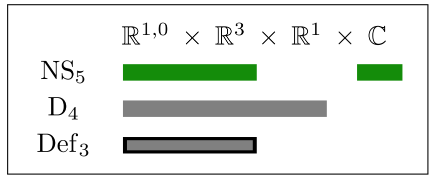
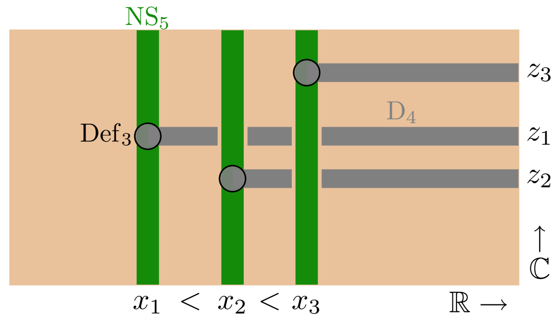
2.3.4 Quantum $\mathrm{M}_5 \perp \mathrm{M}_5$-branes via Gelfand-Raikov theorem.

Now consider moduli of the 3-cohomotopical H_3 flux, which in §2.1 we saw appears on the ambient space[20] of 5-branes. For the moduli space of codim=2 $\mathrm{D}_4 \perp \mathrm{NS}_5 \overset{\mathrm{IIA/M}}{\rightsquigarrow} \mathrm{M}_5 \perp \mathrm{M}_5$ defects inside 5-branes [CHKS21, Fig. 1 & 3][SS23-Dfc1, pp. 28], we are to consider the situation (119) for $n = 3,\ d = 7,\ p = 4$, which yields configurations of ordered points in the transverse $\mathbb{C}$-plane.

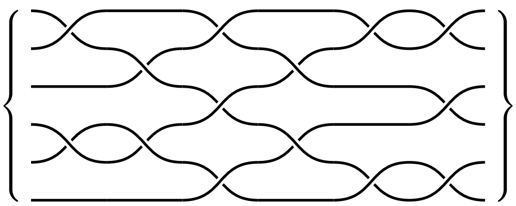




To understand the light-cone quantization (§2.2) of these brane moduli, observe that the homotopy type of this configuration space is the classifying space of the pure[21] **braid group** PBr (cf. [MySS23, pp. 12]), being the *group of motions* of the codim = 2 defects (Def_3) around each other in the transverse M_5-worldvolume $\mathbb{C}$.

$$\pi_1\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}(\mathbb{C})\Big) \simeq \mathrm{PBr}(n)$$

$$=$$

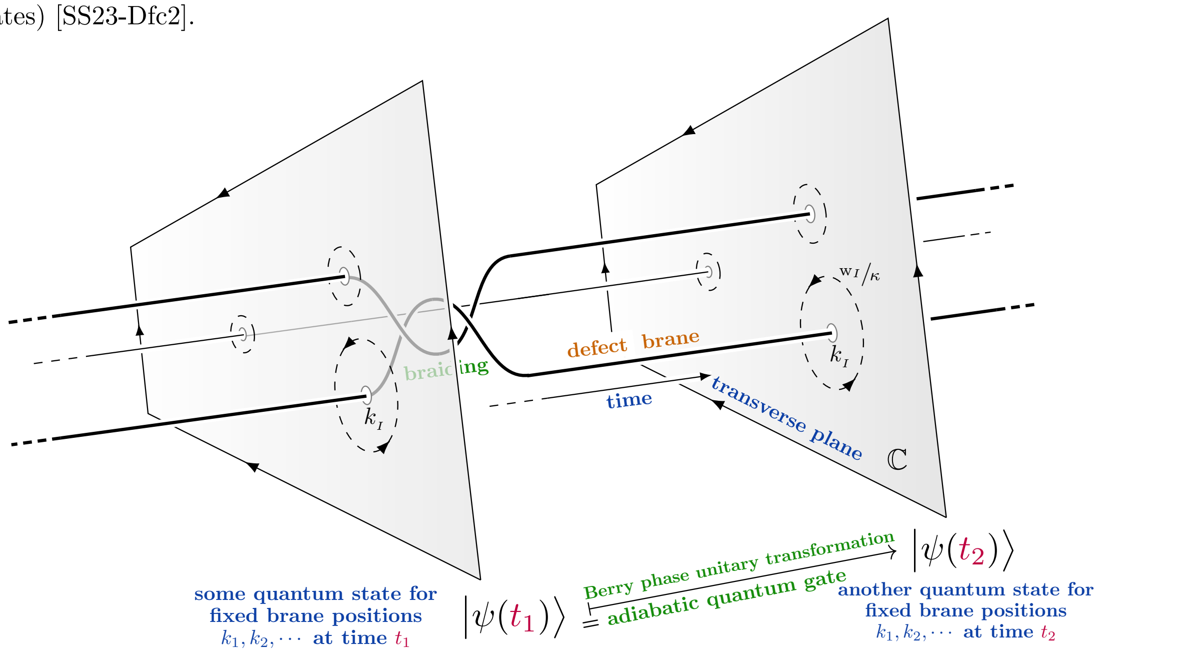


This implies that the light-cone quantum observable algebra (114) is the pure braid group algebra.

$$
\begin{aligned}
&\mathrm{QObsrvbls}_{N\mathrm{M}_5 \perp \mathrm{M}_5} \\
&\equiv H_\bullet\big(\Omega \underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C})\big) \simeq H_\bullet\big(\Omega B \mathrm{PBr}(N)\big) \\
&\simeq H_\bullet\big(\mathrm{PBr}(N)\big) \simeq \underbrace{\mathbb{C}\big[\mathrm{PBr}(N)\big]}_{\text{group algebra}}
\end{aligned}
$$

The Quantum states. Thus with the *Gelfand-Raikov theorem* [Di77, Thm. 13.4.5.(ii)] it follows that the light-cone quantum states are given by unitary pure braid representations, hence are **anyonic states** ("topologically ordered" quantum states) [SS23-Dfc2].

$$
\mathrm{QStates}_{N\mathrm{M}5 \perp \mathrm{M}5} \simeq \left\{ \rho : \mathbb{C}\big[\mathrm{PBr}(N)\big] \to \mathbb{C} \;\middle|\; \begin{array}{l} g \mapsto \rho(g) = \langle \psi | U(g) | \psi \rangle \\ \text{for } U \in \mathrm{PBr}(N) \circlearrowright \mathcal{H},\ |\psi\rangle \in \mathcal{H} \end{array} \right\}.
$$



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[20] Hence with the M-theory circle included, the ambient space of the 5-brane on which we consider the H_3-flux is 8-dimensional.
[21] Our figures show im-pure braids, just for ease of illustration.