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Classification of modular invariant representations of affine algebras.[1]

V.G. Kac and M. Wakimoto

§0. Introduction.

Let $\mathfrak{g}$ be an infinite-dimensional complex Lie algebra and let H be a derivation of $\mathfrak{g}$ (Hamiltonian) which is diagonalizable with finite-dimensional eigenspaces. Consider an irreducible representation of $\mathfrak{g}$ in a vector space V which extends to $\mathfrak{g} \rtimes \mathbb{C}H$. V is called a *positive energy* representation if H is diagonalizable on V with finite-dimensional eigenspaces and $\mathrm{Spec}\, H \subset h + \frac{1}{N}\mathbb{Z}_+$, $h \in \mathrm{Spec}\, H$, for some $N \in \mathbb{N}$ and $h \in \mathbb{R}$, called the *minimal energy* of V. The representation of $\mathfrak{g}$ in V is called *modular invariant* if the function $\mathrm{tr}_V e^{2\pi i\tau(H+a)}$, called the (modified) *character* of V, is a holomorphic modular function in τ (of weight 0) on the upper half-plane, for some $a \in \mathbb{R}$, called the *modular anomaly* of V. (Note that modular invariance implies energy positivity since a modular function must have at worst a pole at $i\infty$.)

It has been clear for some time now, both to mathematicians and to physicists, that the most interesting representations of an infinite-dimensional Lie algebra are the positive energy representations. But it is the modular invariant representations that have played a fundamental role in the recent development of conformally invariant quantum field theory and statistical mechanics.

In the present paper we address the problem of classification of modular invariant representaions of an affine algebra $\mathfrak{g}$. Note that positive energy representations of $\mathfrak{g}$ are nothing else but irreducible highest weight modules $L(\lambda)$. In [7], we proved a character formula for $L(\lambda)$ under the assumptions

(0.1a) $\langle\lambda+\rho,\alpha\rangle \notin \{0,-1,-2,\dots\}$ for all $\alpha \in R_+$, where R_+ is the set of all positive real coroots,

(0.1b) $\mathrm{Re}\langle\lambda+\rho,c\rangle > 0$, where c is the canonical central element of $\mathfrak{g}$.

This formula shows, that, under the assumptions (0.1), the character of $L(\lambda)$ is a modular function of weight $-\frac{1}{2}r(\lambda)$, where $r(\lambda)$ is the codimension of the $\mathbb{Q}$-span of $\{\alpha \in R_+ \mid \langle\lambda+\rho,\alpha\rangle \in \mathbb{Z}\}$ in the $\mathbb{Q}$-span of R_+. Thus, $L(\lambda)$ is modular invariant if λ satisfies (0.1a) and

(0.2) $r(\lambda) = 0$.

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(Note that (0.1a) and (0.2) imply (0.1b).) We call λ satisfying (0.1a) and (0.2) an *admissible weight.*

We conjectured in [7] that if $L(\lambda)$ is modular invariant and $\langle\lambda+\rho,c\rangle \neq 0$, then λ is an admissible weight. This problem is still open. In §§1 and 2 we give a complete classification of admissible weights for all affine algebras (Propositions 1.1 and 1.2 and Theorems 2.1, 2.2 and 2.3).

Furthermore, in §3 we express the character of $L(\lambda)$ for an admissible λ in terms of theta functions (Theorems 3.2 and 3.5). This not only shows that $L(\lambda)$ is modular invariant, but also allows us to calculate the asymptotics of the character at high temperatures (i.e as $T = -i\tau \downarrow 0$)(Theorems 3.3 and 3.4), and find explicit transformation properties of characters (Theorems 3.6 and 3.7).

In §4.1 we prove a complete reducibility theorem (Theorem 4.1 and Corollary 4.1). §§4.2 and 4.3 are complements to our paper [6]. In particular, in §4.2 we prove the uniqueness of the vacuum (Theorem 4.2).

Throughout the paper $\mathbb{Z}_+$ (resp. $\mathbb{N}$) denotes the set of all non–negative (resp. positive) integers.

§1. Admissible simple sets.

Throughout the paper we use notations and basic definitions of the book [4], unless otherwise stated.

Let $A = (a_{ij})_{i,j\in I}$, where $I = \{0,1,\dots,\ell\}$, be a generalized Cartan matrix of affine type $X_N^{(k)}$, listed in Table Aff k of [4, Chapter 4]; k is called the *tier number* of A. Let $(a_0,\dots,a_\ell)$ (resp. $a_0^\vee,\dots,a_\ell^\vee$) denote the *null–vector* (resp. *null–covector*) of A, i.e. the unique vector of relatively prime positive integers such that

$$(a_0,\dots,a_\ell)^t A = 0 \quad (\text{resp. } (a_0^\vee,\dots,a_\ell^\vee)A = 0).$$

Recall that $a_0 = 1$ (resp. $a_0 = 2$) if $A \neq A_{2\ell}^{(2)}$ (resp. $A = A_{2\ell}^{(2)}$) and that $a_0^\vee = 1$ in all cases. The number $g = \sum_{i\in I} a_i^\vee$ is called the *dual Coxeter number* of A.

Let $\mathfrak{h}'$ be an $(\ell+1)$–dimensional vector space with a basis $\Pi^\vee = \{\alpha_0^\vee,\dots,\alpha_\ell^\vee\}$. Introduce another basis $\Pi = \{\alpha_0,\dots,\alpha_\ell\}$ by

$$a_i\alpha_i = a_i^\vee\alpha_i^\vee .$$

The sets $\Pi^\vee$ and Π are called *coroot* and *root* bases respectively.

Introduce the *standard bilinear form* on $\mathfrak{h}'$ by the following formulas [4, Chapter 6]:

$$(\alpha_i^\vee | \alpha_j^\vee) = a_{ij} a_j / a_j^\vee \quad (i,j \in I) .$$

This is a symmetric bilinear form whose kernel consists of all multiples of the element

$$c = \sum_{i \in I} a_i^\vee \alpha_i^\vee$$

called the *canonical central element.*

We have the following homomorphism $\alpha \longmapsto t_\alpha$ of $\mathfrak{h}'$ into $\mathrm{Aut}\, \mathfrak{h}'$ (with kernel $\mathbb{C}c$) [4, Chapter 6]:

(1.1) $$t_\alpha(h) = h + (h|c)\alpha - ((h|\alpha) + \tfrac{1}{2}(\alpha|\alpha)(h|c))c , \quad h \in \mathfrak{h}' .$$

For a subset $L \subset \mathfrak{h}'$, we let $t_L = \{t_\alpha | \alpha \in L\}$.

Let $r_i \in \mathrm{Aut}\, \mathfrak{h}'$ be the *fundamental reflections*, i.e. $r_i(h) = h - \langle \alpha_i, h \rangle \alpha_i^\vee$, $h \in \mathfrak{h}'$, and let $W = \langle r_i | i \in I \rangle$ be the *Weyl group.* Let $R = W(\Pi^\vee)$ be the set of *real coroots* and let R_+ be the set of *positive* real coroots (in [4] they are denoted by $\Delta^{\vee re}$ and $\Delta_+^{\vee re}$). Given a subset $S \subset R$, we denote by A_S the matrix $(2(\alpha|\beta)/(\beta|\beta))_{\alpha,\beta \in S}$ (so that $A = A_{\Pi^\vee}$).

Let $I_0 = \{1,\dots,\ell\} \subset I$ and let $\bar{\mathfrak{h}} = \sum_{i \in I_0} \mathbb{C}\alpha_i^\vee \subset \mathfrak{h}'$. Let $\bar{Q} = \sum_{i \in I_0} \mathbb{Z}\alpha_i$, $\bar{Q}^\vee = \sum_{i \in I_0} \mathbb{Z}\alpha_i^\vee$. Then

(1.2) $$\bar{Q} \supset \bar{Q}^\vee \text{ if } k = 1 \quad \text{and} \quad \bar{Q} \subset \bar{Q}^\vee \text{ if } k > 1,$$

where k is the tier number of A . Let $k^\vee$ denote the tier number of ${}^tA$. Let

$$M = \bar{Q} \text{ if } k^\vee = 1 \, ; \; M = \bar{Q}^\vee \text{ if } k^\vee > 1 .$$

Then [4, Chapter 6}:

(1.3) $$W = \bar{W} \ltimes t_M ,$$

where $\bar{W} = \langle r_i | i \in I_0 \rangle$ is a finite subgroup of W . The set R is invariant with respect to the group $\tilde{W} := W \ltimes t_{\tilde{M}}$ (containing W), where $\tilde{M}$ is a lattice in $\mathfrak{h}'$ defined by [5, §4.8]:

$$\tilde{M} = (\bar{Q} + \bar{Q}^\vee)^* .$$

Here and further, for a subgroup $L \subset \mathfrak{h}'$ we let $L^* = \{\alpha \in \mathbb{Q}L \,|\, (\alpha|L) \subset \mathbb{Z}\}$. Also, given a subset $S \subset \mathfrak{h}'$, $\mathbb{Z}S$ (resp. $\mathbb{Q}S$ or $\mathbb{C}S$) stands for the $\mathbb{Z}$–span (resp. $\mathbb{Q}$– or $\mathbb{C}$–span) of S. Since $(\bar{Q}|\bar{Q}^\vee) \subset \mathbb{Z}$, we have $M \subset \tilde{M} \subset M^*$.

Recall that the group $\tilde{W}_+ := \{w \in \tilde{W} | w(\Pi^\vee) = \Pi^\vee\}$ acts transitively on orbits of Aut $\Pi^\vee$ (and simply transitively on the orbit of $\alpha_0^\vee$) [5, §4.8].

Let $\Lambda_i \in \mathfrak{h}'^*$ $(i \in I)$ be the *fundamental weights*:

$$<\Lambda_i, \alpha_j^\vee> = \delta_{ij} \quad (j \in I),$$

and let $\rho = \sum_{i \in I} \Lambda_i$. Note that $g = <\rho, c>$.

Given $\lambda \in \mathfrak{h}^*$ we let

$$R^\lambda = \{\alpha \in R | <\lambda+\rho, \alpha> \in \mathbb{Z}\}, \ R_+^\lambda = R^\lambda \cap R_+$$

(note that $<\rho,\alpha> \in \mathbb{Z}$ for all $\alpha \in R$). We denote by S^λ the set of *simple roots* in R_+^λ, i.e. the set of $\alpha \in R_+^\lambda$ which do not decompose into a sum of several elements from R_+^λ.

It is easy to prove the following two basic properties of S^λ:

(1.4) $\qquad R_+^\lambda = R_+ \cap \mathbb{Z}S^\lambda$,

(1.5) $\qquad$ if $\alpha, \beta \in S^\lambda$, then $\alpha - \beta \notin R$.

Property (1.5) implies

(1.6) $\qquad A_{S^\lambda}$ decomposes into a finite direct sum of affine type matrices.

We call the *type* of A_{S^λ} or simply the *type* of λ the direct sum of types of these affine matrices.

We call $\lambda \in \mathfrak{h}^*$ an *admissible weight* (for A) if it satisfies the following two properties:

(1.7a) $\qquad <\lambda+\rho, \alpha> \notin -\mathbb{Z}_+$ for all $\alpha \in R_+$,

(1.7b) $\qquad \mathbb{Q}R^\lambda = \mathbb{Q}\Pi^\vee$.

It is clear that (1.7a,b) imply

(1.8) $\qquad <\lambda+\rho, c> > 0$.

Note that for an admissible λ we have:

(1.9) $\qquad \mathbb{Q}S^\lambda = \mathbb{Q}\Pi^\vee$,

(1.10) $\qquad R_+^\lambda = \{\alpha \in R | <\lambda+\rho, \alpha> \in \mathbb{N}\}$.

The set S^λ for an admissible λ is called an *admissible simple set.*

Our first objective is to classify all admissible weights. For this we first describe all, up to the action of $\tilde{W}$, (finite) subsets S of R_+ satisfying properties (1.5) and (1.9).

Given such an S, pick $\alpha \in S$ and let S' be an indecomposable component of S containing α. Then $S' \backslash \{\alpha\}$ is a union of indecomposable sets of finite type $\dot{S}_1, \dots, \dot{S}_r$; let $\gamma_1, \dots, \gamma_r \in R_+$ be such that $\gamma_i \in \mathbb{Z}\dot{S}_i + \mathbb{Z}c$, $\dot{S}_i \cup \{\gamma_i\}$ is a linearly independent set and $\beta - \gamma_i \notin R$ for $\beta \in \dot{S}_i$. Let $\sigma_\alpha^{(\gamma_1, \dots, \gamma_r)}(S) = \{\gamma_1, \dots, \gamma_r\} \cup (S \backslash \{\alpha\})$. We say that this set is obtained from S by an *elementary operation* $\sigma_\alpha^{(\gamma_1, \dots, \gamma_r)}$. For example, we may take $\gamma_i = c -$ (highest root of the finite root system $\mathbb{Z}\dot{S}_i \cap R$). In this case we shall drop the superscript $(\gamma_1, \dots, \gamma_r)$ and call σ_α an elementary operation of the *first kind*. All other are called elementary operations of the *second kind*. In the case $\alpha = \alpha_i^\vee$ we shall often write σ_i (resp. $\sigma_i^{(\gamma_1, \dots, \gamma_r)}$) in place of $\sigma_{\alpha_i^\vee}$ (resp. $\sigma_{\alpha_i^\vee}^{(\gamma_1, \dots, \gamma_r)}$).

We call subsets S and S_1 of R *equivalent* if $S = w(S_1) \bmod \mathbb{Z}c$ for some $w \in \bar{W}$.

Lemma 1.1. (a) All subsets S of R_+ satisfying properties (1.5) and (1.9) are equivalent to subsets obtained from $\Pi^\vee$ by a finite sequence of elementary operations.

(b) If $k^\vee = 1$, we may consider in (a) elementary operations of the 1'st kind only.

Proof. The case $k^\vee = 1$ follows immediately from the Dynkin–Borel–de Siebenthal algorithm [1]. Indeed, $\bar{S} := S \bmod \mathbb{C}c \subset \mathfrak{h}'/\mathbb{C}c$ still satisfies the property $\alpha, \beta \in \bar{S} \Longrightarrow \alpha - \beta \notin \bar{R} := R \bmod \mathbb{C}c$. If $k^\vee > 1$, the latter property does not always hold, and we check the lemma by a casewise analysis. □

Proposition 1.1. Let $k^\vee = 1$. Then an admissible simple set S is equivalent to one of the sets $\sigma_i(\Pi^\vee)$, $i \in I$.

Proof. We prove the lemma be a case–wise discussion using Lemma 1.1. In the case $A = A_\ell^{(1)}$ there is nothing to prove since all sets obtained from $\Pi^\vee$ by an elementary operation are $\tilde{W}$–equivalent to $\Pi^\vee$.

Let now $A = D_\ell^{(1)}$. Then all sets given by Lemma 1.1(b) can be described as follows. Let J be a subset of the set $\{2,3,\dots,\ell-2\}$ such that $|r-s| \geq 2$ if $r,s \in J$, $r \neq s$, and let $J_0 = \{j \mid (\alpha_j^\vee | \alpha_n^\vee) < 0$, for some $n \in J\}$. For $i \in J$ let $\theta_i' = \alpha_0^\vee + \alpha_1^\vee + 2\alpha_2^\vee + \dots + 2\alpha_i^\vee + \alpha_{i+1}^\vee$, $\theta_i'' = \alpha_{i-1}^\vee + 2\alpha_i^\vee + \dots + 2\alpha_{\ell-2}^\vee + \alpha_{\ell-1}^\vee + \alpha_\ell^\vee$. Then S is $\bar{W}$–equivalent to one of the following sets:

$S(J) = \{\alpha_i^\vee + k_i c,\ i \notin J\} \cup \{\theta_i' + k_i' c,\ \theta_i'' + k_i'' c,\ i \in J\}$

$\cup \{-\alpha_i^\vee + k_i'' c,\ i \in J_0\}$, where $k_i, k_i', k_i'' \in \mathbb{Z}$.

Suppose now that there exists $\lambda \in \mathfrak{h}'^*$ such that $S(J) = S^\lambda$. Let $m = <\lambda,c>$; this is a rational number with denominator $u \in \mathbb{N}$ (so that u and um are relatively prime integers). Then we have, using the integrality condition for the first member of $S(J)$:

(1.11) $\qquad k_i m + <\lambda,\alpha_i^\vee> \in \mathbb{Z}$ if $i \notin J$.

Using the integrality condition for the second member of $S(J)$ gives

$(1.12)_i \qquad n_i := p_i m + 2<\lambda,\alpha_i^\vee> \in \mathbb{Z}$ if $i \in J$ for some $p_i \in \mathbb{Z}$.

Adding um to (1.12) we get

(1.13a) $\qquad n_i + um = (p_i + u)m + 2<\lambda,\alpha_i^\vee>$ if $i \in J$;

adding $(1.12)_j$ to (1.13) we get:

(1.13b) $\qquad n_i + n_j + um = (p_i + p_j + u)m + 2<\lambda,\alpha_i^\vee + \alpha_j^\vee>$ if $i,j \in J$.

Suppose now that $|J| \geq 2$ and fix $i,j \in J$ such that $i < j$ and $\{r | i < r < j\} \cap J = \emptyset$. Note that $kc + \alpha_i^\vee$, $kc + \alpha_j^\vee$ and $kc + \sum_{r=i}^{j} \alpha_r^\vee \notin R^\lambda$ since the coefficients of $\alpha_i^\vee$ and $\alpha_j^\vee$ are even for any $\alpha \in S(J)$. Hence we get for all $k \in \mathbb{Z}$:

(1.14) $\qquad km + <\lambda,\alpha_i^\vee> \notin \mathbb{Z}$, $km + <\lambda,\alpha_j^\vee> \notin \mathbb{Z}$, $km + <\lambda,\alpha_i^\vee + \alpha_j^\vee> \notin \mathbb{Z}$.

Comparing $(1.12)_i$ and (1.13) with (1.14), we see that none of the pairs (n_i,p_i) , (n_j,p_j) , $(n_i+n_j,\ p_i+p_j)$, $(n_i+um,\ p_i+u)$, $(n_j+um,\ n_j+u)$, $(n_i+n_j+um,\ p_i+p_j+u)$ are (even,even). But this is impossible since either u or um is odd.

A similar argument works for $A = A_{2\ell-1}^{(2)}$, $D_{\ell+1}^{(2)}$, $D_4^{(3)}$, $E_6^{(1)}$ and $E_6^{(2)}$ and shows that applying more than one elementary operation produces a set which is not admissible. For $A = E_7^{(1)}$ the only possibilities of more than one elementary operation are $\sigma_i\sigma_j(\Pi^\vee)$ with $(i,j) = (4,1)$ or $(2,5)$. But both are $\bar{W}$–equivalent to $\sigma_3(\Pi^\vee)$. Similarly, for $A = E_8^{(1)}$ the only possibilities of more than one elementary operation are $\sigma_i\sigma_j(\Pi^\vee)$ with $(i,j) = (4,1), (4,2), (6,1), (6,2), (8,2)$ or $(8,1)$ or $(2,7), (4,7)$. But these are $\bar{W}$–equivalent to $\sigma_r(\Pi^\vee)$ with $r = 5, 6$ or 3 respectively. □

If S is equivalent to $\Pi^\vee$ (resp. to $\sigma_s(\Pi^\vee)$) we call this case 1 (resp. case 2_s).

Proposition 1.2. Let $k^\vee > 1$, and let λ be an admissible weight of level m whose denominator is u . Then $S = S^{\lambda}$ is equivalent to one of the following sets:

Table 1

A	case	u	S	$A_{S^{\lambda}}$	$\gamma_s, \gamma'_s, \gamma''_s$
$B_{\ell}^{(1)}$	1	odd	$\Pi^{\vee}$	$B_{\ell}^{(1)}$	
	2_s	even	$\sigma_s(\Pi^{\vee})$ $0 \le s \le \ell,\ s \ne 1$	$D_s^{(1)} \oplus D_{\ell-s+1}^{(2)}$	
	3_2	odd	$\sigma_2^{(\gamma_2,\gamma'_2,\gamma''_2)}(\Pi^{\vee})$	$D_2^{(1)} \oplus B_{\ell-2}^{(1)}$	$\alpha_0^{\vee}+\alpha_1^{\vee}+2\alpha_2^{\vee}+\alpha_3^{\vee}$ $c-\alpha_0^{\vee}$ $c-\alpha_1^{\vee}$
	3_s	odd	$\sigma_s^{(\gamma_s,\gamma'_s)}(\Pi^{\vee})$ $3 \le s \le \ell-1$	$B_s^{(1)} \oplus B_{\ell-s}^{(1)}$	$\begin{cases} \alpha_0^{\vee}+\alpha_1^{\vee}+2(\alpha_2^{\vee}+\ldots+\alpha_s^{\vee})+\alpha_{s+1}^{\vee} \\ 2c-\alpha_{\ell}^{\vee} \quad (s=\ell-1) \end{cases}$ $2(\alpha_s^{\vee}+\ldots+\alpha_{\ell-1}^{\vee})+\alpha_{\ell}^{\vee}$
$C_{\ell}^{(1)}$	1	odd	$\Pi^{\vee}$	$C_{\ell}^{(1)}$	
	2_s	even	$\sigma_s(\Pi^{\vee})$ $0 \le s \le \ell$	$A_{2s-1}^{(2)} \oplus A_{2(\ell-s)-1}^{(2)}$	
	3_2	odd	$\sigma_2^{(\gamma_2,\gamma'_2,\gamma''_2)}\sigma_0(\Pi^{\vee})$	$D_2^{(1)} \oplus C_{\ell-2}^{(1)}$	$\alpha_1^{\vee}+2(\alpha_2^{\vee}+\ldots+\alpha_{\ell}^{\vee})$ $\alpha_0^{\vee}+\alpha_1^{\vee}+\alpha_2^{\vee}$ $2c-\alpha_1^{\vee}$
	3_s	odd	$\sigma_s^{(\gamma_s,\gamma'_s)}\sigma_0(\Pi^{\vee})$ $3 \le s \le \ell-1$	$D_s^{(1)} \oplus C_{\ell-s}^{(1)}$	$\alpha_{s-1}^{\vee}+2(\alpha_s^{\vee}+\ldots+\alpha_{\ell}^{\vee})$ $\alpha_0^{\vee}+\alpha_1^{\vee}+\ldots+\alpha_s^{\vee}$
	3_{ℓ}	odd	$\sigma_{\ell}^{(\gamma_{\ell})}\sigma_0(\Pi^{\vee})$	$D_{\ell}^{(1)}$	$\alpha_{\ell-1}^{\vee}+2\alpha_{\ell}^{\vee}$

$A_{2\ell}^{(2)}$	1	odd	$\Pi^{\vee}$	$A_{2\ell}^{(2)}$	
	2_s	even	$\sigma_s(\Pi^{\vee})$ $0 \le s \le \ell$	$D_{s+1}^{(2)} \oplus A_{2(\ell-s)-1}^{(2)}$	
	3_s	odd	$\sigma_s^{(\gamma_s,\gamma_s')}(\Pi^{\vee})$ $1 \le s \le \ell-1$	$B_s^{(1)} \oplus A_{2(\ell-s)}^{(2)}$	$\begin{cases} 2c-\alpha_0^{\vee} & (s=1) \\ \alpha_1^{\vee}+2(\alpha_2^{\vee}+\ldots+\alpha_{\ell-1}^{\vee})+\alpha_\ell^{\vee} & (s=2) \\ \alpha_{s-1}^{\vee}+2(\alpha_s^{\vee}+\ldots+\alpha_\ell^{\vee}) & (s>2) \end{cases}$ $\alpha_0^{\vee}+2(\alpha_1^{\vee}+\ldots+\alpha_s^{\vee})$
	3_ℓ	odd	$\sigma_\ell^{(\gamma_\ell)}(\Pi^{\vee})$	$B_\ell^{(1)}$	$\alpha_{\ell-1}^{\vee}+2\alpha_\ell^{\vee}$
$G_2^{(1)}$	1	$\notin 3\mathbb{Z}$	$\Pi^{\vee}$	$G_2^{(1)}$	
	2_0	$\in 3\mathbb{Z}$	$\sigma_0(\Pi^{\vee})$	$D_4^{(3)}$	
	2_1		$\sigma_1(\Pi^{\vee})$	$A_1^{(1)} \oplus A_1^{(1)}$	
	2_2	$\in 3\mathbb{Z}$	$\sigma_2(\Pi^{\vee})$	$A_2^{(1)}$	
	3	$\notin 3\mathbb{Z}$	$\sigma_1^{(\gamma_1)}\sigma_0(\Pi^{\vee})$	$A_2^{(1)}$	$3\alpha_1^{\vee}+\alpha_2^{\vee}$
$F_4^{(1)}$	1	odd	$\Pi^{\vee}$	$F_4^{(1)}$	
	2_0	even	$\sigma_0(\Pi^{\vee})$	$E_6^{(2)}$	
	2_1	even	$\sigma_1(\Pi^{\vee})$	$A_1^{(1)} \oplus A_5^{(2)}$	
	2_2		$\sigma_2(\Pi^{\vee})$	$A_2^{(1)} \oplus A_2^{(1)}$	
	2_3	even	$\sigma_3(\Pi^{\vee})$	$A_3^{(1)} \oplus A_1^{(1)}$	
	2_4	even	$\sigma_4(\Pi^{\vee})$	$D_5^{(2)}$	

3_1	odd	$\sigma_1^{(\gamma_1)}\sigma_0(\Pi^\vee)$	$C_4^{(1)}$	$c+2\alpha_1^\vee+3\alpha_2^\vee+2\alpha_3^\vee+\alpha_4^\vee$
3_2	odd	$\sigma_2^{(\gamma_2,\gamma_2')}\sigma_0(\Pi^\vee)$	$A_1^{(1)}\oplus A_3^{(1)}$	$c-\alpha_1^\vee$ $2\alpha_1^\vee+4\alpha_2^\vee+2\alpha_3^\vee+\alpha_4^\vee$
3_4	odd	$\sigma_4^{(\gamma_4,\gamma_4')}\sigma_0(\Pi^\vee)$	$B_3^{(1)}\oplus A_1^{(1)}$	$c-\alpha_1^\vee-2\alpha_2^\vee-\alpha_3^\vee$ $2\alpha_1^\vee+4\alpha_2^\vee+3\alpha_3^\vee+2\alpha_4^\vee$

Here we adopt the following conventions: $X_0^{(k)} = 0$, $D_1^{(k)} = 0$, $A_1^{(1)} = B_1^{(1)} = C_1^{(1)} = A_1^{(2)} = D_2^{(2)}$, $D_2^{(1)} = A_1^{(1)} \oplus A_1^{(1)}$, $B_2^{(1)} = C_2^{(1)}$, $A_3^{(k)} = D_3^{(k)}$.

Proof is similar to that of Proposition 1.1. □

§2. Classification of admissible weights.

For $u \in \mathbb{N}$ let

$$S_{(u)} = \{\gamma_0 := (u-1)c + \alpha_0^\vee,\ \gamma_i := \alpha_i^\vee\ (i \in I_0)\}.$$

Given $y \in \tilde{W}$, denote by $P_{u,y}$ the set of all admissible weights λ such that $S^\lambda = y(S_{(u)})$.

Let P (resp. P_+) $= \{\lambda = \sum_{i\in I} n_i\Lambda_i \mid n_i \in \mathbb{Z}$ (resp. $\mathbb{Z}_+$)$\}$ be the sets of all integral (resp. dominant integral) weights. Given $m \in \mathbb{Q}$ we denote by P_+^m , $P_{u,y}^m$, etc. the subsets of P_+ , $P_{u,y}$, etc. consisting of all elements of level m .

Recall the shifted action of $\tilde{W}$:

$$w.\lambda = w(\lambda+\rho)-\rho .$$

Now we can state the first main result of this section.

Theorem 2.1. (a) $P_{u,y}^m \neq \emptyset$ if and only if the triple (m,u,y) satisfies the following properties:

(2.1a) g.c.d$(u,k^\vee) = 1$,

(2.1b) um is an integer relatively prime to u (i.e. u is the denominator of m), (2.1c) $\quad m + g \geq g/u$,

(2.1d) $\quad y(S_{(u)}) \subset R_+$.

(b) If properties (2.4a–d) hold, then

$$P^m_{u,y} = \{y.(\lambda^0-(u-1)(m+g)\Lambda_0) \mid \lambda^0 \in P_+^{u(m+g)-g}\} .$$

(c) The set of all admissible weights λ for A such that $A_{S^\lambda} = A$ is $\bigcup_{u,m,y} P^m_{u,y}$ where (u,m,y) runs over all triples satisfying (2.1a–d).

The theorem follows from Lemmas 2.1–4 proved below.

Lemma 2.1. All S satisfying (1.5) and (1.9) of the same type as A are of the form $S = y(S_{(u)})$, where $y \in \tilde{W}$, $u \in \mathbb{N}$.

Proof. By Lemma 1.1, S is $\tilde{W}$–equivalent to the set $S' = \{\alpha_i^\vee + u_i c\}_{i=0,\dots,\ell}$, where $u_i \in \mathbb{Z}_+$. If $k > 1$, then, by (1.2), $\tilde{M} = (\bar{Q}^\vee)^*$, hence S' is $t_{\tilde{M}}$ – equivalent to $S_{(u)}$. If $k = 1$, then $\tilde{M} = \bar{Q}^*$, and $\alpha_i^\vee = \alpha_i$ if $\alpha_i^\vee$ is short and $\alpha_i^\vee = k^\vee \alpha_i$ if $\alpha_i^\vee$ is long. Since $S' \subset R_+$, it follows from [4, Proposition 6.3] that again S' is $t_{\tilde{M}}$–equivalent to $S_{(u)}$. □

Lemma 2.2. Let λ be an admissible weight and let $w \in \tilde{W}$ be such that $w(R_+^\lambda) \subset R_+$. Then $w.\lambda$ is also an admissible weight and $R_+^{w.\lambda} = w(R_+^\lambda)$.

Proof is obvious. □

Lemma 2.3. Condition (2.1a) is equivalent to

$$\gamma_0 \in R \text{ and } \gamma_0 - \gamma_i \notin R \text{ for all } i \in I_0 .$$

Proof is straightforward using [4, Proposition 6.3]. □

Lemma 2.4. (a) Let u be a positive integer satisfying (2.1a) and let $\lambda \in P^m_{u,1}$. Then (2.1b) and (2.1c) hold and

(2.2) $$\lambda = \lambda^0 - (u-1)(m+g)\Lambda_0 \text{ , where } \lambda^0 \in P_+^{u(m+g)-g} .$$

(b) Conversely, let u and m satisfy conditions (2.1a–c) and let $\lambda^0 \in P_+^{u(m+g)-g}$. Then λ defined by (2.2) lies in $P^m_{u,1}$.

Proof. Let $\lambda \in P^m_{u,1}$; then

$$\langle \lambda+\rho, \gamma_i \rangle = n_i + 1 \text{ , where } n_i \in \mathbb{Z}_+ \ (i \in I).$$

Note that $n_i = <\lambda, \alpha_i^\vee> + (u-1)(m+g)\delta_{i,0}$. It follows that λ is of the form (2.2) with $\lambda^0 = \sum_{i\in I} n_i\Lambda_i$, and that $\mathbb{Z}_+ \ni \sum_{i\in I} a_i^\vee n_i = <\lambda, c> + (u-1)(m+g) = u(m+g)-g$ proving (2.1c) and $um \in \mathbb{Z}$.

To complete the proof of (a), it remains to show that u and um are relatively prime. Let $u' = u/\text{g.c.d.}(u,um)$. Consider the coroot $\gamma = u'c + \gamma_0$ if $A \neq A_{2\ell}^{(2)}$; $\gamma = u'c + \gamma_\ell$ if $A = A_{2\ell}^{(2)}$. Then $<\lambda+\rho, \gamma> \in \mathbb{Z}$, hence γ is a $\mathbb{Z}$–linear combination of γ_i's, hence $u'c = \sum_{i\in I} r_i\gamma_i = r_0(u-1)c + \sum_{i\in I} r_i\alpha_i^\vee$ for some $r_i \in \mathbb{Z}$. It follows that $\sum_{i\in I} r_i\alpha_i^\vee = r_0 c$. Plugging this in the previous formula, we get $u' = r_0 u$. Hence $u = u'$ and (a) is proved.

In order to prove (b) we have to show

$$(\mathbb{Z}S_{(u)} \cap R) \supset R^\lambda .$$

First, consider the case $A \neq A_{2\ell}^{(2)}$. Let $\alpha \in R^\lambda$. Since $\alpha \in R$, we have [4, Proposition 6.3]:

$$\alpha = bc + \bar{\alpha}, \text{ where } b \in \mathbb{Z},\ \bar{\alpha} \in \bar{\Delta}_s^\vee \text{ or } b \in k^\vee\mathbb{Z},\ \bar{\alpha} \in \bar{\Delta}_\ell^\vee .$$

Since $<\lambda, \alpha> \in \mathbb{Z}$, it follows that $(u-1)mb \in \mathbb{Z}$. Hence, by (2.1b), u divides b. Thus, by (2.1a) we have:

$$\alpha = nuc + \bar{\alpha} \text{ if } \bar{\alpha} \in \bar{\Delta}_s^\vee \text{ and } = k^\vee nuc + \bar{\alpha} \text{ if } \alpha \in \bar{\Delta}_\ell^\vee ,$$

where $n \in \mathbb{Z}$, hence $\alpha \in \mathbb{Z}S_{(u)}$.

In the case $A = A_{2\ell}^{(2)}$, the proof is similar. □

The following is an immediate consequence of Lemma 1.1 and Theorem 2.1:

Corollary 2.1. The set of all admissible weights for $A = A_\ell^{(1)}$ is $\bigcup_{u,m,y} P_{u,y}^m$, where (u,m,y) runs over all triples satisfying (2.1a–d).

The following proposition describes the redundancies in Corollary 2.1.

Proposition 2.1. Let (m,u,y) and (m,u,y′) be two triples satisfying conditions (2.1a–d) of Theorem 2.1. Then the following three statements are equivalent:

(i) $P_{u,y}^m \cap P_{u,y'}^m \neq \emptyset$,

(ii) $P_{u,y}^m = P_{u,y'}^m$,

(iii) $y(S_{(u)}) = y'(S_{(u)})$.

Proof. Let $\lambda \in P^m_{u,y} \cap P^m_{u,y'}$; then $y(S_{(u)}) = S^\lambda = y'(S_{(u)})$, proving (i) $\Rightarrow$ (iii).

Suppose that $y(S_{(u)}) = y'(S_{(u)})$ and let $\lambda \in P^m_{u,y}$. We want to show that $\lambda \in P^m_{u,y'}$. Note that by Lemma 2.2 we have

(2.3) $\qquad \lambda \in P^m_{u,y} \Leftrightarrow y^{-1}.\lambda \in P^m_{u,1} \Leftrightarrow \langle y^{-1}(\lambda+\rho),\gamma_i\rangle \in \mathbb{N}$ for all $i \in I$.

Hence, it suffices to show that $\langle (y')^{-1}(\lambda+\rho),\gamma_i\rangle \in \mathbb{N}$ for $i \in I$, which is equivalent to

$$\langle y^{-1}(\lambda+\rho)|w(\gamma_i)\rangle \in \mathbb{N}\ ,\ i \in I\ ,\ \text{where}\ w = y^{-1}y'\ .$$

Note that $w(S_{(u)}) = S_{(u)}$, hence the last condition is equivalent to $\langle y^{-1}(\lambda+\rho),\gamma_i\rangle \in \mathbb{N}$, $i \in I$. Thus by (2.3), (iii) $\Rightarrow$ (ii). □

We proceed now to classify all admissible weights for the affine matrix A , using Theorem 2.1. Let S be an admissible set listed by Propositions 1.1 and 1.2 (i.e. $S = \sigma_i(\Pi^\vee)$ if $k^\vee = 1$ and S is one of the sets from Table 1 if $k^\vee > 1$). Decompose S into a union of orthogonal indecomposable subsets:

(2.4) $\qquad S = \dot{S} \cup \ddot{S} \cup \dots$.

Pick one of the subsets, say, $\dot{S} = \{\dot{\alpha}_0^\vee,\dots,\dot{\alpha}_{\dot{\ell}}^\vee\}$. Let $\dot{c} = \sum_{i=0}^{\dot{\ell}} \dot{a}_i^\vee \dot{\alpha}_i^\vee$ be its canonical central element . We have:

(2.5) $\qquad \dot{c} = \dot{e}c$ for some $\dot{e} \in \mathbb{N}$.

For $u \in \mathbb{N}$ we let

(2.6) $\qquad \dot{u} = u/\text{g.c.d.}(u,\dot{e})$.

Let $(.|.)^\cdot$ be the normalized standard bilinear form on $\dot{\mathfrak{h}}' := \mathbb{C}\dot{S}$; then

(2.7) $\qquad (x|y)^\cdot = \dot{K}^{-1}(x|y)$ for some $\dot{K} \in \mathbb{Q}$ for all $x,y \in \dot{\mathfrak{h}}'$.

Here and further, for subsets $\ddot{S}$, $\ddot{S}^\cdot$,... we use similar notations. We have in the case $k^\vee = 1$:

(2.8) $\qquad \dot{e} = \ddot{e} = \dots = 1$;

(2.9) $\qquad \dot{K} = k^{-1}$ if $\dot{S} \subset R_{\text{short}}$, and = 1 otherwise,....

The values of $\dot{e},\ddot{e},\dots$ and $\dot{K},\ddot{K},\dots$ in the case $k^\vee > 1$ are given in Table 2.

The elements $\dot{\alpha}_0^\vee,\dot{\alpha}_1^\vee,\dots,\dot{\alpha}_{\dot{\ell}}^\vee,\ddot{\alpha}_1^\vee,\dots,\ddot{\alpha}_{\ddot{\ell}}^\vee,\dots$ form a basis of $\mathfrak{h}'$; define $\dot{\Lambda}_i \in \mathfrak{h}'^*$ $(i = 0,\dots,\dot{\ell})$ by $\langle\dot{\Lambda}_i,\dot{\alpha}_j^\vee\rangle = \delta_{ij}$, $\langle\dot{\Lambda}_i,\alpha\rangle = 0$ for all other basis elements α . Let

$$\dot{\rho} = \sum_{i=0}^{\dot{\ell}} \dot{\Lambda}_i\ ,\ \dot{g} = \langle\dot{\rho},\dot{c}\rangle = \Sigma\, \dot{a}_i^\vee\ .$$

As in Section 1, define the lattice $\tilde{M}^{\cdot} \subset \dot{\mathfrak{h}}'$. Choose a set of representatives $\bar{M}(S)$ of $\tilde{M}^{\cdot} + \tilde{M}^{\cdot\cdot} + \ldots \bmod \tilde{M}$. In the case $k^{\vee} = 1$, we have: $|\bar{M}(\sigma_s(\Pi^{\vee}))| = a_s^{\vee}$, and we can choose $\bar{M}(\sigma_s(\Pi^{\vee})) = \{j\bar{\Lambda}_s \,|\, j = 0,\ldots,a_s^{\vee}-1\}$. In the case $k^{\vee} > 1$, the sets $\bar{M}(S)$ are listed in Table 2.

Let $\lambda \in \mathfrak{h}'^{*}$ be of rational level m whose denominator is $u \in \mathbb{N}$. Define $\dot{\lambda}$ by

(2.10a) $\qquad \dot{\lambda} + \dot{\rho} = \lambda + \rho|_{\dot{\mathfrak{h}}'}$,

so that

(2.10b) $\qquad \dot{m} = \dot{e}(m+g) - \dot{g}$.

Then $\dot{\lambda}$ is a weight for $\dot{S}$ of level $\dot{m}$ whose denominator is $\dot{u}$. The weight λ is completely determined by the sequence $(\dot{\lambda},\ddot{\lambda},\ldots)$. This gives us a bijective correspondence between $\lambda \in \mathfrak{h}'^{*}$ of rational level m and the sequences $(\dot{\lambda},\ddot{\lambda},\ldots)$ such that $\dot{\lambda} \in \dot{\mathfrak{h}}'^{*},\ldots$ and (2.10b) holds (λ is then recovered from (2.10a)).

<u>Theorem 2.2</u>. Let S be an admissible simple set listed by Propositions 1.1 and 1.2. Let m be a rational number with denominator $u \in \mathbb{N}$ and suppose that the numbers $\dot{m},\ddot{m},\ldots$ and $\dot{u},\ddot{u},\ldots$ (defined above) satisfy the following conditions:

(2.11a) $\qquad$ g.c.d.$(\dot{u},\dot{k}^{\vee}) = 1$, g.c.d.$(\ddot{u},\ddot{k}^{\vee}) = 1,\ldots$;

(2.11b) $\qquad \dot{m}+\dot{g} \geq \dot{g}/\dot{u}$, $\ddot{m}+\ddot{g} \geq \ddot{g}/\ddot{u}$,... .

Let $\dot{\beta} + \ddot{\beta} + \ldots \in \bar{M}(S)$ and $y \in \tilde{W}$ be such that

(2.11c) $\qquad y t_{\dot{\beta}}\{(\dot{u}-1)\delta_{i,0}\dot{c} + \dot{\alpha}_i^{\vee}\} \subset R_+$,....

Pick $\dot{\lambda}^0 = \sum_i \dot{n}_i \dot{\Lambda}_i \in \dot{P}_+^{\dot{u}(\dot{m}+\dot{g})-\dot{g}}$,..., and let $\dot{\lambda} = \dot{\lambda}^0 - (\dot{u}-1)(\dot{m}+\dot{g})\dot{\Lambda}_0$,...

Then $\lambda = (y\dot{t}_{\dot{\beta}}.\dot{\lambda},\, y\ddot{t}_{\ddot{\beta}}.\ddot{\lambda},\ldots)$ is an admissible weight for A such that S^{λ} is equivalent to S if and only if $\dot{n}_i,\ddot{n}_i,\ldots$ satisfy certain "matching condition". In the case $k^{\vee} = 1$ and $\dot{\beta} = j\dot{\bar{\Lambda}}_s$ $(0 \leq j < a_s^{\vee})$ this condition is

(2.12) $\qquad u(\sum_{i\in\dot{I}} a_i^{\vee}\dot{n}_i + \sum_{i\in\ddot{I}} a_i^{\vee}\ddot{n}_i + \ldots) \not\equiv mu(u-j) \bmod p$ for every prime p dividing $a_s^{\vee}$.

In the case $k^{\vee} > 1$ the matching condition is given in Table 2. All admissible weights λ for A with S^{λ} equivalent to S are obtained in this way. We thus obtain a complete list of admissible sets for A .

<u>Proof</u>. Let λ be an admissible weight for A and let $S^{\lambda} = \dot{S} \cup \ddot{S} \cup \ldots$ be the decomposition (2.4) of S^{λ}. Then $\dot{S} = \dot{S}^{\dot{\lambda}},\ldots$ hence $\dot{\lambda},\ldots$ is an admissible weight of type

$A_{\dot{S}}$ and we can apply Theorem 2.1. Hence λ is obtained as described by Theorem 2.2, except that it remains to show that matching conditions given by Table 2 are necessary and sufficient for λ to be admissible if all $\dot{\lambda},\ddot{\lambda},\ldots$ are admissible for $\dot{S},\ddot{S},\ldots$.

We show this on the example of $A_{2\ell}^{(2)}$, case 3_1. Let λ be an admissible weight of level m whose denominator is u . the corresponding admissible set is $S = \dot{S} \cup \ddot{S}$, where

$$\dot{S} = \{(\dot{u}-1)\dot{c} + \dot{\alpha}_0^\vee, \dot{\alpha}_1^\vee\} = \{2(u-1)c + \alpha_0^\vee, 2c-\alpha_0^\vee\} ,$$

$$\ddot{S} = \{(\ddot{u}-1)\delta_{i,0}\ddot{c} + \ddot{\alpha}_i^\vee\}_{0\le i\le \ell-1} = \{(u-1)c + \alpha_0^\vee + 2\alpha_1^\vee, \alpha_2^\vee,\ldots,\alpha_\ell^\vee\} .$$

(Table 2.2 shows us that $\dot{e} = 2$, $\ddot{e} = 1$ and hence $\dot{u} = \ddot{u} = u$.)
We have: $\lambda = (\dot{\lambda},\ddot{\lambda})$, where

$$\dot{\lambda} = \dot{\lambda}^o - (u-1)(\dot{m}+\dot{g})\dot{\Lambda}_0 , \ \ddot{\lambda} = \ddot{\lambda}^o - (u-1)(\ddot{m}+\ddot{g})\ddot{\Lambda}_0 ,$$

$$\dot{\lambda}^o = \dot{n}_0\dot{\Lambda}_0 + \dot{n}_1\dot{\Lambda}_1 \in \dot{P}_+^{u(\dot{m}+\dot{g})-\dot{g}} , \ \ddot{\lambda}^o = \sum_{i=0}^{\ell-1} \ddot{n}_i\ddot{\Lambda}_i \in \ddot{P}_+^{u(\ddot{m}+\ddot{g})-\ddot{g}} .$$

Note that

$$<\dot{\lambda}+\dot{\rho}, 2(u-1)c + \alpha_0^\vee> = \dot{n}_0 + 1 , <\ddot{\lambda}+\ddot{\rho}, (u-1)c + \alpha_0^\vee + 2\alpha_1^\vee> = \ddot{n}_0 + 1 ,$$

which is equivalent to:

$$<\lambda+\rho, 2(u-1)c + \alpha_0^\vee> = \dot{n}_0 + 1 , <\lambda+\rho, (u-1)c + \alpha_0^\vee + 2\alpha_1^\vee> = \ddot{n}_0 + 1 .$$

Hence we have: $<\lambda+\rho, \frac{1}{2}(1-u)c + \alpha_1^\vee> = \frac{1}{2}(\ddot{n}_0 - \dot{n}_0)$. But $\frac{1}{2}(1-u)c + \alpha_1^\vee \in R_+$ since u is odd. Since $nc + \alpha_1^\vee \notin R_+^\lambda$ for any $n \in \mathbb{Z}$, we deduce that $\ddot{n}_0 - \dot{n}_0$ is odd, proving that the matching condition of Table 2 is necessary. It is also easy to see that this condition is sufficient. □

Remark 2.1. Condition (2.11b) of Theorem 2.2 is equivalent to:

$$m + g \ge \max(\dot{g}/\dot{e}\dot{u},\ddot{g}/\ddot{e}\ddot{u},\ldots) ;$$

It is easy to see from Theorem 2.2 that any rational m satisfying this condition is the level of a modular invariant representation provided that the following additional conditions are satisfied:

$$m + g > \min(\dot{g}/\dot{e}\dot{u},\ddot{g}/\ddot{e}\ddot{u},\ldots) \text{ if } A_S \ne A ;$$

$$m + g > \max(\dot{g}/\dot{e}\dot{u},\ddot{g}/\ddot{e}\ddot{u},\ldots) \text{ in the following cases:}$$

$(A_{2\ell-1}^{(2)}, \sigma_s(\Pi^\vee))$ and $(A_{2\ell}^{(2)}$, type $3_s)$, where $\frac{1}{2}(\ell+1) < s < \ell$,
$(E_6^{(2)}, \sigma_3(\Pi^\vee))$, $(E_7^{(1)}, \sigma_3(\Pi^\vee))$, $(E_8^{(1)}, \sigma_6(\Pi^\vee))$.

Remark 2.2. If $\alpha_i^\vee$ and $\alpha_j^\vee$ lie in the same orbit of Aut $\Pi^\vee$, then the sets of admissible weights λ with S^λ equivalent to $\sigma_i(\Pi^\vee)$ and to $\sigma_j(\Pi^\vee)$ are the same.

Table 2

A	case	$\dot{e},\ddot{e},\ldots$	$\dot{K},\ddot{K},\ldots$	$\bar{M}(S)$	matching condition
$B_\ell^{(1)}$	1	1	1	0	
	2_0	2	2	0	
	2_2	1,1,2	1,1,2	$0,\bar{\dot{\Lambda}}_1$	
	$2_s(3\leq s\leq \ell-1)$	2,2	1,2	$0,\bar{\dot{\Lambda}}_{s-1}$	
	2_ℓ	2	1	0	
	3_2	1,1,1	1,1,1	0	$\dot{n}_0+\ddot{n}_1+\ddot{n}_0^{\cdot}+\ddot{n}_1^{\cdot}$ odd
				$\bar{\dot{\Lambda}}_1$	$\dot{n}_0+\ddot{n}_0+\ddot{n}_0^{\cdot}+\ddot{n}_1^{\cdot}$ odd
	$3_s(3\leq s\leq \ell-2)$	1,1	1,1	0	$\dot{n}_s+\ddot{n}_{\ell-s}$ odd
	$3_{\ell-1}$	1,2	1,2	0	$\dot{n}_{\ell-1}+\ddot{n}_1$ odd
$C_\ell^{(1)}$	1	1	1	0	
	2_0	2	2	0	
	2_1	1,2	1,2	0	
	$2_s(2\leq s\leq \frac{\ell}{2})$	2,2	2,2	0	
	3_2	2,2,1	2,2,1	0	$\dot{n}_0+\ddot{n}_0$ odd
	3_3	2,1	2,1	0	$\dot{n}_0+\dot{n}_2$ odd

	$3_s(4\leq s\leq \ell-1)$	2,1	2,1	0	$\dot{n}_0+\dot{n}_1$ odd
	3_ℓ	2	2	0	$\dot{n}_0+\dot{n}_1$ odd
$A_{2\ell}^{(2)}$	1	1	1	0	
	2_0	2	1	0	
	$2_s(1\leq s\leq \ell-2)$	2,2	2,1	0	
	$2_{\ell-1}$	2,1	$2,\frac{1}{2}$	0	
	2_ℓ	2	2	0	
	3_1	2,1	2,1	0	$\dot{n}_0+\ddot{n}_0$ odd
	3_2	1,1	1,1	0	$\ddot{n}_0+\dot{n}_1$ odd
	$3_s(3\leq s\leq \ell-1)$	1,1	1,1	0	$\dot{n}_0+\ddot{n}_0$ odd
	3_ℓ	1	1	0	$\dot{n}_0+\dot{n}_1$ odd
$G_2^{(1)}$	1	1	1	0	
	2_0	3	3	0	
	2_1	1,3	1,3	0	$u(\dot{n}_0+\ddot{n}_0)\not\equiv u^2m \bmod 2$
				$\bar{\Lambda}_1$	$u(\dot{n}_0+\ddot{n}_0)\not\equiv um(u-1) \bmod 2$
	2_2	3	1	0	
	3	3	3	0	$\dot{n}_1\not\equiv\dot{n}_2 \bmod 3$
$F_4^{(1)}$	1	1	1	0	

2_0	2	2	0	
2_1	2,2	1,2	0	
2_2	1,2	1,2	$j\bar{\Lambda}_1$ (j=0,1,2)	$u(\dot{n}_1-\dot{n}_2-\ddot{n}_1+\ddot{n}_2)\neq -jum \bmod 3$
2_3	1,2	1,2	$0,\ 2\bar{\Lambda}_3$	
2_4	2	2	0	
3_1	1	1	0	$\dot{n}_1+\dot{n}_2$ odd
3_2	1,2	1,2	$0,\bar{\Lambda}_1$	$\ddot{n}_1+\ddot{n}_3$ odd
3_4	1,2	1,2	0	$\dot{n}_2+\ddot{n}_1$ odd

We give now a more explicit version of Proposition 1.1 and Theorem 2.2 in the case $k^\vee = 1$.

Given $s \in I$, let $I_s = I\backslash\{s\}$. The set $\Pi^\vee\backslash\{\alpha_s^\vee\}$ decomposes into a union of orthogonal indecomposable subsets; let $I_s = \dot{I} \cup \ddot{I} \cup \ldots$ be the corresponding decomposition of I_s. The set $\sum_{i\in\dot{I}} \mathbb{Z}\alpha_i^\vee \cap R$ is a finite root system; let $\dot{\theta}^\vee = \sum_{j\in\dot{I}} \dot{a}_i^\vee\alpha_i^\vee$ be its highest root and let $g = 1 + \sum_{i\in\dot{I}} \dot{a}_i^\vee$.

Given a triple $(s,\vec{k},u)$, where $s \in I$, $\vec{k} = (k_i)_{i\in I_s}$ with $k_i \in \mathbb{Z}_+$, $u \in \mathbb{N}$, we let:

$$S_{(s,\vec{k},u)} = \{k_i c + \alpha_i^\vee\ (i \in I_s), uc-\dot{\theta}^\vee, uc-\ddot{\theta}^\vee, \ldots\}.$$

<u>Theorem 2.3</u>. Let $k^\vee = 1$. Then

(a) Every admissible simple subset of R_+ is of the form $y(S_{(s,\vec{k},u)})$ for some $y \in \bar{W}$.

(b) Given a rational number m, the set $P^m_{(s,\vec{k},u,y)}$ of all admissible weights

λ of level m with $S^\lambda = y(S_{(s,\vec{k},u)})$, where $y \in \bar{W}$, is non–empty if and only if

(2.13a) u is the denomiator of m;

(2.13b) $\sum_{j\in\dot{I}} \dot{a}_i^\vee k_i \le u-1$, $\sum_{j\in\ddot{I}} \ddot{a}_i^\vee k_i \le u-1, \ldots$;

(2.13c) $m+g \ge \max\{\dot{g},\ddot{g},\ldots\}/u$;

(2.13d) $m+g > \min\{\dot{g},\ddot{g},\ldots\}/u$ if $a_s^\vee > 1$;

(2.13d′) $m+g > \max\{\dot{g},\ddot{g},\ldots\}/u$ in the following cases: $(A_{2\ell-1}^{(2)}, \sigma_s(\Pi^\vee))$ with $\frac{1}{2}(\ell+1) < s < \ell$, $(E_6^{(2)},\sigma_3(\Pi^\vee))$, $(E_7^{(1)},\sigma_3(\Pi^\vee))$, $(E_8^{(1)},\sigma_6(\Pi^\vee))$.

(2.13e) $y(S_{(s,\vec{k},u)}) \subset R_+$.

(c) If conditions (2.13) hold, then

$$P^m_{(s,\vec{k},u,y)} = y.\{m(a_s^\vee)^{-1}\Lambda_s + \sum_{j\in I_s} (n_j-k_j(m+g))(\Lambda_j-a_j^\vee a_s^{\vee -1}\Lambda_s)\},$$

where the $(n_i)_{i\in I_s}$ are non–negative integers satisfying the following conditions:

(2.14a) $\sum_{i\in\dot{I}} \dot{a}_i^\vee n_i \le u(m+g)-\dot{g}$, $\sum_{i\in\ddot{I}} \ddot{a}_i^\vee n_i \le u(m+g)-\ddot{g}, \ldots$

(2.14b) $\text{g.c.d.}(u(m - \sum_{i\in I_s} a_i^\vee(n_i-k_i(m+g))), a_s^\vee) = 1$.

(d) The set of all admissible weights for A is the union of all $P^m_{(s,\vec{k},u,y)}$ from (c).

Proof. Let λ be an admissible weight for A of level m with denominator u, and let $S^\lambda = \dot{S} \cup \ddot{S} \cup \ldots$ be the decomposition (2.4). As in the proof of Theorem 2.2, $\dot{S} = \dot{S}^{\dot{\lambda}}$, $\ddot{S} = \ddot{S}^{\ddot{\lambda}}, \ldots$, hence $\dot{\lambda},\ldots$ is an admissible weight of type $A_{\dot{S}},\ldots$, and we can apply Theorem 2.1. Also, by Lemma 1.1, S^λ is equivalent to $\sigma_s(\Pi^\vee)$ for some $s \in I$. Hence $\dot{S} = \{k_i c + \alpha_i^\vee\}_{i\in\dot{I}} \cup \{u_1 c-\dot{\theta}^\vee\}$, $S = \{k_i c+ \alpha_i^\vee\}_{i\in\ddot{I}} \cup \{u_2 c-\ddot{\theta}^\vee\},\ldots$. But by Theorem 2.2, $u = u_1 = u_2 = \ldots$. Thus, $S^\lambda = y(S_{(s,\vec{k},u)})$, which proves (a). Similarly, Theorem 2.2 implies the necessity of (2.13) except for (2.13d and d′) (namely, (2.1b) implies (2.13a), (2.1c) implies (2.13c), (2.1d) implies (2.13b) and (2.13e)).

Furthermore, we have, by definition of S^λ:

$(2.15)_i$ $\langle\lambda+\rho,k_i c+\alpha_i^\vee\rangle = n_i + 1$, where $n_i \in \mathbb{Z}_+$ $(i \in I_s)$, (2.16) $$<\lambda+\rho, uc - \sum_{j\in\dot{I}} \dot{a}_j^\vee(k_jc+\alpha_j^\vee)> = \dot{n} + 1 \text{ , where } \dot{n} \in \mathbb{Z}_+ ,\ldots$$

Adding up all $(2.15)_i$, multiplied by $\dot{a}_i^\vee$, for $i \in \dot{I}$, plus (2.16), we get:

$$u(m+g) = \dot{n} + \dot{g} + \sum_{i\in\dot{I}} \dot{a}_i^\vee n_i, \ldots ,$$

which proves the necessity of (2.14a).

Rewriting $(2.15)_i$ as $<\lambda,\alpha_i^\vee> = n_i - k_i(m+g) := \tilde{n}_i$, we see that

$$\lambda = \sum_{i\in I_s} \tilde{n}_i(\Lambda_i - (a_i^\vee/a_s^\vee)\Lambda_s) + (m/a_s^\vee)\Lambda_s ,$$

where the last term arises because $<\lambda,c> = m$.

In order to complete the proof of the "only if" part of (c), it remains to prove the necessity of (2.14b). For this we add up all $(2.15)_i$, multiplied by $a_i^\vee$, for all $i \in I_s$, obtaining:

$$<\lambda+\rho, Kc - a_s^\vee\alpha_s^\vee> = N - a_s^\vee ,$$

where $K = 1 + \sum_{i\in I_s} a_i^\vee k_i$, $N = g + \sum_{i\in I_s} a_i^\vee n_i \in \mathbb{N}$. Adding $ju(m+g)$ $(j \in \mathbb{Z})$ to the last equation, we obtain

$(2.17)_j$ $$<\lambda + \rho, (K + ju)c - a_s^\vee\alpha_s^\vee> = N - a_s^\vee + ju(m + g) .$$

Note that

(2.18) $$\alpha = \sum b_i\alpha_i^\vee \in R^\lambda \Longrightarrow a_s^\vee \mid b_s .$$

We claim that

(2.19) $$\text{g.c.d.}(K + ju, N + ju(m+g), a_s^\vee) = 1 \text{ for all } j \in \mathbb{Z} .$$

Suppose the contrary. If $a_s^\vee$ is a prime number, then we get from $(2.17)_j$ that $(K + ju)(a_s^\vee)^{-1}c - \alpha_s^\vee \in R^\lambda$, which contradicts (2.18). If $a_s^\vee$ is not prime, which happens only for (1) $A = E_7^{(1)}$, $s = 3$; (2) $A = E_8^{(1)}$, $s = 3$, 5 or 6 ; (3) $A = E_6^{(2)}$, $s = 3$, the argument is similar. For example, in case (1) we argue as follows. In this case $a_3^\vee = 4$ and g.c.d.$(K + ju, N + ju(m+g)) = 4$ or 2. In the first case the argument is the same as above. In the second case we have, as above:

$$<\lambda+\rho, ac-2\alpha_s^\vee> \in \mathbb{Z} \text{ , where } a \in \mathbb{Z} .$$

Furthermore:

$$<\lambda+\rho, k_2c+ \alpha_2^\vee>, <\lambda+\rho, k_4c + \alpha_4^\vee>, <\lambda+\rho, k_7c + \alpha_7^\vee> \in \mathbb{Z} .$$

Hence the coroot $(k_2+k_4+k_7-a)c + \alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee + \alpha_7^\vee> \in R^\lambda$, a contradiction.

But (2.19) is equivalent to

(2.20) $\qquad$ $\text{g.c.d.}(uN - u(m+g)K, a_s^\vee) = 1$,

which is (2.14b). Here we have used statement (b) of the following elementary lemma.

Lemma 2.4. Let u and t be coprime integers and let p be a prime integer. Then

(a) $\quad \{(ju,jt) \in (\mathbb{Z}/p\mathbb{Z})^2 \mid j \in \mathbb{Z}\} = \{(K,N) \in (\mathbb{Z}/p\mathbb{Z})^2 \mid K,N \in \mathbb{Z},\ tK \equiv uN \bmod p\}$

(b) $\quad$ Given integers K and N, we have

$$(K + ju, N + jt) \not\equiv (0,0) \bmod p \ \text{ for all } j \in \mathbb{Z}$$

if and only if $tK \not\equiv uN \bmod p$.

Proof. It is clear that (a) implies (b). In order to prove (a), it suffices to show that if $tK \equiv uN \bmod p$ for some $K,N \in \mathbb{Z}$, then $(K,N) \equiv j(u,t) \bmod p$ for some $j \in \mathbb{Z}$. Since t and u are coprime, either t or u is coprime to p, say, t. Then $st \equiv 1 \bmod p$ for some $s \in \mathbb{Z}$, and we let $j = sN$. Then $ju \equiv sNu \equiv stK \equiv K \bmod p$ and $jt \equiv stN \equiv N \bmod p$. □

End of the proof of Theorem 2.3. The necessity of the (2.13d and d′) follows from (2.14b),(d) follows from (a) and (c).

It remains to prove the sufficiency in (b) and (c). Let $\lambda = m(a_s^\vee)^{-1}\Lambda_s + \sum_{j \in I_s} (n_j - k_j(m+g))(\Lambda_j - a_j^\vee (a_s^\vee)^{-1}\Lambda_s)$, where $(s,\vec{k},u)$ satisfy (2.13a–d′) (u is the denominator of m), and $(n_i)_{i \in I_s}$, $n_i \in \mathbb{Z}_+$, satisfy (2.14). Let $\alpha = \sum b_i \alpha_i^\vee \in R^\lambda$.

We have to show that

(2.21) $\qquad$ $a_s^\vee$ divides b_s.

Let $\gamma_i = k_i c + a_i^\vee$ $(i \in I_s)$. Then

(2.22) $\qquad$ $\gamma_i \in R_+^\lambda$

since $<\lambda+\rho,\gamma_i> = n_i + 1$. We also have:

(2.23) $\qquad$ $<\lambda+\rho,\alpha_s^\vee> = (a_s^\vee)^{-1}(a(m+g)-b)$, where $a-g := \sum_{j \in I_s} a_j^\vee k_j^\vee$, $b-g := \sum_{j \in I_s} a_j^\vee n_j^\vee$.

Condition (2.14b) can be rewritten as follows:

(2.24) $\qquad$ $\text{g.c.d.}(-ub + ua(m+g), a_s^\vee) = 1$.

Rewrite α in the following form:

$$\alpha = \sum_{i \in I_s} b_i \gamma_i - (\sum_{i \in I_s} b_i k_i)c + b_s \alpha_s^\vee .$$

Since $\alpha \in R^\lambda$, we have, using (2.23):

$$\mathbb{Z} \ni \langle\lambda+\rho,\alpha\rangle = \sum_{i\in I_s} b_i\langle\lambda+\rho,\gamma_i\rangle - (m+g)\sum_{i\in I_s} b_ik_i + b_s(a_s^\vee)^{-1}(a(m+g)-b)\ .$$

Hence, using (2.22):

$$-m\sum_{i\in I_s} b_ik_i + b_s(a_s^\vee)^{-1}(a(m+g)-b) \in \mathbb{Z}\ .$$

Multiplying this by u we have:

$$b_s(a_s^\vee)^{-1}(-ub + ua(m+g)) \in \mathbb{Z}\ .$$

This together with (2.24) implies (2.21). □

§3. The character formula, the asymptotics of characters and transformation properties of characters.

Let $\mathfrak{g}$ be an affine (Kac–Moody) algebra of type $X_N^{(k)}$ and let $\mathfrak{h}$ be its Cartan subalgebra. We have: $\mathfrak{h} = \mathfrak{h}' \oplus \mathbb{C}d$, where $\mathfrak{h}' = \sum_{i\in I} \mathbb{C}\alpha_i^\vee$ is the space considered in §§1 and 2. The standard bilinear form $(.|.)$ extends from $\mathfrak{h}'$ to $\mathfrak{h}$ to a symmetric non–degenerate bilinear form by

(3.1) $\qquad (d|\alpha_0^\vee) = a_0\ , (d|\alpha_i^\vee) = 0$ for $i \in I_0\ , (d|d) = 0\ .$

Given $\alpha \in R$, we denote by r_α the corresponding reflection.

Then $\mathfrak{h}'$ is r_α–invariant, $r_{\alpha_i^\vee}|_{\mathfrak{h}'} = r_i$ and the restriction map $W_1 := \langle r_\alpha | \alpha \in R\rangle \longrightarrow \mathrm{Aut}\,\mathfrak{h}'$ induces an isomorphism $W_1 \xrightarrow{\sim} W$.

The bilinear form $(.|.)$ induces an isomorphism $\nu\colon \mathfrak{h} \longrightarrow \mathfrak{h}^*$ and we shall identify $\mathfrak{h}$ with $\mathfrak{h}^*$ via this isomorphism. Note that $\nu(d) = a_0\Lambda_0$. The space $\mathfrak{h}'^*$ will be identified with a subspace of $\mathfrak{h}^*$ via extending a linear function on $\mathfrak{h}'$ to $\mathfrak{h}$ by $\langle\lambda,d\rangle = 0$.

Introduce the following domain Y in $\mathfrak{h}$ and coordinates (τ,z,t) on Y:

$$\begin{aligned} Y &= \{h \in \mathfrak{h} \mid \mathrm{Re}\langle\lambda,c\rangle > 0\} \\ &= \{(\tau,z,t) := 2\pi i(-\tau a_0^{-1}d+z+tc) \mid \tau,t \in \mathbb{C},\ \mathrm{Im}\tau > 0,\ z \in \bar{\mathfrak{h}}\}\ . \end{aligned}$$

Define a holomorphic function $Q_{\mathfrak{g}}$ on Y by

$$Q_{\mathfrak{g}}(h) = e^{\langle\rho,h\rangle} \prod_{\alpha\in\Delta_+} (1-e^{-\langle\alpha,h\rangle})^{\mathrm{mult}\,\alpha}\ .$$

Here $\Delta_+ = \Delta_+^{re} \cup \mathbb{N}c$, where $\Delta_+^{re} = \{2\gamma/(\gamma|\gamma) \mid \gamma \in R_+\}$, is the set of positive roots of

$\mathfrak{g}$, mult $\alpha = 1$ if $\alpha \in \Delta_+^{re}$, mult kjc $= \ell$, mult jc $= (N-\ell)/(k-1)$ if $k > 1$ and $j \not\equiv 0$ mod k (see [4,Chapters 6 and 8] for details).

Given $\lambda \in \mathfrak{h}^*$, we let

$$W^\lambda = \langle r_\alpha | \alpha \in R^\lambda \rangle.$$

One knows that, in fact, $W^\lambda = \langle r_\alpha | \alpha \in S^\lambda \rangle$.

Let $L(\lambda)$ denote the irreducible *highest weight* $\mathfrak{g}$*–module*, and let

$$ch_\lambda(h) = tr_{L(\lambda)} e^h , \; h \in Y ,$$

be its *character.*

Let $\theta = \sum_{i \in I_0} a_i \alpha_i$, let

$$D_0 \;(\text{resp. } \bar{D}_0) = \{z \in \bar{\mathfrak{h}} \,|\, \alpha_i(z) > 0 \;(\text{resp.} \geq 0) \text{ for } i \in I_0 ,\ \theta(z) < a_0 \;(\text{resp.} \leq a_0)\} ,$$

and let

$$D \;(\text{resp. } \bar{D}) = \{(\tau,z,t) \in Y \,|\, z \in D_0 \;(\text{resp.} \in \bar{D}_0)\} .$$

By [4, Lemma 10.6b) and (11.10.1)], ch_λ converges in D. The following result is a special case of [7, Theorem 1].

<u>Theorem 3.1</u>. Let λ be an admissible weight. Then ch_λ converges in the domain D and is given in this domain by the following formula:

$$ch_\lambda(h) = \sum_{w \in W^\lambda} \epsilon(w) e^{\langle w(\lambda+\rho),h \rangle} / Q_{\mathfrak{g}}(h) . \qquad \square \tag{3.2}$$

<u>Remark 3.1</u>. Both the numerator and denominator in (3.2) converge in Y [4, Proposition 10.6d] and the denominator $Q_{\mathfrak{g}}$ does not vanish in D.

We proceed to rewrite formula (3.2) in terms of characters of integrable modules. As in §2, we start with the case of S^λ being of the same type as A (see Theorem 2.1). In this case the admissible simple sets are of the form $y\{\gamma_i = (u-1)\delta_{i,0} c + \alpha_i^\vee \; (i \in I)\}$, where $u \in \mathbb{N}$ and $y \in \tilde{W}$. Introduce an automorphism $\phi = \phi_{u,y}$ of $\mathfrak{h}$, denoted by $h \longmapsto \hat{h}$, by the following formulas:

$$\phi(y(\gamma_i)) = \alpha_i^\vee \;(i \in I) ,\ \phi(y(d)) = ud .$$

This is an isomety of $\mathfrak{h}$ which intertwines the actions of $W_{(u,y)} := \langle r_{y(\gamma_i)} \; (i \in I) \rangle$ and

W, i.e. $\phi r_{y(\gamma_i)}\phi^{-1} = r_i$.

Theorem 3.2. Let $\lambda = y.(\lambda^o-(u-1)(m+g)\Lambda_0)$ be an admissible weight from $P^m_{u,y}$, where $\lambda^o = \sum_i u_i\Lambda_i \in P_+^{u(m+g)-g}$. Then

(3.3) $$ch_\lambda(h) = ch_{\lambda^o}(\hat{h})Q_g(\hat{h})/Q_g(h) .$$

Proof. Note that $W^\lambda = W_{(u,y)}$, hence by Theorem 3.1 we have:

$$ch_\lambda(h)Q_g(h) = \sum_{w\in W_{(u,y)}} \epsilon(w)\exp\langle w(\lambda^o+\rho-(u-1)(m+g)\Lambda_0,h\rangle .$$
$$= \sum_{w\in W} \epsilon(w)\exp\langle w(\lambda^o+\rho-(u-1)(m+g)\Lambda_0),\hat{h}\rangle$$

But $\langle\lambda^o+\rho-(u-1)(m+g)\Lambda_0,\gamma_i\rangle = u_i+1$, $i \in I$, hence $\phi(\lambda^o+\rho-(u-1)(m+g)\Lambda_0) = \lambda^o+\rho$. Therefore, $ch_\lambda(h)Q_g(h) = \sum_{w\in W} \epsilon(w)e^{\langle w(\lambda^o+\rho),\hat{h}\rangle}$, and we apply formula (3.2) for $ch_{\lambda^o}(\hat{h})$ to the right-hand side. □

The following result is a straightforward consequence of Theorem 3.2 and §2. Let λ be an admissible weight and let $S = S^\lambda$. Let $S = \dot{S} \cup \ddot{S} \cup \dots$ be the decomposition (2.4), and let $\dot{\mathfrak{h}} = \mathbb{C}\dot{S} + \mathbb{C}d,\dots$. Given $h \in \mathfrak{h}$, we define $\dot{h} \in \dot{\mathfrak{h}}$ by: $(\dot{h}|h') = (h|h')$ for all $h' \in \dot{\mathfrak{h}},\dots$ and define the automorphism $\dot{h} \longmapsto \hat{\dot{h}}$ of $\dot{\mathfrak{h}}$ relative to the set $\dot{S}$ for $\dot{u} \in \mathbb{N}$ and $yt_{\dot{\beta}},\dots$. Define $\dot{\lambda} \in \dot{\mathfrak{h}}^*$ by $\dot{\lambda} + \dot{\rho} = \lambda + \rho|_{\dot{\mathfrak{h}}},\dots$. Let $\dot{\mathfrak{g}}$ be the affine algebra with the coroot basis $\dot{S}$ and let $ch^\cdot$ denotes the character for $\dot{\mathfrak{g}},\dots$. Then we have the following formula for $h \in D$:

(3.4) $$ch_\lambda(h) = Q_g^{-1}(h)(ch^\cdot_{\dot{\lambda}^o}(\hat{\dot{h}})Q_{\dot{g}}(\hat{\dot{h}}))(ch^{\cdot\cdot}_{\ddot{\lambda}^o}(\hat{\ddot{h}})Q_{\ddot{g}}(\hat{\ddot{h}}))\dots$$

We turn now to the calculation of the asymptotic behavior of ch_λ for admissible weights λ. The basic tool for this calculation is the functional equation for theta-functions. Let $\lambda \in P^n$, $n \in \mathbb{N}$, and let $\bar{\lambda}$ denote the restriction of λ to $\bar{\mathfrak{h}}$. The associated theta function is a holomorphic function on Y defined as follows:

$$\theta_\lambda(\tau,z,t) = \theta_{\bar{\lambda},n}(\tau,z,t) = e^{2\pi i n t} \sum_{\gamma\in M+\bar{\lambda}/n} e^{\pi i n\tau(\gamma|\gamma)+2\pi i n(\gamma|z)} .$$

This function satisfies the following functional equation:

$$(3.5)\qquad \theta_{\bar\lambda,n}\left(-\frac{1}{\tau},\frac{z}{\tau},t-\frac{(z|z)}{2\tau}\right) = (-i\tau)^{\frac{\ell}{2}}|M^*/nM|^{-\frac{1}{2}} \sum_{\mu\in M^* \bmod nM} \left(\exp -\frac{2\pi i}{n}(\bar\lambda|\mu)\right)\theta_{\mu,n}(\tau,z,t)\ .$$

The characters are expressed in terms of functions

$$A_\lambda = \sum_{w\in\bar W} \epsilon(w)\,\theta_{w(\lambda)}\ ,\ \lambda\in P_+^n\ .$$

Formula (3.5) gives us the following functional equation for A_λ :

$$(3.6)\qquad A_\lambda(\tau,z,t) = \left(\frac{i}{\tau}\right)^{\frac{\ell}{2}}|M^*/nM|^{-\frac{1}{2}} \sum_{w\in\bar W}\ \sum_{\mu\in M^* \bmod nM} \epsilon(w) \times \left(\exp -\frac{2\pi i}{n}(w(\lambda)|\mu)\right)\theta_{\mu,n}\left(-\frac{1}{\tau},-\frac{z}{\tau},t-\frac{(z|z)}{2\tau}\right).$$

Denote by $d(X_N)$ the dimension of the simple Lie algebra associated to a finite type matrix X_N. Let $T=-i\tau$. Introduce the following (finite) set:

$$\bar R_+ = R_+\cap\bar{\mathfrak h}\ \text{ if }\ k^\vee=1\ ;\ \bar R_+ = \Delta_+\cap\bar{\mathfrak h}\ \text{ if }\ k^\vee>1\ .$$

<u>Lemma 3.1</u>. (cf. [5, Proposition 4.21]) Let $\mathfrak g$ be an affine algebra of type $X_N^{(k)}$ and let λ be a strictly positive integral form (i.e. $\langle\lambda,\alpha_i^\vee\rangle\in\mathbb N$, $i\in I$) of level n. Then one has as $T\downarrow 0$:

$$(3.7)\qquad A_\lambda(iT,-iTz,0) \sim b(\lambda,z)T^{-\frac{\ell}{2}}e^{-\frac{\pi}{12T}\frac{gd(X_N)}{kn}}\ ,$$

where $b(\lambda,z) = n^{-\ell/2}|M^*/M|^{-1/2}\prod_{\alpha\in\bar R_+} 4\sin\frac{\pi(\lambda|\alpha)}{n}\sin\pi(z|\alpha)$;

$$(3.8)\qquad A_\rho(iT,-iTz,0) \sim b(\rho,z)T^{-\frac{\ell}{2}}e^{-\frac{\pi}{12T}\frac{d(X_N)}{k}}\ ,$$

where $b(\rho,z) = \prod_{\alpha\in\bar R_+} 2\sin\pi(z|\alpha)$.

<u>Proof</u>. We calculate the leading term of $A_\lambda(iT,-iTz,t)$ as $T\downarrow 0$ using (3.6). Since non-regular μ (i.e. μ fixed under some $r_\alpha\in\bar W$) give a zero contribution, we may rewrite (3.6) as follows:

(3.9) $$A_\lambda(iT,-iTz,0) = (Tn)^{-\frac{\ell}{2}}|M^*/M|^{-\frac{1}{2}} \times \sum_{w\in\bar{W}} \sum_{\substack{\mu\in M^* \mathrm{mod}\, nM \\ \mu \text{ regular}}} \epsilon(w) e^{-\frac{2\pi i}{n}(w(\lambda)|\mu)} \Theta_{\mu,n}(-\tfrac{1}{iT}, z, -\tfrac{1}{2}\, iT(z|z)).$$

By definition, we have:

$$\Theta_{\mu,n}(-\tfrac{1}{iT}, z, -\tfrac{1}{2}\, iT(z|z)) = e^{\pi n T(z|z)} \sum_{\gamma\in M+\mu/n} e^{-\frac{\pi}{T}n(\gamma|\gamma)+2\pi i n(\gamma|z)}.$$

Plugging this into (3.9), we get:

(3.10) $$A_\lambda(iT,-iTz,0) = (Tn)^{-\frac{\ell}{2}} e^{\pi n T(z|z)} |M^*/M|^{-\frac{1}{2}} \times \sum_{w\in\bar{W}} \sum_{\substack{\mu\in M^* \mathrm{mod}\, nM \\ \mu \text{ regular}}} \sum_{\gamma\in M+\mu/n} \epsilon(w) e^{-\frac{2\pi i}{n}(w(\lambda)|\mu) - \frac{\pi}{T}n(\gamma|\gamma) + 2\pi i n(\gamma|z)}.$$

In order to complete the proof we need the following lemma.

Lemma 3.2. Let $\mu \in M^*$ be regular.

(a) If $k^\vee = 1$, then $(\mu|\mu) \geq (\bar{\rho}^\vee|\bar{\rho}^\vee)$ with equality iff $\mu = \sigma(\bar{\rho}^\vee)$ for some $\sigma \in \bar{W}$.

(b) If $k^\vee > 1$, then $(\mu|\mu) \geq (\bar{\rho}|\bar{\rho})$ with equality iff $\mu = \sigma(\bar{\rho})$ for some $\sigma \in \bar{W}$.

Proof. (a) If $k^\vee = 1$, then $M = \sum_{i\in I_0} \mathbb{Z}\alpha_i$, $M^* = \sum_{i\in I_0} \mathbb{Z}\bar{\Lambda}_i^\vee$. Let $\sigma \in \bar{W}$ be such that $\sigma^{-1}(\mu)$ is (strictly) dominant. Then $\sigma^{-1}(\mu) = \sum_i b_i\bar{\Lambda}_i^\vee = \sum_i b_i'\alpha_i$, where $b_i \in \mathbb{N}$, $b_i' \in \mathbb{Q}_+$. Hence: $(\mu|\mu)-(\bar{\rho}^\vee|\bar{\rho}^\vee) = (\sigma^{-1}(\mu)|\sigma^{-1}(\mu))-(\bar{\rho}^\vee|\bar{\rho}^\vee) = (\sigma^{-1}(\mu)-\bar{\rho}^\vee|\sigma^{-1}(\mu)+\bar{\rho}^\vee) = \sum_{i\in I_0} (b_i-1)(\bar{\Lambda}_i^\vee|\sigma^{-1}(\mu)+\bar{\rho}^\vee)$. Since $\sigma^{-1}(\mu)+\bar{\rho}^\vee$ is a linear combination of the α_i, $i \in I_0$, with positive coefficients, this completes the proof of (a). The proof of (b) is similar. □

End of the proof of Lemma 3.1. Using Lemma 3.2, it is clear from (3.10) that the asymptotic behavior of $A_\lambda(iT,-iTz,0)$ as $T \downarrow 0$ is determined by the terms with $\bar{\mu} = \sigma(\bar{\rho}^\vee)$, $\gamma = n^{-1}\sigma(\bar{\rho}^\vee)$ (resp. $\bar{\mu} = \sigma(\bar{\rho})$, $\gamma = n^{-1}\sigma(\bar{\rho})$) if $k^\vee = 1$ (resp. $k^\vee > 1$), where $\sigma \in \bar{W}$. Thus we have as $T \downarrow 0$:

$$A_\lambda(iT,-iTz,0) \sim (Tn)^{-\ell/2}|M^*/M|^{-\frac{1}{2}} e^{-\frac{\pi}{T}\frac{(\xi|\xi)}{n}}$$
$$\times \sum_{\sigma\in\bar W}\Big(\sum_{w\in\bar W}\epsilon(w)e^{-\frac{2\pi i}{n}(\sigma w(\lambda)|\xi)}\Big)e^{2\pi i(\xi|\sigma(z))},$$

where $\xi = \bar\rho^\vee$ if $k^\vee = 1$ and $\xi = \bar\rho$ if $k^\vee > 1$. (Note that $\bar\rho = \bar\rho^\vee$ in the simply laced case.) Using twice the Weyl denomiator identity and the "strange" formula (cf. [5, Proposition 1.11]):

(3.11) $\qquad \frac{1}{24}\, d(X_N) = \frac{k}{2g}\,(\xi|\xi)$,

we obtain the proof of (3.7). The proof of (3.8) follows from (3.7) and the identity [5, Proposition 4.30]:

$$\prod_{\alpha\in\bar R_+} 2\sin \pi(\rho|\alpha)/g = |M^*/M|^{1/2} g^{\ell/2}. \qquad \square$$

Let now $s_\lambda = \frac{(\lambda+\rho\,|\,\lambda+\rho)}{2\,(m+g)} - \frac{(\rho|\rho)}{2g}$, where m is the level of λ. We have by the Weyl–Kac character formula (which is a special case of (3.2)):

(3.12) $\qquad e^{2\pi i\tau s_\lambda} ch_\lambda = A_{\lambda+\rho}/A_\rho$ if $\lambda\in P_+^m$,

(3.13) $\qquad e^{2\pi i\tau(\rho|\rho)/2g} Q_{\mathfrak g} = A_\rho$.

Remark 3.2. Formula (3.12) holds in the whole domain Y; more precisely the left–hand side converges in Y and the right–hand side has removable singularities along hyperplanes

$$T_{\alpha,n} = \{(\tau,z,t)\in Y\,|\,(\alpha|z) = n\},\ \alpha\in\bar R_+,\ n\in\mathbb{Z}.$$

For general admissible λ not all singularities $T_{\alpha,n}$ are removable. For example, in the right–hand side of (3.3), $T_{\alpha,n}$ is removable if and only if

$$(\beta|\alpha)\in n + u\mathbb{Z},\ \text{where}\ y = t_\beta\bar y,\ \beta\in\tilde M,\ \bar y\in\bar W.$$

The same argument as in [4, Chapter 10] shows that $ch_\lambda(\tau,z,t)$ converges in the interior of $W_\lambda^0(\bar D)$, where $W_\lambda^0 = \langle r_i\,|\,\langle\lambda,\alpha_i^\vee\rangle\in\mathbb{Z}_+\rangle$.

Applying Lemma 3.1 to (3.12), we obtain

Theorem 3.3. Let $\mathfrak g$ be an affine algebra of type $X_N^{(k)}$ and let $\lambda\in P_+^m$, $m\in\mathbb{Z}_+$. Then we have for each $z\in\bar{\mathfrak h}$ as $T\downarrow 0$:

$$(3.14)\qquad \mathrm{tr}_{L(\lambda)} e^{2\pi T(a_0^{-1} d+z)} \sim a(\lambda) e^{\frac{\pi}{12T} g^{(m)}} ,$$

where

$$(3.15)\qquad a(\lambda) = (m+g)^{-\frac{\ell}{2}} |M^*/M|^{-\frac{1}{2}} \prod_{\alpha\in\bar{R}_+} 2 \sin \frac{\pi(\lambda+\rho|\alpha)}{m+g} ,$$

$$(3.16)\qquad g^{(m)} = \frac{m}{k(m+g)} d(X_N) . \qquad \square$$

Now we turn to the calculation of asymptotics of ch_λ for $\lambda \in P^m_{u,y}$. For this we need an explicit expression for $\hat{h}$ (defined before Theorem 3.2).

Lemma 3.3. Let $h = (\tau,-\tau z,t) \in Y$, let $u \in \mathbb{N}$ and let $y = t_\beta \bar{y} \in \tilde{W}$, where $\beta \in \tilde{M}$, $\bar{y} \in \bar{W}$. Then

$$\hat{h} = (\tau u, \tau\bar{y}^{-1}(\beta - z), u^{-1}(t-\tau(z|\beta)+ \tfrac{1}{2}\tau(\beta|\beta)) .$$

Proof. Let h' denote the right–hand side. We check directly that

$$(\alpha_i^\vee|h') = (y(\gamma_i)|h) ,\ i \in I ;\ u(d|h') = (y(d)|h) . \qquad \square$$

Combining Lemma 3.3, Theorem 3.3, formulas (3.3), (3.13) and (3.8) gives the following theorem.

Theorem 3.4. Let $\mathfrak{g}$ be an affine algebra of type $X_N^{(k)}$ and let $\lambda = y.(\lambda^0-(u-1)(m+g)\Lambda_0) \in P^m_{u,y}$ be an admissible weight. Recall that $u \in \mathbb{N}$ is the denominator of the rational number m, $\lambda^0 \in P_+^{u(m+g)-g}$ and $y = t_\beta \bar{y}$, where $\beta \in \tilde{M}$ and $\bar{y} \in \bar{W}$. Then for each $z \in D_0$, we have as $T \downarrow 0$:

$$(3.17)\qquad \mathrm{tr}_{L(\lambda)} e^{2\pi T(a_0^{-1} d+z)} \sim b(\lambda,z) e^{\frac{\pi}{12T} g^{(m)}} ,$$

where:

$$(3.18)\qquad b(\lambda,z) = \epsilon(\bar{y}) u^{-\frac{\ell}{2}} a(\lambda^0) \prod_{\alpha\in\bar{R}_+} \sin \frac{\pi(z-\beta|\alpha)}{u} / \sin \pi(z|\alpha) ,$$

$$(3.19)\qquad g^{(m)} = \frac{m+(1-u^{-2})g}{k(m+g)} d(X_N) . \qquad \square$$

Applying Theorem 3.3 and (3.13) to (3.4), we derive asymptotics of ch_λ for arbitrary admissible λ of level m as $T \downarrow 0$:

(3.20a) $$\operatorname{tr}_{L(\lambda)} e^{2\pi T(a_0^{-1}d+z)} \sim \text{const}\ e^{\frac{\pi}{12T} g_\lambda},$$

where const is equal to the product of the $\dot{b}(\dot{\lambda},\dot{z})$ corresponding to indecomposable components of S^λ and the *growth* g_λ is given by the following formula:

(3.20b) $$g_\lambda = \frac{d(X_N)}{k} - \frac{1}{m+g}\left(\frac{\dot{K}\dot{g}d(\dot{X}_{\dot{N}})}{\dot{k}(\dot{e}\dot{u})^2} + \frac{\ddot{K}\ddot{g}d(\ddot{X}_{\ddot{N}})}{\ddot{k}(\ddot{e}\ddot{u})^2} + \ldots\right).$$

The growth g_λ measures the "size" of the representation $L(\lambda)$. In the integrable case, i.e. the case $\lambda \in P_+$, g_λ grows linearly with ℓ. Using formula (3.20b) it is not difficult to find all other cases of admissible λ with g_λ growing linearly with ℓ (in all other cases g_λ grows quadratically).

Proposition 3.1. Let A be a classical affine matrix of rank ℓ.

(a) Let λ be an admissible weight of level m of type 2_s or 3_s with s as in Table 3. Then

(i) $m \in \mathbb{Z} \Rightarrow g_\lambda$ grows linearly with ℓ,

(ii) $A = C_\ell^{(1)}$ and $m \in \frac{1}{2} + \mathbb{Z} \Rightarrow g_\lambda$ grows linearly with ℓ.

(b) All cases in which λ is non–integrable admisible and g_λ grows linearly with ℓ are given by (a) and are listed explicitly in Table 3 ($r = \ell - s$):

Table 3

A	case	s	m ∈
$B_\ell^{(1)}$	3_s	$2 \le s < \frac{1}{2}\ell$	$-2s + \mathbb{Z}_+$
	3_s	$\frac{1}{2}\ell < s \le \ell-1$	$-2r + \mathbb{Z}_+$
$C_\ell^{(1)}$	2_s	$0 \le s < \frac{1}{2}\ell$	$-s - \frac{1}{2} + \mathbb{Z}_+$
	3_s	$2 \le s < \frac{\ell}{2} + 1$	$-s + \mathbb{Z}_+$
	3_s	$\frac{\ell}{2} + 1 < s \le \ell$	$-r - 2 + \mathbb{Z}_+$
$D_\ell^{(1)}$	2_s	$2 \le s < \frac{1}{2}\ell$	$-2s + \mathbb{Z}_+$

$A_{2\ell}^{(2)}$	3_s	$1 \le s < \frac{\ell+1}{2}$	$-2s + \mathbb{Z}_+$
	3_s	$\frac{\ell+1}{2} < s \le \ell$	$-2r - 1 + \mathbb{Z}_+$
$A_{2\ell-1}^{(2)}$	2_s	$2 \le s < \frac{\ell+1}{2}$	$-2s + \mathbb{Z}_+$
	2_s	$\frac{\ell+1}{2} < s \le \ell$	$-2r - 1 + \mathbb{Z}_+$
$D_{\ell+1}^{(2)}$	2_s	$1 \le s < \frac{1}{2}\ell$	$-2s + \mathbb{Z}_+$

□

Explicit constructions of these representations are known only in cases $(C_\ell^{(1)}, 2_0, m = -\frac{1}{2})$, $(A_{2\ell}^{(2)}, 3_\ell, m = -1)$, $(A_{2\ell-1}^{(2)}, 2_\ell, m = -1)$ [3],[8]. It would be interesting to find explicit constructions of other representations from the list.

For the study of transformation properties another form of the character formula is useful. Define the *normalized character* (cf. (3.12)):

$$\chi_\lambda = e^{2\pi i \tau s_\lambda} \mathrm{ch}_\lambda .$$

Theorem 3.5. Let λ be an admissible weight for A of type A, i.e.

$$\lambda = y.(\lambda^0 - (u-1)(m+g)\Lambda_0),$$

where m is a rational number with denominator $u \in \mathbb{N}$, $\lambda^0 \in P_+^{u(m+g)-g}$ and $y = t_\beta \bar{y}$ with $\beta \in \tilde{M}$, $\bar{y} \in \bar{W}$ and $y(S_{(u)}) \subset R_+$. Then

$$\chi_\lambda(\tau,z,t) = \sum_{w\in\bar{W}} \epsilon(\bar{y}w)\Theta_{uw(\lambda^0+\rho)+u(m+g)\beta}(\tau,z/u,t/u^2)/A_\rho(\tau,z,t) .$$

Proof. Formula (3.2) can be written as follows (see the proof of Theorem 3.2):

$$\chi_\lambda(h) = A_{\lambda^0+\rho}(\hat{h})/A_\rho(h) . \tag{3.21}$$

Furthermore, Lemma 3.3 can be stated as follows:

$$\hat{h} = u\bar{y}^{-1} t_{-\beta/u}(\tau,z/u,t/u^2)$$

Using these and the obvious formulas

$$\Theta_\lambda(h) = \Theta_{\lambda+a\delta}(h) = \Theta_{u\lambda}(h/u),\ \Theta_{t_\beta(\lambda)}(h) = \Theta_\lambda(t_{-\beta}(h)) ,$$

we get the result. □

We turn now to the transformation properties of the characters. We consider here only the case when λ has type A (in other cases formulas are more complicated). Denote the set of such λ having level m by $P^m(A)$; explicitly (Theorem 2.1): $P^m(A) = \{t_\beta \bar{y}.(\lambda^0-(u-1)(m+g)\Lambda_0 \,|\, \beta \in \tilde{M}, \bar{y} \in \bar{W}, \lambda^0 \in P_+^{u(m+g)-g}$ and $t_\beta \bar{y}(S_{(u)}) \subset R_+\}$. (Recall that $u \in \mathbb{N}$ is the denominator of m , and that g.c.d. $(u,k^\vee) = 1$.)

We first consider the case when A is either non–twisted (i.e. has type $X_\ell^{(1)}$) or has type $A_{2\ell}^{(2)}$. Note that in this case (due to (3.11)) s_λ can be rewritten as follows:

$$s_\lambda = \frac{(\bar{\lambda}+2\bar{\rho}|\bar{\lambda})}{2(m+g)} - \frac{1}{24}\frac{m d(X_N)}{k(m+g)},$$

and also that $M = \bar{Q}^\vee$, $\tilde{M} = M^* = \bar{P}$ $(= \{\lambda \in \bar{\mathfrak{h}}^* \,|\, \langle\lambda,\alpha_i^\vee\rangle \in \mathbb{Z}$ for all $i = 1,\ldots,\ell\})$, $(\bar{Q}^\vee|\bar{Q}^\vee) \subset \mathbb{Z}$. Recall that in this case we have [5, Proposition 4.6(c)]:

$$A_\rho(-\tfrac{1}{\tau}, \tfrac{z}{t}, t - \tfrac{(z|z)}{2\tau}) = (-i)^{|\bar{\Delta}_+|}(-i\tau)^{\ell/2} A_\rho(\tau,z,t) . \tag{3.22}$$

For $\lambda = t_\beta \bar{y}.(\lambda^0-(u-1)(m+g)\Lambda_0) \in P^m(A)$, let

$$B_\lambda(\tau,z,t) = \sum_{w \in \bar{W}} \epsilon(\bar{y}w)\, \theta_{uw(\lambda^0+\rho)+u(m+g)\beta}(\tau,z/u,t/u^2) .$$

Then Theorem 3.5 can be stated as follows:

$$\chi_\lambda = B_\lambda/A_\rho . \tag{3.23}$$

Using formula (3.5) we get:

$$B_\lambda(-\tfrac{1}{\tau},\tfrac{z}{\tau},t - \tfrac{(z|z)}{2\tau}) = (-i\tau)^{\ell/2}\, |M^*/u^2(m+g)M|^{-\frac{1}{2}} \sum_{w \in \bar{W}} \epsilon(\bar{y}w) \tag{3.24}$$

$$\times \sum_{\mu \in P^{u^2(m+g)} \bmod u^2(m+g)M} e^{-\frac{2\pi i}{u(m+g)}(w(\lambda^0+\rho)+(m+g)\beta|\bar{\mu})}\, \theta_\mu(\tau,z/u,t/u^2) ,$$

since $M^* = \bar{P}$ in our case. To go further, we need the following lemma (which will be proved later).

<u>Lemma 3.4</u>. Suppose that A is of type $X_\ell^{(1)}$ or $A_{2\ell}^{(2)}$.

(a) Given $\beta \in \tilde{M}$, there exists a unique $\bar{y} \in \bar{W}$ and a unique $\gamma \in M$ such that $t_{\beta+u\gamma}\bar{y}(S_{(u)}) \subset R_+$.

(b) Let u and v be relatively prime positive integers, and let u be relatively prime to $k^\vee a_0^{-1}$.

Then:

(i) any element $\mu \in P^{uv}$ can be written in the form

$(3.25)_v$ $\qquad \mu = uw(\nu) + v\beta$, where $\nu \in P_+^v$, $\beta \in \tilde{M}$ and $w \in \bar{W}$;

(ii) there exists precisely $n := |\tilde{M}/M|$ such solutions mod uM of the equation (3.25) for given μ : (ν_i,β_i,w_i) , $i = 1,\dots,n$;

(iii) for each triple (ν_i,β_i,w_i) there exists a unique $\bar{y}_i \in \bar{W}$ such that $t_{\beta_i}\bar{y}_i(S_{(u)}) \subset R_+$; we let $y_i = t_{\beta_i}\bar{y}_i \in \tilde{W}$ $(i = 1,\dots,n)$;

(iv) $S := y_i(S_{(u)})$ is independent of i ;

(v) the elements $\lambda_i+\rho:=y_i(\nu_i-(1-u^{-1})v\Lambda_0)$ lie in the fundamental chamber for S ;

(vi) $\lambda := \lambda_i$ is independent of i ;

(vii) $\lambda+\rho$ is regular with respect to S iff ν_i is regular with respect to $\Pi^\vee$ for some (resp. all) i ;

(viii) if $\lambda+\rho$ is regular with respect to S , then λ is an admissible weight (of level $u^{-1}v-g$) with $S^\lambda = S$;

(ix) if $\lambda+\rho$ is regular, then $\epsilon(w_i\bar{y}_i)$ is independent of i ;

(x) $\sum_{w\in\bar{W}} \epsilon(\bar{y}_i w)\, \Theta_{uw(\nu_i)+v\beta_i}(\tau,z/u,t/u^2) = A_\rho ch_\lambda'(\tau,z,t)$ if ν_i is regular, and $= 0$ otherwise; in particular, it is independent of i .

Using Lemma 3.4(b)(i), we obtain:

$$(3.26)\quad \sum_{\mu\in P^{u^2(m+g)} \bmod u^2(m+g)M} e^{-\frac{2\pi i}{u(m+g)}(w(\lambda^0+g)+(m+g)\beta|\bar{\mu})}\,\Theta_\mu(t,z/u,t/u^2)$$

$$= \sum_{\substack{w'\in\bar{W}\\ \nu\in P_+^{u(m+g)}\\ \beta'\in\tilde{M} \bmod uM}} e^{-\frac{2\pi i}{m+g}(w'^{-1}w(\lambda^0+\rho)|\nu)-2\pi i((\lambda^0+\rho|\beta')+(\nu|\beta)+(m+g)(\beta|\beta'))}$$

$$\times\,\Theta_{u(w'(\nu)+(m+g)\beta')}(t,z/u,t/u^2)\ ,$$

where the summation is taken over all equivalence classes of triples (w',ν,β') (two such triples are called equivalent if they give the same μ in the equation $(3.25)_{u(m+g)}$). We have used here that $(w(\lambda^0+\rho)|\beta') \equiv (\lambda^0+\rho|\beta') \bmod \mathbb{Z}$ and $(\beta|w'(\nu)) \equiv (\nu|\beta) \bmod \mathbb{Z}$. Plugging (3.26) in (3.24), we can replace $\nu \in P_+^{u(m+g)}$ by $\lambda'^0+\rho$, where

$\lambda'^{o} \in P_{+}^{u(m+g)-g}$ (otherwise ν is non–regular, hence the corresponding contribution is 0), obtaining (the summation is again taken over all equivalence classes of triples):

$$B_{\lambda}(-\frac{1}{\tau},\frac{z}{\tau},t-\frac{(z|z)}{2\tau}) = (-i\tau)^{\ell/2}|M^{*}/u^{2}(m+g)M|^{-\frac{1}{2}} \sum_{\substack{w''\in\bar{W} \\ \lambda'^{o}\in P_{+}^{u(m+g)-g} \\ \beta'\in\tilde{M} \bmod uM}} \epsilon(\bar{y}w'')$$

$$\times e^{-\frac{2\pi i}{m+g}(w''(\lambda^{o}+\rho)|\lambda'^{o}+\rho)-2\pi i((\lambda^{o}+\rho|\beta')+(\lambda'^{o}+\rho|\beta)+(m+g)(\beta|\beta'))}$$

$$\times \sum_{w'\in\bar{W}} \epsilon(w')\Theta_{uw'(\lambda'^{o}+\rho)+u(m+g)\beta'}(\tau,z/u,t/u^{2}) .$$

Using Lemma 3.4(b)(x), the last factor can be written in the form $\epsilon(\bar{y}')B_{\lambda'}(\tau,z,t)$, $y'\in\bar{W}$. Thus, we obtain the following

Theorem 3.6. Suppose that A is either non–twisted or of type $A_{2\ell}^{(2)}$. Then we have the following transformation formula for $\lambda\in P^{m}(A)$:

$$\chi_{\lambda}(-\frac{1}{\tau},\frac{z}{\tau},t-\frac{(z|z)}{2\tau}) = \sum_{\lambda'\in P^{m}(A)} a(\lambda,\lambda')\chi_{\lambda'}(\tau,z,t) ,$$

where

$$a(\lambda,\lambda') = i^{|\bar{\Delta}_{+}|}u^{-\ell}(m+g)^{-\ell/2}|M^{*}/M|^{-1/2}\epsilon(\bar{y}\bar{y}')e^{-2\pi i((\lambda^{o}+\rho|\beta')+(\lambda'^{o}+\rho|\beta)+(m+g)(\beta|\beta'))}$$
$$\times \sum_{w\in\bar{W}} \epsilon(w)\, e^{-\frac{2\pi i}{m+g}(w(\lambda^{o}+\rho)|\lambda'^{o}+\rho)} . \qquad \square$$

Proof of Lemma 3.4. In order to prove (a), note that $S_{(u)}$ is the simple coroot set for $R_{(u)} := \{\alpha\in R|<\Lambda_{0},\alpha>\in u\mathbb{Z}\}$, uc is its canonical central element and $W_{(u)} := \{\bar{w}t_{u\alpha}|\bar{w}\in\bar{W},\ \alpha\in M\}$ is its Weyl group. Let $\xi=\rho/g$ in the case A symmetric or $A_{2\ell}^{(2)}$, and let $\xi=\rho^{\vee}/h$ otherwise. We have to show that, given $\beta\in\bar{P}$, there exists a unique $\gamma\in M$ and a unique $\bar{y}\in\bar{W}$ such that $<\xi,t_{\beta+u\gamma}\bar{y}(\gamma_{i})> > 0$ for all $i\in I$, or, equivalently, such that $\bar{y}^{-1}t_{-\beta-u\gamma}(\xi)$ lies in $C_{(u)}$, the fundamental chamber for $S_{(u)}$. Since $t_{-\beta}(\xi)=\xi-\beta \bmod \mathbb{C}\delta$ and $<\xi,\alpha>\notin\mathbb{Z}$ for all $\alpha\in R$, the element $t_{-\beta}(\xi)$ is regular. Hence there exists a unique $w\in\bar{W}_{(u)}$ such that $wt_{-\beta}(\xi)$ lies in $C_{(u)}$.

Writing $w = \bar{y}^{-1} t_{-u\gamma}$ with $\bar{y} \in \bar{W}$, $\gamma \in M$, we obtain the result.

To prove (b), write $\mu = uv\Lambda_0 + \sum_{i\in I_0} n_i \bar{\Lambda}_i$, $n_i \in \mathbb{Z}$. There exists $n_i', n_i'' \in \mathbb{Z}$ such that $n_i = un_i' + a_0^{-1} k^\vee v n_i''$ $(i \in I_0)$. Hence $\mu = u\nu' + v\beta$, where $\nu' = v\Lambda_0 + \sum_{i\in I_0} n_i' \bar{\Lambda}_i \in P^v$ and $\beta = a_0^{-1} k^\vee \sum_{i\in I_0} n'' \bar{\Lambda}_i \in a_0^{-1} k^\vee \bar{P} \subset \tilde{M}$. Choosing $w \in \bar{W}$ such that $\nu = w^{-1}(\nu')$ $\in P_+^v$, we obtain the decomposition $(3.25)_v$. Statement b(ii) is clear since choosing representatives $\xi_1,\dots,\xi_n$ of $\tilde{M}$ mod uM, elements $\beta+u\xi_1,\dots,\beta+u\xi_n$ are also solutions of $(3.25)_v$ for some uniquely defined $w_1,\dots,w_n \in \bar{W}$. Statement (b)(iii) follows from (b)(ii) and (a).

In order to prove (b)(iv), let

$$\tilde{W}_+ = \{w \in \tilde{W}_{(u)} \,|\, w(\Pi^\vee) = \Pi^\vee\}\ ,\ \tilde{W}_{(u)+} = \{w \in \tilde{W} \,|\, w(S_{(u)}) = S_{(u)}\}\ .$$

We have [5, Proposition 4.27]: $\tilde{W}_+ = \{t_{\xi_i} w^{(i)} \,|\, i = 1,\dots,n\}$, where $\xi_1,\dots,\xi_n$ is certain set of representatives of $\tilde{M}$ mod M and $w^{(i)} \in \bar{W}$. Hence $\tilde{W}_{(u)+} = \{t_{u\xi_i} w^{(i)} \,|\, i = 1,\dots,n\}$.

Fix now a solution (ν,β,w) of $(3.25)_v$. By b(iii) there exists a unique $\bar{y} \in \bar{W}$ such that $t_\beta \bar{y} S_{(u)} \subset R_+$. Then we can construct the following set of representatives of $\tilde{M}$ mod uM: $\beta_i = \beta + u\bar{y}(\xi_i)$, $i = 1,\dots,n$; for each β_i the corresponding $\bar{y}_i$ is $\bar{y}w^{(i)}$, since $t_{\beta_i}(\bar{y}w^{(i)})(S_{(u)}) = t_\beta(t_{u\bar{y}(\xi_i)}\bar{y})w^{(i)}(S_{(u)}) = t_\beta \bar{y} t_{u\xi_i} w^{(i)}(S_{(u)}) = t_\beta \bar{y}(S_{(u)}) \subset R_+$. Using b(iii), we obtain $t_{\beta_i}\bar{y}_i(S_{(u)}) = t_\beta \bar{y}(S_{(u)})$.

b(v) is clear since $S = y_i(S_{(u)})$ and $\nu_i - (1-u^{-1})v\Lambda_0$ is in the fundamental chamber for $S_{(u)}$.

To prove b(vi) note that $W_S := y_i\, W_{(u)}\, y_i^{-1}$ is the Weyl group of S, hence is independent of i. Equation $(3.25)_v$ can be written as follows:

$$u^{-1}\mu - (1-u^{-1})v\Lambda_0 = y_i \bar{y}_i^{-1} w_i(\nu_i - (1-u^{-1})v\Lambda_0)\ .$$

Hence we have:

$$\lambda_i + \rho = y_i(w_i^{-1}\bar{y}_i)y_i^{-1}(u^{-1}\mu - (1-u^{-1})v\Lambda_0)\ , \tag{3.27}$$

i.e. $\lambda_i + \rho$ is W_S–conjugate to $u^{-1}\mu - (1-u^{-1})v\Lambda_0$. Thus, the elements $\lambda_i + \rho$ from the fundamental chamber for W_S (see (b)(v)) are W_S–conjugate, hence they are equal.

b(vii) is clear since ν_i lies in the fundamental chamber. b(viii) is also clear since $\lambda_i = y_i.((\nu_i-\rho) - (1-u^{-1})v\Lambda_0)$. b(ix) follows from (3.27) and b(vi). Finally, b(x) follows from Theorem 3.5. □

We consider now the remaining cases, i.e. A of type $A^{(2)}_{2\ell-1}$, $D^{(2)}_{\ell+1}$, $E^{(2)}_6$ or $D^{(3)}_4$. We let A′ to be of type $D^{(2)}_{\ell+1}$, $A^{(2)}_{2\ell-1}$, $E^{(2)}_6$ and $D^{(3)}_4$ respectively. Let $\tilde{s} = \ell+1-s$ for $s \in I_0$, and let $\bar{\Lambda}'_s = (a_s/a^\vee_s)\bar{\Lambda}_s$ (resp. $= (a_{\tilde{s}}/a^\vee_{\tilde{s}})\bar{\Lambda}_{\tilde{s}}$) if $A = A^{(2)}_{2\ell-1}$ or $D^{(2)}_{\ell+1}$ (resp. $= E^{(2)}_6$ or $D^{(3)}_4$). Let $\Lambda'_i = \bar{\Lambda}'_i + a'^\vee_i\Lambda_0$ $(i \in I_0)$, $\Lambda'_0 = \Lambda_0$, $\rho' = \sum_{i\in I} \Lambda'_i$, $c' = c$, $\bar{P}' = \sum_{i\in I_0} \mathbb{Z}\bar{\Lambda}'_i$, $P' = \sum_{i\in I} \mathbb{Z}\Lambda'_i$, $M' = \bar{Q}^\vee$, $\bar{W}' = \bar{W}$. Then $\bar{P}' = M^*$ and P' is the set of integral weights for A′. As before, we define functions Θ'_λ, A'_λ and χ'_λ for the affine matrix A′.

Then we have the following transformation properties [5] for $\lambda \in P^m$ $(m \in \mathbb{N})$:

$$\Theta_\lambda(-\frac{1}{\tau},\frac{z}{\tau},t-\frac{(z|z)}{2\tau}) = (-i\tau)^{\ell/2}|M^*/mM|^{-1/2} \sum_{\mu\in P'^m \bmod mM'} e^{-\frac{2\pi i}{m}(\bar{\lambda}|\bar{\mu})}\Theta'_\mu(\frac{\tau}{k},\frac{z}{k},t),$$

$$A_\rho(-\frac{1}{\tau},\frac{z}{\tau},t-\frac{(z|z)}{2\tau}) = (-i\tau)^{\ell/2}(-i)^{|\bar{\Delta}_+|}|M'/M|^{-\frac{1}{2}}A'_{\rho'}(\frac{\tau}{k},\frac{z'}{k},t) .$$

Using these transformation formulas and Lemma 3.4 applied to A′, we deduce

<u>Theorem 3.7</u>. Suppose that A is of type $A^{(2)}_{2\ell-1}$, $D^{(2)}_{\ell+1}$, $E^{(2)}_6$ or $D^{(3)}_4$. Then we have for $\lambda \in P^m(A)$:

$$\chi_\lambda(-\frac{1}{\tau},\frac{z}{\tau},t-\frac{(z|z)}{2\tau}) = \sum_{\lambda'\in P^m(A')} a(\lambda,\lambda')\chi'_{\lambda'}(\frac{\tau}{k},\frac{z}{k},t) ,$$

where

$$a(\lambda,\lambda') = i^{|\bar{\Delta}_+|}u^{-\ell}(m+g)^{-\ell/2}|M^*/M'|^{-1/2}\epsilon(\bar{y}\bar{y}')$$
$$\times e^{-2\pi i((\lambda^0+\rho|\beta')+(\lambda'^0+\rho'|\beta)+(m+g)(\beta|\beta'))}\sum_{w\in\bar{W}}\epsilon(w)e^{-\frac{2\pi i}{m+g}(w(\lambda^0+\rho)|\lambda'^0+\rho')} . \quad □$$

We conclude this section with the following interpretation of the matrix $(a(\lambda,\mu))$.

Theorem 3.8. Let A be of type $X_\ell^{(1)}$ or $A_{2\ell}^{(2)}$, and let $\lambda,\mu \in P^m(A)$. Define $\mu^* \in P_+^{u(m+g)-g}$ and $\sigma \in W$ by $u\sigma.\mu = \mu^*$, and let $\epsilon(\mu) = \epsilon(\sigma)$.

Then we have as $T \downarrow 0$:

$$\mathrm{ch}_\lambda(iT, -\frac{\bar\mu+\bar\rho}{m+g}, 0) \sim \epsilon(\mu) a(\lambda,\mu)\, e^{\frac{\pi g}{T}\left|\frac{\bar\mu^*+\bar\rho}{u(m+g)} - \frac{\bar\rho}{g}\right|^2}.$$

Proof. We have for $\Lambda \in P_{++}^n$ (cf. (3.10)):

$$A_\Lambda(iT,z,t) = (nT)^{-\frac{\ell}{2}} |M^*/M|^{-\frac{1}{2}} e^{2\pi i n t} \times \sum_{w\in\bar W} \sum_{\substack{\nu\in M^* \bmod nM \\ \nu \text{ regular}}} \sum_{\gamma\in M+\frac{\nu}{n}} \epsilon(w)\, e^{-\frac{2\pi i}{n}(w(\bar\Lambda)|\nu) - \frac{\pi n}{T}|\gamma+z|^2}. \tag{3.28}$$

Using (3.28) for $\hat h$ given by Lemma 3.3, where $h = (\tau,z,0)$, we have:

$$A_{\lambda^0+\rho}(\hat h) = (u^2(m+g)T)^{-\frac{\ell}{2}} |M^*/M|^{-\frac{1}{2}} e^{\frac{2\pi i(m+g)}{u}(\beta|z+\frac{1}{2}iT\beta)} \times \sum_{w\in\bar W} \sum_{\substack{\nu\in M^* \bmod u(m+g)M \\ \nu \text{ regular}}} \sum_{\gamma\in M+\frac{\nu}{u(m+g)}} \epsilon(w) \times \exp\left(-\frac{2\pi i}{u(m+g)}(w(\lambda^0+\rho)|\nu) - \frac{\pi(m+g)}{T}|\gamma+\bar y^{-1}(iT\beta+z)|^2\right). \tag{3.29}$$

Now we let $z = -(\bar\mu+\bar\rho)/(m+g)$ in (3.29). Since $\mu+\rho \equiv \bar y'(\bar\mu^0+\bar\rho) + (m+g)\beta'$ $\bmod(\mathbb{C}\Lambda_0 + \mathbb{C}\delta)$, we have: $z = -(u\bar y'(\bar\mu^0+\bar\rho) + u(m+g)\beta')/u(m+g)$. Since $\beta' \in \tilde M \subset M^*$, the element $\nu = \bar y^{-1}(u\bar y'(\bar\mu^0+\bar\rho) + u(m+g)\beta')$ lies in M^*, hence the leading term in (3.29) is given by the pair $\nu = u\bar y^{-1}(\bar y'(\bar\mu^0+\bar\rho) + (m+g)\beta')$, $\gamma = \nu/u(m+g)$. Thus, we have as $T \downarrow 0$:

$$A_{\lambda^0+\rho}(\hat h) \sim (u^2(m+g)T)^{-\frac{\ell}{2}} |M^*/M|^{-\frac{1}{2}} e^{-2\pi i(\bar y'(\bar\mu^0+\bar\rho)+(m+g)\beta'|\beta)} \times \sum_{w\in\bar W} \epsilon(w)\, e^{-\frac{2\pi i}{m+g}(\bar y w(\lambda^0+\rho)|\bar y'(\bar\mu^0+\bar\rho)) - 2\pi i(\bar y w(\lambda^0+\rho)|\beta')}. \tag{3.30}$$

Since $\bar\lambda^0+\bar\rho \in \bar P$, we have: $w(\bar\lambda^0+\bar\rho) - (\bar\lambda^0+\bar\rho) \in M$ for any $w \in \bar W$. It follows that $(w(\bar\lambda^0+\bar\rho) - (\bar\lambda^0+\bar\rho)|\gamma) \in \mathbb{Z}$ for any $w \in \bar W, \gamma \in \tilde M \subset M^* = \bar P$. Hence we have as $T \downarrow 0$:

$$(3.31)\qquad A_{\lambda^0+\rho}(\hat{h}) \sim T^{-\frac{\ell}{2}}(-i)^{|\bar{\Delta}_+|}\, a(\lambda,\mu)\ ,\ \text{where}\ h = (\tau, -\tfrac{\bar{\mu}+\bar{\rho}}{m+g}, 0)\ .$$

Furthermore, applying (3.28) to the case $X_\ell = A_1$, $\Lambda = \rho$, we obtain, using (3.13), for $0 < z < 1$, as $T \downarrow 0$:

$$(3.32)\qquad \prod_{n\geq 1} (1 - e^{-2\pi nT-2\pi iz})(1 - e^{-2\pi(n-1)T+2\pi iz}) \sim -ie^{\pi iz} e^{-\frac{\pi}{T}(z^2-z+\frac{1}{6})}\ .$$

Asymptotics (3.32) together with (3.13) and

$$(3.33)\qquad \sum_{\alpha\in\bar{\Delta}_+} (\alpha|\lambda)(\alpha|\lambda') = g(\lambda|\lambda')\ \text{ for }\ \lambda,\lambda' \in \bar{\mathfrak{h}}^*\ ,$$

gives for $z \in D_0$:

$$(3.34)\qquad A_\rho(iT,-z\ ,0) \sim T^{-\frac{\ell}{2}}(-i)^{|\bar{\Delta}_+|} e^{-\frac{\pi g}{T}|z-\frac{\bar{\rho}}{g}|^2}\ .$$

An immediate generalization of (3.34) is the following lemma.

Lemma 3.5. Let $z \in \bar{\mathfrak{h}}^*$ be such that $\Lambda_0 + z$ is regular. Choose $\sigma \in \bar{W}$ and $z' \in D_0$ such that $\sigma(z)-z' \in M$. Then we have as $T \downarrow 0$:

$$A_\rho(iT,-z,0) \sim \epsilon(\sigma)(-i)^{|\bar{\Delta}_+|} T^{-\frac{\ell}{2}} e^{-\frac{\pi g}{T}|z' - \frac{\bar{\rho}}{g}|^2}\ . \qquad \square$$

Theorem 3.8 follows immediately from (3.21), (3.31) and Lemma 3.5. □

§4. Appendix.

4.1. On complete reducibility. We have the following simple general result:

Theorem 4.1. Let $\mathfrak{g}$ be an affine algebra and let V be a $\mathfrak{g}$–module from the category $\mathcal{O}$ such that every its irreducible subquotient $L(\lambda)$ satisfies the properties

$$(4.1)\qquad \langle\lambda+\rho,\alpha\rangle \notin \{-1,-2,...\}\ \text{ for all }\ \alpha \in R_+\ ,$$

$$(4.2)\qquad \mathrm{Re}\langle\lambda+\rho,c\rangle > 0\ .$$

Then V is completely reducible.

Proof. Condition (4.2) means that $\lambda+\rho$ lies in the interior of the Tits cone and condition (4.1) means that $\lambda+\rho$ lies in the fundamental chamber for W^λ . Now we can apply [2, Theorem 5.7]. □

Corollary 4.1. Let $\Lambda \in P_+$ and let λ be an admissible weight. Then the $\mathfrak{g}$–module $L(\Lambda) \otimes L(\lambda)$ decomposes into a direct sum of irreducible $\mathfrak{g}$–modules with admissible highest weights μ with $R^{\mu} = R^{\lambda}$.

Proof. First, c acts on $L(\Lambda) \otimes L(\lambda)$ as a scalar $\Lambda(c) + \lambda(c) > -g$, since $\Lambda(c) \geq 0$ and, by (1.8), $<\lambda+\rho,c> > 0$. Second, we have:

$$Q_{\mathfrak{g}}\, ch_{(L(\Lambda) \otimes L(\lambda))} = ch_{\Lambda} \sum_{w \in W^{\lambda}} \epsilon(w) e^{w(\lambda+\rho)}$$

by (3.2). Since ch_{Λ} is W–invariant, we deduce that $Q_{\mathfrak{g}}\, ch_{(L(\Lambda) \otimes L(\lambda))}$ is W^{λ}–anti–invariant, hence is an (infinite) sum of expressions of the form $\sum_{w \in W^{\lambda}} \epsilon(w) e^{w(\mu+\rho)}$, where $\mu+\rho$ lies in the interior of the fundamental chamber for W^{λ}. Hence $<\mu+\rho,\alpha> \notin \{0,-1,-2,...\}$ and we can apply Theorem 4.1. It is clear that the μ are admissible and that $W^{\lambda} = W^{\mu}$. Applying (3.2) to $\lambda = \mu$ completes the proof. □

Remark 4.1. Applying the arguments of [2] to the Virasoro algebra, we conclude immediately that if V is a Vir–module such that L_0 is diagonalizable with spectre bounded below and all irreducible subquotients are modular invariant, then V is a direct sum of modular invariant Vir–modules.

Remark 4.2. In the conditions of Corollary 4.1 the branching functions are modular functions. In general, branching functions for admissible representations are not modular functions (see [8]). It is an interesting open problem to find their asymptotics and transformation properties. They look very much like Ramanujan's "mock" modular functions.

4.2. On uniqueness of the vacuum. In [6, §2.5] we proved the uniqueness of the vacuum for a conformal subalgebra acting on the direct sum of level 1 modules of an affine algebra. We generalize here this result to the general case. Throughout this and next two subsections we stick to notations of [6].

Theorem 4.2. Let $\mathfrak{p}' \longrightarrow \mathfrak{g}'$ be a homomorphism of affine algebras corresponding to the inclusion of reductive finite–dimensional Lie algebras $\bar{\mathfrak{p}} \longrightarrow \bar{\mathfrak{g}}$ where $\bar{\mathfrak{p}}$ is a Lie algebra of a reductive of a reductive subgroup of the Lie group corresponding to $\bar{\mathfrak{g}}$. Let $L_0^{\mathfrak{g},\mathfrak{p}}$ be the coset Virasoro operator (commuting with $\mathfrak{p}'$), see e.g. [6, §3.1]. Let $V^{(m)} = \bigoplus_{\Lambda \in P_+^m} L(\Lambda)$ be the direct sum of all integrable $\mathfrak{g}'$–modules of level m. One knows that the specter of $L_0^{\mathfrak{g},\mathfrak{p}}$ on $V^{(m)}$ is non–negative ([6, Proposition 3.2(a)]); let $V_0^{(m)}$ be its 0–th eigenspace. Then the multiplicity of the $\mathfrak{p}'$–module $L(\dot{m}\Lambda_0)$ in $V_0^{(m)}$ is 1. In other words the highest weight vector of $L(m\Lambda_0)$ is the unique, up to a constant multiple, non–zero vector in $V^{(m)}$ which is annihilated by $\bar{\mathfrak{p}}$ and $L_0^{\mathfrak{g},\mathfrak{p}}$.

Proof. We consider, for simplicity, the case of semisimple $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{p}}$. We have for $\Lambda \in P_+^m$:

$$\mathrm{ch}_\Lambda = \sum_{\lambda \in \dot{P}_+^{(\dot{m})}} b_\lambda^\Lambda(\tau)\, \mathrm{ch}\, \dot{L}(\lambda) .$$

Hence as $\tau \downarrow 0$, we deduce, using [6, (2.5.1) and (2.5.5)]:

(4.3) $$a(\Lambda) = \sum_{\lambda \in \dot{P}_+^{(\dot{m})}} b(\Lambda,\lambda)\dot{a}(\lambda) .$$

Let $\lambda = \dot{m}\dot{\Lambda}_0 + a\delta$ be a maximal weight $(a \in \mathbb{C})$. From [6, Theorem A(a) and Theorem B(a)] we have asymptotically as $\tau \downarrow 0$:

$$b_\lambda^\Lambda(-\tfrac{1}{\tau}) \sim \sum_{M \in P_+^{(m)}} a(\Lambda,M) \Big(\sum_{\mu \in \dot{P}_+^{(\dot{m})}} \dot{a}(\mu) b(M,\mu)\Big)\, e^{\pi i(z_m - \dot{z}_{\dot{m}})/12\tau} .$$

Using (4.3) we can rewrite this as follows:

$$b_\lambda^\Lambda(-\tfrac{1}{\tau}) \sim \Big(\sum_{M \in P_+^{(m)}} a(\Lambda,M) a(M,m\Lambda_0)\Big) e^{\pi i(z_m - \dot{z}_{\dot{m}})/12\tau} .$$

Using unitarity of the matrix $(a(\Lambda,M))$, we get:

(4.4) $$b_\lambda^\Lambda(-\tfrac{1}{\tau}) \sim \delta_{\Lambda,m\Lambda_0}\, e^{\pi i(z_m - \dot{z}_{\dot{m}})/12\tau} .$$

On the other hand, by [6, (2.5.8)], if $h_{\Lambda,\lambda} := h_\Lambda - \dot{h}_\lambda = 0$, we have:

(4.5) $$b_\lambda^\Lambda(-\tfrac{1}{\tau}) \sim \mathrm{mult}_\Lambda(\lambda;\mathfrak{p}) e^{\pi i(z_m - \dot{z}_{\dot{m}})/12\tau} .$$

Comparing (4.4) and (4.5) we get, provided that $h_{\Lambda,\lambda} = 0$:

$$\mathrm{mult}_\Lambda(\lambda;\mathfrak{p}) = \delta_{\Lambda,m\Lambda_0} . \qquad \square$$

Remark 4.3. The set $S_m := \{(\Lambda;\lambda) \mid \dot{L}(\lambda) \subset L(\Lambda) \text{ and } h_{\Lambda,\lambda} = 0\}$ introduced in [6] can be described as follows:

(4.6) $$S_m = \{(\Lambda,\lambda) \mid \lambda \in P(\Lambda)|_{\dot{\mathfrak{h}}} \cap \dot{P}_+ \text{ and } h_{\Lambda,\lambda} = 0\}.$$

Indeed, in the contrary case, $\dot{L}(\mu) \subset L(\Lambda)$ with $\mu = \lambda + \alpha$, $\alpha \in \dot{Q}_+ \backslash \{0\}$, and we have:

$\dot{h}_\mu - \dot{h}_\lambda = (\mu+\lambda+2\dot\rho\,|\,\alpha)/2(m+g) > 0$, hence $h_{\Lambda,\mu} < 0$, a contradiction with [6, Proposition 3.2(a)]. This argument also implies that $\mathrm{mult}_\Lambda(\lambda)$ is equal to the sum of multiplicities of weights of $L(\Lambda)$ whose restriction to $\dot{\mathfrak{h}}$ is λ .

Remark 4.4. Combining [6, (2.5.8)] with [6, Theorem A(a)] we see that for $(\Lambda;\lambda) \in S_m$ one has:

$$\mathrm{mult}_\Lambda(\lambda;\mathfrak{p}) = \sum_{(M;\mu)\in S_m} a(\Lambda,M)\dot{a}(\mu,{}^t\lambda)\mathrm{mult}_M(\mu;\mathfrak{p}) , \tag{4.7}$$

i.e. the vector $(\mathrm{mult}_\Lambda(\lambda;\mathfrak{p}))_{(\Lambda;\lambda)\in S_m}$ is an eigenvector with eigenvalue 1 of the matrix $(a(\Lambda,M)\dot{a}(\mu,{}^t\lambda))_{(\Lambda;\lambda),(M;\mu)\in S_m}$. The problem (posed in [6]) that all entries of this matrix are non–negative real numbers is still open (by Frobenius theory then (4.6) and (4.7) together with $\mathrm{mult}_{m\Lambda_0}(m\dot{\Lambda}_0;\mathfrak{p}) = 1$ completely determine the above vector).

4.3. On "admissible" subspaces. In [6] we gave a conjecture on classification of admissible subspaces in $CH_{n-2}(A_1^{(1)})$ (Conjecture 4.7.2). Later we found two more series of examples:

(d′) $n = 9r - 2$ $(r = 2,3,...)$:

$CH_n'' = \{ch_j \,|\, 1 \le j \le n+1,\ j \not\equiv 0 \bmod 3 \text{ or } j \equiv 0 \bmod 3r\}$

$\cup \{ch_{3j}+ch_{6r-3j},\ ch_{3r+3j} + ch_{9r-3j} \,|\, 1 \le j \le r-1\}$;

(e″) $n = 18r - 2$ $(r = 1,2,...)$:

$CH_n''' = \{ch_j + ch_{18r-j} \,|\, 1 \le j \le 9r-1,\ j \text{ odd},\ j \not\equiv 0 \bmod 3\}$

$\cup \{ch_{3j} + ch_{12r-3j} + ch_{6r+3j} + ch_{18r-3j} \,|\, 1 \le j \le 2r-1,\ j \text{ odd}\}$ if r is odd;

$CH_n''' = \{ch_j \,|\, 1 \le j \le n+1,\ j \text{ odd},\ j \not\equiv 0 \bmod 3\}$

$\cup \{ch_{3j} + ch_{12r-3j},\ ch_{6r+3j} + ch_{18r-3j} \,|\, 1 \le j \le 2r-1,\ j \text{ odd}\}$

$\cup \{ch_j + ch_{18r-j} \,|\, 1 \le j \le 9r-1,\ j \text{ even},\ j \not\equiv 0 \bmod 6 \text{ or } j \equiv 0 \bmod 6r\}$

$\cup \{ch_{3j} + ch_{12r-3j} + ch_{6r+3j} + ch_{18r-3j} \,|\, 1 \le j \le 2r-1,\ j \text{ even}\}$ if r is even.

Note that (d) and (e) of Conjecture 4.7.2 in [6] are special cases of (d′) and (e′) for $r = 2$ and $r = 1$ respectively. We think that there are no other examples.

Similar new series can be constructed for the Virasoro algebra (see [6, Conjecture 4.7.1]).

4.4. Corrections to [7].

p. 4957 left, ℓ. 7–13: replace λ by $\lambda+\rho$ and μ by $\mu+\rho$.

p. 4957 right, formula [4]: add g.c.d. $(u,k^\vee) = 1$.

p. 4957 right, formula [6]: replace d by $\tilde{d}$ in the formula for b_λ.

p. 4957 right, Theorem 2: conditions on z which guarantee convergence of χ_λ are these: $\langle\alpha_i,z\rangle > 0$, $i = 1,\ldots,\ell$, $\langle\theta,z\rangle < a_0$; replace $\bar{\Delta}$ and $\bar{\Delta}_+$ by $\nu(\bar{R})$ and $\nu(\bar{R}_+)$ if $k^\vee = 1$; replace d by $a_0^{-1}d$; in the formula for $a(\lambda)$ replace $\bar{P}$ by M^*.

p. 4958 left, ℓ. 2,3: should be $g^{(m)} = \dfrac{\dim \mathfrak{g}(X_N)(m+(1-u^{-2})g)}{k(m+g)}$.

p. 4958 left, ℓ. 6: add factor $e^{-\pi m T(z|z)}$ on the right.

p. 4958 left, Conjecture 2: one should assume that $m+g \neq 0$.

p. 4958 right, ℓ. 6: add factor $e^{-\frac{1}{2}\pi m z^2 T}$ on the right.

p. 4958 right, ℓ. 9: replace $L(\lambda)$ by: $L(\lambda)$ of level $\neq -2$.

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V.G. Kac, Department of Mathematics, MIT, Cambridge, MA 02139, USA
M. Wakimoto, Department of Mathematics, Mie University, Tsu 514, Japan