Fractional Quantum Hall States with Conformal Field Theories

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Abstract: Fractional quantum Hall (FQH) states are topological phases with anyonic excitations that form representations of the braid group. In these notes, we give a brief introduction to applications of conformal field theories (CFT) to FQH systems. The wave functions of different FQH states are shown to be conformal blocks of rational CFT, and the anyonic excitations correspond to their primary fields. The fusion and braiding properties of the Pfaffian state and the idea of bulk-edge correspondence are discussed with the aid of CFT.

Contents

1 Introduction

Symmetry, correlation, and topology are three principal themes in contemporary condensed matter theories. No area can exemplify these themes better than the rapidly developing theories of topological phases, especially the phases with intrinsic topological orders. Quasi-particle excitations with fractional charges and fractional statistics are a key characterization of a topologically ordered phase. Famous examples include fractional quantum Hall (FQH) systems, quantum spin liquids, and Kitaev's toric code. These fractional excitations can form a representation of the braid group. They are called abelian anyons if the representation is one dimensional. This type of anyons emerges in most FQH states. Nonabelian anyons, which form a nonabelian representation of the braid group, have been thought to be a cornerstone of fault-tolerant quantum computing, since their topological braiding properties are insensitive to local perturbations [1]. The $\nu = 5/2$ FQH state is generally believed to carry nonabelian anyons. Another well-known example is 2D topological p-wave superconductors which carry Majorana zero modes.

FQH states can be studied using rational conformal field theories (RCFT). These are the conformal field theories (CFT) with a finite number of primary fields. The connection between them was first studied explicitly in the context of FQH systems by Moore and Read [2]. The Moore-Read Pfaffian state is one of the promising candidates for the $\nu = 5/2$

FQH state.[1] The wave functions of these states can be expressed as holomorphic conformal blocks of the $U(1)_2 \times \mathbb{Z}_2$ CFT. The fractional statistics of the nonabelian anyons can be studied using the tools from CFT. Other states with nonabelian anyonic excitation can also be constructed by tensoring or cosetting Wess-Zumino-Witten (WZW) models. For example, Z_3-Read-Rezayi states, which carry Fibonacci anyons, can be constructed by the $SU(2)_3/U(1)$-coset WZW model.

In these notes, we give a short introduction to the application of CFT to FQH states. We first briefly review the FQH effect, anyons, and the braid group. Then we discuss the application of RCFT to general FQH states based on conformal blocks before we move on to the fusion and braiding of nonabelian anyons. Since our focus is on the intuitive physical picture, a brief summary of the more technical CFT language is given in the appendix. We also comment on the bulk-edge correspondence, and then end the notes by pointing out some directions of recent progress in research related to nonabelian anyons.

2 Fractional Quantum Hall States and Anyons

2.1 Fractional Quantum Hall States

A reputed signature of the quantum Hall effect (QHE) is the quantization in the Hall conductance of a two-dimensional electron gas in a strong magnetic field: $\sigma = \nu e^2/h$, where ν is the filling factor of Landau levels. For integral quantum Hall systems, $\nu = 1, 2, 3, \dots$ As for FQH systems, $\nu = 1/3, 2/3, 2/5, \dots$, fractions with odd integral denominators, with the exception of $\nu = 5/2$. This FQH state is exactly where peculiar nonabelian anyons emerge.

The basic physics of a 2D electron gas under an applied magnetic field can be captured by the following Hamiltonian

$$H = \frac{1}{2m}(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})^2 + V_{ee}, \tag{2.1}$$

where we have assumed that the magnetic field is strong enough so that all electrons are polarized.

On the one hand, the integral QHE can be explained just using the first term. Under the symmetric gauge, the eigenfunctions of the Hamiltonian are given by

$$\Phi_n = \exp\left(-\frac{|z|^2}{4}\right)(\tilde{D}_z)^n f(z) \tag{2.2}$$

where n indexes the Landau level, $\tilde{D}_z = \partial_z - \frac{1}{2}\bar{z}$, and $f(z)$ is any holomorphic function. We have taken the magnetic length $l_B = \sqrt{|c\hbar/eB|} = 1$. Every Landau level has the same degeneracy. The N-electron wave function of the $\nu = 1$ state takes the form

$$\Phi(z_1, \dots, z_N) = \prod_{1\le i<j\le N}(z_i - z_j)\exp\left(-\sum_i \frac{|z_i|^2}{4}\right), \tag{2.3}$$

[1] Other more recent candidates include the anti-Pfaffian state [3, 4] and the particle-hole symmetric Pfaffian state [5].

a Slater determinant of concentric wave functions of different radii. The integral QHE is very interesting in its own right, but they do not support anyonic excitations.

On the other hand, the FQHE is strictly a result of strong correlations between electrons; in other words, V_{ee} is an essential player in the game. With a filling factor $\nu = 1/m$, the ground state is well approximated by the celebrated Laughlin wave function

$$\Phi_m(\{z_i\}) = \prod_{1\le i<j\le N} (z_i - z_j)^m \exp\left(-\sum_i \frac{|z_i|^2}{4}\right), \tag{2.4}$$

regardless of the details in interaction. The quasiparticle excitations are gapped. An elementary excitation is a quasihole created by an adiabatic insertion of a flux quantum hc/e, say, at η. Then the wave function is given by

$$\Phi_m^+(\eta; \{z_i\}) = \prod_{j=1}^{N} (z_i - \eta)\Phi_m(\{z_i\}). \tag{2.5}$$

The wave function with two quasiholes was shown by Halperin to be

$$\Phi_m^{++}(u, w; \{z_i\}) = N_0(u - w)^{1/m} \prod_{j=1}^{N} [(u - z_j)(w - z_j)] \exp\left(-\frac{1}{4m}(|u|^2 + |w|^2)\right) \Phi_m(\{z_i\}), \tag{2.6}$$

where N_0 is normalization constant. By using the plasma analogy, it can be shown that quasiholes carry a fractional charge of $q = +e/m$. Combining m quasiholes yields a hole of positive e (as opposed to an electron). It is also known that when two quasiholes are exchanged, they gain a nontrivial phase $\theta = \pi/m$ in the wave function, i.e.,

$$\Phi_m^{++}(u, w; \{z_i\}) = \exp\left(\frac{\mathrm{i}\pi}{m}\right) \Phi_m^{++}(w, u; \{z_i\}). \tag{2.7}$$

This feature can already be seen from the factor $(u - w)^{1/m}$ in Eq. 2.6: Since $m > 1$, transporting a quasiparticle around another will encounter the nonlocal branch cut, a *monodromy*! This is an indication that the quasihole excitations are anyons with nontrivial braiding properties, which we will discuss a bit more now.

2.2 Anyons and the Braid Group

Elementary particles in nature are usually either bosons or fermions, but it is also known that anyons can exist in $(2+1)$D. When the path of circling one particle around another cannot be deformed continuously to a point, there may be a nontrivial Berry phase associated with it from the exchange (or the rotation – two exchanges) similar to that in Eq. 2.7.

Let us be more concrete and introduce the braid group. Imagine that we have N particles with a thread attached to each one of them, and we can swap any two arbitrarily. Unlike pure permutations, however, now we have to distinguish clockwise rotations from counterclockwise ones (see Fig. 1). It is easy to see that these operations form a group, the braid group B_N. This group is generated by $\sigma_i, (1 \le i \le N-1)$, operations that exchange particle i and $i+1$ counterclockwisely with relations indicated in Fig. 1:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for} \quad |i-j| \ge 2, \tag{2.8}$$

and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for} \quad 1 \le i \le N-1. \tag{2.9}$$

If a many-body state $\phi(z_1, \dots, z_i, \dots, z_j, \dots, z_N) \in \mathcal{H}$ transforms like Eq. 2.7, then they form a one-dimensional representation of the braid group: $\rho : B_N \to \mathrm{Aut}(\mathcal{H})$, with

$$\rho(\sigma_{ij})\phi(z_1, \dots, z_i, \dots, z_j, \dots, z_N) = \exp(im\theta)\phi(z_1, \dots, z_j, \dots, z_i, \dots, z_N). \tag{2.10}$$

Here m is the net number of times that the operation σ_{ij} exchanges particle i with particle j counterclockwisely. $\theta = 0, \pi$ correspond to a system of bosons and of fermions, respectively. Other θ's yields the so-called *fractional* statistics, and the particles are abelian anyons because in this case the representation is abelian. More generally, we can consider a multi-dimensional representation

$$\rho(\sigma_{ij})\phi_p(z_1, \dots, z_i, \dots, z_j, \dots, z_N) = \sum_q B_{pq}\phi_q(z_1, \dots, z_i, \dots, z_j, \dots, z_N). \tag{2.11}$$

Now B is a matrix representation that can be nonabelian. The states of the representation necessarily correspond to nonabelian anyons, the unique properties of which are what will be discussed more.

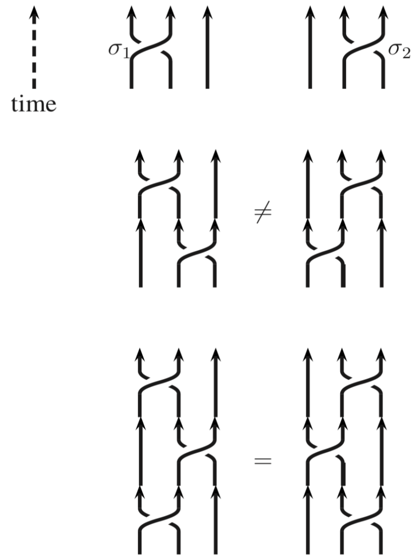

Figure 1. Braid group relations. (a) Two independent operations. (2) $\sigma_1\sigma_2 \neq \sigma_2\sigma_1$. (3) $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.

3 Fractional Quantum Hall States and Conformal Field Theories

There are many ways to analyze a FQH system (see, e.g., Fradkin [6] and Tong [7] for more discussions). In this section, we will only discuss FQH states from a RCFT perspecttive. The approach was first initiated by Moore and Read [2] and it is particularly useful in discussing nonabelian FQH states. We provide a short summary of the basic information of RCFT needed in Appendix.

3.1 Conformal Blocks

Moore and Read's insight came from the realization that Laughlin/Halperin states can be recast as expectation values of the CFT of a compactified chiral boson. Indeed, the wave functions in Eq. 2.4 and Eq. 2.5 can be written, respectively, as

$$\Phi_m(\{z_i\}) = \left\langle \left(\prod_{i=1}^{N} \mathrm{e}^{\mathrm{i}\sqrt{m}\varphi(z_i)} \right) \exp\left(-\int d^2z' \sqrt{m}\rho_0\varphi(z') \right) \right\rangle \tag{3.1}$$

and

$$\Phi_m^+(z_0;\{z_i\}) = \left\langle \exp\left(\frac{\mathrm{i}}{\sqrt{m}}\varphi(z_0) \right) \left(\prod_{i=1}^{N} \mathrm{e}^{\mathrm{i}\sqrt{m}\varphi(z_i)} \right) \exp\left(-\int d^2z' \sqrt{m}\rho_0\varphi(z') \right) \right\rangle, \tag{3.2}$$

where $\rho_0 = 1/2\pi m$ is the density of a uniform neutralizing charge background and $\varphi(z)$ is a chiral bosonic field in 2 dimensional Euclidean space with the correlation

$$\langle \varphi(z_1)\varphi(z_2) \rangle = -\ln(z_1 - z_2). \tag{3.3}$$

Its associated vertex operator $V_e = \exp[\mathrm{i}\sqrt{m}\varphi(z_i)]$ creates an electron, and the vertex operator $V_h = \exp[\mathrm{i}\varphi(z_i)/\sqrt{m}]$ creates a quasihole. The Halperin state, Eq. 2.6, can be obtained if one more quasiparticle vertex operator V_h is inserted into $\Phi_m^+(z_0;\{z_i\})$.

As we have mentioned, the charge of a quasihole in this case is e/m, and therefore m quasiparticles can fuse into a hole of positive charge e, consistent with the fact $V_e = (V_h)^m$. The hole of a positive charge e can be regarded as part of the condensate formed by the underlying electrons. We may regard them as the identity operator in the fusion process. As a result, $(V_h)^i, (i = 0, 1, \ldots, m-1)$ are all the distinct primary operators that we have, which means that we have a RCFT with m primary fields. Moreover, the bosonic field has to be compactified on a circle with radius $R = 1/\sqrt{m}$ since $\varphi \to \varphi + 2\pi(1/\sqrt{m})$ leaves V_e invariant. Identifying the right-hand side of Eq. 3.2 and Eq. 3.3 as a holomorphic conformal block of the $\mathrm{U}(1)_m$ RCFT, we succeed in connecting some FQH states to a RCFT of compacted bosons. Is the converse statement also true? In other words, given a plausible RCFT, can we also construct a different FQH state? Moore and Read shown that this may be possible.

The Moore-Read Pfaffian state, constructed from the Laughlin state, has the form

$$\Phi_{\mathrm{MR}}(\{z_i\}) = \mathrm{Pf}\left(\frac{1}{z_i - z_j} \right) \Phi_m(\{z_i\}). \tag{3.4}$$

The Pfaffian is the the fully antisymmetric product of an square skew matrix, which in this case is given by

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j} \right) = \frac{1}{2^{N/2}(N/2)!} \sum_P \mathrm{sgn}(P) \prod_{r=1}^{N/2} \frac{1}{z_{P(2r-1)} - z_{P(2r)}}. \tag{3.5}$$

We assumed N to be even. Note that if m is odd, Φ_{MR} is totally symmetric and it describes a bosonic FQH state. To describe a fermionic FQH state, m has to be even.

We know from CFT that the Pfaffian is simply the correlation function of free chiral Majorana fermion fields

$$\langle \chi(z_1) \ldots \chi(z_N) \rangle = \mathrm{Pf}\left(\frac{1}{z_i - z_j} \right). \tag{3.6}$$

Therefore, the Pfaffian state is the holomorphic conformal block of the CFT of a free compactified boson field and a free Majorana fermion.

3.2 Nonabelian Anyons

Now let us consider the excitations on the Pfaffian state. The anyons in the state include not only the usual abelian ones, but also Ising anyons, a name received because of their connection with the 2D Ising model (see, e.g., [8]). Indeed, imposing the locality of the operators with respect to the electron operator, which in this case is given by $\phi_e(z) = \chi(z)\exp(\mathrm{i}\sqrt{m}\varphi(z))$, we can show there are four types of allowed operators:

1. The identity (vacuum), I;
2. the σ particle (nonabelian anyon) $\sigma(z)\exp(\mathrm{i}\varphi(z)/2\sqrt{m})$ with charge $e/2m$;
3. the Majorana fermion $\chi(z)$ with a neutral excitation;
4. The Laughlin quasiparticle (vortex) $\exp(\mathrm{i}\varphi(z)/\sqrt{m})$ with charge e/m and abelian fractional statistics $\theta = \pi/m$.

It is found the the usual excitation state

$$\Phi_m^{+}(\eta;\{z_i\}) = \prod_{j=1}^{N}(z_i-\eta)\mathrm{Pf}\left(\frac{1}{z_i-z_j}\right)\Phi_m(\{z_i\}) \tag{3.7}$$

can be interpreted as an excited state with two quasiparticle

$$\Phi_m^{2\mathrm{qh}}(\eta_1,\eta_2;\{z_i\}) = \mathrm{Pf}\left(\frac{(z_i-\eta_1)(z_j-\eta_2)+(z_j-\eta_1)(z_i-\eta_2)}{z_i-z_j}\right)\Phi_m(\{z_i\}), \tag{3.8}$$

for, when $\eta_1 = \eta_2 = \eta$, we retrieve the original state in Eq. 3.7. Now these quasiparticles carry charge $e/2m$ each. They are simply the σ particles mentioned above, and the wave function $\Phi_m^{2\mathrm{qh}}(\eta_1,\eta_2;\{z_i\})$ indeed can be written as a conformal block of the Ising model together with a chiral bosonic CFT:

$$\Phi_m^{2\mathrm{qh}}(\eta_1,\eta_2;\{z_i\}) = \langle\sigma(\eta_1)\sigma(\eta_2)\chi(z_1)\ldots\chi(z_N)\rangle_{\text{Ising CFT}}$$
$$\times\left\langle \mathrm{e}^{\frac{\mathrm{i}}{2\sqrt{m}}\varphi(\eta_1)}\mathrm{e}^{\frac{\mathrm{i}}{2\sqrt{m}}\varphi(\eta_2)}\left(\prod_{i=1}^{N}\mathrm{e}^{\mathrm{i}\sqrt{m}\varphi(z_i)}\right)\exp\left(-\int d^2z'\sqrt{m}\rho_0\varphi(z')\right)\right\rangle_{\mathrm{U}(1)_m}. \tag{3.9}$$

Given four quasiparticles, an excited state can be constructed in three different ways because of different pairing. For example, we can construct a state like

$$\Phi_m^{(12)(34)}(\eta_1,\eta_2,\eta_3,\eta_4;\{z_i\}) = \mathrm{Pf}_{(12),(34)}(z)\Phi_m(\{z_i\}), \tag{3.10}$$

where

$$\mathrm{Pf}_{(12),(34)}(z) = \mathrm{Pf}\left(\frac{(z_i-\eta_1)(z_i-\eta_2)(z_j-\eta_3)(z_j-\eta_4)+(i\leftrightarrow j)}{z_i-z_j}\right). \tag{3.11}$$

We can similarly construct $\mathrm{Pf}_{(13),(24)}(z)$ and $\mathrm{Pf}_{(14),(23)}(z)$ as well. These three states are not totally linearly independent since one can show that

$$\mathrm{Pf}_{(12),(34)} - \mathrm{Pf}_{(14),(23)} = \frac{\eta_{14}\eta_{23}}{\eta_{13}\eta_{24}}(\mathrm{Pf}_{(12),(34)} - \mathrm{Pf}_{(13),(24)}) \tag{3.12}$$

where $\eta_{ij} = \eta_i - \eta_j$. One can actually show that there are 2^{n-1} linearly independent states with $2n$ quasiholes [9]. To confirm that these anyons are nonabelian anyons, let us see how they fuse and braid.

3.3 Fusion and Braiding

Two quasiholes can fuse into other quasiholes. Adiabatically braided at low temperature (to avoid collective excitations and other quasiparticle excitations), they form a nontrivial representation of the braid group. The fusion and braiding of the quasiholes in the Pfaffian state follow those of free chiral bosons and free Majorana fermions in the RCFT we discussed above. In general, the fusion algebra of different anyons ϕ_i can be expressed as

$$\phi_i \times \phi_j = \sum_k N_{ij}^k \phi_k. \tag{3.13}$$

For example, for abelian FQH states, the quasiholes satisfy the fusion rule

$$(V_h)^i \times (V_h)^j = (V_h)^{(i+j) \mod m}, \tag{3.14}$$

with $(V_h)^0$ identified with V_e. In this case, there is only one fusion channel for each pair. Interpret it in a reverse way. A quasihole can split into two quasiholes without help from other quasiholes. If there exists any pair of i and j such that $\dim(\mathcal{H}_{ij}) = \sum_k N_{ij}^k > 1$, then the anyons can form a nonabelian representation of the braid group.

The commutativity and the associativity of fusion require that

$$N_{ij}^k = N_{ji}^k. \tag{3.15}$$

and

$$\sum_l N_{il}^m N_{jk}^l = \sum_n N_{ij}^n N_{nk}^m. \tag{3.16}$$

The fusion of n anyons gives a Hilbert space with dimension

$$\dim(\mathcal{H}_{i_1,i_2,\ldots,i_n}) = \sum_{j_1,\cdots,j_{n-2}} N_{i_1 i_2}^{j_1} N_{j_1 i_3}^{j_2} \ldots N_{j_{n-2} i_n}^{j_{n-1}}. \tag{3.17}$$

In particular, if $i_1 = \ldots = i_n \equiv i$, then $\dim(\mathcal{H}_{i_1,i_2,\ldots,i_n}) = d_i^n$, as $n \to \infty$, where d_i is called the quantum dimension of anyons of type i. The fusion matrix, F, and the braid matrix, B, can be defined as in RCFT, In this case, every primary field corresponds to an anyon, and every conform block to a wave function. The system of consistent equations imposed by pentagon relations and hexagon relations can be used to solve for explicit expressions of the matrices (see, e.g., [9]).

Using the OPE (see Appendix), one can show that, in the Pfaffian state, neglecting the the abelian Laughlin quasiparticle, the three particles left form a closed fusion algebra

$$\sigma \times \sigma = 1 \oplus \chi, \quad \sigma \times \chi = \sigma, \quad \chi \times \chi = I. \tag{3.18}$$

The first fusion rule tells us that the σ anyons have more than one fusion channel. The last rule says that two Majorana fermions can fuse into a condensate. Also, we can see

$$\dim(\mathcal{H}_\sigma^{2n}) = \dim(\mathcal{H}_\sigma^{2n+1}) = 2^n, \quad \dim(\mathcal{H}_\chi^n) = 1, \quad \dim(\mathcal{H}_I^n) = 1. \tag{3.19}$$

The total Hilbert space seems to have a dimension of 2^n. However, the braiding of particles preserves the parity of the number of anyons in a state, thus cutting the dimension of the invariant Hilbert space by half, consistent with what we mentioned earlier. This can be shown using the Bratteli diagram (see [1]). It is an important property that the correlation function of the anyons vanishes unless they fuse into the identity operator, I. The dimension of the Hilbert of the Ising anyons is just the number of channels how N σ fields fuse into I. It is more intuitive to see it if we map the anyons to states in a topological p-wave superconductor, knowing that they both belong to the same universal class. The nonlocalilty of the states and the topological nature of the Hilbert space can also be seen more directly using the Majorana zero-mode interpretation in that context (see [10, 11]).

Let us consider the states with four quasiholes again. In this case, we already know that there are only two linearly independent states. Let us denote them as $\Phi_+^{4\text{qh}}$ and $\Phi_-^{4\text{qh}}$. Nayak and Wilczek [9] showed that they can be written down as

$$\Phi_\pm^{4\text{qh}} = \frac{(\eta_{13}\eta_{24})^{1/4}}{(1+\sqrt{1\mp x})^{1/2}} \left(\Phi_{(13),(24)} \pm \sqrt{1-x}\ \Phi_{(14),(23)}\right)$$

where x is the cross ratio. Once again, we see that both $\Phi_+^{4\text{qh}}$ and $\Phi_-^{4\text{qh}}$ have a branch cut, a sign of fractional statistics. A braiding operation acting on the two dimensional Hilbert space as a unitary and nonabelian matrix rotate the basis formed by the conformal blocks. For example, the action of switching η_1 and η_3 is given by

$$B = \frac{1}{2}\exp\left(\mathrm{i}\pi\left(\frac{1}{8}+\frac{1}{4m}\right)\right)\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{3.20}$$

3.4 Bulk-Edge Correspondence

It is know that the low energy effective field theory of an abelian FQH state is a Chern-Simon (CS) theory, a topological field theory with fractional excitations when coupled with matter. With time reversal invariance explicitly broken, the edge states of an abelian FQH system form a chiral Luttinger liquid, the physics of which is captured by a chiral bosonic WZW model, the same model whose conformal blocks yield the wave functions of different bulk states. Meanwhile, it is also know that there is a one-one correspondence between a (2+1)D CS theory with a boundary and the 2D WZW model that lives on that boundary [12]. This bulk-edge correspondence is reminiscent of the holographic principle.

The correspondence may be generalized to the nonabelian case except now there are some subtleties. For $m = 1$, the bosonic Pfaffian state is still described by a CS theory in the bulk and by $\mathrm{SU}(2)_2$ WZW model on the boundary. We cannot naively generalize this to the case where $m > 1$. It turns out that the correct effective theory is a coset CS theory and a coset WZW model. For example, the fermionic Moore-Read state is described by the chiral $(\mathrm{SU}_2/\mathrm{U}(1)_2) \times \mathrm{U}(1)_8$ CS theory [13]. The corresponding CFT is the simply the $\mathrm{Ising} \times \mathrm{U}(1)_2$ coset WZW model, as $(\mathrm{SU}_2/\mathrm{U}(1)_2) \times \mathrm{U}(1)_8 \simeq \mathbb{Z}_2 \times \mathrm{U}(1)_2$. The edge states of the Pfaffian state consist of a chiral Majorana fermionic field and a free compactified chiral bosonic field, coinciding with the picture we drew from the conformal block representation.

The expectation of a Wilson loop in the CS theory gives a topological invariant that is directly linked to the braiding of the anyons. The correspondence maps Wilson lines in the CS theory to primary operators of the WZW theory, and the braiding of the Wilson lines to that of conformal blocks. Fusion and braid matrices are in one-one correspondence. In general, it is natural to conjecture that: 'The edge states are described physically by the same RCFT that describes the bulk mathematically.' [2]

4 Discussion

We briefly discussed some applications of RCFT to FQH states. All the information of anyons is encoded in the anyon species, fusion rules, the fusion matrix, and the braiding matrix. In particular, we shown that the wave function of the Pfaffian state can be considered as a holomorphic conformal block of the Ising $\times$ $\mathrm{U}(1)_m$ CFT. In this state, each particle is paired with another. We can generalize this state to states with n-particle clusters. These states, called Read-Rezayi states, can be obtained from conformal blocks of the tensor product of a chiral bosonic $\mathrm{U}(1)_m$ CFT and a $\mathbb{Z}_3$ parafermionic CFT. The excitations of these states contain the so-called Fibonacci anyons, the simplest anyons of a single type, τ, satisfying the fusion rule, $\tau \times \tau = I + \tau$. They are an interesting research topic for universal quantum computing [14].

We did not discuss the interpretation of the Pfaffian state as a p-wave condensate of composite fermions. A topological $p_x + \mathrm{i}p_y$ superconductor contains Majorana zero mode in its vortices. These Majorana fermions obey exactly the same nonabelian statistics of the Ising anyons. A similar interpretation of the Read-Rezayi states has also been discussed recently [14]. Recent progress in realizing Majorana fermions in condensed matter systems paves the way for the future of fault-tolerant quantum computing [15].

Another different but certainly interesting perspective is offered by surface states of a symmetry protected phase. It is proposed that the protected gapless surface states can be gapped without breaking the symmetries, provided they become topologically ordered. These surface states, called T-Pfaffian states, also constain nonabelian anyons in their spectrum [16]. The interplay between bulk states and their corresponding surface states is a hot ongoing research topic.

A Appendix: Basic Information of Rational Conformal Field Theory

For a detailed discussion on CFT, we refer to [8, 17].

RCFT contains a finite number of primary fields $\{\phi_i\}$. If $T(z)$ is the holomorphic part of the energy-momentum tensor, then each mode operator L_n is given by

$$L_n = \frac{1}{2\pi \mathrm{i}} \oint_w dz\ (z-w)^{n+1} T(z). \tag{A.1}$$

$\bar{L}_n$ can also be defined similarly. A general descendant field, $\phi_i^{-k_1,\ldots,-k_n;-\bar{k}_1,\ldots,-\bar{k}_m}$, can be written as

$$\phi_i^{\{k,\bar{k}\}}(z,\bar{z}) \equiv \phi_i^{-k_1,\ldots,-k_n;-\bar{k}_1,\ldots,-\bar{k}_m}(z,\bar{z}) = (L_{-k_1}\ldots L_{-k_n}\bar{L}_{-\bar{k}_1}\ldots \bar{L}_{-\bar{k}_m}\phi_i)(z,\bar{z}). \tag{A.2}$$

A primary field, ϕ_i, together with its descendant fields, constitutes a conformal family, $[\phi_i]$.

Recall that conformal invariance tells us that

$$\langle\phi_i(z_i,\bar{z}_i)\phi_j(z_j,\bar{z}_j)\rangle = \frac{\delta_{ij}\delta_{h_i,h_j}\delta_{\bar{h}_i,\bar{h}_j}}{z_{ij}^{h_i+h_j}\bar{z}_{ij}^{\bar{h}_i+\bar{h}_j}}, \tag{A.3}$$

where $(h_i,\bar{h}_i)$ is the conformal dimension (weight) of ϕ_i and $z_{ij} = z_i - z_j$. Note that we have chosen the basis of the primary fields such that $C_{ij} = \delta_{ij}$.

The three-point correlation function takes the form

$$\langle\phi_i(z_i,\bar{z}_i)\phi_j(z_j,\bar{z}_j)\phi_k(z_k,\bar{z}_k)\rangle = \frac{C_{ijk}}{z_{ij}^{h_i+h_j-h_k}z_{jk}^{h_j+h_k-h_i}z_{ki}^{h_k+h_i-h_j}\bar{z}_{ij}^{\bar{h}_i+\bar{h}_j-\bar{h}_k}\bar{z}_{jk}^{\bar{h}_j+\bar{h}_k-\bar{h}_i}\bar{z}_{ki}^{\bar{h}_k+\bar{h}_i-\bar{h}_j}}, \tag{A.4}$$

where C_{ijk} is the structure constant. One can show that the OPE of two primary fields has a general form

$$\phi_i(z_i,\bar{z}_i)\phi_j(z_j,\bar{z}_j) = \sum_p \sum_{\{k,\bar{k}\}} C_{ij}^p \frac{\beta_{ij}^{p,\{k\}}\bar{\beta}_{ij}^{p,\{\bar{k}\}}\phi_p^{\{k,\bar{k}\}}(z_j,\bar{z}_j)}{z_{ij}^{h_i+h_j-h_p-K}\bar{z}_{ij}^{\bar{h}_i+\bar{h}_j-\bar{h}_p-\bar{K}}}, \tag{A.5}$$

where $K = \sum_i k_i$ and $\bar{K} = \sum_i \bar{k}_i$. We also have introduced coefficients $\beta_{ij}^{p,\{k\}}$ and $\bar{\beta}_{ij}^{p,\{\bar{k}\}}$ which are functions of the central charges and the conformal dimensions. They are totally fixed by requiring that both sides match. Therefore, the three-point correlation functions of the operator algebra can be determined by the central charges, the conformal dimensions, and C_{ij}^p. In general, the OPE of any two conformal fields implies the following fusion algebra

$$[\phi_i]\times[\phi_j] = \sum_k N_{ij}^k[\phi_k], \quad N_{ij}^k \in \mathbb{Z}_0^+. \tag{A.6}$$

In principle, any n-point correlation functions can then be calculated by using the OPE of primary fields recursively. In some CFTs, the structure constants are fully determined by the constraint equations imposed by the crossing symmetries.

The conformal invariance requires the four-point function

$$\langle\phi_i(z_1,\bar{z}_1)\phi_j(z_2,\bar{z}_2)\phi_k(z_3,\bar{z}_3)\phi_l(z_4,\bar{z}_4)\rangle \tag{A.7}$$

to depend only on the cross ratios

$$x = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \quad \bar{x} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}}. \tag{A.8}$$

We can map the four coordinates of the primary fields conformally to $z_1 = 0, z_2 = x, z_3 = 1$ and $z_4 = \infty$, and similarly for $\bar{z}_i$. By applying the OPE to different pairs, we obtain different forms of the correlation functions, but they should be consistent. These cross symmetries therefore lead to constraint equations on the structure constant. For example, if we use OPE on $\phi_i(\bar{z}_1,\bar{z}_1)\phi_j(\bar{z}_2,\bar{z}_2)$ and $\phi_k(\bar{z}_3,\bar{z}_3)\phi_l(\bar{z}_4,\bar{z}_4)$, we obtain

$$\langle\phi_i(z_1,\bar{z}_1)\phi_j(z_2,\bar{z}_2)\phi_k(z_3,\bar{z}_3)\phi_l(z_4,\bar{z}_4)\rangle = \sum_p C_{ij}^p C_{kl}^p \mathcal{F}_{ij}^{kl}(p|x)\bar{\mathcal{F}}_{ij}^{kl}(p|\bar{x}), \tag{A.9}$$

where the intermediate index p runs over all primary fields, and $\mathcal{F}_{ij}^{kl}(p|x)$ and $\bar{\mathcal{F}}_{ij}^{kl}(p|\bar{x})$ are called conformal blocks. Similarly, if we use OPE on $\phi_j\phi_k$ and $\phi_i\phi_l$, the conformal blocks are $\mathcal{F}_{il}^{jk}(p|1-x)$ and $\bar{\mathcal{F}}_{il}^{jk}(p|1-\bar{x})$; if we choose $\phi_j\phi_l$ and $\phi_i\phi_k$, the conformal blocks are $\mathcal{F}_{ik}^{jl}(p|1/x)$ and $\bar{\mathcal{F}}_{ik}^{jl}(p|1/\bar{x})$.

Since the number of primary fields is finite, there are only a finite number of intermediate p's. $\mathcal{F}_{ij}^{kl}(p|x)$'s, therefore, form a finite dimensional vector space. $\mathcal{F}_{il}^{jk}(p|1-x)$ and $\mathcal{F}_{ik}^{jl}(p|1/x)$ are simply other choices of basis. The expansions

$$\mathcal{F}_{ij}^{kl}(p|x) = \sum_q F_{pq}\begin{pmatrix} j & k \\ i & l \end{pmatrix} \mathcal{F}_{il}^{jk}(q|1-x) \tag{A.10}$$

and

$$\mathcal{F}_{ij}^{kl}(p|x) = \sum_q B_{pq}\begin{pmatrix} j & k \\ i & l \end{pmatrix} \mathcal{F}_{ik}^{jl}(q|\frac{1}{x}) \tag{A.11}$$

define the fusion matrix, F, and braiding matrix, B, respectively. These matrices also satisfy the so-called pentagon relations and hexagon relations.

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