's *geometric cobordism-theoretic* e-invariant (189). Our Toda-bracket-theoretic construction (80) makes manifest (with a glance at Prop. 188) that this is *equal* to the refined $\widehat{e}_{KU}$-invariant (71) on the induced H_3^{KU}-fluxes, as shown by the commuting diagram at the top of (80). But, by Prop. 2.11, this implies that modulo integers both coincide with the Adams e-invariant; which is Conner-Floyd's classical theorem (Prop. 3.77):

$$
\begin{tikzcd}
\underset{\substack{\text{universal Toda-bracket observable} \\ \text{on } H_3\text{-flux measured in } MSU}}{\widehat{e}_{KU}\big(H_3^{MSU}(c)\big)} \arrow[r, equal] \arrow[d, mapsto] & \underset{\substack{\text{actual } H_3\text{-flux} \\ \text{through punctures (83)}}}{\mathrm{Td}\big[M^4_{SU,\mathrm{Fr}}\big]} \arrow[d, mapsto] & \in & \mathbb{Q} \arrow[d] \\
\underset{\substack{\text{Adams} \\ \text{e-invariant}}}{\mathrm{e}_{\mathrm{Ad}}(c)} \arrow[r, equal] & \underset{\substack{\text{Conner-Floyd} \\ \text{e-invariant}}}{\mathrm{e}_{\mathrm{CF}}\big(\partial M^4_{SU,\mathrm{Fr}}\big)} & \in & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}
\tag{82}
$$

Green-Schwarz mechanism for NS5**-Branes and Twisted 3-Cohomotopy.** The identification (82) reveals differential-geometric content underlying the abstract assignment $\widehat{e}_{KU}$ of rational charges to H_3^{KU}-fluxes (from Prop. 2.11): In the case that $M^4_{SU,\mathrm{Fr}}$ is a a punctured KK-compactification space (77) with non-empty framed boundary being a disjoint union of 3-spheres around transversal 5-branes (p. 19), it carries (by Prop. 3.73) a differential 3-form H_3 whose closure is twisted by the Euler-form and whose boundary integral equals $\frac{1}{12}$ of the the rational Todd number (82)

$$
\underset{\substack{\text{KK-compactification} \\ \text{space}}}{M^4_{SU,\mathrm{Fr}}}, \quad \text{s.t.} \quad \partial M^4_{SU} = \underbrace{\underset{1\leq k\leq n}{\sqcup} S^3_k}_{\substack{\text{spheres around} \\ \text{transversal 5-branes}}} \quad \Rightarrow \quad \underset{\substack{\text{ordinary NS} \\ \text{3-form flux}}}{\exists H_3} \in \Omega^3_{\mathrm{dR}}\big(M^4_{SU,\mathrm{Fr}}\big), \quad \text{s.t.} \quad \left\{ \begin{array}{l} \overset{\substack{\text{Green-Schwarz Bianchi identity} \\ \text{(for vanishing gauge flux)}}}{d\,H_3 = \chi_4\big(\nabla^{TM^4}\big)}, \\ \mathrm{Td}[M^4_{U,\mathrm{Fr}}] = \frac{1}{12} \underset{1\leq k\leq n}{\sum} \underbrace{\int_{S^3_k} H_3}_{\substack{\text{ordinary charge of } k\text{th stack} \\ \text{of transversal 5-branes}}}. \end{array} \right.
\tag{83}
$$

Notice that the proof of this implication, in Prop. 3.73 below, proceeds (via Lemma 3.72) by regarding H_3 as the Cohomotopical character of a cocycle in J-twisted 3-Cohomotopy on M^4; which is made possible by the Poincaré-Hopf theorem (just as discussed for H_7-flux in [FSS19b, §2.5]). This is in line with the emergence, from *Hypothesis H*, of H_3-flux quantization in twisted 3-Cohomotopy seen in [FSS20a][SS20c].

In conclusion, so far:

> *When the Cohomotopy charge* (52) *near black M2-branes* (23) *is a multiple of 12 (in particular a multiple of 24), then the integer values of the H_3^{KU}-flux that are abstractly observed by the $\widehat{e}_{KU}$-invariant (Prop. 2.11) equals ordinary $H_3^{H\mathbb{R}}$-flux (i.e. periods of a differential 3-form H_3) through the 3-spheres around transversal brane insertion points in the complex surface on which spacetime has spontaneously compactified via* (28).

Specifically, when $M^4_{SU,\mathrm{Fr}}$ in (83) is a punctured K3-surface (28), now regarded with its SU-structure, then this boundary integral (83) is (still by Prop. 3.73) the Euler number of K3:

$$
M^4_{SU,\mathrm{Fr}} = \mathrm{K3} \setminus \underset{1\leq k\leq n}{\sqcup} D^4_k \quad \Rightarrow \quad \underbrace{\underset{1\leq k\leq n}{\sum} \int_{S^3_k} H^3}_{\substack{\text{total charge of} \\ \text{transversal 5-branes}}} = 24\,.
\tag{84}
$$

However, by comparison to the traditional string theory lore (e.g. [Sch97, p. 50]), we see that:

- (83) is the Green-Schwarz mechanism for the ordinary 3-flux H_3 of heterotic NS5-brane sources (as on p. 2.2) transversal to the compactification space (here in the case of vanishing gauge flux);
- (84) is Green-Schwarz mechanism specified to K3-compactifications, implying that transversal heterotic NS5-branes must appear in multiples of 24 in order to cancel to total flux in the compactification space (heterotic tadpole cancellation).

In further conclusion, we find the following correspondence, under *Hypothesis H*, between (a) the Green-Schwarz mechanism for heterotic NS5-branes transversal to K3-compactifications and (b) the construction and properties of the Conner-Floyd e-invariant:

GS anomaly cancellation in KK-compactification on K3	$\longleftrightarrow$	**Vanishing of $\frac{1}{2}\mathrm{e}_{\mathrm{CF}}$-invariant on Cohomotopy charge** $S^7 \xrightarrow{c} S^4$	
Asymptotic boundary of n p-branes of codim $= 4$ carrying charges $\{Q_k \in \ltimes\}_{k=1}^{n}$	$\mathbb{R}^{p,1} \times \big(\underset{n}{\sqcup}\, S^3\big)$	Class in framed Cobordism $\begin{array}{ccc} \Big[\overset{n}{\underset{k=1}{\sqcup}} S^3_{\mathrm{nfr}=Q_k}\Big] & \mapsto & \Big[S^7 \xrightarrow{\sum_{i=k}^{N} Q_k \cdot h_{\mathbb{H}}} S^4\Big] \\ \cap & & \cap \\ (M\mathrm{Fr})_3 & \underset{\simeq}{\xrightarrow{\mathrm{Src}}} & \mathbb{S}_3 \end{array}$	§2.2
Spacetime compactified on generalized CY manifold Y^4 around n branes of codim $= 4$	$\mathbb{R}^{p,1} \times \big(Y^4 \setminus \underset{n}{\sqcup} D^4\big)$	Lift to $\big[Y^4 \setminus \underset{n}{\sqcup} D^4\big] \in (MS\mathrm{U}/\mathbb{S})_4$ $\downarrow \partial$ of $\Big[\overset{N}{\underset{i=1}{\sqcup}} S^3_{\mathrm{nfr}=Q_i}\Big] \in \mathbb{S}_3 \;\simeq\; (M\mathrm{Fr})_3$	(82)
D-brane charge in compact space	$\sqrt{\mathrm{Td}}\big[Y^4 \setminus \underset{N}{\sqcup} D^4\big] = \frac{1}{2} \int\limits_{Y^4 \setminus \underset{N}{\sqcup} D^4} \frac{1}{12} c_2(\nabla)$ $= \frac{1}{24} \sum_{i=1}^{N} Q_i$	Conner-Floyd bordism formula for $\frac{1}{2}\mathrm{e}_{\mathrm{Ad}}$-invariant	
3-flux density on $Y^4 \setminus \underset{n}{\sqcup} D^4$ satisfying GS Bianchi identity	$d H_3 = \frac{1}{2} p_1(\nabla)$ $= c_2(\nabla) = \chi_4(\nabla)$ (by SU-structure)	J-twisted 3-cohomotopical character	(180)
Charge of kth stack of branes	$\int_{S^3_k} H_3 \;=\; \deg\big(h\vert_{S^3_k}\big)$	Poincaré-Hopf index	(178)
GS-anomaly-free 5-brane charge	$\int_{\sqcup_{1 \le k \le n} S^3_k} H_3 \overset{!}{=} \chi_4[Y^4]$	Poincaré-Hopf theorem	(181)
The case of $Y^4 = \mathrm{K3}$	$\int_{\sqcup_{1 \le k \le n} S^3_k} H_3 \overset{!}{=} 24$	Vanishing $\frac{1}{2}\mathrm{e}_{\mathrm{Ad}}$-invariant, Vanishing d-invariant in $E = \mathbb{S}$	(74) (52)

2.7 M2-Brane Page charge and the Hopf invariant

Page charge and the E-Steenrod-Whitehead integral formula. We consider again the case that spacetime is homotopy equivalent to a 7-sphere, $X \simeq S^7$, as in the vicinity of M2-branes (23), hence with H_3^E-flux (48) of the form shown on the left here:

$$
\begin{tikzcd}
\underset{\text{of M2-brane}}{\text{near-horizon}}\; S^7 \arrow[r] \arrow[d, "c"', "\text{general Cohomotopy charge}"'] & * \arrow[d] \arrow[dl, phantom, "\Swarrow\, d\left(H_3^E(c)\right) = G_4^E(c)"] \\
S^4 \arrow[r, "G^E_{4,\mathrm{unit}}"] & E^4 .
\end{tikzcd}
\qquad
\begin{tikzcd}
S^4 \arrow[r, "G^E_{4,\mathrm{unit}} = \Sigma^4(1^E)"] \arrow[d] & E^4 \arrow[d, "(-)^2"] \arrow[dl, phantom, "\Swarrow\, d\left(2G^E_{7,\mathrm{unit}}\right) = -G^E_{4,\mathrm{unit}} \cup G^E_{4,\mathrm{unit}}"] \\
* \arrow[r, "0"'] & E^8
\end{tikzcd}
\tag{85}
$$

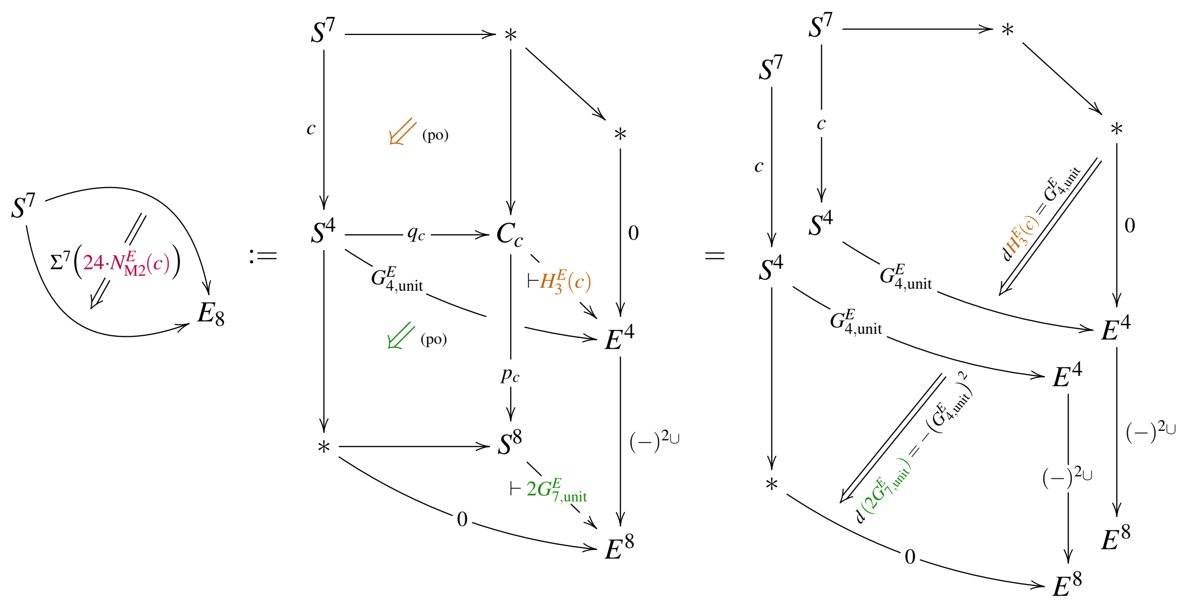
for any Cohomotopy charge c, measured in any multiplicative cohomology theory E (35).

The diagram on the right of (85) recalls the general trivialization (38) of the cup square of the unit $G^E_{4,\mathrm{unit}}$-flux. This induces the corresponding Toda-bracket observable (56) on H_3^E, given as the pasting diagram of the two diagrams in (85). We now denote this Toda-bracket observable induced from the trivialization of $(G_4)^2$ by

$$
24\, N^E_{\mathrm{M2}}(c) \;\in\; \pi_1 \mathrm{Maps}\big(S^7, E^8\big) \;\simeq\; E_0 \,,
$$

where a formal normalization factor of 24 is included for reasons discussed in Remark 4.1. Hence this element is, by definition, given by the following pasting composite:

$$
\underset{\substack{E\text{-number of M2s =}\\ E\text{-Hopf invariant}}}{24\, N^E_{\mathrm{M2}}(c)} \quad := \quad \underset{\substack{E\text{-Page charge =}\\ E\text{-Steenrod-Whitehead integral}}}{\int_{S^7} \Big(H_3^E(c) \cup G_4^E(c) \;+\; 2G_7^E(c) \Big)} \qquad \text{for: } \begin{cases} d\, H_3^E(c) = G_4^E(c) \\ d\left(2G_7^E(c)\right) = -\big(G_4^E(c)\big)^2 \end{cases}
\tag{86}
$$



Here
(a) on the right we highlight the two homotopies (85) and (38) that are being composed, while
(b) in the middle we are showing their classifying maps via (58), and **(c)** on the left we highlight that the composite is a single self-homotopy of the 0-cocycle on S^7 in $\widetilde{E}^8(-)$.

This construction clearly exists for every multiplicative cohomology theory E (and it generalizes in the evident way to maps of the form $S^{4n-1} \to S^{2n}$). We observe now that it subsumes as special cases all of the following classical constructions:

- In ordinary cohomology $E = H\mathbb{Z}$...
  - ... the middle diagram in (86) is the diagrammatic expression of the **classical Hopf invariant** of c: Namely, the homotopy-commutativity of the part

$$
\begin{tikzcd}
C_c \arrow[d, "p_c"'] \arrow[r, "\vdash H_3^{H\mathbb{Z}}(c)"] & K(\mathbb{Z},4) \arrow[d, "(-)^{2\cup}"] \\
S^8 \arrow[r, "\vdash G_7^{H\mathbb{Z}}(c)"'] & K(\mathbb{Z},8)
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{aligned}
\big[\vdash H_3^{H\mathbb{Z}}(c)\big]^2 & = p_c^*\big[\vdash G_{7,\mathrm{unit}}^{H\mathbb{Z}}\big] \\
& = p_c^*\Big(h(c)\cdot \Sigma^8\big(1^{H\mathbb{Z}}\big)\Big) \;\in\; \widetilde{H\mathbb{Z}}^8(C_c)
\end{aligned}
\tag{87}
$$

  expresses that the cup square of the degree-4 cohomology generator $\big[\vdash H_3^{H\mathbb{Z}}\big]$ must be *some* integer multiple $h(c)$ of the degree-8 cohomology generator, which in turn must equal the pullback of that same multiple of the canonical generator on S^8, the latter multiple being $\big[\vdash G_7^{H\mathbb{Z}}\big]$. That multiple is the classical definition of the Hopf invariant $h(c)$ (e.g. [MT86, p. 33]).

  But since, by the pasting law (59), the map $\vdash G_7^{H\mathbb{Z}}$ classifies also the total homotopy filling the total rectangle in (86), also that total rectangle and hence the left hand side of (86) expresses the classical Hopf invariant.
  - ... the right hand diagram in (86) is manifestly the diagrammatic expression of **Steenrod's functional cup product** ([St49]). If it happens on representatives that $G_7^{H\mathbb{Z}}(c) = 0$ then this reduces, evidently, to **Whitehead's integral formula** for the Hopf invariant ([Wh47], review in [BT82, Prop. 7.22]). That this specialization should not be necessary was already suggested in [Ha78, p. 17]; the more general formula found renewed attention in [SW08, Ex. 1.9]. We gave a proof that the general formula (86) computes the Hopf invariant in [FSS19c, Prop. 4.6], using rational homotopy theory (as a special case of the generalization of this statement to classes in tangentially J-twisted Cohomotopy). The above pasting diagram decomposition is another proof.
  - ...under Hypothesis H – whereby the terms $G_4^E(c)$ (31), $G_7^E(c)$ (38), $H_3^E(c)$ (48), in the *Steenrod-Whitehead homotopy period integral formula for the Hopf invariant* (86) are interpreted as M-theory fluxes, and using Examples 2.5, 2.6, 2.14 for their incarnation in ordinary cohomology – the formula (86) manifestly expresses [FSS19c] the **Page charge** carried by M2-branes surrounded by the 7-sphere (23) according to [Pa83, (8)][DS91, (43)] (review in [BLMP13, (1.5.2)]).

  Just as the total electric flux through a 2-sphere around a collection of electrons, hence their total electric charge (as on p. 3) is proportional to their number, with the proportionality constant being the unit of electric charge, so the Page charge of M2-branes should be proportional to their number N_{M2}. In (86) we declared that proportionality constant, under Hypothesis H, to be 24, due to the arguments listed in Remark 4.1.

- In complex K-theory $E = K\mathrm{U}$...
  - ... the middle diagram in (86) is the diagammatic expression of the **K-theoretic Hopf invariant** of [AA66]: Namely, the homotopy-commutativity of

$$
\begin{tikzcd}
C_c \arrow[d, "p_c"'] \arrow[r, "\vdash H_3^{K\mathrm{U}}(c)"] & B\mathrm{U} \arrow[d, "(-)^{2\cup}"] \\
S^8 \arrow[r, "\vdash G_7^{K\mathrm{U}}(c)"'] & B\mathrm{U}
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{aligned}
\big[\vdash H_3^{K\mathrm{U}}(c)\big]^2 & = p_c^*\big[\vdash G_{7,\mathrm{unit}}^{K\mathrm{U}}\big] \\
& = p_c^*\Big(h(c)\cdot \Sigma^8\big(1^{K\mathrm{U}}\big)\Big) \;\in\; \widetilde{K\mathrm{U}}^8(C_c)
\end{aligned}
\tag{88}
$$

  now expresses (proceeding just as in the discussion of the Adams e-invariant around (158)) the *choice* of a generator $\big[\vdash H_3^{K\mathrm{U}}(c)\big]$ lifting the canonical generator on S^4, whose cup square is necessarily an integer multiple $h_{K\mathrm{U}}(c)$ of the canonical generator $\big[\vdash H^{K\mathrm{U}}{}_{7,\mathrm{unit}}\big]$ pulled back from S^8. This multiple is the K-theoretic Hopf invariant (review in [Wir12, p. 50][Qu14, §26.1]).

2.8 M2-brane horizons and Ravenel E-orientations

In further specialization of the above discussion of the fluxless (§2.4) vicinity of M2-branes (23) we consider now the case that H_3^E-flux (48) is chosen for the *unit* Cohomotopy charge $[c] = [h_{\mathbb{H}}]$ (52).

This case is of interest because it gives a *universal* choice of H_3^E-flux (48), in that it implies $H_3^E(c)$-flux for any integral Cohomotopy charge multiple $c := n \cdot c_{\mathrm{unit}}$, by pullback along $[n] \in \pi^7(S^7)$, hence by the pasting composite shown on the right here:

$$
\begin{array}{c}\text{3-flux for}\\ \text{integer multiple}\\ \text{Cohomotopy charge}\end{array}
\begin{tikzcd}
S^7 \arrow[r] \arrow[d, "{c := n\cdot h_{\mathbb{H}}}"'] & * \arrow[d, "0"] \arrow[dl, Rightarrow, "H_3^E(c)"] \\
S^4 \arrow[r, "G^E_{4,\mathrm{unit}}"'] & E^4
\end{tikzcd}
:=
\begin{tikzcd}
 & \text{multiple winding} & \\
S^7 \arrow[r, "n"'] \arrow[d, "{c := n\cdot h_{\mathrm{unit}}}"'] & S^7 \arrow[r] \arrow[d, "{c_{\mathrm{unit}} := h_{\mathbb{H}}}"'] & * \arrow[d, "0"] \arrow[dl, Rightarrow, "H_3^E(c_{\mathrm{unit}})"] \\
S^4 \arrow[r, equal] & S^4 \arrow[r, "G^E_{4,\mathrm{unit}}"'] & E^4
\end{tikzcd}
\begin{array}{c}\text{universal 3-flux for}\\ \text{unit (quat. Hopf fib.)}\\ \text{Cohomotopy charge}\end{array}
\tag{89}
$$

Universal M2-brane backgrounds and Quaternionic orientation in E-Cohomology. In the vicinity of M2-branes, hence for $X \simeq S^7$ (23), and given unit Cohomotopy charge $[c] := 1 \cdot [h_{\mathbb{H}}]$ (52), the corresponding homotopy cofiber space (57) is the quaternionic projective plane $C_{h_{\mathbb{H}}} \simeq \mathbb{H}P^2$ (Remark 3.85). Therefore, the choice of universal 3-flux H_3^E (89) is equivalently (63) the choice of a class

$$
\left[\tfrac{1}{2}p_1^E\right] \;:=\; \left[\vdash H_3^E(c_{\mathrm{unit}})\right] \;\in\; E^4\big(\mathbb{H}P^2\big)
$$

fitting into the diagram shown on the right here:

$$
\begin{array}{c}\text{near-horizon}\\ \text{of M2-brane}\\[2em] \text{unit}\\ \text{Cohomotopy charge}\end{array}
\begin{tikzcd}
 & \text{universal 3-flux seen in } E\text{-cohomology} & \\
S^7 \arrow[rr] \arrow[d, "c_{\mathrm{unit}}"'] & & * \arrow[d] \arrow[dll, Rightarrow, "{d\left(H_3^E(c_{\mathrm{unit}})\right) = G_4^E(c_{\mathrm{unit}})}"'] \\
S^4 \arrow[rr, "G^E_{4,\mathrm{unit}}"'] & & E^4
\end{tikzcd}
\simeq
\begin{tikzcd}
 & \text{quaternionic orientation in } E\text{-cohomology to 2nd stage} & \\
S^7 \arrow[r] \arrow[d, "h_{\mathbb{H}}"'] & * \arrow[d] \arrow[dl, Rightarrow, "\text{(po)}"] \arrow[ddr, bend left, "0"] & \\
S^4 \arrow[r, "q"] \arrow[drr, bend right, "\Sigma^4 1"'] & \mathbb{H}P^2 \arrow[dr, dashed, "\frac{1}{2}p_1^E"] & \\
 & & E^4
\end{tikzcd}
\tag{90}
$$

But such a $\frac{1}{2}p_1^E$ is equivalently the choice of a universal *10-dimensional quaternionic orientation* on E-cohomology, (Def. 3.92).

More widely familiar than quaternionic oriented cohomology theory is complex oriented cohomology theory (Def. 3.91). But in fact, ($4k+2$-dimensional) complex orientations c_1^E *induce* ($4k+2$-dimensional) quaternionic orientations (Theorem 3.99 below) by taking the first E-Pontrjagin class of a quaternionic vector bundle to be the second Conner-Floyd E-Chern class c_2^E (Prop. 3.97) of the underlying complex vector bundle (3.99)

$$
\tfrac{1}{2}p_1^E(-) \;:=\; c_2^E\big((-)_{\mathbb{C}}\big)\,. \tag{91}
$$

In conclusion:

> *Under Hypothesis H, with M-brane charge measured in E-cohomology* (35)*, the class of a universal 3-flux* $H_3^E(c_{\mathrm{unit}})$ (89) *near the horizon of M2-branes* (23) *is equivalent to a choice of 10-dimensional quaternionic E-orientation, such as is induced from a 10-dimensional complex E-orientation.*

$$
\begin{array}{ccccc}
\begin{array}{c}\textbf{\textit{E} is a complex oriented}\\ \textbf{multiplicative cohomology}\\ \text{(in 10 dimensions)}\end{array}
& \overset{\text{Thm. 3.99}}{\Longrightarrow} &
\begin{array}{c}\textbf{\textit{E} is a quaternionic oriented}\\ \textbf{multiplicative cohomology}\\ \text{(in 10 dimensions)}\end{array}
& \overset{(90)}{\Longleftrightarrow} &
\begin{array}{c}\textbf{\textit{E}-cohomology detects}\\ \textbf{unit charges near M2-branes}\\ \text{(under Hypothesis H)}\end{array}
\\[1em]
\begin{tikzcd}
\mathbb{C}P^1 \arrow[r, "\Sigma^2(1^E)"] \arrow[d, hook] & E_2 \\
\mathbb{C}P^5 \arrow[ur, dashed, "c_1^E"'] &
\end{tikzcd}
& &
\begin{tikzcd}
\mathbb{H}P^1 \arrow[r, "\Sigma^4(1^E)"] \arrow[d, hook] & E^4 \\
\mathbb{H}P^2 \arrow[ur, dashed, "{\frac{1}{2}p_1^E \; (= c_2^E)}"'] &
\end{tikzcd}
& &
\begin{tikzcd}
S^7 \arrow[r] \arrow[d, "c_{\mathrm{unit}}"'] & * \arrow[d] \arrow[dl, Rightarrow, dashed, "{d\left(H_3^E(c_{\mathrm{unit}})\right) = G_4^E(c_{\mathrm{unit}})}"'] \\
S^4 \arrow[r, "G^E_{4,\mathrm{unit}}"'] & E^4
\end{tikzcd}
\end{array}
\tag{92}
$$

Remark 2.13 (Emergence of line bundle orientations in 10 dimensions)**.** The 10-dimensional complex E-orientations appearing in (92) are *exactly* the data necessary to (fiberwise) orient general complex line bundles over 10-dimensional manifolds X^{10} (as discussed in §3.8), hence are *exactly* the data needed to orient general heterotic line bundles in heterotic M-theory (discussed in §2.9) on globally hyperbolic 11-dimensional spacetimes [9] of the form $\mathbb{R}^{0,1} \times X^{10}$. This seems remarkable, as the number 10 *emerges* here, via Theorem 3.99, from just *Hypothesis H* and the consideration of black codimension-8 brane backgrounds in (89) – see Remark 3.100.

Example 2.14 (3-Flux in ordinary cohomology)**.** When $E = HR$ is ordinary cohomology, there is a unique quaternionic orientation (by Example 3.95), equivalently a unique choice of 3-flux $H^{HA}_{3,\mathrm{unit}}$ (by (50) and because $(\widetilde{HA})^3(S^7) \simeq \pi_4(HA) = 0$). In the case $A = \mathbb{R}$, via Example 2.5 and under the fundamental theorem of rational homotopy theory (see [FSS20c, Prop. 3.60]) this corresponds to the degree-3 generator in the relative Sullivan model for the quaternionic Hopf fibration [FSS19b, Prop. 3.20] ([FSS19c, Thm. 3.4]).

[9] Note that the heterotic theory viewed as a cobordism is studied in [Sa11c].

2.9 M5$_{\mathrm{HET}}$-brane charge and Conner-Floyd classes

The C-field in heterotic M-theory and the Twistor fibration. The (class of the) quaternionic Hopf fibration (23), representing unit Cohomotopy charge in the vicinity of M2-brane horizons (89), naturally factors through $\mathbb{C}P^3$, via the complex Hopf fibration followed by the Atiyah-Penrose *twistor fibration* $t_{\mathbb{H}}$ (Remark 3.88, see [FSS20b, §2] for more pointers):

$$
\begin{tikzcd}
 & \mathbb{C}P^3 \arrow[d, "t_{\mathbb{H}}", "\substack{\text{twistor}\\\text{fibration}}"'] \\
S^7 \arrow[ur, "h_{\mathbb{C}}", "\substack{\text{complex}\\\text{Hopf fibration}}"'] \arrow[r, "h_{\mathbb{H}}", "\substack{\text{quaternionic}\\\text{Hopf fibration}}"'] & S^4 .
\end{tikzcd}
\tag{93}
$$

We may think also of $\mathbb{C}P^3$ as the classifying space of a non-abelian cohomology theory (6), which is thus a close twistorial cousin of 4-Cohomotopy (12), called *twistorial Cohomotopy* in [FSS20b, §3.2][FSS20c, Ex. 2.44][SS20c, Def. 2.48]:

$$
\begin{array}{rcl}
\text{twistorial Cohomotopy } \widetilde{\mathcal{T}}(X) & := & \pi_0 \mathrm{Maps}^{*/}\big(X\,,\,\mathbb{C}P^4\big) \\
\substack{\text{pushforward along}\\\text{twistor fibration}} \quad (t_{\mathbb{H}})_* \big\downarrow \quad & & \\
\text{4-Cohomotopy } \widetilde{\pi}^4(X) & := & \pi_0 \mathrm{Maps}^{*/}\big(X\,,\,S^4\big)\,.
\end{array}
\tag{94}
$$

Evidence discussed in [FSS20b][FSS20c][SS20c] suggests that *Hypothesis H* extends from M-theory to *heterotic M-theory* (Hořava-Witten theory [HW95][Wi96][HW96][DOPW99][DOPW00][Ov02]) by lifting M-brane charges from 4-Cohomotopy (12) to twistorial Cohomotopy (94) (generally tangentially J-twisted, but here considered on homotopically flat spacetimes and hence for vanishing twist, see Remark 2.2).

Concretely, the lift of the Cohomotopical character map (13) through (94) turns out [FSS20b, Prop. 3.9][SS20c, Thm. 1.1][FSS20c, Ex. 5.23] to correspond to the appearance of the rational second Chern class

$$
c_2\big(\mathscr{L} \oplus \mathscr{L}^*\big) \;=\; c_1(\mathscr{L}) \wedge c_1(\mathscr{L}^*) \;=\; -\big(c_1(\mathscr{L})\big)^2 \;\in\; H^4(X;\mathbb{R})
$$

of a $\big(\mathrm{U}(1) \hookrightarrow S\mathrm{U}(2) \hookrightarrow E_8\big)$-bundle (a "heterotic line bundle" [AGLP12][AGLP12][BBL17][FSS20c], here of rank 2 as considered in [ADO20a, §4.2][ADO20b, §2.2]):

$$
\begin{tikzcd}
\overset{\text{complex line bundle}}{\mathscr{L}} \arrow[r, mapsto, "\substack{\text{heterotic line bundle}\\\text{of rank 2}}"'] & \overset{\substack{\text{rank-2 complex vector bundle}\\\text{with vanishing first Chern class}}}{\mathscr{L} \oplus \mathscr{L}^*} \\
\mathrm{U}(1) \arrow[r, hook, "c \mapsto \mathrm{diag}(c,c^*)"'] & S\mathrm{U}(2) \;\; \text{structure group}
\end{tikzcd}
\tag{95}
$$

together with a 3-form $H_3^{H\mathbb{R}}$ satisfying

$$
\begin{aligned}
d\,H_3^{H\mathbb{R}} &= G_4^{H\mathbb{R}} - c_1(\mathscr{L}) \wedge c_1(\mathscr{L}) \\
&= G_4^{H\mathbb{R}} + c_2\big(\mathscr{L} \oplus \mathscr{L}^*\big).
\end{aligned}
\tag{96}
$$

This *Bianchi identity* is just that of the Hořava-Witten Green-Schwarz mechanism in heterotic M-theory near MO9-planes [HW96, (1.13)][DFM03, (3.9)][SS20c, (161)], specialized to rank-2 heterotic line bundles (95) and shown here for vanishing $\frac{1}{2}p_1$ (following Remark 2.2, see [FSS20b][SS20c][FSS20c] for the general statement including the summand of $\frac{1}{2}p_1$).

We now observe that the form of the M-theoretic Green-Schwarz mechanism (96) appears, from *Hypothesis*, not just in the rational approximation $E = H\mathbb{R}$, but is visible for measurement (34) of G_4-flux in any multiplicative generalized cohomology theory E:

Heterotic line bundles and the E-Whitney sum rule. Given $H_3^E(c)$-flux (48) for the unit 4-Cohomotopy charge $c_{\mathrm{unit}} = [h_{\mathbb{H}}]$ near the horizon of black M2-branes (89), which comes from a 10d complex E-orientation (92), we obtain a homotopy

$$
\left.\begin{array}{c} H_3^E(c_{\mathrm{unit}})\text{-flux induced from} \\ \text{10d complex } E\text{-orientation (92)}\end{array}\right\} \Rightarrow
\begin{tikzcd}
 & \mathbb{C}P^3 \arrow[r, "c_1^E \cdot c_{1*}^E"] \arrow[d, "t_{\mathbb{H}}"'] & E^4 \arrow[d, equal] \arrow[dl, phantom, "\Leftarrow\, \mathrm{d}\, H^E_{3,\mathrm{het}} = G^E_{4,\mathrm{unit}} - c_1^E\cdot c_{1*}^E"] \\
S^7 \arrow[ur, "t_{\mathbb{C}}"', "\substack{\text{lift to}\\ \text{twistorial Cohomotopy}}"] \arrow[r, "h_{\mathbb{H}}", "\text{unit Cohomotopy charge}"'] & S^4 \arrow[r, "G^E_{4,\mathrm{unit}}"'] & E^4
\end{tikzcd}
\quad \substack{\text{heterotic } H_3\text{-flux}\\ \text{measured in}\\ E\text{-cohomology}}
\tag{97}
$$

by co-restricting the given $H_3^E(c_{\mathrm{unit}})$-flux from the quaternionic Hopf fibration $h_{\mathbb{H}}$ to the twistor fibration $t_{\mathbb{H}}$ (93) as follows:

$$
\begin{tikzcd}
S^7 \arrow[rr] \arrow[d, "h_{\mathbb{C}}"'] & & * \arrow[d] \\
\mathbb{C}P^3 \arrow[rr, "c_1^E\cdot c_{1*}^E"] \arrow[d, "t_{\mathbb{H}}"'] & & E^4 \arrow[d, equal] \arrow[dll, phantom, "\Leftarrow H^E_{3,\mathrm{het}}"] \\
\mathbb{H}P^1 \arrow[rr, bend right, "G^E_{4,\mathrm{unit}}"'] & & E^4
\end{tikzcd}
:=
\begin{tikzcd}
S^7 \arrow[r] \arrow[d, "h_{\mathbb{C}}"'] & * \arrow[r] \arrow[d] \arrow[dl, phantom, "(\mathrm{po})"] & * \arrow[d] \\
\mathbb{C}P^3 \arrow[r, hook] \arrow[d, "t_{\mathbb{H}}"'] & \mathbb{C}P^5 \arrow[r, "c_1^E\cdot c_{1*}^E"] \arrow[d, "\substack{\mathscr{L}\\ \downarrow \\ \mathscr{L}\oplus\mathscr{L}^*}"'] \arrow[dl, phantom, "(\mathrm{po})"] & E^4 \arrow[d, equal] \arrow[dl, phantom, "\Leftarrow \mathrm{WSR}"] \\
\mathbb{H}P^1 \arrow[r, hook] \arrow[rr, bend right, "G^E_{4,\mathrm{unit}}"'] & \mathbb{H}P^2 \arrow[r, "c_2^E"] & E^4
\end{tikzcd}
$$

$$
\begin{tikzcd}
S^7 \arrow[rr] \arrow[d, "h_{\mathbb{H}}"'] & & * \arrow[d] \arrow[dll, phantom, "\Leftarrow H_3^E(c_{\mathrm{unit}})"] \\
S^4 \arrow[rr, bend right, "G^E_{4,\mathrm{unit}}"'] & & E^4
\end{tikzcd}
=
\begin{tikzcd}
S^7 \arrow[r] \arrow[d, "h_{\mathbb{H}}"', "t_{\mathbb{H}}"] & * \arrow[r] \arrow[d] \arrow[dl, phantom, "(\mathrm{po})"] & * \arrow[d] \\
\mathbb{H}P^1 \arrow[r, hook] \arrow[rr, bend right, "G^E_{4,\mathrm{unit}}"'] & \mathbb{H}P^2 \arrow[r, "\frac{1}{2}p_1^E := c_2^E"] & E^4
\end{tikzcd}
$$

Here, going counter-clockwise from the bottom left:

- The bottom left homotopy is the given H_3^E-flux (89).
- The bottom right pasting decomposition is that through the corresponding cofiber space (64), which here is a 10d quaternionic E-orientation $\frac{1}{2}p_1^E$ (90) which, by assumption in (97), equals c_2^E (92).
- The top right pasting decomposition is
  - on the left the pasting law (59) applied to the twistor factorization (93) with the induced appearance (211) of complex projective 5-space and the classifying map (216) for the assignment of heterotic line bundles (95);
  - on the right the homotopy WSR exhibiting on classifying maps the E-Whiteney sum rule (225) for evaluation of the second Conner-Floyd E-Chern class c_2^E on these heterotic line bundles.
- The top left square is obtained from this by forming horizontal pasting composites of these homotopy squares.

In conclusion:

Hypothesis H implies not only the usual Hořava-Witten Green-Schwarz Bianchi identity (96) *in ordinary cohomology $E = H\mathbb{R}$ (for rank-2 heterotic line bundles* (95)*), but lifts* (97) *the structure of this relation to any multiplicative 10d complex-oriented cohomology theory E in which* (92) *cohomotopical M-brane charge near black M2-branes is being measured* (89).

3 The proofs

Here we provide rigorous mathematical background and proofs for the right hand side of the correspondence that is discussed in §2 and summarized in §4.

While a fair bit of the following background material is classical in algebraic topology, it has not previously found its connection to and application in mathematical physics and string theory. Moreover, much of this classical material seems to not have received modernized discussion before; for instance:

(i) Toda brackets are traditionally still discussed in the style of Toda's original article [Tod62] from 1962. While writing our diagrammatic re-formulation of Toda brackets in §3.4, we discovered that this perspective had been observed already by Hardie et al. in 1999 (see Remark 3.45 below); but their treatment, in turn, does not seem to have received attention nor application.

(ii) The essential reference for Conner-Floyd's geometric e-invariant (§3.6) (and related constructions such as the Conner-Floyd K-orientation, Prop. 3.66 below) has remained their original account [CF66] from 1966. We believe that our reformulation (80) via diagrammatic Toda-brackets makes usefully transparent not only the general relation to observables on trivializations of d-invariants ("H-fluxes", in our correspondence entry §2.4) but also the construction itself, notably the proof of its equivalence to the classical Adams e-invariant (Prop. 3.77 below).

Moreover, the fact (Prop. 3.73 below) that the Conner-Floyd construction is really a special case of Chern-Simons-invariants computed via "extended worldvolumes" in the style of [Wi83] (relating it to the Green-Schwarz mechanism in our dictionary entry §2.6), seems to not have been touched upon before.

(iii) The bounded-dimensional generalization of complex orientations on generalized E-cohomology originates with [Ra84, §3], we call them *Ravenel orientations* in §3.8. However, this reference, with most of its followups, focuses on properties of the associated *Ravenel spectra*. The main reference with explicit discussion of their role in bounded-dimensional complex orientation remains Hopkins' thesis [Ho84, §2.1]. Since any application of complex oriented cohomology to physics (such as to D-brane charge and elliptic partition functions) is necessarily bounded by spacetime dimension, it is really these bounded-dimensional E-orientations that matter here.

Besides providing streamlined accounts of these and related topics, our main mathematical results here are the following two theorems, crucial for the progression of the correspondence in §2.5, §2.6 and §2.8, §2.9, respectively:

(a) Theorem 3.62, confirms that the Toda-bracket formulation of the refined e-invariant (Def. 3.56) does reproduce the classical Adams e-invariant;

(b) Theorem 3.99, shows how bounded-dimensional complex E-orientations induce bounded-dimensional quaternionic E-orientations.

While these statements are not surprising and may be known to experts in the field, they are not in the literature (to the best of our knowledge), and there is some amount of fiddly detail involved in their proofs, which we aim to spell out in detail. Hence we hope that this self-contained and streamlined account will be useful.

3.1 Borsuk-Spanier Cohomotopy

We briefly set up basics of homotopy theory to fix concepts and notation. For more details and further pointers see [FSS20c, §A].

Categories. While we give a fairly self-contained account of the more advanced notions in the following, we make free use of elementary concepts in category theory and homotopy theory: for joint introduction see [Rie14][Ri20] and for concise review in our context see [FSS20c, §A]. Here just our notation conventions:

Notation 3.1 (Categories)**.** For $\mathscr{C}$ a category and for $X, A \in \mathscr{C}$ a pair of its objects, we write

$$\mathscr{C}(X,A) \in \mathrm{Sets} \tag{98}$$

for the hom-set of morphisms in $\mathscr{C}$ from X to A. This extends, of course, to a contravariant functor on $\mathscr{C}$ in its first argument, which we denote as

$$\begin{array}{rccc} \mathscr{C}(-,A) \;: & \mathscr{C}^{\mathrm{op}} & \longrightarrow & \mathrm{Sets} \\ & X & \longmapsto & \mathscr{C}(X,A)\,. \end{array} \tag{99}$$

Elementary as this is, these hom-functors are of profound interest in the case that $\mathscr{C}$ is the *homotopy category* $\mathrm{Ho}(\mathbf{H})$ of (the pointed objects in) a *model topos* (∞-topos) $\mathbf{H}$ (see [FSS20c, A.43][SS20b, §2.1] for more pointers), such as is the case for the classical homotopy category (Notation 3.9 below): Then the contravariant hom-functors (99) are *non-abelian generalized cohomology theories* (6), see Def. 3.16 below.

Notation 3.2 (Adjunctions)**.** For $\mathscr{C}, \mathscr{D} \in$ Categories (Notation 3.1) a pair of functors $L : \mathscr{D} \longleftrightarrow \mathscr{C} : R$ is an *adjunction* with L *left adjoint* and R *right adjoint* if there is a natural isomorphism

$$\mathscr{C}\big(L(X)\,,A\big) \;\simeq\; \mathscr{D}\big(X\,,R(A)\big)\,, \quad \text{to be denoted:} \qquad \begin{tikzcd} \mathscr{C} \arrow[r, shift right=1.5ex, "R"'] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, shift right=1.5ex, "L"'] \end{tikzcd}. \tag{100}$$

The point at infinity. In applications to physics, the role of *pointed* topological spaces, so fundamental to (stable) homotopy theory, is to encode the crucial concept of *fields vanishing at infinity*. To bring this out, we recall some basics of pointed topology (e.g. [Br93, p. 199]; a good account is in [Cu20]).

Notation 3.3 (Pointed topological spaces)**.** We write

$$\mathrm{Spaces}\,, \mathrm{Spaces}^{*/} \;\in\; \mathrm{Categories} \tag{101}$$

for a convenient category of topological spaces (convenient in the technical sense of [St67]), such as D-topological (i.e. Delta-generated) spaces (e.g. [SS20b, Def. 2.2]), and for its pointed version, respectively. The objects of the latter are spaces X equipped with a "base"-point $\infty_X \in X$ and the morphisms are continuous functions $f : X \to Y$ respecting that choice: $f(\infty_X) = \infty_Y$. We refer to $\infty_X \in X$ as the *point at infinity* in X, because that is the role these points play in application to physics (17):

$$\mathrm{Spaces}^{*/}\big((X,\infty_X)\,,(Y,\infty_Y)\big) \;=\; \left\{ \begin{tikzcd} X \arrow[r, "\text{continuous function}", "f"'] & Y \\ \{\infty_X\} \arrow[u, hook] \arrow[r] & \{\infty_Y\} \arrow[u, hook] \end{tikzcd} \right\}$$

The *convenience* of Spaces (101) means, in particular, that this category is Cartesian closed, in that we have adjunctions (100) of the following form:

$$\begin{tikzcd} \mathrm{Spaces} \arrow[r, shift right=1.5ex, "\underset{\text{mapping space}}{\mathrm{Maps}(X,-)}"'] \arrow[r, phantom, "\perp"] & \mathrm{Spaces} \arrow[l, shift right=1.5ex, "\overset{\text{product space}}{X\times(-)}"'] \end{tikzcd}, \qquad \text{for all } X \in \mathrm{Spaces}\,, \tag{102}$$

while $\mathrm{Spaces}^{*/}$ is closed symmetric monoidal, in that we have an adjunction

$$\mathrm{Spaces}^{*/} \underset{\underset{\text{pointed mapping space}}{\mathrm{Maps}^{*/}\big((X,\infty_X),-\big)}}{\overset{\overset{\text{smash product space}}{(X,\infty_X)\wedge(-)}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Spaces}^{*/}, \quad \text{for all } (X,\infty_X) \in \mathrm{Spaces}^{*/}, \tag{103}$$

where $\mathrm{Maps}^{*/}\big((X,\infty_X),(Y,\infty_Y)\big) \subset \mathrm{Maps}(X,Y)$ is the subspace of maps that preserves the base points, and where

$$\big(X,\infty_X\big) \wedge \big(Y,\infty_Y\big) \quad := \quad \left(\frac{X \times Y}{\{\infty_X\} \times Y \,\cup\, X \times \{\infty_Y\}} \,,\, \underbrace{\big[\{\infty_X\} \times Y \,\cup\, X \times \{\infty_Y\}\big]}_{\infty_{(X \wedge Y)}} \right)$$

is the *smash product* of pointed spaces: the quotient space of the product space where all those points that are at infinity in X or in Y are identified with a single point at infinity. This yields the structure of a symmetric monoidal category, with tensor unit the 0-sphere $S^0 := * \sqcup \infty$:

$$\big(\mathrm{Spaces}^{*/}, S^0, \wedge\big) \;\in\; \mathrm{SymmetricMonoidalCategories}. \tag{104}$$

Notation 3.4 (Locally compact Hausdorff space with proper maps)**.** We write $\mathrm{LCHSpaces}^{\mathrm{prop}} \subset \mathrm{Spaces}$ for the non-full subcategory of locally compact Hausdorff topological spaces with *proper* maps between them. We regard this as a symmetric monoidal category

$$\big(\mathrm{LCHSpaces}^{\mathrm{prop}}, *, \times\big) \;\in\; \mathrm{SymmetricMonoidalCategories}\,. \tag{105}$$

with tensor product given by forming topological product spaces (which, beware, is no longer the *cartesian* product in $\mathrm{LCHSpaces}^{\mathrm{prop}}$, due to the restriction of the morphisms to proper maps).

Proposition 3.5 (Functorial one-point compactification (e.g. [Cu20, Prop. 1.5]))**.** *On* $\mathrm{LCHSpaces} \subset \mathrm{Spaces}$ *(Notation 3.4) the construction of one-point compactifications* (17) *extends to a functor:*

$$\mathrm{LCHSpaces}^{\mathrm{prop}} \xrightarrow{\;(-)_{\mathrm{cpt}}\;} \mathrm{Spaces}^{*/}. \tag{106}$$

Example 3.6 (Adjoining infinity to a compact space)**.** If $X \in \mathrm{LCHSpaces}$ (Notation 3.4) is already compact, then its one-point compactification (17) is given by disjoint union with the singleton space on the point at infinity, often denoted X_+:

$$X \in \mathrm{Spaces} \text{ is compact} \quad \Rightarrow \quad X_{\mathrm{cpt}} \;=\; X_+ \;:=\; X \sqcup \{\infty\}\,. \tag{107}$$

Hence: *In a compact space, a continuous trajectory does not escape to infinity* (unless it starts there, in which case it stays there).

Example 3.7 (Spheres and Thom spaces as one-point compactifications)**.**
(i) For $X = \mathbb{R}^n$ the n-dimensional Euclidean space, its one-point compactification (17) is the n-sphere:

$$\big(\mathbb{R}^n\big)_{\mathrm{cpt}} \simeq S^n \quad \in \mathrm{Spaces}^{*/}. \tag{108}$$

(ii) Notice, for $n = 0$, that the 0-sphere is the two-point space, by (107):

$$S^0 \;\simeq\; *_{\mathrm{cpt}} \;\simeq\; *_+ \;\simeq\; * \sqcup \{\infty\}\,. \tag{109}$$

(iii) More generally, for $\mathscr{V}_X \to X$ a real vector bundle, over a *compact* topological space, its one-point compactification is equivalently its *Thom space* $\mathrm{Th}(\mathscr{V}_X)$, the quotient space of the unit disk bundle by the unit sphere bundle in $\mathscr{V}_X$ (with respect to any choice of fiber metric).

$$X \text{ compact} \;\;\Rightarrow\;\; \big(\mathscr{V}_X\big)_{\mathrm{cpt}} \;\simeq\; \mathrm{Th}\big(\mathscr{V}_X\big) \;:=\; D\big(\mathscr{V}_X\big)/S\big(\mathscr{V}_X\big) \quad \in \mathrm{Spaces}^{*/}. \tag{110}$$

Proposition 3.8 (One-point compactification is strong monoidal (e.g. [Cu20, Prop. 1.6]))**.** *The one-point compactification functor* (106) *is strong monoidal with respect to* (104) *and* (105)*:*

$$(-)_{\mathrm{cpt}} \;:\; \big(\mathrm{LCHSpaces}^{\mathrm{prop}}, *, \times\big) \longrightarrow \big(\mathrm{Spaces}^{*/}, S^0, \wedge\big)$$

in that there is a natural isomorphism between the one-point compactification of a product space and the smash product of the compactification of the two factors:

$$X, Y \in \mathrm{LCHSpaces}^{\mathrm{prop}} \subset \mathrm{Spaces} \quad \Rightarrow \quad \left\{ \begin{array}{l} \big(X \times Y\big)_{\mathrm{cpt}} \simeq X_{\mathrm{cpt}} \wedge Y_{\mathrm{cpt}} \\ \big(X \sqcup Y\big)_{\mathrm{cpt}} \simeq X_{\mathrm{cpt}} \vee Y_{\mathrm{cpt}} \end{array} \right. \in \mathrm{Spaces}^{*/}. \tag{111}$$

Homotopy types.

Notation 3.9 (Classical homotopy category)**.** We write

$$\mathrm{Spaces} \xrightarrow{\gamma} \mathrm{Ho}\big(\mathrm{Spaces}\big) \;,\;\; \mathrm{Spaces}^{*/} \xrightarrow{\gamma^{*/}} \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big) \quad \in \mathrm{Categories} \tag{112}$$

for the classical homotopy categories of (pointed) topological spaces (e.g. [FSS20c, (338)]), i.e. the localization of the categories of spaces (Def. 3.3) at the weak homotopy equivalences. We say that an object of (112) is a (pointed) *homotopy type*.

Example 3.10 (Pointed mapping spaces are homotopy fiber of evaluation map)**.** For $X, Y \in \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big)$ (Notation 3.9) their pointed mapping space (103) is the homotopy fiber (e.g. [FSS20c, Def. A.23]) of the evaluation map out of the plain mapping space (102) at the base-point:

$$\mathrm{Maps}^{*/}(X, Y) \xrightarrow{\;\;F\;\;} \mathrm{Maps}(X, Y) \xrightarrow{\;\mathrm{ev}_{*_X}\;} Y$$

Moreover, if $\pi_0(Y) = *$ and $\pi_1(Y) = 1$, then the forgetful map F is a weak homotopy equivalence (e.g. [Ha01, Prop. A4.2]).

Example 3.11 (Homotopy groups as homs in the classical homotopy category)**.** For $n \in \mathbb{N}$, with the n-sphere S^n (108) regarded as a pointed topological space (Def. 3.9), and for any $X \in \mathrm{Spaces}^{*/}$ we have a natural identification of the nth homotopy group/set of X (at its base-point) as the hom-set from S^n to X in the pointed homotopy category (Notation 3.9):

$$\begin{aligned} \pi_{n \geq 2}(X) &\simeq \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big)\big(S^n, X\big) && \in \mathrm{AbelianGroups} \\ \pi_1(X) &\simeq \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big)\big(S^1, X\big) && \in \mathrm{Groups} \\ \pi_0(X) &\simeq \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big)\big(S^0, X\big) && \in \mathrm{Sets} \end{aligned} \tag{113}$$

Example 3.12 (Suspension/Looping adjunction)**.** The smash/hom-adjunction (103) descends to its derived adjunction (e.g. [SS20b, Prop. A.20]) on the classical homotopy category (112),

$$\begin{tikzcd} \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big) \arrow[r, phantom, "\perp"] \arrow[r, shift right=2, "\underset{\text{derived pointed mapping space}}{\mathbb{R}\mathrm{Maps}^{*/}(X,-)}"'] & \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big) \arrow[l, shift right=2, "\overset{\text{derived smash product}}{\mathbb{L}(X \wedge (-))}"'] \end{tikzcd}, \qquad \text{for all } X \in \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big). \tag{114}$$

For X pointed n-sphere (108) this is the derived *suspension/looping adjunction*:

$$\begin{tikzcd} \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big) \arrow[r, phantom, "\perp"] \arrow[r, shift right=2, "\underset{\text{based loop space}}{\Omega(-) := \mathbb{R}\mathrm{Maps}^{*/}(S^1,-)}"'] & \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big) \arrow[l, shift right=2, "\overset{\text{reduced suspension}}{\Sigma(-) := \mathbb{L}\big(S^1 \wedge (-)\big)}"'] \end{tikzcd}, \qquad \begin{tikzcd} \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big) \arrow[r, phantom, "\perp"] \arrow[r, shift right=2, "\underset{\text{based } n\text{th loop space}}{\Omega^n := [S^n,-]}"'] & \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big) \arrow[l, shift right=2, "\overset{\text{reduced } n\text{th suspension}}{\Sigma^n(-) := S^n \wedge (-)}"'] \end{tikzcd}. \tag{115}$$

Beware that we will *notationally suppress* the derived-functor notation $\mathbb{L}(-)$, $\mathbb{R}(-)$.

Notation 3.13 (Delooping)**.** For $A \in \mathrm{Ho}(\mathrm{Spaces}^{*/})$ (Notation 3.9) and $n \in \mathbb{N}$, we say that the *n-fold delooping* of A is (if it exists) the homotopy type $B^n A \in \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big)$ such that:
(i) $\pi_{\bullet \leq n}\big(B^n A\big) \simeq *$ (via Example 3.11);
(ii) $A \simeq \Omega^n B^n A$ (via Example 107).

Example 3.14 (Classifying spaces)**.** For G a compact topological group, regarded as a pointed homotopy type $G \in \mathrm{Ho}(\mathrm{Spaces}^{*/})$ (Notation 3.9) via its neutral element, its delooping (Notation 3.13) exists and is given by the classical classifying space BG. If G is discrete this is also called the Eilenberg-MacLane space $K(G,1)$.

Example 3.15 (Higher Eilenberg-MacLane spaces)**.** If the group in Example 3.14 is a discrete abelian group A, then it admits arbitrary deloopings, called its higher Eilenberg-MacLane spaces, denoted:

$$B^n A \;\simeq\; K(A,n)\,.$$

Non-abelian cohomology. As in [FSS20c, Def. 2.1] we say:

Definition 3.16 (Reduced non-abelian cohomology, vanishing at infinity)**.** For $A \in \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big)$ (Notation 3.9):
(i) We say that the functor on pointed spaces that it represents

$$\begin{array}{ccccc} \mathrm{Spaces}^{*/} & \xrightarrow{\;\gamma\;} & \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big)^{\mathrm{op}} & \xrightarrow{\;\widetilde{A}(-)\;} & \mathrm{Sets} \\ X & \longmapsto & & & \mathrm{Ho}(\mathrm{Spaces}^{*/})(X,A) \end{array} \tag{116}$$

is the *non-abelian A-cohomology theory* in its *reduced* version, i.e. *vanishing it infinity*.
(ii) This takes values in graded sets via looping (3.12): For $n \in \mathbb{N}$ we write

$$\widetilde{A}^{-n}(-) \;:=\; \widetilde{\big(\Omega^n A\big)}(-) \;:=\; \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big)\big(-\,,\Omega^n A\big) \;\simeq\; \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big)\big(\Sigma^n(-)\,,A\big)$$

for reduced A-cohomology in negative degrees; and if there exist n-fold deloopings (Notation 3.13) $B^n A \in \mathrm{Spaces}^{*/}$, then we write

$$\widetilde{A}^n(-) \;:=\; \widetilde{\big(B^n A\big)}(-) \;:=\; \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big)\big(-\,,B^n A\big)$$

for reduced A-cohomology in the respective positive degrees.

Example 3.17 (Ordinary non-abelian cohomology)**.** For $A = G$ a compact topological group, with delooping given by its classifying space BG (Example 3.14), the resulting non-abelian cohomology in degree 1 (Def. 3.16) is the classical non-abelian G-cohomology which on compact spaces with a disjoint base point (107) classifies principal G-bundles:

$$\widetilde{(BG)}(X_+) \;\simeq\; H^1(X;G) \;\simeq\; G\mathrm{Bund}(X)_{/\sim} \;\;\in\; \mathrm{Sets}\,. \tag{117}$$

Notice that in degree 0 the classical non-abelian cohomology of the point is the group of connected components of G, by (113):

$$\widetilde{G}(*_+) \;\simeq\; H^0(*;G) \;\simeq\; \pi_0(G) \;\;\in\; \mathrm{Groups}\,. \tag{118}$$

Definition 3.18 (Reduced Cohomotopy)**.** For $n \in \mathbb{N}$ and $A := S^n$ the homotopy type of the pointed n-sphere (108), the corresponding reduced non-abelian cohomology theory (Def. 3.16) is reduced *Cohomotopy theory* in degree n, denoted:

$$\widetilde{\pi}^n(-) \;:=\; \widetilde{S}^n(-) \;:=\; \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big)(-,S^n)\,. \tag{119}$$

Remark 3.19 (Comparing reduced and unreduced Cohomotopy)**.** By Example 3.10, reduced and unreduced Cohomotopy sets differ at most in degree 0: For X a topological space equipped with a point $\infty \in X$, the forgetful map from the reduced Cohomotopy of (X,x) – hence the Cohomotopy of X that *vanishes at infinity* – to the plain Cohomotopy of X is an isomorphism:

$$\widetilde{\pi}^{n\geq 1}\big((X,\infty)\big) \xrightarrow{\;\simeq\;} \pi^{\geq 1}(X)\,. \tag{120}$$

Example 3.20 (Cohomotopy of flat space vanishing at infinity is homotopy groups of spheres)**.** By combining Example 3.11 with Example 3.7 and Remark 3.19, we see that the Cohomotopy vanishing at infinity (Example 3.18) of Euclidean spaces is given by homotopy groups of spheres:

$$\underset{\substack{\text{reduced Cohomotopy}\\ \text{of Euclidean spaces}\\ \text{vanishing at infinity}}}{\widetilde{\pi}^n\Big(\big(\mathbb{R}^d\big)_{\mathrm{cpt}}\Big)} \;\underset{n\geq 1}{\simeq}\; \pi^n\Big(\big(\mathbb{R}^d\big)_{\mathrm{cpt}}\Big) \;\simeq\; \pi^n(S^d) \;=\; \underset{\substack{\text{homotopy groups}\\ \text{of spheres}}}{\pi_d(S^n)}\,. \tag{121}$$

3.2 Pontrjagin-Thom construction

Cohomotopy charge of Cobordism classes of normally-framed submanifolds. We briefly recall *Pontrjagin's theorem* (Prop. 3.24 below) identifying the n-Cohomotopy of a smooth manifold with the cobordism classes of its normally framed submanifolds in codimension n. Ponrjagin's construction has come to be mostly known as the *Pontrjagin-Thom construction*, after Thom considered the variant for orientation structure and Lashof generalized both to any choice of tangential structure (see Remark 3.25 for historical references). Review of the original Pontrjagin theorem (Prop. 3.24) may be found, apart from the original exposition [Po55], in [FU91, §B][Kos93, §IX] [Mil97, §7][BK02, §16][Ke11][Sad13, §1][Cs20]; for illustration see Figure D.

Definition 3.21 (Normally framed cobordism [Po55, §6][Kos93, §IX.2])**.** Let M^D be a smooth manifold.
(i) A *normally-framed submanifold* of M^d is a smooth submanifold Σ^d of M^D equipped with a trivialization of its normal bundle $N\Sigma^d := T_\Sigma M/_\Sigma T\Sigma$:

$$\overset{\text{smooth submanifold}}{\Sigma^d \hookrightarrow X^D}\,, \qquad \underset{\text{normal bundle}}{N\Sigma^d} \xrightarrow[\text{fr}]{\overset{\text{normal framing}}{\simeq}} \underset{\substack{\text{trivial vector bundle} \\ \text{of rank } n := D-d}}{\Sigma \times \mathbb{R}^n} \tag{122}$$

(ii) A *bordism*

$$(\Sigma_0^d, \text{fr}_0) \xRightarrow{(\widehat{\Sigma}, \widehat{\text{fr}})} (\Sigma_1^d, \text{fr}_1) \tag{123}$$

between a pair of *closed* (i.e. compact without boundary) such normally-framed submanifolds (122) is a normally-framed submanifold of the cylinder $M^D \times \mathbb{R}^1$ whose boundary at $i = 0, 1 \in \mathbb{R}^1$ coincides with (Σ_i, fr_i) (illustration on p. 12).
The *relation* on normally framed submanifolds of having a bordism (123) between them is called *cobordism*. [10]
(iii) We denote the set of equivalence classes of the cobordism relation (123) between normally framed submanifolds of codimension $n = D - d$ by:

$$\text{Cob}^n_{\text{Fr}}\big(M^D\big) \;:=\; \left\{ \begin{array}{c} \text{Closed submanifolds } \Sigma^d \hookrightarrow X^D \\ \text{of dimension } d = D - n \\ \text{and equipped with normal framing} \end{array} \right\}\Big/_{\text{cobordism}} . \tag{124}$$

Definition 3.22 (Cohomotopy charge sourced by submanifolds [Po55, p. 48][Kos93, p. 179])**.** Let $\Sigma^d \hookrightarrow M^D$ be a smooth submanifold embedding.
(i) A *tubular neighborhood* of the submanifold is an extension $\exp_\Sigma$ to an embedding of the normal bundle $N\Sigma$ into M^D. Its inverse $\exp_\Sigma^{-1}$, when extended to the one-point compactifications (17) where it is known as the *Pontrjagin-collapse map* – regards all points in M^D not inside this tubular neighborhood as being *at infinity*:

$$\begin{tikzcd} & \Sigma \arrow[dl, hook', "\text{submanifold}"'] \arrow[dr, hook] & \\ M^D & & N\Sigma^d \;\; \substack{\text{normal}\\\text{bundle}} \arrow[ll, hook', "\exp_\Sigma", "\text{tubular neighborhood}"'] \end{tikzcd}$$

$$\begin{array}{ccc} \big(M^D\big)_{\text{cpt}} & \xrightarrow[\exp_\Sigma^{-1}]{\text{Pontrjagin collapse}} & \big(N\Sigma\big)_{\text{cpt}} \\ x & \longmapsto & \left\{ \begin{array}{ll} \exp_\Sigma^{-1}(x) & \mid x \in \text{image}(\exp_\Sigma) \\ \infty & \mid \text{otherwise} \end{array} \right. \end{array} \tag{125}$$

(ii) Given, moreover, a normal framing fr on Σ (Def. 3.21), the *Cohomotopy charge* of the normally-framed submanifold (Σ, fr) is the homotopy class of the composite of the collapse (125) with the normal framing (122), regarded as a class in reduced n-Cohomotopy (Def. 3.18):

$$\Big[\underbrace{\big(M^D\big)_{\text{cpt}} \xrightarrow{\exp_\Sigma^{-1}} \big(N\Sigma\big)_{\text{cpt}} \xrightarrow[\simeq]{(\text{fr})_{\text{cpt}}} \big(\Sigma \times \mathbb{R}^n\big)_{\text{cpt}} \xrightarrow{(\text{pr}_2)_{\text{cpt}}} (\mathbb{R}^n)_{\text{cpt}} \xlongequal{\quad} S^n}_{\text{Cohomotopy charge of } (\Sigma, \text{fr})} \Big] \;\in\; \widetilde{\pi}^n\Big(\big(M^D\big)_{\text{cpt}}\Big)\,, \tag{126}$$

---

[10] In [Po55, p. 42, Def. 3] cobordant submanifolds are called "homologous", thinking of Cobordism as a variant of singular homology. The modern term "cobordant" is due to [Th54, p. 64].

where we used the functoriality of one-point compactification (Prop. 3.5) on proper maps (since Σ is assumed to be closed, hence compact) and the fact (108) that the one-point compactification of Euclidean n-space is the n-sphere (Example 3.7).

(iii) This construction is clearly independent, up to homotopy of the resulting Cohomotopy charge cocycle (126), of (a) the choice $\exp_\Sigma$ of tubular neighborhood (125) and (b) the deformation of (Σ, fr) along bordisms (123), and hence constitutes a function on the Cobordism set (124):

$$\begin{array}{c}\textbf{normally-framed}\\ \textbf{Cobordism}\end{array} \quad \mathrm{Cob}^n_{\mathrm{Fr}}\big(M^D\big) \xrightarrow{\begin{array}{c}\textbf{Cohomotopy charge}\\ \textbf{assignment}\end{array}} \widetilde{\pi}^n\Big(\big(M^D\big)_{\mathrm{cpt}}\Big) \quad \begin{array}{c}\textbf{reduced}\\ \textbf{Cohomotopy}\end{array} \tag{127}$$

Definition 3.23 (Submanifolds sourcing Cohomotopy charge [Po55, p. 44, Def. 4])**.** For M^D a smooth manifold and $[c] \in \widetilde{\pi}^n\big(\big(M^D\big)_{\mathrm{cpt}}\big)$ a class in the reduced n-Cohomotopy set (Def. 3.18) of its one-point compactification (17), let $M^d \xrightarrow{c_{\mathrm{reg}}} S^n$ be a representative for which $0 \in \big(\mathbb{R}^n\big)_{\mathrm{cpt}} = S^n$ is a regular value. This means that the pre-image $\Sigma := c^{-1}(0) \subset M^D$ is a smooth submanifold, and that there is an open ball neighborhood $\mathbb{R}^n \simeq N\{0\} \subset \big(\mathbb{R}^n\big)_{\mathrm{cpt}} = S^n$ of 0 whose pre-image is a tubular neighborhood (125) of Σ, thereby equipped with a normal framing (122), as exhibited by the following pasting composite of pullbacks of smooth manifolds:

$$\begin{tikzcd}
\textbf{submanifold} & \Sigma \arrow[rr] \arrow[d, hook, "{(\mathrm{id},0)}"'] & \arrow[d, phantom, "\text{(pb)}"] & \{0\} \arrow[d, hook] \\
& \Sigma \times \mathbb{R}^n \arrow[rr, "\mathrm{pr}_2"] \arrow[d, "\mathrm{fr}^{-1}"', "\simeq"] & \arrow[d, phantom, "\text{(pb)}"] & \mathbb{R}^n \arrow[d, "\simeq"] \\
\begin{array}{c}\textbf{normal}\\ \textbf{framing}\end{array} & N\Sigma \arrow[rr] \arrow[d, hook, "\exp_\Sigma"'] & \arrow[d, phantom, "\text{(pb)}"] & N\{0\} \arrow[d, hook, "\exp_{\{0\}}"] \\
\begin{array}{c}\textbf{tubular}\\ \textbf{neighborhood}\end{array} \quad \begin{array}{c}\textbf{regularized}\\ \textbf{Cohomotopy coccyle}\end{array} & M^D \arrow[rr, "c_{\mathrm{reg}}"] \arrow[rr, bend right, "c"'] & \arrow[d, phantom, "\Downarrow \simeq"] & S^n
\end{tikzcd} \tag{128}$$

It is clear that the normally framed submanifold (Σ, fr) obtained this way gets deformed along a bordism (123) when c is deformed by a homotopy, so that this construction constitutes a function

$$\begin{array}{c}\textbf{normally-framed}\\ \textbf{Cobordism}\end{array} \quad \mathrm{Cob}^n_{\mathrm{Fr}}\big(M^D\big) \xleftarrow{\ \mathrm{Src}\ } \widetilde{\pi}^n\Big(\big(M^D\big)_{\mathrm{cpt}}\Big) \quad \begin{array}{c}\textbf{reduced}\\ \textbf{Cohomotopy}\end{array} \tag{129}$$

to the Cobordism set (124).

Proposition 3.24 (Pontrjagin's Theorem [Po55, p. 48, Thm. 10][Kos93][Mil97, p. 50])**.** *For M^D a smooth manifold, the construction of*

(a) *Cohomotopy charge of normally framed submanifolds (Def. 3.22), and*

(b) *of submanifolds sourcing Cohomotopy charge (Def. 3.23)*

are inverse isomorphisms between

(i) *the Cobordism set* (124) *of normally framed submanifolds of codimension n, and*

(ii) *the reduced n-Cohomotopy (Def. 3.18) for all* $0 \leq n \leq D$*:*

$$\begin{tikzcd}
\begin{array}{c}\textbf{normally-framed}\\ \textbf{Cobordism}\end{array} \quad \mathrm{Cob}^n_{\mathrm{Fr}}\big(M^D\big) \arrow[r, shift left, "\begin{array}{c}\textbf{Cohomotopy charge}\\ \textbf{assignment}\end{array}"] \arrow[r, phantom, "\simeq"] & \widetilde{\pi}^n\Big(\big(M^D\big)_{\mathrm{cpt}}\Big) \quad \begin{array}{c}\textbf{reduced}\\ \textbf{Cohomotopy}\end{array} \arrow[l, shift left, "\mathrm{Src}"]
\end{tikzcd} \tag{130}$$

Remark 3.25 (Attributions for Pontrjagin's theorem)**.** The statement of Prop. 3.24 underlies already the announcement [Po36]; in print it is made explicit in [Po38], briefly and for the case $M^D = S^D$. But it is only after Thom's variant of the construction (for normally-oriented instead of normally-framed submanifolds) appears much later [Th54] that Pontrjagin gives a comprehensive account of the construction [Po55], still focusing on $M^D = S^D$. Later, most authors consider the theorem for the case that M^D is any closed manifold (e.g. [Kos93, §IX.5]). The further generalization in Prop. 3.24, where M^D need not be closed, readily follows by the same kind of proof; it appears stated this way in, e.g., [Mil97, p. 50][Mor13, p. 1][Cs20, p. 12-13]. The first statement of the construction in the generality of arbitrary tangential structure ("(B,f)-structure" [La63, p. 258][Koc96, §1.4]) is [La63, Thm. C]: This general theorem by Lashof is the actual *Pontrjagin-Thom theorem* in modern parlance.

Example 3.26 (Cohomotopy charge of signed points (e.g. [Kos93, §IX.4]))**.** A normally-framed closed submanifold (Def. 3.21) in maximal codimension $n = D$ of an orientable manifold M^D is a finite subset of points

$$\Sigma^0 \;=\; \underset{1 \leq k \leq n}{\sqcup} \{x_k\} \;\subset\; M^D\,, \qquad \mathrm{fr}^0 \;=\; (\sigma_1, \cdots, \sigma_k) \;\in\; \{\pm 1\}^n$$

each equipped with a signed *unit charge* $\sigma_k \in \{\pm 1\}$ (the orientation of open balls around each point relative to the ambient orientation). If M^D is, moreover, connected, then any such configuration is cobordant, via pair creation/annihilation bordisms as illustrated in (2.2), to a subset of points whose framing labels are either all $+$ or all $-$; whence the Cobordism classes (124) are, manifestly, classified by the integers:

$$M^D \text{ is connected \& orientable} \quad \Rightarrow \quad \begin{tikzcd} \mathrm{Cob}^D_{\mathrm{Fr}}(M^D) \arrow[rr, leftrightarrow, "\text{Prop. 3.24}"] \arrow[dr, equal, "\text{sum of charges}"'] & & \widetilde{\pi}^D\Big(\big(M^D\big)_{\mathrm{cpt}}\Big). \arrow[dl, equal, "\text{Hopf winding degree}"] \\ & \mathbb{Z} & \end{tikzcd}$$

Under the Pontrjagin theorem (Prop. 3.24), this says that the D-Cohomotopy of a connected and orientable D-manifold is given by *winding number*: which is the statement of *Hopf degree theorem* (see also Example 2.5). For more discussion of this situation in our context see [SS19a, §2.2].

Example 3.27 (Cohomotopy charge of 3-Sphere in $\mathbb{R}^7$)**.** The normally-framed submanifold of $\mathbb{R}^7 \subset S^7$ which sources (Def. 3.23) the 4-Cohomotopy charge represented by the quaternionic Hopf fibration $h_{\mathbb{H}}$ is the 3-sphere $S^3 \simeq S\mathrm{U}(2)$ equipped with the normal framing induced from its $S\mathrm{U}(2)$-invariant tangential framing:

$$\begin{array}{ccc} \mathrm{Cob}^4_{\mathrm{Fr}}(\mathbb{R}^7) & \xleftarrow{\;\;\mathrm{Src}\;\;} & \widetilde{\pi}^4(S^7) \\ S^3_{\mathrm{nfr}=1} & \longleftarrow\!\!\shortmid & \big[S^7 \xrightarrow{h_{\mathbb{H}}} S^4\big]\,, \end{array} \qquad \begin{tikzcd} S^3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & S^7 \arrow[loop above, "S\mathrm{U}(2)"] \arrow[d, "h_{\mathbb{H}}"] & \text{quaternionic Hopf fibration} \\ * \arrow[r] & S^4 & \end{tikzcd} \tag{131}$$

This follows from the pullback construction (128) by the fact that the quaternionic Hopf fibration is an $S\mathrm{U}(2)$-principal bundle.

Remark 3.28 (Circle-reduction of Cohomotopy charge in $\mathbb{R}^7$)**.** The 7-sphere S^7, besides being a $S\mathrm{U}(2)$-principal bundle over S^4 via the quaternionic Hopf fibration, is also a $\mathrm{U}(1)$-principal bundle over complex projective 3-space $\mathbb{C}P^3$, via the complex Hopf fibration in 7d as reflected in the following pasting diagram of pullbacks of manifolds (the remaining factorization shown at the bottom is through the Atiyah-Penrose twistor fibration $t_{\mathbb{H}}$, see Remark 3.88, and see [FSS20b, §2] for more pointers):

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & S^3 \arrow[loop above, "\mathrm{U}(1)"] \arrow[r] \arrow[d, "h_{\mathbb{C}}"] \arrow[dr, phantom, "\text{(pb)}"] & S^7 \arrow[loop above, "\mathrm{Sp}(1)"] \arrow[d, "h_{\mathbb{C}}"] \arrow[dd, bend left, "h_{\mathbb{H}}"] & \text{complex Hopf fibration} \\ * \arrow[r] & S^2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathbb{C}P^3 \arrow[d, "t_{\mathbb{H}}"] & \text{quaternionic Hopf fibration} \\ & * \arrow[r] & S^4 & \text{Atiyah-Penrose twistor fibration} \end{tikzcd}$$

Under restriction to the $S\mathrm{U}(2) \simeq S^3$-fiber of the quaternionic Hopf fibration, this $\mathrm{U}(1)$-action becomes that of the standard complex Hopf fibration in 3d. This means that (a) the present discussion has an enhancement to $\mathrm{U}(1)$-equivariant cohomology (as in [SS20b]) where a circle-action is understood throughout, where (b) the 3-sphere in Example 3.27 appears equipped with its complex Hopf-fibration structure. We will discuss this in more detail elsewhere.

Product in Cohomotopy.

Proposition 3.29 (Product of source manifolds from product in Cohomotopy [Kos93, §6.1])**.** *Under Pontrjagin's theorem (Prop. 3.24), composition of Cohomotopy classes*

$$
\begin{array}{ccc}
\widetilde{\pi}^{n_1}\big(S^{n_2+n_1}\big) \times \widetilde{\pi}^{n_2+n_1}(X) & \longrightarrow & \pi^{n_1}(X) \\
\big([c_1],[c_2]\big) & \longmapsto & [c_1 \circ c_2]
\end{array}
$$

corresponds to Cartesian product of normally-framed submanifolds, in that

$$
\mathrm{Src}\big([c_1 \circ c_2]\big) \;\simeq\; \Big[\mathrm{Src}(c_2) \times \mathrm{Src}(c_1) \;\subset\; N\mathrm{Src}(c_2) \;\subset\; X\Big].
$$

Proof. Using the pullback-description (128), this follows by the pasting law (59), applied to the following pasting diagram:

$$
\begin{tikzcd}
 & & \text{submfd. class. by } c_1 & \\
\text{submanifold classif. by } c_1 \circ c_2 & \Sigma_2 \times \Sigma_1 \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\text{(pb)}"] & \Sigma_1 \times \{0\} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\text{(pb)}"] & \{0\} \arrow[d, hook] \\
\text{normal bundle of submfd. class. by } c_1 \circ c_2 & \Sigma_2 \times \Sigma_1 \times \mathbb{R}^{n_1} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\text{(pb)}"] & \Sigma_1 \times \mathbb{R}^{n_1} \arrow[r, "\mathrm{pr}_2"] \arrow[d, hook] \arrow[ddr, phantom, "\text{(pb)}"] & \mathbb{R}^{n_1} \arrow[dd, hook] \\
\text{normal bundle of submfd. class. by } c_2 & \Sigma_2 \times \mathbb{R}^{n_2+n_1} \arrow[r, "\mathrm{pr}_2"] \arrow[d, hook] \arrow[dr, phantom, "\text{(pb)}"] & \mathbb{R}^{n_2+n_1} \arrow[d, hook] & \\
 & X \arrow[r, "(c_2)_{\mathrm{reg}}", "\text{regularized Cohomotopy cocycle}"'] & S^{n_2+n_1} \arrow[r, "(c_1)_{\mathrm{reg}}", "\text{regularized Cohomotopy cocycle}"'] & S^{n_1}
\end{tikzcd}
$$

□

As an example we have:

Example 3.30 (3-Folds in $\mathbb{R}^7$ carrying integer Cohomotopy charge)**.** The normally framed submanifolds of $\mathbb{R}^7$ which source integer-valued Cohomotopy charge (Def. 3.23) $N \in \mathbb{N} \hookrightarrow \widetilde{\pi}^4(S^7)$ (52) are those bordant (123) to N disjoint copies of the Lie-framed 3-sphere (131):

$$
\begin{array}{ccccc}
\mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z}_{12} & \simeq & \widetilde{\pi}^4\big(S^7\big) & \xrightarrow{\ \mathrm{Src}\ } & \mathrm{Cob}^4_{\mathrm{Fr}}\big(\mathbb{R}^7\big) \\
N \overset{(89)}{\longmapsto} & & \Big[S^7 \xrightarrow{N} S^7 \xrightarrow{h_{\mathbb{H}}} S^4\Big] & \longmapsto & \Big[\coprod_N S^3_{\mathrm{nfr}=1}\Big]
\end{array}
$$

With Prop. 3.29, this follows from Example 3.27 and Example 3.26.

3.3 Adams d-invariant

From the point of view of stable homotopy theory and generalized cohomology theory, the *d-invariant* of a map (Def. 3.36 below) is an elementary concept whose interest is mainly in it being the first in a sequence of more interesting stable invariants, proceeding with the *e-invariant* (§3.5) and the *f-invariant*. However, from the broader perspective of unstable homotopy theory and of non-abelian cohomology theory (§3.1), passage to the d-invariant of a map is, conversely, synonymous with *stabilization* (135) and with evaluation in multiplicative cohomology. It is in this form that the d-invariant appears in §2.3. Therefore, we begin by quickly reviewing some basics of stable homotopy theory, streamlined towards our applications.

Stable homotopy theory. Non-abelian cohomology (Def. 3.16) is so named since its cohomology groups are in general not abelian, as in (118); in fact they are in general just sets bare of group structure, as in (117). By Example 3.11 this is related to the suspension/looping adjunction (Example 3.12) not being an equivalence of categories, hence of the classical homotopy category (Notation 3.9) *not being stable under looping*; for if it were then the nth connective cover of the nth suspension of any space would serve as its n-fold delooping (Notation 3.13).

This situation reflects the rich nature of the classical homotopy category: It is *non-linear* in a sense that is made precise by *Goodwillie calculus* [Go90]. Ultimately we are interested in exploring the full non-linear structure of (co)homotopy theory. But just as in ordinary calculus and in ordinary perturbation theory, where the first step towards fully understanding a non-linear object is to understand its linear approximations, so here the first step is to consider the homotopy-linear version of the classical homotopy category.

Since, by the above, this involves making it *stable under looping*, it is known as the *stable homotopy category*, for short:

Notation 3.31 (Stable homotopy category)**.** We write $\mathrm{Ho}\big(\mathrm{Spectra}\big)$ for the *stable homotopy category* of spectra (8) (see [FSS20c, (350)] for pointers). By suitable (somewhat subtle) use of the adjunction (103), this is again closed symmetric monoidal, with unit the sphere spectrum $\mathbb{S}$:

$$\big(\mathrm{Ho}(\mathrm{Spectra}), \mathbb{S}, \wedge\big) \quad \in \ \mathrm{SymmetricMonoidalCategories} \tag{132}$$

in that we have adjunctions

$$\begin{tikzcd}[column sep=large] \mathrm{Ho}(\mathrm{Spectra}) \arrow[r, phantom, "\perp"] \arrow[r, <-, shift left=8pt, "{\overset{\text{smash product spectrum}}{E\wedge(-)}}"] \arrow[r, shift right=8pt, "{\underset{\text{mapping spectrum}}{[E,-]}}"'] & \mathrm{Ho}(\mathrm{Spectra}), \end{tikzcd} \qquad \text{for all } E \in \mathrm{Ho}(\mathrm{Spectra}). \tag{133}$$

Moreover, we have the *stabilization adjunction* (e.g. [FSS20c, Ex. A.41]) between the classical homotopy category (Def. 3.9) and the stable homotopy category (Def. 3.31):

$$\begin{tikzcd} \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big) \arrow[r, phantom, "\perp"] \arrow[r, <-, shift left=8pt, "\Sigma^\infty"] \arrow[r, shift right=8pt, "\Omega^\infty"'] & \mathrm{Ho}(\mathrm{Spectra}), \end{tikzcd} \qquad \mathbb{S} = \Sigma^\infty S^0, \tag{134}$$

which stabilizes the suspension/looping adjunction (Example 3.12) in that the following diagram commutes (both that of right adjoint functors as well as, and equivalently, that of left adjoint functors)

$$\begin{tikzcd}[column sep=huge, row sep=huge] \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big) \arrow[r, phantom, "\perp"] \arrow[r, <-, shift left=8pt, "{\overset{\text{adjunction}}{\Sigma^n}}"] \arrow[r, shift right=8pt, "\Omega^n"'] \arrow[d, shift right=8pt, "{\underset{\text{stabilization adjunction}}{\Sigma^\infty}}"'] \arrow[d, phantom, "\dashv"] \arrow[d, <-, shift left=8pt, "\Omega^\infty"] \arrow[dr, phantom, "{\Sigma^{\infty+n} \quad \Omega^{\infty+n}}"] & \mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big) \arrow[d, shift right=8pt, "\Sigma^\infty"'] \arrow[d, phantom, "\dashv"] \arrow[d, <-, shift left=8pt, "\Omega^\infty"] \arrow[dl, dashed, shift right=4pt] \\ \mathrm{Ho}\big(\mathrm{Spectra}\big) \arrow[r, phantom, "\simeq"] \arrow[r, <-, shift left=8pt, "\Sigma^n"] \arrow[r, shift right=8pt, "{\underset{\text{adjoint equivalence}}{\Omega^n}}"'] \arrow[ur, dashed, <-, shift right=4pt] & \mathrm{Ho}\big(\mathrm{Spectra}\big) \end{tikzcd} \tag{135}$$

Example 3.32 (Stable homotopy groups and Generalized cohomology of spheres)**.** The hom-isomorphisms (100) of the stabilization square of adjunctions (135) gives the identification of homotopy groups of spectra with the reduced generalized Cohomology of spheres: (9):

$$
\begin{array}{ccc}
 & & \underset{\text{of spectra}}{\text{homotopy groups}}\;\pi_\bullet(E) \\
 & & \Big\| \; \forall\, k+\bullet \geq 0 \\
\underset{\text{from stable Cohomotopy}}{\text{cohomology operations}}\;\mathrm{Ho}(\mathrm{Spectra})\big(\Sigma^k \mathbb{S}, \Omega^{\bullet - k} E\big) & = & \mathrm{Ho}(\mathrm{Spectra})\big(\Sigma^\infty S^{\bullet + k}, \Sigma^k E\big) \\
 & & \Big\| \\
 & & \mathrm{Ho}(\mathrm{Spaces}^{*/})\big(S^{\bullet + k}, \Omega^{\infty - k} E\big) = \widetilde{E}^k\big(S^{\bullet + k}\big)\;\underset{\text{of spheres}}{\text{reduced } E\text{-cohomology}} \\
 & & \Big\| \\
 & & \pi_{\bullet + k}\big(E^k\big)\;\underset{\text{component spaces}}{\text{homotopy groups of}}
\end{array}
$$

We will need the following deep fact about stable homotopy theory:

Proposition 3.33 (Fiber sequences of spectra coincide with cofiber sequences (e.g. [LMS86, §III, Thm. 2.4]))**.** *A sequence*

$$\cdots \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow \cdots$$

of spectra is a homotopy fiber sequence if and only if it is a homotopy cofiber sequence:

$$
\begin{tikzcd}
E \arrow[r] \arrow[d, "\mathrm{fib}"'] \arrow[dr, phantom, "\Swarrow\,(\mathrm{pb})"] & * \arrow[d] \\
F \arrow[r] & G
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd}
E \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow\,(\mathrm{po})"] & * \arrow[d] \\
F \arrow[r, "\mathrm{cofib}"'] & G
\end{tikzcd}
\tag{136}
$$

It follows in particular that Ω^∞ (134), being a derived right adjoint (see [FSS20c, Prop. A.21] for pointers), takes homotopy co-fiber sequences of spectra to homotopy fiber sequences of pointed spaces.

Generalized cohomology.

Definition 3.34 (Whitehead-generalized cohomology theory)**.** Given a stable homotopy type $E \in \mathrm{Ho}\big(\mathrm{Spectra}\big)$ (Def. 3.31), the corresponding reduced *generalized cohomology theory* in the sense of Whitehead is the functor

$$
\begin{array}{ccc}
\mathrm{Ho}\big(\mathrm{Spaces}^{*/}\big)^{\mathrm{op}} & \xrightarrow{\widetilde{E}^\bullet(-)} & \mathbb{Z}\mathrm{GradedAbelianGroups} \\
X & \longmapsto & \mathrm{Ho}(\mathrm{Spectra})\big(\Sigma^\infty X, \Sigma^\bullet E\big)\,.
\end{array}
$$

The *suspension isomorphism* in E-cohomology is the natural isomorphism

$$\widetilde{E}^\bullet(X) \underset{\simeq}{\xrightarrow{\Sigma^n}} \mathrm{Ho}(\mathrm{Spectra})\big(\Sigma^{\infty + n} X, \Sigma^{\bullet + n} E\big) \;\simeq\; \mathrm{Ho}(\mathrm{Spectra})\big(\Sigma^\infty S^n \wedge X, \Sigma^{\bullet + n} E\big) \;\simeq\; \widetilde{E}^{\bullet + n}(S^n \wedge X)\,. \tag{137}$$

Definition 3.35 (Multiplicative cohomology theory [CF66, p. 23][Ad74, §III.10])**.** A *homotopy-commutative ring spectrum* is a commutative monoid object $(E, 1, \cdot)$ in the smash-monoidal stable homotopy category (Def. 3.31):

$$\big(E, 1, \cup\big) \;\in\; \mathrm{CommutativeMonoids}\big(\mathrm{Ho}(\mathrm{Spectra}),\, 1,\, \wedge\big)\,. \tag{138}$$

The Whitehead-generalized cohomology theory $\widetilde{E}(-)$ (9) that is represented by a homotopy-commutative ring spectrum E (138) inherits cup-product operations that make it a *multiplicative cohomology theory*.

In particular, the homotopy-ring structure (138) induces on the coefficient groups $E_\bullet$ (9) (the reduced E-cohomology of S^0, Example 3.32) the structure of a graded-commutative ring ([Ad74, §III.10], review in [Bo95, §3][TK06, §C]):

$$\big(E_\bullet, 1, \cdot\big) \;\in\; \mathrm{CommutativeMonoids}\big(\mathbb{Z}\mathrm{GradedAbelianGroups}\big)\,. \tag{139}$$

The d-invariant.

Definition 3.36 (d-invariant on Cohomotopy)**.** For E a multiplicative cohomology theory we say that the *d-invariant* of a Cohomotopy class $\big[X \xrightarrow{c} S^n\big] \in \pi^n(X)$ (for any $n \in \mathbb{N}$) seen in E-cohomology is the class of the pullback along c of the n-fold suspended E-unit (32):

$$d_E(c) \;:=\; \big[c^*\big(\Sigma^n 1\big)\big] \;=\; \Big[X \xrightarrow{c} S^n \xrightarrow{\Sigma^n(1^E)} E^n\Big] \;\in\; \widetilde{E}^n(X)\,.$$

Remark 3.37 (d-invariant for mapping spaces)**.** More generally, for $X \xrightarrow{f} Y$ a map with any codomain space, its *Adams d_E-invariant* is the element

$$[f^*] \;\in\; \mathrm{Hom}_{E_{-\bullet}}\Big(\widetilde{E}^\bullet(Y)\,,\widetilde{E}^\bullet(X)\Big).$$

For the case $Y = S^n$ this reduces to Def. 3.36, since now the suspension isomorphism identifies

$$\widetilde{E}^\bullet\big(\underbrace{S^n}_{Y}\big) \;\simeq\; E_{n-\bullet} \;\simeq\; E_{-\bullet}\big\langle \Sigma^n(1^E)\big\rangle$$

with the free $E_{-\bullet}$-module on a single generator in degree n. This follows [Ad66, §3].

The name "d-invariant" in Def. 3.36 alludes to its realization in ordinary cohomology:

Example 3.38 (Hopf degree is d-invariant in ordinary cohomology)**.** For X a connected closed oriented manifold of dimension n, the d-invariant (Def. 3.36) of a map $\phi : X \to S^n$ seen in ordinary integral cohomology $E = H\mathbb{Z}$ is the *Hopf degree* in $H^n(X;\mathbb{Z}) \simeq \mathbb{Z}$.

In the other extreme:

Example 3.39 (d-Invariant in stable Cohomotopy is stable class)**.** The d-invariant in stable Cohomotopy $E = \mathbb{S}$ is equivalently the stable class of a map:

$$\begin{aligned} d_{\mathbb{S}}(c) \;&\simeq\; \Big[X \xrightarrow{c} S^n \xrightarrow{\Sigma^n(1^{\mathbb{S}})} \mathbb{S}^n = \Omega^\infty\Sigma^\infty S^n\Big] \\ &\simeq\; \Big[\Sigma^\infty X \xrightarrow{\Sigma^\infty c} \Sigma^\infty S^n\Big]\,. \end{aligned}$$

Trivializations of the d-invariant. The d-invariant is a "primary" homotopy invariant, in that it is an invariant of the (stable) homotopy class of a map of pointed spaces, hence of a *1-morphism* in the homotopy theory (∞-category) of pointed homotopy types. When this primary invariant vanishes, then "secondary" homotopy invariants appear, namely higher-homotopy invariants of (null) homotopies witnessing this vanishing of the primary invariant, hence invariants of *2-morphisms* in the ∞-category of pointed homotopy types:

Definition 3.40 (Trivializations of the d-invariant)**.** For E a multiplicative cohomology theory and $n, d \in \mathbb{N}$, write

$$H^E_{n-1}\mathrm{Fluxes}\big(S^{n+d-1}\big) \;:=\; \coprod_{[c] \in \mathbb{S}_{d-1}} \pi_0 \mathrm{Paths}_0^{c^*(1^E)}\Big(\mathrm{Maps}^{*/}\big(S^{n+d-1}\,,E^n\big)\Big)$$

$$= \coprod_{[c]\in\mathbb{S}_{d-1}} \left\{ \begin{tikzcd} S^{n+d-1} \arrow[rr] \arrow[d, "c"'] \arrow[dr, "G^{\mathbb{S}}_n(c)"] & & * \arrow[d, "0"] \arrow[dl, Rightarrow, dashed, "H^E_{n-1}(c)"'] \\ S^n \arrow[r, "\Sigma^n(1^{\mathbb{S}})"] \arrow[rr, bend right, "\Sigma^n(1^E)"'] & \mathbb{S}^n \arrow[r, "(e^E)^n"] & E^n \end{tikzcd} \right\}_{\big/ \text{2-homotopy}} \tag{140}$$

for the set of tuples consisting of the stable Cohomotopy class $\big[G^{\mathbb{S}}_n(c)\big]$ of a map $S^{n+d-1} \xrightarrow{c} S^n$ and the 2-homotopy class $\big[H^E_{n-1}(c)\big]$ of a trivialization (if any) of its d-invariant $c^*(1^E)$ (35) in E-cohomology. This set is canonically fibered over the underlying $G^{\mathbb{S}}_n\mathrm{Fluxes}(X)$ (37).

Remark 3.41 (Trivialization for vanishing classes)**.** In the special case that $[c] = 0 \in \mathbb{S}_{d-1}$, the trivialization $\big[H^E_{n-1}(c)\big]$ in (140) may be identified with a class in E_{d-1}. For general $[c]$, however, the diagram (140) indicates that $\big[H^E_{n-1}(c)\big]$ is like a class in *$G^{\mathbb{S}}_n(c)$-twisted E-cohomology*.

We make this precise in §3.4 by understanding H^E_{n-1}-fluxes as refined Toda brackets (Def. 3.44) of (1) the Cohomotopy charge $[c]$, with (2) the unit $\Sigma^n(1^E)$, and (3) the canonical projection to the classifying space of the Adams cofiber cohomology theory (Def. 3.48).

3.4 Toda brackets

The notion of *Toda brackets* ([Tod62]) is central to MFr-theory. [11] Nevertheless, their definition is traditionally stated in a somewhat intransparent way (Remark 3.45 below), and their *indeterminancy* is often treated more as a nuisance than as a feature. We use the occasion to bring out the hidden elegance of the notion (Def. 3.44) and highlight how the Toda bracket, via its apparent "indeterminancy", is really a *fibered moduli space* (albeit discrete): the *Toda bundle* (Remark 3.47 below), a special case of which are the moduli spaces of H_3^E-fluxes for given G_4^E-fluxes (51); this is Prop. 3.53 below.

Further below in §3.5, we see how the Adams e-invariant and the Hopf invariant are special cases of "refined" Toda brackets, namely of points in fibers of the Toda bundle.

Toda brackets.

Definition 3.42 (Zero-map and null-homotopy)**.** Let $X, Y \in \text{PointedTopologicalSpaces}$.
(a) The *zero map* or *zero morphism* between them is the unique function that is constant on the base point of Y. In particular, for $X = *$, the base point inclusion itself is the zero map, and we write

$$0 : \underbrace{X \longrightarrow * \xrightarrow{\;0\;} Y}_{\text{zero morphism}} . \tag{141}$$

(b) For any pointed map $X \xrightarrow{f} Y$ a *null-homotopy* is (if it exists) a pointed homotopy from (or to) the zero-map (141)

$$\begin{tikzcd}
X \arrow[rr, bend left, "0"] \arrow[rr, bend right, "f"'] \arrow[rr, phantom, "\overset{\Vert}{\underset{\Downarrow}{\text{null-homotopy}}}"] && Y
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& * \arrow[dr, "0"] \arrow[d, phantom, "\Downarrow"] & \\
X \arrow[ur] \arrow[rr, bend right, "f"'] & & Y
\end{tikzcd}$$

Lemma 3.43 (Torsor of null homotopies)**.** *For $X \xrightarrow{f} Y$ a map of pointed topological spaces, its set of 2-homotopy classes of null-homotopies (Def. 3.42) is either empty or is a torsor (a principal bundle over the point) for the group*

$$\underbrace{\pi_1 \text{Maps}^{*/}\big(X, Y\big)}_{\simeq \pi_0 \text{Maps}^{*/}\big(X, \Omega Y\big)} \;=\; \left\{ [g] \;\middle|\; \begin{tikzcd} X \arrow[r, bend left, "0"] \arrow[r, bend right, "0"'] \arrow[r, phantom, "\Downarrow g"] & Y \end{tikzcd} \right\} \tag{142}$$

under its canonical action by composition of homotopies:

$$\begin{tikzcd}
X \arrow[rr, "0"] \arrow[rr, bend right=50, "f"'] \arrow[rr, phantom, shift right=3, "\Downarrow\phi"] && Y
\end{tikzcd}
\quad \overset{g}{\longmapsto} \quad
\begin{tikzcd}
X \arrow[rr, bend left=50, "0"] \arrow[rr, "0"] \arrow[rr, bend right=50, "f"'] \arrow[rr, phantom, shift left=3, "\Downarrow g"] \arrow[rr, phantom, shift right=3, "\Downarrow\phi"] && Y
\end{tikzcd} .$$

Proof. It is clear that this is an action. That this action is principal means that for evert pair $[\phi_1], [\phi_2]$ of 2-homotopy classes of null-homotopies of f, there is a *unique* $[g] := [\phi_1]^{-1}[\phi_2]$ which takes $[\phi_1]$ to $[\phi_2]$. This is clearly given by the 2-homotopy class of the composite of ϕ_2 with the inverse of ϕ_1:

$$[\phi_1]^{-1}[\phi_2] \;=\; \left[\begin{tikzcd}
X \arrow[rr, bend left=50, "0"] \arrow[rr, "f"] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, shift left=3, "\Downarrow\phi_2"] \arrow[rr, phantom, shift right=3, "\Downarrow\phi_1^{-1}"] && Y
\end{tikzcd}\right] \;\in\; \pi_1 \text{Maps}^{*/}\big(X, Y\big),$$

establishing the statement. ∎

[11] [IWX20, p. 17]: "Our philosophy is that the stable homotopy groups of spheres are not really understood until the Toda bracket structure is revealed."

Definition 3.44 (Toda brackets)**.** Given three consecutive maps of pointed spaces, equipped in pairs with null-homotopies (Def. 3.42)

$$
\begin{tikzcd}
X_0 \arrow[r, "f_1"] \arrow[rr, bend left, "0"] & X_1 \arrow[r, "f_2"] \arrow[rr, bend right, "0"'] \arrow[u, phantom, "\Downarrow \phi_1"] & X_2 \arrow[r, "f_3"] \arrow[d, phantom, "\Downarrow \phi_2"] & X_3 \,,
\end{tikzcd}
$$

then the pasting composite of these null-homotopies is a loop on the zero-map between X_0 and X_3,

$$
\begin{tikzcd}
X_0 \arrow[r, bend left=60, "0"] \arrow[r, bend right=60, "0"'] \arrow[r, phantom, "\Downarrow (\phi_2\cdot f_1)\circ(f_3\cdot\phi_1)"] & X_3
\end{tikzcd}
\;:=\;
\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "f_1"'] \arrow[dr, phantom, "\swarrow \phi_1"] & * \arrow[d, "0"] \\
X_1 \arrow[r, "f_2"] \arrow[d] \arrow[dr, phantom, "\swarrow \phi_2"] & X_2 \arrow[d, "f_3"] \\
* \arrow[r, "0"'] & X_3
\end{tikzcd}
\quad
\begin{array}{l}
\in \;\; \pi_1 \mathrm{Maps}^{*/}(X_0, X_3) \\
= \pi_0 \mathrm{Maps}^{*/}(X_0, \Omega X_3) \\
= \pi_0 \mathrm{Maps}^{*/}(\Sigma X_0, X_3)
\end{array}
$$

$$
=\;
\begin{tikzcd}
X_0 \arrow[r, dashed, "\langle f_1, f_2, f_3\rangle_{(\phi_1,\phi_2)}"'] \arrow[rr, bend left] \arrow[d, "f_1"'] & \Omega X_3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\,(\mathrm{pb})"] & * \arrow[d, "0"] \\
X_1 \arrow[r, dashed, "\vdash \phi_2"'] \arrow[rr, bend left, "f_2"] \arrow[d] & F_{f_3} \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\,(\mathrm{pb})"] & X_2 \arrow[d, "f_3"] \\
* \arrow[r, equal] \arrow[rr, bend left, "0"] & * \arrow[r, "0"'] & X_3
\end{tikzcd}
\;=\;
\begin{tikzcd}
X_0 \arrow[r] \arrow[rr, bend right] \arrow[d, "f_1"'] \arrow[dr, phantom, "(\mathrm{po})\,\swarrow"] & * \arrow[r, equal] \arrow[d] & * \arrow[d, "0"] \\
X_1 \arrow[r] \arrow[rr, bend right, "f_2"'] \arrow[d] \arrow[dr, phantom, "(\mathrm{po})\,\swarrow"] & C_{f_1} \arrow[r, dashed, "\vdash \phi_1"] \arrow[d] & X_2 \arrow[d, "f_3"] \\
* \arrow[r] \arrow[rr, bend right, "0"'] & \Sigma X_0 \arrow[r, dashed, "\langle f_1, f_2, f_3\rangle_{(\phi_1,\phi_2)}"] & X_3
\end{tikzcd}
$$

$$
=\;
\begin{tikzcd}
X_0 \arrow[r, hook] \arrow[rr, bend right] \arrow[d, "f_1"'] \arrow[dr, phantom, "(\mathrm{po})"] & C_{X_0} \arrow[r] \arrow[d] & * \arrow[d, "0"] \\
X_1 \arrow[r, hook] \arrow[rr, bend right, "f_2"'] \arrow[d, hook] \arrow[dr, phantom, "(\mathrm{po})"] & C_{f_1} \arrow[r, dashed, "\vdash \phi_1"] \arrow[d] & X_2 \arrow[d, "f_3"] \\
C_{X_1} \arrow[r] \arrow[rr, bend right, "0"'] & \Sigma X_0 \arrow[r, dashed, "\langle f_1, f_2, f_3\rangle_{(\phi_1,\phi_2)}"] & X_3 \,,
\end{tikzcd}
$$

which is classified, via the universal property of the homotopy (co)fiber (58), by a map from the suspension of X_0 or equivalently to the loop space of X_3.
(i) The homotopy class of this classifying map

$$
\langle f_1, f_2, f_3\rangle_{(\phi_1,\phi_2)} \;:=\; \Big[\vdash \big((\phi_2\cdot f_1)\circ(f_3\cdot\phi_1) \big) \Big] \;\;\in\;\; \begin{array}{l} \pi_0 \mathrm{Maps}^{*/}(\Sigma X_0, X_3) \\ = \pi_0 \mathrm{Maps}^{*/}(X_0, \Omega X_3) \end{array} \tag{143}
$$

is the *refined Toda bracket* of the triple of maps (f_1, f_2, f_3), depending on the chosen null-homotopies.
(ii) The *plain Toda bracket* is the *set* of all refined Toda brackets (143) obtained as the choice of the pair (ϕ_1, ϕ_2) of null-homotopies is varied:

$$
\langle f_1, f_2, f_3\rangle \;:=\; \left\{ \langle f_1, f_2, f_3\rangle_{(\phi_1,\phi_2)} \;\middle|\; \begin{array}{rcl} 0 & \overset{\phi_1}{\Rightarrow} & f_2\circ f_1\,, \\ f_3\circ f_2 & \underset{\phi_2}{\Rightarrow} & 0 \end{array} \right\} \;\subset\; \begin{array}{l} \pi_0 \mathrm{Maps}^{*/}(\Sigma X_0, X_3) \\ = \pi_0 \mathrm{Maps}^{*/}(X_0, \Omega X_3)\,. \end{array} \tag{144}
$$

Remark 3.45 (Perspectives on the Toda bracket)**.** **(i)** The traditional way to state the definition of the Toda bracket [Tod62] is essentially the last diagram shown in Def. 3.44, where the homotopy cofibers are modeled by topological cone constructions and where the Toda bracket appears as a "consecutive extension of functions over cones", these being the dashed morphisms in our diagram.
(ii) The more intrinsic description of the Toda bracket as a homotopy-coherent pasting composite, which we amplify at the beginning of the sequence of diagrams in Def. 3.44, is made more explicit in [HKK99, (0.2), (0.3)][HMO01, §3][HKM02, (2.2)].
(iii) The Toda bracket in homotopy is the analogue of the *Massey product* in cohomology (being Eckman-Hilton duals), where composition of maps corresponds to cup products and null-homotopies to coboundaries in cohomology. The corresponding diagrammatic formulation of refined Massey products is discussed in [GS17, §3.2].

The perspective of the Toda bracket as the homotopy-pasting composite makes immediate the characterization of the "indeterminancy" in the plain Toda bracket:

Proposition 3.46 (Toda bracket is torsor over Toda group)**.** *The plain Toda bracket* (144) *is either the empty set or is a torsor (a principal bundle over the point) for the direct product group*

$$\begin{aligned}\mathrm{TodaGroup}(X_0,X_1,X_2,X_3) &:= \pi_1\mathrm{Maps}^{*/}(X_0,X_2)\times\pi_1\mathrm{Maps}^{*/}(X_1,X_3)^{\mathrm{op}}\\ &= \pi_0\mathrm{Maps}^{*/}(X_0,\Omega X_2)\times\pi_0\mathrm{Maps}^{*/}(X_1,\Omega X_3)^{\mathrm{op}}\end{aligned}$$

acting via composition of homotopies:

$$\langle f_1,f_2,f_3\rangle_{(\phi_1,\phi_2)} \;=\; \begin{tikzcd} X_0 \arrow[d, "f_1"'] \arrow[r] & 0 \arrow[dr] & \\ X_1 \arrow[ur, phantom, "\Swarrow\phi_1"] \arrow[rr, "f_2"] \arrow[dr] & & X_2 \arrow[d, "f_3"] \arrow[dl, phantom, "\Swarrow\phi_2"] \\ & 0 \arrow[r] & X_3 \end{tikzcd} \;\longmapsto\; \begin{tikzcd} X_0 \arrow[d, "f_1"'] \arrow[r] \arrow[rr, bend left, "0"] & 0 \arrow[dr] \arrow[u, phantom, "\Swarrow g_1"] & \\ X_1 \arrow[ur, phantom, "\Swarrow\phi_1"] \arrow[rr, "f_2"] \arrow[dr] \arrow[rrd, bend right, "0"'] & & X_2 \arrow[d, "f_3"] \arrow[dl, phantom, "\Swarrow\phi_2"] \\ & 0 \arrow[r] \arrow[d, phantom, "\Swarrow g_2"] & X_3 \end{tikzcd} \;=\; \langle f_1,f_2,f_3\rangle_{(\phi_1\circ g_1,\, g_2\circ\phi_2)}$$

Proof. This follows as in Lemma 3.43. □

Remark 3.47 (The Toda bundle)**.** In conclusion, for a fixed sequence (X_0,X_1,X_2,X_3) of pointed spaces, the Toda bracket in Def. 3.44 is *not a function but a bundle*, which *instead of values has fibers*, namely the plain Toda brackets (144), whose elements are the refined Toda blackets (143):

$$\begin{tikzcd} \overset{\substack{\text{element of fiber of Toda bundle:}\\ \text{a refined Toda bracket}}}{\langle f_1,f_2,f_3\rangle_{(\phi_1,\phi_2)}} \;\in\; \overset{\substack{\text{fiber of Toda bundle:}\\ \text{a Toda bracket}}}{\langle f_1,f_2,f_3\rangle} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \overset{\substack{\text{structure group of Toda bundle:}\\ \text{indeterminacy of Toda brackets}}}{\mathrm{TodaGroup}(X_0,\cdots,X_3)} \circlearrowright \underset{\substack{\text{Toda bundle:}\\ \text{all Toda brackets}}}{\mathrm{TodaBundle}(X_0,\cdots,X_3)} \arrow[d] \\ \underset{\substack{\text{point in base space:}\\ \text{a triple of arguments for Toda bracket}}}{\big\{\big([f_1],[f_2],[f_3]\big)\big\}} \arrow[r, hook] & \underset{\substack{\text{base space of Toda bundle:}\\ \text{possible arguments of Toda brackets}}}{\prod_{i\in\{0,1,2\}}\pi_0\mathrm{Maps}^{*/}(X_i,X_{i+1})} \end{tikzcd} \tag{145}$$

Away from the empty fibers, this bundle is principal for the Toda group (by Prop. 3.46).

Adams cofiber cohomology. Since a choice of trivialization $H^E_{n-1}(c)$ of a d-invariant $d_E(c)=G^E_n(c)$ (Def. 3.40) gives "half" of the arguments of a refined Toda bracket (Def. 3.44), namely

$$\langle c,\Sigma^n(1^E),-\rangle_{(H^E_{n-1}(c),-)}\,,$$

it is natural to consider the *universal* completion of the remaining two arguments. This is provided by the boundary map in the *Adams cofiber cohomology theory* $E/\mathbb{S}$:

Definition 3.48 (Unit cofiber cohomology)**.** For E any multiplicative cohomology theory, the homotopy cofiber (57) of the its unit morphism (32) deserves to be denoted $E/\mathbb{S}$ (but is often abbreviated $\Sigma\overline{E}$, following [Ad74, Thm 15.1, p. 319], or just $\overline{E}$, as in [Ho99, Cor. 5.3]). By the pasting law (59) this comes equipped with a cohomology operation ∂^E from $E/\mathbb{S}$-cohomology to stable Cohomotopy in one degree higher:

$$
\begin{tikzcd}
\overset{\text{stable Cohomotopy}}{\mathbb{S}} \arrow[rr, "e^E", "\text{unit}"'] \arrow[d] & & \overset{E\text{-cohomology}}{E} \arrow[rr] \arrow[d, "i^E"] \arrow[dll, phantom, "\text{(po)}"] & & {*} \arrow[d] \arrow[dll, phantom, "\text{(po)}"] \\
{*} \arrow[rr] & & \underset{\substack{E\text{-unit cofiber}\\ \text{cohomology}\\ \text{theory}}}{E/\mathbb{S}} \arrow[rr, "\partial^E", "\text{boundary operation}"'] & & \underset{\substack{\text{shifted stable}\\ \text{Cohomotopy}}}{\Sigma\mathbb{S}} .
\end{tikzcd}
\quad \in \ \text{Spectra} . \tag{146}
$$

By Prop. 3.33, this means that for any $n \in \mathbb{N}$ we have the following homotopy fiber sequence on component spaces:

$$
\begin{tikzcd}
(E/\mathbb{S})^{n-1} \arrow[r] \arrow[d, "(\partial^E)^{n-1}"'] & {*} \arrow[d] \arrow[dl, phantom, "\text{(pb)}"] \\
S^n \arrow[r, "\Sigma^n(1^E)"] \arrow[d] & E^n \arrow[d, "(i^E)^n"] \arrow[dl, phantom, "\text{(pb)}"] \\
{*} \arrow[r] & (E/\mathbb{S})^n
\end{tikzcd}
\tag{147}
$$

Proposition 3.53 below shows that this *cofiber theory* $E/\mathbb{S}$ is the cohomology theory that classifies H_3^E-fluxes (48).

Lemma 3.49 (Cofiber E-cohomology as extension of stable Cohomotopy by E-cohomology)**.** *For E a multiplicative theory and X a space, assume that the E-Boardman homomorphism $\widetilde{\mathbb{S}}^\bullet(X) \xrightarrow{\beta_X^\bullet} \widetilde{E}^\bullet(X)$ (33) is zero in degrees n and $n+1$ – for instance in that $X \simeq S^{k\geq 2}$, $n = 0$ and the groups $\widetilde{E}^0(S^k) = \pi_k(E)$ have no torsion – then the cohomology operations i^E, ∂^E in* (146) *form a short exact sequence of cohomology groups:*

$$
0 \longrightarrow \widetilde{E}(X) \overset{i}{\hookrightarrow} (\widetilde{E/\mathbb{S}})^n(X) \overset{\partial}{\twoheadrightarrow} \widetilde{\mathbb{S}}^{n+1}(X) \longrightarrow 0 . \tag{148}
$$

(This is in generalization of [Sto68, p. 102], which in turn follows [CF66, Thm. 16.2].)

Proof. Generally, the long cofiber sequence of cohomology theories (146) induces a long exact sequence of cohomology groups (e.g. [Ad74, p. 197]):

$$
\cdots \longrightarrow \widetilde{\mathbb{S}}^n(X) \xrightarrow{1_X^n} \widetilde{E}^n(X) \xrightarrow{i_X^n} (\widetilde{E/\mathbb{S}})^n(X) \xrightarrow{\partial_X^n} \widetilde{\mathbb{S}}^{n+1}(X) \xrightarrow{1_X^{n+1}} \widetilde{E}^{n+1}(X) \longrightarrow \cdots
$$

Under the given assumption the two outermost morphisms shown are zero, and hence the sequence truncates as claimed. $\square$

Definition 3.50 (Induced cohomology operations on cofiber cohomology)**.** Let $E \xrightarrow{\phi} F$ be a multiplicative cohomology operation, so that in particular it preserves the units, witnessed by a homotopy-commutative square on the left here:

$$
\begin{tikzcd}
\mathbb{S} \arrow[r, "1^E"] \arrow[d, equal] & E \arrow[r] \arrow[d, "\phi"'] & E/\mathbb{S} \arrow[r, "\partial^E"] \arrow[d, "\phi/\mathbb{S}"'] & \Sigma\mathbb{S} \arrow[d, equal] \\
\mathbb{S} \arrow[r, "1^F"] & F \arrow[r] & F/\mathbb{S} \arrow[r, "\partial^F"] & \Sigma\mathbb{S} .
\end{tikzcd}
$$

Then passing to homotopy cofibers yields the induced cohomology operation $\phi/\mathbb{S}$ on cofiber theories (Def. 3.48).

Example 3.51 (Chern character on cofiber of K-theory)**.** The Chern character on $K\mathrm{U}$ and the Pontrjagin character on $K\mathrm{O}$ are multiplicative, hence descend to operations on cofiber K-theories via Def. 3.50:

$$
\begin{tikzcd}
K\mathrm{O} \arrow[rr, "\mathrm{ph}"] \arrow[dr, "i^{K\mathrm{O}}"'] \arrow[dd, "\mathrm{cplx}"'] & & H^{\mathrm{ev}}\mathbb{Q} \arrow[dr, "i^{H^{\mathrm{ev}}\mathbb{Q}}"] \arrow[dd] & \\
& K\mathrm{O}/\mathbb{S} \arrow[rr, "\mathrm{ph}/\mathbb{S}"] \arrow[dd, "\mathrm{cplx}/\mathbb{S}"'] & & (H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S} \arrow[dd, equal] \\
K\mathrm{U} \arrow[rr, "\mathrm{ch}"] \arrow[dr, "i^{K\mathrm{U}}"'] & & H^{\mathrm{ev}}\mathbb{Q} \arrow[dr, "i^{H^{\mathrm{ev}}\mathbb{Q}}"] & \\
& K\mathrm{U}/\mathbb{S} \arrow[rr, "\mathrm{ch}/\mathbb{S}"] & & (H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}
\end{tikzcd}
$$

Remark 3.52 (Canonical splitting of cofiber of periodic rational cohomology)**.** For $n,d \in \mathbb{N}$ with $d \geq 1$,
(i) we have a canonically split short exact sequence of the form

$$
\begin{tikzcd}
0 \arrow[r] & \big(\widetilde{H^{\mathrm{ev}}\mathbb{Q}}\big)^{2n}\big(S^{2(n+d)}\big) \arrow[r] \arrow[d, equal] & \big(\widetilde{(H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}}\big)^{2n}\big(S^{2(n+d)}\big) \arrow[r] \arrow[d, "\mathrm{spl}_0"', "\simeq"] & \widetilde{\mathbb{S}}^{2n+1}\big(S^{2(n+d)}\big) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Q} \arrow[r, hook] & \mathbb{Q} \oplus \mathbb{S}_{2d-1} \arrow[r, two heads] & \mathbb{S}_{2d-1} \arrow[r] & 0\,.
\end{tikzcd}
\tag{149}
$$

Here **(a)** the short exact sequence at the top follows with Lemma 3.49, using that the stable stems $\mathbb{S}_\bullet$ are finite in positive degrees (Serre's finiteness theorem), hence in particular pure torsion, hence admit only the 0-morphism into $\mathbb{Q}$. Moreover **(b)** the canonical splitting spl_0 shown is that induced from the inclusion $H\mathbb{Q} \hookrightarrow H^{\mathrm{ev}}\mathbb{Q}$ (which is multiplicative, hence in particular preserves the units):

$$
\begin{tikzcd}
0 \arrow[r] & \big(\widetilde{H\mathbb{Q}}\big)^{2n}\big(S^{2(n+d)}\big) \arrow[r] \arrow[d, equal] & \big(\widetilde{(H\mathbb{Q})/\mathbb{S}}\big)^{2n}\big(S^{2(n+d)}\big) \arrow[r] \arrow[d, "\simeq"] & \widetilde{\mathbb{S}}^{2n+1}\big(S^{2(n+d)}\big) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & 0 \arrow[r, hook] & \mathbb{S}_{2d-1} \arrow[r, equal] & \mathbb{S}_{2d-1} \arrow[r] & 0\,,
\end{tikzcd}
$$

where on the left we used again the assumption that $d \neq 0$.
(ii) We will use this canonical splitting together with the canonical splitting of the even rational cohomology of a cofiber space $S^{2(n+d)-1} \xrightarrow{c} S^{2n} \to C_c$ given by the splitting into degrees $2(n+d)$ and $2n$

$$
\begin{tikzcd}
0 \arrow[r] & \big(\widetilde{H^{\mathrm{ev}}\mathbb{Q}}\big)^{2n}\big(S^{2(n+d)}\big) \arrow[r] \arrow[d, equal] & \big(\widetilde{H^{\mathrm{ev}}\mathbb{Q}}\big)^{2n}\big(C_c\big) \arrow[r] \arrow[d, "\mathrm{spl}_0"', "\simeq"] & \big(\widetilde{H^{\mathrm{ev}}\mathbb{Q}}\big)^{2n}\big(S^{2n}\big) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Q} \arrow[r, hook] & \mathbb{Q} \oplus \mathbb{Q} \arrow[r, two heads] & \mathbb{Q} \arrow[r] & 0\,,
\end{tikzcd}
\tag{150}
$$

which is induced in the same fashion from the inclusion $H\mathbb{Q} \hookrightarrow H^{\mathrm{ev}}\mathbb{Q}$.
(iii) Both of these splittings are hence characterized by the fact that their corresponding retraction is *projection onto rational cohomology in degree* $2(n+d)$.

H^E_{n-1}-fluxes as Toda brackets.

Proposition 3.53 (Trivializations of d-invariant lift to classes in cofiber theory via refined Toda bracket)**.** *For E a multiplicative cohomology theory and $n,d \in \mathbb{N}$*
(i) *there exists a lift from the set* (140) *of trivializations of the d_E-invariant of maps $S^{n+d-1} \to S^n$ through the boundary map ∂^E* (146) *on coefficient groups* (9) *of the unit cofiber theory $E/\mathbb{S}$* (146)*:*

$$
\begin{tikzcd}
& \underset{\text{trivialization of a } d_E\text{-invariant}}{\big([G^{\mathbb{S}}_n(c)]\,,\,[H^E_{n-1}(c)]\big)} \arrow[r, mapsto] & \underset{\text{refined Toda bracket}}{\big\langle c, \Sigma^n(1^E), (i^E)^n \big\rangle_{\left(H^E_{n-1}(c),\,(\mathrm{pb})\right)}} \\
\big([G^{\mathbb{S}}_n(c)]\,,\,[H^E_{n-1}(c)]\big) \arrow[d, mapsto] & H^E_{n-1}\mathrm{Fluxes}\big(S^{n+d-1}\big) \arrow[r, dashed, "O^{E/\mathbb{S}}"] \arrow[d] & \big(E/\mathbb{S}\big)_d \arrow[d, "\partial"] \\
{[G^{\mathbb{S}}_n(c)]} & G^{\mathbb{S}}_n\mathrm{Fluxes}\big(S^{n+d-1}\big) \arrow[r, "\simeq"] & \mathbb{S}_{d-1}
\end{tikzcd}
\tag{151}
$$

(ii) *given by sending an H^E_{n-1}-flux to its refined Toda bracket (Def. 3.44) with the universal null homotopy* (147)*, denoted "(pb)" in* (151)*, as made fully explicit in diagram* (66)*.*

Proof. Regarding **(i)**: Since homotopy cofiber sequences of spectra are also homotopy fiber sequences (Prop. 3.33), the universal property of the defining cofiber sequence in (146) says that the homotopy diagram in the definition (140), when equivalently seen under the stabilization adjunction (135), factors uniquely, up to homotopy of cones, as shown here:

$$
\begin{tikzcd}
\Sigma^{n+d-1}\mathbb{S} \arrow[r] \arrow[d, "\Sigma^\infty c"'] & \ast \arrow[d, "0"] \arrow[dl, phantom, "\Swarrow H^E_{n-1}(c)"] \\
\Sigma^n\mathbb{S} \arrow[r, "\Sigma^n(e^E)"'] & \Sigma^n E
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
\Sigma^{n+d-1}\mathbb{S} \arrow[dr, dashed, "\vdash (G^E_n(c), H^E_{n-1}(c))"] \arrow[drr, bend left] \arrow[ddr, bend right, "\Sigma^\infty c"'] & & \\
& \Sigma^{n-1}(\mathbb{S}/E) \arrow[r] \arrow[d, "\partial"] \arrow[dr, phantom, "\Swarrow (\mathrm{pb})"] & \ast \arrow[d, "0"] \\
& \Sigma^n\mathbb{S} \arrow[r, "\Sigma^n(e^E)"] & \Sigma^n E
\end{tikzcd}
\tag{152}
$$

Now the dashed morphism on the right of (152) represents an element in $(E/\mathbb{S})_d$ and the homotopy-commutativity of the left triangle on the right shows that this bijection $\left[H^E_{n-1}(c)\right] \mapsto \left[\vdash H^E_{n-1}(c)\right]$ makes the square in (151) commute.

Regarding **(ii)**: Let $\left[S^{n+d-1} \xrightarrow{c} S^n\right] \in \pi^n\left(S^{n+d-1}\right)$ be a given class in Cohomotopy and consider the following construction of a homotopy pasting diagram in Spectra, all of whose cells are homotopy pushouts:

$$
\begin{tikzcd}
\Sigma^\infty S^{n+d-1} \arrow[r] \arrow[d, "\Sigma^\infty c"'] & \ast \arrow[d] & & \\
\Sigma^\infty S^n \arrow[r] \arrow[d] \arrow[rr, bend right, "\Sigma^n(e^E)"] & \Sigma^\infty C_c \arrow[r, dashed, "\vdash H^E_{n-1}(c)"] \arrow[d] \arrow[rr, bend left] & \Sigma^n E \arrow[r] \arrow[d] & \ast \arrow[d] \\
\ast \arrow[r] & \Sigma^{\infty+1} S^{n+d-1} \arrow[r, "M^d"] \arrow[rr, bend right, "\Sigma^{\infty+1} c"'] & \Sigma^n(E/\mathbb{S}) \arrow[r, "\partial"] & \Sigma^{\infty+1} S^n
\end{tikzcd}
\tag{153}
$$

For given $H^E_{n-1}(c)$, this diagram is constructed as follows (where we say "square" for any *single* cell and "rectangle" for the pasting composite of any adjacent *pair* of them):

- The two squares on the left are the stabilization of the homotopy pushout squares defining the cofiber space C_c (63) and the suspension of S^{n+d-1} (60).
- The bottom left rectangle (with $\Sigma^n(e^E)$ at its top) is the homotopy pushout (146) defining $\Sigma^n(E/\mathbb{S})$.
- The classifying map $\vdash H^E_{n-1}(c)$ for the given $(n-1)$-flux (63), shown as a dashed arrow, completes a co-cone under the bottom left square. Thus the map M^d forming the bottom middle square is uniquely implied by the homotopy pushout property (58) of the bottom left square. Moreover, the pasting law (59) implies that this bottom middle square is itself homotopy cartesian.
- The bottom right square is the homotopy pushout (146) defining ∂.
- By the pasting law (59) it follows that also the bottom right rectangle is homotopy cartesian, hence that, after the two squares on the left, it exhibits the third step in the long homotopy cofiber sequence (61) of $\Sigma^\infty c$. This means that its total bottom morphism is $\Sigma^{\infty+1}c$, and hence that $\partial\left[M^d\right] = [c]$.

In conclusion, these construction steps yield a map map $H^E_{n-1}(c) \mapsto M^d$ over the stable class of c.

We claim that this assignment is bijective over any $\Sigma^\infty c$: To that end, assume conversely that M^d is given, and with it the diagram (153) except for the dashed arrow. But since the bottom right square is a homotopy pushout (146) it is also a homotopy pullback (by Prop. 3.33), whence a dashed morphism is uniquely implied. By its uniqueness, this reverse assignment $M^{2d} \mapsto H^E_{n-1}(c)$ must be the inverse of the previous construction.

Finally, comparison of the diagrams (66) and (153) shows that M^d corresponds to the refined Toda bracket in question under the hom-isomorphism of the stabilization adjunction (134)

$$
\left[M^d\right] \leftrightarrow \left[\vdash (G^E_n(c), H^E_{n-1}(c))\right] = \left\langle c, \Sigma^n(1^E), (i^E)^n \right\rangle_{\left(H^E_{n-1}(c), (\mathrm{pb})\right)}.
$$

□

Remark 3.54 (Higher homotopies)**.** We may improve the lift (151) in Prop. 3.53 to a bijection by retaining the 2-homotopy class of the the left homotopy on the right of (152) (using the homotopy-universal property of ∞-limits, e.g. [Lu06, Def. 1.2.13.4, Thm. 4.2.4.1]). For example, in the extreme case when $E = \mathbb{S}$ we have $E/\mathbb{S} \simeq 0$, so that there is no information in the lift (151) itself, while *all* the information about the H_{n-1}-flux is solely in the cone homotopies on the right of(152), these being the very homotopies that define $H^{\mathbb{S}}_{n-1}$ in Def. 3.40, Example 2.7.

H_3^E-Fluxes and extraordinary flat differential Cohomotopy.

Remark 3.55 (H_3^E-Fluxes as extraordinary flat differential Cohomotopy)**.** Proposition 3.53 implies that the set of H_3^EFluxes(X) is a variant of the flat stable *differential Cohomotopy* of X:

(i) Recall from [FSS20c, Ex. 434] that *differential stable Cohomotopy* in degree 4 (just for definiteness) is the cohomology classified by the homotopy pullback

$$
\begin{tikzcd}
\underset{\text{differential stable 4-Cohomotopy}}{\overset{\text{classifying space for}}{}} \mathbb{S}^4_{\mathrm{conn}} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] && \Omega^4_{\mathrm{dR}}(-)_{\mathrm{cl}} \overset{\text{classifying stack for differential 4-forms}}{} \arrow[d, "\text{de Rham theorem}"] \\
\underset{\text{classifying space for stable 4-Cohomotopy}}{\mathbb{S}^4} \arrow[rr, "\text{Chern-Dold character map}", "{\mathrm{ch}_{\mathbb{S}^4} = \Sigma^4(e^{H\mathbb{R}})}"', "\text{here: unit map}"'] && \underset{\text{classifying space for ordinary real 4-cohomology}}{(H\mathbb{R})^4}
\end{tikzcd}
\tag{154}
$$

formed in the homotopy theory of smooth ∞-stacks, where the smooth classifying space $\Omega^4_{\mathrm{dR}}(-)_{\mathrm{cl}}$ of closed differential 4-forms exists and serves to capture the G_4-flux as a differential form.

(ii) While smooth ∞-stacks are beyond the scope of our discussion here, this technicality goes away as we focus on *flat* differential Cohomotopy where the 4-flux form vanishes, as befits the fluxless backgrounds that we are focusing on in §2. In that case the above diagram (154) reduces to the one shown on the left here:

$$
\begin{tikzcd}
\overset{\text{classifying space for flat differential stable 4-Cohomotopy}}{\mathbb{S}^4_{\mathrm{flat}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \ast \arrow[d, "0"] \\
\underset{\text{classifying space for stable 4-Cohomotopy}}{\mathbb{S}^4} \arrow[r, "{\mathrm{ch}_{\mathbb{S}^4} = \Sigma^4(e^{H\mathbb{R}})}", "H\mathbb{R}\text{-unit map}"'] & \underset{\text{classifying space for ordinary real 4-cohomology}}{(H\mathbb{R})^4}
\end{tikzcd}
\quad \text{special case of:} \quad
\begin{tikzcd}
\overset{\text{classifying space for } H_3^E\text{-fluxes}}{(E/\mathbb{S})^3} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \ast \arrow[d, "0"] \\
\underset{\text{classifying space for stable 4-Cohomotopy}}{\mathbb{S}^4} \arrow[r, "\Sigma^4(e^E)", "E\text{-unit map}"'] & \underset{\text{classifying space for degree-4 } E\text{-cohomology}}{E^4}\,.
\end{tikzcd}
\tag{155}
$$

(iii) While in general the Chern-Dold character on a generalized cohomology theory A is the rationalization map [FSS20c, §4.1], given on spectra as the smash product with the unit $e^{H\mathbb{R}}$ (32) of the Eilenberg-MacLane spectrum

$$
\mathrm{ch}_A \;:\; A \xrightarrow{\;\simeq\;} A \wedge \mathbb{S} \xrightarrow{\mathrm{id}_A \wedge e^{H\mathbb{R}}} A \wedge H\mathbb{R}\,,
$$

here for $A = \mathbb{S}$ this is reduces to the unit map itself, as shown in (155). But this means (by Def. 3.48 with Prop. 3.33) that the classifying spectrum for flat differential Cohomotopy is the unit cofiber spectrum (146) of $H\mathbb{R}$, as highlighted on the right of (155).

(iv) It follows by Prop. 3.53 that flat differential Cohomotopy is exactly the theory of $H_3^{H\mathbb{R}}$-fluxes (51) in the sense of Def. 3.40:

$$
\mathbb{S}^4_{\mathrm{flat}} \;\simeq\; \big((H\mathbb{R})/\mathbb{S}\big)^3\,, \qquad \overset{\text{flat differential 4-Cohomotopy of } X}{\widetilde{\mathbb{S}}^4_{\mathrm{flat}}(X)} \;\simeq\; H_3^{H\mathbb{R}}\mathrm{Fluxes}(X)\,.
$$

(v) Conversely, this means that for Whitehead-generalized cohomology theories E, the H_3^EFluxes (51) from Def. 3.40: constitute a generalization of flat differential Cohomotopy where the trivialization of the G_4-flux by the H_3-flux happens not in ordinary real cohomology, but in a Whitehead-generalized ("extraordinary") cohomology theory.

3.5 Adams e-invariant

While the Adams e-inavariant of a map (recalled below as Def. 3.58) exists whenever the d-invariant vanishes, its classical constructions proceeds through a refined quantity $\widehat{e}$ which is an invariant of the given map *equipped with a choice of trivialization* of its d-invariant (Def. 3.40). Since it is these choices of trivializations that are identified with H-fluxes in §2.4, we now re-cast the construction of the e-invariant from the more abstract perspective of Toda brackets (Def. 3.44) that makes this refined homotopy-dependence explicit (Def. 3.56, Theorem 3.62). We find that this perspective renders key properties of the e-invariant transparently manifest, notably it reduces Conner-Floyd's cobordism interpretation to an immediate corollary (discussed in §3.6 below).

The refined e-invariant as a Toda-bracket.

Definition 3.56 (The $\widehat{e}_{K\mathrm{U}}$-invariant)**.** We define the $\widehat{e}_{K\mathrm{U}}$-*invariant* to be the composite of

(i) the refined Toda-bracket $O^{K\mathrm{U}/\mathbb{S}}$ (151) from Prop. 3.53 for $E = K\mathrm{O}$,

(ii) with the cofiber Chern character from Example 3.51, yielding the refined Toda-bracket (71),

(iii) regarded as a rational number under the canonical splitting from Remark 3.52:

$$
\begin{tikzcd}
H^{K\mathrm{U}}_{2n-1}\mathrm{Fluxes}\big(S^{2(n+d)-1}\big) \arrow[r, "O^{K\mathrm{U}/\mathbb{S}}"'] \arrow[rr, bend left, "O^{(H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}}"] & (K\mathrm{U}/\mathbb{S})_{2d} \arrow[r, "\mathrm{ch}/\mathbb{S}"'] & \big((H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}\big)_{2d} \arrow[r, "\simeq", "\mathrm{spl}_0"'] & \mathbb{S}_{2(n+d)-1} \oplus \mathbb{Q}\,. \\
\big([c], \big[H^{K\mathrm{U}}_{2n-1}(c)\big]\big) \arrow[rrr, mapsto] & & & \big([c], \widehat{e}_{K\mathrm{U}}(c)\big)
\end{tikzcd}
\tag{156}
$$

Proposition 3.57 (Refined $\widehat{e}_{K\mathrm{U}}$-invariant)**.** *The refined $\widehat{e}_{K\mathrm{U}}$-invariant (from Def. 3.56) on $H^{K\mathrm{U}}_{n-1}\mathrm{Fluxes}\big(S^{2(n+d)-1}\big)$ descends to a $\mathbb{Q}/\mathbb{Z}$-valued function $e_{K\mathrm{U}}$ on $G^{\mathbb{S}}_n\mathrm{Fluxes}\big(S^{2(n+d)-1}\big)$ (37):*

$$
\begin{tikzcd}
H^{K\mathrm{U}}_{n-1}\mathrm{Fluxes}\big(S^{2(n+d)-1}\big) \arrow[r, "\widehat{e}_{K\mathrm{U}}"] \arrow[d] & \mathbb{Q} \arrow[d] \\
G^{\mathbb{S}}_n\mathrm{Fluxes}\big(S^{2(n+d)-1}\big) \arrow[r, "e_{K\mathrm{U}}"] & \mathbb{Q}/\mathbb{Z}\,.
\end{tikzcd}
\tag{157}
$$

Proof. Using the same three ingredients that enter Def. 3.56, we obtain the following commuting diagram of abelian groups, where the middle vertical composite is $\widehat{e}_{K\mathrm{U}}$:

$$
\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d, equal] & H^{K\mathrm{U}}_{2n-1}\mathrm{Fluxes}\big(S^{2(n+d)}\big) \arrow[r] \arrow[d, equal] & \widetilde{\mathbb{S}}^{2n+1}\big(S^{2(n+d)}\big) \arrow[d, equal] \\
\widetilde{K\mathrm{U}}^{2n}\big(S^{2(n+d)}\big) \arrow[r] \arrow[d, hook, "\mathrm{ch}"'] & \big(\widetilde{(K\mathrm{U}/\mathbb{S})}\big)^{2n}\big(S^{2(n+d)}\big) \arrow[r] \arrow[d, "\mathrm{ch}/\mathbb{S}"'] & \widetilde{\mathbb{S}}^{2n+1}\big(S^{2(n+d)}\big) \arrow[d, equal] \\
\big(\widetilde{H^{\mathrm{ev}}\mathbb{Q}}\big)^{2n}\big(S^{2(n+d)}\big) \arrow[r] \arrow[d, equal] & \big(\widetilde{(H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}}\big)^{2n}\big(S^{2(n+d)}\big) \arrow[r] \arrow[d, "\mathrm{spl}_0"', "\simeq"] & \widetilde{\mathbb{S}}^{2n+1}\big(S^{2(n+d)}\big) \arrow[d, equal] \\
\mathbb{Q} \arrow[r, hook] & \mathbb{Q} \oplus \mathbb{S}_{2d-1} \arrow[r, two heads] & \mathbb{S}_{2d-1}
\end{tikzcd}
$$

Now **(a)** Lemma 3.49 shows that the middle horizontal rows are exact; and **(b)** the left vertical morphism is the canonical injection (since the Chern character preserves the unit). Therefore (a) any two choices of $H^{K\mathrm{U}}_{n-1}(c)$ differ by an integer in the top row, and (b) this translates to an integer difference as we pass down to the bottom row. $\square$

Recovering the classical e-invariant. We prove (Theorem 3.62 below) that the diagrammatically defined e-invariant $\widehat{e}_{K\mathrm{U}}$ from Prop. 3.57 coincides with Adams's classical e_{Ad}-invariant. First we recall the classical construction ([Ad66, §3,7], review in [Qu14, §29]):

Definition 3.58 (Classical construction of Adams e-invariant)**.** For $n, d \in \mathbb{N}$, and $\big[S^{2(n+d)-1} \xrightarrow{c} S^{2n}\big] \in \pi^{2n}\big(S^{2(n+d)-1}\big)$, consider the following construction. Since $K\mathrm{U}_{2(n+d)-1} = 0$ (so that the $d_{K\mathrm{U}}$-invariant of all such c vanishes) the long exact sequence in $K\mathrm{U}$-cohomology along the cofiber sequence of c

$$\cdots \longleftarrow \underbrace{\widetilde{K\mathrm{U}}^0\big(S^{2(n+d)-1}\big)}_{=0} \xleftarrow{c^*} \widetilde{K\mathrm{U}}^0\big(S^{2d}\big) \xleftarrow{q_c^*} \widetilde{K\mathrm{U}}^0\big(C_c\big) \xleftarrow{p_c^*} \widetilde{K\mathrm{U}}^0\big(S^{2(n+d)}\big) \longleftarrow \underbrace{\widetilde{K\mathrm{U}}^{-1}\big(S^{2n}\big)}_{=0} \longleftarrow \cdots$$

truncates to a short exact sequence, which we may identify as follows:

$$\begin{array}{ccccccccc}
0 \longleftarrow & \widetilde{K\mathrm{U}}^0\big(S^{2n}\big) & \xleftarrow{\quad q_c^* \quad} & \widetilde{K\mathrm{U}}^0\big(C_c\big) & \xleftarrow{\quad p_c^* \quad} & \widetilde{K\mathrm{U}}^0\big(S^{2(n+d)}\big) & \longleftarrow 0 \\
& \| & & \| & & \| & \\
0 \longleftarrow & \mathbb{Z} & \twoheadleftarrow & \mathbb{Z} \oplus \mathbb{Z} & \hookleftarrow & \mathbb{Z} & \longleftarrow 0 \\
& \| & & \| & & \| & \\
& \Big\langle \big[G^{K\mathrm{U}}_{2n,\mathrm{unit}}\big] \Big\rangle & & \Big\langle \big[\vdash C^{K\mathrm{U}}_{2n-1}(c)\big]\,,\, \big[\vdash H^{K\mathrm{U}}_{2n-1,\mathrm{unit}}\big] \Big\rangle & & \Big\langle \big[\vdash H^{K\mathrm{U}}_{2n-1,\mathrm{unit}}\big] \Big\rangle &
\end{array}
\tag{158}$$

$$\big[G^{K\mathrm{U}}_{2n,\mathrm{unit}}\big] \xmapsto{\ \mathrm{spl}^{K\mathrm{U}}\ } \big[\vdash C^{K\mathrm{U}}_{2n-1}(c)\big]$$

Here we have observed that the two outer groups are the integers, by the suspension isomorphism and Bott-periodicity. Such short exact sequences split, and any choice of splitting spl identifies the middle group as a direct sum $\mathbb{Z} \oplus \mathbb{Z}$, whose canonical generators we have indicated by suggestive symbols.

Now, using that the Adams operations ψ^k (41) over the $(2r)$-sphere are given (46) by multiplication with k^r, and that they commute with pullback (43), it follows from (158) that

$$\begin{array}{ccc}
\widetilde{K\mathrm{U}}^0\big(C_c\big) & \xrightarrow{\qquad \psi^k \qquad} & \widetilde{K\mathrm{U}}^0\big(C_c\big) \\
\big[\vdash C^{K\mathrm{U}}_{2n-1}(c)\big] & \longmapsto & k^n \cdot \big[\vdash C^{K\mathrm{U}}_{2n-1}(c)\big] + \mu_k(c) \cdot \big[\vdash H^{K\mathrm{U}}_{2n-1,\mathrm{unit}}\big] \\
\big[\vdash H^{K\mathrm{U}}_{2n-1,\mathrm{unit}}\big] & \longmapsto & k^{n+d} \cdot \big[\vdash H^{K\mathrm{U}}_{2n-1,\mathrm{unit}}\big]
\end{array}
\tag{159}$$

for *some* assignment

$$(c,k) \longmapsto \mu_k(c) \in \mathbb{Z}\,. \tag{160}$$

Then the *classical Adams* e*-invariant* of c is

$$\underset{\text{Adams e-invariant}}{\mathrm{e}_{\mathrm{Ad}}(c)} := \big[\widehat{\mathrm{e}_{\mathrm{Ad}}}(c)\big]_{\mathrm{mod}\,\mathbb{Z}} \in \mathbb{Q}/\mathbb{Z}\,, \quad \text{for} \quad \underset{\substack{\text{refined} \\ \text{Adams e-invariant}}}{\widehat{\mathrm{e}_{\mathrm{Ad}}}(c)} := \frac{\mu_k(c)}{k^n\big(k^d - 1\big)} \in \mathbb{Q}\,. \tag{161}$$

It is readily checked that this is well-defined in that it is...

- ...independent of the choice of $k \geq 1$ – this follows by imposing the commutativity of the Adams operations $\psi^{k_1} \circ \psi^{k_2} = \psi^{k_2} \circ \psi^{k_1}$ on the transformation law (159);
- ...independent of the choice of splitting of (158) in that

  $$\big[\vdash C^{K\mathrm{U}}_{2n-1}(c)\big] \longmapsto \big[\vdash C^{K\mathrm{U}}_{2n-1}(c)\big] + n \cdot \big[\vdash H^{K\mathrm{U}}_{2n-1,\mathrm{unit}}\big] \qquad \Rightarrow \qquad \widehat{\mathrm{e}_{\mathrm{Ad}}} \longmapsto \widehat{\mathrm{e}_{\mathrm{Ad}}} + n\,.$$

  This follows by direct inspection of (159).

Alternatively, the number $\widehat{\mathrm{e}_{\mathrm{Ad}}}$ may be defined without reference to the Adams operations. Since the Chern character map ch respects (44) the Adams operations, the effect of (159) on Chern characters in $\widetilde{H^{\mathrm{ev}}\mathbb{Q}}^0(C_c)$ is given, in matrix notation, by:

$$\psi^k \begin{pmatrix} \mathrm{ch}\big[\vdash C^{K\mathrm{U}}_{2n-1}(c)\big] \\ \mathrm{ch}\big[\vdash H^{K\mathrm{U}}_{2n-1,\mathrm{unit}}\big] \end{pmatrix} = \begin{pmatrix} k^n & \widehat{\mathrm{e}_{\mathrm{Ad}}}(c) \cdot k^n\big(k^d - 1\big) \\ 0 & k^{n+d} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{ch}\big[\vdash C^{K\mathrm{U}}_{2n-1}(c)\big] \\ \mathrm{ch}\big[\vdash H^{K\mathrm{U}}_{2n-1,\mathrm{unit}}\big] \end{pmatrix}.$$

This admits diagonalization, with eigenvectors

$$
\begin{array}{rcl}
\mathrm{ch}\big[\vdash C^{KU}_{2n-1}(c) - \widehat{\mathrm{e}_{\mathrm{Ad}}}(c)\cdot \vdash H^{KU}_{3,\mathrm{unit}}\big] & \longmapsto & k^n \cdot \quad \mathrm{ch}\big[\vdash C^{KU}_{2n-1}(c) - \widehat{\mathrm{e}_{\mathrm{Ad}}}(c)\cdot \vdash H^{KU}_{2n-1,\mathrm{unit}}\big]\,, \\
\mathrm{ch}\big[\vdash H^{KU}_{2n-1,\mathrm{unit}}\big] & \longmapsto & k^{n+d} \cdot \quad \mathrm{ch}\big[\vdash H^{KU}_{2n+1,\mathrm{unit}}\big]\,.
\end{array}
\tag{162}
$$

Since the component of the Chern character transforming with k^r is (45) exactly the component of cohomological degree $2r$, this means that the refined e-invariant $\widehat{\mathrm{e}_{\mathrm{Ad}}}(c)$ (161) is equivalently the degree-$2(n+d)$-component of the Chern character of the choice of splitting (158):

$$
\begin{tikzcd}
\widetilde{KU}^{2n}\big(S^{2n}\big) \arrow[r, "\text{choose lift}", "\mathrm{spl}^{KU}\circ i_r"'] & \widetilde{KU}^{2n}\big(C_c\big) \arrow[r, "\text{Chern character}", "\mathrm{ch}"'] & \widetilde{H^{\mathrm{ev}}\mathbb{Q}}^{2n}\big(C_c\big) \arrow[r, "\substack{\text{projection onto}\\ \text{degree } 2(n+d)}", "p_l\circ \mathrm{spl}_0^{H^{\mathrm{ev}}\mathbb{Q}}"'] & \widetilde{H\mathbb{Q}}^{2(n+d)}\big(C_c\big) \arrow[r, equal] & \mathbb{Q} \\
\big[G^{KU}_{2n,\mathrm{unit}}\big] \arrow[r, mapsto] & \big[\vdash C^{KU}_{2n-1}(c)\big] \arrow[r, mapsto] & \mathrm{ch}\big[\vdash C^{KU}_{2n-1}(c)\big] \arrow[r, mapsto, "(162)"'] & \widehat{\mathrm{e}_{\mathrm{Ad}}}(c)\cdot \mathrm{ch}\big[\vdash H^{KU}_{2n-1,\mathrm{unit}}\big] \arrow[r, mapsto] & \widehat{\mathrm{e}_{\mathrm{Ad}}}(c)
\end{tikzcd}
\tag{163}
$$

Example 3.59 (Adams e-invariant on 3rd stable stem)**.** On the third stem $\mathbb{S}_3 \simeq \mathbb{Z}_{24} := \mathbb{Z}/24$ (52), the classical Adams e-invariant (Def. 3.58) is:

$$
\begin{tikzcd}
\mathbb{S}_\bullet \arrow[d, "e_{\mathbb{R}}", "\simeq"'] & \ni & \big[h_{\mathbb{H}}\big] \arrow[dd, mapsto] \\
\mathbb{Z}_{24} \arrow[d, hook] & & \\
\mathbb{Q}/\mathbb{Z} & \ni & \big[\tfrac{1}{24}\big]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{S}_\bullet \arrow[d, "e_{\mathbb{C}}", "\simeq"'] & \ni & \big[h_{\mathbb{H}}\big] \arrow[dd, mapsto] \\
\mathbb{Z}_{12} \arrow[d, hook] & & \\
\mathbb{Q}/\mathbb{Z} & \ni & \big[\tfrac{1}{12}\big]
\end{tikzcd}
$$

This is due to [Ad66, Prop. 7.14, Ex. 7.17].

Remark 3.60. Observe that both the classical construction of the e_{Ad}-invariant in the form (163) as well as our diagrammatic construction in Def. 3.56 produce a number by reference to one of the *canonical splittings* of Remark 3.52, given by projection onto the rational cohomology in degree $2(n+2)$.

Hence to relate the two constructions we need to relate these two canonical splittings:

Lemma 3.61 (Compatibility of the two canonical splittings)**.** *Given non-trivial* $\big[S^{2(n+d)-1} \xrightarrow{c} S^{2n}\big] \in \mathbb{S}_{2d-1}$ *for any* $n, d \in \mathbb{N}$ *with* $d \geq 1$, *we have a commuting diagram*

$$
\begin{tikzcd}
 & \big(\widetilde{(H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}}\big)^{2n}\big(S^{2(n+d)}\big) \arrow[d, "p^*"'] \arrow[r, equal, "\mathrm{spl}_0"] & \mathbb{Q}\oplus\mathbb{S}_{2d-1} \arrow[d, "\left(\begin{smallmatrix}\mathrm{id} & \cdot \\ 0 & \cdot\end{smallmatrix}\right)"] \\
\big(\widetilde{H^{\mathrm{ev}}\mathbb{Q}}\big)^{2n}\big(C_c\big) \arrow[r, "i_*"] \arrow[d, equal, "\mathrm{spl}_0"'] & \big(\widetilde{(H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}}\big)^{2n}\big(C_c\big) \arrow[r, equal] & \mathbb{Q}\oplus\big(\mathbb{Q}/\mathbb{Z}\oplus'\mathbb{S}_{2d-1}/\mathbb{Z}\big)\,, \\
\mathbb{Q}\oplus\mathbb{Q} \arrow[urr, "\left(\begin{smallmatrix}\mathrm{id} & \cdot \\ 0 & \cdot\end{smallmatrix}\right)"'] & &
\end{tikzcd}
$$

Here:

- the rear morphisms are pushforward along the projection $E \xrightarrow{i} E/\mathbb{S}$ (146) and pullback along $C_c \xrightarrow{p} S^{2(n+d)}$, respectively;
- spl_0 denotes the canonical isomorphisms from Remark 3.52;
- the matrices act on row vectors by multiplication from the right;
- $\mathbb{S}_{d-1} := \mathrm{cof}\big(\mathbb{Z} \xrightarrow{c} \mathbb{S}_{d-1}\big)$;
- the symbol $\oplus'$ denotes some possibly non-trivial extension, left undetermined, of $\mathbb{Q}/\mathbb{Z}$ by $\mathbb{S}_{d-1}/\mathbb{Z}$ and vice versa.

Proof. Consider the diagram

$$
\pi_0 \mathrm{Maps}\left(
\begin{tikzcd}[row sep=small]
\vdots \\
S^{2(n+d)} \arrow[u] \\
C_c \arrow[u] \\
S^{2n} \arrow[u] \\
\vdots \arrow[u]
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
(H^{\mathrm{ev}}\mathbb{Q})^{2n} \arrow[r] & \big((H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}\big)^{2n} \arrow[r] & \mathbb{S}^{2n-1}
\end{tikzcd}
\right)
$$

all of whose rows and columns are exact (since $\mathrm{Maps}^{*/}$ sends both homotopy cofiber sequences in the first argument as well as homotopy fiber sequences in the second argument to homotopy fiber sequences, and using that these induce long exact sequences on homotopy groups).

By definition of reduced cohomology groups, this diagram is equal to the following (in the sector shown):

$$
\begin{tikzcd}[column sep=small]
\widetilde{\mathbb{S}}^{2n}(S^{2n+1}) \arrow[r] \arrow[d] & \widetilde{(H^{\mathrm{ev}}\mathbb{Q})}^{2n}(S^{2n+1}) \arrow[r] \arrow[d] & ((\widetilde{H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}})^{2n}(S^{2n+1}) \arrow[r] \arrow[d] & \widetilde{\mathbb{S}}^{2n+1}(S^{2n+1}) \arrow[d] & \\
\widetilde{\mathbb{S}}^{2n}(S^{2(n+d)}) \arrow[r] \arrow[d] & \widetilde{(H^{\mathrm{ev}}\mathbb{Q})}^{2n}(S^{2(n+d)}) \arrow[r] \arrow[d] & ((\widetilde{H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}})^{2n}(S^{2(n+d)}) \arrow[r] \arrow[d] & \widetilde{\mathbb{S}}^{2n+1}(S^{2(n+d)}) \arrow[r] \arrow[d] & \widetilde{(H^{\mathrm{ev}}\mathbb{Q})}^{2n+1}(S^{2(n+d)}) \arrow[d] \\
\widetilde{\mathbb{S}}^{2n}(C_c) \arrow[r] \arrow[d] & \widetilde{(H^{\mathrm{ev}}\mathbb{Q})}^{2n}(C_c) \arrow[r] \arrow[d] & ((\widetilde{H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}})^{2n}(C_c) \arrow[r] \arrow[d] & \widetilde{\mathbb{S}}^{2n+1}(C_c) \arrow[r] \arrow[d] & \widetilde{(H^{\mathrm{ev}}\mathbb{Q})}^{2n+1}(C_c) \\
\widetilde{\mathbb{S}}^{2n}(S^{2n}) \arrow[r] \arrow[d] & \widetilde{(H^{\mathrm{ev}}\mathbb{Q})}^{2n}(S^{2n}) \arrow[r] \arrow[d] & ((\widetilde{H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}})^{2n}(S^{2n}) \arrow[r] & \widetilde{\mathbb{S}}^{2n+1}(S^{2n}) & \\
\widetilde{\mathbb{S}}^{2n}(S^{2(n+d)-1}) \arrow[r] & \widetilde{(H^{\mathrm{ev}}\mathbb{Q})}^{2n}(S^{2(n+d)-1}) & & &
\end{tikzcd}
$$

Evaluating all the cohomology groups on spheres yields:

$$
\begin{tikzcd}[column sep=small]
\mathbb{S}_1 \oplus 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & ((\widetilde{H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}})^{2n}(S^{2n+1}) \arrow[r] \arrow[d] & 0 \oplus \mathbb{Z} \arrow[d, "c"'] & \\
\mathbb{S}_{2d} \oplus 0 \arrow[r, "0"] \arrow[d] & \mathbb{Q} \oplus 0 \arrow[r] \arrow[d] & ((\widetilde{H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}})^{2n}(S^{2(n+d)}) \arrow[r] \arrow[d] & 0 \oplus \mathbb{S}_{2d-1} \arrow[r] \arrow[d] & 0 \arrow[d] \\
\widetilde{\mathbb{S}}^{2n}(C_c) \arrow[r] \arrow[d] & \widetilde{(H^{\mathrm{ev}}\mathbb{Q})}^{2n}(C_c) \arrow[r] \arrow[d] & ((\widetilde{H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}})^{2n}(C_c) \arrow[r] \arrow[d] & \widetilde{\mathbb{S}}^{2n+1}(C_c) \arrow[r] \arrow[d] & \widetilde{(H^{\mathrm{ev}}\mathbb{Q})}^{2n+1}(C_c) \\
0 \oplus \mathbb{Z} \arrow[r, "1 \mapsto 1"] \arrow[d, "c"'] & 0 \oplus \mathbb{Q} \arrow[r] \arrow[d] & ((\widetilde{H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S}})^{2n}(S^{2n}) \arrow[r] & 0 & \\
0 \oplus \mathbb{S}_{2d-1} \arrow[r] & 0 & & &
\end{tikzcd}
$$

From this we recognize various split exact sequences, using that $\mathrm{Ext}^1(-,\mathbb{Q}) = 0$ and $\mathrm{Ext}^1(\mathbb{Z},-) = 0$:

$$
\begin{tikzcd}
\mathbb{S}_1 \oplus 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \oplus \mathbb{Z} \arrow[r] \arrow[d] & 0 \oplus \mathbb{Z} \arrow[d, "c"'] & \\
\mathbb{S}_{2d} \oplus 0 \arrow[r, "0"] \arrow[d] & \mathbb{Q} \oplus 0 \arrow[r] \arrow[d] & \mathbb{Q} \oplus \mathbb{S}_{2d-1} \arrow[r] \arrow[d] & 0 \oplus \mathbb{S}_{2d-1} \arrow[r] \arrow[d] & 0 \arrow[d] \\
(\mathbb{S}_{2d}/\mathbb{S}_1) \oplus \mathbb{Z} \arrow[r] \arrow[d, "0 \oplus \mathrm{ord}(c)"'] & \mathbb{Q} \oplus \mathbb{Q} \arrow[r] \arrow[d] & \big((\widetilde{H^{\mathrm{ev}}\mathbb{Q})}/\mathbb{S}\big)^{2n}(C_c) \arrow[r] \arrow[d] & 0 \oplus \mathbb{S}_{2d-1}/\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \oplus \mathbb{Z} \arrow[r, "1 \mapsto 1"] \arrow[d, "c"'] & 0 \oplus \mathbb{Q} \arrow[r] \arrow[d] & 0 \oplus \mathbb{Q}/\mathbb{Z} \arrow[r] & 0 & \\
0 \oplus \mathbb{S}_{2d-1} \arrow[r] & 0 & & &
\end{tikzcd}
$$

Here the splittings shown in purple we may *choose to be the canonical ones* from Remark 3.52.
Further from this, and using

1. again that $\mathrm{Ext}^1(-,\mathbb{Q}) = 0$;
2. that $\mathrm{Ext}^1(-,A \oplus B) \simeq \mathrm{Ext}^1(-,A) \oplus \mathrm{Ext}^1(-,B)$;
3. commutativity of the middle square with the two purple entries

the remaining entry and the maps into it must be as claimed:

$$
\begin{tikzcd}
\mathbb{S}_1 \oplus 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \oplus \mathbb{Z} \arrow[r] \arrow[d] & 0 \oplus \mathbb{Z} \arrow[d, "c"'] & \\
\mathbb{S}_{2d} \oplus 0 \arrow[r, "0"] \arrow[d] & \mathbb{Q} \oplus 0 \arrow[r] \arrow[d] & \mathbb{Q} \oplus \mathbb{S}_{2d-1} \arrow[r] \arrow[d, "\left(\begin{smallmatrix} \mathrm{id} & \cdot \\ 0 & \cdot \end{smallmatrix}\right)"'] & 0 \oplus \mathbb{S}_{2d-1} \arrow[r] \arrow[d] & 0 \arrow[d] \\
(\mathbb{S}_{2d}/\mathbb{S}_1) \oplus \mathbb{Z} \arrow[r] \arrow[d, "0 \oplus \mathrm{ord}(c)"'] & \mathbb{Q} \oplus \mathbb{Q} \arrow[r, "\left(\begin{smallmatrix} \mathrm{id} & \cdot \\ 0 & \cdot \end{smallmatrix}\right)"] \arrow[d] & \mathbb{Q} \oplus \big(\mathbb{Q}/\mathbb{Z} \oplus' \mathbb{S}_{2d-1}/\mathbb{Z}\big) \arrow[r] \arrow[d] & 0 \oplus \mathbb{S}_{2d-1}/\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \oplus \mathbb{Z} \arrow[r, "1 \mapsto 1"] \arrow[d, "c"'] & 0 \oplus \mathbb{Q} \arrow[r] \arrow[d] & 0 \oplus \mathbb{Q}/\mathbb{Z} \arrow[r] & 0 & \\
0 \oplus \mathbb{S}_{2d-1} \arrow[r] & 0 & & &
\end{tikzcd}
$$

□

Now we may conclude:

Theorem 3.62 (Identification of e-invariants)**.** *The refined $\widehat{e}_{KU}$-invariant from Def. 3.56 coincides with the quantity $\widehat{\mathrm{e}_{\mathrm{Ad}}}$ in* (161)*:*

$$
\widehat{e}_{K\mathrm{U}}(-,-) \;=\; \widehat{\mathrm{e}_{\mathrm{Ad}}}(-,-)\,. \tag{164}
$$

In particular, the diagrammatic $e_{K\mathrm{U}}$-invariant from Prop. 3.57 coincides with the classical Adams e *invariant (Def. 3.58):*

$$
e_{K\mathrm{U}}(-) \;=\; \mathrm{e}_{\mathrm{Ad}}(-)\,.
$$

Proof. The homotopy-commuting rectangle in the bottom right part of the defining diagram (71) says that

$$
p^*\Big(\widehat{e}_{K\mathrm{U}}(c),[c]\Big) \;=\; i_*\Big(\mathrm{ch}\big[\vdash C^{K\mathrm{U}}_{2n-1}(c)\big]\Big)\,.
$$

By Lemma 3.61 this means that the image of both sides along their canonical retractions (Remark 3.52) onto degree=$2(n+d)$ rational cohomology $\simeq \mathbb{Q}$ coincide. But by definitions (156) and (163) this is the statement of equality (164). □

3.6 Conner-Floyd e-invariant

Calabi-Yau manifolds and SU-Cobordism. The special unitary Cobordism ring is tightly related to the complex geometry of Calabi-Yau manifolds:

Proposition 3.63 (Torsion in the SU-Cobordism ring [CLP19, Thm. 5.8b, Thm. 5.11a])**.** *The Cobordism ring $(MS\mathrm{U})_\bullet$ of SU-manifolds has only 2-torsion, and that is concentrated in degrees* $1,2 \,\mathrm{mod}\, 8$.

Proposition 3.64 (Non-torsion SU-Cobordism ring is generated by Calabi-Yau manifolds [LLP17, Thm. 2.4])**.** *Away from its 2-torsion (Prop. 3.63) the SU-cobordism ring is multiplicatively generated by Calabi-Yau manifolds, in that every element in $(MS\mathrm{U})_\bullet[\frac{1}{2}]$ is equal to a polynomial over $\mathbb{Q}$ of SU-Cobordism classes of Calabi-Yau manifolds.*

In particular:

Proposition 3.65 (K3 spans SU-Cobordism in degree 4 [LLP17, Ex. 3.1][CLP19, Thm. 13.5a])**.** *The SU-Cobordism ring in degree 4 consists precisely of integer multiples of the class* [K3] *of any non-toroidal K3-surface:*

$$(MS\mathrm{U})_4 \;\simeq\; \mathbb{Z}\big\langle [\mathrm{K3}] \big\rangle \,.$$

The Conner-Floyd K-orientation.

Proposition 3.66 (Conner-Floyd K-orientation [CF66, §5, p. 29])**.** *There is a homotopy-commutative diagram*

$$\begin{tikzcd} MS\mathrm{U} \arrow[r, "\sigma_{S\mathrm{U}}"] \arrow[d] & K\mathrm{O} \arrow[d] \\ M\mathrm{U} \arrow[r, "\sigma_{\mathrm{U}}"] & K\mathrm{U} \end{tikzcd} \tag{165}$$

of homotopy-commutative ring spectra, where the vertical morphism are the canonical ones.

Remark 3.67. The Conner-Floyd K-orientation of Prop. 3.66 is directly analogous to the more widely known *Atiyah-Bott-Shapiro orientation* [ABS64]

$$\begin{aligned} M\mathrm{Spin} &\xrightarrow{\;\sigma_{\mathrm{Spin}}\;} K\mathrm{O} \\ M\mathrm{Spin}^c &\xrightarrow{\;\sigma_{\mathrm{Spin}^c}\;} K\mathrm{U}\,. \end{aligned} \tag{166}$$

(All these horizontal maps are "orientations" in the sense of Example 3.96.)

The rational Todd class.

Definition 3.68 (Todd class)**.** For $M^{\leq 6}_{\mathrm{U}}$ a U-manifold of real dimension ≤ 6, its *Todd class* is the following rational combination of Chern classes of the given stable U-structured tangent bundle:

$$\mathrm{Td}\big(M^{\leq 6}_{\mathrm{U}}\big) \;\coloneqq\; 1 + \tfrac{1}{2}c_1 + \tfrac{1}{12}\big(c_1^2 + c_2\big) + \tfrac{1}{24}c_1 c_2 \quad \in H^\bullet\big(M^{\leq 6}_{\mathrm{U}};\mathbb{Q}\big)\,, \tag{167}$$

where we abbreviate $c_i \coloneqq c_i\big(T_{\mathrm{st}}X^{\leq 6}_{\mathrm{U}}\big)$. [12] For closed M_{U}, their *Todd number*, i.e. the evaluation

$$\mathrm{Td}[M_{\mathrm{U}}] \;\coloneqq\; \mathrm{Td}(M_{\mathrm{U}})[M] \quad \in \mathbb{Q} \tag{168}$$

of the Todd class of the stable tangent bundle on the fundamental homology class $[M]$ of the underlying manifold, is a cobordism invariant and hence constitutes a function on the complex cobordism ring, which happens to be an integer (Prop. 3.70):

$$\mathrm{Td}\,:\; (M\mathrm{U})_\bullet \longrightarrow \mathbb{Z} \hookrightarrow \mathbb{Q}\,.$$

[12] Of course the Todd class is defined more abstractly and in all dimensions (see e.g. [SUZ19]); but we restrict attention as above for focus of the exposition.

Example 3.69 (Square root of Todd class of SU-manifolds)**.** On SU-manifolds $[M^{\leq 6}_{S\mathrm{U}}] \in (MS\mathrm{U})_{\leq 6} \to (M\mathrm{U})_{\leq 6}$, where the first Chern class vanishes $c_1(M_{S\mathrm{U}}) = 0$, the Todd class (167) reduces to

$$\mathrm{Td}(M^{\leq 6}_{S\mathrm{U}}) \;=\; 1 + \tfrac{1}{12} c_2 \,, \tag{169}$$

and hence the *square root* of the Todd class (in the sense of formal power series of Chern classes) on these manifolds is

$$\begin{aligned} \sqrt{\mathrm{Td}}\big(M^{\leq 6}_{S\mathrm{U}}\big) \;&=\; 1 + \tfrac{1}{2}\big(\tfrac{1}{12}c_2\big) + \underbrace{-\tfrac{1}{8}\big(\tfrac{1}{12}c_2\big)^2 + \tfrac{1}{16}\big(\tfrac{1}{12}c_2\big)^3 + \cdots}_{=0} \\ &=\; 1 + \tfrac{1}{24} c_2 \qquad \in\; H^\bullet\big(X^{\leq 6}_{S\mathrm{U}};\mathbb{Q}\big)\,. \end{aligned} \tag{170}$$

Yet more specifically, if $M^{\leq 6}_{S\mathrm{U}} = M^4_{S\mathrm{U}}$ is a *complex curve* with vanishing first Chern class, such as a Calabi-Yau 2-fold/K3-surface, then the second Chern class equals the Euler class (being the top Chern class),

$$c_2(M^4_{\mathrm{U}}) \;=\; \chi_4(M^4) \quad \in\; H^4(M^4;\mathbb{Q})\,, \tag{171}$$

so that (170) reduces further to

$$\sqrt{\mathrm{Td}}\big(M^4_{S\mathrm{U}}\big) \;=\; 1 + \tfrac{1}{24}\chi_4 \quad \in H^\bullet\big(X;\mathbb{Q}\big)\,. \tag{172}$$

Finally, in terms of characteristic numbers (168), i.e. after evaluation on the fundamental homology class $[M^4]$, this means that the evaluation of the square root of the Todd class is half the evaluation of the Todd class itself:

$$\sqrt{\mathrm{Td}}\,[M^4] \;=\; \tfrac{1}{24}\chi_4[M^4] \;=\; \tfrac{1}{2}\mathrm{Td}(M^4_{S\mathrm{U}})[M^4] \quad \in\; \mathbb{Z} \hookrightarrow \mathbb{Q}\,. \tag{173}$$

Proposition 3.70 (Quantization of Todd numbers)**.** *The Todd number* (168) *of any closed* U*-manifold is integer, and that of an S*U*-manifold of dimension* 4 mod 8 *is moreover divisible by 2.*

Remark 3.71 (Rational Todd numbers via Chern-Weil theory)**.** By Chern-Weil theory (review in [FSS20c]) the Todd number (Def. 3.68) of a smooth closed U-manifold M_{U} may equivalently be computed as the integral of a differential form: Choosing any unitary connection $\nabla := \nabla^{T_{\mathrm{st}}M_{\mathrm{U}}}$ on the stable tangent bundle $T_{\mathrm{st}}M_{\mathrm{U}}$ and evaluating its curvature 2-form F_∇ in the invariant polynomials $\langle\cdots\rangle : \mathrm{Sym}^\bullet(\mathfrak{u}) \to \mathbb{R}$ on the unitary Lie algebra $\mathfrak{u}$ yields the *Chern forms* $c_{2k}(\nabla)$ and hence the *Todd form* $\mathrm{Td}(\nabla)$, by inserting the Chern forms in place of the Chern classes in the defining polynomial expression (167):

$$\nabla := \nabla^{T_{\mathrm{st}}M_{\mathrm{U}}}\,, \qquad c_k(\nabla) \;:=\; \langle\underbrace{F_\nabla \wedge \cdots \wedge F_\nabla}_{k \text{ factors}}\rangle \;\in\; \Omega^{2i}_{\mathrm{dR}}\big(M_{\mathrm{U}}\big)\,, \qquad \mathrm{Td}(\nabla) \;:=\; \mathrm{Td}(c_i(\nabla)) \;\in\; \Omega^\bullet_{\mathrm{dR}}\big(M_{\mathrm{U}}\big)\,.$$

Then the Todd number (168) is equal to the integral of this differential Todd form over the manifold:

$$\mathrm{Td}[M_{\mathrm{U}}] \;=\; \int_{M_{\mathrm{U}}} \mathrm{Td}(\nabla^{TM})\,. \tag{174}$$

An advantage of this differential re-formulation of the Todd number is that it generalizes to (compact) U-manifolds *with boundaries*. Specifically for U-manifolds $M_{\mathrm{U,Fr}}$ with framed boundaries it yields again an invariant, now generally taking rational values:

$$\begin{array}{ccc} \underset{\text{with Fr-boundaries}}{\overset{\text{Cobordism ring of}}{\text{U-manifolds}}}\;(M\mathrm{U}/\mathbb{S})_\bullet & \xrightarrow[\text{rational Todd number}]{\mathrm{Td}} & \mathbb{Q} \\ [M_{\mathrm{U,Fr}}] & \longmapsto & \int_M \mathrm{Td}(\nabla^{T_{\mathrm{st}}M_{\mathrm{U,Fr}}})\,. \end{array} \tag{175}$$

This differential-geometric formulation of the Todd class allows it to be further expressed by boundary data alone:

Lemma 3.72 (Euler class of 4-manifolds with boundary via twisted 3-forms)**.** *Let M^4 be a closed manifold of real dimension 4, and let*

$$\underset{1\leq k\leq n}{\sqcup} D^4_k \hookrightarrow M^4$$

be an embedding of a positive number $n \geq 1 \in \mathbb{N}$ of disjoint open balls. Then, for any choice of tangent connection ∇ on the complement manifold $M^4 \setminus (\underset{1\leq k\leq n}{\sqcup} D^4_k)$ with boundary $\underset{1\leq k\leq n}{\sqcup} S^3_k$, there exists a differential 3-form H_3 which
(a) *trivializes the Euler-form in de Rham cohomology, and*
(b) *integrates over the boundary 3-spheres to the Euler number of M^4:*

$$\exists\, H_3 \in \Omega^3_{\mathrm{dR}}\big(M^4 \setminus (\sqcup_n D^4)\big) \quad \text{s.t.} \quad \begin{cases} d\,H_3 \;=\; \chi_4(\nabla) \\ \displaystyle\int_{\sqcup_n S^3} H_3 \;=\; \chi_4[M^4]\,. \end{cases} \tag{176}$$

Proof. We use the standard argument of choosing any vector field with isolated zeros on M^4 and then deforming that continuously to move all of these into the open balls. With this, the complement manifold $M^4 \setminus \big(\sqcup_n D^4\big)$ carries a nowhere-vanishing smooth vector field v and hence, after re-scaling, a section h of the 3-sphere fiber bundle associated with the tangent bundle

$$v \in \Gamma_{M^4}\Big(T\big(M^4 \setminus \sqcup_n D^4\big)\Big)\,, \qquad h := v/|v| \;\in\; \Gamma_{M^4}\Big(S\big(T(M^4 \setminus \sqcup_n D^4)\big)\Big)\,. \tag{177}$$

After choosing trivializations of the tangent bundle, and hence its 3-sphere bundle, on an open neighborhood of each D^4_k, this section restricts over each boundary component to a map $h|_{S^3_k} : S^3 \to S^3$, with Hopf winding degree

$$\underset{\text{of } h \text{ at } x_k}{\overset{\text{Poincaré-Hopf index}}{}} \quad \deg\big(h|_{S^3_k}\big) \;=\; \int_{S^3_k} \big(h^*_{S^3_k}\mathrm{vol}_{S^3}\big) \quad \underset{\text{of } h|_{S^3_k}}{\overset{\text{Hopf degree}}{}} \tag{178}$$

By the Poincaré-Hopf theorem (e.g. [DNF85, Sec. 15.2], see [FSS19b, (83)] for review in our context), the sum of these degrees is the Euler characteristic of M^4:

$$\sum_{1\leq k\leq n} \deg\big(h|_{S^3_k}\big) \;=\; \chi_4[M^4]\,. \tag{179}$$

Now, the section h (177) constitutes a cocycle in the $\tau := TM^4$-twisted 4-Cohomotopy of $M^4 \setminus \sqcup_n D^4$ ([FSS19b, Def. 2.1]); and by [FSS19b, Prop. 2.5 (i)], the image of this cocycle under the twisted cohomotopical character map ([FSS20c])

$$\begin{array}{ccc} \overset{\text{J-twisted 3-Cohomotopy}}{\pi^\tau\big(M^4\big)} & \xrightarrow[\mathrm{ch}^\tau_\pi]{\text{twisted 3-cohomotopical character map}} & \overset{\text{twisted non-abelian de Rham cohomology}}{H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\big(M^4; \mathfrak{l}S^3\big)} \\ & & \big\| \\ & & \Big\{ H_3 \in \Omega^3_{\mathrm{dR}}\big(M^4\big) \;\Big|\; d\,H_3 = \chi_4\big(\nabla^{TM^4}\big) \Big\} \\[1em] \begin{tikzcd} & S(TM^4) \arrow[d] \\ X \arrow[ur, dashed, "h"] \arrow[r, equal] & X \end{tikzcd} & \longmapsto & \begin{tikzcd} & \widehat{H}_3 \arrow[dl, "h^*"'] \\ H_3 := h^*\widehat{H}_3 & \end{tikzcd} \end{array} \tag{180}$$

is a differential form H_3 satisfying $d\,H_3 = \chi_4(\nabla)$. Moreover, this H_3 is [FSS19b, (45)] the pullback along h of a fiber-wise unit volume form $\widehat{H}_3$ on the spherical fibration. This implies the claim.

More concretely: There exists (e.g. [Wa04, §6.6, Thm. 6.1]) a differential 3-form $\widehat{H}_3$ on the total space of the 3-sphere bundle of the tangent bundle of M^4, which (a) trivializes the pullback of the Euler-form along the bundle projection p and (b) restricts to the unit volume form on each 3-sphere fiber S^3_x:

$$\exists\, \widehat{H}_3 \in \Omega^3_{\mathrm{dR}}\Big(S\Big(T\big(M^4 \setminus (\sqcup_n D^4)\big)\Big)\Big) \quad \text{s.t.} \quad \begin{cases} d\,\widehat{H}_3 \;=\; p^*\chi_4(\nabla) \\ \displaystyle\int_{S^3_x} \widehat{H}_3 \;=\; 1\,. \end{cases} \tag{181}$$

Therefore, the pullback form

$$H_3 \;\coloneqq\; h^* \widehat{H}_3$$

satisfies

$$dH_3 \;=\; dh^*\widehat{H}_3 \;=\; h^* d\widehat{H}_3 \;=\; h^* p^* \chi_4(\nabla) \;=\; \chi_4(\nabla)$$

and

$$\int_{\sqcup_n S^3} H_3 \;=\; \int_{\sqcup_n S^3} h^*\widehat{H}_3 \;=\; \sum_{1\leq k\leq n} \int_{S^3_k} \big(h|_{S^3_k}\big)^* \mathrm{vol}_{S^3} \;=\; \sum_{1\leq k\leq n} \deg\big(h|_{S^3_k}\big) \;=\; \chi_4[M^4]\,,$$

where in the last step we used (179). □

In conclusion we highlight the following special case:

Proposition 3.73 (Rational Todd number of punctured complex curve as boundary integral)**.** *Let*

$$M^4_{\mathrm{SU,Fr}} \quad \text{s.t.} \quad \partial M^4 \;=\; \underset{1\leq k\leq n}{\sqcup} S^3_k$$

be a real 4-dimensional SU*-manifold whose non-empty framed boundary is a disjoint union of 3-spheres. Then its rational Todd number* (175) *equals* $\frac{1}{12}$*th of the boundary integral of a differential 3-form* H_3 *whose de Rham differential equals the Euler form:*

$$\exists H_3 \in \Omega^3_{\mathrm{dR}}\big(M^4\big)\,, \quad \text{s.t.} \quad d\,H_3 \;=\; \chi_4(\nabla^{TM^4_{\mathrm{U,Fr}}}) \;\; \text{and} \;\; \mathrm{Td}\,[M^4_{\mathrm{U,Fr}}] \;=\; \tfrac{1}{12} \sum_{1\leq k\leq n} \int_{S^3_k} H^3\,. \tag{182}$$

Moreover, if $M^4_{\mathrm{SU,Fr}}$ *is the complement of* $n \geq 1$ *disjoint open balls in a closed* SU*-manifold* $M^4_{S\mathrm{U}}$*, then this equals* $\frac{1}{12}$*th of the Euler number of that closed manifold:*

$$M^4_{S\mathrm{U,Fr}} \;=\; M^4_{S\mathrm{U}} \setminus \underset{1\leq k\leq n}{\sqcup} \qquad \Rightarrow \qquad \mathrm{Td}\,[M^4_{S\mathrm{U,Fr}}] \;=\; \chi_4[M^4_{S\mathrm{U}}]\,. \tag{183}$$

Proof. The existence of the differential 3-form and the final statement (183) follows by Lemma 3.72. To see (182) we compute as follows:

$$\begin{aligned} \mathrm{Td}\,[M^4_{\mathrm{U,Fr}}] &= \tfrac{1}{12} c_2 [M^4_{\mathrm{U,Fr}}] \\ &= \tfrac{1}{12} \int_{M^4} c_2(\nabla^{TM^4_{\mathrm{U,Fr}}}) \\ &= \tfrac{1}{12} \int_{M^4} \chi_4(\nabla^{TM^4_{\mathrm{U,Fr}}}) \\ &= \tfrac{1}{12} \int_{M^4} d\,H_3 \\ &= \tfrac{1}{12} \int_{\partial M^4} H^3\,. \end{aligned}$$

Here the first line is (169), the second line uses (174), the third line is (171), the fourth line is item (a) in (181) and the last line is the Stokes theorem. □

The Todd character.

Proposition 3.74 (Todd character on U-manifolds is Chern character of KU-Thom class)**.** *The composite of the* σ_{U}*-orientation of* KU (165) *with the Chern character is the Todd character*

$$\begin{tikzcd} M\mathrm{U} \arrow[r, "\sigma_{\mathrm{U}}"] \arrow[rr, bend right, "\mathrm{Td}"'] & K\mathrm{U} \arrow[r, "\mathrm{ch}"] & H^{\mathrm{ev}}\mathbb{Q}\,, \end{tikzcd} \tag{184}$$

i.e. the morphism of homotopy commutative ring spectra whose induced morphism of coefficient rings is the Todd number (Def. 3.68)

$$\begin{aligned} (M\mathrm{U})_\bullet &\xrightarrow{\mathrm{Td}_\bullet} (K\mathrm{U})_\bullet \simeq \mathbb{Z}[\beta_2] \hookrightarrow \mathbb{Q}[\beta_2] \\ [M^{2d}_{\mathrm{U}}] &\longmapsto \mathrm{Td}[M^{2d}_{\mathrm{U}}] \cdot \beta_2^d\,, \end{aligned} \tag{185}$$

This goes back to [CF66, §6] and is formulated explicitly as above in [Sm73, p. 303] (review in [Spi13, §2.3.2]). In components, (184) encodes the fact that the Todd class of a complex vector bundle in rational cohomology equals, under the Thom isomorphism in complex K-theory, the Chern character of a $K\mathrm{U}$-Thom class; in which form the statement is given in [Ka78, §V, Thm. 4.4].

We record the following two consequences:

Proposition 3.75 (Todd character on SU-manifolds is Pontrjagin character of KO-Thom class)**.** *After restriction from* U*-manifolds to* SU*-manifolds, the rational Todd character factors as the Conner-Floyd K-orientation* $\sigma_{S\mathrm{U}}$ (3.66) *on* KO *followed by the Pontrjagin character:*

$$\begin{tikzcd} MS\mathrm{U} \arrow[r, "\sigma_{S\mathrm{U}}"] \arrow[rr, bend right, "\mathrm{Td}"'] & K\mathrm{O} \arrow[r, "\mathrm{ph}"] & H^{\mathrm{ev}}\mathbb{Q}\,. \end{tikzcd} \tag{186}$$

Proof. Observe that we have the following homotopy-commutative pasting diagram:

$$\begin{tikzcd} MS\mathrm{U} \arrow[r, "\sigma_{S\mathrm{U}}"] \arrow[d] & K\mathrm{O} \arrow[r, "\mathrm{ph}"] \arrow[d] & H^{\mathrm{ev}}\mathbb{Q} \arrow[d, equal] \\ M\mathrm{U} \arrow[r, "\sigma_{\mathrm{U}}"] \arrow[rr, bend right, "\mathrm{Td}"'] & K\mathrm{U} \arrow[r, "\mathrm{ch}"] & H^{\mathrm{ev}}\mathbb{Q} \end{tikzcd} \tag{187}$$

Here the left square is from Prop. 3.66, the bottom triangle is Prop. 3.74, while the right square is the defining relation (70) between the Chern- and Pontrjagin character. Thus the homotopy-commutativity of the total outer diagram implies the claim. □

Proposition 3.76 (Chern-, Pontrjagin- and Todd-character on Adams cofiber theories)**.** *All of (a) the Conner-Floyd K-orientations (Prop. 3.66), (b) the Chern- and Pontrjagin-character* (70) *and (c) the Todd character* (185) *descend to the Adams cofiber theories (Def. 3.48) where they form the following homotopy-commutative diagram:*

$$\begin{tikzcd} (MS\mathrm{U})/\mathbb{S} \arrow[r, "\sigma_{S\mathrm{U}}/\mathbb{S}"] \arrow[d] & (K\mathrm{O})/\mathbb{S} \arrow[r, "\mathrm{ph}/\mathbb{S}"] \arrow[d] & (H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S} \arrow[d, equal] \\ (M\mathrm{U})/\mathbb{S} \arrow[r, "\sigma_{\mathrm{U}}/\mathbb{S}"] \arrow[rr, bend right, "\mathrm{Td}/\mathbb{S}"'] & (K\mathrm{U})/\mathbb{S} \arrow[r, "\mathrm{ch}/\mathbb{S}"] & (H^{\mathrm{ev}}\mathbb{Q})/\mathbb{S} \end{tikzcd} \tag{188}$$

Proof. This follows with Def. 3.50 from (187), once we know that all maps involved are multiplicative: For the Chern- and Pontrjagin character this is Example 3.51, while for the K-orientations σ_{U} and $\sigma_{S\mathrm{U}}$ this is Prop. 3.66. □

The Conner-Floyd e-invariant.

Proposition 3.77 (The Conner-Floyd e-invariant [CF66, Thm. 16.2])**.** *The Adams* e_{Ad}*-invariant (Def. 3.58) on stable Cohomotopy elements* $[c] \in \mathbb{S}_\bullet$ *is expressed geometrically as the rational Todd class* (175)*, modulo integers, of any compact* U*-manifold* $M^{\bullet+1}_{\mathrm{U,Fr}}$ *whose framed boundary is the manifold corresponding to* $[c]$ *under the Pontrjagin-Thom isomorphism*

$$\underset{\substack{\text{Adams'}\\ \text{e-invariant}}}{\mathrm{e}_{\mathrm{Ad}}(c)} \;=\; \underset{\substack{\text{Conner-Floyd's}\\ \text{e-invariant}}}{\mathrm{e}_{\mathrm{CF}}\big(M^d_{\mathrm{Fr}}\big)} \;:=\; \mathrm{Td}\big[M^{d+1}_{\mathrm{U,Fr}}\big]\,, \qquad \text{for } \partial M^{d+1}_{\mathrm{U,Fr}} = M^d_{\mathrm{Fr}} \xleftrightarrow[\text{PT-iso}]{} [c]. \tag{189}$$

In more detail, consider the following diagram:

$$\begin{tikzcd}[column sep=large] 0 \arrow[r] & \overset{\substack{\text{cobordism classes of}\\ \text{compact U-manifolds}\\ \text{without boundary}}}{(M\mathrm{U})_{\bullet+1}} \arrow[r, hook] \arrow[d, "\mathrm{Td}"'] \arrow[dr, phantom, "\text{Todd class}"] & \overset{\substack{\text{cobordism classes of}\\ \text{compact U-manifolds}\\ \text{with framed boundary}}}{(M\mathrm{U}/\mathbb{S})_{\bullet+1}} \arrow[r, "\partial", "\text{boundary map}"'] \arrow[d, "\mathrm{Td}"'] & \overset{\substack{\text{cobordism classes of}\\ \text{framed manifolds}}}{(M\mathrm{Fr})_\bullet} \arrow[r, leftrightarrow, "\simeq", "\substack{\text{Pontrjagin-Thom}\\ \text{isomorphism}}"'] \arrow[d, dashed, "\mathrm{e}_{\mathrm{CF}}"'] & \overset{\substack{\text{stable Cohomotopy}\\ \text{ground ring}}}{\mathbb{S}_\bullet} \arrow[d, "e_{\mathrm{Ad}}"', "\substack{\text{Adams}\\ \text{e-invariant}}"] \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, hook] & \mathbb{Q} \arrow[r] & \mathbb{Q}/\mathbb{Z} \arrow[r, equal] & \mathbb{Q}/\mathbb{Z}\,, \end{tikzcd} \tag{190}$$

where the top row is exact, by Lemma 3.49, (so that, in particular, the boundary map is surjective and hence does admit the lifts assumed in (189)) while the two left vertical morphisms are from Remark 3.71, thus inducing the dashed morphism e_{CF}. The claim is that the square on the right commutes.

This is, at its heart, a consequence of the fact that the Todd character is the Chern character of any Thom class (Prop. 3.74), as made fully manifest by our diagrammatic construction (80).

Remark 3.78 (Conner-Floyd e-invariant on SU-manifolds [CF66, p. 104])**.** Since the Todd number on SU-manifolds of real dimension 4 mod 8 is divisible by 2 (Prop. 3.70), it follows that in this situation also the Conner-Floyd e-invariant (Def. 3.77) is divisible by 2.
(i) Hence if follows that (190) refines to the following diagram:

$$
\begin{tikzcd}
0 \arrow[r] & \underset{\substack{\text{cobordism classes of}\\ \text{compact } SU \text{ manifolds}\\ \text{without boundary}}}{(MSU)_{8\bullet+4}} \arrow[rr, hook] \arrow[d, "\frac{1}{2}\mathrm{Td}"'] \arrow[rr, phantom, shift right=8ex, "\text{half Todd class}"] && \underset{\substack{\text{cobordism classes of}\\ \text{compact } SU\text{-manifolds}\\ \text{with framed boundary}}}{(MU/\mathbb{S})_{8\bullet+4}} \arrow[r, two heads, "\partial", "\text{boundary map}"'] \arrow[d, "\frac{1}{2}\mathrm{Td}"'] & \underset{\substack{\text{cobordism classes of}\\ \text{framed manifolds}}}{(M\mathrm{Fr})_{8\bullet+3}} \arrow[r, leftrightarrow, "\simeq", "\substack{\text{Pontrjagin-Thom}\\ \text{isomorphism}}"'] \arrow[d, dashed, "\frac{1}{2}\mathrm{e}_{\mathrm{CF}}"'] & \underset{\substack{\text{stable Cohomotopy}\\ \text{ground ring}}}{\mathbb{S}_{8\bullet+3}} \arrow[d, "\frac{1}{2}e_{\mathrm{Ad}}"', "\substack{\text{Adams}\\ e_{\mathbb{R}}\text{-invariant}}"] \\
0 \arrow[r] & \mathbb{Z} \arrow[rr, hook] && \mathbb{Q} \arrow[r] & \mathbb{Q}/\mathbb{Z} \arrow[r, equal] & \mathbb{Q}/\mathbb{Z}\,.
\end{tikzcd}
\tag{191}
$$

(ii) The divided Adams e-invariant on the right coincides with the invariant called $\mathrm{e}_{\mathbb{R}}$ in [Ad66], which is defined in terms of Adams operation in KO by the same formula (161) that defines e_{Ad} by Adams operations on KU.
(iii) Beware that the alternative formula (163) for e_{Ad} in terms of the Chern character does *not* have a direct analogue that computes $e_{\mathbb{R}}$: Replacing the Chern character on KU with the Pontrjagin character on KO still computes $e_{\mathrm{Ad}} = 2e_{\mathbb{R}}$. Since it is the Chern character formula (163) for the Adams e-invariant that is captured (according to Theorem 3.62) by the Toda bracket observable $\widehat{e}_{KU}$ (Def. 3.56) the analogous Toda-bracket observable $\widehat{e}_{KO}$ built from the Pontrjagin character on KO still lifts e_{Ad} and not $\mathrm{e}_{\mathbb{R}}$.

3.7 Hopf invariant

We give an abstract discussion of the *Hopf invariant* in generalized cohomology, streamlined to make manifest its nature as a refined Toda-bracket (Def. 3.44), on the same footing as the refined Adames e-invariant §3.5 and the refined Conner-Floyd e-invariant §3.6, all unified by the notion of observables on trivializations ("H_{n-1}-fluxes") of the d-invariant in §2.4.

The cup-square cohomology operation and its trivialization.

Definition 3.79 (Cup square cohomology operation)**.** For E a multiplicative cohomology theory (Def. 3.35) and $n \in \mathbb{N}$, we say that the *cup-square operation* in E-cohomology of degree n is the unstable cohomology operation

$$\big[E^n \xrightarrow{(-)^2} E^{2n}\big] \;\in\; \widetilde{E}^{2n}\big(E^n\big)$$

which represents (via the Yoneda lemma) over each $X \in \mathrm{Ho}\big(\mathrm{Spaces}^{*/}_{\mathrm{Qu}}\big)$ the operation

$$\big[X,(-)^2\big] \;:\; \big[X,E^n\big] = \widetilde{E}^n(X) \xrightarrow{\;\Delta_{\widetilde{E}^n(X)}\;} \widetilde{E}^n(X)\times\widetilde{E}^n(X) \xrightarrow{\;\cup^E_X\;} \widetilde{E}^{2n}(X) \;=\; \big[X,E^{2n}\big]\;.$$

Of course, this exists more generally for kth cup powers for any $k \in \mathbb{N}$. Review for the case of $E = K\mathrm{U}$ is in [Wir12, p. 44].

Lemma 3.80 (Connective covers of ring spectra)**.** *For E a ring spectrum with E_∞-structure, its connective cover $E\langle 0\rangle$ inherits the structure of a ring spectrum such that the coreflection $E\langle 0\rangle \xrightarrow{\varepsilon_E} E$ is a homomorphism of ring spectra.*

This is due to [Ma77, Prop. VII 4.3][Lu17, Prop. 7.1.3.13].

Proposition 3.81 (Canonical trivialization of cup square over n-sphere)**.** *For E a multiplicative cohomology theory (Def. 3.35) and $n \in \mathbb{N}$ with $n \geq 1$, the cup square of every element $[c] \in \widetilde{E}^n\big(S^n\big)$ is trivial; and if $n \geq 2$ then it is canonically trivialized.*

Proof. By the suspension isomorphism, c is the n-fold suspension of an element in E_0, and hence is represented by a cocycle that factors through the connective cover cohomology theory $E\langle 0\rangle$. But since connective covers of ring spectra are equipped with compatible ring spectrum structure (by Lemma 3.80) so does its cup square, in that we have a solid diagram as follows, with the vertical morphisms on the right representing the cup square cohomology operation via Def. 3.79

$$
\begin{tikzcd}
S^n \arrow[r, "c"] \arrow[d] & E^n \arrow[d, "(-)^2"] \arrow[dl, Rightarrow] \\
* \arrow[r] & E^{2n}
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
S^n \arrow[r] \arrow[d] & (E\langle 0\rangle)^n \arrow[r, "(\varepsilon_E)^n"] \arrow[d, "(-)^2"'] \arrow[dl, Rightarrow, dashed, "\exists !"] & E^n \arrow[d, "(-)^2"] \\
* \arrow[r] & E^{2n} \arrow[r, "(\varepsilon_E)^{2n}"] & (E)^{2n}.
\end{tikzcd}
\tag{192}
$$

But, since $(E\langle 0\rangle)^{2n}$ is $2n$-connective (because the stable homotopy groups $\pi_k\big(E\langle 0\rangle\big) \,=\, \pi_{k+n}\big((E\langle 0\rangle)^n\big)$ of a connective spectrum vanish for $k < 0$, by definition), we have (using with $n \geq 1$ that $n < 2n$) $\pi_n\big((E\langle 0\rangle)^{2n}\big) \;=\; 0$, meaning that there exists a dashed homotopy, as shown. If $n \geq 2$ then also the next higher homotopy group is trivial and the dashed homotopy exists uniquely, thus inducing a canonical homotopy filling the full diagram. $\square$

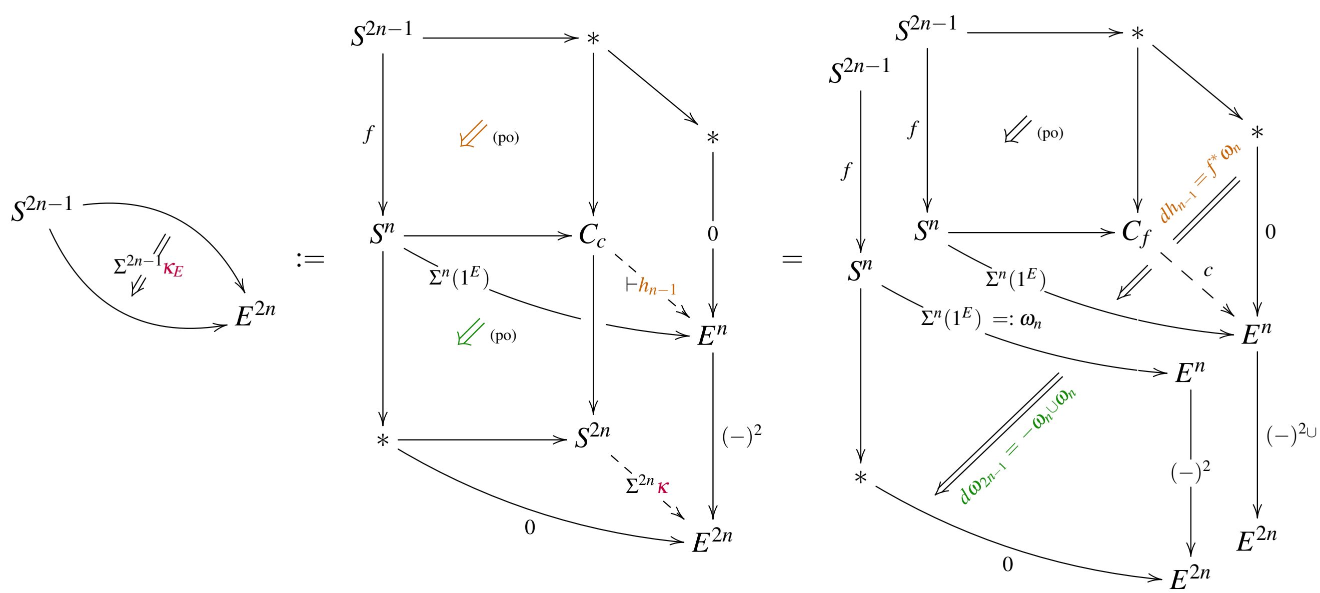
The E-Hopf invariant.

Definition 3.82 (E-Hopf invariant)**.** For E a multiplicative cohomology theory (Def. 3.35) and $n \in \mathbb{N}$ with $n \geq 1$ we say that the *refined E-Hopf invariant* is the function $H^E_{n-1}\mathrm{Fluxes}(S^{2n-1}) \xrightarrow{\kappa_E} E_0$, which sends any H^E_{n-1}-flux (Def. 3.40, (64))

$$\begin{tikzcd} S^{2n-1} \arrow[r] \arrow[d, "f"'] & \ast \arrow[d] \arrow[dl, Rightarrow, "h_{n-1}"] \\ S^n \arrow[r, "\Sigma^n(1^E)"'] & E^n \end{tikzcd}$$

to the class in $\pi_1 \mathrm{Maps}^{*/}\big(S^{2n-1}, E^{2n}\big) \simeq \pi_0 \mathrm{Maps}^{*/}\big(S^{2n}, E^{2n}\big) \simeq E_0$ of its pasting composite with the canonical trivialization of the cup square from Prop. 3.81:

$$\underset{E\text{-Hopf invariant}}{\kappa_E} \quad := \quad \underset{\substack{\text{homotopy Whitehead integral /} \\ \text{functional cup product in } E}}{\int_{S^{2n-1}} \Big(h_{n-1} \cup f^* \omega_n \; + \; f^* \omega_{2n-1} \Big)} \qquad \text{for: } \begin{cases} d\, h_{n-1} = f^* \omega_n \\ d\, \omega_{2n-1} = -\omega_n \cup \omega_n \end{cases} \tag{193}$$



The two ways to read this diagram yield *two component definitions of the E-Hopf invariant*:

(1) Hopf-Adams/Atiyah-style definition. On the left in (193), the choice of classifying map $\vdash h_{n-1}$ is equivalently a choice of extension of the pullback of the canonical generator from S^n; and the homotopy-commuting square repeated on the right says that the cup square of this lift is an E_0-multiple κ of the canonical generator on S^{2n}.

$$\begin{tikzcd} C_c \arrow[r, dashed, "\vdash h_{n-1}"] \arrow[d, "p_c"'] & E^n \arrow[d, "(-)^2"] \\ S^{2n} \arrow[r, "\Sigma^{2n}\kappa"'] & E^{2n} \end{tikzcd} \qquad \Leftrightarrow \qquad \big[\vdash h_{n-1}\big]^2 = \kappa \cdot p_c^* \big[\Sigma^{2n}(1^E)\big].$$

For $E = H\mathbb{F}_2$ this gives Hopf's original definition ([MT86, p. 33]), while for $E = K\mathrm{U}$ this gives Adams-Atiyah's K-theoretic definition of the Hopf invariant [AA66] (review in [Wir12, p. 50]).

(2) Whitehead-Steenrod-Haefliger-type definition. On the right in (193) we see, in the case that $E = H\mathbb{R}$ is ordinary real cohomology, and translating to differential forms via the fundamental theorem of rational homotopy theory, we get the classical homotopy Whitehead-Steenrod integral formula [Wh47] (review in [BT82, Prop. 7.22]) [St49][Ha78, p. 17] (review in [SW08, Ex. 1.9]) as discussed in our context in [FSS19c, Prop. 4.6][FSS19b, Prop. 3.20]. (The symbols in (193) are chosen such as to match the notation used in [FSS19b][FSS19c] in this rational case.)

Hence the *proof* that these two formulations of the classical Hopf invariant are equivalent is trivialized by the diagrammatic Definition 3.82, as is its generalization to coefficients in any multiplicative cohomology theory E.

3.8 Ravenel orientations

Finite-dimensional E-orientations. The traditional term *complex-oriented E-cohomology* (e.g. [Ho99]), reviewed in a moment (the formal definition is Def. 3.91 below) is short for *universal orientation of fibers of complex vector bundles in the Whitehead-generalized cohomology theory E*; analogously there is real- and quaternionic oriented cohomology (Def. 3.92 below). Here "universal" means that a fiber-wise orientation is chosen for *all* vector bundles over *all* (paracompact) topological spaces at once, and compatibly so under pullback – which means to choose an orientation on the universal vector bundles over the corresponding infinite-dimensional (as cell complexes) classifying spaces:

$$
\begin{tikzcd}
\text{\textit{any} G-vector bundle} & \mathcal{V}_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathcal{V}_{BG} \arrow[d] & \text{universal G-vector bundle} \\
\text{\textit{any} topological space} & X \arrow[r, "\vdash \mathcal{V}_X", "\text{classifying map}"'] & BG & \text{classifying space for G-vector bundles over any space}
\end{tikzcd}
\qquad
E\text{-orientation of } \mathcal{V}_{BG} \;\Rightarrow\; \underset{\mathcal{V}_X}{\forall}\; E\text{-orientation of } \mathcal{V}_X .
$$

However, in applications, such as to physics, there is often an upper bound both on the dimension of relevant base spaces (spacetimes) as well as on the rank of the vector bundle (field content), so that the standard notion of oriented E-cohomology involves an infinite tower of redundant data. Therefore we turn attention to *finite-dimensional E-orientations*, namely to E-orientations universally chosen (only) on vector bundles of rank $\leq r+1$ over $\leq d$-dimensional base spaces and those pulled back from these, for some fixed upper bound on $d+2r$ (for complex vector bundles) or on $d+4r$ (for quaternionic vector bundles). For complex vector bundles this finite-dimensional notion of E-orientations appears briefly in [Ho84, §1.2][Ra86, §. 6.5], but does not seem to have found attention in bundle theory (beyond formal investigation into the associated Thom spectra, now known as *Ravenel's spectra* [Ra84, §3][Ra86, §6.5][DHJ88, Thm. 3][Bea19]).

In particular, our discussion in §2.8 shows (with Lemma 3.90 below) that *Hypothesis H* leads to the appearance of E-orientations on 10-dimensional quaternionic line bundles, in this sense, namely of quaternionic line bundles whose classifying map factors through the 11-skeleton of the full classifying space:

$$
\begin{tikzcd}
 & & \text{classifying space for $\mathbb{H}$-line bundles over 10-dimensional spaces} \arrow[r] & \mathbb{H}P^2 \arrow[d, hook, "\text{11-skeleton}"] \\
\underset{\text{spacetime}}{\mathbb{R}^{0,1} \times X^{10}} \arrow[r, two heads] & \underset{\text{10-dim space}}{X^{10}} \arrow[urr, dashed] \arrow[rr, "\vdash \mathcal{V}_{X^{10}}", "\text{classifying map}"'] & & \mathbb{H}P^\infty = B\mathrm{Sp}(1) .
\end{tikzcd}
$$

While quaternionic E-orientations have not found much attention yet by themselves, we know that they subsume the widely-studied complex E-orientations, in that every complex E-orientation induces a quaternionic E-orientation (see Prop. 3.98 below). Here we generalize this relation to finite-dimensional orientations and observe (Theorem 3.99 below) that $(4n+2)$-dimensional complex orientations induce $4n+2$-dimensional quaternionic orientations (which is not as immediate as it may seem from the coincidence of the two dimensions; see Remark 3.100).

The traditional terminology of "oriented cohomology theory" makes use of a number of tacit identifications, beginning with the standard definition of "complex E-orientation" (see Def. 3.91 below) being all phrased in terms of a first E-Chern class with a superficially ad-hoc condition on it, and no mentioning of any notion of orientation. Therefore, we first give a brief exposition of the relevant concept formation (the oriented reader may want to skip ahead):

Orientations of Euclidean space. An ordinary *orientation* of a differentiable manifold Y is traditionally defined to be the choice of a volume form (a nowhere vanishing top degree differential form) up to pointwise positive rescaling. Assuming that Y is closed, hence compact, we may equivalently remember just the class of this form in de Rham cohomology, up to positive rescaling:

$$
\mathrm{vol}_Y \;\in\; \underbrace{\Omega^{\dim(Y)}_{\mathrm{dR}}(Y)}_{\text{differential forms}} \xrightarrow[\;[-]\;]{\text{pass to total volume}} \underbrace{H^{\dim(Y)}_{\mathrm{dR}}(Y)}_{\text{de Rham cohomology}} \,.
$$

If Y were already oriented otherwise, we could ask that $[\mathrm{vol}_Y]$ be rescaled to a *unit* $\pm 1 \in \mathbb{Z} \subset \mathbb{R}$, *with respect to* the reference orientation. That number $\pm 1 \in \mathbb{Z}$ would be the choice of *relative orientation*. But since we want to define that reference orientation in the first place, the normalization demand on it is, conversely, that it be **(a)** integral and **(b)** minimally so, such that any other integral class is a unique integral multiple. This means to demand that
(a) the orienting volume class lifts to integral cohomology:

$$
\underbrace{[\mathrm{vol}_Y]}_{\substack{\text{integral}\\ \text{volume class}}} \;\in\; \underbrace{H^{\dim(Y)}\big(Y;\mathbb{Z}\big)}_{\text{integral cohomology}} \xrightarrow{\substack{\text{extension}\\ \text{of scalars}}} \underbrace{H^{\dim(Y)}\big(Y;\mathbb{R}\big)}_{\text{real cohomology}} \xrightarrow[\;\simeq\;]{\substack{\text{de Rham}\\ \text{theorem}}} \underbrace{H^{\dim(Y)}_{\mathrm{dR}}\big(Y\big)}_{\text{de Rham cohomology}}
$$

(b) every other integral cohomology n-class of Y is obtained by external cup product of $[\mathrm{vol}_Y]$ with an integral class on the point:

$$
\underbrace{H^{\dim(Y)}\big(Y;\,\mathbb{Z}\big)}_{\text{top integral cohomology}} \;\simeq\; \underbrace{H^0\big(*;\,\mathbb{Z}\big)}_{\substack{\text{generated}\\ \text{under cupping}}}\underbrace{\big\langle[\mathrm{vol}_Y]\big\rangle}_{\substack{\text{from normalized}\\ \text{volume class}}} \;\in\; H^0\big(*;\,\mathbb{Z}\big)\,\mathrm{Modules}. \tag{194}
$$

If Y is not compact, we apply this logic to those volume forms that *vanish at infinity* on Y – their classes are naturally identified with classes as above, defined on the one-point compactification space $X := Y_{\mathrm{cpt}}$.

In the special case that $Y = V^n$ is a Euclidean space of finite dimension n, its one-point compactification $(V^n)_{\mathrm{cpt}} = S^n$ is the sphere of that dimension. Since the *reduced* integral cohomology of the n-sphere is all concentrated in degree n, and since the integral cohomology of the point is concentrated in degree 0, the above condition (194) on an orienting volume class on V^n may equivalently be stated in terms of the full graded cohomology ring as

$$
\underbrace{\widetilde{H}^\bullet\big((V^n)^{\mathrm{cpt}};\mathbb{Z}\big)}_{\text{reduced cohomology}} \;\simeq\; \underbrace{H^\bullet(*;\mathbb{Z})}_{\substack{\text{generated over}\\ \text{ground ring}}}\underbrace{\big\langle[\mathrm{vol}_{S^n}]\big\rangle}_{\substack{\text{from normalized}\\ \text{volume class}}} \;\in\; H^\bullet(*;\mathbb{Z})\,\mathrm{Modules}\,. \tag{195}
$$

Any two choices of $[\mathrm{vol}_Y]$ that satisfy this condition (195) differ by multiplication by a *unit in the graded ground ring*, which here is either of $\pm 1 \in \mathbb{Z} \simeq H^0(*;\mathbb{Z}) \simeq H^\bullet(*;\mathbb{Z})$. While (195) is a heavy way of speaking about the 2-element set of choices of $[\mathrm{vol}_{S^n}]$, it has the advantage that it readily generalizes to any Whitehead-generalized multiplicative cohomology theory E:

In direct generalization of (195), one says that an orientation of the Euclidean space V^n in E-cohomology is the choice of a class $[\mathrm{vol}^E_{S^n}] \in \widetilde{E}^\bullet(S^n)$ which is a *generator* of the reduced cohomology as a module over the graded ground ring:

$$
\underbrace{\widetilde{E}^\bullet(S^n)}_{\substack{\text{reduced}\\ E\text{-cohomology}}} \;\simeq\; \underbrace{E^\bullet(*)}_{\substack{\text{generated over}\\ E\text{-ground ring}}}\underbrace{\big\langle\,[\mathrm{vol}^E_{S^n}]\,\big\rangle}_{\substack{\text{from normalized}\\ E\text{-volume class}}} \;\in\; E^\bullet(*)\,\mathrm{Modules}\,. \tag{196}
$$

Now any two choices of E-orientations, hence of classes $[\mathrm{vol}^E_{S^n}]$ that solve (196), differ by multiplication with a unit (a multiplicatively invertible element) in the ground ring $E_{-\bullet} := E^\bullet(*)$. This may be a set of choices much larger than the two orientations possible in ordinary cohomology. In particular, the characteristic property of Whitehead-generalized cohomology, namely that it allows the violation of the *dimension axiom* satisfied by ordinary cohomology, means that relative orientations may have non-vanishing degree.

On the other hand, there is one bit of extra information already provided with a Whitehead-generalized cohomology theory E, namely a choice of natural *suspension isomorphisms* $\widetilde{E}^\bullet(X) \xrightarrow[\simeq]{\sigma^E} \widetilde{E}^{\bullet+n}(\Sigma^n X)$. For multiplicative theories this choice implies a canonical choice of E-orientation of V^n, namely that given by the image under n-fold suspension of the *canonical unit element* 1^E in the ground ring:

$$
\begin{array}{ccc}
E^0(*) \xrightarrow[\simeq]{} \widetilde{E}^0(S^0) \xrightarrow[\simeq]{\sigma^E} & \widetilde{E}^n(S^n) \\
\underbrace{1^E \longmapsto}_{\substack{\text{suspension isomorphism in}\\ \text{multiplicative } E\text{-cohomology induces...}}} & \underbrace{[\mathrm{vol}^E_{S^n}] := \Sigma^n(1^E)}_{\substack{\text{...canonical } E\text{-orientation class}\\ \text{of Euclidean } n\text{-space/the } n\text{-sphere}}}
\end{array} \tag{197}
$$

Orientation of parametrized Euclidean spaces. Given not just one, but a *bundle* of Euclidean spaces over a parameter space X, hence a V^n-fiber bundle $\mathscr{V}_X \to X$, then a choice of E-orientations of each fiber $\mathscr{V}_{\{x\}} \simeq V^n$, continuously varying in the parameter $x \in X$, is naturally defined to be a single class in the one-point compactification of the bundle – its *Thom space* $\mathrm{Th}(\mathscr{V}_X) = (\mathscr{V}_X)_{\mathrm{cpt}}$ if X is compact, which we shall assume – whose restriction to any fiber $\mathscr{V}_{\{x\}}$ is an E-orientation (196) of that Euclidean space:

$$\begin{array}{ccc} \mathscr{V}_X \xleftarrow{\;\; i_x \;\;} \mathscr{V}_{\{x\}} \xleftarrow{\;\;\simeq\;\;} V^n \\ \widetilde{E}^\bullet\big((\mathscr{V}_X)_{\mathrm{cpt}}\big) \xrightarrow{\qquad (i_x)^* \qquad} \widetilde{E}^\bullet\big((V^n)_{\mathrm{cpt}}\big) \\ \underset{= \text{"Thom class" in } E\text{-cohomology}}{\text{X-parametrized E-orientation class}} \quad [\mathrm{vol}^E_{\mathscr{V}_X}] \xmapsto{\text{restriction to fiber}} [\mathrm{vol}^E_{S^n}] \quad \underset{\text{of Euclidean n-space/the n-sphere}}{\text{E-orientation class}} \end{array} \tag{198}$$

Notice here, by the homotopy invariance of Whitehead-generalized cohomology theory, that the fiber restrictions to any pair of points $x, y \in X$ in the same connected component are isomorphic

$$[x] = [y] \in \pi_0(X) \qquad \Rightarrow \qquad \widetilde{E}^\bullet(\mathscr{V}_{\{y\}}) \xrightarrow{\simeq} \widetilde{E}^\bullet(\mathscr{V}_{\{x\}}),$$

and *uniquely* isomorphic if X is simply connected, $\pi_1(X) = 1$.

Therefore, over connected and simply-connected base spaces X, it is sufficient to ask that $[\mathrm{vol}^E_{\mathscr{V}_X}]$ restricts to an E-orientation class on any one fiber; and without restriction of generality, this may be taken to be the canonical E-orientation (197) induced by the suspension isomorphism provided with E.

Orientation of parametrized vector spaces. If we equip Euclidean space V^n with the structure of a $\mathbb{K}$-vector space, and consider parametrizations that respect this linear structure, then we are dealing, of course, with fiberwise E-orientation of $\mathbb{K}$-*vector bundles* $\mathscr{V}_X$. A key point of the theory of E-orientations is that, in this case, the orientation classes/Thom classes (198) on $\mathscr{V}_X$ pull back, along the zero-section, to *characteristic classes* on the base, and in fact bijectively so, if the resulting system of characteristic classes is characterized appropriately:

In the case where $E = H\mathbb{Z}$ is ordinary integral cohomology and $\mathbb{K} = \mathbb{C}$ is the complex numbers, this is just the classical fact that top-degree *Chern classes* are equivalently the pullback of any orientation class/Thom class along the zero-section. The generalization of this statement to Whitehead-generalized cohomology theories E is the theory of *Conner-Floyd E-chern classes* (see Prop. 3.97 below):

$$\begin{tikzcd} \underset{\text{vector bundle}}{\text{complex}}\ \mathscr{V}_X \arrow[d] & \widetilde{E}^\bullet(\mathscr{V}_X) \arrow[d, "\substack{\text{pullback}\\\text{along 0-section}}"'] & \ni & {[\mathrm{vol}^E_{\mathscr{V}_X}]} \arrow[d, leftrightarrow] & \underset{= \text{"Thom class" in } E\text{-cohomology}}{\text{X-parametrized E-orientation class}} \\ X & E^\bullet(X) & \ni & c^E_{\mathrm{rnk}(\mathscr{V}_X)} & \underset{\text{Conner-Floyd E-Chern class}}{\text{top-degree}} \end{tikzcd}$$

It is this equivalence between **(a)** universal fiberwise E-orientations of complex vector bundles and **(b)** universal E-Chern classes that makes "complex oriented E-cohomology theory" often look more like "Conner-Floyd E-Chern class theory". It is from the latter perspective that the notion naturally emerges from *Hypothesis H* in §2.8.

We now turn to formal discussion of these matters.

Projective spaces. To set up notation and to be fully explicit about some fine-print needed later on, we briefly recall projective spaces and their tautological line bundles over topological skew-fields, i.e. including the non-commutative quaternionic case, which requires some care and leads to the all-important effect of Prop. 3.89 below.

Notation 3.83 (Ground fields)**.** We write $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ for the topological (skew-)fields of real numbers, complex numbers or quaternions, respectively.
(i) We write

$$\mathbb{K}^\times := (\mathbb{K}, \cdot) \setminus \{0\} \quad \in \text{Groups}$$

for their multiplicative groups (their *groups of units*).

(ii) As topological groups these are homotopy equivalent to the groups of unit-norm elements

$$S(\mathbb{H}) \;:=\; \big\{ v \in \mathbb{H} \,\big|\, v \cdot v^* = 1 \big\} \quad \in \text{Groups}\,,$$

which, specifically, are these compact Lie groups:

$$S(\mathbb{R}) \;=\; \mathbb{Z}/2\,, \qquad S(\mathbb{C}) \;=\; \mathrm{U}(1)\,, \qquad S(\mathbb{H}) \;=\; \mathrm{Sp}(1) \simeq \mathrm{SU}(2)\,.$$

(iii) In particular, their classifying spaces are weakly homotopy equivalent:

$$B\mathbb{K}^\times \;\simeq\; B\big(S(\mathbb{K})\big)\,. \tag{199}$$

Notation 3.84 (Projective spaces and their tautological line bundles)**.** For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and $n \in \mathbb{N}$ we have:

(i) the $\mathbb{K}$-*projective space*:

$$\mathbb{K}P^n := P\big(\mathbb{K}^{n+1}\big) := \big(\mathbb{K}^{n+1} \setminus \{0\}\big)/\mathbb{K}^\times \tag{200}$$

(ii) its *tautological* $\mathbb{K}$*-line bundle*:

$$\begin{tikzcd} \mathscr{L}_{\mathbb{K}P^n} := \dfrac{\big(\mathbb{K}P^{n+1}\setminus\{0\}\big)\times\mathbb{K}^*}{\mathbb{K}^\times} \arrow[r, "{[v,t]\,\mapsto\, v\cdot t}"] \arrow[d, "{[v,t]\mapsto[v]}"] & \mathbb{K}^{n+1} \\ \mathbb{K}P^n = \dfrac{\big(\mathbb{K}P^{n+1}\setminus\{0\}\big)\times *}{\mathbb{K}^\times} \end{tikzcd} \tag{201}$$

(iii) its *dual tautological* $\mathbb{K}$*-line bundle*:

$$\begin{tikzcd} \mathscr{L}^*_{\mathbb{K}P^n} := \dfrac{\big(\mathbb{K}P^{n+1}\setminus\{0\}\big)\times\mathbb{K}}{\mathbb{K}^\times} \arrow[r, "{[v,t]\,\mapsto\,[(v,t)]}"] \arrow[d, "{[v,t]\mapsto[v]}"] & \mathbb{K}P^{n+1} \\ \mathbb{K}P^n = \dfrac{\big(\mathbb{K}P^{n+1}\setminus\{0\}\big)\times *}{\mathbb{K}^\times} \end{tikzcd} \tag{202}$$

Here:

(i) we write elements of $\mathbb{K}^{r+1}$ as lists $w = (v, v_{r+1}) = (v_1, \cdots, v_r, v_{r+1})$;

(ii) we regard $\mathbb{K}^r$ as equipped with the right $\mathbb{K}^\times$-action by multiplication from the *right*

$$\begin{array}{ccc} \mathbb{K}^r \times \mathbb{K}^\times & \longrightarrow & \mathbb{K}^r \\ \big((v_1, \cdots, v_r), q\big) & \longmapsto & (v_1 \cdot q, \cdots, v_r \cdot q) \end{array} \tag{203}$$

(iii) in contrast, $\mathbb{K}^*$ denotes $\mathbb{K}$ equipped with the right $\mathbb{K}^\times$-action by *inverse multiplication* from the *left*;

$$\begin{array}{ccc} \mathbb{K}^* \times \mathbb{K}^\times & \longrightarrow & \mathbb{K}^* \\ (t, q) & \longmapsto & q^{-1} \cdot t \end{array} \tag{204}$$

(iv) $\frac{(-)\times(-)}{\mathbb{K}^\times}$ denotes the quotient space of a product of right $k^\times$-spaces by their diagonal action, and $[-]$ denotes its elements as equivalence classes of those of the original space (so $[(v_1, \cdots, v_n)] = [v_1 : \cdots : v_n]$).

These quotient spaces become $\mathbb{K}$-vector bundles (line bundles) with respect to the remaining left actions: For $\mathscr{L}^*_{\mathbb{K}P^n}$ (202) this is by left multiplication on $\mathbb{K}$, but for $\mathscr{L}_{\mathbb{K}P^n}$ (201) this is by *conjugate* multiplication from the *right*:

$$\begin{array}{ccc} \mathbb{K}^\times \times \mathscr{L}_{\mathbb{K}P^n} & \longrightarrow & \mathscr{L}_{\mathbb{K}P^n} \\ \big(q, [v,t]\big) & \longrightarrow & [v, t\cdot q^*] \end{array} \qquad \begin{array}{ccc} \mathbb{K}^\times \times \mathscr{L}^*_{\mathbb{K}P^n} & \longrightarrow & \mathscr{L}^*_{\mathbb{K}P^n} \\ \big(q, [v,t]\big) & \longrightarrow & [v, q\cdot t]\,. \end{array} \tag{205}$$

While the horizontal map in (201) exhibits the tautological line bundle as the "blow-up" of the origin in $\mathbb{K}^{n+1}$, the horizontal map in (202) embeds the dual tautological line bundle into $\mathbb{K}P^{n+1}$, as the complement of the single remaining point $[(v,\infty)] := [(0,1)]$. Hence the embedding (202) extends over this point to become a homeomorphism between the Thom space (Example 3.7) of the dual tautological line bundle and the next projective space

$$\begin{tikzcd} \underset{\text{projective } n\text{-space}}{\mathbb{K}P^n} \arrow[r, hook, "{[v]\mapsto[(v,0)]}", "\text{zero-section}"'] & \underset{\substack{\text{Thom space of}\\ \text{tautological line bundle}\\ \text{over projective } n\text{-space}}}{\mathrm{Th}\big(\mathscr{L}^*_{\mathbb{K}P^n}\big)} \arrow[r, "{[v,t]\mapsto \left\{ \begin{array}{lcl} [(0,1)] & | & t=\infty \\ {[(v,t)]} & | & \text{else} \end{array}\right.}", "\underset{\text{homeomorphism}}{\simeq}"'] & \underset{\text{projective } (n+1)\text{-space}}{\mathbb{K}P^{n+1}} \end{tikzcd} \tag{206}$$

(compare to [TK06, §III, Lemma 3.8], where the dualization $\mathscr{L}^*$ is actually implicit in the use of an inner product). The colimit (in topological spaces) over the sequence of inclusions of projective spaces (the coordinate ordering convention here is crucial for (209) below):

$$
\underset{\longrightarrow}{\lim}
\left(
\begin{tikzcd}
 & {[v,t]} \arrow[r, mapsto] & {[(0,v),t]} & \\
\cdots \arrow[r, hook] & \mathscr{L}^*_{\mathbb{K}P^n} \arrow[r, hook] \arrow[d] & \mathscr{L}^*_{\mathbb{K}P^n} \arrow[r, hook] \arrow[d] & \cdots \\
\cdots \arrow[r, hook] & \mathbb{K}P^n \arrow[r, hook] & \mathbb{K}P^{n+1} \arrow[r, hook] & \cdots \\
 & {[v]} \arrow[r, mapsto] & {[(0,v)]} &
\end{tikzcd}
\right)
\;\simeq\;
\begin{tikzcd}
\mathscr{L}^*_{\mathbb{K}P^\infty} \arrow[r, equal] \arrow[d] & \overset{\text{universal } \mathbb{K}\text{-line bundle}}{E\mathbb{K}^\times \underset{\mathbb{K}^\times}{\times} \mathbb{K}} \arrow[d] \\
\mathbb{K}P^\infty \arrow[r, equal] & B\mathbb{K}^\times
\end{tikzcd}
\tag{207}
$$

yields the *universal* $\mathbb{K}$*-line bundle* over the infinite projective spaces, homotopy equivalent to the classifying spaces (199):

$$
\mathbb{K}P^\infty \simeq B\big(S(\mathbb{K})\big), \; i.e.: \qquad \mathbb{R}P^\infty \simeq B\mathbb{Z}_2\,, \qquad \mathbb{C}P^\infty \simeq B\mathrm{U}(1)\,, \qquad \mathbb{H}P^\infty \simeq B\mathrm{SU}(2)\,. \tag{208}
$$

In summary, there is the following commuting diagram of sequences of projective spaces, their dual tautological line bundles and their Thom spaces, whose colimit is the *universal* $\mathbb{K}$*-line bundle* over the classifying space:

$$
\begin{tikzcd}[column sep=small]
S^{\dim_{\mathbb{R}}(\mathbb{K})} = \mathbb{K}P^1 \arrow[r, hook] & \mathbb{K}P^2 \arrow[r, hook] \arrow[d, equal] & \cdots \arrow[r, hook] & \mathbb{K}P^{n+1} \arrow[r, hook] \arrow[d, equal] & \cdots \arrow[r] & \mathbb{K}P^\infty \arrow[d, equal] & \\
 & \mathrm{Th}\big(\mathscr{L}^*_{\mathbb{K}P^1}\big) \arrow[r, hook] & \cdots \arrow[r, hook] & \mathrm{Th}\big(\mathscr{L}^*_{\mathbb{K}P^n}\big) \arrow[r, hook] & \cdots \arrow[r] & \mathrm{Th}\big(\mathscr{L}^*_{\mathbb{K}P^\infty}\big) \arrow[r, equal] & \mathrm{Th}\Big(E\mathbb{K}^\times \underset{\mathbb{K}^\times}{\times} \mathbb{K}\Big) \\
 & S^{\dim_{\mathbb{R}}(\mathbb{K})} = \mathbb{K}P^1 \arrow[r, hook] \arrow[u, "0_{\mathrm{Th}(\mathscr{L}^*_{\mathbb{K}P^1})}"] & \cdots \arrow[r, hook] & \mathbb{K}P^n \arrow[rr, hook, "\cdots"] \arrow[u, "0_{\mathrm{Th}(\mathscr{L}^*_{\mathbb{K}P^n})}"] & & \mathbb{K}P^\infty \arrow[r, equal] \arrow[u, "0_{\mathrm{Th}(\mathscr{L}^*_{\mathbb{K}P^\infty})}"] & B\mathbb{K}^\times \arrow[u, "\simeq"]
\end{tikzcd}
\tag{209}
$$

Notice that, in the above conventions, the horizontal inclusions (207) are by adjoining a zero-coordinate to the left of the list, while the vertical inclusions, being the zero-sections under the identification (206), are by adjoining a zero-coordinate to the right of the list, so that the squares in (209) indeed commute.

Lemma 3.85 (Cell structure of projective spaces)**.** *For $n \in \mathbb{N}$, the projective spaces (Notation 3.84) are equivalently quotient spaces of k-spheres*

$$
\mathbb{R}P^n \simeq S^n/\mathbb{Z}_2\,, \qquad \mathbb{C}P^n \simeq S^{2n+1}/\mathrm{U}(1)\,, \qquad \mathbb{H}P^n \simeq S^{4n+3}/\mathrm{Sp}(1)\,, \tag{210}
$$

and are related by homotopy pushouts of the following form (cell attachments):

$$
\begin{tikzcd}
S^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle (\mathrm{po})"] & \mathbb{R}P^n \arrow[d] \\
\ast \arrow[r] & \mathbb{R}P^{n+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
S^{2n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle (\mathrm{po})"] & \mathbb{C}P^n \arrow[d] \\
\ast \arrow[r] & \mathbb{C}P^{n+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
S^{4n+3} \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle (\mathrm{po})"] & \mathbb{H}P^n \arrow[d] \\
\ast \arrow[r] & \mathbb{H}P^{n+1}\,,
\end{tikzcd}
\tag{211}
$$

where the top morphisms are the quotient projections from (210)*.*

As indicated on the far right of (209), we have the following basic fact, of crucial importance in orientation-theory (e.g. [Ad74, §I, Ex. 2.1][Koc96, Lem. 2.6.5]):

Proposition 3.86 (Zero-section into Thom space of universal $\mathbb{K}$-line bundle is weak equivalence)**.** *The 0-section into the Thom space of the universal* $\mathbb{K}$*-line bundle, on the far right of* (209)*, is a weak homotopy equivalence:*

$$
\underset{\text{classifying space}}{B\mathbb{K}^\times} \xrightarrow[\text{zero-section}]{\simeq} \underset{\substack{\text{Thom space of} \\ \text{universal } \mathbb{K}\text{-line bundle}}}{\mathrm{Th}\big(\mathscr{L}^*_{\mathbb{K}P^\infty}\big)}
$$

Proof. Observing that, in the present case, the sphere bundle is contractible:

$$
S\big(\mathscr{L}^*_{\mathbb{K}P^\infty}\big) \simeq S\big(E(S(\mathbb{K})) \underset{S(\mathbb{K})}{\times} \mathbb{K}\big) \simeq E(S(\mathbb{K})) \underset{S(\mathbb{K})}{\times} S(\mathbb{K}) \simeq E(S(\mathbb{K})) \simeq \ast\,,
$$

and that the zero-section of the unit disk bundle is, manifestly and generally, a homotopy equivalence $X \xrightarrow{\simeq} D(\mathscr{V}_X)$, we have the solid part of the following commuting diagram:

$$
\begin{tikzcd}
\ast \arrow[d, "\simeq"'] \arrow[r, hook] & D\big(\mathscr{L}^\ast_{\mathbb{K}P^\infty}\big) \arrow[d, equal] \arrow[r] & D\big(\mathscr{L}^\ast_{\mathbb{K}P^\infty}\big) \arrow[d, dashed] & B\mathbb{K}^\times \arrow[l, "0"'] \arrow[dl] \\
S\big(\mathscr{L}^\ast_{\mathbb{K}P^\infty}\big) \arrow[r, hook, "\in \mathrm{Cofibrations}"'] & D\big(\mathscr{L}^\ast_{\mathbb{K}P^\infty}\big) \arrow[r, "\mathrm{cofiber}", "\Rightarrow\ \mathrm{homotopy\ cofiber}"'] & \mathrm{Th}\big(\mathscr{L}^\ast_{\mathbb{K}P^\infty}\big)
\end{tikzcd}
$$

Now observing that the inclusion of the sphere bundle into the disk bundle is a relative cell complex inclusion, (using that our projective spaces are CW-complexes, by Lemma 3.85), so that its cofiber is in fact a model for its homotopy cofiber, the claim follows by the respect of homotopy cofibers for weak equivalences (see. e.g., [FSS20c, Ex. A.24]). □

In parametrized generalization of Notation 3.84 we have:

Notation 3.87 (Projective bundles)**.** For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and $\mathscr{V}_X \in \mathbb{K}\mathrm{VectorBundles}_{/X}$, we have the *projective bundle*

$$
P\big(\mathscr{V}_X\big) \;:=\; \Big(\mathscr{L}_X \setminus \overbrace{X \times \{0\}}^{\text{0-section}}\Big)/\mathbb{K}^\times\,, \tag{212}
$$

with respect to the evident projection map to X. This is a fiber bundle with typical fiber $\mathbb{K}P^{r-1}$, where $r := \mathrm{rnk}_{\mathbb{K}}\big(\mathbb{V}_X\big)$

Relating complex-projective to quaternionic-projective spaces. Fixing an orthonormal basis for the quaternions we have an induced fixed star-algebra inclusion of the complex numbers, and hence a real-linear identification of the quaternions with two copies of the complex numbers:

$$
\begin{tikzcd}
\mathbb{R}\langle 1, \mathrm{i}\rangle \arrow[r, hook] \arrow[d, equal] & \mathbb{R}\langle 1, \mathrm{i}, \mathrm{j}, \mathrm{k}\rangle \arrow[d, equal] \\
\mathbb{C} \arrow[r, hook] & \mathbb{H}
\end{tikzcd}
\qquad \Rightarrow \qquad \mathbb{H} \;\simeq_{\mathbb{R}}\; \mathbb{C} \oplus \mathrm{j}\cdot\mathbb{C} \tag{213}
$$

With this choice of ordering (the factor of j being on the left) the multiplication action of $\mathbb{C} \hookrightarrow \mathbb{H}$ on $\mathbb{H}$ leads to the following U(1)-module structures (since $z \cdot \mathrm{j} = \mathrm{j} \cdot z^\ast$ for $z \in \mathbb{C} \hookrightarrow \mathbb{H}$ via (213)):

$$
\mathbb{H} \;\simeq_{\mathbb{C}}\; \mathbb{C} \oplus \mathbb{C}^\ast \;\in\; \mathrm{U}(1)\mathrm{LeftModules}\,, \qquad\qquad \mathbb{H} \;\simeq_{\mathbb{C}}\; \mathbb{C} \oplus \mathbb{C} \;\in\; \mathrm{U}(1)\mathrm{RightModules}\,, \tag{214}
$$

where the right action is the canonical one while the *left* action is the sum of the canonical one and its dual. In particular the quotient of $\mathbb{H}^\times$ by the right $\mathbb{C}^\times$ action, under (213), is $\mathbb{H}^\times/\mathbb{C}^\times \;=\; \mathbb{C}P^1$. Hence:

Remark 3.88 (Complex projective spaces over Quaternionic-projective spaces)**.** For each $n \in \mathbb{N}$, quotienting along the inclusion (213) yields S^2-fibration of complex projective spaces over quaternionic projective spaces (Notation 3.84) of this form:

$$
\begin{tikzcd}[column sep=small]
S^2 \arrow[dr] & = & \mathbb{H}^\times/\mathbb{C}^\times \arrow[dr] & & \\
& \mathbb{C}P^{2n+1} \arrow[d] & = & \big(\mathbb{C}^{2n+2}\setminus\{0\}\big)/\mathbb{C}^\times \arrow[d] & \ni\; v\cdot\mathbb{C}^\times \arrow[d, mapsto] \\
& \mathbb{H}P^n & = & \big(\mathbb{H}^{n+1}\setminus\{0\}\big)/\mathbb{H}^\times & \ni\; v\cdot\mathbb{H}^\times
\end{tikzcd}
\tag{215}
$$

For $n = 1$ this is also known as the *twistor fibration* $t_{\mathbb{H}}$, see [FSS20b, §2] for pointers. As n varies, these form a sequence of S^2-fibrations:

$$
\begin{tikzcd}[column sep=small]
\mathbb{C}P^1 \arrow[r, hook] \arrow[d] & \mathbb{C}P^2 \arrow[r, hook] & \mathbb{C}P^3 \arrow[r, hook] \arrow[d, "t_{\mathbb{H}}"] & \mathbb{C}P^4 \arrow[r, hook] & \mathbb{C}P^5 \arrow[r, hook] \arrow[d] & \mathbb{C}P^6 \arrow[r, hook] & \mathbb{C}P^7 \arrow[r, hook, dotted] \arrow[d] & \mathbb{C}P^\infty \arrow[d, "{\scriptstyle \mathscr{L} \,\mapsto\, \mathscr{L}\oplus\mathscr{L}^\ast}"'] & \simeq & B\mathrm{U}(1) \arrow[d, "B(z \mapsto \mathrm{diag}(z, z^\ast))"] \\
\mathbb{H}P^0 \arrow[rr, hook] & & \mathbb{H}P^1 \arrow[rr, hook] & & \mathbb{H}P^2 \arrow[rr, hook] & & \mathbb{H}P^3 \arrow[r, hook, dotted] & \mathbb{H}P^\infty & \simeq & B\mathrm{SU}(2)
\end{tikzcd}
\tag{216}
$$

Proposition 3.89 (Maps from $\mathbb{C}P$ to $\mathbb{H}P$ converge to skew diagonal map)**.** *In the colimit $n \to \infty$, the morphism of classifying spaces on the right of* (216) *is that induced from the group inclusion* $\mathrm{U}(1) \xhookrightarrow{z \mapsto \mathrm{diag}(z,z^*)} \mathrm{SU}(2)$, *as indicated.*

Proof. The claim is equivalent to the statement that the complex vector bundle underlying the dual tautological quaternionic line bundle (201) pulls back to the Whitney sum of the tautological complex line bundle (201) with its dual (202), hence that we have a pullback diagram of spaces

$$
\begin{tikzcd}
\mathscr{L}^*_{\mathbb{C}P^n} \oplus_{\mathbb{C}P^n} \mathscr{L}_{\mathbb{C}P^n} \arrow[rr, "\substack{\text{fiberwise} \\ \text{left } \mathbb{C}\text{-linear isomorphism}}"] \arrow[d] \arrow[drr, phantom, "(\mathrm{pb})"] & & \mathscr{L}^*_{\mathbb{H}P^n} \arrow[d] \\
\mathbb{C}P^n \arrow[rr] & & \mathbb{H}P^n \\
v \cdot \mathbb{C}^\times & \longmapsto & v \cdot \mathbb{H}^\times
\end{tikzcd}
\quad \Leftrightarrow \quad \underset{v}{\forall} \quad
\begin{tikzcd}
\mathbb{C}^* \oplus \mathbb{C} \arrow[r, "\text{left } \mathbb{C}\text{-linear isomorphism}"] \arrow[d] & \mathbb{H} \arrow[d] \\
\{v \cdot \mathbb{C}^\times\} \arrow[r] & \{v \cdot \mathbb{H}^\times\}
\end{tikzcd}
$$

such that the top morphism is fiberwise complex linear for the *left* $\mathbb{C}$-actions (205). On the typical fiber this means equivalently that $\mathbb{H} \simeq_{\mathbb{C}} \mathbb{C} \oplus \mathbb{C}^*$, which is the case by (214). $\square$

Bounded-dimensional complex and quaternionic orientation in E-cohomology.

Lemma 3.90 (Bounded-dimensional universal $\mathbb{K}$-line bundles)**.** *Let $n \in \mathbb{N}$.*
(a) *The dual tautological $\mathbb{C}$-line bundle $\mathscr{L}^*_{\mathbb{C}P^n}$* (201) *over $\mathbb{C}P^n$ is universal for complex line bundles over $d \leq 2n$-dimensional manifolds X^{2n} (more generally: over $\leq 2n$-dimensional cell complexes) in that every such has a classifying map that factors through $\mathbb{C}P^n \hookrightarrow \mathbb{C}P^\infty = B\mathrm{U}(1)$ and is hence isomorphic to a pullback of $\mathscr{L}^*_{\mathbb{C}P^n}$:*

$$
\begin{tikzcd}
\text{complex line bundle} & \mathscr{V}^*_X \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & \mathscr{L}^*_{\mathbb{C}P^n} \arrow[rr] \arrow[d] \arrow[drr, phantom, "(\mathrm{pb})"] & & E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C} \arrow[d] \\
\substack{\text{base space of} \\ \text{bounded dimension } \leq 2n} & X^{2n} \arrow[r, dashed] \arrow[rrr, bend right, "\vdash \mathscr{V}_X"'] & \mathbb{C}P^n \arrow[r, hook] & \mathbb{C}P^\infty \arrow[r, "\simeq"] & B\mathrm{U}(1) \,.
\end{tikzcd}
$$

and any two such factorizations are homotopic to each other.
(b) *The dual tautological $\mathbb{H}$-line bundle $\mathscr{L}^*_{\mathbb{H}P^n}$* (201) *over $\mathbb{H}P^n$ is universal for quaternionic line bundles over $d \leq 4n+2$-dimensional manifolds X^{4n+2} (more generally: over $\leq 4n+2$-dimensional cell complexes) in that every such has a classifying map that factors through $\mathbb{H}P^n \hookrightarrow \mathbb{H}P^\infty = B\mathrm{Sp}(1)$ and is hence isomorphic to a pullback of $\mathscr{L}^*_{\mathbb{H}P^n}$:*

$$
\begin{tikzcd}
\text{quaternionic line bundle} & \mathscr{V}^*_X \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & \mathscr{L}^*_{\mathbb{H}P^n} \arrow[rr] \arrow[d] \arrow[drr, phantom, "(\mathrm{pb})"] & & E\mathrm{Sp}(1) \underset{\mathrm{Sp}(1)}{\times} \mathbb{H} \arrow[d] \\
\substack{\text{base space of} \\ \text{bounded dimension } leq 4n+2} & X^{4n+2} \arrow[r, dashed] \arrow[rrr, bend right, "\vdash \mathscr{V}_X"'] & \mathbb{H}P^n \arrow[r, hook] & \mathbb{H}P^\infty \arrow[r, "\simeq"] & B\mathrm{Sp}(1) \,.
\end{tikzcd}
$$

and any two such factorizations are homotopic to each other.

This goes back to [Ch51, p. 45, 67]. For our purposes it will be useful to see this as a transparent consequence of cellular approximation within the familiar infinite classifying space:

Proof. On the one hand, the topological space underlying a real d-dimensional manifold X admits the structure of a d-dimensional CW-complex (e.g. [FSS20c, Ex. A.37]). On the other hand, the sequence of finite dimensional projective spaces (209) form the stages of a CW-complex structure on infinite projective space, by Lemma 3.85. Therefore the cellular approximation theorem applies [Sp66, p. 404] and says that any classifying map $X \to \mathbb{K}P^\infty$ factors, up to homotopy, through the d-skeleton of $\mathbb{K}P^\infty$, while any homotopy between such classifying maps factors still through its $d+1$-skeleton, hence through largest projective space of real dimension $\leq d+1$. $\square$

Definition 3.91 (Bounded-dimensional complex orientation in E-cohomology)**.** For E be a multiplicative cohomology theory (Def. 3.35), and $n \in \mathbb{N}_+ \sqcup \{\infty\}$, a *universal* $2n$*-dimensional complex* E*-orientation* is a class $c_1^E \in E^2(\mathbb{C}P^n)$ (to be called the *universal first* E*-Chern class* in dimension $2n$) whose restriction to $\mathbb{C}P^1$ is the suspended unit, hence such that we have a homotopy-commutative diagram of this form:

$$
\begin{tikzcd}
\mathbb{C}P^1 \simeq S^2 \arrow[r, "\Sigma^2(1^E)"] \arrow[d, hook] & E^2 \\
\mathbb{C}P^n \arrow[ur, dashed, "c_1^E"', "\substack{\text{complex orientation}\\ \text{to degree } n}"]
\end{tikzcd}
\tag{217}
$$

For $n = \infty$, this definition is classical ([CF66, Thm. 7.6][Ad74, §II.2][Ra86, §4.1][Koc96, §. 4.3][TK06, §3.2][Lu10, §4]). For finite n, this has been considered in [Ho84, §1.2][Ra86, Def. 6.5.2]. Analogously:

Definition 3.92 (Bounded-dimensional quaternionic oriented E-cohomology)**.** For E be a multiplicative cohomology theory, and $n \in \mathbb{N} \sqcup \{\infty\}$, a *universal* $(4n+2)$*-dimensional quaternionic* E*-orientation* is a choice of a class $\frac{1}{2}p_1^E \in E^4(\mathbb{H}P^n)$ (to be called the *universal first fractional* E*-Pontrjagin class*[13] in dimension $4n+2$) whose restriction to $\mathbb{H}P^1$ is the suspended unit, hence such that we have a homotopy-commutative diagram of this form:

$$
\begin{tikzcd}
\mathbb{H}P^1 \simeq S^4 \arrow[r, "\Sigma^4(1^E)"] \arrow[d, hook] & E^4 \\
\mathbb{H}P^n \arrow[ur, dashed, "\frac{1}{2}p_1^E"', "\substack{\text{quaternionic orientation}\\ \text{to degree } n}"]
\end{tikzcd}
\tag{218}
$$

For $n = \infty$, this definition appears explicitly in [Lau08, Expl. 2.2.5], almost explicitly in [CF66, Thm. 7.5][TK06, §3.9], and somewhat implicitly in [Ba92, p. 2]. The general Def. 3.92 in finite dimension seems not to have received attention.

Proposition 3.93 (Complex-oriented cohomology of complex projective spaces)**.** *Given a* $2n$*-dimensional complex* E*-orientation* c_1^E *(Def. 3.91), we have for all* $k \leq n$ *an identification of the (un-reduced)* E*-cohomology ring of* $\mathbb{C}P^k$ *as the quotient polynomial algebra over the coefficient ring* $E^\bullet(*)$ *generated by the* $2n$*-dimensional first* E*-Chern class* (217) *and quotiented to make its* $(k+1)$*st power vanish:*

$$
E^\bullet(\mathbb{C}P^k) \simeq E_\bullet\big[c_1^E\big]\big/\big(c_1^E\big)^{k+1}. \tag{219}
$$

This is classical, e.g. [Koc96, Prop. 4.3.2(b)][Lu10, §4, Example 8]. The vanishing of $[c_1^E]^{k+1}$ follows as in Prop. 3.81 (which gives the case $k = 1$).

Remark 3.94 (First E-Chern classes of bounded-dimensional complex line bundles)**.** Since $\mathbb{C}P^n$ (208) is a classifying space for complex line bundles over $d \leq 2n$-dimensional base spaces (Lemma 3.90), a choice of $2n$-dimensional E-orientation (Def. 3.91) induces assignment to any $\mathscr{L}_X^* \in \mathbb{C}\mathrm{LineBundles}_{/X}$ of an E-cohomology classs

$$
c_1^E\big(\mathscr{L}_X^*\big) := \Big[X \xrightarrow{\vdash \mathscr{L}_X^*} \mathbb{C}P^n \xrightarrow{c_1^E} E_2\Big] \in E^2(X)\,, \tag{220}
$$

to be called the *first* E*-Chern class* of the complex line bundle. In particular, the generating class in (219) is that of the dual tautological line bundle (201), classified by the identity map: $c_1^E = c_1^E\big(\mathscr{L}^*_{\mathbb{C}P^k}\big) \in E^2\big(\mathbb{C}P^k\big)$.

Example 3.95 ($2n$-Dimensional complex orientations in ordinary cohomology)**.** For $E = HR$ ordinary cohomology over a ring R, $(HR)_\bullet \simeq \mathbb{Z}$ is concentrated in degree=0. Therefore, Prop. 3.93 implies that the restriction morphisms $H^2(\mathbb{C}P^n; R) \xrightarrow{\simeq} H^2(\mathbb{C}P^1; R) \simeq \mathbb{Z}$ are isomorphisms for all $n \in \mathbb{N}_+ \sqcup \{\infty\}$. This means that there is a *unique* stage=n complex HR-orientation (Def. 3.91). For this the HR-Chern class from Remark 3.94 is the ordinary Chern class

$$
c_1^{HR} = c_1\,.
$$

Analogously, there is a unique quaternionic orientation on ordinary cohomology.

[13] In [Ba92] the "$\frac{1}{2}$" is omitted from the notation. But since this class does give the generator of $E^\bullet(\mathbb{H}P^\infty) \simeq E^\bullet(B\mathrm{Spin}(3))$ we include that factor, for compatibility with the standard notation in the case that $E = H\mathbb{Z}$ is ordinary cohomology.

Example 3.96 (Universal orientations on MU and MSp)**.** The weak equivalences from Prop. 3.86, regarded as maps

$$\mathbb{C}P^1 \xrightarrow[\simeq]{(c_1^{M\mathrm{U}})^{\mathrm{univ}}} (M\mathrm{U})^2 \;, \qquad \mathbb{H}P^1 \xrightarrow[\simeq]{(\frac{1}{2}p_1^{M\mathrm{Sp}})^{\mathrm{univ}}} (M\mathrm{Sp})^4$$

constitute an unbounded complex orientation (Def. 3.91) of MU and an unbounded quaternionic orientation (Def. 3.92) of MSp, respectively.

In fact these orientations are *initial* among all complex/quaternionic orientations: Given a multiplicative cohomology theory E, there is a bijection between complex/quaternionic orientations on E and homotopy-classes of homomomorphisms of homotopy-commutative ring spectra (138) from MU (e.g. [Ra86, §4, Lem. 4.1.13][Lu10, §6, Thm. 8]) or from MSp (e.g. [Lau08, Ex. 2.2.9]), respectively:

$$\underbrace{\big[\mathbb{C}P^\infty \xrightarrow{c_1^E} E^2\big]}_{\text{complex orientation}} \;\leftrightarrow\; \underbrace{\big[M\mathrm{U} \xrightarrow{\mathrm{mult}} E\big]}_{\substack{\text{homotopy-multiplicative}\\ \text{map of ring spectra}}}, \qquad \underbrace{\big[\mathbb{H}P^\infty \xrightarrow{\frac{1}{2}p_1^E} E^2\big]}_{\text{quaternionic orientation}} \;\leftrightarrow\; \underbrace{\big[M\mathrm{Sp} \xrightarrow{\mathrm{mult}} E\big]}_{\substack{\text{homotopy-multiplicative}\\ \text{map of ring spectra}}}. \tag{221}$$

3.9 Conner-Floyd classes

We recall the construction of Conner-Floyd E-Chern classes in complex-oriented cohomology, working out the minimal data needed to construct them in the case of bounded-dimensional complex orientations (Prop. 3.97 below). Then we use this to prove that the second Conner-Floyd E-Chern class of a $4k+2$-dimensional complex E-orientation induces (the first fractional E-Pontrjagin class of) a $4k+2$-dimenional quaternionic E-orientation (Theorem 3.99 below).

Proposition 3.97 (Bounded-dimensional Conner-Floyd E-Chern classes)**.** *For $r \leq n \in \mathbb{N}$, let c_1^E be a $2n$-dimensional complex orientation in E-cohomology (Def. 3.91) and X a CW-complex of bounded dimension,*

$$\dim(X) = 2(n-r)\,, \qquad \mathrm{rnk}_{\mathbb{C}}\big(\mathscr{V}_X\big) \;\leq\; r+1\,. \tag{222}$$

Then there is an assignment to each $\mathscr{V}_X \in \mathbb{C}\mathrm{VectorBundles}_{/X}$ with bounded rank (222) *of an E-cohomology classes*

$$\Big\{ c_n^E\big(\mathscr{V}_X^*\big) \;\in\; E^{2n}(X) \Big\}_{n\in\mathbb{N}}$$

such that the total class

$$c^E \;:=\; 1 + c_1^E + c_2^E + \cdots$$

satisfies the same kind of conditions that characterize the usual Chern classes in ordinary cohomology:

(i) *For $X \xrightarrow{f} Y$ a map, we have:*
$$c^E\big(f^*\mathscr{V}_X^*\big) \;=\; f^*\big(c^E\big(\mathscr{V}_X^*\big)\big)\,. \tag{223}$$

(ii) *For $\mathscr{L}_X$ a line bundle we have:*
$$c^E\big(\mathscr{L}_X^*\big) \;=\; 1 + \underbrace{c_1^E\big(\mathscr{L}_X^*\big)}_{via\ (220)}. \tag{224}$$

(iii) *The Whitney sum rule holds:*
$$c^E\big(\mathscr{V}_X^* \oplus_X \mathscr{W}_X^*\big) \;=\; c^E\big(\mathscr{V}_X^*\big)\cdot c^E\big(\mathscr{W}_X^*\big)\,. \tag{225}$$

(iv) *The top degree is bounded by the rank:*
$$c^E\big(\mathscr{V}_X^*\big) \;=\; \sum_{k=0}^{\mathrm{rnk}(\mathscr{V}_X)} c_k^E\big(\mathscr{V}_X^*\big)\,. \tag{226}$$

Proof. Without the bound on dimension and rank, this is [CF66, Thm. 7.6], see also [Ad74, I.4]. We just need to recall the construction of these *Conner-Floyd Chern classes* with attention to the required dimension in each step. So consider the pullback of the dual vector bundle $\mathscr{V}_X^*$ to the total space of the projective bundle $P\big(\mathscr{V}_X\big)$ (212). This pullback splits as a direct sum of:
(a) the dual tautological bundle $\mathscr{L}^* := \mathscr{L}^*_{P(\mathscr{V}_X)}$, which is fiber-wise the dual tautological bundle (202) over $\mathbb{C}P^r$,
(b) a remaining orthogonal vector bundle $\mathscr{V}^\perp := \big\{ (\eta_x, [v_x]) \in \mathscr{V}_X^* \times_X P(\mathscr{V}_X) \,|\, \eta_x(v_x) \,=\, 0 \big\}$ of rank r,

as shown at the top of the following homotopy-commutative diagram:

$$\begin{tikzcd}
 & \mathscr{L}^*\oplus\mathscr{V}^\perp \arrow[r] \arrow[d] & \mathscr{V}_X^* \arrow[d] & \\
\mathbb{C}P^r \arrow[r, "\mathrm{fib}(\pi)"] & P\big(\mathscr{V}_X\big) \arrow[r, "\pi"] \arrow[ur, phantom, "(\mathrm{pb})"] \arrow[dr, dashed, "(\vdash\mathscr{L}^*{,}\vdash\mathscr{V}^\perp)"'] & X^{2(n-r)} \arrow[r, "\vdash\mathscr{V}_X^*"] & B\mathrm{U}(r+1)\,. \\
 & & \mathbb{C}P^{2n}\times B\mathrm{U}(r) \arrow[r] & B\mathrm{U}(1)\times B\mathrm{U}(r) \arrow[u]
\end{tikzcd} \tag{227}$$

Shown at the bottom of the diagram (227) is a classifying map for this direct sum bundle, where we noticed that

$$\dim_{\mathbb{C}}\big(P\big(\mathscr{V}_X\big)\big) \;=\; \dim(X) + 2\big(\mathrm{rnk}_{\mathbb{C}}\big(\mathscr{V}_X\big) - 1\big) \;=\; 2(n-r) + 2r \;=\; 2n\,, \tag{228}$$

and then used Lemma 3.90 to factor the classifying map $P(\mathscr{V}_X) \xrightarrow{\vdash\mathscr{L}^*} B\mathrm{U}(1)$ of $\mathscr{L}^*$ through $\mathbb{C}P^{2n}$, up to homotopy. Via this factorization, we obtain the class (220)

$$c_1^E\big(\mathscr{L}^*\big) \;:=\; \big[P(\mathscr{V}_X) \xrightarrow{\vdash\mathscr{L}^*} \mathbb{C}P^{2n} \xrightarrow{c_1^E} E^2\big] \;\in\; E^2\big(P(\mathscr{V}_X)\big),$$

whose pullback along $\mathrm{fib}(\pi)$, yields the give first E-Chern class c_1^E (by Remark 3.94) and hence, by Prop. 3.93, the polynomial generator of the E-cohomology ring (219). Therefore, the Leray-Hirsch theorem in its E-cohomology version ([CF66, Thm. 7.4][TK06, Thm. 3.1]) applies and says that

$$E^\bullet\big(P(\mathscr{V}_X)\big) \;\simeq\; E^\bullet(X)\Big\langle 1, c_1^E(\mathscr{L}), \big(c_1^E(\mathscr{L})\big)^2, \cdots, \big(c_1^E(\mathscr{L})\big)^n \Big\rangle \;\in\; E^\bullet(X)\mathrm{Modules}\,. \tag{229}$$

Now, by the required properties (223), (224), (225) of the E-Chern classes, they need to satisfy

$$\pi^*\big(c^E(\mathscr{V}_X^*)\big) \;=\; c^E(\mathscr{V}^\perp)\cdot\big(1 + c_1^E(\mathscr{L}^*)\big) \quad\in\quad E^\bullet\big(P(\mathscr{V}_X)\big)\,,$$

hence equivalently:

$$\begin{aligned} c^E\big(\mathscr{V}^\perp\big) \;&=\; \pi^*\big(c^E(\mathscr{V}_X^*)\big)\cdot\big(1 + c_1^E(\mathscr{L}^*)\big)^{-1} \\ &:=\; \pi^*\big(c^E(\mathscr{V}_X^*)\big)\cdot\sum_k(-1)^k\big(c_1^E(\mathscr{L}^*)\big)^k\,. \end{aligned} \tag{230}$$

But since $\mathrm{rnk}\big(\mathscr{V}_X^\perp\big) = r < r+1$, the last condition (226) implies that (230) reduces in degree $2(r+1)$ to

$$\begin{aligned} 0 = \pi^*\big(c_{r+1}^E(\mathscr{V}_X^*)\big) - \pi^*\big(c_r^E(\mathscr{V}_X^*)\big)\cdot c_1^E(\mathscr{L}^*) + \cdots + (-1)^r\pi^*\big(c_1^E(\mathscr{V}_X^*)\big)\cdot\big(c_1^E(\mathscr{L}^*)\big)^r \\ + (-1)^{r+1}\big(c_1^E(\mathscr{L}^*)\big)^{r+1}\,. \end{aligned} \tag{231}$$

By (229), this equation (231) has a unique solution for classes $c_k^E(\mathscr{V}_X^*) \in E^\bullet(X)$. These are the Conner-Floyd E-Chern classes of $\mathscr{V}_X^*$ for the given complex orientation c_1^E, usually defined for c_1^E given on $\mathbb{C}P^\infty$, but here seen to depend only on $c_1^E \in E^2(\mathbb{C}P^n)$, under the bounds (222). □

Quaternionic orientations induced from complex orientations.

Proposition 3.98 (Complex orientation induces quaternionic orientation)**.** *Given a complex E-orientation c_1^E (Def. 3.91, i.e. unbounded, ∞-dimensional), the second Conner-Floyd E-Chern class (Prop. 3.97) defines a quaternionic E-orientation (Def. 3.92) by setting:*

$$\tfrac{1}{2}p_1^E \;:=\; c_2^E\big((-)_{\mathbb{C}}\big) \qquad \textit{i.e.:} \qquad \begin{tikzcd} \mathbb{H}P^1 \arrow[rrr, dashed, "\Sigma^4(1^E)"] \arrow[d, hook] & & & E^4 \\ \mathbb{H}P^\infty \arrow[urrr, "\frac{1}{2}p_1^E"] & \simeq & B\mathrm{SU}(2) \arrow[r, hook] & B\mathrm{U}(2) \arrow[u, "c_2^E"'] \end{tikzcd} \;\; \begin{array}{c}\textbf{quaternionic orientation}\\ \textbf{from complex orientation}\end{array} \tag{232}$$

Proof. We need to show that the top triangle in (232) homotopy-commutes, hence that the restriction of c_2^E along $S^4 \simeq \mathbb{H}P^1 \hookrightarrow \mathbb{H}P^\infty \simeq B\mathrm{SU}(2)$ is the suspended unit class $\Sigma^4(1^E)$.

But the nth Conner-Floyd Chern class c_n^E (induced by the given choice of c_1^E) is the pullback of an E-Thom class $\big[\mathrm{vol}^E_{E\mathrm{U}(n)\underset{\mathrm{U}(n)}{\times}\mathbb{C}^n}\big]$ along the 0-section of the universal complex vector bundle over $B\mathrm{U}(n)$ (e.g. [TK06, p. 61]). For the case $n = 2$ and restricted along $\mathbb{H}P^\infty \simeq B\mathrm{SU}(2) \to B\mathrm{U}(2)$, this means that c_2^E is the restriction along the 0-section of an E-Thom class $\big[\mathrm{vol}^E_{\mathscr{L}^*_{\mathbb{H}P^\infty}}\big]$ on the universal $\mathbb{H}$-line bundle.

Now the weak equivalence from Prop. 3.86 in the diagram (209) means that the restriction of this Thom class $\big[\mathrm{vol}^E_{\mathscr{L}^*_{\mathbb{H}P^\infty}}\big]$ to the base fiber – which is $\Sigma^4(1^E)$ by definition of Thom classes – equals the restriction of c_2^E along $\mathbb{H}P^1 \hookrightarrow \mathbb{H}P^\infty$:

$$\begin{tikzcd} \widetilde{E}^\bullet\big(S^4\big) \arrow[d, equal] & \widetilde{E}^\bullet\big(\mathrm{Th}(\mathscr{L}^*_{\mathbb{H}P^\infty})\big) \arrow[l, "\textbf{restriction to}"', "\textbf{fiber}"] \arrow[d, "\simeq"] \arrow[r, phantom, "="] & \widetilde{E}^\bullet\big(\mathrm{Th}(E\mathrm{SU}(2)\underset{\mathrm{SU}(2)}{\times}\mathbb{C}^2)\big) \arrow[d, "\simeq"] & \Sigma^4(1^E) \arrow[d, mapsto] & \big[\mathrm{vol}^E_{\mathscr{L}^*_{\mathbb{H}P^1}}\big] \arrow[l, mapsto] \arrow[r, phantom, "="] & \big[\mathrm{vol}^E_{E\mathrm{SU}(2)\underset{\mathrm{SU}(2)}{\times}\mathbb{C}^2}\big] \\ \widetilde{E}^\bullet\big(S^4\big) & \widetilde{E}^\bullet\big(\mathbb{H}P^\infty\big) \arrow[l, "\textbf{restriction to}"', "\mathbb{H}P^1 \subset \mathbb{H}P^\infty"] \arrow[r, equal] & \widetilde{E}^\bullet\big(B\mathrm{SU}(2)\big) & \Sigma^4(1^E) & \tfrac{1}{2}p_1^E \arrow[l, mapsto, dashed] \arrow[r, mapsto] & c_2^E \arrow[u, mapsto] \end{tikzcd} \tag{233}$$

□

This situation generalizes to finite-stage orientations:

Theorem 3.99 (Finite-dimensional quaternionic orientation from finite-dimensional complex orientation)**.** *For $k \in \mathbb{N}$, given a $2(2k+1) = 4k+2$-dimensional complex E-orientation c_1^E (Def. 3.91), the second E-Chern class (Prop. 3.97) defines a $4k+2$-dimensional quaternionic orientation (Def. 3.92) by setting*

$$
\tfrac{1}{2}p_1^E \;:=\; c_2^E\big((-)_{\mathbb{C}}\big) \qquad \textit{i.e.:} \qquad
\begin{tikzcd}
\mathbb{H}P^1 \arrow[rr, dashed, "\Sigma^4(1^E)"] \arrow[d, hook] & & E^4 \\
\mathbb{H}P^n \arrow[urr, "\frac{1}{2}p_1^E"] \arrow[r, phantom, "\simeq"] & \mathrm{sk}_{4n}B\mathrm{SU}(2) \arrow[r, hook] & \mathrm{sk}_{4n}B\mathrm{U}(2) \arrow[u, "c_2^E"']
\end{tikzcd}
\quad
\begin{array}{c}\textbf{finite-dimensional}\\ \textbf{quaternionic orientation}\\ \textbf{from finite-dimensional}\\ \textbf{complex orientation}\end{array}
\tag{234}
$$

Proof. By Prop. 3.98 the statement is true for complex orientations at sufficiently large dimension. We need to see in which dimension the complex orientation needs to be defined in order to provide the quaternionic orientation at the required dimension. In order to run the proof (233) with $\mathbb{H}P^\infty$ replaced by $\mathbb{H}P^k$, we need the E-Chern class c_2^E to be defined on the complex vector bundle of rank $r+1=2$ which underlies a quaternionic line bundle on $\mathbb{H}P^k$. By Prop. 3.97 and using that $\dim\big(\mathbb{H}P^k\big) = 2(2k)$ this is given by a complex orientation of dimension $2(2k+r) = 2(2k+1) = 4k+2$. □

Remark 3.100 (Relation of dimensions for complex- and quaternionic orientations)**.** In summary, the matching of the dimensions for complex- and quaternionic orientations in Theorem 3.99 comes about from the interaction of two slightly subtle effects:

(i) The construction of the *second* E-Chern class on $\mathbb{H}P^k$ requires the *first* E-Chern class to be defined in dimension

$$
\dim_{\mathbb{R}}\big(P_{\mathbb{C}}\big(\mathscr{L}^*_{\mathbb{H}P^k}\big)\big) \;=\; 4k+2 \;=\; \dim\big(\mathbb{C}P^{2k+1}\big)
$$

(by (228) in the proof of Prop. 3.97).

(ii) The finite-dimensional projective spaces $\mathbb{H}P^k$ and $\mathbb{C}P^{2k+1}$ are universal for $\mathbb{H}$- and $\mathbb{C}$-line bundles, respectively, over spaces of coinciding dimension

$$
\dim_{\mathbb{R}}\big(\mathbb{H}P^{k+1}\big) - 2 \;=\; 4k+2 \;=\; \dim_{\mathbb{R}}\big(\mathbb{C}P^{(2k+1)+1}\big) - 2
$$

(by Lemma 3.90).

Example 3.101 (Orientation and near horizons of M2-branes)**.** By Theorem 3.99, a universal orientation of complex line bundles over base spaces of dimension 10 induces a universal orientation of quaternionic line bundles over 10-dimensional base space, which equivalently is (90) a choice of universal H_3^E-flux near M2-brane horizons (89):

$$
\begin{tikzcd}
\mathbb{C}P^1 \arrow[r, "\Sigma^2(1^E)"] \arrow[d, hook] & E^2 \\
\mathbb{C}P^5 \arrow[ur, dashed, "c_1^E"']
\end{tikzcd}
\qquad \Rightarrow \qquad
\begin{tikzcd}
S^7 \arrow[r] \arrow[d, "h_{\mathbb{H}}"'] & * \arrow[d] \arrow[dl, phantom, "\Swarrow H_3^E"] \\
S^4 \arrow[r, "\Sigma^4(1^E)"'] & E^4
\end{tikzcd}
$$

4 Conclusions

M-Brane charge quantization in Cobordism cohomology. Since the Pontrjagin theorem (19) identifies Cohomotopy theory with framed Cobordism cohomology, the statement of *Hypothesis H* is equivalently:

M-Brane charge is quantized in framed Cobordism.

We have seen in §2.2 that this is not just an abstract mathematical identification, but that Cobordism representatives of brane charges in Cohomotopy are naturally identified with polarized brane worldvolumes (2.2), and their trivializations with brane-punctured Kaluza-Klein compactification of spacetimes $\mathbb{R}^{D,1}$, near the probe brane charges (p. 2.2). This is a consequence of how the *unstable* Pontrjagin-theorem concerns (a) actual embedded submanifolds Σ^{D-n} equipped with (b) normal framings and hence Cartesian-product neighborhoods $\Sigma^{D-n} \times \mathbb{R}^n$, as befits a KK-compactification on Σ^{D-n} (near Σ^{D-n}).

But since Cohomotopy/framed Cobordism are, stably, the *initial* multiplicative cohomology theory (36), every other multiplicative cohomology serves to approximate brane charge quantization in Cohomotopy.

Specifically, existence of universal H_3-fluxes near M2-branes forces (90) brane charge observation through quaternionic-orientated – in particular (92), complex-orientated – multiplicative cohomology theories. But initial among oriented cohomology theories are again Cobordism cohomology theories, namely (221) quaternionic Coordism $M\mathrm{Sp}$ and complex Cobordism $M\mathrm{U}$, respectively. And indeed, one may recover observations of charges in Cohomotopy, hence in framed Cobordism, from their observations in unitary Cobordism – this is the statement of the Adams-Novikov spectral sequence (p. 5):

M-Brane charge quantization may be observed in complex Cobordism.

Which is to say that, with Hypothesis H, M-brane charge is not only *fundamentally* in framed Cobordism, but, in the presence of black M2-brane sources, M-brane charges are canonically approximated/perceived through other flavors of Cobordism cohomology Mf, such as quaternionic and unitary cobordism, but also, for instance, special unitary Cobordism $MS\mathrm{U}$ (78), (which connects with Calabi-Yau compactifications, Prop. 3.64).

In summary and in the form of a broad slogan[14], Hypothesis H implies that:

M-brane charge quantization is seen in Cobordism theories.

hence that

M-theory is controlled by Mf-theory.

This is a direct analogue – a refinement – of the traditional conjecture that D-brane charge is quantized in topological K-theory (p. 1). Indeed, if M-brane charge is in Cobordism theory, then perceiving it through the lens of K-orientations (165) means to *approximate* it by K-theory.

One place where brane charge quantization laws become important is in discussing tadpole cancellations (see [SS19]), hence the condition that total brane charge on compact spaces is required to vanish: It is the charge quantization law in generalized cohomology which determines what it actually means for brane charges to vanish. For example, the cancelling of charges of 24 transverse NS5/D7-branes in M/F theory on K3 is faithfully reflected by vanishing in the framed Cobordism group $M\mathrm{Fr}_{7-4} \simeq \mathbb{S}_3 \simeq \mathbb{Z}/24$ (28), Rem. 4.1.

These charge-less or *cohomologically fluxless* backgrounds (Rem. 2.2) are witnessed by H_3-fluxes (48), which encode *how* the total charge is cancelling, observed through refined e-invariants (71). Seen in Cobordism cohomology, such a brane charge cancellation is represented by an actual cobordism between the vicinities of the source branes. For instance, the Conner-Floyd e-invariant (77) manifestly sees the Green-Schwarz mechanism on K3-surfaces with transversal NS5-brane punctures (83).

We close by tabulating, on pp. 86, the correspondences between M-theory and Mf-theory obtained in §2.

[14] The idea that brane charge should be quantized/vanish in Cobordism cohomology may be compared to the discussion in [McVa19, §5.2, 2nd paragraph]

A correspondence of concepts between M/F-Theory and ***M*****Fr-Theory.** The following tables survey the correspondence, that we have obtained in §2, between phenomena expected in M-theory and made precise under Hypothesis H, and definitions/theorems in stable homotopy theory and generalized cohomology, specifically revolving around Cobordism cohomology $M\mathrm{Fr}$, $M\mathrm{SU}$ and topological K-theory:

Non-perturbative physics	$\longleftrightarrow$	**Algebraic Topology**	**§2.1**
Fields vanishing at ∞	$\mathrm{LCHSpaces}^{\mathrm{prop}} \xrightarrow{(-)_{\mathrm{cpt}}} \mathrm{Spaces}^{*/}$	(One-point compactification to) Pointed topological spaces	(17)
Quantized charges in spacetime X including ∞	$\big[X \xrightarrow{c} A\big] \in \widetilde{A}\big(X\big) := \pi_0 \mathrm{Maps}^{*/}\big(X, A\big)$	Reduced non-abelian generalized cohomology	(6)

M/F-Theory	$\overset{\textbf{Hypothesis H}}{\longleftrightarrow}$	MFr**-Theory**	**§2.2**
Full M-brane charge on parallelizable spacetime	$\big[X \xrightarrow{c} S^4\big] \in \widetilde{\pi}^4(X)$	Unstable Cohomotopy theory	(14)
Polarized worldvolume of branes sourcing the charge	$\big[c^{-1}\big(0_{\mathrm{reg}}\big) \subset X\big] \in \mathrm{Cob}^4_{\mathrm{Fr}}(X)$	Unstable Fr-Cobordism via Pontrjagin construction	(19)
Charge of probe p-branes near black b-branes	$\big[\big(\mathbb{R}^{b-p} \times S^{9-b}\big)_{\mathrm{cpt}} \xrightarrow{c} S^4\big]$	Unstable Homotopy groups of spheres	(121) (22)
Charge of probe p-branes in bulk around black b-branes	$\big[\big(\mathbb{R}^{b-p} \times S^{9-b}\big)_{\mathrm{cpt}} \xrightarrow{c} S^4\big] \wedge (\mathbb{R}^1_{\mathrm{rad}})_{\mathrm{cpt}}$	1-Stable Homotopy groups of spheres	(27)
Linearized M-brane charge on parallelizable spacetime	$\big[\Sigma^\infty X \xrightarrow{\Sigma^\infty c} \Sigma^\infty S^4\big] \in \widetilde{\mathbb{S}}^4(X)$	Stable Cohomotopy theory	(36)
Polarized worldvolume class of branes sourcing linear charge	$\mathrm{Src}(\Sigma^\infty c) \in \widetilde{M\mathrm{Fr}}^4(X)$	Stable Fr-Cobordism cohomology via Pontrjagin-Thom isomorphism	(27)

M/F-Theory	$\overset{\textbf{Hypothesis H}}{\longleftrightarrow}$	MFr**-Theory**	**§2.3**
Unit M5-brane charge	$G^E_{4,\mathrm{unit}} := \Sigma^4(1^E) \in \widetilde{E}^4\big(S^4\big)$	Unital cohomology theory	(30)
Total M5-brane charge in given units of charge	$[G^E_4(c)] := [c^* G^E_{4,\mathrm{unit}}] \in \widetilde{E}^4(X)$	Adams d-invariant	(35)
Dual M5-brane flux and Topological equation of motion	$G^E_4 \cdot G^E_4 \overset{2G^E_7}{\Longrightarrow} 0 \in P\mathrm{Maps}^{*/}\big(X, E^8\big)$	Multiplicative cohomology theory	(38)
Computing M-brane charge from its measurement in multipl. units	$\mathrm{Ext}^\bullet_{\widetilde{E}^\bullet(E)}\big(\widetilde{E}^\bullet(S^4), \widetilde{E}^\bullet(X)\big) \Rightarrow \widetilde{\mathbb{S}}^4(X)^{\wedge_E}$	Adams-Novikov spectral sequence	p. 5

M/F-Theory	$\overset{\textbf{Hypothesis H}}{\longleftrightarrow}$	MFr**-Theory**	**§2.4**
H_3-Flux in given units	$0 \overset{H^E_3(c)}{\Longrightarrow} G^E_4(c) \in P\mathrm{Maps}^{*/}\big(X, E^4\big)$	Trivialization of d-invariant	(48) (140)
Observable on H_3-flux	$O^{(-)}(H^E_3(c)) := \big\langle c, G^E_{4,\mathrm{unit}}, - \big\rangle_{(H^E_3(c), -)}$	Refined Toda bracket	(56)
Universal observable on H^E_3-flux	$O^{E/\mathbb{S}}(H^E_3) := \big\langle c, G^E_{4,\mathrm{unit}}, \mathrm{cof}(e^E)\big\rangle_{(H^E_3(c), \mathrm{po})}$	Refined Toda bracket in Adams cofiber cohomology	(66)
Observation of H^E_3-flux through multiplicative operation	$(E \xrightarrow{\phi} F) \rightsquigarrow O^{F/\mathbb{S}} := \phi/\mathbb{S} \circ O^{E/\mathbb{S}}$	Box operation on Toda brackets	(68)

M/F-Theory	$\overset{\textbf{Hypothesis H}}{\longleftrightarrow}$	*M*Fr-**Theory**	**§2.5**
Observation of H_3^{KU}-flux through Chern character	$\widehat{\mathrm{e}}_{KU}\big(H_3^{KU}\big) := \mathrm{spl}_0 \circ \mathrm{ch}/\mathbb{S} \circ O^{KU/\mathbb{S}}(H_3^{KU})$	Refined Adams e-invariant	(71)
Obstruction to observation of integral H_3-flux	$\mathrm{e}_{\mathrm{Ad}}(c) = \widehat{\mathrm{e}}_{KU}\big(H_3^{KU})(c)\big) \bmod \mathbb{Z}$	Classical Adams e-invariant	(72)
Existence of C_3-field	$[H_3^{KU}(c)] = [C_3^{KU}(c)] + n \cdot [H_{3,\mathrm{unit}}^{KU}]$ $\Leftrightarrow\ \mathrm{e}_{\mathrm{Ad}}(c) = 0$	Vanishing of classical Adams e-invariant	(73)

M/F-Theory	$\overset{\textbf{Hypothesis H}}{\longleftrightarrow}$	*M*Fr-**Theory**	**§2.6**
Spontaneous KK-compactification on K3 with 24 transversal branes	$24 \cdot [S^3_{\mathrm{fr}=1}] \xRightarrow{\mathrm{K3} \setminus \sqcup_{24} D^4} 0 \in P\mathrm{Maps}^{*/}\big(S^7, M\mathrm{Fr}^4\big)$	24-punctured K3 witnesses 3rd stable stem $M\mathrm{Fr}_3 = \mathbb{Z}/24$	(28)
Observation of ordinary H_3-flux sourced by transversal branes	$\widehat{e}_{KU}\big(H_3^{MSU}(nh_{\mathbb{H}})\big) \ = \ \mathrm{Td}\big[M^4_{\mathrm{SU}} \setminus \sqcup_n D^4\big]$	Conner-Floyd's e-invariant via Todd character	(28)
Green-Schwarz mechanism for transversal 5-branes	$d\,H_3 = \chi_4(\nabla), \quad \widehat{e}\big(H_3^{MSU}\big) = \frac{1}{12}\int_{S^3_k} H_3$	Conner-Floyd's e-invariant via J-twisted 3-Cohomotopy	(83)

M/F-Theory	$\overset{\textbf{Hypothesis H}}{\longleftrightarrow}$	*M*Fr-**Theory**	**§2.7**
M2-Brane Page charge	$N^E_{\mathrm{M2}}(c) \ = \ \frac{1}{24} H\big(c\big)$	Hopf invariant	(86)
M2-Brane Page charge as observable on H_3-flux	$N^E_{\mathrm{M2}}(c) \ = \ \frac{1}{12}\int_{S^7}\big(\frac{1}{2}H_3 \wedge G_4 + G_7\big)$	Whitehead integral formula/ Steenrod functional cup product	

M/F-Theory	$\overset{\textbf{Hypothesis H}}{\longleftrightarrow}$	*M*Fr-**Theory**	**§2.8** **§2.9**
H_3-flux in given units universal near black M2-branes	$0 \xRightarrow{H_3^E(h_{\mathbb{H}})} G_4^E(h_{\mathbb{H}}) \ \in P\mathrm{Maps}^{*/}\big(S^7, E^4\big)$	10d quaternionic orientation in E-cohomology	(90)
	$c_2^E \xRightarrow{H^E_{3,\mathrm{het}}} G^E_{4,\mathrm{unit}} \ \in P\mathrm{Maps}^{*/}\big(S^4, E^4\big)$	10d complex orientation in E-cohomology	(90)
Hořava-Witten Green-Schwarz mechanism for heterotic line bundles	$c_1^E \cdot c_{1^*}^E \xRightarrow{H^E_{3,\mathrm{het}}} G_4^E(t_{\mathbb{H}}) \ \in P\mathrm{Maps}^{*/}\big(S^4, E^4\big)$	Factorization through twistor fibration	(90)

Remark 4.1 (Unit M2-brane charge and the Order of the third stable stem.). A crucial subtlety of *Hypothesis H* is that it unifies M2/M5-brane charge in a single non-abelian cohomology theory, in that the Borsuk-Spanier Cohomotopy $\pi^4(X)$ (12)

- quantizes *both* G_4-flux *and* its dual G_7-flux (14); hence
- measures *both* M5-brane charge (p. 15) *and* M2-brane Page charge (§2.7).

$$
\begin{tikzcd}
 & & S^7 \arrow[d, "h_{\mathbb{H}}"', "\substack{\text{quaternionic}\\ \text{Hopf fibration}}"] \\
\overbrace{\mathbb{R}^{2,1} \times S^3}^{\substack{\text{probe M5-brane wordvolume}\\ \text{given by } \mathrm{Src}(c)}} \times \mathbb{R}_+ \arrow[r, hook] & \underbrace{\mathbb{R}^{2,1} \times S^7 \times \mathbb{R}^+}_{\substack{\text{vicinity of}\\ \text{black M2-branes}}} \arrow[ur, "N_{\mathrm{M2}}"', "\substack{\text{M2-brane Page charge}\\ \text{in 7-Cohomotopy}}"] \arrow[r, dashed, "c", "\substack{\text{induced M5-brane charge}\\ \text{in 4-Cohomotopy}}"'] & S^4
\end{tikzcd}
$$

For example, it is this interplay which, under *Hypothesis H* and the Pontrjagin isomorphism (19), gives rise to the phenomenon of M2-branes polarizing into M5-branes as discussed on p. 16. But already ordinary polarization effects in electromagnetic fields change the number of charged particles per unit flux, at least locally.

Therefore, a key subtlety in identifying the physics content of Hypothesis H is to disentangle these two kinds of brane charges: According to [FSS19b, p. 13, §3.8], near the horizon of black M2-branes (23) we are to regard the 24th multiple $24 \cdot c_{\mathrm{unit}} = 24 \cdot [h_{\mathbb{H}}]$ (52) of the non-torsion unit Cohomotopy charge (89) as the unit of charge of a single black M2-brane, where $24 = |\mathbb{S}_3|$ arises as the order of the third stable stem (52).

Here in this article we have found further perspectives that support this conclusion:

Rationale for identifying: $\overbrace{N_{\mathrm{M2}}\big(\underbrace{n \cdot h_{\mathbb{H}}}_{\substack{\text{non-torsion}\\ \text{Cohomotopy charge}}}\big)}^{\substack{\text{number of black M2-branes/}\\ \text{M2-brane Page charge §2.7}}} := n/\underbrace{24}_{=\lvert\mathbb{S}_3\rvert}$		**References**
Existence of H_3-flux and M2-brane Page charge in stable Cohomotopy $\mathbb{S}$ and hence in the bulk	With Cohomotopy charges seen in the bulk spacetime and hence (27) in the initial multiplicative cohomology theory $E = \mathbb{S}$ (stable Cohomotopy), H_3-fluxes near M2-branes – and with them the very notion of M2-brane charge (86) – exist precisely for n a multiple of 24.	[FSS19b, (32) & §3.8], §2.4, §2.7
Expected number of 24 NS5/D7 branes in $\mathrm{M_{HET}}$/F theory on K3	Under the Pontrjagin-isomorphism, the multiple $n = 24$ corresponds to a spontaneous Kaluza-Klein compactification of M-theory spacetime on a K3-fiber, with 24 transversal 5-branes, dual to the 24 D7-branes thought to be required in F-theory on elliptically fibered K3.	§2.2
Expected coefficient in tadpole cancellation for M-theory on 8-manifolds	In the generalization to M-brane charge on curved spacetimes measured in tangentially J-twisted Cohomotopy (discussed in [FSS19b][FSS19c]) the proportionality factor $1/24$ between N_{M2} and the Page charge makes Hypothesis H imply the expected tadpole cancellation formula for M2-branes on 8-manifolds.	[FSS19b, Prop. 3.22]
Existence of C_3-field and integral H_3-flux in KU- and KO-theory	With H_3-flux measured in KU-cohomology, the universal stable observable O^{MSU} of 3-flux, hence the $\widehat{e}_{KU}$-invariant, sees a well-defined C_3-field and with it integrally quantized H_3-flux precisely when n is a multiple of 12.	§2.5
Emergence of expected GS-mechanism behind 24 5-branes transverse to K3	For n a multiple of 24 the observable on H_3^{MSU}-flux induced by the Todd character witnesses the above appearance of 24 5-branes transversal to a K3-fiber explicitly as the solution to the Green-Schwarz anomaly cancellation condition on this compact space.	§2.6

References

[AFCS99] B. Acharya, J. Figueroa-O'Farrill, C. Hull, and B. Spence, *Branes at conical singularities and holography*, Adv. Theor. Math. Phys. **2** (1999), 1249-1286, [`arXiv:hep-th/9808014`].

[Ad60] J. Adams, *On the Non-Existence of Elements of Hopf Invariant One* Bull. Amer. Math. Soc. **64** (1958) 279-282, Ann. Math. 72 (1960) 20-104 [`doi:10.2307/1970147`]

[Ad62] J. Adams, *Vector fields on spheres*, Bull. Amer. Math. Soc. **68** (1962), 39-41, [`euclid:bams/1183524456`].

[Ad66] J. F. Adams, *On the groups $J(X)$ IV*, Topology **5** (1966), 21-71, [`doi:10.1016/0040-9383(66)90004-8`].

[Ad74] J. F. Adams, *Stable homotopy and generalized homology*, The University of Chicago Press, 1974, [`ucp:bo21302708`].

[AA66] J. F. Adams and M. Atiyah, *K-theory and the Hopf invariant*, Quart. J. Math. Oxford **17** (1966), 31-38, [`doi:10.1093/qmath/17.1.31`].

[AJTZ10] O. Aharony, D. Jafferis, A. Tomasiello, and A. Zaffaroni, *Massive type IIA string theory cannot be strongly coupled*, J. High Energy Phys. **1011** (2010), 047, [`arXiv:1007.2451`].

[Al24] P. Aleksandrov, *Über die Metrisation der im Kleinen kompakten topologischen Räume*, Math. Ann. **92** (1924) 294-301 [`dml:159072`]

[Al85] O. Alvarez, *Topological quantization and cohomology*, Comm. Math. Phys. **100** (1985), 279-309, [`euclid:1103943448`].

[AGZ98] L. Alvarez-Gaumé and F. Zamora, *Duality in Quantum Field Theory (and String Theory)*, AIP Conference Proceedings **423** (1998), 46, [`arXiv:hep-th/9709180`].

[AGLP11] L. Anderson, J. Gray, A. Lukas, and E. Palti, *Two Hundred Heterotic Standard Models on Smooth Calabi-Yau Threefolds*, Phys. Rev. **D 84** (2011), 106005, [`arXiv:1106.4804`].

[AGLP12] L. Anderson, J. Gray, A. Lukas, and E. Palti, *Heterotic Line Bundle Standard Models*, J. High Energy Phys. **06** (2012), 113, [`arXiv:1202.1757`].

[ACGLP14] L. Anderson, A. Constantin, J. Gray, A. Lukas, and E. Palti, *A Comprehensive Scan for Heterotic* $\mathrm{SU}(5)$ *GUT models*, J. High Energy Phys. **01** (2014), 047, [`arXiv:1307.4787`].

[AMS08] M. Arai, C. Montonen, and S. Sasaki, *Vortices, Q-balls and Domain Walls on Dielectric M2-branes*, J. High Energy Phys. **0903** (2009), 119, [`arXiv:0812.4437`].

[AC67] M. Arkowitz and C. R. Curjel, *On maps of H-spaces*, Topology **6** (1967), 137-148, [`doi:10.1016/0040-9383(67)90030-4`].

[Arl04] D. Arlettaz, *The generalized Boardman homomorphisms*, Cent. Eur. J. Math. **2** (2004), 50-56, [`doi:10.2478/BF02475949`].

[ADO20a] A. Ashmore, S. Dumitru, and B. Ovrut, *Line Bundle Hidden Sectors for Strongly Coupled Heterotic Standard Models*, [`arXiv:2003.05455`].

[ADO20b] A. Ashmore, S. Dumitru, and B. Ovrut, *Explicit Soft Supersymmetry Breaking in the Heterotic M-Theory MSSM*, [`arXiv:2012.11029`].

[AM97] P. Aspinwall and D. Morrison, *Point-like Instantons on K3 Orbifolds*, Nucl. Phys. **B503** (1997), 533-564, [`arXiv:hep-th/9705104`].

[At61] M. Atiyah, *Bordism and Cobordism*, Math. Proc. Cambridge Phil. Soc. **57** (1961), 200-208, [`doi:10.1017/S0305004100035064`].

[At64] M. Atiyah, *K-theory*, Harvard Lecture 1964, notes by D. Anderson, Benjamin 1967 [ncatlab.org/nlab/files/AtiyahKTheory.pdf]

[At68] M. Atiyah, *Bott periodicity and the index of elliptic operators*, Quart. J. Math. **19** (1968), 113-140, [`doi:10.1093/qmath/19.1.113`].

[At89] M. Atiyah, *Topological quantum field theories*, Pub. Math. l'IHÉS **68** (1989), 175-186, [`numdam:PMIHES_1988__68__175_0`].

[ABS64] M. F. Atiyah, R. Bott, and A. Shapiro, *Clifford modules*, Topology **3** (1964), 3-38, [`doi:10.1016/0040-9383(64)90003-5`].

[AH61] M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Sympos. Pure Math., Vol. III pp. 7–38 AMS 1961 [9789814401319_0008]

[AH88] M. Atiyah and N. Hitchin, *The geometry and dynamics of magnetic monopole*, Princeton University Press, Princeton, 1988, [jstor:j.ctt7zv206].

[BDS00] C. Bachas, M. Douglas, and C. Schweigert, *Flux Stabilization of D-branes*, J. High Energy Phys. **0005** (2000), 048, [arXiv:hep-th/0003037].

[BLMP13] J. Bagger, N. Lambert, S. Mukhi, and C. Papageorgakis, *Multiple Membranes in M-theory*, Phys. Rep. **527** (2013), 1-100, [doi:10.1016/j.physrep.2013.01.006], [arXiv:1203.3546].

[Ba92] A. Baker, *Some chromatic phenomena in the homotopy of* MSp, in: N. Ray, G. Walker (eds.), Adams Memorial Symposium on Algebraic Topology, Vol. 2 editors, Cambridge University Press (1992), 263-280, [www.maths.gla.ac.uk/∼ajb/dvi-ps/bpmsp.pdf]

[BaS10] A. P. Bakulev and D. Shirkov, *Inevitability and Importance of Non-Perturbative Elements in Quantum Field Theory*, Proc. 6th Mathematical Physics Meeting, Sep. 14-23, 2010, Belgrade, Serbia, pp. 27-54, [ISBN:978-86-82441-30-4], [arXiv:1102.2380].

[Bea19] J. Beardsley, *A Theorem on Multiplicative Cell Attachments with an Application to Ravenel's $X(n)$ Spectra*, J. Homotopy Relat. Struct. **14** (2019), 611-624, [doi:10.1007/s40062-018-0222-6], [arXiv:1708.03042].

[BBS06] K. Becker, M. Becker and J. Schwarz, *String Theory and M-Theory: A Modern Introduction*, Cambridge University Press, 2006, [doi:10.1017/CBO9780511816086].

[Be00] I. Bena, *The M-theory dual of a 3 dimensional theory with reduced supersymmetry*, Phys. Rev. **D62** (2000), 126006, [arXiv:hep-th/0004142].

[BN00] I. Bena and A. Nudelman, *Warping and vacua of* $(\mathrm{S})\mathrm{YM}_{3+1}$, Phys. Rev. **D62** (2000), 086008, [arXiv:hep-th/0005163].

[BW04] I. Bena and N. Warner, *A harmonic family of dielectric flow solutions with maximal supersymmetry*, J. High Energy Phys. **0412** (2004), 021, [arXiv:hep-th/0406145].

[BGKM14] I. Bena, M. Graña, S. Kuperstein and S. Massai, *Tachyonic Anti-M2 Branes*, J. High Energy Phys. **1406** (2014), 173, [arXiv:1402.2294].

[BZ14] D. Ben-Zvi, *$\mathscr{X}$ and Geometric Representation Theory*, talks at *Mathematical Aspects of Six-Dimensional Quantum Field Theories* IHES 2014 [https://math.berkeley.edu/~qchu/Notes/6d/].

[BGH99] O. Bergman, E. G. Gimon and P. Hořava, *Brane Transfer Operations and T-Duality of Non-BPS States*, JHEP 9904 (1999) 010 [arXiv:hep-th/9902160] [doi:10.1088/1126-6708/1999/04/010]

[BST87] E. Bergshoeff, E. Sezgin, and P. K. Townsend, *Supermembranes and Eleven Dimensional Supergravity*, Phys. Lett. **B189** (1987), 75-78, [doi:10.1016/0370-2693(87)91272-X].

[BK02] M. Bestvina and A. Keenan, *Differentiable Topology and Geometry*, Lecture notes, 2002, [ncatlab.org/nlab/files/BestvinaKeenanDifferentialTopology.pdf]

[Bo95] J. M. Boardman, *Stable Operations in Generalized Cohomology* in: I. M. James (ed.), *Handbook of Algebraic Topology*, Oxford, 1995, [doi:10.1016/B978-0-444-81779-2.X5000-7].

[Bo36] K. Borsuk, *Sur les groupes des classes de transformations continues*, CR Acad. Sci. Paris **202** (1936), 1400-1403, [doi:10.24033/asens.603].

[BT82] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics 82, Springer, 1982, [doi:10.1007/BFb0063500].

[Br04] V. Braun, *Twisted K-Theory of Lie Groups*, JHEP 0403 (2004) 029 [sarXiv:hep-th/0305178] [doi:10.1088/1126-6708/2004/03/029]

[BBL17] A. Braun, C. R. Brodie, and A. Lukas, *Heterotic Line Bundle Models on Elliptically Fibered Calabi-Yau Three-folds*, J. High Energy Phys. **04** (2018), 087, [arXiv:1706.07688].

[BBLR18] A. Braun, C. Brodie, A. Lukas, and F. Ruehle, *NS5-Branes and Line Bundles in Heterotic/F-Theory Duality*, Phys. Rev. D **98** (2018), 126004, [arXiv:1803.06190].

[BMSS19] V. Braunack-Mayer, H. Sati and U. Schreiber, *Gauge enhancement of Super M-Branes via rational parameterized stable homotopy theory*, Comm. Math. Phys. **371** (2019), 197-265, [10.1007/s00220-019-03441-4], [arXiv:1806.01115].

[Br93] G. Bredon, *Topology and Geometry*, Graduate Texts in Mathematics 139, Springer, 1993, [doi:10.1007/978-1-4757-6848-0].

[BMRS08] J. Brodzki, V. Mathai, J. Rosenberg, and R. Szabo, *D-Branes, RR-Fields and Duality on Noncommutative Manifolds*, Commun. Math. Phys. **277** (2008), 643-706, [doi:10.1007/s00220-007-0396-y], [arXiv:hep-th/0607020].

[BSS19] S. Burton, H. Sati, and U. Schreiber, *Lift of fractional D-brane charge to equivariant Cohomotopy theory*, J. Geom. Phys. **161** (2020), 104034, [doi:10.1016/j.geomphys.2020.104034], [arXiv:1812.09679].

[CJM04] A. Carey, S. Johnson and M. Murray, *Holonomy on D-Branes*, J. Geom. Phy. Volume 52, Issue 2, October 2004, Pages 186-216 [arXiv:hep-th/0204199]

[CDF91] L. Castellani, R. D'Auria, and P. Fré, *Supergravity and Superstrings – A Geometric Perspective*, World Scientific, 1991, [doi:doi:10.1142/0224].

[CGNT05] M. Cederwall, U. Gran, B. Nilsson, and D. Tsimpis, *Supersymmetric Corrections to Eleven-Dimensional Supergravity*, J. High Energy Phys. **0505** (2005), 052, [arXiv:hep-th/0409107].

[Cha00] S. J. Chapman, *A Hierarchy of Models for Type-II Superconductors*, IAM Review **42** (2000), 555-598, [jstor:2653134s].

[Ch51] S.-S. Chern, *Topics in Differential Geometry*, Institute for Advanced Study, Princeton, 1951, [ncatlab.org/nlab/files/Chern-IASNotes1951.pdf]

[CLP19] G. Chernykh, I. Limonchenko, and T. Panov, *SU-bordism: structure results and geometric representatives*, Russian Math. Surveys **74** (2019), 461-524, [arXiv:1903.07178].

[CK19] K.-S. Choi and T. Kobayashi, *Transitions of Orbifold Vacua*, J. High Energy Phys. **07** (2019), 111, [arXiv:1901.11194].

[CGHP08] A. A. Coley, G. Gibbons, S. Hervik, and C. Pope, *Metrics with vanishing quantum corrections*, Class. Quantum Grav., **25** (2008), 145017, [arXiv:0803.2438].

[CF66] P. Conner and E. Floyd, *The Relation of Cobordism to K-Theories*, Lecture Notes in Mathematics 28, Springer 1966, [doi:10.1007/BFb0071091].

[CS69] P. Conner, and L. Smith, *On the complex bordism of finite complexes*, Pub. Math. l'IHÉS **37** (1969), 117-221, [numdam:PMIHES_1969__37__117_0].

[CJS78] E. Cremmer, B. Julia, and J. Scherk, *Supergravity in theory in 11 dimensions*, Phys. Lett. **76B** (1978), 409-412, [doi:10.1016/0370-2693(78)90894-8].

[Cs20] A. Csépai, *Stable Pontrjagin-Thom construction for proper maps*, Period. Math. Hungarica **80** (2020), 259-268, [doi:10.1007/s10998-020-00327-0].

[Cu20] T. Cutler, *The category of pointed topological spaces*, 2020, [ncatlab.org/nlab/files/CutlerPointedTopologicalSpaces.pdf]

[dBD$^+$02] J. de Boer, R. Dijkgraaf, K. Hori, A. Keurentjes, J. Morgan, D. Morrison, and S. Sethi, *Triples, Fluxes, and Strings*, Adv. Theor. Math. Phys. **4** (2002), 995-1186, [arXiv:hep-th/0103170].

[D'AF82] R. D'Auria and P. Fré, *Geometric supergravity in* $D = 11$ *and its hidden supergroup*, Nucl. Phys. **B 201** (1982), 101–140, [doi:10.1016/0550-3213(82)90376-5].

[De08] F. Denef, *Les Houches Lectures on Constructing String Vacua*, Les Houches 87 (2008) 483-61, [spire:780946], [arXiv:0803.1194].

[DHJ88] E. Devinatz, M. Hopkins, and J. Smith, *Nilpotence and Stable Homotopy Theory I*, Ann. Math. **128** (1988), 207-241, [jstor:1971440].

[DFM03] E. Diaconescu, D. S. Freed, and G. Moore, *The M-theory 3-form and* E_8 *gauge theory*, In: Elliptic Cohomology, 44-88, Cambridge University Press, 2007, [arXiv:hep-th/0312069].

[DMW03] D. Diaconescu, G. Moore, and E. Witten, E_8*-gauge theory and a derivation of K-theory from M-theory*, Adv. Theor. Math. Phys **6** (2003), 1031–1134, [arXiv:hep-th/0005090].

[Di31] P.A.M. Dirac, *Quantized Singularities in the Electromagnetic Field*, Proc. Royal Soc. **A133** (1931), 60-72 [doi:10.1098/rspa.1931.0130].

[DOPW99] R. Donagi, B. Ovrut, T. Pantev, and D. Waldram, *Standard Models from Heterotic M-theory*, Adv. Theor. Math. Phys. **5** (2002), 93-137, [arXiv:hep-th/9912208].

[DOPW00] R. Donagi, B. Ovrut, T. Pantev, and D. Waldram, *Standard Model Vacua in Heterotic M-Theory*, Strings'99, Potsdam, Germany, 19 - 24 Jul 1999, [arXiv:hep-th/0001101].

[DPS14] M. Douglas, D. S. Park, and C. Schnell, *The Cremmer-Scherk Mechanism in F-theory Compactifications on K3 Manifolds*, J. High Energy Phys. **05** (2014), 135, [arXiv:1403.1595].

[DNF85] B. A. Dubrovin, S. P. Novikov, and A. T. Fomenko, *Modern Geometry – Methods and Applications - Part II: The Geometry and Topology of Manifolds*, Springer 1985, [ISBN:978-0-387-96162-0].

[Du96] M. J. Duff, *M-Theory (the Theory Formerly Known as Strings)*, Int. J. Mod. Phys. **A11** (1996), 5623-5642, [arXiv:hep-th/9608117].

[Du98] M. J. Duff, *A Layman's Guide to M-theory*, Abdus Salam Memorial Meeting, Trieste, Italy, 19 - 22 Nov 1997, pp.184-213, [arXiv:hep-th/9805177].

[Du99] M. J. Duff (ed.), *The World in Eleven Dimensions: Supergravity, Supermembranes and M-theory*, Institute of Physics Publishing, Bristol, 1999, [ISBN 9780750306720].

[Du18] M. J. Duff, *Thirty years of Erice on the brane*, International Symposium "60 Years of Subnuclear Physics at Bologna", U. Bologna, Nov. 2018, [arXiv:1812.11658][hep-th].

[Du19] M. J. Duff, in: G. Farmelo, *The Universe Speaks in numbers*, interview 14, 2019, [grahamfarmelo.com/the-universe-speaks-in-numbers-interview-14].

[DGT95] M. Duff, G. Gibbons, and P. Townsend, *Macroscopic superstrings as interpolating solitons*, Phys. Lett. **B332** (1994), 321-328, [arXiv:hep-th/9405124].

[DS91] M. J. Duff and K. S. Stelle, *Multimembrane solutions of $D = 11$ Supergravity*, Phys. Lett. **B253** (1991), 113-118, [doi:10.1016/0370-2693(91)91371-2].

[Ei40] S. Eilenberg, *Cohomology and Continuous Mappings*, Ann. Math. **41** (1940), 231-251, [jstor:1968828].

[EML54b] S. Eilenberg and S. Mac Lane, *On the Groups $H(\Pi, n)$, III: Operations and Obstructions*, Ann. Math. **60** (1954), 513-557, [jstor:1969849].

[EGH80] T. Eguchi, P. Gilkey,and A. Hanson, *Gravitation, gauge theories and differential geometry*, Phys. Rep. **66** (1980), 213-393, [doi:10.1016/0370-1573(80)90130-1].

[Er13] T. Erler, *Analytic Solution for Tachyon Condensation in Berkovits' Open Superstring Field Theory*, J. High Energy Phys. **1311** (2013), 007, [arXiv:1308.4400].

[Er19] T. Erler, *Four Lectures on Analytic Solutions in Open String Field Theory*, [arXiv:1912.00521].

[Ev06] J. Evslin, *What Does(n't) K-theory Classify?* [arXiv:hep-th/0610328].

[ES06] J. Evslin and H. Sati, *Can D-Branes Wrap Nonrepresentable Cycles?*, J. High Energy Phys. **0610** (2006), 050, [arXiv:hep-th/0607045].

[FSS15] D. Fiorenza, H. Sati, U. Schreiber, *The WZW term of the M5-brane and differential cohomotopy*, J. Math. Phys. **56** (2015), 102301, [doi:10.1063/1.4932618], [arXiv:1506.07557].

[FSS17] D. Fiorenza, H. Sati, and U. Schreiber, *Rational sphere valued supercocycles in M-theory and type IIA string theory*, J. Geom. Phys. **114** (2017), 91-108, [doi:10.1016/j.geomphys.2016.11.024], [arXiv:1606.03206].

[FSS19a] D. Fiorenza, H. Sati, and U. Schreiber, *The rational higher structure of M-theory*, Proc. LMS-EPSRC Durham Symposium *Higher Structures in M-Theory*, Aug. 2018, Fortsch. Phys., 2019, [doi:10.1002/prop.201910017] [arXiv:1903.02834].

[FSS19b] D. Fiorenza, H. Sati, and U. Schreiber, *Twisted Cohomotopy implies M-theory anomaly cancellation on 8-manifolds*, Commun. Math. Phys. **377** (2020), 1961-2025, [doi:10.1007/s00220-020-03707-2], [arXiv:1904.10207].

[FSS19c] D. Fiorenza, H. Sati, and U. Schreiber, *Twisted Cohomotopy implies M5 WZ term level quantization*, Commun. Math. Phys. **384** (2021) 403–432 [arXiv:1906.07417] [doi:10.1007/s00220-021-03951-0].

[FSS20a] D. Fiorenza, H. Sati, and U. Schreiber, *Twisted cohomotopy implies twisted String structure on M5-branes*, J. Math. Phys. **62** 042301 (2021) [arXiv:2002.11093] [doi:10.1063/5.0037786].

[FSS20b] D. Fiorenza, H. Sati, and U. Schreiber, *Twistorial Cohomotopy implies Green-Schwarz anomaly cancellation*, Reviews in Mathematical Physics **34** 05 (2022) 2250013 [arXiv:2008.08544] [doi:10.1142/S0129055X22500131].

[FSS20c] D. Fiorenza, H. Sati, and U. Schreiber, *The character Nonabelian Cohomology – Twisted, Differential and Generalized*, World Scientific (2023, in print) [arXiv:2009.11909].

[FS01] S. Fredenhagen and V. Schomerus, *Branes on Group Manifolds, Gluon Condensates, and twisted K-theory*, JHEP 0104 (2001) 007 [arXiv:hep-th/0012164]

[FW99] D. Freed and E. Witten, *Anomalies in String Theory with D-Branes*, Asian J. Math. 3:819, 1999 [arXiv:hep-th/9907189]

[Fr12] D. Freed, *4-3-2 8-7-6*, talk at ASPECTS of Topology, Oxford, 2012, [people.maths.ox.ac.uk/tillmann/ASPECTSfreed.pdf]

[Fr13] D. Freed, *The cobordism hypothesis*, Bull. Amer. Math. Soc. **50** (2013), 57-92, [arXiv:1210.5100].

[FT14] D. Freed and C. Teleman, *Relative quantum field theory*, Commun. Math. Phys. **326** (2014), 459-476, [arXiv:1212.1692].

[FU91] D. Freed and K. Uhlenbeck, *Instantons and Four-Manifolds*, MSRI Publications, Springer 1991, [doi:10.1007/978-1-4613-9703-8].

[Fred08] S. Fredenhagen, *Physical Background to the K-Theory Classification of D-Branes: Introduction and References* [doi:10.1007/978-3-540-74956-1_1], chapter in: D. Husemoeller et al., *Basic Bundle Theory and K-Cohomology Invariants*, Springer 2008, [doi:10.1007/978-3-540-74956-1].

[FQ05] S. Fredenhagen and T. Quella, *Generalised permutation branes*, J. High Energy Phys. **0511** (2005), 004, [arXiv:hep-th/0509153].

[Fr00] D. Freed, *Dirac charge quantization and generalized differential cohomology*, Surveys in Differential Geometry, Int. Press, Somerville, MA, 2000, pp. 129-194, [doi:10.4310/SDG.2002.v7.n1.a6], [arXiv:hep-th/0011220].

[FMS07] D. Freed, G. W. Moore, and G. Segal, *The Uncertainty of Fluxes*, Commun. Math. Phys. **271** (2007), 247-274, [doi:10.1007/s00220-006-0181-3], [arXiv:hep-th/0605198].

[FQ05] S. Fredenhagen and T. Quella, it Generalised permutation branes, J. High Energy Phys. **0511** (2005), 004, [arXiv:hep-th/0509153].

[FR80] P. Freund and M. A. Rubin, *Dynamics Of Dimensional Reduction*, Phys. Lett. **B97** (1980), 233-235, [doi:10.1016/0370-2693(80)90590-0].

[GG04] M. R. Gaberdiel and T. Gannon, *D-brane charges on non-simply connected groups*, JHEP 0404 (2004) 030 [arXiv:hep-th/0403011] [doi:10.1088/1126-6708/2004/04/030]

[GS00] M. R. Gaberdiel and B. Stefanski Jr, *Dirichlet Branes on Orbifolds*, Nucl. Phys. B578:58-84, 2000 [arXiv:hep-th/9910109] [doi:10.1016/S0550-3213(99)00813-5]

[GCHS$^+$08] H. García-Compeán, W. Herrera-Suárez, B. A. Itzá-Ortiz and O. Loaiza-Brito, *D-Branes in Orientifolds and Orbifolds and Kasparov KK-Theory*, JHEP 0812 (2008) 007 [arXiv:0809.4238] [doi:10.1088/1126-6708/2008/12/007]

[Ga86] K. Gawedzki, *Topological Actions in two-dimensional Quantum Field Theories*, in: Cargese 1987 proceedings, *Nonperturbative quantum field theory* (1986) [spire:257658]

[GeSn08] D. Gepner and V. Snaith, *On the motivic spectra representing algebraic cobordism and algebraic K-theory*, Documenta Math. 2008 [arXiv:0712.2817]

[GRRV08] J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk, and H. Verlinde, *A Massive Study of M2-brane Proposals*, J. High Energy Phys. **0809** (2008), 113, [arXiv:0807.1074].

[Go90] T. Goodwillie, *The differential calculus of homotopy functors*, Proceedings of the ICM Kyoto 1990, Vol. I, Math. Soc. Japan, 1991, pp. 621-626, [ncatlab.org/nlab/files/GoodwillieICM1990.pdf]

[GS17] D. Grady and H. Sati, *Massey products in differential cohomology via stacks*, J. Homotopy Relat. Struct. **13** (2017), 169-223, [doi:10.1007/s40062-017-0178-y], [arXiv:1510.06366].

[GS19] D. Grady and H. Sati, *Ramond-Ramond fields and twisted differential K-theory*, Adv. Theor. Math. Phys. **26** 5 (2022) [arXiv:1903.08843] [].

[GS20] D. Grady and H. Sati, *Differential cohomotopy versus differential cohomology for M-theory and differential lifts of Postnikov towers*, J. Geom. and Phys. **165** (2021) 104203 [arXiv:2001.07640] [doi:10.1016/j.geomphys.2021.104203].

[Gr01] J. Greenlees, *Equivariant formal group laws and complex oriented cohomology theories*, Homology Homotopy Appl. **3** (2001), 225-263, [euclid:hha/1139840255].

[Gu00] S. Gukov, *K-Theory, Reality, and Orientifolds*, Commun. Math. Phys. **210** (2000) 621-639 [arXiv:hep-th/9901042] [doi:10.1007/s002200050793]

[Gü92] R. Güven, *Black p-brane solutions of $D = 11$ supergravity theory*, Phys. Lett. **B276** (1992), 49-55, [doi:10.1016/0370-2693(92)90540-K].

[Ha78] A. Haefliger, *Whitehead products and differential forms*, Differential topology, foliations and Gelfand-Fuks cohomology, pp. 13-24, Lec. Notes in Math. **652**, Springer, Berlin, 1978, [doi:10.1007/BFb0063500].

[HKK99] K. Hardie, K. H. Kamps, and R. Kieboom, *Higher homotopy groupoids and Toda brackets*, Homology Homotopy Appl. **1** (1999), 117-134, [euclid:hha/1139840198].

[HKM02] K. Hardie, K. H. Kamps, and H. Marcum, *The Toda bracket in the homotopy category of a track bicategory*, J. Pure Appl. Algeb. **175** (2002), 109-133, [doi:10.1016/S0022-4049(02)00131-7].

[HMO01] K. Hardie, H. Marcum, and N. Oda, *Bracket operations in the homotopy theory of a 2-category*, Rend. Ist. Mat. Univ. Trieste **33** (2001), 19-70, [rendiconti.dmi.units.it/volumi/33/02.pdf]

[Ha01] A. Hatcher, *Algebraic Topology*, 2001 [pi.math.cornell.edu/∼hatcher/AT/ATpage.html].

[Ho84] M. Hopkins, *Stable decompositions of certain loop spaces*, PhD Dissertation, Northwestern U., 1984, [ncatlab.org/nlab/files/HopkinsStableDecompositions.pdf]

[Ho99] M. Hopkins, *Complex oriented cohomology theories and the language of stacks*, course notes, 1999, [ncatlab.org/nlab/files/HopkinsComplexOrientedCohomology.pdf]

[HM12] M. Hopkins and A. Mathew, *Spectra and stable homotopy theory*, 2012, [math.uchicago.edu/∼amathew/256y.pdf]

[Ho99] P. Hořava, *Type IIA D-Branes, K-Theory, and Matrix Theory*, Adv. Theor. Math. Phys. 2 (1999) 1373-1404 [arXiv:hep-th/9812135]

[HW95] P. Hořava and E. Witten, *Heterotic and Type I string dynamics from eleven dimensions*, Nucl. Phys. **B460** (1996), 506-524, [arXiv:hep-th/9510209].

[HW96] P. Hořava and E. Witten, *Eleven dimensional supergravity on a manifold with boundary*, Nucl. Phys. **B475** (1996), 94-114, [arXiv:hep-th/9603142].

[HSS19] J. Huerta, H. Sati, and U. Schreiber, *Real ADE-equivariant (co)homotopy of super M-branes*, Commun. Math. Phys. **371** (2019) 425, [doi:10.1007/s00220-019-03442-3], [arXiv:1805.05987].

[HT95] C. Hull and P. Townsend, *Unity of Superstring Dualities*, Nucl. Phys. **B438** (1995), 109-137, [arXiv:hep-th/9410167].

[Hun95] J. Hunton, *The Boardman homomorphism*, The Čech centennial (Boston, MA, 1993), 251-262, Contemp. Math. 181, Amer. Math. Soc., Providence, RI, 1995.

[Huy16] D. Huybrechts, *Lectures on K3-surfaces*, Cambridge University Press, 2016, [doi:10.1017/CBO9781316594193].

[IU12] L. Ibáñez and A. Uranga, *String Theory and Particle Physics: An Introduction to String Phenomenology*, Cambridge University Press, 2012, [arXiv:10.1017/CBO9781139018951].

[IWX20] D. Isaksen, G. Wang, and Z. Xu, *Stable homotopy groups of spheres*, Proc. Nat. Acad. Sci. **117** (2020), 24757-24763, [doi:10.1073/pnas.2012335117], [arXiv:2001.04247].

[IWX20] D. Isaksen, G. Wang, and Z. Xu, *More stable stems*, [arXiv:2001.04511].

[Jo98] C. Johnson, *Études on D-Branes*, in: M. Duff et. al. (eds.) *Nonperturbative aspects of strings, branes and supersymmetry*, Proceedings, 1998, [spire:481393], [arXiv:hep-th/9812196].

[Ka78] M. Karoubi, *K-Theory – An introduction*, Springer 1978 [doi:10.1007/978-3-540-79890-3].

[Ke55] J. Kelly, *General Topology*, 1955, re-printed as Graduate Texts in Mathematics, Springer 1975 [ISBN:9780387901251]

[Ke11] M. Kervaire, *La méthode de Pontrjagin pour la classification des applications sur une sphère*, in: E. Vesentini (ed.), *Topologia Differenziale*, CIME Summer Schools, vol. 26, Springer, 2011, [doi:10.1007/978-3-642-10988-1_3].

[Kob16] A. Kobin, *Algebraic Topology*, 2016, [https://ncatlab.org/nlab/files/KobinAT2016.pdf]

[Koc96] S. Kochman, *Bordism, Stable Homotopy and Adams Spectral Sequences*, Fields Institute Monographs, Amer. Math. Soc., 1996, [cds:2264210].

[Kos93] A. Kosinski, *Differential manifolds*, Academic Press, 1993, [ISBN:ISBN:978-0-12-421850-5].

[KS04] I. Kriz and H. Sati, *M Theory, Type IIA Superstrings, and Elliptic Cohomology*, Adv. Theor. Math. Phys. **8** (2004), 345-395, [arXiv:hep-th/0404013].

[KS05a] I. Kriz and H. Sati, *Type IIB String Theory, S-Duality, and Generalized Cohomology*, Nucl. Phys. **B715** (2005), 639-664, [arXiv:hep-th/0410293].

[KS05b] I. Kriz and H. Sati, *Type II string theory and modularity*, J. High Energy Phys. **0508** (2005) 038, [arXiv:hep-th/0501060].

[KX07] I. Kriz and H. Xing, *On effective F-theory action in type IIA compactifications*, Int. J. Mod. Phys. **A22** (2007), 1279-1300, [arXiv:hep-th/0511011].

[Kr73] W. M. Krueger, *Generalized Steenrod-Hopf Invariants for Stable Homotopy Theory*, Proc. Amer. Math. Soc. **39** (1973), 609-615, [jstor:2039603].

[La63] R. Lashof, *Poincaré duality and cobordism*, Trans. Amer. Math. Soc. **109** (1963), 257-277, [doi:10.1090/S0002-9947-1963-0156357-4].

[Lau08] C. Laughton, *Quasitoric manifolds and cobordism theory*, PhD Dissertation, U. Manchester, 2008, [ncatlab.org/nlab/files/LaughtonQuasitoric.pdf]

[La00] G. Laures, *On cobordism of manifolds with corners*, Trans. Amer. Math. Soc. 352 (2000) [doi:10.1090/S0002-9947-00-02676-3]

[Le99] W. Lerche, *On the Heterotic/F-Theory Duality in Eight Dimensions*, in *Progress in String Theory and M-Theory*, NATO Science Series, vol 564, Springer 2001, [arXiv:hep-th/9910207].

[LMS86] L. G. Lewis, P. May, and M. Steinberger *Equivariant stable homotopy theory*, Springer Lecture Notes in Mathematics vol. **1213**, 1986, [doi:10.1007/BFb0075778].

[LLP17] I. Limonchenko, Z. Lu, and T. Panov, *Calabi-Yau hypersurfaces and SU-bordism*, Proc. Steklov Inst. Math. **302** (2018), 270-278, [arXiv:1712.07350].

[Lu06] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies 170, Princeton University Press, 2009 [arXiv:math/0608040] [pup:8957].

[Lu08] J. Lurie *On the Classification of Topological Field Theories* Current Developments in Mathematics vol. 2008, 129-280, [doi:10.4310/CDM.2008.v2008.n1.a3], [arXiv:0905.0465].

[Lu09] J. Lurie, *A Survey of Elliptic Cohomology*, in: *Algebraic Topology*, Abel Symposia **4** (2009), 219-277, [doi:10.1007/978-3-642-01200-6_9].

[Lu10] J. Lurie, *Chromatic Homotopy Theory*, Lecture series 2010, [people.math.harvard.edu/∼lurie/252x.html].

[Lu17] J. Lurie, *Higher Algebra*, www.math.ias.edu/∼lurie/papers/HA.pdf

[MR87] M. Mahowald and D. Ravenel, *Towards a Global Understanding of the Homotopy Groups of Spheres*, in: S. Gitler (ed.), *The Lefschetz Centennial Conference: Proceedings on Algebraic Topology II*, Contemp. Math. 58, Amer. Math. Soc., 1987, [ISBN:978-0-8218-5063-3].

[MMS01] J. Maldacena, G. Moore and N. Seiberg, *D-Brane Instantons and K-Theory Charges*, J. High Energy Phys. **0111** (2001), 062, [arXiv:hep-th/0108100].

[Mat] A. Mathew, *Snaith's construction of complex K-theory* [http://math.uchicago.edu/~amathew/snaith.pdf]

[Ma77] P. May, *E_∞-Ring spaces and E_∞ ring spectra*, Lecture Notes in Mathematics 577, Springer, 1977, [cds:1690879].

[Ma99] P. May, *A concise course in algebraic topology*, University of Chicago Press, 1999, [ISBN:9780226511832].

[McVa19] J. McNamara and C. Vafa, *Cobordism Classes and the Swampland*, [arXiv:1909.10355].

[MiSc06] A. Miemiec and I. Schnakenburg, *Basics of M-Theory*, Fortsch. Phys. **54** (2006), 5-72, [arXiv:hep-th/0509137].

[Mil97] J. Milnor, *Topology from the differential viewpoint*, Princeton University Press, 1997, [ISBN:9780691048338].

[MM00] R. Minasian and G. Moore, *K-theory and Ramond-Ramond charge*, J. High Energy Phys. **9711** (1997), 002, [arXiv:hep-th/9710230].

[MV03] G. Mislin and A. Valette, *Proper Group Actions and the Baum-Connes Conjecture*, Advanced Courses in Mathematics CRM Barcelona, Springer 2003 [doi:10.1007/978-3-0348-8089-3]

[MoSe06] G. Moore and G. Segal, *D-branes and K-theory in 2D topological field theory*, [arXiv:hep-th/0609042].

[Mo14] G. Moore, *Physical Mathematics and the Future*, talk at Strings 2014, http://www.physics.rutgers.edu/~gmoore/PhysicalMathematicsAndFuture.pdf

[Mor13] F. Moràn, *Homology Stability for Spaces of Surfaces*, 2013, www.tdx.cat/bitstream/handle/10803/129511/FCM_THESIS.pdf

[MV96] D. R. Morrison and C. Vafa, *Compactifications of F-Theory on Calabi-Yau Threefolds – I*, Nucl. Phys. **B473** (1996), 74-92, [arXiv:hep-th/9602114].

[MT86] R. Mosher and M. Tangora, *Cohomology operations and applications in homotopy theory*, Dover Books, 2008, [ISBN13:9780486466644].

[Mü20] L. Müller, *Extended Functorial Field Theories and Anomalies in Quantum Field Theories*, [arXiv:2003.08217].

[My03] R. Myers, *Nonabelian Phenomena on D-branes*, Class. Quant. Grav. **20** (2003), 347-372, [arXiv:hep-th/0303072].

[Na03] M. Nakahara, *Geometry, Topology and Physics*, IOP 2003 [doi:10.1201/9781315275826]

[NPR09] H. Nastase, C. Papageorgakis and S. Ramgoolam, *The fuzzy S^2 structure of M2-M5 systems in ABJM membrane theories*, J. High Energy Phys. **0905** (2009), 123, [arXiv:0903.3966].

[NO73] H. B. Nielsen and P. Olesen, *Vortex-line models for dual strings*, Nucl. Phys. **B61** (1973), 45-61, [doi:10.1016/0550-3213(73)90350-7].

[Nov67] S. Novikov, *The methods of algebraic topology from the viewpoint of cobordism theories*, Izv. Akad. Nauk. SSSR. Ser. Mat. **31** (1967), 855-951, [ncatlab.org/nlab/files/Novikov67.pdf]

[Nov86] S. Novikov, *Topology I – General survey*, in: *Encyclopedia of Mathematical Sciences* Vol. 12, Springer 1986, [doi:10.1007/978-3-662-10579-5].

[OS99] K. Olsen, R. Szabo, *Constructing D-Branes from K-Theory*, Adv. Theor. Math. Phys. 3 (1999) 889-1025 [arXiv:hep-th/9907140]

[Ov02] B. Ovrut, *Lectures on Heterotic M-Theory*, TASI 2001, [doi:10.1142/9789812702821_0007], [arXiv:hep-th/0201032].

[Pa83] D. Page, *Classical stability of round and squashed seven-spheres in eleven-dimensional supergravity*, Phys. Rev. **D28** (1983), 2976, [doi:10.1103/PhysRevD.28.2976].

[Pe56] F. P. Peterson, *Some Results on Cohomotopy Groups*, Amer. J. Math. **78** (1956), 243-258, [jstor:2372514].

[Po36] L. Pontrjagin, *Sur les transformations des sphères en sphères*, in: *Comptes Rendus du Congrès International des Mathématiques*, Oslo 1936, [ncatlab.org/nlab/files/PontrjaginSurLesTransformationDesSpheres.pdf]

[Po38] L. Pontrjagin, *Classification of continuous maps of a complex into a sphere*, Communication I, Doklady Akademii Nauk SSSR **19** (1938), 147-149.

[Po55] L. Pontrjagin, *Smooth manifolds and their applications in Homotopy theory*, Trudy Mat. Inst. im Steklov, No 45, Izdat. Akad. Nauk. USSR, Moscow, 1955, (AMS Translation Series 2, Vol. 11, 1959), [doi:10.1142/9789812772107_0001].

[PTW15] K. Pilch, A. Tyukov and N. Warner, *Flowing to Higher Dimensions: A New Strongly-Coupled Phase on M2 Branes*, J. High Energy Phys. **11** (2015), 170, [arXiv:1506.01045].

[Pr15] D. Prins, *On flux vacua, SU-structures and generalised complex geometry*, Lyon, 2015, [tel:01280717], [arXiv:1602.05415].

[Qu14] G. Quick, *Advanced algebraic topology*, 2014, [folk.ntnu.no/gereonq/Math231br.html].

[Qu71] D. Quillen, *Elementary Proofs of Some Results of Cobordism Theory Using Steenrod Operations*, Adv. Math. **7** (1971), 29-56, [doi:10.1016/0001-8708(71)90041-7].

[Ra78] D. Ravenel, *A Novice's guide to the Adams-Novikov spectral sequence*, in: *Geometric Applications of Homotopy Theory II* Lecture Notes in Mathematics, vol 658, Springer 1978, [doi:10.1007/BFb0068728].

[Ra84] D. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math. **106** (1984), 351-414, [jstor:2374308].

[Ra86] D. Ravenel, *Complex cobordism and stable homotopy groups of spheres* Academic Press Orland, 1986; reprinted as: AMS Chelsea Publishing, vol. 347, 2004, [ISBN:978-0-8218-2967-7].

[Re06] S. Reffert, *Toroidal Orbifolds: Resolutions, Orientifolds and Applications in String Phenomenology*, PhD Thesis, U. Munich, 2006, [spire:725406], [arXiv:hep-th/0609040].

[Ri20] B. Richter, *From categories to homotopy theory*, Cambridge University Press, 2020, [doi:10.1017/9781108855891].

[Rie14] E. Riehl, *Categorical Homotopy Theory* Cambridge University Press, 2014, [doi:10.1017/CBO9781107261457].

[Ro04] J. Rognes, *The sphere spectrum*, 2004, [ncatlab.org/nlab/files/RognesSphereSpectrum.pdf]

[Sad13] R. Sadykov, *Elements of Surgery Theory*, Lecture notes 2013, [https://www.math.ksu.edu/~sadykov/LectureNotes/SurgeryTheory.pdf]

[Sa05a] H. Sati, *M-theory and Characteristic Classes*, J. High Energy Phys 0508 (2005) 020, [doi:10.1088/1126-6708/2005/08/020], [arXiv:hep-th/0501245].

[Sa05b] H. Sati, *Flux Quantization and the M-Theoretic Characters*, Nucl. Phys. **B727** (2005), 461-470, [doi:10.1016/j.nuclphysb.2005.09.008], [arXiv:hep-th/0507106].

[Sa06a] H. Sati, *Duality symmetry and the form fields of M-theory*, J. High Energy Phys. **0606** (2006), 062, [doi:10.1088/1126-6708/2006/06/062], [arXiv:hep-th/0509046].

[Sa06b] H. Sati, *The Elliptic curves in gauge theory, string theory, and cohomology*, J. High Energy Phys. **0603** (2006), 096, [doi:10.1088/1126-6708/2006/03/096], [arXiv:hep-th/0511087].

[Sa08] H. Sati, *An approach to anomalies in M-theory via KSpin*, J. Geom. Phys. **58** (2008), 387-401, [doi:10.1016/j.geomphys.2007.11.010], [arXiv:0705.3484].

[Sa10] H. Sati, *Geometric and topological structures related to M-branes*, in: R. Doran, G. Friedman and J. Rosenberg (eds.), *Superstrings, Geometry, Topology, and C^*-algebras*, Proc. Symp. Pure Math. **81**, AMS, Providence, 2010, pp. 181-236, [doi:10.1090/pspum/081], [arXiv:1001.5020].

[Sa11a] H. Sati, *Geometric and topological structures related to M-branes, Part II: Twisted* String *and* String^c *structures*, J. Aust. Math. Soc. 90 (2011), 93-108, [doi:10.1017/S1446788711001261], [arXiv:1007.5419].

[Sa11b] H. Sati, *Twisted topological structures related to M-branes*, Int. J. Geom. Meth. Mod. Phys. **8** (2011), 1097-1116, [doi:10.1142/S0219887811005567], [arXiv:1008.1755].

[Sa11c] H. Sati, *Constraints on heterotic M-theory from s-cobordism*, Nucl. Phys. **B853** (2011), 739-759, [doi:10.1016/j.nuclphysb.2011.08.006], [arXiv:1102.1171].

[Sa13] H. Sati, *Framed M-branes, corners, and topological invariants*, J. Math. Phys. **59** (2018), 062304, [doi:10.1063/1.5007185], [arXiv:1310.1060].

[Sa14] H. Sati, *M-Theory with Framed Corners and Tertiary Index Invariants*, SIGMA **10** (2014), 024, 28 pages, [doi:10.3842/SIGMA.2014.024], [arXiv:1203.4179] [hep-th].

[SS19a] H. Sati and U. Schreiber, *Equivariant Cohomotopy implies orientifold tadpole cancellation*, J. Geom. Phys. **156** (2020) 103775, [10.1016/j.geomphys.2020.103775], [arXiv:1909.12277].

[SS19b] H. Sati and U. Schreiber, *Differential Cohomotopy implies intersecting brane observables via configuration spaces and chord diagrams*, Adv. Theor. Math. Phys. **26** 4 (2022) 957-1051 [arXiv:1912.10425] [doi:10.4310/ATMP.2022.v26.n4.a4].

[SS20a] H. Sati and U. Schreiber, *Twisted Cohomotopy implies M5-brane anomaly cancellation*, Lett. Math. Phys. **111** 120 (2021) [arXiv:2002.07737] [doi:10.1007/s11005-021-01452-8].

[SS19] H. Sati and U. Schreiber, *Equivariant Cohomotopy implies orientifold tadpole cancellation*, J. Geom. Phys. **156** (2020) 103775, [arXiv:1909.12277] [doi:10.1016/j.geomphys.2020.103775].

[SS20b] H. Sati and U. Schreiber, *Proper Orbifold Cohomology*, [arXiv:2008.01101].

[SS20c] H. Sati and U. Schreiber, *The character map in equivariant twistorial Cohomotopy implies the Green-Schwarz mechanism with heterotic M5-branes*, [arXiv:2011.06533].

[SSS12] H. Sati, U. Schreiber, and J. Stasheff, *Twisted differential string and fivebrane structures*, Commun. Math. Phys. **315** (2012), 169-213, [doi:10.1007/s00220-012-1510-3], [arXiv:0910.4001].

[SUZ19] H. Sati, S.-M. Udrescu, and E. Zogla, *Computations of characteristic classes and genera: A practical toolkit for beginners and practitioners*, Grad. J. Math. **3** (2018), 60-93.

[SP17] C. Schommer-Pries, *Nilpotence of the stable Hopf map via framed cobordism*, 2017, [MO:a/218053/381].

[Sch13] U. Schreiber, *Differential cohomology in a cohesive infinity-topos*, [arXiv:1310.7930] [math-ph].

[ScSh10] M. Schütt and T. Shioda, *Elliptic surfaces*, Adv. Stud. Pure Math. Algebraic Geometry in East Asia, Mathematical Society of Japan, 2010, 51-160, [euclid:aspm/1543085637], [arXiv:0907.0298].

[Sch96] J. Schwarz, *The Second Superstring Revolution*, in: *Proc. COSMION 96: 2nd International Conference on Cosmo Particle Physics*, Moscow 1996, [spire:969846], [arXiv:hep-th/9607067].

[Sch97] J. Schwarz, *Lectures on Superstring and M Theory Dualities*, Nucl. Phys. Proc. Suppl. **B55** (1997), 1-32, [arXiv:hep-th/9607201].

[Schw01] J. H. Schwarz, *TASI Lectures on Non-BPS D-Brane Systems*, pp. 809-846 in:
J. Harvey, S. Kachru, E. Silverstein (eds.) *Strings, Branes and Gravity*, World Scientific 2001
[arXiv:hep-th/9908144] [doi:10.1142/9789812799630_0010]

[Se74] G. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293-312,
[doi:10.1016/0040-9383(74)90022-6].

[Se04] G. Segal, *The definition of conformal field theory* in: *Topology, geometry and quantum field theory*, London Math. Soc. Lect. Note Ser., Vol. 308, Cambridge University Press, 2004, pp. 421-577,
[doi:10.1017/CBO9780511526398].

[Sen96] A. Sen, *F-theory and Orientifolds*, Nucl. Phys. **B475** (1996), 562-578, [arXiv:hep-th/9605150].

[Sen97] A. Sen, *A Note on Enhanced Gauge Symmetries in M- and String Theory*, J. High Energy Phys. **9709** (1997), 001, [arXiv:hep-th/9707123].

[Sen98] A. Sen, *Tachyon Condensation on the Brane Antibrane System*, J. High Energy Phys. **9808** (1998), 012, [arXiv:hep-th/9805170].

[Sen05] A. Sen, *Tachyon Dynamics in Open String Theory*, Int. J. Mod. Phys. **A20** (2005), 5513-5656,
[arXiv:hep-th/0410103].

[SW08] D. Sinha and B. Walter, *Lie coalgebras and rational homotopy theory II: Hopf invariants*, Trans. Amer. Math. Soc. **365** (2013), 861-883, [doi:10.1090/S0002-9947-2012-05654-6], [arXiv:0809.5084].

[Sm71] L. Smith, *On characteristic numbers of almost complex manifolds with framed boundaries*, Topology Volume 10, Issue 3, 1971, Pages 237-256 [doi:10.1016/0040-9383(71)90008-5]

[Sm73] L. Smith, *The Todd character and the integrality theorem for the Chern character*, Illinois J. Math. **17** (1973), 301-310, [euclid:ijm/1256051760].

[Sn79] V. Snaith, *Algebraic Cobordism and K-theory*, Mem. Amer. Math. Soc. no 221 (1979)

[Sp49] E. Spanier, *Borsuk's Cohomotopy Groups*, Ann. Math. **50** (1949), 203-245, [jstor:1969362].

[Sp66] E. Spanier, *Algebraic topology*, Springer, 1966, [doi:10.1007/978-1-4684-9322-1].

[Spi13] M. J. Spiegel, *K-Theory of Intersection Spaces*, 2013, [doi:10.11588/heidok.00015738].

[St49] N. Steenrod, *Cohomology Invariants of Mappings*, Ann. of Math. **50** (1949), 954-988, [jstor:1969589].

[St67] N. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. **14** (1967), 133-152,
[euclid:mmj/1028999711].

[St72] N. Steenrod, *Cohomology operations, and obstructions to extending continuous functions*, Adv. Math. **8** (1972), 371-416, [doi:10.1016/0001-8708(72)90004-7].

[ST11] S. Stolz and P. Teichner, *Supersymmetric field theories and generalized cohomology* in: H. Sati, U. Schreiber (eds.), *Mathematical Foundations of Quantum Field and Perturbative String Theory*, Proceedings of Symposia in Pure Mathematics, AMS, 2011, [doi:10.1090/pspum/083], [arXiv:1108.0189].

[Sto68] R. Stong, *Notes on Cobordism theory*, Princeton University Press, 1968, [ISBN:9780691649016].

[Su74] D. Sullivan, *Genetics of homotopy theory and the Adams conjecture*, Ann. of Math. **100** (1974), 1-79, [doi:10.2307/1970841].

[TK06] D. Tamaki and A. Kono, *Generalized Cohomology*, American Mathematical Society, 2006, [ISBN:978-0-8218-3514-2].

[Th54] R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17-86, [doi:10.1007/BF02566923].

[Tod62] H. Toda, *Composition Methods in Homotopy Groups of Spheres*, Annals of Mathematics Studies Volume 49, Princeton University Press, 1962, [jstor:j.ctt1bgzb5t].

[To09] D. Tong, *Quantum Vortex Strings: A Review*, Annals Phys. **324** (2009), 30-52, [arXiv:0809.5060].

[Va96] C. Vafa, *Evidence for F-theory*, Nucl. Phys. **B469** (1996), 403-418, [arXiv:hep-th/9602022].

[Wa04] G. Walschap, *Metric Structures in Differential Geometry*, Springer, 2004, [doi:10.1007/978-0-387-21826-7].

[WX10] G. Wang and Z. Xu, *A survey of computations of homotopy groups of Spheres and Cobordisms*, 2010, [ncatlab.org/nlab/files/WangXuHomotopyGroupsOfSpheres2010.pdf]

[Wh47] J. H. C. Whitehead, *An expression of Hopf's invariant as an integral*, Proc. Nat. Acad. Sci. USA **33** (1947), 117-123, [jstor:87688].

[Wh62] G. Whitehead, *Generalized homology theories*, Trans. Amer. Math. Soc. **102** (1962), 227-283, [jstor:1993676].

[Wil13] D. Wilson, *Spectral Sequences from Sequences of Spectra: Towards the Spectrum of the Category of Spectra*, lecture at 2013 Pre-Talbot Seminar, March 2013, [ncatlab.org/nlab/files/DylanWilsonOnANSS.pdf]

[Wir12] K. Wirthmüller, *Vector Bundles and K-theory*, 2012, [wirthmueller-vector-bundles-and-k-theory.pdf]

[Wi83] E. Witten, *Global aspects of current algebra*, Nucl. Phys. **B223** (1983), 422-432, [spire:13234].

[Wi95] E. Witten, *String Theory Dynamics In Various Dimensions*, Nucl. Phys. **B443** (1995), 85-126, [arXiv:hep-th/9503124].

[Wi96] E. Witten, *Strong Coupling Expansion Of Calabi-Yau Compactification*, Nucl. Phys. **B 471** (1996), 135-158, [arXiv:hep-th/9602070].

[Wi97a] E. Witten, *On Flux Quantization In M-Theory And The Effective Action*, J. Geom. Phys. **22** (1997), 1-13, [arXiv:hep-th/9609122].

[Wi97b] E. Witten, *Five-Brane Effective Action In M-Theory*, J. Geom. Phys. **22** (1997), 103-133, [arXiv:hep-th/9610234].

[Wi98] E. Witten, *D-Branes And K-Theory*, J. High Energy Phys. **9812** (1998), 019, [arXiv:hep-th/9810188].

[Wi01] E. Witten, *Overview Of K-Theory Applied To Strings*, Int. J. Mod. Phys. **A16** (2001), 693-706, [arXiv:hep-th/0007175].

[Wi10] E. Witten, *Geometric Langlands From Six Dimensions*, in: *A Celebration of the Mathematical Legacy of Raoul Bott*, CRM Proceedings & Lecture Notes Vol. 50, Amer. Math. Soc., Providence, RI, 2010, [arXiv:0905.2720].

[Wi19] E. Witten, in: G. Farmelo, *The Universe Speaks in numbers*, interview 5, 2019, [grahamfarmelo.com/the-universe-speaks-in-numbers-interview-5].

Hisham Sati, *Mathematics, Division of Science, New York University Abu Dhabi, UAE.*
hsati@nyu.edu

Urs Schreiber, *Mathematics, Division of Science, New York University Abu Dhabi, UAE*
us13@nyu.edu