AARHUS UNIVERSITET

TENSOR PRODUCTS OF C*-ALGEBRAS

PART II.

INFINITE TENSOR PRODUCTS

by

A. Guichardet

Juni 1969

Lecture Notes Series

No. 13

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INTRODUCTION

The basic idea of these Lectures Notes is to consider the tensor product of an arbitrary family of vector spaces $(E_i)_{i\in I}$ as the inductive limit of the finite tensor products $\underset{i\in J}{\otimes} E_i$, J finite subset of I ; to this end we suppose that we are given for each i, a non zero vector t_i in E_i and we define, for $J\subset K$, a mapping $L_{J,K} : \underset{i\in J}{\otimes} E_i \longrightarrow \underset{i\in K}{\otimes} E_i$ by writing

$$\underset{i\in K}{\otimes} E_i = (\underset{i\in J}{\otimes} E_i) \otimes (\underset{i\in K-J}{\otimes} E_i)$$

$$L_{J,K}(x) = x \otimes (\underset{i\in K-J}{\otimes} t_i) \quad ;$$

we thus obtain an inductive limit which we denote by $\otimes^t E_i$. If each E_i is a Banach space t_i must have norm one ; if E_i is a $*$ - algebra t_i must be hermitian and idempotent.

If now we have C^*- algebras A_i with non zero projections e_i we can define two tensor products $\overset{\vee}{\otimes}{}^e A_i$ and $\overset{*}{\otimes}{}^e A_i$, which are identical if the A_i are postliminar. Our main results concern the irreducible representations and the characters of $\overset{\alpha}{\otimes}{}^e A_i$ where $\alpha = \vee$ or $*$; for instance if each e_i is central every finite character of $\overset{\alpha}{\otimes}{}^e A_i$ is a tensor product of characters and we get a precise description of the topological space $C_1(\overset{\alpha}{\otimes}{}^e A_i)$ (see n.14.3). As for the irreducible representations of $\overset{\alpha}{\otimes}{}^e A_i$ we examine two thoroughly different particular cases : if e_i is "large" in the sense that A_i admits sufficiently many irreducible representations π with rank $\pi(e_i) \geqslant 2$, $\overset{*}{\otimes}{}^e A_i$ is antiliminar and, with some further assumptions, admits an irreducible representation which is not a tensor product (see § 11). On the contrary if e_i is "small", i.e. if for each π in $\hat{A}_i$ we have rank $\pi(e_i) \leq 1$ and if moreover each A_i is postliminar, then $\overset{*}{\otimes}{}^e A_i$ is postliminar, each irreducible representation of it is a tensor product and we get a precise description of the topological space $\widehat{\overset{*}{\otimes}{}^e A_i}$ (see n.13.2).

In § 8 we introduce the notion of infinite tensor product of Hilbert algebras, which gives us a very simple method to determine the type of certain infinite tensor products of type I factors (see § 9).

In § 1 we introduce a notion which plays a basic role throughout these lectures : the restricted product of a family of sets X_i with respect to a family of subsets Y_i ; this is the subset $\prod^{(Y_i)} X_i$ of $\prod X_i$ consisting of all families (x_i) such that $x_i \in Y_i$ except for a finite number of indices i ; if each X_i is a topological space and Y_i is open in X_i , the restricted product becomes a topological space in a natural way.

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Given a set I we denote by $\mathcal{F}(I)$ the set of all finite subsets of I ; we say that a property P of an element i of I holds <u>for almost every i</u> or <u>almost everywhere</u> if P holds for every i lying outside some finite subset. If i is an element of I we denote by δ_i the real function on I which takes the value 1 at i and 0 at each other point.

If X is a topological space we denote by $\mathcal{K}(X)$ the space of all continuous complex functions on X with compact support.

If A is a $*$ - algebra every hermitian idempotent element in A will be called a <u>projection</u>. We recall the following result concerning infinite products of complex numbers : given a family $(x_i)_{i\in I}$ of non zero complex numbers, the product $\prod_{i\in I} x_i$ is convergent and non zero if and only if we have $\sum_{i\in I} |x_i - 1| < \infty$.

We finally recall the associativity property for finite tensor products : given a finite family $(E_i)_{i\in I}$ of Banach spaces and a partition $I = \bigcup_{\lambda\in L} I_\lambda$, there exists an isomorphism F of $\hat{\bigotimes}_{i\in I} E_i$ onto $\hat{\bigotimes}_{\lambda\in L}(\hat{\bigotimes}_{i\in I_\lambda} E_i)$ such that $F(\bigotimes_{i\in I} x_i) = \bigotimes_{\lambda\in L}(\bigotimes_{i\in I_\lambda} x_i)$ for each family (x_i) in $\prod_{i\in I} E_i$. Similar results hold for the $\vee$ and $*$ tensor products of C^*- algebras.

§ 1. Restricted products of sets and topological spaces.

Definition 1. Given a family $(X_i)_{i \in I}$ of sets and for each i a subset Y_i of X_i, we call restricted product of the family (X_i) with respect to the family (Y_i) the set of all families $(x_i) \in \prod_{i \in I} X_i$ such that $x_i \in Y_i$ for almost every i ; and we denote it by $\prod_{i \in I}^{(Y_i)} X_i$. If Y_i is reduced to some point a_i we write $\prod^{(a_i)} X_i$ or $\prod^{a} X_i$ instead of $\prod^{(\{a_i\})} X_i$.

For each finite subset J of I we denote by $X_{(J)}$ the set of all families $(x_i) \in \prod X_i$ such that $x_i \in Y_i$ for $i \notin J$, i.e.

$$X_{(J)} = \prod_{i \in J} X_i \times \prod_{i \in I-J} Y_i \ ;$$

then the $X_{(J)}$ are subsets of $\prod^{(Y_i)} X_i$ which is their union, and $X_{(J)} \subset X_{(K)}$ if $J \subset K$.

Suppose now that each X_i is a topological space and that Y_i is open in X_i ; we shall define a topology on $\prod^{(Y_i)} X_i$ in the following way : we endow each $X_{(J)}$ with the product topology ; then for $J \subset K$, $X_{(J)}$ is an open topological subspace of $X_{(K)}$; we say that a subset U of $\prod^{(Y_i)} X_i$ is open iff for each J , $U \cap X_{(J)}$ is open in $X_{(J)}$; we get a topology which is the inductive limit of those of the $X_{(J)}$, and is stronger than the product topology ; each $X_{(J)}$ appears as an open topological subspace of $\prod^{(Y_i)} X_i$.

Particular cases.

(i) If $Y_i = X_i$, the restricted product is identical to the ordinary product.

(ii) If for each i, X_i is locally compact and Y_i compact, the restricted product is locally compact since each $X_{(J)}$ is locally compact.

(iii) If X_i is discrete and Y_i is reduced to some point a_i, the restricted product is discrete since each $X_{(J)}$ is discrete.

(iv) If X_i is a locally compact group and Y_i a compact open subgroup, the restricted product is, in a natural way, a locally compact group ; this construction is used in order to define the so called " adele groups " : in the simplest case I is the set of all prime numbers, the X_i are the p-adic fields Q_p and the Y_i - there rings of integers Z_p (see for instance [37], ch. III, § 1).

(v) If X_i is a discrete group and Y_i is reduced to the neutral element, $\prod^{(Y_i)} X_i$ is nothing but the usual restricted product $\prod' X_i$.

(vi) If X_i is a compact group and $Y_i = X_i$ the restricted product is identical with the ordinary product $\prod X_i$.

Restricted products of Borel spaces.

We now suppose that each X_i is a Borel space and Y_i a Borel subset of X_i ; put on each $X_{(J)}$ the product Borel structure ; then for $J \subset K$, $X_{(J)}$ is a Borel subspace of $X_{(K)}$; we define a Borel structure on $X = \prod^{(Y_i)} X_i$ by saying that a subset U of X is Borel iff for each J, $U \cap X_{(J)}$ is Borel in $X_{(J)}$; then each $X_{(J)}$ appears as a Borel subspace of X.

Restricted products of measures.

Suppose that each X_i is locally compact, Y_i compact and open, and that we have a positive Radon measure μ_i on X_i with $\mu_i(Y_i) = 1$; set $\nu_i = \mu_i / Y_i$; for each $J \in \mathcal{F}(I)$ we can form the product measure

$$\mu_{(J)} = (\bigotimes_{i \in J} \mu_i) \otimes (\bigotimes_{i \in I-J} \nu_i) \quad ;$$

if $J \in K$ we have $\mu_{(K)} / X_{(J)} = \mu_{(J)}$, hence there exists a unique positive Radon measure μ on $\prod^{(Y_i)} X_i$ such that $\mu / X_{(J)} = \mu_{(J)}$; it will be called restricted product of the measures μ_i .

If in particular X_i is a locally compact group, Y_i a compact open subgroup and μ_i a left Haar measure on X_i, μ is a left Haar measure on $\prod^{(Y_i)} X_i$.

Finally if X_i is a Borel space, Y_i a Borel subset and μ_i a positive Borel measure on X_i with $\mu_i(Y_i) = 1$, the same construction applies and yields a positive Borel measure .

§ 2. Infinite tensor products of vector spaces.

Let us consider an arbitrary family $(E_i)_{i\in I}$ of vector spaces and for each i a non zero element t_i in E_i ; denote by t the family (t_i) ; for each finite subset J of I we set $E_{(J)} = \bigotimes_{i\in J} E_i$; for $J\subset K$ we define a linear mapping

$$L_{J,K} : E_{(J)} \longrightarrow E_{(K)}$$

by writing

$$E_{(K)} = E_{(J)} \otimes E_{(K-J)}$$

$$L_{J,K}(x) = x \otimes (\bigotimes_{i\in K-J} t_i) \qquad \forall\ x \in E_{(J)} ;$$

the mappings $L_{J,K}$ are injective and form an inductive system, which means that for $J\subset K \subset M$ we have $L_{J,M} = L_{K,M} \circ L_{J,K}$.

Definition 2. We shall denote by $\bigotimes_{i\in I}^{(t_i)} E_i$ or $\bigotimes_{i\in I}^{t} E_i$ the inductive limit of the above inductive system, and by L_J the canonical mapping $E_{(J)} \longrightarrow \otimes^t E_i$; the L_J will sometimes allow us to consider the $E_{(J)}$ as subspaces of $\otimes^t E_i$, which is then their union ; if J is reduced to a point i we shall write L_i instead of $L_{\{i\}}$. For each family $(x_i) \in \Pi^t E_i$ we denote by $\otimes x_i$ the element $L_J(\bigotimes_{i\in J} x_i)$ where J is an arbitrary finite subset verifying $x_i = t_i\ \ \forall\ i \notin J$; every element of $\otimes^t E_i$ is a linear combination of elements of the form $\otimes x_i$; the mapping $(x_i) \longmapsto \otimes x_i$ is multilinear.

Properties of the infinite tensor product.

(i) Universal property.

Proposition 1. For every multilinear mapping u of $\prod^t E_i$ into a vector space F there exists a unique linear mapping $v : \otimes^t E_i \longrightarrow F$ such that $v(\otimes x_i) = u((x_i))$ for each $(x_i) \in \prod^t E_i$; in this way we get a bijective correspondance between the multilinear mappings $\prod^t E_i \longrightarrow F$ and the linear mappings $\otimes^t E_i \longrightarrow F$.

Proof. Choose a finite subset J of I ; for each family $(x_i)_{i\in J}$ define a family $(x'_i)_{i\in I}$ where

$$x'_i = \begin{cases} x_i & \text{if} \quad i \in J \\ t_i & \text{if} \quad i \notin J \end{cases} ;$$

the multilinear mapping

$$\prod_{i\in J} E_i \longrightarrow F$$

$$(x_i)_{i\in J} \longmapsto u((x'_i))$$

gives rise to a linear mapping

$$v_J : E_{(J)} \longrightarrow F$$

$$\bigotimes_{i\in J} x_i \longmapsto u((x'_i)) ;$$

the v_J form an inductive system and v is their inductive limit.

(ii) Associativity.

For each partition $I = \bigcup_{\lambda\in L} I_\lambda$ there exists an isomorphism $\bigotimes^t_{i\in I} E_i \longrightarrow \bigotimes^{(v_\lambda)}_{\lambda\in L} (\bigotimes^{u_\lambda}_{i\in I_\lambda} E_i)$ taking each element of the form x_i into $\otimes_{\lambda \in L} (\otimes_{i \in I_\lambda} x_i)$; here we have set $u_\lambda = (t_i)_{i \in I_\lambda}$ and $v_\lambda = \otimes_{i \in I_\lambda} t_i$.

(iii) Functorial property.

Let us also consider vector spaces F_i and non zero elements u_i in F_i ; let v_i be a linear mapping $E_i \longrightarrow F_i$ with $v_i(t_i) = u_i$; there exists a unique linear mapping $\otimes v_i$: $\otimes^t E_i \longrightarrow \otimes^u F_i$ such that

$$(\otimes v_i)(\otimes x_i) = \otimes v_i(x_i) \qquad \forall (x_i) \in \prod^t E_i \; ;$$

if the v_i are injective, $\otimes v_i$ is injective too.

(iv) Bases of $\otimes^t E_i$.

Suppose that for each i we have a basis of E_i of the form (e_{i,x_i}) where the index x_i runs over some set X_i, and that X_i contains an element y_i with $e_{i,y_i} = t_i$; for each $x = (x_i)$ in $\prod^{(y_i)} X_i$ we set $e_{(x)} = \otimes_{i \in I} e_{i,x_i}$; then it is easy to verify that the vectors $e_{(x)}$ constitute a basis of $\otimes^t E_i$.

Bibliography [9].

N.B. There is no pages 7,8,9.

§ 3. Infinite tensor products of algebras.

Let us consider a family of algebras $(A_i)_{i\in I}$ and for each i , a non zero idempotent e_i in A_i ; the finite tensor products $A_{(J)}$ are algebras and the mappings $L_{J,K}$ are morphisms of algebras ; by endowing $\otimes^e A_i$ with the inductive limit structure we get an algebra whose multiplication is characterized by

$$\otimes a_i \cdot \otimes b_i = \otimes a_i b_i \qquad \forall (a_i) , (b_i) \in \Pi^e A_i .$$

Let us now suppose that for each i, A_i is a $*$- algebra and e_i a projection (i.e. hermitian idempotent) ; then the $A_{(J)}$ are $*$ - algebras and the $L_{J,K}$ are morphisms of $*$ - algebras ; we get a structure of $*$ - algebra on $\otimes^e A_i$ characterized by

$$(\otimes a_i)^* = \otimes a_i^* .$$

The reader will easily state the properties similar to (ii) and (iii) of § 2.

<u>A particular case</u>. Suppose that e_i is a unit element for A_i ; we then write $\otimes A_i$ instead of $\otimes^e A_i$; $\otimes e_i$ is the unit element of $\otimes A_i$; the L_J are mutually commuting morphisms of unitary $*$ - algebras ; moreover $\otimes A_i$ has the following universal property :

Given a unitary $*$ - algebra B there exists a bijective correspondance between the morphisms (of unitary $*$ - algebras) u :

$\otimes A_i \longrightarrow B$ and the families of mutually commuting morphisms $u_i : A_i \longrightarrow B$; this correspondance is given by

$$u(\otimes a_i) = \prod u_i(a_i) \qquad \forall (a_i) \in \Pi^e A_i .$$

Infinite tensor products of representations.

Proposition 2. Let, for each i, A_i an algebra, e_i a non zero idempotent in A_i, E_i a vector space, t_i a non zero element of E_i, π_i a representation of A_i in E_i such that $\pi_i(e_i).t_i = t_i$. Then there exists a unique representation π of $\otimes^e A_i$ in the space $\otimes^t E_i$ such that

$$\pi(\otimes a_i).\otimes x_i = \otimes \pi_i(a_i).x_i \quad \forall (a_i) \in \Pi^e A_i, (x_i) \in \Pi^t E_i .$$

Proof. Take a family $(a_i) \in \Pi^e A_i$ and a finite subset J with $a_i = e_i \ \forall\ i \in I-J$; write

$$\bigotimes_{i\in I}^t E_i = (\bigotimes_{i\in J} E_i) \otimes (\bigotimes_{i\in I-J}^t E_i) ;$$

we have an operator $\bigotimes_{i\in J} \pi_i(a_i)$ in the first factor and, by property (iii) § 2, an operator $\bigotimes_{i\in I-J} \pi_i(a_i)$ in the second factor ; whence an operator in $\bigotimes_{i\in I}^t E_i$ which we denote by $u((a_i))$:

$$u((a_i)).\otimes x_i = \otimes \pi_i(a_i).x_i \qquad \forall (x_i) \in \Pi^t E_i ;$$

the mapping $u : \Pi^e A_i \longrightarrow \mathcal{L}(\otimes^t E_i)$ is multilinear, hence defines a linear mapping

$$\pi : \otimes^e A_i \longrightarrow \mathcal{L}(\otimes^t E_i)$$

$$\pi(\otimes a_i)\otimes x_i = \otimes \pi_i(a_i).x_i ;$$

finally it is easily seen that π is a representation.

§ 4. Infinite tensor products of Banach spaces.

Let us consider a family of Banach spaces E_i and for each i, a unit vector t_i in E_i ; let us endow each finite tensor product $E_{(J)}$ with the $\wedge$ crossnorm ; if $J \subset K$ the isomorphism $E_{(K)} \sim E_{(J)} \otimes E_{(K-J)}$ is isometric, thus the $L_{J,K}$ are isometric ; we put on $\otimes^t E_i$ the inductive limit norm, so that each mapping L_J becomes isometric ; we have

$$\| \otimes x_i \| = \prod \| x_i \| \qquad \forall (x_i) \in \prod^t E_i .$$

Definition 3. We shall denote by $\hat{\otimes}^t E_i$ the completion of $\otimes^t E_i$ with respect to the norm defined above ; this is also the inductive limit of the Banach spaces $\hat{\bigotimes}_{i \in J} E_i$.

Definition of $\otimes x_i$ for certain families (x_i).

Proposition 3. Let (x_i) be a family of vectors $x_i \in E_i$ such that $\sum \| x_i - t_i \| < \infty$; the product $\prod \| x_i \|$ exists, and it is null iff one of the x_i is null ; the family of the vectors $L_J(\bigotimes_{i \in J} x_i)$ has a limit in $\hat{\otimes}^t E_i$, whose norm is equal to $\prod \| x_i \|$.

Proof. The proposition being trivial if one x_i is null, we can suppose $x_i \neq 0 \ \forall i$; then

$$\sum | \, \| x_i \| - 1 \, | = \sum | \, \| x_i \| - \| t_i \| \, |$$

$$\leq \sum \| x_i - t_i \| < \infty$$

whence $\prod \|x_i\|$ exists and is not null ; the finite products $\prod_{i\in J} \|x_i\|$ are bounded by some constant $k \geqslant 1$. We must now prove that the vectors $X_J = L_J(\bigotimes_{i\in J} x_i)$ form a Cauchy family, i.e. that for every $\varepsilon > 0$ there exists $K \in \mathcal{F}(I)$ with the following property :

$$J_1 , J_2 \supset K \Longrightarrow \| X_{J_2} - X_{J_1} \| \leqslant \varepsilon \quad ; \tag{1}$$

take K such that

$$J \cap K = \emptyset \Longrightarrow \sum_{i\in J} \| x_i - t_i \| \leqslant \varepsilon / k \quad ;$$

in order to prove (1) we can suppose $J_1 \subset J_2$; set $J = J_2 - J_1$; we have

$$X_{J_2} - X_{J_1} = L_{J_2}(\bigotimes_{i\in J_2} x_i - (\bigotimes_{i\in J_1} x_i) \otimes (\bigotimes_{i\in J} t_i))$$

$$\| X_{J_2} - X_{J_1} \| = \prod_{i\in J_1} \| x_i \| . \| \bigotimes_{i\in J} x_i - \bigotimes_{i\in J} t_i \| \; ;$$

denoting by $i_1, \dots i_n$ the elements of J we can write

$$\bigotimes_{i\in J} x_i - \bigotimes_{i\in J} t_i = x_{i_1} \otimes \dots \otimes x_{i_n} - t_{i_1} \otimes \dots \otimes t_{i_n}$$

$$= (x_{i_1} - t_{i_1}) \otimes x_{i_2} \otimes \dots \otimes x_{i_n} + \dots + t_{i_1} \otimes \dots t_{i_{n-1}} \otimes (x_{i_n} - t_{i_n})$$

$$\| \bigotimes_{i\in J} x_i - \bigotimes_{i\in J} t_i \| \leqslant k \| x_{i_1} - t_{i_1} \| + \dots + k \| x_{i_n} - t_{i_n} \|$$

$$\leqslant \varepsilon / k$$

whence

$$\| X_{J_2} - X_{J_1} \| \leqslant k . \varepsilon / k = \varepsilon .$$

Definition 4. Given a family (x_i) we set

$$\otimes x_i = \lim L_J(\underset{i\in J}{\otimes} x_i)$$

whenever the righthand side makes sense ; this is the case if $\sum \| x_i - t_i \| < \infty$; in any case we have $\| \otimes x_i \| = \prod \| x_i \|$.

Bibliography [9].

§ 5. Infinite tensor products of Banach $*$ - algebras.

Suppose we are given a family of Banach $*$ - algebras A_i and for each i, a projection e_i of norm 1 in A_i ; then $\hat{\otimes}^e A_i$ is a Banach $*$ - algebra, the inductive limit of the finite tensor products $\hat{\bigotimes}_{i \in J} A_i$. If in particular e_i is a unit element for A_i, $\hat{\otimes}^e A_i$, which will be denoted $\hat{\otimes} A_i$, has the following universal property : given a unitary Banach $*$ - algebra B and mutually commuting morphisms (of unitary $*$ - algebras) continuous and with norm 1, $u_i : A_i \longrightarrow B$, there exists a unique continuous morphism $u : \hat{\otimes} A_i \longrightarrow B$ such that

$$u(\otimes a_i) = \prod u_i(a_i) \qquad \forall (a_i) \in \prod^e A_i .$$

<u>Example</u> 1. We consider the situation of § 1, (iv), set $A_i = L^1(X_i)$ and denote by e_i the characteristic function of Y_i ; this is a projection of norm 1 if the Haar measure μ on X_i is chosen so that $\mu(Y_i) = 1$.

<u>Theorem</u> 1. There exists an isometric isomorphism w of $\hat{\otimes}^e A_i$ onto $L^1(\prod^{(Y_i)} X_i)$ with the following property : for every family (a_i) in $\prod^e A_i$, $w(\otimes a_i)$ is the function f on $\prod^{(Y_i)} X_i$ defined by

$$f(x) = \begin{cases} \prod a_i(x_i) & \text{if} \quad x = (x_i) \in X_{(J)} \\ 0 & \text{in the opposite case} \end{cases}$$

(we have set $J = \{ i \mid a_i \neq e_i \}$).

Proof. ~~The reader will notice the similarity of this proof~~ [redacted]. For every $J \in \mathcal{F}(I)$ we have an isometric isomorphism

$$u_J : \hat{\bigotimes}_{i \in J} A_i \longrightarrow L^1(\prod_{i \in J} X_i)$$

which transforms each element $\bigotimes_{i \in J} a_i$ into the function $(x_i)_{i \in J} \longmapsto \prod_{i \in J} a_i(x_i)$; then an isometric morphism

$$v_J : \hat{\bigotimes}_{i \in J} A_i \longrightarrow L^1(\prod_{i \in J} X_i) \hat{\otimes} L^1(\prod_{i \in I-J} Y_i) \sim L^1(X_{(J)})$$

$$a \longmapsto u_J(a) \otimes 1 \quad ;$$

finally extending $v_J(a)$ to a function on $X = \prod^{(Y_i)} X_i$ which is zero outside of $X_{(J)}$, we get an isometric morphism

$$w_J : \hat{\bigotimes}_{i \in J} A_i \longmapsto L^1(X) \ .$$

As easily cheked the w_J form an inductive system and we get an isometric morphism

$$w : \hat{\otimes}^e A_i \longrightarrow L^1(X)$$

which transforms $\otimes a_i$ as indicated in the statement. It remains to be shown that $\operatorname{Im} w$ is dense in $L^1(X)$, or that each function f in $\mathcal{K}(X)$ is a limit of elements in $\operatorname{Im} w$; the support of f is included in some $X_{(J)}$; by the Stone-Weierstrass theorem we can suppose that f depends only on a finite number of coordinates, i.e. that there exists some $J' \in \mathcal{F}(I)$ such that $f = g \otimes 1$ with $g \in \mathcal{K}(\prod_{i \in J'} X_i)$;

we can also suppose that $J' > J$, but in this case $f \in \mathrm{Im}\, w_{J'}$.

<u>Corollary</u> 1. The L^1 algebra of the restricted product of discrete groups G_i is canonically isomorphic to $\hat{\otimes} L^1(G_i)$.

<u>Corollary</u> 2. The L^1 algebra of a product of compact groups G_i is canonically isomorphic to $\hat{\otimes}^e L^1(G_i)$ where e_i is the function 1 on G_i.

<u>Example</u> 2. We first define the <u>symmetric algebra of a Banach space</u> Consider a Banach space E and set, for each integer $n > o$

$$E^{\hat{\otimes} n} = E \hat{\otimes} \dots \hat{\otimes} E \quad \text{n-times} ;$$

every permutation s of the set $\{1, \dots n\}$ gives rise to an automorphism $U_{s,n}$ of $E^{\hat{\otimes} n}$ such that

$$U_{s,n}(x_1 \otimes \dots \otimes x_n) = x_{s(1)} \otimes \dots \otimes x_{s(n)} ;$$

the operator $P_n = (n!)^{-1} \sum_s U_{s,n}$ is a projection of norm 1 ; we set $S^nE = \mathrm{Im}\, P_n =$ the set of all elements in $E^{\hat{\otimes} n}$ which are invariant by all $U_{s,n}$; finally we denote by SE the Banach direct sum of all S^nE for $n = 0,1,2,\dots$, i.e. the set of all sequences $x = (x_n)$ where $x_n \in S^nE$ and $\|x\| = \sum \|x_n\| < \infty$; by definition $S^oE = \mathbb{C}$.

It is proved in the courses of Algebra that there exists on the algebraic direct sum $A = \bigoplus_{n=o}^{\infty} E^{\otimes n}$ a structure of commutative algebra such that

$$(x\,y)_n = \sum_{p=0}^{n} P_n(x_p \otimes y_{n-p}) \qquad (2)$$

for every $x = (x_n)$ and $y = (y_n)$ in A ; we then have

$$\| (x\,y)_n \| \leq \sum_{p=0}^{n} \| x_p \| \cdot \| y_{n-p} \|$$

$$\| x\,y \| = \sum_n \| (x\,y)_n \| \leq \sum_{p,q} \| x_p \| \cdot \| y_q \| = \| x \| \cdot \| y \|$$

hence the multiplication can be extended to SE which becomes a <u>commutative Banach algebra</u> ; (2) is still valid for x and $y \in$ SE ; SE admits a unit element $\varepsilon = (1,0,0,\dots)$.

For each $a \in E$ it will be convenient to denote by exp a the following element of SE :

$$\exp a = (1, a, a^{\otimes 2}/2!, \dots, a^{\otimes n}/n!, \dots) ;$$

we have

$$\| \exp a \| = e^{\| a \|}$$

and

$$\exp(a+b) = \exp a \cdot \exp b.$$

Each Banach space S^nE is generated by the particular tensors $x^{\otimes n}$; then the algebra SE is generated by ε and E identified with the set of all elements $(0, x, 0,0,\dots)$.

<u>Proposition</u> 4. The Banach algebra SE possesses the following universal property : given a commutative Banach algebra B with unit, by associating with each morphism of unitary algebras

$v : SE \longrightarrow B$ with $\| v \| \leqslant 1$, its restriction to E one gets a bijective correspondance between such morphisms v and all the continuous linear mappings $E \longrightarrow B$ of norm $\leqslant 1$.

Proof. We have to show that every continuous linear mapping $u : E \longrightarrow B$ of norm $\leqslant 1$ can be extended to a morphism v ; we have for each n a multilinear mapping of norm $\leqslant 1$

$$E^n \longrightarrow B$$

$$(a_1, \dots a_n) \longmapsto u(a_1) \dots u(a_n)$$

whence a linear mapping of norm $\leqslant 1$

$$v_n : E^{\hat{\otimes} n} \longrightarrow B$$

$$a_1 \otimes \dots \otimes a_n \longmapsto u(a_1) \dots u(a_n) \ ;$$

it suffices to set, for each $x = (x_n) \in SE$:

$$v(x) = \sum_{n=0}^{\infty} v_n(x_n) \ .$$

QED

Note that $v(\exp a) = e^{u(a)}$.

Corollary 3. Let $(E_i)_{i \in I}$ be a family of Banach spaces, E its Banach direct sum, i.e. the set of all families $x = (x_i) \in \prod E_i$ with $\| x \| = \sum \| x_i \| < \infty$. Then SE is canonically isomorphic to $\hat{\otimes} SE_i$; this isomorphism carries each $\exp x$ into $\otimes \exp x_i$.

The proof is purely categorical : it suffices to remark that SE and $\hat{\otimes}\, SE_i$ are solutions of the same universal problem ; note that $\otimes \exp x_i$ exists because $\sum \| \exp x_i - \varepsilon_i \| < \infty$ since $\| \exp x_i - \varepsilon_i \| = e^{\| x_i \|} - 1 \sim \| x_i \|$.

<u>Remark</u> 0. For $a \in E$, exp a is nothing but $\sum_{n=0}^{\infty} a^n / n!$, the image of a in the exponential map which can be defined in any Banach algebra.

§ 6. Infinite tensor products of Hilbert spaces.

n.6.1. Definition and first properties.

Let us consider a family $(H_i)_{i \in I}$ of Hilbert spaces and for each i, a unit vector t_i in H_i ; endow each $H_{(J)}$ with its usual prehilbert structure ; the mappings $L_{J,K}$ are isometric and we can put on $\otimes^t H_i$ the inductive limit prehilbert structure ; each $H_{(J)}$ appears as a subprehilbert space of $\otimes^t H_i$ and we have

$$(\otimes x_i \mid \otimes y_i) = \Pi (x_i \mid y_i) \qquad \forall\ (x_i),(y_i) \in \Pi^t H_i .$$

Definition 5. We shall denote by $\overset{h}{\otimes}{}^t H_i$ the Hilbert completion of the prehilbert space $\otimes^t H_i$; it is also the inductive limit of the finite tensor products $\overset{h}{\underset{i \in J}{\otimes}} H_i$.

It is easy to construct orthonormal bases of $\overset{h}{\otimes}{}^t H_i$: choose for each i an orthonormal basis $(e_{i,j})_{j \in J_i}$ of H_i with $e_{i,o} = t_i$; for each element $f = (f(i))$ in $\Pi^o J_i$ set $e_f = \otimes e_{i,f(i)}$; then it is easy to verify that the e_f constitute an orthonormal basis of $\overset{h}{\otimes}{}^t H_i$.

Associativity. For each partition $I = \bigcup_{\lambda \in L} I_\lambda$ there exists an isomorphism of $\overset{h}{\underset{i \in I}{\otimes}}{}^t H_i$ onto $\overset{h}{\underset{\lambda \in L}{\otimes}}{}^{(v_\lambda)} (\overset{h}{\underset{i \in I_\lambda}{\otimes}}{}^{u_\lambda} H_i)$ with the same properties as in § 2 (ii).

Bibliography [9] .

n.6.2. Definition of $\otimes x_i$ for certain families (x_i).

As in § 4 we set $\otimes x_i = \lim L_J(\underset{i\in J}{\otimes} x_i)$ whenever this limit exists ; one can prove exactly as in prop. 3 that it does exist if $\sum \| x_i - t_i \| < \infty$; but it still exists under more general conditions :

Proposition 5. Let (x_i) be a family of vectors satisfying

$$\sum | \| x_i \| - 1 | < \infty \qquad (3)$$

$$\sum |(x_i \mid t_i) - 1| < \infty \quad ; \qquad (4)$$

then $\prod \| x_i \|$ exists, and it is null iff one of the x_i is null. The family of the vectors $L_J(\underset{i\in J}{\otimes} x_i)$ has a limit in $\overset{h}{\otimes}{}^t H_i$ whose norm is $\prod \| x_i \|$; moreover we have

$$\lim_J \| \underset{i\in I-J}{\otimes} x_i - \underset{i\in I-J}{\otimes} t_i \| = 0 .$$

Proof. As in prop. 3 we can suppose $x_i \neq 0 \ \forall\, i$; then $\prod \| x_i \|$ exists and is non null by virtue of (3) ; set $c = \prod \| x_i \|^2$; the finite products $\prod_{i\in J} \| x_i \|$ are bounded by some $k > 0$; on the other hand we have $(x_i \mid t_i) \neq 0$ almost everywhere and we can suppose $(x_i \mid t_i) \neq 0 \ \forall\, i$; by (4), $\prod (x_i \mid t_i)$ has a value $d \neq 0$.

Take a number $\varepsilon > 0$; there exists $J \in \mathcal{F}(I)$ such that $K \supset J$ implies

$$\left| \prod_{i\in K} \| x_i \|^2 - c \right| \leqslant \varepsilon c / (4k+\varepsilon)$$

and

$$\Big|\prod_{i\in K}(x_i \mid t_i) - d\Big| \leq \varepsilon|d|\,(8k+\varepsilon) \quad ;$$

setting $L = K - J$ we have

$$\Big\|\bigotimes_{i\in L} x_i - \bigotimes_{i\in L} t_i\Big\|^2 = \Big\|\bigotimes_{i\in L} x_i\Big\|^2 + 1 - 2\,\mathrm{Re}\Big(\bigotimes_{i\in L} x_i \,\Big|\, \bigotimes_{i\in L} t_i\Big)$$

$$\leq \Big|\prod_{i\in L}\|x_i\|^2 - 1\Big| + 2\Big|\prod_{i\in L}(x_i \mid t_i) - 1\Big|$$

$$\Big|\prod_{i\in L}\|x_i\|^2 - 1\Big| = \Big|\prod_{i\in K}\|x_i\|^2 - c + c - \prod_{i\in J}\|x_i\|^2\Big| \Big/ \prod_{i\in J}\|x_i\|^2$$

$$\leq (2\varepsilon c/(4k+\varepsilon))/(c - \varepsilon c/(4k+\varepsilon)) = \varepsilon/2k$$

$$\Big|\prod_{i\in L}(x_i \mid t_i) - 1\Big| = \Big|\prod_{i\in K}(x_i \mid t_i) - d + d - \prod_{i\in J}(x_i \mid t_i)\Big| \Big/ \Big|\prod_{i\in J}(x_i \mid t_i)\Big|$$

$$\leq (2\varepsilon|d|/(8k+\varepsilon))/(d - \varepsilon|d|\,(8k+\varepsilon)) = \varepsilon/4k$$

$$\Big\|\bigotimes_{i\in L} x_i - \bigotimes_{i\in L} t_i\Big\|^2 \leq \varepsilon/k \quad ; \qquad (5)$$

then

$$\Big\|L_K\Big(\bigotimes_{i\in K} x_i\Big) - L_J\Big(\bigotimes_{i\in J} x_i\Big)\Big\| = \Big\|\bigotimes_{i\in K} x_i - L_{J,K}\Big(\bigotimes_{i\in J} x_i\Big)\Big\|$$

$$= \prod_{i\in J}\|x_i\| \cdot \Big\|\bigotimes_{i\in L} x_i - \bigotimes_{i\in L} t_i\Big\|$$

$$\leq k\,(\varepsilon/k)^{\frac{1}{2}} = (\varepsilon k)^{\frac{1}{2}} \;;$$

this proves that $(L_J(\bigotimes_{i\in J} x_i))$ is a Cauchy family. Finally our last assertion is a consequence of (5).

n.6.3. <u>Relations between the various tensor products</u> $\overset{h}{\otimes}{}^{t} H_i$.

We shall prove that if two families (t_i), (u_i) are sufficiently close, $\overset{h}{\otimes}{}^{t} H_i$ and $\overset{h}{\otimes}{}^{u} H_i$ are canonically isomorphic.

<u>Lemma</u> 1. The following relations between families $(t_i),(u_i)$ of unit vectors

$$\sum_i |1 - (t_i | u_i)| < \infty \tag{6}$$

$$\sum_i (1 - |(t_i | u_i)|) < \infty \tag{7}$$

are equivalence relations ; if we write them respectively $t \approx u$ and $t \sim u$, we have $t \sim u$ if and only if there exists a family of complex numbers α_i with $|\alpha_i| = 1$ and $(u_i) \approx (\alpha_i\ t_i)$.

<u>Proof</u>. These relations are trivially reflexive and symmetric ; let us show that the first one is ~~symmetric~~ transitive : suppose $(u_i) \approx (v_i)$; then

$$1 - (t_i | v_i) = 1 - (t_i | u_i) + 1 - (u_i | v_i) + (t_i - u_i | u_i - v_i)$$

$$\sum \| t_i - u_i \|^2 = \sum (2 - 2\,\mathrm{Re}(t_i | u_i))$$

$$\leqslant \sum 2 |1 - (t_i | u_i)| < \infty$$

and similarly

$$\sum \| u_i - v_i \|^2 < \infty$$

hence

$$\sum |(t_i - u_i | u_i - v_i)| \leqslant \sum \| t_i - u_i \| \cdot \| u_i - v_i \| < \infty$$

$$\sum |1 - (t_i | v_i)| < \infty .$$

We now prove the last assertion ; if $(u_i) \approx (\alpha_i\ t_i)$ we have

$$\Sigma(1-|(t_i|u_i)|) = \Sigma(1-|(\alpha_i t_i|u_i)|)$$

$$\leqslant \Sigma|1-(\alpha_i t_i|u_i)| < \infty ;$$

conversely suppose $t \sim u$ and set

$$\alpha_i = \begin{cases} |(t_i|u_i)|/(t_i|u_i) & \text{if } (t_i|u_i) \neq 0 \\ 1 & \text{in the opposite case ;} \end{cases}$$

then

$$\Sigma|1-(\alpha_i t_i|u_i)| = \Sigma(1-|(t_i|u_i)|) < \infty .$$

The transitivity of $\sim$ is now immediate.

<u>Theorem</u> 2. Let us suppose $t \sim u$ and more precisely $(u_i) \approx (\alpha_i t_i)$; there exists a unique isomorphism $F : \overset{h}{\otimes}{}^t H_i \longrightarrow \overset{h}{\otimes}{}^u H_i$ with the following property : if $\otimes x_i$ exists in the first space, $\otimes \alpha_i x_i$ exists in the second space and is equal to $F(\otimes x_i)$.

<u>Proof</u>. The unicity is clear since the $\otimes x_i$ with $(x_i) \in \Pi^t H_i$ generate the first space. For each $J \in \mathcal{F}(I)$ we define a multilinear mapping

$$\prod_{i\in J} H_i \longrightarrow \overset{h}{\underset{i\in I}{\otimes}}{}^u H_i = (\overset{h}{\underset{i\in J}{\otimes}} H_i) \overset{h}{\otimes} (\overset{h}{\underset{i\in I-J}{\otimes}}{}^u H_i)$$

$$(x_i)_{i\in J} \longmapsto (\underset{i\in J}{\otimes} \alpha_i x_i) \otimes (\underset{i\in I-J}{\otimes} \alpha_i t_i)$$

which makes sense by prop. 5 ; it gives rise to a linear mapping

$$F_J : \overset{\ell}{\underset{i\in J}{\otimes}} H_i \longrightarrow \overset{h}{\underset{i\in I}{\otimes}}{}^{u} H_i$$

$$\underset{i\in J}{\otimes} x_i \longmapsto (\underset{i\in J}{\otimes} \alpha_i x_i) \otimes (\underset{i\in I-J}{\otimes} \alpha_i t_i)$$

which is easily seen to be isometric ; since the F_J form an inductive system we get an isometric linear mapping

$$F : \overset{h}{\otimes}{}^{t} H_i \longrightarrow \overset{h}{\otimes}{}^{u} H_i$$

$$\otimes x_i \longmapsto \otimes \alpha_i x_i \qquad \forall (x_i) \in \Pi^{t} H_i .$$

Let us prove that F is onto ; if $(y_i) \in \Pi^{u} H_i$ we have, for J sufficiently large

$$\otimes y_i = (\underset{i\in J}{\otimes} y_i) \otimes (\underset{i\in I-J}{\otimes} u_i)$$

$$= \lim (\underset{i\in J}{\otimes} y_i) \otimes (\underset{i\in I-J}{\otimes} \alpha_i t_i)$$

by the last assertion of prop. 5 ; since

$$(\underset{i\in J}{\otimes} y_i) \otimes (\underset{i\in I-J}{\otimes} \alpha_i t_i) \in \mathrm{Im}\, F ,$$

we see that Im F is dense, hence equal to the whole space. Let us now suppose that $\otimes x_i$ exists in $\overset{h}{\otimes}{}^{t} H_i$; then

$$F(\otimes x_i) = F(\lim L_J(\underset{i\in J}{\otimes} x_i))$$

$$= \lim F_J(\underset{i\in J}{\otimes} x_i)$$

$$= \lim (\underset{i\in J}{\otimes} \alpha_i x_i) \otimes (\underset{i\in I-J}{\otimes} \alpha_i t_i) ;$$

and this is equal to $\lim (\underset{i\in J}{\otimes} \alpha_i x_i) \otimes (\underset{i\in I-J}{\otimes} u_i)$ since

$$\| (\underset{i\in J}{\otimes} \alpha_i x_i) \otimes (\underset{i\in I-J}{\otimes} \alpha_i t_i) - (\underset{i\in J}{\otimes} \alpha_i x_i) \otimes (\underset{i\in I-J}{\otimes} u_i) \| =$$

$$\prod_{i\in J} \| x_i \| \cdot \| \bigotimes_{i\in I-J} \alpha_i t_i - \bigotimes_{i\in I-J} u_i \|$$

which tends to 0 by the last assertion of prop. 5 .

Remark 1. The infinite tensor products of Hilbert spaces have been introduced by von Neumann in [44] ; the space $\overset{h}{\otimes}{}^{t} H_i$ is denoted by him $\prod \otimes^{\mathcal{C}}_{\alpha\in I} H_\alpha$ where $\mathcal{C}$ is the class of t with respect to the relation $\approx$; if we take one element t in each class and the sum of the corresponding tensor products, we get $\prod \otimes_{\alpha\in I} H_\alpha$; finally if we take only elements t which lie in some class with respect to $\sim$, we get $\prod \otimes^{\mathcal{C}_w}_{\alpha\in I} H_\alpha$.

n.6.4. Infinite tensor products of operators.

Proposition 6. Suppose we have for each i a continuous linear operator T_i in H_i such that $\prod \| T_i \|$ exists and

$$\sum | \| T_i t_i \| - 1 | < \infty \tag{8}$$

$$\sum | (T_i t_i | t_i) - 1 | < \infty \; ; \tag{9}$$

there exists a unique continuous linear operator T in $\overset{h}{\otimes}{}^{t} H_i$ with the following property : if $\otimes x_i$ exists, $\otimes T_i x_i$ exists and is equal to $T(\otimes x_i)$.

Proof. First take an element x of the algebraic tensor product $\otimes^t H_i$ and write

$$\otimes^t H_i = (\bigotimes_{i\in J} H_i) \otimes (\bigotimes^t_{i\in I-J} H_i)$$

$$x = x_J \otimes \left(\bigotimes_{i \in I-J} t_i \right) \quad ;$$

by prop. 5 we can consider the vector

$$T x = \left(\bigotimes_{i \in J} T_i . x_J \right) \otimes \left(\bigotimes_{i \in I-J} T_i t_i \right)$$

and we get a linear operator $\mathcal{T}$ in $\otimes^t H_i$; T is continuous since

$$\| T x \| \leq \prod_{i \in J} \| T_i \| . \| x_J \| . \prod_{i \in I-J} \| T_i t_i \|$$

$$\leq \| x \| . \prod \| T_i \| \quad ;$$

hence it can be extended to a continuous linear operator T in $\overset{h}{\otimes}{}^t H_i$. Suppose now that $\otimes x_i$ exists ; then

$$T(\otimes x_i) = T(\lim_J L_J(\bigotimes_{i \in J} x_i))$$

$$= \lim_J T((\bigotimes_{i \in J} x_i) \otimes (\bigotimes_{i \in I-J} t_i))$$

$$= \lim_J (\bigotimes_{i \in J} T_i x_i) \otimes (\bigotimes_{i \in I-J} T_i t_i)$$

and this is equal to $\lim_J (\bigotimes_{i \in J} T_i x_i) \otimes (\bigotimes_{i \in I-J} t_i)$ by the same reasoning as in the end of th. 2.

Definition 6. Given a family (T_i) of continuous linear operators in H_i, we set

$$\overset{h}{\otimes}{}^t T_i = \text{str.lim.} \left(\bigotimes_{i \in J} T_i \right) \otimes I$$

whenever this limit exists ; it does exist under the hypothesis of prop. 6.

Proposition 7. In the situation of th. 2, if $\overset{h}{\otimes}{}^{t}\, T_i$ exists then $\overset{h}{\otimes}{}^{u}\, T_i$ exists too and is equal to $F.\overset{h}{\otimes}{}^{t}\, T_i.F^{-1}$.

In fact it is easy to see that for each finite J, F carries $(\bigotimes_{i\in J} T_i)\otimes I$ into the analogous operator in $\overset{h}{\otimes}{}^{u}\, H_i$.

n.6.5. Distributivity of tensor products with respect to Hilbert sums and integrals.

In the following theorem we suppose I countable.

Theorem 3. Let us consider for each i, a standard Borel space X_i, a Borel subset Y_i, a positive Borel measure μ_i on X_i with $\mu_i(Y_i) = 1$, a μ_i - measurable field of Hilbert spaces $x_i \longmapsto H_{i,x_i}$, and a square integrable vector field $t_{i,x_i} \in H_{i,x_i}$ where t_{i,x_i} is of norm 1 if $x_i \in Y_i$ and 0 in the opposite case. For each i let us set

$$H_i = \int_{X_i}^{\oplus} H_{i,x_i}.d\mu_i(x_i)$$

$$t_i = \int_{X_i}^{\oplus} t_{i,x_i}.d\mu_i(x_i) \quad \text{(unit vector in } H_i\text{)} ;$$

let us set $X = \prod^{(Y_i)} X_i$ and define μ on X as in § 1 ;

finally for each $x = (x_i) \in X$ we set

$$H_{(x)} = (\overset{h}{\bigotimes_{i\in J}} H_{i,x_i}) \overset{h}{\otimes} (\overset{h\,(t_{i,x_i})}{\bigotimes_{i\in I-J}} H_{i,x_i})$$

where $J = \{ i \mid x_i \notin Y_i \}$. Then one can put on the field $x \longmapsto H_{(x)}$ a structure of μ - measurable field such that

$\overset{h}{\underset{i \in J}{\otimes}}{}^{t} H_i$ is canonically isomorphic to $\int_X^{\oplus} H_{(x)} \cdot d\mu(x)$.

Sketch of the proof. For $J \in \mathcal{F}(I)$ we have, by § 1.3 of Part I, an isomorphism

$$u_J : \overset{h}{\underset{i \in J}{\otimes}} H_i \longrightarrow K = \int_{\prod_{i \in J} X_i}^{\oplus} \overset{h}{\underset{i \in J}{\otimes}} H_{i,x_i} \cdot d(\underset{i \in J}{\otimes} \mu_i)(x)$$

$$a_i \longmapsto \int^{\oplus} \underset{i \in J}{\otimes} a_{i,x_i} \cdot d(\underset{i \in J}{\otimes} \mu_i)(x)$$

where $a_i = \int_{X_i}^{\oplus} a_{i,x_i} \cdot d\mu_i(x_i)$; then we have an isometric mapping

$$u_J' : K \longrightarrow L = K \overset{h}{\otimes} \int_{\prod_{i \in I-J} Y_i}^{\oplus} \overset{h}{\underset{i \in I-J}{\otimes}}{}^{(t_{i,x_i})} H_{i,x_i} \cdot d(\underset{i \in I-J}{\otimes} \mu_i)(x)$$

$$b \longmapsto b \otimes \int^{\oplus} \underset{i \in I-J}{\otimes} t_{i,x_i} \cdot d(\underset{i \in I-J}{\otimes} \mu_i)(x) \quad ;$$

then an isomorphism

$$u_J'' : L \longrightarrow \int_{X_{(J)}}^{\oplus} H_{(x)} \cdot d\mu_{(J)}(x) \quad ;$$

and finally an isometric mapping

$$u_J''' : \int_{X_{(J)}}^{\oplus} H_{(x)} \cdot d\mu_{(J)}(x) \longrightarrow \int_X^{\oplus} H_{(x)} \cdot d\mu(x)$$

consisting in extending each vector field by 0 outside of $X_{(J)}$; the mappings $u_J''' \circ u_J'' \circ u_J' \circ u_J$ form an inductive system, hence define an isometric mapping

$$u : \overset{h}{\otimes}{}^{t} H_i \longrightarrow \int_X^{\oplus} H_{(x)} \cdot d\mu(x) \quad ;$$

one proves that it is surjective by a reasoning similar to that of theorem 1.

Corollary 4. If μ_i has total mass 1 and $Y_i = X_i$ we get an isomorphism

$$\overset{h}{\underset{i\in I}{\otimes}}{}^{t} \int_{X_i}^{\oplus} H_{i,x_i}\cdot d\mu_i(x_i) \sim \int_{\Pi X_i}^{\oplus} \overset{h}{\underset{i\in I}{\otimes}}{}^{(t_{i,x_i})} H_{i,x_i}\cdot d(\otimes \mu_i)(x).$$

Corollary 5. If $H_{i,x_i} = \mathbb{C}$ and $t_{i,x_i} = 1$ or 0 depending on whether x_i belongs to Y_i or not, we get an isomorphism

$$\overset{h}{\otimes}{}^{t} L^2(X_i, \mu_i) \sim L^2(X, \mu)$$

where t_i is the characteristic function of Y_i.

Corollary 6. Assuming the hypotheses of both corollaries 4 and 5 we have an isomorphism

$$\overset{h}{\otimes}{}^{t} L^2(X_i, \mu_i) \sim L^2(\Pi X_i, \otimes\mu_i)$$

where t_i is the function 1 on X_i.

Suppose now that each μ_i has the mass 1 at each point and that Y_i is reduced to some point a_i ; the reader will be able to state a result similar to th. 3 ; strictly speaking this is not a corollary of th. 3 since in our particular case we have not to assume I countable and X_i standard ; we shall only state the following corollary, analogous to cor. 5 :

Corollary 7. Suppose we have for each i, a set X_i and a point a_i of X_i ; then $\ell^2(\prod^{(a_i)} X_i)$ is canonically isomorphic to $\overset{h}{\otimes}{}^{(\delta_{a_i})} \ell^2(X_i)$.

n.6.6. The Hilbert symmetric space of a Hilbert space.

We shall introduce a notion similar to that of " symmetric algebra of a Banach space " (see § 5, ex. 2), but conveniently adapted to the category of Hilbert spaces. Let H be some Hilbert space ; for each integer $n > 0$ we can consider the Hilbert space

$$H^{\overset{h}{\otimes} n} = H \overset{h}{\otimes} \cdots \overset{h}{\otimes} H \quad n - \text{times}$$

and then the closed subspace S^nH consisting of those elements which are invariant by all permutations ; we denote by SH the Hilbert sum of all S^nH , $n = 0,1,2,\ldots$; an element of SH is a sequence $X = (X_n)$ with $X_n \in S^nH$ and we have

$$\|X\|^2 = \Sigma \|X_n\|^2 < \infty$$

$$(X \mid Y) = \Sigma (X_n \mid Y_n) .$$

For every x in H we denote by exp x the following element of SH :

$$\exp x = (1, x, x^{\otimes 2}/(2!)^{\frac{1}{2}}, \ldots x^{\otimes n}/(n!)^{\frac{1}{2}}, \ldots)$$

so that we have

$$(\exp x \mid \exp y) = e^{(x \mid y)}$$

$$\| \exp x \|^2 = e^{\| x \|^2}$$

whence it follows that the mapping exp is continuous.

Lemma 2. The elements exp x are total in SH.

It is sufficient to prove that each element of the form $a^{\otimes n}$ belongs to the closed linear subspace K generated by the exp x ; let us set for each real number t :

$$f(t) = \exp t\, a \quad ;$$

an easy computation shows that

$$f^{(n)}(0) = (n!)^{\frac{1}{2}}\, a^{\otimes n} \quad ;$$

now the relation

$$f^{(n)}(0) = \lim_{t=0} n!.t^{-n} \left(f(t) - f(0) - \ldots - t^{n-1}((n-1)!)^{-1} f^{(n-1)}(0)\right)$$

proves by induction that $f^{(n)}(0) \in K$.

Proposition 8. Let H be the Hilbert sum of a family of Hilbert spaces H_i ; there exists a unique isomorphism F of SH onto $\overset{h\,(\varepsilon_i)}{\otimes}\, SH_i$ with the following property : for each $x = (x_i) \in H$, $\otimes \exp x_i$ exists in $\overset{h\,(\varepsilon_i)}{\otimes}\, SH_i$ and is equal to F(exp x).

Proof. The unicity is clear. Now if $(x_i) \in H$ we have

$$\| \exp x_i \| - 1 = e^{\frac{1}{2}\| x_i \|^2} - 1 \sim \tfrac{1}{2} \| x_i \|^2$$

$$(\exp x_i \mid \varepsilon_i) - 1 = 0$$

so that $\exp x_i$ exists by prop. 5 ; we have

$$\begin{aligned}(\otimes \exp x_i \mid \otimes \exp y_i) &= \prod \exp x_i \mid \exp y_i) \\ &= \prod e^{(x_i \mid y_i)} \\ &= e^{\sum (x_i \mid y_i)} = e^{(x \mid y)} \\ &= (\exp x \mid \exp y) \ ;\end{aligned}$$

thus there exists an isomorphism F of SH onto the closed linear subspace of $\overset{h}{\otimes}{}^{\varepsilon} SH_i$ generated by the elements $\otimes \exp x_i$ with $F(\exp x) = \otimes \exp x_i$; but the $\otimes \exp x_i$ are total in $\overset{h}{\otimes}{}^{\varepsilon} SH_i$.

<u>Remark</u> 2. The space SH is used in Quantum Field Theory for the so called Representations of Commutation Relations ; see for instance [32],[34],[39].

§ 7. Infinite tensor products of von Neumann algebras.

n.7.1. The concrete tensor product.

Let us consider a family of Hilbert spaces H_i and for each i, a unit vector t_i in H_i and a von Neumann algebra $\mathcal{A}_i$ in H_i.

Definition 7. We shall denote by $\overset{c}{\otimes}{}^{t} \mathcal{A}_i$ the von Neumann algebra in the space $H = \overset{h}{\otimes}{}^{t} H_i$ which is generated by all operators of the form $\otimes T_i$ where $T_i \in \mathcal{A}_i$ and $T_i = I$ almost everywhere.

Clearly $\overset{c}{\otimes}{}^{t} \mathcal{A}_i$ also contains every operator $\otimes T_i$ with $T_i \in \mathcal{A}_i$ in the sense of definition 6 ; and in particular every operator $\otimes T_i$ where $T_i \in \mathcal{A}_i$, $\Pi \| T_i \|$ exists, $\sum | \| T_i t_i \| - 1 | < \infty$ and $\sum | (T_i t_i \mid t_i) - 1 | < \infty$. In the situation of th. 2, $\overset{c}{\otimes}{}^{t} \mathcal{A}_i$ and $\overset{c}{\otimes}{}^{u} \mathcal{A}_i$ are spatially isomorphic ; more precisely we have

$$F . \overset{c}{\otimes}{}^{t} \mathcal{A}_i . F^{-1} = \overset{c}{\otimes}{}^{u} \mathcal{A}_i \ ;$$

we shall see later (see § 9) that the type of $\overset{c}{\otimes}{}^{t} \mathcal{A}_i$ depends strongly on the choice of t.

Proposition 9. We have $\overset{c}{\otimes}{}^{t} \mathcal{A}'_i = (\overset{c}{\otimes}{}^{t} \mathcal{A}_i)'$, ~~the equality holds if all $\mathcal{A}_i$ are semi-finite~~.

First assertion : if $T_i \in \mathcal{A}_i$, $T'_i \in \mathcal{A}'_i$ and $T_i = T'_i = I$ almost everywhere we have

$$\otimes T_i \cdot \otimes T'_i \;=\; \otimes T_i T'_i \;=\; \otimes T'_i T_i \;=\; \otimes T'_i \cdot \otimes T_i \,.$$

<u>Second assertion</u> : take some operator S in $(\overset{c}{\otimes}{}^t \mathcal{A}_i)'$, some weak neighbourhood V of S :

$$V = \{ S' \mid |((S'-S).x_n|y_n)| < 1 \;,\; n = 1,\dots N \}$$

and some $\varepsilon > 0$. There exists $J \in \mathcal{F}(I)$ with the following property :

$$\| P.x_n - x_n \| \leqslant \varepsilon \;,\; \| P.y_n - y_n \| \leqslant \varepsilon \;,\; n = 1,\dots N$$

where P is the projection onto the subspace $K = (\overset{\ell}{\underset{i\in J}{\otimes}} H_i) \otimes (\underset{i\in I-J}{\otimes} t_i)$; we can write $P = I \otimes Q$ where Q is the projection onto the vector $\underset{i\in I-J}{\otimes} t_i$.

We claim that $S_P \in (\overset{c}{\underset{i\in J}{\otimes}} \mathcal{A}_i)' \otimes I$; in fact for $T_i \in \mathcal{A}_i$ and $x_i \in H_i$ with $i \notin J \Longrightarrow T_i = I,\ x_i = t_i$, we have

$$\begin{aligned} S_P \cdot \otimes T_i \cdot \otimes x_i &= P.S. \otimes T_i \cdot \otimes x_i \\ &= P. \otimes T_i .S. \otimes x_i \\ &= I \otimes Q. \otimes T_i .S. \otimes x_i \\ &= \otimes T_i . I \otimes Q.S. \otimes x_i \\ &= \otimes T_i .P.S. \otimes x_i \\ &= \otimes T_i .S_P. \otimes x_i \,. \end{aligned}$$

Since our assertion is true for finite tensor products (see
[42] [illegible]
~~[1], prop. 14, p. 102~~), this implies $S_P \in (\overset{c}{\underset{i\in J}{\otimes}} \mathcal{A}'_i) \otimes I$; now S_P, operator in K, has the form $R \otimes I$ where $R \in \overset{c}{\underset{i\in J}{\otimes}} \mathcal{A}'_i$;

we can consider the operator $R \otimes I$ in $\overset{h}{\otimes}{}^{t} H_i$ where I is the identity operator in $\overset{h}{\underset{i \in I-J}{\otimes}}{}^{t} H_i$; we have

$$R \otimes I \in \left(\overset{c}{\underset{i \in J}{\otimes}} \mathcal{A}_i' \right) \otimes I \subset \overset{c}{\otimes}{}^{t} \mathcal{A}_i' \quad ;$$

on the other hand if ε is sufficiently small we have $R \otimes I \in V$ since

$$|((R \otimes I - S).x_n | y_n)| \leqslant |(R \otimes I.x_n | y_n) - (R \otimes I.Px_n | Py_n)|$$

$$+ |(R \otimes I.Px_n | Py_n) - (S.Px_n | Py_n)|$$

$$+ |(S.Px_n | Py_n) - (S.x_n | y_n)| \quad ;$$

the second member of the righthand side is null while the other two are less than $\varepsilon \|S\|.(\|x_n\| + \|y_n\|)$.

Theorem 4. The von Neumann algebra $\overset{c}{\otimes}{}^{t} \mathcal{A}_i$ is a factor if and only if each $\mathcal{A}_i$ is a factor ; it is equal to $\mathcal{L}(H)$ if and only if $\mathcal{A}_i = \mathcal{L}(H_i)$ $\forall i$.

Second assertion : if for some j, $\mathcal{A}_j'$ contains a non scalar operator T_j, $(\overset{c}{\otimes}{}^{t} \mathcal{A}_i)'$ contains the non scalar operator $\otimes T_i$ where $T_i = T_j$ if $i = j$, $T_i = I$ if $i \neq j$.

Conversely suppose $\mathcal{A}_i = \mathcal{L}(H_i)$; by the preceding proposition we have

$$(\overset{c}{\otimes}{}^{t} \mathcal{A}_i)' = \overset{c}{\otimes}{}^{t} \mathcal{A}_i' = \text{scalars.}$$

First assertion : if for some j the center of $\mathcal{A}_j$ contains a non scalar operator T_j, the center of $\overset{c}{\otimes}{}^t \mathcal{A}_i$ contains the above operator $\otimes T_i$. Conversely suppose $\mathcal{A}_i$ is a factor ; by the preceding proposition the von Neumann $\mathcal{A}$ generated by $\overset{c}{\otimes}{}^t \mathcal{A}_i$ and $(\overset{c}{\otimes}{}^t \mathcal{A}_i)'$ contains all operators $\otimes T_i T_i'$ where $T_i \in \mathcal{A}_i$, $T_i' \in \mathcal{A}_i'$, $T_i = T_i' = I$ almost everywhere, hence all operators $\otimes S_i$ where $S_i \in \mathcal{L}(H_i)$, $S_i = I$ almost everywhere ; by the first part of the proof we have $\mathcal{A} = \mathcal{L}(H)$, and $\overset{c}{\otimes}{}^t \mathcal{A}_i$ is a factor.

n.7.2. <u>The abstract tensor product.</u>

Civen a family $(\mathcal{A}_i)_{i \in I}$ of von Neumann algebras and, for each i, a projection e_i in $\mathcal{A}_i$, we shall construct a von Neumann algebra $\overset{a}{\otimes}{}^e \mathcal{A}_i$ which admits as quotients the various concrete tensor products $\overset{c}{\otimes}{}^t \mathcal{A}_i$.

Let us first define the <u>inductive limit</u> of an inductive system of von Neumann algebras ; let I be a filtering ordered set, $(\mathcal{A}_i)_{i \in I}$ a family of von Neumann algebras, and for $i \leqslant j$, $M_{i,j}$ a normal morphism $\mathcal{A}_i \longrightarrow \mathcal{A}_j$ (by convention all morphisms of von Neumann algebras preserve the unit elements) such that $M_{j,k} \circ M_{i,j} = M_{i,k}$ for $i \leqslant j \leqslant k$; denote by $\mathcal{A}$ the algebraic inductive limit of this inductive system, by M_i the canonical morphisms $\mathcal{A}_i \longrightarrow \mathcal{A}$ and by π the direct sum of all cyclic representations ρ of $\mathcal{A}$ such that $\rho \circ M_i$ is normal for each i ; the von Neumann $\mathcal{A}$ generated by $\pi(\mathcal{A})$ will be called the <u>inductive limit</u> of our inductive system ; note that it exists if and only if there exist representations ρ of the above type ; it has the following universal property : given a von Neumann algebra $\mathcal{B}$, by associating to each normal morphism $v : \mathcal{A} \longrightarrow \mathcal{B}$ the family $(v \circ u_i)$, we get a bijective correspondance between the normal morphisms v and the families of normal morphisms v_i : $\mathcal{A}_i \longrightarrow \mathcal{B}$ such that $v_j \circ M_{i,j} = v_i$ for $i \leqslant j$ (inductive systems of normal morphisms).

We are now in a position to define $\overset{a}{\otimes}{}^{e}{}_{i}$; by realizing each $\mathcal{A}_i$ in some Hilbert space we can define the finite tensor products $\overset{c}{\underset{i \in J}{\otimes}} \mathcal{A}_i$, which are independant of the chosen realizations and form an inductive system : for $J \subset K$ we write

$$\overset{c}{\underset{i \in K}{\otimes}} \mathcal{A}_i = (\overset{c}{\underset{i \in J}{\otimes}} \mathcal{A}_i) \overset{c}{\otimes} (\overset{c}{\underset{i \in K-J}{\otimes}} \mathcal{A}_i)$$

and define $M_{J,K}$ by

$$M_{J,K}(a) = a \otimes (\underset{i \in K-J}{\otimes} e_i) .$$

<u>Definition</u> 8. We denote by $\overset{a}{\underset{i \in I}{\otimes}}{}^{e} \mathcal{A}_i$ the inductive limit of the above inductive system. If e = I we write $\overset{a}{\underset{i \in I}{\otimes}} \mathcal{A}_i$.

<u>Proposition</u> 10. Let, for each i, H_i a Hilbert space, t_i a unit vector in H_i, π_i a normal representation of $\mathcal{A}_i$ in H_i

with $\pi_i(e_i).t_i = t_i$; there exists a normal representation π of $\overset{a}{\otimes}{}^{e} \mathcal{A}_i$ in $\overset{h}{\otimes}{}^{t} H_i$ such that $\pi(\otimes a_i) = \otimes \pi_i(a_i)$ for each family $(a_i) \in \Pi^e \mathcal{A}_i$. Moreover $\operatorname{Im} \pi = \overset{c}{\otimes}{}^{t} \operatorname{Im} \pi_i$.

<u>Proof</u>. Set $H = \overset{h}{\otimes}{}^{t} H_i$, $\mathcal{B}_i = \operatorname{Im} \pi_i$ = von Neumann algebra in H_i ; take some J in $\mathcal{F}(I)$; by [1], p. 60 we have a normal morphism

$$u_J : \overset{c}{\underset{i \in J}{\otimes}} \mathcal{A}_i \longrightarrow \overset{c}{\underset{i \in J}{\otimes}} \mathcal{B}_i$$

$$\otimes a_i \longmapsto \otimes \pi_i(a_i) \quad ;$$

on the other hand we can define a normal morphism

$$u'_J : \overset{c}{\underset{i \in J}{\otimes}} \mathcal{B}_i \longrightarrow \mathcal{L}(H)$$

by writing

$$H = (\overset{h}{\underset{i \in J}{\otimes}} H_i) \overset{h}{\otimes} (\overset{h}{\underset{i \in I-J}{\otimes}}{}^{t} H_i)$$

$$u'_J(b) = b \otimes (\underset{i \in I-J}{\otimes} \pi_i(e_i)) \quad ;$$

we get normal morphisms

$$v_J = u'_J \circ u_J : \overset{c}{\underset{i \in J}{\otimes}} \mathcal{A}_i \longrightarrow \mathcal{L}(H)$$

which form an inductive system and define a normal morphism

$\pi : \overset{a}{\otimes}{}^{e} \mathcal{A}_i \longrightarrow \mathcal{L}(H)$ such that

$$\pi(\otimes a_i) = \otimes \pi_i(a_i) \qquad \forall (a_i) \in \Pi^e \mathcal{A}_i .$$

Last assertion : clearly we have $\operatorname{Im} \pi \subset \overset{c}{\otimes}{}^{t} \mathcal{B}_i$; to prove the converse inclusion it suffices to prove that $\otimes b_i \in \operatorname{Im} \pi$ for each family (b_i) with $b_i \in \mathcal{B}_i$ and $b_i = I$ almost everywhere, i.e. for $i \notin J$; for each $i \in J$ there exists a_i in $\mathcal{A}_i$ with $\pi_i(a_i) = b_i$; take $K \supset J$ and define an element $a_{(K)}$ in $\overset{c}{\underset{i \in K}{\otimes}} \mathcal{A}_i$ by

$$a_{(K)} = (\underset{i \in J}{\otimes} a_i) \otimes (\underset{i \in K-J}{\otimes} I) \;;$$

since $v_K(a_{(K)})$ belongs to $\operatorname{Im} \pi$, it suffices to show that $v_K(a_{(K)})$ converges strongly to $\otimes b_i$, i.e. that

$$v_K(a_{(K)}).x \longrightarrow \otimes b_i.x$$

for each x in H ; by equicontinuity and linearity we can suppose $x = \otimes x_i$, $x_i = t_i$ for $i \notin K$; then

$$\begin{aligned} v_K(a_{(K)}).x &= (\underset{i \in J}{\otimes} \pi_i(a_i).x_i) \otimes (\underset{i \in K-J}{\otimes} x_i) \otimes (\underset{i \in I-K}{\otimes} \pi_i(e_i).t_i) \\ &= (\underset{i \in J}{\otimes} b_i.x_i) \otimes (\underset{i \in K-J}{\otimes} x_i) \otimes (\underset{i \in I-K}{\otimes} t_i) \\ &= \otimes b_i.x_i = \otimes b_i.x \;. \end{aligned}$$

§ 8. Infinite tensor products of Hilbert algebras.

Let us consider for each i, a Hilbert algebra $\mathcal{A}_i$ with Hilbert completion H_i, and a projection of norm one $e_i \in \mathcal{A}_i$; then the left multiplication operator U_{e_i} is a non zero projection. The algebraic tensor product $\mathcal{A} = \otimes^e \mathcal{A}_i$ is a $*$ - algebra and at the same time a prehilbert space whose Hilbert completion is $H = \overset{h}{\otimes}{}^e H_i$; we claim that $\mathcal{A}$ is a Hilbert algebra : the axioms (i),(ii),(iv) of [1], p. 66 are trivially verified ; as for (iii), take an element $a = \otimes a_i$ in $\mathcal{A}$; by prop. 6 we can form the continuous operator $\overset{h}{\otimes}{}^e U_{a_i}$ in H and we have, for $(b_i) \in \mathcal{A}$:

$$(\overset{h}{\otimes}{}^e U_{a_i})(\otimes b_i) = \otimes a_i b_i = \otimes a_i \cdot \otimes b_i \quad ;$$

this proves that $\overset{h}{\otimes}{}^e U_{a_i} = U_{\otimes a_i}$ and that the mapping $b \longmapsto ab$ of $\mathcal{A}$ into itself is continuous for each a of the form $\otimes a_i$; the same property holds by linearity for each $a \in \mathcal{A}$.

We shall denote by $\mathcal{U}_i$, $\mathcal{V}_i$, $\mathcal{U}$, $\mathcal{V}$ the von Neumann algebras canonically associated with $\mathcal{A}_i$ and $\mathcal{A}$.

<u>Proposition</u> 11. We have $\mathcal{U} = \overset{c}{\otimes}{}^e \mathcal{U}_i$, $\mathcal{V} = \overset{c}{\otimes}{}^e \mathcal{V}_i$.

We have $\mathcal{U} \subset \overset{c}{\otimes}{}^e \mathcal{U}_i$ since $\mathcal{U}$ is generated by the operators $U_{\otimes a_i} = \overset{h}{\otimes}{}^e U_{a_i}$ which belong to $\overset{c}{\otimes}{}^e \mathcal{U}_i$; in the same manner $\mathcal{V} \subset \overset{c}{\otimes}{}^{e} \mathcal{V}_i$; then we have

$$\mathcal{U} \subset \overset{c}{\otimes}{}^{e} \mathcal{U}_i = \overset{c}{\otimes}{}^{e} \mathcal{V}_i' \subset (\overset{c}{\otimes}{}^{e} \mathcal{V}_i)' \subset \mathcal{V}' = \mathcal{U} .$$

Example 3. We take for $\mathcal{A}_i$ the algebra of all Hilbert-Schmidt operators in some Hilbert space K_i whose dimension r_i is finite or infinite but > 1 ; define the scalar product in $\mathcal{A}_i$ by

$$(a \mid b) = s_i^{-1} \operatorname{Tr} ab^*$$

where s_i is some integer verifying $0 < s_i \leqslant r_i$; finally we take for e_i a projection of rank s_i.

Proposition 12. If I is infinite, the type of the factor $\mathcal{U}$ is

(i) I_∞ if $s_i = 1$

(ii) II_1 if $s_i = r_i$ (which implies $r_i < \infty$)

(iii) II_∞ if $1 < s_i < r_i$.

Proof. We first remark that r_i infinite implies $\mathcal{U}$ infinite. Now choose an orthonormal basis (ξ_α) of K_i such that Im e_i is the subspace generated by $\xi_{\alpha_1}, \dots \xi_{\alpha_{s_i}}$; there exists an isomorphism $F : \mathcal{A}_i \longrightarrow K_i \overset{h}{\otimes} K_i$ with the following properties : for each a in $\mathcal{A}_i$ with matrix $(a_{\alpha\beta})$ we have

$$F(a) = s_i^{-\frac{1}{2}} \sum_{\alpha,\beta} a_{\alpha\beta} \cdot \xi_\alpha \otimes \xi_\beta ;$$

$$F(e_i) = s_i^{-\frac{1}{2}} \sum_{n=1}^{s_i} \xi_{\alpha_n} \otimes \xi_{\alpha_n} ;$$

$$F . U_a . F^{-1} = {}^t a \otimes I .$$

Suppose now $s_i = 1$; we can write

$$H_i = \mathcal{A}_i = H_{i,1} \overset{h}{\otimes} H_{i,2} \qquad \text{with } H_{i,j} = K_i$$

$$e_i = e_{i,1} \otimes e_{i,2}$$

then using the associativity of the tensor products :

$$\overset{h}{\otimes}{}^{e} H_i = (\overset{h}{\otimes}{}^{(e_{i,1})} H_{i,1}) \overset{h}{\otimes} (\overset{h}{\otimes}{}^{(e_{i,2})} H_{i,2})$$

$$\overset{h}{\otimes}{}^{e} U_{a_i} = (\overset{h}{\otimes}{}^{(e_{i,1})} t_{a_i}) \otimes I$$

$$\mathcal{U} = (\overset{h}{\otimes}{}^{(e_{i,1})} H_{i,1}) \otimes I$$

which proves (i).

Now suppose $s_i > 1$; denote by c_i the projection onto ξ_{α_1} ; we have

$$(c_i \mid c_i) = s_i^{-1} < 1 \quad ;$$

then for each $J \in \mathcal{F}(I)$

$$\| (\underset{i\in J}{\otimes} c_i) \otimes (\underset{i\in I-J}{\otimes} e_i) \|^2 = \prod_{i\in J} s_i^{-1} \quad ;$$

the left multiplication operator corresponding to the element $(\underset{i\in J}{\otimes} c_i) \otimes (\underset{i\in I-J}{\otimes} e_i)$ is a trace class projection in $\mathcal{U}$, whose trace is arbitrarily small ; hence $\mathcal{U}$ is continuous.

In case (ii), e_i is a unit element for $\mathcal{A}_i$, $\otimes^e \mathcal{A}_i$ has a unit element, $\mathcal{U}$ is finite and consequently of type II_1.

Finally consider the situation (iii) ; if r_i is infinite, $\mathcal{U}$ is infinite, continuous and also semi-finite, hence of type

II_∞ ; if r_i is finite, $\mathcal{A}_i$ has a unit element 1_i with

$$(1_i \mid 1_i) = r_i / s_i ;$$

by a reasoning quite analogous to the above we can construct projections in $\mathfrak{U}$ whose trace is finite and arbitrarily large, so that $\mathfrak{U}$ is of type II_∞ .

§ 9. The type of certain infinite tensor products of von Neumann algebras.

We suppose I countable ; for each $i \in I$ we set $H_i = H_{i,1} \overset{h}{\otimes} H_{i,2}$ where $H_{i,1}$ and $H_{i,2}$ are Hilbert spaces having the same dimension r_i, $1 < r_i \leqslant \aleph_0$; every element t_i of H_i can be written

$$t_i = \sum_{n=0}^{r_i - 1} \alpha_{i,n} \cdot t_{i,1,n} \otimes t_{i,2,n}$$

where $(t_{i,j,n})$ is an appropriate basis of $H_{i,j}$ and $\alpha_{i,n}$ a positive number with

$$\alpha_{i,o} \geqslant \alpha_{i,1} \geqslant \ldots\ldots \quad \text{and} \quad \sum_n \alpha^2_{i,n} = 1 .$$

The aim of this paragraph is to determine the type of the factor $\overset{c}{\otimes}\, {}^{t}\mathcal{a}_i$ where $\mathcal{a}_i = \mathcal{L}(H_{i,1}) \otimes I$.

n.9.1. First method.

We shall use prop. 12 ; we first suppose that for each i, the strictly positive $\alpha_{i,n}$ are equal, let

$$\alpha_{i,o} = \ldots\ldots = \alpha_{i,s_i-1} = s_i^{-\frac{1}{2}}$$

where s_i is some integer $\leqslant r_i$, and

$$\alpha_{i,n} = 0 \qquad \text{for} \quad n \geqslant s_i .$$

Denote by $\mathcal{A}_i$ the Hilbert algebra of the Hilbert-Schmidt operators in $H_{i,1}$, endowed with the scalar product

$$(a \mid b) = s_i^{-1} \cdot \operatorname{Tr} ab^* \quad ;$$

and by e_i the projection corresponding to the subspace of $H_{i,1}$ generated by $t_{i,1,o}, \ldots, t_{i,1,s_i-1}$; the isomorphism $\mathcal{A}_i \longrightarrow H_{i,1} \overset{h}{\otimes} H_{i,2}$ described in § 8 gives rise to an isomorphism of the Hilbert completion of $\otimes^e \mathcal{A}_i$ onto $\overset{h}{\otimes}{}^t H_i$ which carries $\mathcal{U}(\otimes^e \mathcal{A}_i)$ into $\overset{c}{\otimes}{}^t \mathcal{a}_i$; thus the type of $\overset{c}{\otimes}{}^t \mathcal{a}_i$ is given by prop.12.

Suppose now the $\alpha_{i,n}$ arbitrary ; we can replace the family (t_i) by an equivalent family without changing the type of $\overset{c}{\otimes}{}^t \mathcal{a}_i$ (see § 7 after defin. 7) ; if we set

$$t'_i = t_{i,1,o} \otimes t_{i,2,o}$$

we have $(t_i \mid t'_i) = \alpha_{i,o}$; thus if $\sum_i (1 - \alpha_{i,o})$ is finite, t is equivalent to t' and $\overset{c}{\otimes}{}^t \mathcal{a}_i$ is of type I_∞ ; one can get in a similar way the other results contained in the

<u>Theorem</u> 5. The type of $\overset{c}{\otimes}{}^t \mathcal{a}_i$ is

(i) I_∞ if $\sum_i (1 - \alpha_{i,o}) < \infty$

(ii) II_1 if $r_i < \infty$ and $\sum_i (1 - r_i^{-\frac{1}{2}} \sum_{n=o}^{r_i - 1} \alpha_{i,n}) < \infty$

(iii) II_∞ if there exist integers s_i with $1 < s_i < r_i$ and $\sum_i (1 - s_i^{-\frac{1}{2}} \sum_{n=o}^{s_i - 1} \alpha_{i,n}) < \infty$.

n.9.2. <u>Second method</u>.

In this number we shall prove that $\overset{c}{\otimes}{}^t \mathcal{a}_i$ can be obtained by the method used by Murray and von Neumann to construct examples of factors (see [1], p. 132), which will allow us to establish the converses of (i) and (ii) in th. 5. It will be more convenient to write

$$t_i = \sum_{n\in N_i} \alpha_{i,n} \cdot t_{i,1,n} \otimes t_{i,2,n}$$

where N_i is equal to $\mathbb{Z}$ if $r_i = \aleph_0$ and to $\mathbb{Z}/r_i\mathbb{Z}$ if r_i is finite ; we can suppose

$$\alpha_{i,o} \geqslant \alpha_{i,1} \geqslant \alpha_{i,-1} \geqslant \alpha_{i,2} \geqslant \alpha_{i,-2} \geqslant \dots \qquad \text{if} \quad r_i = \aleph_0$$

$$\alpha_{i,o} \geqslant \alpha_{i,1} \geqslant \dots \geqslant \alpha_{i,r_i-1} \qquad \text{if} \quad r_i < \aleph_0 .$$

a) <u>Particular case</u>.

We suppose $\alpha_{i,n} > 0$ for each i and n . Let us denote by μ_i the measure on N_i having the mass $\alpha^2_{i,n}$ at each point n , by $\delta_{i,n}$ the Dirac function at n ; set

$$K_{i,1} = L^2(N_i, \mu_i)$$

$$K_{i,2} = \ell^2(N_i) \quad ;$$

the elements $\alpha^{-1}_{i,n} \cdot \delta_{i,n} \otimes \delta_{i,n-p}$ constitute an orthonormal basis of $K_{i,1} \overset{h}{\otimes} K_{i,2}$; define an isomorphism

$$M_i \;:\; H_{i,1} \overset{h}{\otimes} H_{i,2} \longrightarrow K_{i,1} \overset{h}{\otimes} K_{i,2}$$

by

$$M_i(t_{i,1,n} \otimes t_{i,2,p}) = \alpha^{-1}_{i,n} \cdot \delta_{i,n} \otimes \delta_{i,n-p} \quad ;$$

then

$$M_i(t_i) = \sum_n \delta_{i,n} \otimes \delta_{i,o} = 1 \otimes \delta_{i,o} .$$

The von Neumann algebra $\mathcal{L}(H_{i,1})$ is generated by, on the one hand, the diagonal operators with respect to the basis $(t_{i,1,n})$, and on the other hand the shift operator

$$V_i \; : \; t_{i,1,n} \longmapsto t_{i,1,n+1} \;;$$

if T_f is the diagonal operator of multiplication by some function f, M_i carries $T_f \otimes I$ into $T_f \otimes I$; on the other hand M_i carries $V_i \otimes I$ into W_i defined by

$$W_i(\delta_{i,n} \otimes \delta_{i,p}) = \alpha_{i,n} / \alpha_{i,n+1} \cdot \delta_{i,n+1} \otimes \delta_{i,p+1} \,.$$

The restricted product $\Pi' N_i$ acts in the space ΠN_i by componentwise addition ; consequently it acts in $L^2(\Pi N_i, \otimes \mu_i)$ by unitary operators U_m for $m \in \Pi' N_i$; similarly $\Pi' N_i$ acts in $\ell^2(\Pi' N_i)$ by unitary operators U'_m . Now we have an isomorphism

$$\overset{h}{\otimes} M_i \; : \; \overset{h}{\otimes}{}^{t} H_i = \overset{h}{\otimes}{}^{t}(H_{i,1} \overset{h}{\otimes} H_{i,2}) \longrightarrow \overset{h}{\otimes}{}^{(1 \otimes \delta_{i,o})}(K_{i1} \otimes K_{i2}) \;;$$

then by the associativity property, an isomorphism

$$\overset{h}{\otimes}{}^{(1 \otimes \delta_{i,o})}(K_{i,1} \overset{h}{\otimes} K_{i,2}) \longrightarrow (\overset{h}{\otimes}{}^{1} K_{i,1}) \overset{h}{\otimes} (\overset{h}{\otimes}{}^{(\delta_{i,o})} K_{i,2}) \;;$$

finally by corollaries 6 and 7 an isomorphism

$$(\overset{h}{\otimes}{}^{1} K_{i,1}) \overset{h}{\otimes} (\overset{h}{\otimes}{}^{(\delta_{i,o})} K_{i,2}) \longrightarrow L^2(\Pi N_i, \otimes \mu_i) \overset{h}{\otimes} \ell^2(\Pi' N_i) \;;$$

by composing these three isomorphisms we get a new isomorphism

$$M \; : \; \overset{h}{\otimes}{}^{t} H_i \longrightarrow L^2(\Pi N_i, \otimes \mu_i) \overset{h}{\otimes} \ell^2(\Pi' N_i) \;;$$

M carries each operator of the form $\otimes(T_{f_i} \otimes I)$ into $T_{\otimes f_i} \otimes I$;

each operator $\otimes(V_i^{m_i} \otimes I)$, where $m = (m_i) \in \Pi' N_i$, into the operator $U_m \otimes U'_m$; hence M carries $\overset{c}{\otimes}\, {}^t\mathcal{a}_i$ into the von Neumann algebra generated by the operators $T_f \otimes I$ with $f \in L^\infty(\Pi N_i, \otimes\mu_i)$, and $U_m \otimes U'_m$ with $m \in \Pi' N_i$.

Suppose $\overset{c}{\otimes}\, {}^t\mathcal{a}_i$ is of type I ; by [43], $\otimes\mu_i$ is atomic ; since the point in ΠN_i which has the largest measure is the point with all components null, we must have $\Pi \alpha_{i,o} > 0$, or equivalently $\sum (1 - \alpha_{i,o}) < \infty$.

Suppose now $\overset{c}{\otimes}\, {}^t\mathcal{a}_i$ is of type II_1 ; by [43], $\otimes\mu_i$ is equivalent to some finite Borel measure, invariant under $\Pi' N_i$; on the other hand all r_i are finite , ΠN_i is a compact group for the product topology, and $\Pi' N_i$ is a dense subgroup acting by translations ; each finite Borel measure on ΠN_i is a Radon measure, and being invariant under $\Pi' N_i$, it will be invariant by all translations, i.e. equivalent to the Haar measure ; the Haar measure is the product of the measures ν_i which have a mass r_i^{-1} at each point ; by [41] $\otimes\mu_i \sim \otimes \nu_i$ implies $\sum_i (1 - r_i^{-\frac{1}{2}} \sum_{n=0}^{n_i-1} \alpha_{i,n}) < \infty$.

b) <u>General case</u>.

The previous results still hold since we can make all $\alpha_{i,n}$ strictly positive by modifying them sufficiently little to not change the equivalence class of (t_i) nor the nature of the families $(1 - \alpha_{i,o})$ and $(1 - r_1^{-\frac{1}{2}} \Sigma \alpha_{i,n})$; thus we have proved the following

Theorem 6. With the notations of the beginning of § 9, the type of the factor $\overset{c}{\otimes}\, {}^t\mathcal{A}_i$ is

(i) I_∞ if and only if $\sum_i (1 - \alpha_{i,o}) < \infty$

(ii) II_1 if and only if $r_i < \infty$ and $\sum_i (1 - r_i^{-\frac{1}{2}} \sum_{n=o}^{r_i - 1} \alpha_{i,n}) < \infty$.

Remark 3. It is more difficult to distinguish the types II_∞ and III , C.C.Moore proves in [42] the f llowing result, by means of a deeper analysis of the infinite products of measures : suppose $\alpha_{i,o} \geqslant k \quad \forall\, i$ for some $k > 0$, then $\overset{c}{\otimes}\, {}^t\mathcal{A}_i$ is of type III if and only if

$$\sum_{i,n} \alpha^2_{i,n} \left(\inf(\alpha^2_{i,o}/\alpha^2_{i,n} - 1 , c)\right)^2 = \infty$$

for some (or equivalently for all) constants $c > 0$; he also states without proof on p. 458 a result equivalent to the third assertion of our th 5 . E.Størmer has proved by another method (see [46]) some results contained in th. 5 and also the following (contained in Moore's theorem) : suppose r_i and $\alpha_{i,n}$ independant of i ; then $\overset{c}{\otimes}\, {}^t\mathcal{A}_i$ is of type III iff there exist at least two distinct and non null $\alpha_{i,n}$.

<u>Remark</u> 4 (On the isomorphisms between the various factors $\overset{c}{\otimes}\, {}^t\mathcal{a}_i$).

In [35] Araki and Woods study systematically the set E of the isomorphism classes of the factors $\overset{c}{\otimes}\, {}^t\mathcal{a}_i$ which can be obtained by varying the r_i and t_i ; E can be divided into five mutually disjoint subsets $E_1, \dots E_5$:

E_1 contains only one factor, which is of type I .

E_2 contains only one factor, which is of type II_1 and hyperfinite.

E_3 contains exactly the factors $\mathcal{A}_\lambda$, $\lambda \in]0,\frac{1}{2}[$, constructed in the following way : $\mathcal{A}_\lambda = \overset{c}{\otimes}\, {}^t\mathcal{a}_i$ where $r_i = 2$, $\lambda_{i,0} = \lambda$, $\alpha_{i,1} = 1 - \lambda$; these factors are of type III ; it had been proved earlier that they are mutually non isomorphic ([45]).

E_4 contains factors of various types and among others an uncontable family of type III factors.

E_5 contains exactly one factor, which is of type III.

§ 10. Infinite tensor products of C^*- algebras. Definition and first properties.

Let us consider a family $(A_i)_{i\in I}$ of C^*- algebras and for each i, a non zero projection e_i in A_i ; if we endow each finite tensor product $\bigotimes_{i\in J} A_i$ with the $\vee$ crossnorm (resp. the $*$ crossnorm), the morphisms $L_{J,K}$ are isometric, so that we can define the $\vee$ and $*$ norms on $\otimes^e A_i$; clearly these are respectively the largest and the smallest C^*- crossnorms ; the C^* completions will be denoted by $\check{\otimes}^e A_i$ and $\overset{*}{\otimes}{}^e A_i$; they can also be considered as the inductive limits of the finite tensor products $\check{\bigotimes}_{i\in J} A_i$ and $\overset{*}{\bigotimes}_{i\in J} A_i$; if each A_i has property (T), $\check{\otimes}^e A_i$ and $\overset{*}{\otimes}{}^e A_i$ are identical.

The tensor products $\check{\otimes}^e A_i$ and $\overset{*}{\otimes}{}^e A_i$ possess properties of associativity similar to that of § 2.

If e_i is a unit element for A_i, we write $\check{\otimes} A_i$ and $\overset{*}{\otimes} A_i$ instead of $\check{\otimes}^e A_i$ and $\overset{*}{\otimes}{}^e A_i$; $\check{\otimes} A_i$ has the following universal property : given a C^*- algebra B with unit, there is a bijective correspondance between the unitary morphisms $u : \check{\otimes} A_i \longrightarrow B$ and the families of commuting unitary morphisms $u_i : A_i \longrightarrow B$; it is given by $u(\otimes a_i) = \prod u_i(a_i)$ for each (a_i) in $\prod^e A_i$.

Proposition 13. Consider a family of Banach $*$ - algebras A_i with projections e_i of norm 1 ; denote by e'_i the canonical image of e_i in $C^*(A_i)$ and suppose $e'_i \neq 0$. Then $C^*(\hat{\otimes}^e A_i)$ is canonically isomorphic to $\check{\otimes}^{e'} C^*(A_i)$.

In fact we have

$$\check{\otimes}^{e'} C^*(A_i) = \varinjlim \underset{i \in J}{\otimes} C^*(A_i) = \varinjlim C^*(\underset{i \in J}{\hat{\otimes}} A_i)$$

and it is easy to see that the functor $*$ commutes with the inductive limits.

Corollary 8. Let (X_i) be a family of locally compact groups with compact open subgroups Y_i ; then $C^*(\Pi^{(Y_i)} X_i)$ is canonically isomorphic to $\check{\otimes} C^*(X_i)$ where e_i is the characteristic function of Y_i.

This is a consequence of th. 1 and prop. 13.

Example . Let (X_i) be a family of locally compact topological spaces with compact open subsets Y_i ; then $\mathscr{C}_o(\Pi^{(Y_i)} X_i)$ is canonically isomorphic to $\overset{*}{\otimes}^e \mathscr{C}_o(X_i)$ where e_i is the characteristic function of Y_i. The proof is quite similar to that of th. 1.

§ 11. Infinite tensor products of representations of C^*-algebras.

Proposition 14. Let us consider for each i, a C^*- algebra A_i, a non zero projection e_i in A_i, a Hilbert space H_i, a unit vector t_i in H_i and a representation π_i of A_i in H_i such that

$$\sum_i (1 - \| \pi_i(e_i).t_i \|) < \infty . \qquad (10)$$

Then there exists a unique representation π of $\overset{*}{\otimes}{}^e A_i$ in $\overset{h}{\otimes}{}^t H_i$ such that $\pi(\otimes a_i) = \otimes \pi_i(a_i)$ for each family (a_i) in $\prod^e A_i$.

Proof The unicity is clear. To prove the existance take an element in $\otimes^e A_i$ of the form

$$a = a_J \otimes (\underset{i \in I-J}{\otimes} e_i)$$

where $J \in \mathfrak{F}(I)$ and $a_J \in \underset{i \in J}{\otimes} A_i$; write

$$\overset{h}{\otimes}{}^t H_i = (\underset{i\, J}{\overset{h}{\otimes}} H_i) \overset{h}{\otimes} (\underset{i \in I-J}{\overset{h}{\otimes}}{}^t H_i) ;$$

(10) implies $\sum_i | 1 - (\pi_i(e_i) t_i \mid t_i) | < \infty$ and by prop. 6 we can consider the following continuous linear operator in $\overset{h}{\otimes}{}^t H_i$:

$$\pi(a) = (\underset{i \in J}{\otimes} \pi_i)(a_J) \otimes (\underset{i \in I-J}{\overset{h}{\otimes}}{}^t \pi_i(e_i)) ;$$

one can easily check that π is a representation and moreover by prop 6 we have

$$\| \pi(a) \| \leqslant \| a_J \|_* = \| a \|_* ;$$

then π extends to a representation of $\overset{*}{\otimes}{}^{e} A_i$ which has the required properties.

<u>Definition</u> 10. The representation π defined in prop. 14 will be denoted by $\overset{*}{\otimes}{}^{e,t} \pi_i$. The von Neumann algebra it generates is identical with $\overset{c}{\otimes}{}^{t} \pi_i(A_i)''$ (the proof is the same as for prop. 10) ; thus $\overset{*}{\otimes}{}^{e,t} \pi_i$ is factorial or irreducible if and only if each π_i has the same property.

The kernel of $\overset{*}{\otimes}{}^{e,t} \pi_i$ depends only on the kernels of the π_i ; in particular $\overset{*}{\otimes}{}^{e,t} \pi_i$ is faithful if and only if each π_i is faithful.

On the other hand by prop. 7 if $t \sim t'$, $\overset{*}{\otimes}{}^{e,t} \pi_i$ and $\overset{*}{\otimes}{}^{e,t'} \pi_i$ are equivalent ; the following proposition is a partial converse of this result.

<u>Proposition</u> 15. We suppose each π_i is irreducible ; then $\overset{*}{\otimes}{}^{e,t} \pi_i$ and $\overset{*}{\otimes}{}^{e,t'} \pi_i$ are equivalent if and only if we have $t \sim t'$.

<u>Proof</u>. We suppose $t \not\sim t'$ and prove that the two representations are not equivalent.

a) <u>Particular case</u>.

We suppose $A_i = \mathcal{L}(H_i)$, π_i = identity. Since $t \not\sim t'$, we have $\sum_i (1 - |(t_i\ t_i')|) = \infty$; there exists a countable subset $I_o \subset I$ such that $\sum_{i \in I_o} (1 - |(t_i | t_i')|) = \infty$; it is sufficient to prove that the representations $\overset{*}{\underset{i \in I_o}{\otimes}}{}^{e,t} \pi_i$ and $\underset{i \in I_o}{\otimes}{}^{e,t'} \pi_i$ are not equivalent, so that we are led back to the case where I is countable, say $I = \{1, 2, \ldots\}$. Denote by P_i the projection operator in H_i onto t_i and set

$$T_n = P_1 \otimes P_2 \otimes \ldots \otimes P_n \otimes e_{n+1} \otimes e_{n+2} \otimes \ldots \quad ;$$

we shall prove that $(\overset{*}{\otimes}{}^{e,t'} \pi_i)(T_n)$ converges strongly to 0 while $(\overset{*}{\otimes}{}^{e,t} \pi_i)(T_n)$ does not, which will establish our result.

To prove that

$$(\overset{*}{\otimes}{}^{e,t'} \pi_i)(T_n).x \longrightarrow 0 \qquad \forall \; x \in \overset{h}{\otimes}{}^{t'} H_i$$

we can take x in the algebraic tensor product $\otimes^{t'} H_i$ since our operators have norms $\leqslant 1$; then by linearity we can take $x = \otimes x_i$ where $x_i = t_i'$ if i is larger than some number j ; then for $n \geqslant j$:

$$(\overset{*}{\otimes}{}^{e,t'} \pi_i)(T_n).x = (\overset{j}{\underset{i=1}{\otimes}} P_i x_i) \otimes (\overset{\sim}{\underset{i=j+1}{\otimes}} P_i t_i') \otimes (\underset{i>n}{\otimes} t_i')$$

$$\| (\overset{*}{\otimes}{}^{e,t'} \pi_i)(T_n).x \| \leqslant \| x \| . \prod_{i=j+1}^{\sim} |(t_i | t_i')| \quad ;$$

since $\sum_{i=j+1}^{\infty} (1 - |(t_i | t_i')|) = \infty$, $\prod_{i=j+1}^{\infty} |(t_i | t_i')| = 0$,

$\prod_{i=j+1}^{\sim} |(t_i | t_i')| \longrightarrow 0$ and we are done.

b) <u>General case</u>.

Set $f_i = \pi_i(e_i)$; $\otimes^f \mathcal{L}(H_i)$ is included and weakly dense in the tensor product $\overset{a}{\otimes}{}^{f}\mathcal{L}(H_i)$ defined in n.7.2 ; we have a morphism $\otimes\pi_i : \otimes^e A_i \longrightarrow \otimes^f \mathcal{L}(H_i)$ whose image is weakly dense since each π_i is irreducible ; by prop. 10 we have representations u and u' of $\overset{a}{\otimes}{}^{f}\mathcal{L}(H_i)$ in $\overset{h}{\otimes}{}^{t} H_i$ and $\overset{h}{\otimes}{}^{t'} H_i$; by the first part of the proof the restrictions of u and u' to $\otimes^e A_i$ are not equivalent, but these restrictions are nothing but $\otimes^{e,t}\pi_i$ and $\otimes^{e,t'}\pi_i$.

<u>Theorem</u> 7. If I is infinite and if each A_i admits sufficiently many irreducible representations π with rank $\pi(e_i) \geqslant 2$, then $\overset{*}{\otimes}{}^{e} A_i$ is antiliminar.

<u>Proof</u>. For each i there exist a set X_i and for each point x_i of X_i an irreducible representation π_{i,x_i} of A_i in some Hilbert space H_{i,x_i} such that rank $\pi_{i,x_i}(e_i) \geqslant 2$ and $\bigoplus_{x_i \in X_i} \pi_{i,x_i}$ is faithful. We choose a point y_i in X_i and vectors t_{i,x_i} , t'_{i,x_i} in $\operatorname{Im}\pi_{i,x_i}(e_i)$ such that

$$(t_{i,x_i} \mid t'_{i,x_i}) = 0$$

$$\| t_{i,x_i} \| = \| t'_{i,x_i} \| = \begin{cases} 1 & \text{if } x_i = y_i \\ 0 & \text{otherwise} \end{cases} ;$$

for each $x = (x_i)$ in $X = \prod^{(y_i)} X_i$ we set

$$\pi(x) = \overset{*}{\otimes}{}^{e\,(t_{i,x_i})}\pi_i \ , \quad \pi'(x) = \overset{*}{\otimes}{}^{e\,(t'_{i,x_i})}\pi_i$$

which makes sense because t_{i,x_i} and t'_{i,x_i} are unit vectors for almost all i ; $\pi_{(x)}$ and $\pi'_{(x)}$ are irreducible representations of $\overset{*}{\otimes}{}^{e} A_i$, have the same kernel and are unequivalent by the preceding proposition ; it suffices now to prove that $\underset{x \in X}{\oplus} \pi_{(x)}$ and $\underset{x \in X}{\oplus} \pi'_{(x)}$ are faithful. Consider the first : as indicated in n.6.5 we have an isomorphism

$$\underset{x \in X}{\oplus} H_{(x)} \longrightarrow \overset{h}{\underset{i \in I}{\otimes}}{}^{t} \left(\underset{x_i \in X_i}{\oplus} H_{i,x_i} \right) ;$$

as easily verified this isomorphism carries $\underset{x \in X}{\oplus} \pi_{(x)}$ into $\overset{*}{\underset{i \in I}{\otimes}}{}^{e,t} \left(\underset{x_i \in X_i}{\oplus} \pi_{i,x_i} \right)$; since the second representation is faithful, so is the first.

Corollary 9. If I is infinite, e_i = unit element, and A_i has no nonzero commutative two sided closed ideal, $\overset{*}{\otimes} A_i$ is antiliminar.

Corollary 10. We suppose that I is infinite countable, A_i is postliminar separable and admits sufficiently many irreducible representations π with rank $\pi(e_i) \geqslant 2$; then $\overset{*}{\otimes}{}^{e} A_i$ admits an irreducible representation which is not equivalent to a tensor product of representations.

Proof. Take a partition $I = I_1 \cup I_2$ with I_1 and I_2 infinite and write $\overset{*}{\otimes}{}^{e} A_i = B_1 \overset{*}{\otimes} B_2$ where $B_j = \overset{*}{\underset{i \in I_j}{\otimes}}{}^{e} A_i$;

by Part I, prop. 7, B_1 and B_2 have property (T) and we can also write $\overset{*}{\otimes}{}^{e} A_i = B_1 \overset{\vee}{\otimes} B_2$; B_1 and B_2 being separable and antiliminar, by Part I, th. 6 the algebra $B_1 \overset{\vee}{\otimes} B_2$ admits an irreducible representation which is not a tensor product of representations of B_1 and B_2 , and consequently it is not a tensor product of representations of the A_i .

<u>Bibliography</u> [9] .

§ 12. Infinite tensor products of positive functionals on C^*- algebras.

We consider C^*- algebras A_i with non zero projections e_i.

Proposition 16. Let f_i be a positive functional on A_i such that $f_i(e_i) = 1$ and $\prod \|f_i\| < \infty$ (note that $\|f_i\| \geqslant 1$). Then there exists a unique positive functional f on $\overset{*}{\otimes}{}^e A_i$ verifying

$$f(\otimes a_i) = \prod f_i(a_i) \qquad \forall (a_i) \in \prod{}^e A_i \quad ;$$

its norm is equal to $\prod \|f_i\|$; finally the representation associated with f is equivalent to $\overset{*}{\otimes}{}^{e,t} \pi_i$ where π_i is the representation associated with f_i and t_i the corresponding cyclic vector multiplied by $\|f_i\|^{-\frac{1}{2}}$.

Proof. The unicity being trivial we shall prove the existence of f ; we have a multilinear functional on $\prod{}^e A_i$: $(a_i) \longmapsto \prod f_i(a_i)$, whence a linear functional f on $\otimes^e A_i$ such that $f(\otimes a_i) = \prod f_i(a_i)$; f is positive since its restriction to any finite tensor product is positive ; moreover if a is an element of $\underset{i\in J}{\otimes} A_i$ we have

$$|f(a)| \leqslant \prod_{i\in J} \|f_i\| \cdot \|a\|_* \leqslant \prod_{i\in I} \|f_i\| \cdot \|a\|_* ; \qquad (11)$$

hence f extends to a positive functional on $\overset{*}{\otimes}{}^e A_i$.

We now denote by H_i the space of π_i, by x_i the corresponding cyclic vector and set $t_i = x_i \cdot \| f_i \|^{-\frac{1}{2}}$; we have

$$f_i(a_i) = (\pi_i(a_i).x_i \mid x_i) \qquad \forall\ a_i \in A_i$$

$$\| t_i \|^2 = \| x_i \|^2 . \| f_i \|^{-1} = 1$$

$$\| \pi_i(e_i).t_i \|^2 = (\pi_i(e_i).t_i \mid t_i)$$

$$= \| f_i \|^{-1}.(\pi_i(e_i).x_i \mid x_i) = \| f_i \|^{-1} ;$$

since $\prod \| f_i \|^{-\frac{1}{2}} > 0$ we have

$$\sum (1 - \| f_i \|^{-\frac{1}{2}}) < \infty$$

$$\sum (1 - \| \pi_i(e_i).t_i \|) < \infty \quad ;$$

by prop. 14 we can form the representation $\pi = \overset{*}{\otimes}{}^{e,t} \pi_i$;

since

$$\| x_i \| = (x_i \mid t_i) = \| f_i \|^{\frac{1}{2}}$$

we have

$$\sum (\| x_i \| - 1) < \infty$$

$$\sum ((x_i \mid t_i) - 1) < \infty$$

and we can consider the vector $x = \otimes x_i$; it is cyclic for π and we have $f(a) = (\pi(a).x \mid x)$ for each a of the form $\otimes a_i$, hence for each a in $\overset{*}{\otimes}{}^{e} A_i$.

We finally prove that $\| f \| = \prod \| f_i \|$; by (11) we have $\| f \| \leq \prod \| f_i \|$; to prove the converse inequality take an $\varepsilon > 0$, a $J \in \mathcal{F}(I)$ such that

$$\prod_{i \in J} \| f_i \| \geq \prod \| f_i \| . (1+\varepsilon)^{-1} ,$$

and for each $i \in J$ an element a_i in A_i such that

$$\|a_i\| = 1 \quad \text{and} \quad |f_i(a_i)| \geq \|f_i\| . (1+\varepsilon)^{-1/n}$$

where $n = \text{card } J$; set

$$a = (\underset{i \in J}{\otimes} a_i) \otimes (\underset{i \in I-J}{\otimes} e_i) \quad ;$$

then $\| a \| = 1$ and

$$\begin{aligned} \| f \| \geq |f(a)| &= \prod_{i \in J} |f_i(a_i)| \\ &\geq (1+\varepsilon)^{-1} . \prod_{i \in J} \|f_i\| \\ &\geq (1+\varepsilon)^{-2} . \prod \|f_i\| . \end{aligned}$$

<u>Definition</u> 11. The positive functional f defined above will be denoted by $\overset{*}{\otimes}{}^e f_i$; it is factorial or pure if and only if each f_i has the same prpperty ; it is a state if and only if each f_i is a state. In particular if e_i is the identity of A_i one can form the tensor product of an arbitrary family of states.

<u>Proposition</u> 17. Let us consider a C^*- algebra A_o , a nonzero projection e_o in A_o and two distinct pure states f_o and g_o on A_o with $f_o(e_o) = g_o(e_o) = 1$; let us set $A = \overset{*}{\otimes}{}^e A_i$ where $A_i = A_o$, $e_i = e_o$, $f = \overset{*}{\otimes}{}^e f_i$ where $f_i = f_o$, $g = \overset{*}{\otimes}{}^e g_i$ where $g_i = g_o$. Then the representations associated with f and g are unequivalent.

Proof. Suppose they are equivalent ; denote by H_i, π_i, x_i and K_i, ρ_i, y_i the objects associated with f_i and g_i respectively ; by hypothesis there exists an isomorphism

$$F : \overset{h}{\otimes}{}^{x} H_i \longrightarrow \overset{h}{\otimes}{}^{y} K_i$$

with

$$F.\otimes\pi_i(a_i).F^{-1} = \otimes\rho_i(a_i) \qquad \forall\ (a_i)\in \Pi^e A_i \ ;$$

if $a_i = e_i$ except for one index j, $\otimes\pi_i(a_i)$ and $\otimes\rho_i(a_i)$ are multiples of $\pi_j(a_j)$ and $\rho_j(a_j)$, so that π_j and ρ_j have a common multiple ; since they are irreducible they must be equivalent and we can realize them in some common Hilbert space H_o with two vectors x_o and y_o which are nonproportional since $f_o \neq g_o$; by prop. 15, $\overset{*}{\otimes}{}^{e,x}\pi_i$ and $\overset{*}{\otimes}{}^{e,y}\rho_i$ are unequivalent, which is a contradiction.

Remark 5. E.Störmer proves in [46] that the above result still holds when f_o and g_o are not pure, and that one can replace in the conclusion the word " unequivalent " by " non quasi-equivalent ". In the same paper he also studies the states of $\overset{*}{\otimes} A_i$ which are " symmetric ", i.e. invariant by all the automorphisms of $\overset{*}{\otimes} A_i$ determined by permutations of I ; he proves in particular that the extremal symmetric states are exactly the states $\overset{*}{\otimes} f_i$ where $f_i = f_o$, f_o a state of A_o ; and he determines the type of such a state $\overset{*}{\otimes} f_i$ when f_o is factorial.

In [40] A.Hulanicki and R.R.Phelps prove the following result : consider some group G_o of automorphisms of A_o ; then G_o^I acts by automorphisms in $\overset{*}{\otimes} A_i$; let G be the group of automorphisms of $\overset{*}{\otimes} A_i$ generated by G_o^I and the permutation automorphisms ; then the extremal G-invariant states are exactly the states $\overset{*}{\otimes} f_i$ where $f_i = f_o$, f_o a G_o-invariant state of A_o.

§ 13. Study of the case where e_i has rank ≤ 1 .

n.13.1. Definitions and examples.

In the preceding paragraphs we have seen several properties of $\overset{*}{\otimes}^e A_i$ in the case where e_i is "large" in a certain sense (for instance prop. 17, th. 7, cor. 9 and 10); in this paragraph we shall be concerned with the thoroughly different case where e_i is "small".

Definition 12. Given a C^*- algebra A, a projection e in A is said to have rank ≤ 1 if for every irreducible representation π of A the projection $\pi(e)$ has rank ≤ 1.

By [2], 4.2.6 each projection of rank ≤ 1 is contained in the largest liminar ideal of A ; consequently it must be 0 if A is antiliminar. On the other hand if a projection e lies in some closed two sided ideal I of A, it has rank ≤ 1 in I iff it has rank ≤ 1 in A (in fact for each irreducible representation π of A, $\pi | I$ is either null or irreducible); finally if f is a projection of rank ≤ 1 in A, its canonical image in A/I has also rank ≤ 1 .

Example 5.

(i) If A is commutative every projection has rank ≤ 1.

(ii) If A is elementary (i.e. of the form $\mathcal{LC}(H)$ with H a Hilbert space), every projection which has rank 1 in the usual sense has rank $\leqslant 1$ in our sense. If we set $A_i = \mathcal{L}\mathcal{C}(H_i)$ and take e_i of rank 1, $\overset{*}{\otimes}{}^e A_i$ is nothing but $\mathcal{L}\mathcal{C}(\overset{h}{\otimes}{}^t H_i)$ where $t_i \in \operatorname{Im} e_i$; in fact if $(a_i) \in \prod^e A_i$, $\otimes a_i$ belongs to $\mathcal{L}\mathcal{C}(\overset{h}{\otimes}{}^t H_i)$ since it is of the form $(\bigotimes_{i \in J} a_i) \otimes (\bigotimes_{i \in I-J} e_i)$ where both factors are compact ; it follows that $\overset{*}{\otimes}{}^e A_i$ is included in $\mathcal{L}\mathcal{C}(\overset{h}{\otimes}{}^t H_i)$, but being irreducible it must be equal to it.

(iii) Let A be the C^*- algebra defined by a continuous field of C^*- algebras $(A(t),\Theta)$ (see[2], 10.4.1); $e = (e(t))$ an element of A such that each e(t) is a projection of rank $\leqslant 1$ in $A(t)$. Then e is a projection of rank $\leqslant 1$; in fact one obtains all irreducible representations π of A in the following manner : taking an index t and an irreducible representation ρ of $A(t)$, and setting

$$\pi(a) = \rho(a(t)) \qquad \text{for each } a \in A .$$

In particular if the $A(t)$ are elementary one can take $e = (e(t))$ where rank $e(t) = 0$ or 1.

(iv) Let G be a locally compact group containing a compact open subgroup K with the following property : for each irreducible representation π of G in a space H, the space of all vectors in H invariant by $\pi(K)$ has dimension $\leqslant 1$. Then the characteristic function e of K has rank $\leqslant 1$ in $C^*(G)$; in fact it is known that $\pi(e) = \int_K \pi(k).dk$ is the projection onto the space of all vectors invariant under $\pi(K)$. Here dk is the normalized Haar measure of K.

(v) let p and q be two projections in a Hilbert space H ; then the sub-C^*- algebra A of $\mathcal{L}(H)$ generated by p and q is postliminar and p and q have rank $\leqslant 1$ in A (G.K.Pedersen, Oral communication).

n.13.2. <u>Irreducible representations of</u> $\overset{*}{\otimes}{}^e A_i$.

In this number we consider a tensor product $A = \overset{*}{\otimes}{}^e A_i$ where each e_i has rank $\leqslant 1$ in A_i ; we denote by Y_i the set of all π in $\hat{A}_i$ such that $\pi(e_i) \neq 0$; it is open in A_i.

We define a mapping

$$F : \prod^{(Y_i)} \hat{A}_i \longrightarrow \hat{A}$$

in the following manner : take an element $\Pi = (\pi_i)$ in $\prod^{(Y_i)} \hat{A}_i$ and a finite subset $J \subset I$ such that $i \notin J$ implies $\pi_i \in Y_i$; then for $i \notin J$, $\pi_i(e_i)$ has rank 1, we can take a unit vector t_i in $\operatorname{Im} \pi_i(e_i)$ and form the representation $\underset{i \in I-J}{\otimes}{}^{e,t} \pi_i$, which is independant of the choice of t_i in $\operatorname{Im} \pi_i(e_i)$; now by writing $A = (\underset{i\in J}{\overset{*}{\otimes}} A_i) \overset{*}{\otimes} (\underset{i\in I-J}{\overset{*}{\otimes}}{}^e A_i)$

we can form the representation $(\overset{*}{\bigotimes}_{i \in J} \pi_i) \overset{*}{\otimes} (\overset{*}{\bigotimes}_{i \in I-J}^{e,t} \pi_i)$ of A ; this is an element of $\hat{A}$ which is independant of the choice of J (easy verification) and which we shall denote by $F(\Pi)$.

<u>Lemma</u> 3. Denote by A a C^*- algebra, by e a nonzero projection in A, by S the set of all pure states f on A verifying $f(e) = 1$, by Y the set of all π in $\hat{A}$ verifying $\pi(e) \neq 0$, by M the canonical mapping $f \longmapsto \pi_f$ of $P(A)$ onto $\hat{A}$. Then $M|S$ maps S onto Y and is open.

<u>Proof</u>. We have $M(S) \subset Y$ since

$$1 = f(e) = (\pi_f(e).x_f \mid x_f) \Longrightarrow \pi_f(e) \neq 0 \;;$$

we have $M(S) = Y$: in fact if π belongs to Y we can take a unit vector x in $\operatorname{Im} \pi(e)$ and setting $f = \omega_x \circ \pi$ we have $f(e) = 1$ and $\pi = \pi_f$. To prove that $M|S$ is open, denote by T the set of all f in $P(A)$ verifying $f(e) \neq 0$; to each f in T we associate the state $L(f)$ defined by

$$L(f)(a) = f(eae) / f(e) \;;$$

then L is a continuous mapping of T into S ; if f is in S we have $L(f) = f$ since writing $f = \omega_x \circ \pi$ we have

$$f(e) = 1 = (\pi(e).x|x)$$

$$\pi(e).x = x$$

$$L(f)(a) = (\pi(eae).x|x) = (\pi(a).x|x) = f(a) \;;$$

this proves that L maps T onto S. Let U be an open set in S ; $L^{-1}(U)$ is open in T, hence in P(A) since T is open ; $M(L^{-1}(U))$ is equal to M(U) since we have $M(f) = M(L(f))$ for each f in T ; since M is open, M(U) is open and $M|S$ is open.

Proposition 18. The mapping $F : \prod^{(Y_i)} \hat{A}_i \longrightarrow \hat{A}$ is injective and bicontinuous.

Proof of the injectivity. Suppose $F(\pi) = F(\pi')$ and take j in I ; there is $J \in \mathcal{F}(I)$ such that $j \in J$ and

$$i \notin J \Longrightarrow \pi_i \text{ and } \pi'_i \in Y_i \quad ;$$

we can write

$$F(\pi) = \Big(\mathop{\overset{*}{\otimes}}_{i \in J} \pi_i \Big) \overset{*}{\otimes} \Big(\mathop{\overset{*}{\otimes}^{e,t}}_{i \in I-J} \pi_i \Big)$$

and similarly for $F(\pi')$; since the assertion is true for the finite tensor products we have $\mathop{\overset{*}{\otimes}}_{i \in J} \pi_i = \mathop{\overset{*}{\otimes}}_{i \in J} \pi'_i$ and then $\pi_j = \pi'_j$.

Proof of the continuity. It is sufficient to prove that for each J, $F \mid X_{(J)}$ is continuous ; but $F \mid X_{(J)}$ is the composition of the following mappings :

$$(\pi_i)_{i \in I} \overset{a}{\longmapsto} ((\pi_i)_{i \in J}, (\pi_i)_{i \in I-J}) \overset{b}{\longmapsto} \Big(\mathop{\overset{*}{\otimes}}_{i \in J} \pi_i, \mathop{\overset{*}{\otimes}^{e,t}}_{i \in I-J} \pi_i \Big)$$

$$\overset{c}{\longmapsto} \Big(\mathop{\overset{*}{\otimes}}_{i \in J} \pi_i \Big) \overset{*}{\otimes} \Big(\mathop{\overset{*}{\otimes}^{e,t}}_{i \in I-J} \pi_i \Big) \quad ;$$

a is trivially continuous, c is continuous by Part I, prop. 5 ; as for b, $(\pi_i)_{i\in J} \longmapsto \overset{*}{\underset{i\in J}{\otimes}} \pi_i$ is continuous by the same result, and it remains to be shown that $(\pi_i)_{i\in I-J} \longmapsto \overset{*}{\underset{i\in I-J}{\otimes}}{}^{e,t} \pi_i$ is continuous ; in other words we are led back to prove the continuity of the mapping F in the case where $\pi_i(e_i) \neq 0 \quad \forall\, i$. Denote by S_i the set of all pure states f of A_i verifying $f(e_i) = 1$; by lemma 3 the mapping M_i : $S_i \longrightarrow Y_i$ is open ; then the mapping $M = (M_i) : \Pi S_i \longrightarrow \Pi Y_i$ is open ; consider the following diagramm

$$\begin{array}{ccc} \Pi S_i & \xrightarrow{\;G\;} & P(A) \\ {\scriptstyle M}\downarrow & & \downarrow \\ \Pi Y_i & \xrightarrow[\;F\;]{} & \hat{A} \end{array}$$

since it is commutative we have only to show that the mapping $G : (f_i) \longmapsto \overset{*}{\otimes}{}^{e} f_i$ is continuous, i.e. that for each a in A the mapping $(f_i) \longmapsto (\overset{*}{\otimes}{}^{e} f_i)(a)$ is continuous ; since all our positive functionals have norm 1, by equicontinuity we can suppose $a \in \otimes^e A_i$, then by linearity $a = \otimes a_i$, $(a_i) \in \Pi^e A_i$; then the assertion is trivial.

<u>Proof of the bicontinuity</u>. Take $J \in \mathcal{F}(I)$; an element Π of $\Pi^{(Y_i)} A_i$ belongs to $X_{(J)}$ iff $F(\Pi)$ is not identically zero on $\overset{*}{\underset{i\in J}{\otimes}} A_i$; thus we see that $F(X_{(J)})$ is open in Im F ;

it suffices to prove that F^{-1} is continuous on $F(X_{(J)})$, or that each mapping $F(\Pi) \longmapsto \pi_j$ is continuous on $F(X_{(J)})$; we can suppose $j \in J$ and write

$$F(\Pi) = \left(\overset{*}{\underset{i \in J}{\otimes}} \pi_i\right) \overset{*}{\otimes} \left(\overset{*}{\underset{i \in I-J}{\otimes}}{}^{e,t} \pi_i\right) ;$$

by Part I, th. 5, $\overset{*}{\underset{i \in J}{\otimes}} \pi_i$ is a continuous function of $F(\Pi)$ and π_j a continuous function of $\overset{*}{\underset{i \in J}{\otimes}} \pi_i$.

<u>Theorem</u> 8. If each A_i is postliminar and e_i has rank $\leqslant 1$, the mapping F is a homeomorphism of $\Pi^{(Y_i)} \hat{A}_i$ onto $\widehat{\overset{*e}{\otimes} A_i}$; moreover $\overset{*e}{\otimes} A_i$ is postliminar.

<u>Proof</u>. We shall prove that each factor representation π of A is of type I and also that if π is irreducible, it is equivalent to a tensor product of representations. There exists a $J \in \mathcal{F}(I)$ such that $\pi \,|\, \overset{*}{\underset{i \in J}{\otimes}} A_i \neq 0$, i.e.

$$\pi\left(a_{(J)} \otimes \left(\underset{i \in I-J}{\otimes} e_i\right)\right) \neq 0$$

for some $a_{(J)}$ in $\overset{*}{\underset{i \in J}{\otimes}} A_i$; since this algebra is postliminar we can write by Part I, prop. 2, $\pi = \pi_1 \overset{*}{\otimes} \pi_2$ where π_1 is some factor representation of $\overset{*}{\underset{i \in J}{\otimes}} A_i$ and π_2 some factor representation of $\overset{*}{\underset{i \in I-J}{\otimes}}{}^{e} A_i$; π_1 is of type I and if moreover π is irreducible, π_1 is irreducible too and is a tensor product of irreducible representations of the A_i, $i \in J$; on the other hand

$$\pi(a_{(J)} \otimes (\bigotimes_{i \in I-J} e_i)) = \pi_1(a_{(J)}) \otimes \pi_2(\bigotimes_{i \in I-J} e_i)$$

implies $\pi_2(\bigotimes_{i \in I-J} e_i) \neq 0$; we are thus led to prove the following assertion :

If $\pi(\otimes e_i) \neq 0$, π is of type I ; if π is irreducible it is equivalent to a tensor product of representations.

We denote by H the space of π and choose a unit vector u in $\operatorname{Im} \pi(\otimes e_i) \subset H$; for each $j \in I$ we can write (since A_j is postliminar) :

$$H = K_j \overset{\ell}{\otimes} K'_j$$

$$\pi = \rho_j \overset{*}{\otimes} \rho'_j$$

$$\pi(\otimes e_i) = \rho_j(e_j) \otimes \rho'_j(\bigotimes_{i \neq j} e_i)$$

where ρ_j is a factor representation of A_j in K_j ; ρ_j is a multiple of some irreducible representation π_j in a space H_j and we can write

$$H = H_j \overset{\ell}{\otimes} L_j \overset{\ell}{\otimes} K'_j$$

$$\pi = \pi_j \overset{*}{\otimes} I \overset{*}{\otimes} \rho'_j$$

$$\pi(\otimes e_i) = \pi_j(e_j) \otimes I \otimes \rho'_j(\bigotimes_{i \neq j} e_i) \quad ;$$

setting $H'_j = L_j \overset{\ell}{\otimes} K'_j$ and $\pi'_j = I \otimes \rho'_j$ we obtain

$$H = H_j \overset{\ell}{\otimes} H'_j$$

$$\pi = \pi_j \overset{*}{\otimes} \pi'_j$$

$$\pi(\otimes e_i) = \pi_j(e_j) \otimes \pi'_j(\bigotimes_{i \neq j} e_i) \quad ;$$

since $\pi_j(e_j)$ has rank 1, u has the form $t_j \otimes t'_j$ where t_j is some unit vector in $\operatorname{Im} \pi_j(e_j)$.

Consider now a finite subset J of I ; by the same procedure as before we can write

$$H = H_{(J)} \overset{h}{\otimes} H'_{(J)}$$

$$\pi = \pi_{(J)} \overset{*}{\otimes} \pi'_{(J)}$$

$$\pi(\otimes e_i) = \pi_{(J)}\left(\bigotimes_{i \in J} e_i\right) \otimes \pi'_{(J)}\left(\bigotimes_{i \in I-J} e_i\right)$$

where $\pi_{(J)}$ is an irreducible representation of $\overset{*}{\bigotimes}_{i \in J} A_i$ in $H_{(J)}$; $\pi_{(J)}$ is a tensor product of irreducible representations σ_j of the A_j, $j \in J$; for each $j \in J$ the restriction σ'_j of $\pi_{(J)}$ to A_j is a multiple of σ_j ; write

$$A = A_j \overset{*}{\otimes} \left(\overset{*}{\bigotimes}_{i \in J-j} A_i\right) \overset{*}{\otimes} \left(\overset{*}{\bigotimes}{}^{e}_{i \in I-J} A_i\right)$$

and choose approximate identities (u_s) and (v_t) of the second and the third factors in the righthand side ; for each a_j in A_j we have

$$\pi(a_j \otimes u_s \otimes v_t) = \pi_j(a_j) \otimes \pi'_j(u_s \otimes v_t)$$

which converges strongly to $\pi_j(a_j) \otimes I$ in $H_j \overset{h}{\otimes} H'_j$; we have also

$$\pi(a_j \otimes u_s \otimes v_t) = \pi_{(J)}(a_j \otimes u_s) \otimes \pi'_{(J)}(v_t)$$

which converges strongly to $\sigma'_j(a_j) \otimes I$ in $H_{(J)} \overset{h}{\otimes} H'_{(J)}$;

this proves that π_j and σ'_j have a common multiple ; consequently π_j is equivalent to σ_j ; we thus can write

$$H = \left(\overset{h}{\bigotimes_{i \in J}} H_i \right) \overset{h}{\otimes} H'_{(J)} \tag{12}$$

$$\pi = \left(\overset{*}{\bigotimes_{i \in J}} \pi_i \right) \overset{*}{\otimes} \pi'_{(J)} \tag{13}$$

$$\pi(\otimes e_i) = \left(\bigotimes_{i \in J} \pi_i(e_i) \right) \otimes \pi'_{(J)}\left(\bigotimes_{i \in I-J} e_i \right) ;$$

$$u = \left(\bigotimes_{i \in J} t_i \right) \otimes t'_{(J)}$$

where $t'_{(J)}$ is some unit vector in $\operatorname{Im} \pi'_{(J)}\left(\bigotimes_{i \in I-J} e_i \right)$.

If $K \supset J$ we have

$$u = \left(\bigotimes_{i \in K} t_i \right) \otimes t'_{(K)}$$

$$= \left(\bigotimes_{i \in J} t_i \right) \otimes \left(\bigotimes_{i \in K-J} t_i \right) \otimes t'_{(K)}$$

whence

$$t'_{(J)} = \left(\bigotimes_{i \in K-J} t_i \right) \otimes t'_{(K)} . \tag{14}$$

Define an isometric linear mapping $U_J : \overset{h}{\bigotimes_{i \in J}} H_i \longrightarrow H$ by writing (12) and

$$U_J(x) = x \otimes t'_{(J)} ;$$

the various U_J form an inductive system : in fact denoting by $L_{J,K}$ the canonical mapping $\overset{h}{\bigotimes_{i \in J}} H_i \longrightarrow \overset{h}{\bigotimes_{i \in K}} H_i$ we have

$$U_K(L_{J,K}(x)) = x \otimes \left(\bigotimes_{i \in K-J} t_i \right) \otimes t'_{(K)}$$

$$= x \otimes t'_{(J)} \qquad \text{(by (14))}$$

$$= U_J(x) ;$$

this inductive system gives rise to an isometric linear mapping $U : \overset{h}{\otimes}{}^{t} H_i \longrightarrow H$. We now prove that U intertwines the representations $\overset{*}{\otimes}{}^{e,t}\pi_i$ and π , i.e. that

$$U((\overset{*}{\otimes}{}^{e,t}\pi_i)(a).x) = \pi(a).U.x$$

for each a in A and x in $\overset{h}{\otimes}{}^{t} H_i$; we can take $a = \otimes a_i$ with $(a_i) \in \Pi^e A_i$ and $x = \otimes x_i$ with $(x_i) \in \Pi^t H_i$; we have $a_i = e_i$ and $x_i = t_i$ if i belongs to the complement of some finite J , then

$$U((\overset{*}{\otimes}{}^{e,t}\pi_i)(a).x) = U(\otimes \pi_i(a_i).x_i)$$

$$= (\underset{i\in J}{\otimes} \pi_i(a_i).x_i) \otimes t'_{(J)}$$

$$= ((\underset{i\in J}{\otimes} \pi_i)(\underset{i\in J}{\otimes} a_i).\underset{i\in J}{\otimes} x_i) \otimes \pi'_{(J)}(\underset{i\in I-J}{\otimes} e_i).t'_{(J)}$$

$$= \pi(a).((\underset{i\in J}{\otimes} x_i)\otimes t'_{(J)}) \qquad \text{(by (13))}$$

$$= \pi(a).U.x.$$

Thus U intertwines $\overset{*}{\otimes}{}^{e,t}\pi_i$ and π ; since the first representation is irreducible and the second is factorial, it is of type I ; if moreover π is irreducible, it is equivalent to $\overset{*}{\otimes}{}^{e,t}\pi_i$.

<u>Corollary</u> 11. Let for each i, G_i be a postliminar locally compact group containing a compact open subgroup K_i with the property indicated in example 5 (iv). Then the locally compact group $\Pi^{(K_i)} G_i$ is postliminar and its spectrum

is homeomorphic to $\prod^{(Y_i)} \hat{G}_i$ where Y_i is the set of all π in $\hat{G}_i$ such that the space of all vector invariant by $\pi(K_i)$ has dimension 1.

Interesting applications of this result to adele groups can be found in [37], ch.III, § 3, n.3.

Corollary 12. If G_i is a commutative locally compact group and K_i a compact open subgroup, the dual group of $\prod^{(K_i)} G_i$ is isomorphic and homeomorphic to $\prod^{(L_i)} \hat{G}_i$ where L_i is the subgroup orthogonal to K_i .

In fact a character x of G_i verifies $\int_{K_i} x(k).dk = 0$ if and only if it is trivial on K_i.

Another corollary has been stated in example 4.

n.13.3. <u>The Plancherel measure class of</u> $\prod^{(K_i)} G_i$.

In this number we suppose I countable and consider for each i a separable postliminar locally compact group G_i with compact open subgroup K_i such that for each π in $\hat{G}_i$ the space of all vectors invariant under $\pi(K_i)$ has dimension $\leqslant 1$; we denote by Y_i the set of all π such that the above space has dimension 1 ; we set $G = \prod^{(K_i)} G_i$, $X = \prod^{(Y_i)} \hat{G}_i$. We recall that given a separable postliminar group G, the Plancherel measure class of G is the measure class on $\hat{G}$ corresponding to the central desintegration of the left regular representation of G .

<u>Proposition</u> 19. One can choose for each i , a measure λ_i in the Plancherel measure class of G_i in such a way that $\lambda_i(Y_i) = 1$ and that the homeomorphism $F : X \longrightarrow \hat{G}$ of corollary 11 carries the restricted product of the λ_i (see definition in § 1) into a measure belonging to the Plancherel measure class of G .

We shall need the following lemma :

<u>Lemma</u> 4. Let G be a separable locally compact group, K a compact open subgroup, e the characteristic function of K considered as an element of $L^1(G)$, t the same function considered as an element of $L^2(G) = H$, π the left regular

representation of G in H ; let us write the central desintegration of π .

$$H = \int_X^{\oplus} H_x . d\mu(x)$$

$$\pi = \int^{\oplus} \pi_x . d\mu(x)$$

where μ is some Borel measure on some standard Borel space X ; t admits a decomposition $\int^{\oplus} t_x . d\mu(x)$; then the sets

$$X_1 = \{ x \mid \pi_x(e) \neq 0 \}$$

$$X_2 = \{ x \mid t_x \neq 0 \}$$

are identical up to negligible sets.

Proof of the lemma. Since $\pi(e).t = t$ we have $\pi_x(e).t_x = t_x$ almost everywhere and X_2 is almost contained in X_1. To prove the converse inclusion denote by $\mathfrak{Z}$ the algebra of all diagonalizable operators, by $\mathfrak{Z}'$ the algebra of all decomposable operators, by $\mathfrak{A}$ the von Neumann algebra generated by $\pi(G)$, by ρ the right regular representation of G in H :

$$(\rho(g).f)(g') = \Delta(g)^{\frac{1}{2}}.f(g'g) \quad ;$$

set $L = \operatorname{Im} \pi(e)$; we can write

$$L = \int^{\oplus} L_x . d\mu(x) \qquad \text{with} \quad L_x = \operatorname{Im} \pi_x(e) \ ;$$

L is the set of all f in H which are constant on the right cosets Kg ; since G is separable and K open these cosets form a countable set, say $Kg_o, Kg_1, \ldots$ with g_o = neutral element ; L has an orthonormal basis $w_o, w_1, \ldots$ where w_n is the characteristic function of Kg_n, $w_o = t$; w_n admits a decomposition $\int^{\oplus} w_n(x).d\mu(x)$; we have

$$\Delta(g_n)^{\frac{1}{2}} \cdot \rho(g_n)^{-1} \in \mathcal{A}' \subset \mathcal{Z}'$$

hence $\Delta(g_n)^{\frac{1}{2}} \cdot \rho(g_n)^{-1}$ admits a decomposition $\int^{\oplus} T_n(x).d\mu(x)$; on the other hand we have

$$\Delta(g_n)^{\frac{1}{2}} \cdot \rho(g_n)^{-1} \cdot w_o = w_n$$

hence for almost every x we have

$$T_n(x).t_x = w_n(x) \qquad \forall\ n ;$$

since for almost every x the $w_n(x)$ generate L_x, we see that for almost every x

$$t_x = 0 \Longrightarrow L_x = 0 \Longrightarrow \pi_x(e) = 0 .$$

Proof of the proposition.

Denote by π_i and π the left regular representations of G_i in $H_i = L^2(G_i)$ and of G in $H = L^2(G)$, by t_i the characteristic function of K_i considered as an element of H_i, by μ_i a left Haar measure on G_i with $\mu_i(K_i) = 1$; by § 1 the restricted product μ of the μ_i is a left Haar measure on G ; by corollary 5 we have an isomorphism

$$U : \overset{h}{\otimes}{}^{t} H_i \longrightarrow H$$

with the following property : if $f_i \in H_i$ and $f_i = t_i$ almost everywhere, Uf is the function defined by $Uf(g) = \prod f_i(g_i)$; it is easy to check that U carries $\otimes \pi_i(g_i)$ into $\pi(g)$ for each $g = (g_i)$ in G .

Take an arbitrary measure λ_i in the Plancherel class og G_i ; we can write the central desintegration of π_i :

$$H_i = \int_{\hat{G}_i}^{\oplus} H_{i,\rho_i} \cdot d\lambda_i(\rho_i)$$

$$\pi_i = \int^{\oplus} \pi_{i,\rho_i} \cdot d\lambda_i(\rho_i)$$

where π_{i,ρ_i} is some multiple of ρ_i ; by lemma 4, t_i admits a decomposition $\int^{\oplus} t_{i,\rho_i} \cdot d\lambda_i(\rho_i)$ such that $t_{i,\rho_i} \neq 0$ iff $\rho_i \in Y_i$; then we can replace λ_i by an equivalent measure which we still denote by λ_i, and suppose $\| t_{i,\rho_i} \| = 1$ for each $\rho_i \in Y_i$; since $\| t_i \| = 1$ we have $\lambda_i(Y_i) = 1$ and we can form the restricted product λ of the λ_i .

By theorem 3 we have an isomorphism

$$V : \overset{h}{\otimes}{}^t H_i \longrightarrow \int_X^{\oplus} \overset{h}{\otimes}{}^{(t_{i,\rho_i})} H_{i,\rho_i} \cdot d\lambda(\rho) \; ;$$

as easily seen V carries the representation $\otimes^t \pi_i$ into $\int_X^{\oplus} \otimes^{(t_{i,\rho_i})} \pi_{i,\rho_i} \cdot d\lambda(\rho)$, the proof will be complete if we show that for each $\rho \in X$, $\otimes^{(t_{i,\rho_i})} \pi_{i,\rho_i}$ is a multiple of $F(\rho)$. Since π_{i,ρ_i} is a multiple of ρ_i we can write

$$H_{i,\rho_i} = K_{i,\rho_i} \overset{h}{\otimes} K'_{i,\rho_i}$$

$$\pi_{i\,\rho_i} = \rho_i \otimes I$$

then

$$\pi_{i,\rho_i}(e_i) = \rho_i(e_i) \otimes I \quad ;$$

since $t_{i,\rho_i} \in \operatorname{Im} \pi_{i,\rho_i}(e_i)$ and rank $\pi_i(e_i) \leqslant 1$ we can write

$$t_{i,\rho_i} = s_{i,\rho_i} \otimes s'_{i,\rho_i}$$

with $s_{i,\rho_i} \in \operatorname{Im} \rho_i(e_i)$, then by virtue of the associativity

$$\overset{h}{\otimes}{}^{(t_{i,\rho_i})} H_{i,\rho_i} = \left(\overset{h}{\otimes}{}^{(s_{i,\rho_i})} K_{i,\rho_i} \right) \overset{h}{\otimes} \left(\overset{h}{\otimes}{}^{(s'_{i,\rho_i})} K'_{i,\rho_i} \right)$$

$$\otimes^{(t_{i,\rho_i})} \pi_{i\,\rho_i} = \left(\otimes^{(s_{i,\rho_i})} \rho_i \right) \otimes I$$

$$= F(\rho) \otimes I \ .$$

§ 14. Infinite tensor products of traces on C^*- algebras.

n.14.1. Definition.

Let us consider for each i a C^*- algebra A_i , a non zero projection e_i in A_i and a semi-finite lower semi-continuous (s.f.l.s.c.) trace f_i on A_i such that $f_i(e_i) = 1$; denote by m_i, n_i, N_i the associated ideals (cf.[2], 6.1.2 and 6.2.1), by $\mathcal{A}_i$ the Hilbert algebra n_i / N_i , by $\dot{a}$ the canonical image in $\mathcal{A}_i$ of any element a in n_i , by H_i the Hilbert completion of $\mathcal{A}_i$, by $\mathcal{U}_i$ the left von Neumann algebra associated with $\mathcal{A}_i$, by t_i the natural trace on $\mathcal{U}_i$, by π_i the representation of A_i in H_i defined by f_i ; $\dot{e}_i$ is a projection of norm 1 in $\mathcal{A}_i$ and we can form the Hilbert algebra $\mathcal{A} = \otimes^{(\dot{e}_i)} \mathcal{A}_i$; its Hilbert completion is $H = \overset{h}{\otimes}{}^{(\dot{e}_i)} H_i$ and the left von Neumann algebra of $\mathcal{A}$ is $\mathcal{U} = \overset{c}{\otimes}{}^{(\dot{e}_i)} \mathcal{U}_i$; denote by t its natural trace and by $\mathcal{N}$ the ideal of Hilbert-Schmidt operators for t ; since $\pi_i(e_i).\dot{e}_i = \dot{e}_i$ we can form the representation $\pi = \otimes^{e(\dot{e}_i)} \pi_i$ which generates the von Neumann algebra $\mathcal{U}$; for each family (a_i) in $\Pi^e A_i$ with $a_i \in n_i$ we have

$$\pi(\otimes a_i) = U_{\otimes \dot{a}_i} \in \pi(A) \cap \mathcal{N} \quad ;$$

since the operators $\pi(\otimes a_i)$ generate $\mathcal{U}$, we see that the pair (π, t) is a traced representation ; hence it defines a s.f.l.s.c. trace f on $A = \overset{*}{\otimes}{}^e A_i$: $f = t \circ \pi$; if $(a_i) \in \Pi^e A_i$ and $a_i \in m_i^+$ we can write $a_i = b_i^2$ where $b_i \in n_i^+$ and we have

$$f(\otimes a_i) = f((\otimes b_i)^2) = t(\pi((\otimes b_i)^2))$$

$$= t((U_{\otimes \dot{b}_i})^2) = (\otimes \dot{b}_i | \otimes \dot{b}_i)$$

$$= \Pi (\dot{b}_i | \dot{b}_i) = \Pi t_i((U_{\dot{b}_i})^2)$$

$$= \Pi t_i(\pi_i(b_i)^2)$$

$$= \Pi t_i(\pi_i(a_i)) = \Pi f_i(a_i) .$$

Let us now suppose that the f_i are finite and $\Pi \| f_i \| < \infty$; the definition ideal m of f contains each element $\otimes a_i$ with $(a_i) \in \Pi^e A_i^+$, hence contains $\otimes^e A_i$; on the other hand, by prop. 16, f is continuous on $\otimes^e A_i$; since it is l.s.c., it must be finite and hence equal to the positive functional $\overset{*}{\otimes}{}^e f_i$. Thus we have proved the following

<u>Proposition</u> 20. Given for each i a s.f.l.s.c. trace f_i on A_i such that $f_i(e_i) = 1$ we can construct canonically a s.f.l.s.c. trace f on $\overset{*}{\otimes}{}^e A_i$ with the following properties :

(i) $f(\otimes a_i) = \Pi f_i(a_i)$ if $a_i \in A_i^+$, $a_i = e_i$ almost everuwhere and $f_i(a_i) < \infty$

(ii) the representation associated with f is quasi-equivalent to $\overset{*}{\otimes}^{e,\dot{e}} \pi_i$ where π_i is the representation associated with f_i and $\dot{e}_i$ the canonical image of e_i in the space of π_i.

If each f_i is finite and $\prod \| f_i \| < \infty$, f is nothing but the central positive functional $\overset{*}{\otimes}^e f_i$.

Suppose now that $f_i(e_i) = 1$ only for almost all i ; taking J in $\mathcal{F}(I)$ such that $i \notin J \Longrightarrow f_i(e_i) = 1$ we can write

$$A = (\overset{*}{\underset{i \in J}{\otimes}} A_i) \overset{*}{\otimes} (\overset{*}{\underset{i \in I-J}{\otimes}}{}^{e} A_i)$$

and consider the tensor product of the traces $\overset{*}{\underset{i \in J}{\otimes}} f_i$ (defined in Part I, prop. 21) and f (defined in prop. 20).

<u>Definition</u> 13. The above s.f.l.s.c. trace on $\overset{*}{\otimes}^e A_i$ will be denoted by $\overset{*}{\otimes}^e f_i$; it is a character if and only if each f_i is a character. If each f_i is finite and $\prod \| f_i \| < \infty$, $\overset{*}{\otimes}^e f_i$ is nothing but the central positive functional $\overset{*}{\otimes}^e f_i$.

By composing with the canonical morphism $\overset{\vee}{\otimes}^e A_i \longrightarrow \overset{*}{\otimes}^e A_i$ we also get a trace $\overset{\vee}{\otimes}^e f_i$ on $\overset{\vee}{\otimes}^e A_i$ which has the same properties.

In the remainder of this paragraph we shall prove that certain s.f.l.s.c. traces on $\overset{*}{\otimes}^e A_i$ or $\overset{\vee}{\otimes}^e A_i$ are tensor products of traces.

n.14.2. Type I characters of $\overset{*}{\otimes}{}^{e} A_i$.

Theorem 9. We suppose each A_i postliminar ; let f be a character on $\overset{*}{\otimes}{}^{e} A_i$ which is of type I and satisfies the following condition : there exists a family (a_i) in $\prod^{e} A_i^{+}$ such that $0 < f(\otimes a_i) < \infty$. Then f is a tensor product.

Proof. Let $J = \{ i \mid a_i \neq e_i \}$; we can write

$$A = \overset{*}{\otimes}{}^{e} A_i = \Big(\overset{*}{\underset{i \in J}{\otimes}} A_i \Big) \overset{*}{\otimes} \Big(\overset{*}{\underset{i \in I-J}{\otimes}}{}^{e} A_i \Big)$$

$$a_i = \Big(\underset{i \in J}{\otimes} a_i \Big) \otimes \Big(\underset{i \in I-J}{\otimes} e_i \Big) \quad ;$$

by Part I, prop. 22, f is the tensor product of two characters f_1 , f_2 of $\overset{*}{\underset{i \in J}{\otimes}} A_i$ and $\overset{*}{\underset{i \in I-J}{\otimes}}{}^{e} A_i$ respectively, and f_1 is a tensor product of characters of the A_i, $i \in J$; on the other hand we have $0 < f_2\big(\underset{i \in I-J}{\otimes} e_i \big) < \infty$, so that we are led back to prove the theorem in the case where $0 < f(\otimes e_i) < \infty$.

Let π be an irreducible representation of A in a space H such that $f = Tr \circ \pi$ where Tr is the usual trace in H ; for each j we can write

$$H = H_j \overset{h}{\otimes} H'_j$$

$$\pi = \pi_j \overset{*}{\otimes} \pi'_j$$

where π_j and π'_j are irreducible representations of A_j and

$\overset{*e}{\underset{i\neq j}{\otimes}} A_i$; then

$$\pi(\otimes e_i) = \pi_j(e_j) \otimes \pi'_j(\underset{i\neq j}{\otimes} e_j) \qquad (15)$$

$$\mathrm{Tr}\,\pi(\otimes e_i) = \mathrm{Tr}\,\pi_j(e_j).\mathrm{Tr}\,\pi'_j(\underset{i\neq j}{\otimes} e_j) \quad ;$$

hence $\mathrm{Tr}\,\pi_j(e_j)$ is a strictly positive integer.

Consider now a finite subset J of I ; by the same procedure as in th. 8 we can write

$$H = (\overset{h}{\underset{i\in J}{\otimes}} H_i) \overset{h}{\otimes} H'_{(J)}$$

$$\pi = (\overset{*}{\underset{i\in J}{\otimes}} \pi_i) \overset{*}{\otimes} \pi'_{(J)} \quad ;$$

then

$$\mathrm{Tr}\,\pi(\otimes e_i) = \prod_{i\in J} \mathrm{Tr}\,\pi_i(e_i).\ \mathrm{Tr}\,\pi'_{(J)}(\underset{i\in I-J}{\otimes} e_i) \ ;$$

the second factor in the righthand side is a strictly positive integer, so that

$$\prod_{i\in J} \mathrm{Tr}\,\pi_i(e_i) \leqslant \mathrm{Tr}\,\pi(\otimes e_i) \quad ;$$

since J is arbitrary $\mathrm{Tr}\,\pi_i(e_i)$ must be equal to 1 for almost every i ; by taking off again a finite set of indices we can suppose $\mathrm{Tr}\,\pi_i(e_i) = 1 \ \forall i$. Then $\pi_i(e_i)$ is a one dimensional projection ; we choose a unit vector u in $\mathrm{Im}\,\pi(\otimes e_i)$; by (15) u is of the form $u = t_j \otimes t'_j$ where t_j is a unit vector in $\mathrm{Im}\,\pi_j(e_j)$; now the same reasoning as in th. 8 applies to prove that π is equivalent to $\overset{*}{\otimes}^{e,t} \pi_i$; each pair (π_i,Tr) is a traced representation since $\mathrm{Tr}\,\pi_i(e_i)$

$= 1$; set $f_i = \mathrm{Tr} \circ \pi_i$; we shall prove that $f = \overset{*}{\otimes}{}^{e} f_i$; f_i defines a Hilbert algebra $\mathcal{A}_i$, and a representation ρ_i of A_i in $K_i = \overline{\mathcal{A}}_i$; let $\dot{e}_i$ be the canonical image of e_i in $\mathcal{A}_i$; we can identify $\mathcal{A}_i$ with a dense subalgebra of the algebra of all Hilbert-Schmidt operators in H_i, K_i with the space $H_i \overset{h}{\otimes} H_i$, $\dot{e}_i$ (the projection onto t_i) with $t_i \otimes t_i$, ρ_i with $\pi_i \otimes I$; then $\overset{h}{\otimes}{}^{e} K_i$ is canonically isomorphic to $(\overset{h}{\otimes}{}^{t} H_i) \overset{h}{\otimes} (\overset{h}{\otimes}{}^{t} H_i)$; the representation associated with $\overset{*}{\otimes}{}^{e} f_i$ is quasi-equivalent to $\overset{*}{\otimes}{}^{e,\dot{e}} \rho_i$, hence to $(\overset{*}{\otimes}{}^{e,t} \pi_i) \otimes I$, then to $\overset{*}{\otimes}{}^{e,t} \pi_i$ and to π ; but the representation associated with f is also quasi-equivalent to π , and this shows that $f = \overset{*}{\otimes}{}^{e} f_i$.

<u>Corollary</u> 13. Consider a family of postliminar locally compact groups G_i with compact open subgroups K_i, and an irreducible representation π of $\prod^{(K_i)} G_i$; suppose that there exists an integrable function f on $\prod^{(K_i)} G_i$ of the form $f = \otimes f_i$ such that $0 < \mathrm{Tr}\, \pi(f^* f) < \infty$ and that for almost every i, f_i is the characteristic function of K_i . Then π is equivalent to a tensor product of irreducible representations π_i and for almost every i the space of all vectors invariant by $\pi_i(K_i)$ has dimension 1.

Interesting applications of this result to adele groups can be found in [37], ch.III, § 3, n.5.

n.14.3. Characters of $\overset{\alpha}{\otimes}^{e} A_i$ when e_i is central.

In this number we suppose that for each i, e_i belongs to the center of A_i ; then if π is a factor representation of A_i, $\pi(e_i)$ must be equal to 0 or I ; if f is a character of A_i and $0 < f(e_i) < \infty$, f is finite ; if moreover f is normed we have $f(e_i) = 1$.

Example 6. If G is a locally compact group and K an invariant compact open subgroup, its characteristic function is central in $C^*(G)$.

Proposition 21. Let f be a character of $\overset{\alpha}{\otimes}^{e} A_i$ ($\alpha = \gamma$ or $*$) with the following property : there exists a family (a_i) in $\prod^{e} A_i^+$ such that $0 < f(\otimes a_i) < \infty$. Then f is a tensor product in the sense of definition 13.

Proof. For the same reason as in th. 9 we can suppose that $0 < f(\otimes e_i) < \infty$; since $\otimes e_i$ belongs to the center of $A = \overset{\alpha}{\otimes}^{e} A_i$, f is finite and we can suppose it is normed ; it defines a representation π in a space H and a finite normed trace t on the factor $\mathcal{A} = \pi(A)''$; we have $\pi(\otimes e_i) = I$. The canonical morphisms $L_j : A_j \longrightarrow A$ are commuting ; set $\pi_j = \pi \circ L_j$, representation of A_j ; the von Neumann algebras $\pi_j(A_j)'' = \mathcal{A}_j$ are included in $\mathcal{A}$ and commuting .

Consider a family (a_i) in $\prod^e A_i$ with $a_i = e_i$ for $i \notin J$; we have

$$\prod_{i \in J} L_i(a_i) = (\underset{i \in J}{\otimes} a_i e_i) \otimes (\underset{i \in I-J}{\otimes} e_i)$$

$$= \otimes a_i . \otimes e_i$$

hence

$$\underset{i \in J}{\otimes} \pi_i(a_i) = \pi(\otimes a_i). \pi(\otimes e_i) = \pi(\otimes a_i) ;$$

then the $\mathcal{A}_i$ generate $\mathcal{A}$ and consequently are factors ; the von Neumann algebra $\mathcal{A}_{(J)}$ generated by the $\mathcal{A}_i$ with $i \in J$ is also a factor ; set $t_i = t \mid \mathcal{A}_i$, $t_{(J)} = t \mid \mathcal{A}_{(J)}$, $f_i = t_i \circ \pi_i$, character of A_i ; we want to prove that $f = \overset{*}{\otimes}{}^e f_i$; it suffices to prove that $f(\otimes a_i) = \prod f_i(a_i)$ with $a_i = e_i$ for $i \notin J$; then

$$f(\otimes a_i) = t(\pi(\otimes a_i)) = t(\prod_{i \in J} \pi_i(a_i))$$

$$= t_{(J)}(\prod_{i \in J} \pi_i(a_i)) ;$$

by Part I, lemma 13 we get

$$f(\otimes a_i) = \prod_{i \in J} t_i(\pi_i(a_i)) = \prod_{i \in J} f_i(a_i) .$$

QED

We shall now investigate the finite characters of A ; we denote by U_i the set of all $f \in C_1(A_i)$ such that $f(e_i) = 1$; U_i is open since for each f in $C_1(A_i)$ we have $f(e_i) = 0$ or 1 . For each family $F = (f_i)$ in $\prod^{(U_i)} C_1(A_i)$, we denote by $T(F)$ the character $\overset{\alpha}{\otimes}{}^{e} f_i$.

<u>Theorem</u> 10. The mapping T is a homeomorphism of $\prod^{(U_i)} C_1(A_i)$ onto $C_1(\overset{\alpha}{\otimes}{}^{e} A_i)$.

<u>T is injective</u> : suppose $T(F) = T(F')$, take j in I and J in $\mathcal{F}(I)$ such that $j \in J$ and $f_i(e_i) = f_i'(e_i) = 1$ for $i \notin J$; for each a in $\overset{\alpha}{\underset{i \in J}{\otimes}} A_i$ we have

$$\Big(\overset{\alpha}{\underset{i\in J}{\otimes}} f_i\Big)(a) = (T(F))\Big(a \otimes \Big(\underset{i\in I-J}{\otimes} e_i\Big)\Big)$$

$$= (T(F'))\Big(a \otimes \Big(\underset{i\in I-J}{\otimes} e_i\Big)\Big)$$

$$= \Big(\underset{i\in J}{\otimes} f_i'\Big)(a)$$

and this implies $f_j = f_j'$.

<u>T is surjective</u> by prop. 21.

<u>T is continuous</u> : we must prove that for each $J \in \mathcal{F}(I)$, T is continuous on $X_{(J)} = \prod_{i\in J} C_1(A_i) \times \prod_{i\in I-J} U_i$; then T is the composition of the following mappings :

$$F \overset{a}{\longmapsto} ((f_i)_{i\in J}, (f_i)_{i\in I-J}) \overset{b}{\longmapsto} \Big(\overset{\alpha}{\underset{i\in J}{\otimes}} f_i , \overset{\alpha\, e}{\underset{i\in I-J}{\otimes}} f_i\Big)$$

$$\overset{c}{\longmapsto} \Big(\overset{\alpha}{\underset{i\in J}{\otimes}} f_i\Big) \overset{\alpha}{\otimes} \Big(\overset{\alpha\, e}{\underset{i\in I-J}{\otimes}} f_i\Big) \;;$$

a is clearly continuous ; b is the direct product of two mappings b_1 , b_2 ; b_1 and c are continuous by Part I, prop. 5 ;

the proof of the continuity of b_2 is the same as for the continuity of G in prop. 18.

T is bicontinuous : $T(X_{(J)})$ is open in $C_1(A)$ since a character $f = \overset{\alpha}{\otimes}{}^e f_i$ belongs to $T(X_{(J)})$ iff $f_i(e_i) = 0$ $\forall$ $i \in I-J$, which is equivalent to f non zero on the subalgebra $(\overset{\alpha}{\underset{i\in J}{\otimes}} A_i) \overset{\alpha}{\otimes} (\underset{i\in I-J}{\otimes} e_i)$. Thus it is sufficient to prove that F^{-1} is continuous on $T(X_{(J)})$, i.e. that for each j, the mapping $\overset{\alpha}{\otimes}{}^e f_i \longmapsto f_j$ is continuous on this subset ; we can suppose $j \in J$; then our mapping is the composition of the following ones :

$$\overset{\alpha}{\otimes}{}^e f_i = (\overset{\alpha}{\underset{i\in J}{\otimes}} f_i) \overset{\alpha}{\otimes} (\overset{\alpha}{\underset{i\in I-J}{\otimes}} f_i) \longmapsto \overset{\alpha}{\underset{i\in J}{\otimes}} f_i \longmapsto f_j$$

and both are continuous by Part I, th. 10.

Corollary 14. If e_i is the identity of A_i, $C_1(\overset{\alpha}{\otimes} A_i)$ is canonically isomorphic to $\prod C_1(A_i)$.

Corollary 15. We suppose A_i separable and I countable ; then $\widehat{(\overset{\alpha}{\otimes}{}^e A_i)}_f$ is Borel isomorphic to $\prod^{(Y_i)} \widehat{(A_i)}_f$ where Y_i is the set of all π in $\widehat{(A_i)}_f$ with $\pi(e_i) \neq 0$.

In fact it is easy to see that the mapping $\widehat{(\overset{\alpha}{\otimes}{}^e A_i)}_f \longrightarrow \prod^{(Y_i)} \widehat{(A_i)}_f$ is Borel ; on the other hand both spaces are standard.

QED

Given a locally compact group G we denote by E(G) the set of all extremal continuous positive definite functions φ on G with $\varphi(e_o) = 1$, e_o = neutral element ; there is a bijection $E(G) \longleftrightarrow C_1(C^*(G))$, to each φ in E(G) corresponding the character $a \longmapsto f_\varphi(a) = \int a(g).\varphi(g).dg.$

If K is a compact open subgroup of G and e the characteristic function of K, we have $f_\varphi(e) = 1$ iff $\int \varphi(k).dk = 1$ where dk is the normaliaed Haar measure of K ; and this is equivalent to $\varphi(k) = 1 \ \forall \ k \in K$ since $|\varphi(g)| \leq 1$ for each g in G.

<u>Corollary</u> 16. Consider for each i a locally compact group G_i and an invariant compact open subgroup K_i ; then $E(\prod^{(K_i)} G_i)$ is in a canonical bijection with $\prod^{(Y_i)} E(G_i)$ where Y_i is the set of all φ in $E(G_i)$ verifying $\varphi(k) = 1 \ \forall \ k \in K_i$.

<u>Corollary</u> 17. If G_i is compact and $K_i = G_i$, $E(\prod G_i)$ is in a canonical bijection with $\prod^{(\varepsilon_i)} E(G_i)$ where ε_i is the function 1 .

<u>Corollary</u> 18. If G_i is discrete and K_i is reduced to the neutral element, $E(\prod' G_i)$ is in a canonical bijection with $\prod E(G_i)$.

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