Lecture 1: Introduction

Overview

Vector bundles arise in many parts of geometry, topology, and physics. The tangent bundle $TM \to M$ of a smooth manifold M is the first example one usually encounters. The tangent space T_pM is the linearization of the nonlinear space M at the point $p \in M$. Similarly, a nonlinear map between smooth manifolds has a linearization which is a map of their tangent bundles. Vector bundles needn't be tied to the intrinsic geometry of their base space. In quantum field theory, for example, extrinsic vector bundles are used to model subatomic particles. Sections of vector bundles are generalized vector-valued functions. For example, sections of the tangent bundle $TM \to M$ are vector fields on the manifold M.

The set $\operatorname{Vect}(X)$ of isomorphism classes of complex vector bundles on a topological space X is a *homotopy invariant* of X. It is a commutative monoid: a set with a commutative associative composition law with unit, represented by the zero vector bundle. The universal abelian group formed out of this monoid is $K(X)$, the *K-theory group* of X. It is a homotopy invariant of X. Topological K-theory was introduced in the late 1950s by Atiyah-Hirzebruch [AH], following Grothendieck's ideas [BS] in the sheaf theory context related to the Riemann-Roch problem (which in turn was solved in 1954 by Hirzebruch [Hi]). The abelian group $K(X) = K^0(X)$ is part of a generalized cohomology theory, and standard computational techniques in algebraic topology can be brought to bear. K-theory, which is constructed directly from linear algebra, is in many ways more natural than ordinary cohomology and turns out to be more powerful in many situations. Also, because of its direct connection to linear algebra it appears often in geometry and physics. One of the first notable sightings is in the 1963 Atiyah-Singer index theorem for elliptic operators [AS1].

Over the past few decades there has been renewed interest in K-theory, in large part due to its appearance in quantum field theory and string theory. For example, K-theory is the generalized cohomology theory which quantizes D-brane charges in superstring theory [MM], [W1]. (A truncation of real K-theory plays the same role for the B-field [DFM1], [DFM2].) The Atiyah-Singer index theorem, refined to include differential-geometric data, expresses the *anomaly* in the partition function of fermionic fields [AS2]. These two occurrences of K-theory combine [F1] to refine the original Green-Schwarz anomaly cancellation [GS], which catalyzed the first superstring revolution. In a different direction, K-theory enters into "geometric quantization in codimension two". These ideas are surveyed in [F2]. Modern applications often involve *twistings* of complex K-theory,[1] and fleshing out the theory of twisted K-theory has been one focus of recent mathematical activity. Twistings date from the 1960s in work of Donovan-Karoubi [DK], and from an operator point of view somewhat later in work of Rosenberg [Ro]. There are many modern treatments which develop a wide variety of models, ranging from the operator-theoretic to the geometric to the abstract homotopy-theoretic. In geometry twisted K-theory appears in the representation theory of loop groups [FHT1, FHT2, FHT3]: the fusion ring of positive energy representations of the loop group $LG = \operatorname{Map}(S^1, G)$ of a compact Lie group G at a fixed level is a twisted version of $K_G(G)$,

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[1] Real K-theory can be viewed as a particular twisting of complex K-theory, for example.

the equivariant K-theory of G acting on itself by conjugation. Recently K-theory has appeared in condensed matter physics as part of the classification of phases of matter [Ho, Ki, FM].

The course may cover a bit of this recent activity, but we will begin for awhile with basics and some classical results, especially Bott periodicity.

Vector spaces and linear representations

(1.1) *The "trivial" vector space $\mathbb{C}^n$.* Let n be a nonnegative integer. As a set $\mathbb{C}^n = \{(\xi^1, \dots, \xi^n) : \xi^i \in \mathbb{C}\}$. The vector space $\mathbb{C}^n$ has a canonical basis $(1, 0, 0, \dots, 0)$, $(0, 1, 0, \dots, 0)$, … If we view the basis as part of the structure, then $\mathbb{C}^n$ is *rigid*: the only linear symmetry of $\mathbb{C}^n$ which fixes the canonical basis is the identity map $\mathrm{id}_{\mathbb{C}^n}$.

(1.2) *Abstract finite dimensional vector spaces.* In general vector spaces are not equipped with canonical bases, and so have nontrivial linear symmetries. Let $\mathbb{E}$ be a complex vector space, assumed finite dimensional. Its group of linear symmetries is denoted $\mathrm{Aut}(\mathbb{E}) = GL(\mathbb{E})$. The automorphism group $\mathrm{Aut}(\mathbb{C}^n)$ of $\mathbb{C}^n$ (without necessarily fixing the canonical basis) is identified with the group of invertible $n \times n$ matrices. A basis of $\mathbb{E}$ is an isomorphism $b\colon \mathbb{C}^n \to \mathbb{E}$, where $n = \dim \mathbb{E}$. The set $\mathcal{B}(\mathbb{E})$ of bases of $\mathbb{E}$ is a *right torsor*[2] for $\mathrm{Aut}(\mathbb{C}^n)$. There is a unique Hausdorff topology on $\mathbb{E}$ for which vector addition $+\colon \mathbb{E} \times \mathbb{E} \to \mathbb{E}$ is continuous. A basis determines a homeomorphism to the standard topology on $\mathbb{C}^n$. The vector space $\mathrm{End}(\mathbb{E})$ of all linear operators on $\mathbb{E}$ is finite dimensional, so also has a unique vector topology. The subset $\mathrm{Aut}(\mathbb{E})$ of invertible endomorphisms is open and is a topological group in the induced topology. In a natural way $\mathbb{E}$ can be given the structure of a smooth manifold and $\mathrm{Aut}(\mathbb{E})$ the structure of a Lie group.

A geometric structure on $\mathbb{E}$ cuts down the group $\mathrm{Aut}(\mathbb{E})$ of symmetries to a subgroup G. The Kleinian[3] point of view is that the subgroup G defines the geometric structure. We take a slightly more general point of view and take a geometric structure on $\mathbb{E}$ to be a homomorphism $G \to \mathrm{Aut}(\mathbb{E})$ from a Lie group G, which then acts linearly on $\mathbb{E}$.

Example 1.4. The most familiar examples occur for *real* vector spaces $\mathbb{E}$. For example, an inner product $\langle -, - \rangle$ on $\mathbb{E}$ has the orthogonal group $O(\mathbb{E}) \subset \mathrm{Aut}(\mathbb{E})$ as its group of symmetries. A symplectic form on $\mathbb{E}$ determines the symplectic subgroup of symmetries. On the other hand, a spin structure on $\mathbb{E}$ has a group of symmetries which does not act effectively on $\mathbb{E}$: the map $\mathrm{Spin}(\mathbb{E}) \to \mathrm{Aut}(\mathbb{E})$ is a 2:1 covering of a subgroup of $\mathrm{Aut}(\mathbb{E})$.

Remark 1.5 (Infinite dimensional vector spaces). There is not a unique vector topology in infinite dimensions, but rather very different sorts of topological vector spaces: Hilbert spaces, Banach spaces, Fréchet spaces, …In this course we mostly deal with Hilbert spaces. Recall that a Hilbert space $\mathcal{H}$ is a complex vector spaces equipped with a *complete* Hermitian inner product; it can be finite or infinite dimensional. There are many possible topologies on its group $\mathrm{Aut}(\mathcal{H})$ of

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[2] The map

$$\begin{aligned} \mathcal{B}(\mathbb{E}) \times \mathrm{Aut}(\mathbb{C}^n) &\longrightarrow \mathcal{B} \times \mathcal{B} \\ (b, g) &\longmapsto (b, b \circ g) \end{aligned} \tag{1.3}$$

is an isomorphism, which is to say that $\mathrm{Aut}(\mathbb{C}^n)$ acts simply transitively on $\mathcal{B}(\mathbb{E})$.

[3] Felix, that is: *Erlanger Programm*.

automorphisms (and also on the subgroup $U(\mathcal{H})$ of *unitary* automorphisms—automorphisms which fix the inner product). The natural topology for us is the compact-open topology, on which we comment more in subsequent lectures.

Families of vector spaces; vector bundles

(1.6) *Spaces and smooth manifolds.* In this course we move back and forth between topological spaces $X, Y, \dots$ and smooth manifolds $M, N, \dots$ It is important in both cases that partitions of unity exist. Thus we assume all topological spaces are paracompact, and Hausdorff. In particular, we assume all smooth manifolds are paracompact and Hausdorff. These assumptions hold throughout.

Definition 1.7. Let X be a space. A *family of vector spaces parametrized by* X is a space E, a continuous surjection $\pi\colon E \to X$, and a finite dimensional vector space structure on each fiber $\pi^{-1}(x)$, $x \in X$ compatible with the topology of E.

To spell out what this compatibility is, recall that the data[4] for a single vector space consists of a set $\mathbb{E}$, a distinguished element $0 \in \mathbb{E}$, the operation of vector addition $+\colon \mathbb{E} \times \mathbb{E} \to \mathbb{E}$, and the operation of scalar multiplication $m\colon \mathbb{C} \times \mathbb{E} \to \mathbb{E}$. (For definiteness we consider complex vector spaces; an analogous discussion holds in the real case.) The data for the family is a zero section $z\colon X \to E$, vector addition[5] $+\colon E \times_X E \to E$, and scalar multiplication $m\colon \mathbb{C} \times E \to E$. The maps are required to be compatible with π: they preserve fibers. The compatibility in Definition 1.7 is that $z, +, m$ are all continuous maps.

Definition 1.8. Let $\pi\colon E \to X$ be a family of vector spaces parametrized by X. Its *rank* is the function $\operatorname{rank} E\colon X \to \mathbb{Z}^{\geqslant 0}$ defined by $(\operatorname{rank} E)(x) = \dim \pi^{-1}(x)$.

Heuristically (for the moment), we can imagine a parameter space $\mathcal{V}$ of all vector spaces, the components of $\mathcal{V}$ labeled by the dimension of the vector space. Then a family of vector spaces parametrized by X is a map $X \to \mathcal{V}$.

Example 1.9 (constant vector bundle)**.** Let $\mathbb{E}$ be a finite dimensional vector space. The constant (trivial) bundle with fiber $\mathbb{E}$ is $p_1\colon X \times \mathbb{E} \to X$, projection onto the first factor, with the constant vector space structure on fibers. We use the notation $\underline{\mathbb{E}} \to X$ for the trivial bundle with fiber $\mathbb{E}$.

Example 1.10 (tangent bundle)**.** The tangent bundle $\pi\colon TS^2 \to S^2$ is a non-constant family: the tangent spaces to the sphere at different points are not naturally identified with each other. In fact, the *hairy ball theorem* asserts that there does not exist a global nonzero section of π; every vector field on S^2 has a zero. A manifold whose tangent bundle admits a global basis of sections is termed *parallelizable* and such a basis is a global parallelism. The circle S^1 and 3-sphere S^3 are examples of parallelizable manifolds. (The 0-sphere S^0 is also parallelizable, trivially.) A theorem of Kervaire, Bott, and Milnor (independently) states that the only other parallelizable sphere is S^7. These four parallelizable spheres correspond to the four division algebras (reals, complexes, quaternions, octonions).

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[4] As opposed to conditions, or axioms, of which there are many.

[5] Here $E \times_X E = \{(e_1, e_2) : \pi(e_1) = \pi(e_2)\}$ denotes the fiber product of π with itself: pairs of points in a fiber.

Example 1.11 (family associated to a linear operator)**.** Let $T\colon \mathbb{E} \to \mathbb{E}$ be a linear operator on a finite dimensional vector space. Define the family of vector spaces parametrized by $\mathbb{C}$ whose fiber at $x \in \mathbb{C}$ is $\ker(xI - T)^n$, where $I = \mathrm{id}_{\mathbb{E}}$ and $n = \dim \mathbb{E}$. The *support* of the family—the set of $x \in X$ where the fiber is not the zero vector space—consists of the eigenvalues of T and the corresponding fiber is the generalized eigenspace. The direct sum of the fibers is canonically isomorphic to $\mathbb{E}$. In this example the rank is not a continuous function.

If $\pi\colon E \to X$ and $\pi'\colon E' \to X$ are two families of vector spaces parametrized by X, then a morphism of the families is a continuous map $T\colon E \to E'$ such that $\pi' \circ T = \pi$ and T is linear on each fiber. It is an isomorphism if it is bijective with continuous inverse.

Definition 1.12. A family of vector spaces $\pi\colon E \to X$ parametrized by X is a *vector bundle* if it is locally trivial, i.e., if for each $x \in X$ there exists an open set $U \subset X$ containing x and an isomorphism of $\pi\big|_{\pi^{-1}(U)}\colon \pi^{-1}(U) \to U \times \mathbb{E}$ with a constant vector bundle with fiber some vector space $\mathbb{E}$.

For a vector bundle $\operatorname{rank} E\colon X \to \mathbb{Z}^{\geqslant 0}$ is a locally constant function.

Remark 1.13. We emphasize that local triviality is a condition, not data: the local trivializations are not part of the structure of a vector bundle. Local triviality is a key property of vector bundles, and more generally of *fiber bundles*, as exposed in the influential book of Norman Steenrod [St]. One of the earliest appearances in the literature is perhaps a 1935 paper of Whitney [Wh]. We will see in the next lecture that local triviality leads to homotopy invariance, consequently to topological invariants.

Example 1.10 is a vector bundle whereas Example 1.11 is not. The latter can be given the structure of a *sheaf*.

Remark 1.14. We will consider infinite rank vector bundles as well. The definition is the same, but we need to be careful about the topology on the vector spaces (Remark 1.5).

Definition 1.15. Let M be a smooth manifold. A vector bundle $\pi\colon E \to M$ is *smooth* if E is a smooth manifold, π is a smooth map, and the structure maps $z, +, m$ are smooth.

See the text following Definition 1.7 for the structure maps of a family of vector spaces. The tangent bundle of a smooth manifold (Example 1.10) is a smooth real vector bundle.

Clutching construction of vector bundles

(1.16) *Clutching on* S^2. The homotopy invariance we prove in the next lecture implies that any vector bundle over a contractible space is trivializable. Write the 2-sphere S^2 as the union $S^2 = B_+ \cup B_-$ of two balls $B_+ = S^2\backslash\{p_+\}$, $B_- = S^2\backslash\{p_-\}$, where $p_+ \neq p_-$ are distinct points on S^2. Any smooth complex *line*[6] bundle $L \to S^2$ can be trivialized over $B_\pm$. Fix isomorphisms $L\big|_{B_\pm} \xrightarrow{\cong} B_\pm \times \mathbb{C}$, for example by stereographic projection as illustrated in Figure 3. The ratio of the isomorphisms over $B_+ \cap B_-$ is a smooth map to $\mathbb{C}^\times$, the nonzero complex numbers. Fix a diffeomorphism

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[6] A line is a vector space of dimension 1, so a line bundle is a vector bundle of constant rank 1.

$B_+ \cap B_- \cong \mathbb{C}^\times$, so that overlap data is identified with a smooth function $\phi\colon \mathbb{C}^\times \to \mathbb{C}^\times$. Another consequence of homotopy invariance is that the isomorphism class of the line bundle $L \to S^2$ depends only on the homotopy class of ϕ. The homotopy class is determined by the winding number of ϕ, and so isomorphism classes of complex line bundles $L \to S^2$ correspond to $\mathbb{Z}$; the integer invariant is called the *degree* of the line bundle. The tangent bundle has degree 2.

Remark 1.17. This generalizes to higher dimensional spheres and higher rank bundles. Rank N bundles on S^n are classified by homotopy classes of maps $S^{n-1} \to GL_N\mathbb{C}$. So the topology of this Lie group is fundamental for K-theory, and we will see shortly that it is the *stable* topology—the topology as $N \to \infty$—which is relevant. The *Bott periodicity theorem* determines the stable homotopy groups; see Theorem 1.31 below. It is the cornerstone of topological K-theory, and so we may end up giving 3 independent proofs in the course.

(1.18) *More general clutching; groupoids.* This gluing construction has a vast generalization. First, if X is a space and $\{U_i\}_{i\in I}$ an open cover, then we can imagine X as constructed from the disjoint union $\coprod_{i\in I} U_i$ by identifying $p_i \in U_i$ and $p_j \in U_j$ if they correspond to the same point of X. The situation is depicted in Figure 1. Double-headed arrows connect points which are glued. The clutching data for a vector bundle on X, then, is a vector bundle on the disjoint union together with isomorphisms of the fibers for each double-headed arrow. Furthermore, the isomorphisms must satisfy a consistency condition for pairs of arrows which share a vertex, as in the figure.

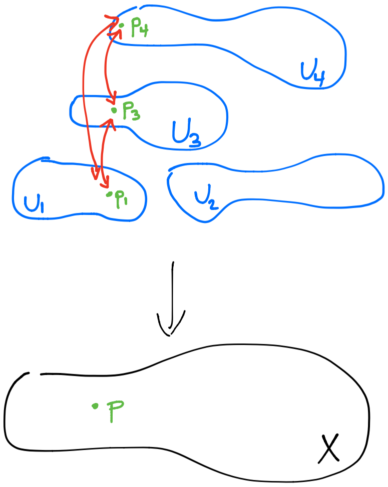

FIGURE 1. The groupoid of an open cover

The geometric structure of points with arrows is a *groupoid*, and it is a more general notion of space which we will use. Above we used a groupoid to present a topological space, but not every groupoid represents a space. At another extreme we can consider a groupoid with a single point equipped with a *group* G of self-arrows, as in Figure 2. A vector bundle over that groupoid is a single vector space $\mathbb{E}$ over the point, and an automorphism of $\mathbb{E}$ for each $g \in \mathbb{E}$. These automorphisms compose according to the group law (which is the "consistency condition" of the previous paragraph in this context), and so we simply have a linear representation of G on $\mathbb{E}$. There is a groupoid representing the action of a group G on a space X, and a vector bundle over it is a G-*equivariant* vector bundle over X. More general groupoids need not come from actions.

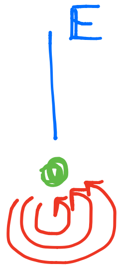

FIGURE 2. A vector bundle over a groupoid with one point

Variations

(1.19) *Families of linear maps.* The tangent space T_pM of a smooth manifold is the best linear approximation at p to the nonlinear space M. Similarly, if $f\colon M \to N$ is a smooth map of manifolds, at each p its differential

$$df_p\colon T_pM \longrightarrow T_{f(p)}N \tag{1.20}$$

is the best linear approximation at p to the nonlinear map f. The differential df is a family of linear maps parametrized by X, mapping between the families of vector spaces in the domain and codomain of (1.20). As a map of vector bundles

$$df\colon TM \longrightarrow f^*TN, \tag{1.21}$$

where f^*TN is the *pullback* of the vector bundle $TN \to N$ via the map $f\colon M \to N$.

The infinite dimensional case is particularly interesting. It happens in interesting circumstances that the differential (1.20), which is say a linear map between infinite dimensional Hilbert spaces, is almost an isomorphism in the sense that the kernel and cokernel have finite dimension. Such an operator is termed *Fredholm*, and in that case the differential (1.21) is a family of Fredholm operators. Families of Fredholms, which may or may not arise as linear approximations to a nonlinear Fredholm map, carry interesting topological information which is measured in K-theory.

Example 1.22. Nonlinear Fredholm maps are a key ingredient in Donaldson's gauge-theoretic approach to the topology of 4-manifolds [Do] and in the various flavors of Floer theory. A specific example of the latter is the Chern-Simons-Dirac functional [KM, §4.1].

Example 1.23. Let Σ denote a closed[7] oriented 2-manifold and $\mathcal{M}$ the space of conformal structures on Σ. Each conformal structure defines a linear operator

$$\bar\partial\colon \Omega^{0,0}(\Sigma) \longrightarrow \Omega^{0,1}(\Sigma) \tag{1.24}$$

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[7] A manifold is *closed* if it is compact without boundary.

on complex functions. It is a *first-order elliptic differential operator* and as such extends to a Fredholm operator on various Hilbert space completions.

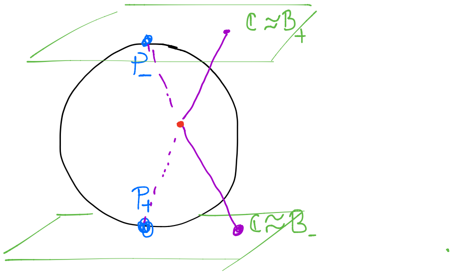

Figure 3. Overlap of two coordinate charts on the 2-sphere

(1.25) *$\mathbb{Z}$-gradings and complexes.* Another variation of the notion of a vector space is a *complex* of vector spaces. If we use cohomological grading conventions, then it is a sequence of linear maps

$$\cdots \longrightarrow E^{-2} \xrightarrow{d} E^{-1} \xrightarrow{d} E^{0} \xrightarrow{d} E^{1} \xrightarrow{d} E^{2} \longrightarrow \cdots \tag{1.26}$$

such that $d \circ d = 0$. Such complexes arise naturally in geometry. Examples include the de Rham complex of a smooth manifold and the Dolbeault complex of a complex manifold, a particular example of which is (1.24).[8] If the *differential* d vanishes, then the complex is a *$\mathbb{Z}$-graded vector space.*

Remark 1.27. The marriage of complexes of vector bundles and Fredholm operators gives *Fredholm complexes* [Se1].

Example 1.28. We have already seen a family of operators on a finite dimensional vector space in Example 1.11, namely the family $x \mapsto xI - T$ parametrized by $x \in \mathbb{C}$. The kernels jump in dimension as x varies, and they do not form a vector bundle. However, the family of operators, which can be viewed as a family of complexes (1.26) of vector spaces with $E^n = 0,\ n \neq 0,1$, does represent an element of K-theory.

K-theory

Let X be a compact[9] space. Denote by $\mathrm{Vect}^{\cong}(X)$ the set of equivalence classes of finite rank complex vector bundles over X. The operation of direct sum passes to $\mathrm{Vect}^{\cong}(X)$, where it is a commutative, associative composition law with identity element represented by the zero vector bundle. In short, $\mathrm{Vect}^{\cong}(X)$ is a commutative monoid. The universal abelian group associated to $\mathrm{Vect}^{\cong}(X)$ is the *K-theory group* $K(X)$, the eponym of this course. It is a homotopy invariant of the space X, which is one face of a cohomological invariant. The other is invariance under suspension, after shifting degree, and for that we will proceed formally at first, roughly defining

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[8] In that example $E^i = 0$ for $i \neq 0, 1$.

[9] We will give more general constructions later, which include noncompact spaces.

$K^{-q}(X)$ as the K-theory of the q^{th} suspension of X. In this way we will construct a generalized cohomology theory. All the apparatuses of algebraic topology—Mayer-Vietoris, spectral sequences, etc.—can be brought to bear on computations.

(1.29) *Variant geometric models.* The geometric variants of vector bundles listed above—Fredholm operators, complexes of vector bundles, Fredholm complexes—define K-theory classes. It is very important to have flexible geometric models for topological objects, since this is how they appear in geometry and physics.

(1.30) *Bott periodicity.* The basic theorem in the subject is the periodicity of the K-theory groups.

Theorem 1.31 (Bott)**.** *There is a natural isomorphism* $K^{q+2}(X) \cong K^q(X)$*.*

The theorem Bott actually proved is that the stable homotopy groups of the unitary groups are periodic, which is related to Theorem 1.31 by the clutching construction (**1.16**), extended to spheres of arbitrary dimension as in Remark 1.17. We will prove Theorem 1.31 in a few different ways, for example using Fredholm operators and the periodicity of Clifford algebras, following Atiyah-Singer [AS3].

Twistings and twisted vector bundles (Bonus material)

The modern incarnations of K-theory often occur in twisted form, as mentioned at the beginning of the lecture. We briefly catalog twisted notions of a complex vector space and linear representation of a group; there are corresponding twistings over a space (or groupoid) and twisted vector bundles. They are all "1-dimensional" in the sense that ordinary vector spaces and linear representations act by tensor product and, in some vague sense, the twisted versions for a fixed twisting are generated by a single object. From these twisted geometric objects we will extract twisted K-theory groups.

(1.32) *$\mathbb{Z}/2\mathbb{Z}$-gradings.* A *$\mathbb{Z}/2\mathbb{Z}$-graded vector space* $\mathbb{E} = \mathbb{E}^0 \oplus \mathbb{E}^1$ is simply a direct sum of two vector spaces. Homogeneous elements of $\mathbb{E}^0$ are termed *even*, homogeneous elements of $\mathbb{E}^1$ are termed *odd*. The word 'super' is used synonymously with '$\mathbb{Z}/2\mathbb{Z}$-graded', an inheritance from supersymmetry in quantum field theory. While the replacement the $\mathbb{Z}$-gradings of (**1.25**) by $\mathbb{Z}/2\mathbb{Z}$-gradings is not strictly a twisting, it does open the way for more twistings than are possible in the $\mathbb{Z}$-graded world. In the super situation the differential in a complex (1.26) is replaced by an odd endomorphism

$$T = \begin{pmatrix} 0 & T'' \\ T' & 0 \end{pmatrix} \tag{1.33}$$

of the vector space $\mathbb{E}$. We do not require $T^2 = 0$, so this is already a twisted version of a differential.[10]

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[10] It is called a *curved differential* in the context of A^∞ modules; T^2 is the *curving*.

(1.34) *$\mathbb{Z}/2\mathbb{Z}$-graded groups.* Let G be a Lie group or topological group. A continuous homomorphism $\epsilon\colon G \to \{\pm 1\}$ is a *$\mathbb{Z}/2\mathbb{Z}$-grading* of the group G. It twists the notion of a linear representation on a super vector space $\mathbb{E} = \mathbb{E}^0 \oplus \mathbb{E}^1$. Namely, in an *$\epsilon$-twisted representation* an element $g \in G$ with $\epsilon(g) = +1$ acts by an even automorphism and an element $g \in G$ with $\epsilon(g) = -1$ acts by an odd automorphism.

(1.35) *Central extensions and projective representations.* Let G be a Lie group and suppose

$$1 \longrightarrow \mathbb{T} \longrightarrow G^\tau \longrightarrow G \longrightarrow 1 \tag{1.36}$$

is a group extension with $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ central. A *τ-twisted representation* of G is a representation of G^τ on a complex vector space $\mathbb{E}$ such that $\lambda \in \mathbb{T}$ acts as scalar multiplication by λ. This induces an action of G on the projective space $\mathbb{P}\mathbb{E}$ of lines (1-dimensional subspaces) in $\mathbb{E}$, a projective representation.

Two examples: Let $G = SO_n$ be the special orthogonal group, and set $G^\tau = \mathrm{Spin}^c_n$ with its spin representation. This occurs in Riemannian geometry. The second example is typically infinite-dimensional and occurs in quantum physics. Namely, the space of pure states of a quantum system is the projective space $\mathbb{P}\mathcal{H}$ of a complex Hilbert space, so the symmetries of a quantum system are projective. A fundamental theorem of Wigner asserts that they lift to be linear or antilinear symmetries of $\mathcal{H}$, determined up to multiplication by a phase, so a group of quantum symmetries gives rise to an extension[11] (1.36).

(1.37) *Antilinearity.* Let G be a Lie group and $\phi\colon G \to \{\pm 1\}$ a $\mathbb{Z}/2\mathbb{Z}$-grading. Then a twisted form of a linear action on a (super) vector space $\mathbb{E}$ has elements $g \in G$ with $\phi(g) = +1$ acting linearly and elements $g \in G$ with $\phi(g) = -1$ acting antilinearly. A particular example is $G = \mathbb{Z}/2\mathbb{Z}$ with the nontrivial grading, in which case a twisted representation on $\mathbb{E}$ is a *real structure*. In this way real vector spaces appear as "twisted" forms of complex vector spaces. Combining with **(1.35)** we can similarly fit a quaternionic structure on a vector space in this framework.

We remark that time-reversing symmetries of a spacetime typically act antilinearly on a quantum mechanical system.

(1.38) *Central simple algebras.* A vector space is a module over the algebra $\mathbb{C}$ of complex numbers, and another form of twisting is to replace $\mathbb{C}$ with a more elaborate algebra A. To obtain a "1-dimensional" notion we require that the algebra A be invertible in the Morita sense. This is equivalent to requiring that A be central simple, and in our current context we use $\mathbb{Z}/2\mathbb{Z}$-graded algebras. These were studied by Wall [Wa]. Each Morita class is represented by a Clifford algebra. We will see that this sort of twisting induces a degree shift in K-theory.

In geometry Clifford modules [ABS] occur in connection with the spin representation and Dirac operator. For example, on an n-dimensional Riemannian spin manifold the natural Dirac operator [LM] has the form (1.33) and acts on a module over the Clifford algebra with n generators. In quantum physics there are algebras of observables, and the quantum Hilbert space is naturally a module over that algebra. In systems with fermions this leads to Clifford modules.

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[11] It is not necessarily central due to possible antilinear symmetries. Antilinearity as a twist is discussed next.

(1.39) *Twistings over groupoids.* Each form of twisting can be defined on a groupoid, not just on a group, and they can appear in combination. We will develop a general model of twistings which is sufficiently flexible to develop a general theory and which covers most appearances in geometry and physics. We end this lecture with a specific example related to **(1.34)**.

(1.40) *The twisting of a double cover.* Let X be a space and $\widetilde{X} \to X$ a double cover with deck transformation $\sigma\colon \widetilde{X} \to \widetilde{X}$. Descent data for a vector bundle $E \to \widetilde{X}$ is an isomorphism $\sigma^*E \to E$ which squares to the identity. Let $\Pi E = E^1 \oplus E^0$ denote the oppositely graded bundle; it represents the negative of E in K-theory.[12] Then one form of twisted bundle on X is a $\mathbb{Z}/2\mathbb{Z}$-graded bundle $E \to \widetilde{X}$ together with an isomorphism $\sigma^*E \to \Pi E$ which squares to the identity.

Remark 1.41. Double covers twist any cohomology theory since -1 always acts as an automorphism. Here is its incarnation in de Rham cohomology. Let $\widetilde{M} \to M$ be a double cover of a smooth manifold with deck transformation σ. Differential forms $\omega \in \Omega^\bullet(\widetilde{M})$ which satisfy $\sigma^*\omega = \omega$ descend to differential forms on M. On the other hand differential forms $\omega \in \Omega^\bullet(\widetilde{M})$ which satisfy $\sigma^*\omega = -\omega$ are twisted differential forms on M. If $\widetilde{M} \to M$ is the orientation double cover, and ω has top degree, then a twisted form is a *density* on M, the natural objects one can integrate.

References

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Lecture 2: Homotopy invariance

We give two proofs of the following basic fact, which allows us to do topology with vector bundles. The basic input is local triviality of vector bundles (Definition 1.12).

Theorem 2.1. *Let* $E \to [0,1] \times X$ *be a vector bundle. Denote by* $j_t \colon X \to [0,1] \times X$ *the inclusion* $j_t(x) = (t,x)$*. Then there exists an isomorphism*

$$j_0^*E \xrightarrow{\cong} j_1^*E. \tag{2.2}$$

The idea of both proofs is to construct parametrized trivializations along the axes $[0,1] \times \{x\}$, $x \in X$, of the cylinder $[0,1] \times X$. For the first proof we assume that X is a smooth manifold and that the vector bundle is smooth. Then we write a differential equation (parallel transport via a covariant derivative) which gives infinitesimal trivializations. The solution to the differential equation gives the global isomorphism. For the second proof we only assume continuity, so X is a (paracompact, Hausdorff) space, and use the local triviality of vector bundles in place of an (infinitesimal) differential equation. Then a patching argument constructs the global isomorphism. Partitions of unity are used as a technical tool in both situations.

Differential equations are used throughout differential geometry to prove global theorems. In this case we use an *ordinary* differential equation, for which there is a robust general theory. For *partial* differential equations the global questions are more delicate. (Think, for example, of the Ricci flow equations which "straighten out" the metric on a Riemannian 3-manifold to one of constant curvature.)

(2.3) *Partitions of unity.* The definition of 'paracompact' varies in the literature. Sometimes it includes the Hausdorff condition. The usual definition is that that every open cover of X has a locally finite refinement, or one can take the following theorem as a definition. Recall that a *partition of unity* is a set A of continuous functions $\rho_\alpha \colon X \to [0,1]$, $\alpha \in A$, with locally finite supports such that $\sum\limits_\alpha \rho_\alpha = 1$. It is *subordinate to an open cover* $\{U_i\}_{i\in I}$ if there exists a map $i \colon A \to I$ such that $\operatorname{supp} \rho_\alpha \subset U_{i(\alpha)}$.

Theorem 2.4. *Let* X *be a paracompact Hausdorff space and* $\{U_i\}_{i\in I}$ *an open cover.*

(*i*) *There exists a partition of unity* $\{\rho_i\}_{i\in I}$ *subordinate to* $\{U_i\}_{i\in I}$ *such that at most countably many* ρ_i *are not identically zero.*

(*ii*) *There exists a partition of unity* $\{\sigma_\alpha\}_{\alpha\in A}$ *subordinate to* $\{U_i\}_{i\in I}$ *such that each* σ_α *is compactly supported.*

(*iii*) *If* X *is a smooth manifold, then we can take the functions* ρ_i, σ_α *to be smooth.*

For a proof, see [War, §1].

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Covariant derivatives

(2.5) *Differentiation of vector-valued functions.* Let M be a smooth manifold and $\mathbb{E}$ a complex[1] vector space. The differential of smooth $\mathbb{E}$-valued functions is a linear map

$$d\colon \Omega^0_M(\mathbb{E}) \longrightarrow \Omega^1_M(\mathbb{E}) \tag{2.6}$$

which satisfies the Leibniz rule

$$d(f\cdot e) = df\cdot e + f\cdot de, \qquad f\in\Omega^0_M(\mathbb{C}), \quad e\in\Omega^0_M(\mathbb{E}), \tag{2.7}$$

where '$\cdot$' is pointwise scalar multiplication. However, it is not the unique map with those properties. Any other has the form

$$d + A, \qquad A\in\Omega^1_M(\operatorname{End}\mathbb{E}). \tag{2.8}$$

It acts as the first order differential operator

$$\begin{aligned} d+A\colon \Omega^0_M(\mathbb{E}) &\longrightarrow \Omega^1_M(\mathbb{E}) \\ e &\longmapsto de + A(e) \end{aligned} \tag{2.9}$$

The last evaluation is the pairing $\operatorname{End}\mathbb{E}\otimes\mathbb{E}\to\mathbb{E}$. The directional derivative in a direction $\xi\in T_pM$ at some point $p\in M$ is

$$de_p(\xi) + A_p(\xi)(e). \tag{2.10}$$

Observe that the space of differentiations of $\mathbb{E}$-valued functions is the infinite dimensional vector space $\Omega^1_M(\operatorname{End}\mathbb{E})$.

(2.11) *Differentiation of vector bundle-valued functions.* Let $\pi\colon E\to M$ be a smooth vector bundle. The local triviality (Definition 1.12) implies the existence of an open cover $\{U_i\}_{i\in I}$ of M and vector bundle isomorphisms $\varphi_i\colon U_i\times\mathbb{E}_i\to\pi^{-1}(U_i)$ for some vector spaces $\mathbb{E}_i$. Using φ_i we identify sections of E over U_i with $\mathbb{E}_i$-valued functions on U_i, and so transport the differentiation operator (2.6) to a differentiation operator[2]

$$\nabla_i\colon \Omega^0_{U_i}(E) \longrightarrow \Omega^1_{U_i}(E), \tag{2.12}$$

that is, a linear map satisfying the Leibniz rule (2.7), where now $e\in\Omega^0_{U_i}(E)$. Let $\{\rho_i\}_{i\in I}$ be a partition of unity satisfying Theorem 2.4(i,iii). Let $j_i\colon U_i\hookrightarrow M$ denote the inclusion. Then

$$\begin{aligned} \nabla\colon \Omega^0_M(E) &\longrightarrow \Omega^1_M(E) \\ e &\longmapsto \sum_i \rho_i\nabla_i(j_i^*e) \end{aligned} \tag{2.13}$$

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[1] The discussion applies without change to real vector spaces and, below, real vector bundles.

[2] '∇' is pronounced 'nabla'.

defines a global differentiation on sections of E. The first-order differential operator (2.13) is called a *covariant derivative*, and the argument given proves their existence on any smooth vector bundle.

If ∇, ∇' are covariant derivatives on E, then the difference $\nabla' - \nabla$ is linear over functions, as follows immediately from the difference of their Leibniz rules, and so is a tensor $A \in \Omega^1_M(\operatorname{End} E)$. Therefore, the set of covariant derivatives is an affine space over the vector space $\Omega^1_M(\operatorname{End} E)$.

Remark 2.14. The averaging argument with partitions of unity works to average geometric objects which live in a convex space, or better sections of a bundle whose fibers are convex subsets of affine spaces. For example, it is used to prove the existence of hermitian metrics on complex vector bundles, and also the existence of splittings of short exact sequences of vector bundles (**2.28**).

(2.15) *Parallel transport.* Let $\gamma\colon [0,1] \to M$ be a smooth parametrized path. The covariant derivative pulls back to a covariant derivative on the pullback bundle $F := \gamma^*E \to [0,1]$. We use the covariant derivative to construct an isomorphism

$$\rho\colon F_0 \longrightarrow F_1 \tag{2.16}$$

from the fiber over 0 to the fiber over 1, called parallel transport. A section $s\colon [0,1] \to F$ of $F \to [0,1]$ is *parallel* if $\nabla_{\partial/\partial t} s = 0$.

Lemma 2.17. *Let P denote the vector space of parallel sections. Then the restriction map $P \to F_0$ which evaluates a parallel section at $0 \in [0,1]$ is an isomorphism.*

Proof. Assume first that $F \to [0,1]$ is trivializable and fix a basis of sections $e_1, \dots, e_n$, where $n = \operatorname{rank} F$. Define functions $A^i_j\colon [0,1] \to \mathbb{C}$ by

$$\nabla_{d/dt} e_j = A^i_j e_i. \tag{2.18}$$

(Here and forever we use the summation convention to sum over indices repeated once upstairs and once downstairs.) Then the section $f^j e_j$ is parallel if and only if

$$\frac{df^i}{dt} + A^i_j f^j = 0, \qquad i = 1, \dots, n. \tag{2.19}$$

The fundamental theorem of ordinary differential equations asserts that there is a unique solution f^j with given initial values $f^j(0)$, which is equivalent to the assertion in the lemma.

In general, by the local triviality of vector bundles and the compactness of $[0,1]$, we can find $0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = 1$ such that $F\big|_{[t_i-1,t_i]} \to [t_{i-1}, t_i]$ is trivializable. Make the argument in the preceding paragraph on each interval and compose the resulting parallel transports to construct (2.16). ☐

(2.20) *Parametrized parallel transport.* We turn now to Theorem 2.1.

Proof of Theorem 2.1—smooth case. Let ∇ be a covariant derivative on $E \to [0,1] \times M$. Use parallel transport along the family of paths $[0,1] \times \{x\}$, $x \in M$, to construct an isomorphism (2.2). ☐

The ordinary differential equation of parallel transport (2.19) is now a family of equations: the coefficient functions A^i_j vary smoothly with x. Therefore, we need a parametrized version of the fundamental theorem of ODEs: we need to know that the solution varies smoothly with parameters. One reference for a proof is [La, §IV.1].

Proof for continuous bundles

Now we turn to the case when X is a space, and we follow [Ha, §1.2] closely; we defer to that reference for details. Choices of local trivializations replace choices of local covariant derivatives in this proof.

Proof of Theorem 2.1—continuous case. Observe first that if $\varphi\colon U\times\mathbb{E}\xrightarrow{\cong}E$ is a trivialization of a vector bundle $E\to U$, then for any $p,q\in U$ the trivialization gives an isomorphism $E_p\to E_q$ of the fibers which varies continuously in p,q. Thus a trivialization of a vector bundle $E\to[a,b]$ over an interval in $\mathbb{R}$ gives an isomorphism $E_a\to E_b$.

Now if $E\to[0,1]\times X$ is a vector bundle, we can find an open cover of $[0,1]\times X$ such that the bundle is trivializable on each open set; then by compactness of $[0,1]$ an open cover $\{U_i\}_{i\in I}$ of X such that the bundle is trivializable on each $[0,1]\times U_i$; and, choosing trivializations, continuous isomorphism $E\big|_{\{a\}\times U_i}\to E\big|_{\{b\}\times U_i}$ for any $0\leqslant a\leqslant b\leqslant 1$. (We use the observation in the previous paragraph.) Choose a partition of unity $\{\rho_i\}_{i\in I}$ subordinate to $\{U_i\}_{i\in I}$, and order the countable set of functions which are not identically zero: $\rho_1,\rho_2,\dots$. Define $\psi_n=\rho_1+\dots+\rho_n$, $n=1,2,\dots$, and set $\psi_0\equiv 0$. Let $\Gamma_n\subset[0,1]\times X$ be the graph of ψ_n. The trivialization on $[0,1]\times U_n$ gives an isomorphism $\tilde{\psi}_n\colon E\big|_{\Gamma_{n-1}}\xrightarrow{\cong}E\big|_{\Gamma_n}$. The composition $\cdots\circ\tilde{\psi}_2\circ\tilde{\psi}_1$ is well-defined by the local finiteness of $\{\rho_i\}$ and gives the desired isomorphism (2.2). ☐

Remark 2.21. If X is a smooth manifold, then we choose ρ_i to be smooth and this proof produces a smooth isomorphism.

Consequences

We prove some standard corollaries of Theorem 2.1.

Corollary 2.22. *Let $f\colon[0,1]\times X\to Y$ be a continuous map between topological spaces, $f_t\colon X\to Y$ its restriction to $\{t\}\times X$, and let $E\to Y$ be a vector bundle. Then $f_0^*E\cong f_1^*E$.*

Proof. Apply Theorem 2.1 to $f^*E\to[0,1]\times X$. ☐

Corollary 2.23. *Let X be a contractible space and $E\to X$ a vector bundle. Then $E\to X$ is trivializable.*

Proof. The identity map id_X is homotopic to a constant map $c\colon X\to X$, and the pullback $c^*E\to X$ is a constant vector bundle with fiber E_c. Now apply Corollary 2.22. ☐

Corollary 2.24. *Let $X=U_1\cup U_2$ be the union of two open sets, $E_i\to U_i$ vector bundles, and $\alpha\colon[0,1]\times U_1\cup U_2\to\mathrm{Iso}\big(E_1\big|_{U_1\cap U_2},E_2\big|_{U_1\cap U_2}\big)$ a homotopy of clutching data. Then the vector bundles $\mathcal{E}_0\to X$ and $\mathcal{E}_1\to X$ obtained by clutching with α_0, α_1 are isomorphic.*

Here $\mathrm{Iso}(-,-)$ is the set of isomorphisms between the indicated vector bundles.

Proof. Clutch over $[0,1]\times X$ and apply Theorem 2.1. ☐

Let $\mathrm{Vect}^{\cong}(X)$ denote the set of isomorphism classes of vector bundles over X. It is a *commutative monoid*: the sum operation is defined by direct sum

$$[E]+[E']=[E\oplus E'] \tag{2.25}$$

and the zero element is represented by the constant vector bundle with fiber the zero vector space. In fact, $\mathrm{Vect}^{\cong}(X)$ is a *semiring*, with multiplication defined by

$$[E]\times[E']=[E\otimes E'] \tag{2.26}$$

Corollary 2.27. *Let $f\colon X\to Y$ be a continuous map. Then the induced pullback $f^*\colon \mathrm{Vect}^{\cong}(Y)\to \mathrm{Vect}^{\cong}(X)$ depends only on the homotopy class of f.*

We write $\mathrm{Vect}^{\cong}_{\mathbb{R}}(X)$ and $\mathrm{Vect}^{\cong}_{\mathbb{C}}(X)$ to indicate the ground field explicitly.

Further applications of partitions of unity

(2.28) *Short exact sequences of vector bundles.* Let

$$0\longrightarrow E'\xrightarrow{\;i\;}E\xrightarrow{\;j\;}E''\longrightarrow 0 \tag{2.29}$$

be a short exact sequence of vector bundles over a space X.[3] A *splitting* of (2.29) is a linear map $E''\xrightarrow{\;s\;}E$ such that $j\circ s=\mathrm{id}_{E''}$. A splitting determines an isomorphism

$$E''\oplus E'\xrightarrow{\;s\oplus i\;}E. \tag{2.30}$$

Lemma 2.31. *The space of splittings is a nonempty affine space over the vector space* $\mathrm{Hom}(E'',E')$.

Let's deconstruct that statement, and in the process prove parts of it. First, if s_0,s_1 are splittings, then the difference $\phi=s_1-s_0$ is a linear map $E''\to E$ such that $j\circ\phi=0$. The exactness of (2.29) implies that ϕ factors through a map $\tilde{\phi}\colon E''\to E'$: in other words, $\phi=i\circ\tilde{\phi}$. This, then, is the affine structure. But we must prove that the space of splittings is nonempty. First, we observe that any short exact sequence of vector spaces splits, and so using local trivializations we deduce that splittings of (2.29) exist locally on X. Now we use a partition of unity argument. Remember that partitions of unity can be used to average sections of a fiber bundle whose fibers are convex subsets of affine spaces. Of course, an affine space is a convex subset of itself. I leave the details to the reader.

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[3] These can be real, complex, or quaternionic.

(2.32) *Inner products on vector bundles.* Recall that if $\mathbb{E}$ is a complex vector space, then an inner product is a bilinear map

$$\langle -,-\rangle \colon \overline{\mathbb{E}} \times \mathbb{E} \longrightarrow \mathbb{C} \tag{2.33}$$

which satisfies

$$\langle \bar{\xi}_1, \xi_2\rangle = \overline{\langle \bar{\xi}_2, \xi_1\rangle}, \qquad \xi_1, \xi_2 \in \mathbb{E}, \tag{2.34}$$

$$\langle \xi, \xi\rangle \in \mathbb{R}^{>0}, \qquad \xi \in \mathbb{E}, \quad \xi \neq 0 \quad . \tag{2.35}$$

Here $\overline{\mathbb{E}}$ denotes the conjugate vector space, which is the same abelian group as $\mathbb{E}$ but with scalar multiplication conjugated. The space of inner products on $\mathbb{E}$ is a subset of the vector space of bilinear maps (2.33) which are symmetric in the sense of (2.34); it is the convex cone of elements which satisfy the positivity condition (2.35).

Lemma 2.36. *A complex vector bundle $E \to X$ admits a positive definite hermitian inner product. The space of inner products is contractible.*

The proof is similar to that of Lemma 2.31 and is left to the reader. We emphasize the importance of the convexity of the set of inner products on a single vector space.

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Lecture 3: Group completion and the definition of K-theory

The goal of this lecture is to give the basic definition of K-theory. The process of *group completion*, which "completes" a commutative monoid M to an abelian group $K(M)$, loses information in general. In our topological setting what is retained is the *stable* equivalence class of a vector bundle. The notion of stability occurs in many guises, and they are all different facets of K-theory.

We begin with a basic proposition which allows us to replace the noncompact general linear groups with compact groups of isometries. This is convenient in many arguments. After describing group completion, we define K-theory and prove one of the basic theorems about the existence of inverses (Proposition 3.15). The basic building blocks of any cohomology theory is the value of that theory on spheres, and we prove in Proposition 3.26 that for K-theory those values are homotopy groups of the stable unitary group (stable orthogonal group in the real case).

A deformation retraction from Gram-Schmidt

Proposition 3.1. *There are deformation retractions*

$$\begin{aligned} GL_n\mathbb{C} &\longrightarrow U_n \\ GL_n\mathbb{R} &\longrightarrow O_n \end{aligned} \tag{3.2}$$

Here $U_n \subset GL_n\mathbb{C}$ is the subgroup of unitary matrices and $O_n \subset GL_n\mathbb{R}$ is the subgroup of orthogonal matrices. The reader should supply pictures for the case $n = 1$; the deformation retractions in that case is the first step in the general proof.

Proof. The proof is the same in both cases; for convenience, we use the notation of the complex version. Identify $GL_n\mathbb{C}$ with the space of bases of $\mathbb{C}^n$: the columns of an invertible $n\times n$ matrix form a basis. Then U_n is the subspace of orthonormal bases. The Gram-Schmidt process, which converts an arbitrary basis into an orthonormal basis, is a composition of deformation retractions. The first takes a basis $e_1, \dots, e_n$ and constructs one in which $|e_1| = 1$. The deformation fixes $e_2, \dots, e_n$ and at time $t \in [0, 1]$ has first vector $\bigl((1 - t) + t/|e_1|\bigr)e_1$. The second step we move e_2 only and make it orthogonal to e_1 via the path $e_2 - t\langle e_2, e_1\rangle e_1$. Now repeat. Move e_2 to have unit norm and then move e_3 to be orthogonal to both e_1 and e_2. After $2n - 1$ steps we are done. ☐

Group completion and universal properties

(3.3) *The group completion of a commutative monoid.* Recall that a commutative monoid M is a set with a commutative, associative composition law $M \times M \to M$ and a unit $0 \in M$.

Definition 3.4. Let M be a commutative monoid. A *group completion* (A, i) of M is an abelian group A and a homomorphism $i\colon M \to A$ of commutative monoids which satisfies the following

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K-Theory (M392C, Fall '15), Dan Freed, September 8, 2015 universal property: If B is an abelian group and $f\colon M \to B$ a homomorphism of commutative monoids, then there exists a unique group homomorphism $\tilde{f}\colon A \to B$ which makes the diagram

$$\begin{tikzcd} M \arrow[rr, "i"] \arrow[dr, "f"'] & & A \arrow[dl, dashed, "\tilde{f}"] \\ & B & \end{tikzcd} \tag{3.5}$$

commute.

The definition does not prove the existence of the group completion—we must provide a proof—but the universal property does imply a strong uniqueness property. Namely, if (A, i) and (A', i') are group completions of M, then there is a unique isomorphism $\phi\colon A \to A'$ of groups which makes the diagram

$$\begin{tikzcd} M \arrow[rr, "i"] \arrow[dr, "i'"'] & & A \arrow[dl, "\phi"] \\ & A' & \end{tikzcd} \tag{3.6}$$

commute. The proof uses four applications of the universal property (to $f = i$ and $f = i'$ to construct the isomorphism and its inverse, and then two more to prove the compositions are identity maps). To construct an explicit group completion, define A as the quotient of $M \times M$ in which (m_1, m_2) is identified with $(m_1 + n, m_2 + n)$ for all $m_1, m_2, n \in M$. Addition in A is defined component-wise in $M \times M$, the unit is $[0, 0]$, and $-[m_1, m_2] = [m_2, m_1]$. (The square brackets denote the equivalence class.)

Example 3.7. If $M = \mathbb{Z}^{\geqslant 0}$ under addition, then the group completion is $\mathbb{Z}$ under addition. If $M = \mathbb{Z}^{>0}$ under multiplication, then the group completion is $\mathbb{Q}^{>0}$ under multiplication.

Example 3.8. If $M = \mathbb{Z}^{\geqslant 0}$ under multiplication, then the group completion $(K(M), i)$ is the trivial group. For there exists $x \in K(M)$ such that $x \cdot i(0) = 1$, and so for any $n \in M$ we have

$$i(n) = \bigl(x \cdot i(0)\bigr) \cdot i(n) = x \cdot \bigl(i(0) \cdot i(n)\bigr) = x \cdot i(0 \cdot n) = x \cdot i(0) = 1. \tag{3.9}$$

Now apply *uniqueness* of the factorization.

This example is the first illustration of how information may be lost in passing to the group completion.

The abelian group $K(X)$ for X compact

Let X be a compact[1] Hausdorff space. Let $\mathrm{Vect}^{\cong}(X)$ denote the set of isomorphism classes of complex vector bundles $E \to X$. Then the operation of direct sum on vector bundles induces a commutative, associative composition law on $\mathrm{Vect}^{\cong}(X)$; the equivalence class of the zero vector bundle is a unit.

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[1] The definitions work for any paracompact Hausdorff X, but for noncompact spaces may give the "wrong" group. We give a more general definition later.

Definition 3.10. $K(X)$ is the group completion of the commutative monoid $\mathrm{Vect}^{\cong}(X)$.

Remark 3.11. For the empty set we have $\mathrm{Vect}^{\cong}(\varnothing) = K(\varnothing) = 0$, since there is a unique vector bundle over $\varnothing$.

(3.12) *Functorial property.* Let Top_c denote the category of compact Hausdorff spaces and continuous maps. Then $X \mapsto K(X)$ is a contravariant functor from Top_c to the category of abelian groups. For a continuous map $f\colon X \to X'$ induces a pullback on bundles. Furthermore, it follows immediately from Corollary 2.22 that K is a *homotopy* functor: homotopic maps $f_0 \simeq f_1$ induce equal maps on K-groups.

Remark 3.13. The functor $X \mapsto \mathrm{Vect}^{\cong}(X)$ to commutative monoids is also a homotopy functor. However, it is more difficult to compute, which is why we pass to the group completion. The group completion loses information in principle, but experience shows that the trade-off for increased computability is a good deal.

(3.14) *Real K-theory.* The proofs work equally for real vector bundles. The group completion of $\mathrm{Vect}^{\cong}_{\mathbb{R}}(X)$, the commutative monoid of real vector bundles, is denoted $KO(X)$.

Proposition 3.15. *Let X be a compact Hausdorff space and $\pi\colon E \to X$ a vector bundle. Then there exists a vector bundle $E' \to X$ such that $E \oplus E' \to X$ is trivializable.*

Observe that if $X \neq \varnothing$, then $\mathbb{Z} \hookrightarrow K(X)$ as the group completion of the trivial bundles. Prove this by choosing applying K to the unique map $X \to \mathrm{pt}$ and a section $\mathrm{pt} \to X$ obtained by choosing a point of X. Proposition 3.15 implies that for any $\alpha \in K(X)$ there exists $\alpha' \in K(X)$ such that $\alpha + \alpha' = N$ for some $N \in \mathbb{Z}$.

Proof. Let $\{U_1, \dots, U_K\}$ be a finite cover for which the restriction of E to each U_i is trivializable. Let $\{\rho_1, \dots, \rho_K\}$ be a partition of unity subordinate to $\{U_1, \dots, U_K\}$. For each open set U_j choose a basis of sections $e_j^1, \dots, e_j^n$ of $E^*\big|_{U_i} \to U_i$. Then $S = \{\rho_1 e_1^1, \rho_1 e_1^2, \dots, \rho_1 e_1^n, \rho_2 e_2^1, \rho_2 e_2^2, \dots\}$ is a set of nK sections of $E^* \to X$. Let $V = (\mathbb{C}S)^*$ be dual to the vector space with basis S. Evaluation defines a map of vector bundles $E \to \underline{V}$ of E to the bundle with constant fiber V, and this is an injective map of vector bundles. Let $E' = \underline{V}/E$ be the quotient bundle, so that we have a short exact sequence

$$0 \longrightarrow E \longrightarrow \underline{V} \longrightarrow E' \longrightarrow 0 \tag{3.16}$$

of vector bundles over X. Any short exact sequence of vector bundles splits (**2.28**), and a splitting determines a trivialization of $E \oplus E'$. □

(3.17) *Reduced K-theory.* Any space X has a unique map $X \to \mathrm{pt}$ to the 1-point space, and the induced map $\mathbb{Z} = K(\mathrm{pt}) \to K(X)$ is injective if X is nonempty, in which case it is the injection mentioned after the statement of Proposition 3.15. (If X is nonempty the map $X \to \mathrm{pt}$ admits sections.)

Definition 3.18. The *reduced K-theory group* is the quotient $\widetilde{K}(X) = K(X)/K(\mathrm{pt})$.

Proposition 3.19. *Let X be a compact Hausdorff space, $E, E' \to X$ vector bundles. Then $[E] = [E'] \in \widetilde{K}(X)$ if and only if $E \oplus \underline{\mathbb{C}^r} \cong E' \oplus \underline{\mathbb{C}^{r'}}$ for some r, r'.*

Proof. If $[E] = [E']$ then there exists $s \in \mathbb{Z}^{\geqslant 0}$ and a vector bundle $F \to X$ such that $E \oplus F \cong E' \oplus F \oplus \underline{\mathbb{C}^s}$. By Proposition 3.15 there exists $F' \to X$ such that $F \oplus F' \cong \underline{\mathbb{C}^r}$. The forward implication follows; the backward implication is immediate from Definition 3.18. ☐

Bundles E, E' which satisfy the hypothesis of Proposition 3.19 are said to be *stably equivalent*, and the reduced K-theory is the group of stable equivalence classes of vector bundles.

Fiber bundles

The definition of a fiber bundle is simpler than that (Definition 1.12) of a vector bundle: the fibers of a fiber bundle are topological spaces with no additional structure. Thus a vector bundle is a special case of a fiber bundle.

Definition 3.20. A *fiber bundle* is a surjective continuous map $\pi\colon \mathcal{E} \to X$ of topological spaces which admits local trivializations: every point $x \in X$ has an open neighborhood U containing x and a topological space F such that there exists a homeomorphism $\varphi\colon U \times F \to \mathcal{E}\big|_U$ which makes the diagram

$$\begin{tikzcd} U \times F \arrow[rr, "\varphi"] \arrow[dr, "pr_1"'] & & \mathcal{E}\big|_U \arrow[dl, "\pi"] \\ & U & \end{tikzcd} \tag{3.21}$$

commute.

Example 3.22. Let $\pi\colon E \to X$ be a complex vector bundle. There are many associated fiber bundles; we indicate the total space. $\mathbb{P}E$ is a fiber bundle whose fibers are the projectivizations of the fibers of E. More generally, for $k \geqslant 0$ we have the bundle of Grassmannians $Gr_k E$; the projectivization is $k = 1$. If E has a metric then we can form the sphere bundle $S(E)$. There is a bundle of groups $\operatorname{Aut}(E)$. If $\operatorname{rank}\colon E \to \mathbb{Z}$ is the constant function r, then $\operatorname{Iso}(\mathbb{C}^n, E)$ is the bundle of frames (bases) of E. It is a *principal fiber bundle*, or *principal bundle* for short: its fibers are right torsors over the group $GL_n\mathbb{C}$.

Fiber bundles satisfy the *homotopy lifting property*—they are fibrations. Assume that $\mathcal{E}, X$ are pointed spaces with basepoints $e, \pi(e) = b$. A fibration is characterized by the *homotopy lifting property.*

Definition 3.23. $p\colon \mathcal{E} \to X$ is a *fibration* if for every pointed space S, continuous map $f\colon [0,1] \times S \to X$ and lift $\tilde{f}_0\colon S \to \mathcal{E}$ of f_0 there exists an extension $\tilde{f}\colon [0,1] \times S \to \mathcal{E}$ lifting f.

The lift is encoded in the diagram

$$\begin{tikzcd} \{0\} \times S \arrow[r, "\tilde{f}_0"] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\ {[0,1]} \times S \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & X \end{tikzcd} \tag{3.24}$$

For a proof that fiber bundles are fibrations, see [Ha2, p. 375].

Recall that a topological space F is *m-connected* if every continuous map $\phi\colon S \to F$ with domain a CW complex S of dimension $\leqslant m$ is null homotopic.

Proposition 3.25. *Let X be a CW complex of dimension $\leqslant n$, $\pi\colon \mathcal{E} \to X$ be a fiber bundle, and suppose the fibers of π are $(n-1)$-connected. Then π admits a section.*

Proof. We give the proof for a finite CW complex by inducting over the skeleta $X_k \subset X$. Since X_0 is a discrete set of points, there is a section of $\mathcal{E}\big|_{X_0} \to X_0$. Now suppose we have a section over the $(k-1)$-skeleton and consider a k-cell with characteristic map $\Phi\colon D^k \to X_k$. Since D^k is contractible there is a trivialization[2] $\underline{F} \xrightarrow{\cong} \Phi^*\mathcal{E}$. The section on the $(k-1)$-skeleton then pulls back via $\partial\Phi$ to a map $S^{k-1} \to F$. By hypothesis this map is null homotopic, so extends over D^k and, unwinding with the trivialization and Φ, extends the section over this k-cell. ☐

Bott periodicity

We express the reduced K-theory of spheres as homotopy groups of unitary groups.

Proposition 3.26. *Let n be a nonnegative integer and $N \geqslant n/2$. Then there is an isomorphism*

$$\pi_{n-1}U_N \longrightarrow \widetilde{K}(S^n). \tag{3.27}$$

Proof. We construct (3.27) as a composition of isomorphisms of *sets*

$$\pi_{n-1}U_N \xrightarrow{i} [S^{n-1}, U_N] \xrightarrow{j} \mathrm{Vect}_N^{\cong}(S^n) \xrightarrow{k} \widetilde{K}(S^n) \tag{3.28}$$

Here $\mathrm{Vect}_N^{\cong}(S^n)$ is the set of isomorphism classes of complex vector bundles of rank N over S^n.

The homotopy group is the set of homotopy classes of pointed maps $S^{n-1} \to U_N$ which send a basepoint $* \in S^{n-1}$ to the identity $e \in U_N$. The first map i forgets basepoints; its inverse sends a map $f\colon S^{n-1} \to U_N$ to $f(*)^{-1}f$.

The second map j is the clutching construction (**1.16**), where we write the sphere as the union $S^n = D^n \cup_{S^{n-1}} D^n$ of two closed hemispheres along the equator. The proof that j is an isomorphism has three ingredients. First, any vector bundle admits a hermitian metric, so the clutching map can be assumed an isometry. Second, to show j is well-defined we apply Corollary 2.24. We need to prove that homotopic clutching maps lead to isomorphic bundles. A homotopy of clutching maps leads to a bundle over $[0,1] \times S^n$, and then Theorem 2.1 applies. Third, distinct homotopy classes of clutching maps construct non-isomorphic bundles, which follows from the fact that there is a unique homotopy class of trivializations on D^n.

To show that k is an isomorphism we need to show that a complex vector bundle $E \to S^n$ of rank $> N$ is stably equivalent to a bundle of rank N (surjectivity of k) and that stably isomorphic bundles of rank N are isomorphic (injectivity of k). To prove the first statement it suffices to construct a *nonzero* section of $E \to S^n$. For such a section spans a *trivial* line subbundle $L \subset E$,

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[2] The homotopy invariance arguments in Lecture 2 apply to general fiber bundles, not just vector bundles.

and by splitting the short exact sequence $0 \to L \to E \to E/L \to 0$ we see that E is stably equivalent to E/L, which has rank one less and we can iterate. To construct a nonzero section, fix a hermitian metric on $E \to S^n$ and consider the sphere bundle $S(E) \to S^n$, a fiber bundle with fiber S^{2N-1}. If $2N-1 \geqslant n$ it follows from Proposition 3.25 that $S(E) \to S^n$ admits a section. For the second statement suppose $E_0, E_1 \to S^n$ have rank N and for some $r > 0$ there exists an isomorphism $\varphi\colon E_0 \oplus \underline{\mathbb{C}^r} \xrightarrow{\cong} E_1 \oplus \underline{\mathbb{C}^r}$. Choose metrics (Lemma 2.36) and homotop φ to an isometry (Proposition 3.1). Consider the fiber bundle[3]

$$p\colon \operatorname{Isom}(E_0 \oplus \underline{\mathbb{C}^r}, E_1 \oplus \underline{\mathbb{C}^r}) \longrightarrow S(E_1 \oplus \underline{\mathbb{C}^r}) \tag{3.29}$$

where p maps an isometry to the image of $(0, \dots, 0, 1) \in \mathbb{C}^r$, which lies in the unit sphere bundle. The isometry φ defines a section of the bundle $\operatorname{Isom}(E_0 \oplus \underline{\mathbb{C}^r}, E_1 \oplus \underline{\mathbb{C}^r}) \to S^n$; its composition with p is a section s of the bundle

$$S(E_1 \oplus \underline{\mathbb{C}^r}) \to S^n. \tag{3.30}$$

Now we apply a relative version of Proposition 3.25 to homotop s to a constant section with value $(0, \dots, 0, 1) \in \mathbb{C}^r$: pull (3.30) back over $[0,1] \times S^n$ and extend the section which at $\{0\} \times S^n$ is φ and at $\{1\} \times S^n$ is the constant. Here the base is $(n+1)$-dimensional and the fiber $(2(N+r)-2)$-connected. Finally, use the homotopy lifting property of (3.29) (see (3.24)) to construct a homotopy of φ to a family of isomorphisms which is the identity on the last copy of $\mathbb{C}$, and so restricts to an isomorphism $E_0 \oplus \underline{\mathbb{C}^{r-1}} \xrightarrow{\cong} E_1 \oplus \underline{\mathbb{C}^{r-1}}$. □

Corollary 3.31. *The inclusion* $U_N \hookrightarrow U_{N+1}$ *induces an isomorphism* $\pi_{n-1}U_N \xrightarrow{\cong} \pi_{n-1}U_{N+1}$ *if* $N \geqslant n/2$.

Remark 3.32. The common value of $\pi_{n-1}U_N$ for N large is the *stable homotopy group* of the unitary group. It is the homotopy group of a topological group $U = U_\infty$ which can be constructed as the colimit of $U_1 \hookrightarrow U_2 \hookrightarrow U_3 \hookrightarrow \cdots$. There are prettier geometric models for the same homotopy type, even Banach Lie group models.

Theorem 3.33 (Bott)**.** *There are isomorphisms*

$$\pi_{n-1}U \cong \widetilde{K}(S^n) \cong \begin{cases} \mathbb{Z}, & n \text{ even;} \\ 0, & n \text{ odd.} \end{cases} \tag{3.34}$$

We will give several proofs of Theorem 3.33 as well as stronger forms of Bott periodicity.

For complex vector bundles and unitary groups the periodicity has period 2. There is an analogous 8-periodic statement in the real case; the stable unitary group U is replaced by the stable orthogonal group O.

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[3] The map p is also a map of fiber bundles over S^n.

Theorem 3.35 (Bott)**.** *There are isomorphisms*

$$\pi_{n-1}O \cong \widetilde{KO}(S^n) \cong \begin{cases} \mathbb{Z}, & n \equiv 0 \pmod 8; \\ \mathbb{Z}/2\mathbb{Z}, & n \equiv 1 \pmod 8; \\ \mathbb{Z}/2\mathbb{Z}, & n \equiv 2 \pmod 8; \\ 0, & n \equiv 3 \pmod 8; \\ \mathbb{Z}, & n \equiv 4 \pmod 8; \\ 0, & n \equiv 5 \pmod 8; \\ 0, & n \equiv 6 \pmod 8; \\ 0, & n \equiv 7 \pmod 8; \end{cases} \tag{3.36}$$

A vocal rendition of $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, 0 , $\mathbb{Z}$, 0 , 0 , 0 , $\mathbb{Z}$ is known as the *Bott song*.

References

[Ha2] Allen Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

Lectures 9 & 10: Fredholm operators

Let X be a compact Hausdorff space. Recall (Definition 3.10) that $K(X)$ is defined as the group completion of the commutative monoid of equivalence classes of complex (finite rank) vector bundles on X. So a general element is represented as the formal difference of two vector bundles. In this lecture we develop another model for K-theory which is more flexible and useful in geometric applications. For example, it allows us to include Example 1.11, but it will allow quite a bit more.

The basic idea is as follows. Suppose H^0, H^1 are complex vector spaces and $T\colon H^0 \to H^1$ a linear map. Then there is an exact sequence

$$0 \longrightarrow \ker T \longrightarrow H^0 \xrightarrow{T} H^1 \longrightarrow \operatorname{coker} T \longrightarrow 0 \tag{9.1}$$

The exactness means that, choosing splittings, there is an isomorphism

$$H^0 \oplus \operatorname{coker} T \cong H^1 \oplus \ker T, \tag{9.2}$$

and so formally we identify the difference $H^1 - H^0$ with $\ker T - \operatorname{coker} T$ in $K(\mathrm{pt}) \cong \mathbb{Z}$. This equality of dimensions is a basic theorem in linear algebra. Now imagine that we have a continuous family of linear operators parametrized by X, and perhaps the vector spaces H^0, H^1 also vary in a locally trivial way and so form vector bundles over X. Then their formal difference defines an element of $K(X)$. The kernels and cokernels of T, however, in general are not locally trivial. In fact their dimensions typically jump. (As a simple example take $X = \mathbb{R}$, $H^0 = H^1 = \mathbb{C}$, and T_x the linear map multiplication by $x \in \mathbb{R}$.) In some sense we control the jumping by considering not the kernel and cokernel, but the entire vector spaces H^0, H^1. The new idea is to allow H^0, H^1 to be infinite dimensional while requiring that $\ker T, \operatorname{coker} T$ be finite dimensional. An operator with this property is called *Fredholm*. In continuous families there is jumping of kernels and cokernels, but that is controlled by by considering finite dimensional subspaces containing the kernels where there is no jumping. In this way we make sense of the formal difference between kernels and cokernels. A canonical open cover of the space of Fredholm operators implements this idea universally.

The infinite dimensional vector spaces H^0, H^1 have a topology, and there are many species of infinite dimensional topological vector spaces. We use Hilbert spaces: the topology is induced from a Hermitian metric, and this retains the usual Euclidean notions of length and angle. The theory is equally smooth for Banach spaces [Pa1, §VII]. We also need to topologize the space of continuous linear maps $\operatorname{Hom}(H^0, H^1)$. In this lecture we use the *norm topology*, which makes this a Banach space. However, other choices are possible and we will see later that the *compact-open topology* is a more flexible and applicable choice [ASe1], [FM, Appendix B]. We remark that the Hilbert spaces H^0, H^1 needn't be infinite dimensional, though much of the theory becomes trivial if not.

Some functional analysis

We remind of some basics. Let H^0, H^1 be Hilbert spaces. A linear map $T\colon H^0 \to H^1$ is continuous if and only if it is *bounded*, i.e., there exists $C > 0$ such that

$$\|T\xi\| \leqslant C\|\xi\|, \qquad \xi \in H^0. \tag{9.3}$$

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In that case the infimum over all C which satisfy (9.3) is the *operator norm* $\|T\|$. Let $\operatorname{Hom}(H^0,H^1)$ denote the linear space of continuous linear maps. The operator norm is complete and makes $\operatorname{Hom}(H^0,H^1)$ a Banach space. The operator norm satisfies

$$\|T_2\circ T_1\| \leqslant \|T_1\|\,\|T_2\| \tag{9.4}$$

whenever the composition makes sense. Let $\operatorname{Hom}(H^0,H^1)^\times \subset \operatorname{Hom}(H^0,H^1)$ denote the subspace of invertible linear operators.

Theorem 9.5.

(*i*) *If* $T\colon H^0\to H^1$ *is continuous and bijective, then* T^{-1} *is continuous.*
(*ii*) $\operatorname{Hom}(H^0,H^1)^\times \subset \operatorname{Hom}(H^0,H^1)$ *is an open subspace.*
(*iii*) $\operatorname{Hom}(H^0,H^1)^\times$ *is contractible in the norm topology.*

(i) is the open mapping theorem. (ii) is proved by constructing a ball of invertible operators around any given invertible using the power series for $1/(1+x)$. (iii) is a theorem of Kuiper [Ku] which we prove soon.

We remark that if $V\subset H$ is a finite dimensional subspace of a Hilbert space H, then V is closed and $V^\perp$ is a closed complement. For any closed subspace $V\subset H$ the quotient H/V inherits a Hilbert space structure by identifying it with $V^\perp$ via the quotient map $V^\perp\hookrightarrow H\twoheadrightarrow H/V$.

Fredholm operators

Definition 9.6. Let H^0,H^1 be Hilbert spaces. A continuous linear map $T\colon H^0\to H^1$ is *Fredholm* if its range $T(H^0)\subset H^1$ is closed and if $\ker T,\operatorname{coker} T$ are finite dimensional. Let $\operatorname{Fred}(H^0,H^1)\subset \operatorname{Hom}(H^0,H^1)$ denote the subset of Fredholm operators, topologized with the norm topology.

The closed range condition is redundant [Pa1, §VII], but as $\operatorname{coker} T$ is not Hausdorff if T is not closed range it seems sensible to include it as part of the definition. The *numerical index* of a Fredholm operator is defined as

$$\operatorname{ind} T = \dim\ker T - \dim\operatorname{coker} T. \tag{9.7}$$

Remark 9.8. From some point of view this definition has the wrong sign! For if H^0,H^1 are finite dimensional we identify $T\colon H^0\to H^1$ as an element of $H^1\otimes (H^0)^*$. It is the domain which is dualized, not the codomain, so we expect the minus sign in (9.7) on the subspace $\ker T$ of the domain. This sign mistake causes minor headaches in certain parts of index theory; for example, see [Q, §2].

We give several examples.

Example 9.9. If H is separable it has a countable basis $e_1,e_2,\dots$ in the sense that any element of H can be written as $\sum_n a_ne_n$ where the complex coefficients satisfy $\sum_n |a_n|^2<\infty$. For each $k\in\mathbb{Z}$ define the shift operator T_k which on the basis is

$$T_k(e_j) = \begin{cases} e_{j-k}, & j>k;\\ 0, & j\leqslant k.\end{cases} \tag{9.10}$$

Then T_k is Fredholm of index k. This shows there exist Fredholm operators of any index.

Example 9.11. The differential operator d/dx is Fredholm acting on complex-valued functions on the circle S^1 with coordinate x. We use Hilbert space completions—Sobolev spaces—of the space of smooth functions:

$$\frac{d}{dx}\colon L^2_1(S^1) \longrightarrow L^2(S^1) \tag{9.12}$$

The index is 0: the kernel and cokernel are each 1-dimensional. This example generalizes to elliptic operators on compact manifolds.

The canonical open cover

Recall that a linear map $T\colon H^0 \to H^1$ is *transverse* to a subspace $W \subset H^1$, written $T \pitchfork W$, if $T(H^0) + W = H^1$. Fix Hilbert spaces H^0, H^1. For each *finite dimensional* subspace $W \subset H^1$ define

$$\mathcal{O}_W = \{T \in \mathrm{Fred}(H^0,H^1) : T \pitchfork W\}. \tag{9.13}$$

Observe that $\mathcal{O}_W \subset \mathcal{O}_{W'}$ if $W \subset W'$.

Proposition 9.14.

(*i*) $\mathcal{O}_W \subset \mathrm{Hom}(H^0,H^1)$ *is open.* $\mathrm{Fred}(H^0,H^1) \subset \mathrm{Hom}(H^0,H^1)$ *is open.*
(*ii*) $\{\mathcal{O}_W\}_W$ *is an open cover of* $\mathrm{Fred}(H^0,H^1)$.
(*iii*) *If* X *is compact and* $T\colon X \to \mathrm{Fred}(H^0,H^1)$ *continuous, then* $T(X) \subset \mathcal{O}_W$ *for some finite dimensional* $W \subset H^1$.

Proof. Fix $T_0 \in \mathcal{O}_W$. Observe that $T \pitchfork W$ if and only if $H^0 \xrightarrow{T} H^1 \twoheadrightarrow H^1/W$ is surjective, and the latter is true if the composition

$$(T_0^{-1}W)^{\perp} \hookrightarrow H^0 \xrightarrow{T} H^1 \twoheadrightarrow H^1/W \tag{9.15}$$

is an isomorphism. That is true for $T = T_0$, and since (Theorem 9.5(ii)) isomorphisms are open in $\mathrm{Hom}\big(T_0^{-1}W, H^1/W\big)$ and the map $\mathrm{Hom}(H^0,H^1) \to \mathrm{Hom}\big((T_0^{-1}W, H^1/W\big)$ is continuous, the space of transverse maps (9.13) is open as well. This proves (i). That every Fredholm operator is transverse to a finite dimensional subspace follows directly from the finite dimensionality of its cokernel, which proves (ii). For (iii) the cover $\{T^{-1}\mathcal{O}_W\}_W$ of X has a finite subcover indexed by finite dimensional subspaces $W_1, W_2, \dots, W_N$. Define $W = W_1 + W_1 + \dots + W_N$. □

On O_W we have the parametrized family of vector spaces $K_W \to \mathcal{O}_W$ whose fiber at $T \in \mathcal{O}_W$ is the finite dimensional subspace $T^{-1}W \subset H^0$.

Lemma 9.16. $K_W \to \mathcal{O}_W$ *is a locally trivial vector bundle.*

Proof. Fix $T_0 \in \mathcal{O}_W$ and let $p\colon H^0 \to T_0^{-1}W$ be orthogonal projection. On the open set U of $T \in \mathcal{O}_W$ for which (9.15) is an isomorphism the restriction of p to $T^{-1}W$ is an isomorphism $T^{-1}W \to T_0^{-1}W$, as is easily verified. Topologize K_W as a subspace of $\mathrm{Hom}(H^0, H^1) \times H^0$. Since the map $(T, \xi) \mapsto (T, p(\xi))$ is continuous, we have a local trivialization of the restriction of $K_W \to \mathcal{O}_W$ to U with the constant vector bundle with fiber $T_0^{-1}W$. □

Corollary 9.17. *The function*

$$\begin{aligned} \mathrm{ind}\colon\ \mathrm{Fred}(H^0, H^1) &\longrightarrow \mathbb{Z} \\ T &\longmapsto \dim\ker T - \dim\operatorname{coker} T \end{aligned} \tag{9.18}$$

is locally constant.

Proof. Observe that for $T \in \mathcal{O}_W$ the sequence

$$0 \longrightarrow \ker T \longrightarrow T^{-1}W \xrightarrow{\ T\ } W \longrightarrow \operatorname{coker} T \longrightarrow 0 \tag{9.19}$$

is exact, from which

$$\mathrm{ind}\, T = \dim\ker T - \dim\operatorname{coker} T = \dim T^{-1}W - \dim W. \tag{9.20}$$

The right hand side is locally constant on $\mathcal{O}_W$, by Lemma 9.16. □

Fredholms and the K-theory of a compact space

As a preliminary we prove that the composition of Fredholms is Fredholm and that the numerical index behaves well under composition. For convenience we now consider Fredholm operators on a fixed Hilbert space H.

Lemma 9.21. *If $T_1, T_2 \in \mathrm{Fred}(H)$, then $T_2 \circ T_1 \in \mathrm{Fred}(H)$ and $\mathrm{ind}\, T_2 \circ T_1 = \mathrm{ind}\, T_1 + \mathrm{ind}\, T_2$.*

Proof. If $T_2T_1 \pitchfork W$, then $T_2 \pitchfork W$ and $T_1 \pitchfork T_2^{-1}W$. Thus

$$\begin{aligned} \mathrm{ind}\, T_2T_1 &= \dim T_1^{-1}T_2^{-1}W - \dim W \\ &= \dim T_1^{-1}T_2^{-1}W - \dim T_2^{-1}W + \dim T_2^{-1}W + \dim W \\ &= \mathrm{ind}\, T_1 + \mathrm{ind}\, T_2. \end{aligned} \tag{9.22}$$

□

Suppose X is compact Hausdorff and $T\colon X \to \mathrm{Fred}(H)$ is continuous. By Proposition 9.14(iii) there exists a finite dimensional subspace $W \subset H$ such that $T_x \pitchfork W$ for all $x \in X$. Then $T^*K_W \to X$ is a vector bundle, and we define

$$[T^*K_W] - [\underline{W}] \in K(X). \tag{9.23}$$

Theorem 9.24 (Atiyah-Jänich)**.** *Assume X is compact Hausdorff. Then $T \mapsto [T^*K_W] - [\underline{W}]$ is a well-defined map*

$$i\colon [X, \mathrm{Fred}(H)] \longrightarrow K(X) \tag{9.25}$$

which is an isomorphism of abelian groups.

The map i sends a family of Fredholm operators to its *index* in K-theory.

Corollary 9.26. *The numerical index*

$$\mathrm{ind}\colon \pi_0\,\mathrm{Fred}(H) \longrightarrow \mathbb{Z} \tag{9.27}$$

is an isomorphism.

This follows from Theorem 9.24 by taking $X = \mathrm{pt}$.

Proof. We first prove that i is well-defined. If W_1, W_2 are finite dimensional subspaces for which $T_x \pitchfork W_i$ for all $x \in X$, then the same holds for $W_1 + W_2$, so it suffices to check well-definedness of (9.23) for subspaces $W \subset W'$. In that case there is a short exact sequence

$$0 \longrightarrow T^*K_W \longrightarrow T^*K_{W'} \longrightarrow \underline{W'}/\underline{W} \longrightarrow 0 \tag{9.28}$$

of vector bundles over X. Choosing a splitting we construct an isomorphism $\underline{W'}/\underline{W} \oplus T^*K_W \cong T^*K_{W'}$ and then, adding $\underline{W}$ to both sides, we obtain an isomorphism $\underline{W'} \oplus T^*K_W \cong \underline{W} \oplus T^*K_{W'}$. It follows that $[T^*K_{W'}] - [\underline{W'}] = [T^*K_W] - [\underline{W}] \in K(X)$.

To check that the index (9.25) is invariant under homotopy, suppose that $H\colon [0,1] \times X \to \mathrm{Fred}(H)$ is a continuous map. Choose $W \subset H$ such that $H_{(t,x)} \pitchfork W$ for all (t,x). Then by Theorem 2.1 the restrictions of $H^*K_W \to [0,1] \times X$ to the ends of $[0,1] \times X$ are isomorphic, and it follows that the K-theory classes (9.23) on the two ends agree.

The domain $[X, \mathrm{Fred}(H)]$ of (9.25) is a monoid by pointwise composition. To see that i is a homomorphism of monoids, begin as in the proof of Lemma 9.21 by choosing $W \subset H$ such that $(T_2T_1)_x \pitchfork W$ for all $x \in X$. Let $E'_x = (T_2)_x^{-1}(W)$. For each $x_0 \in X$ there is an open neighborhood of $x \in X$ such that the orthogonal projection of E'_x to E'_{x_0} is an isomorphism and $(T_1)_x \pitchfork E'_{x_0}$. Cover X by a finite set of such neighborhoods and let $V \subset H$ be a subspace containing the sum of the corresponding E'_{x_0} such that $(T_1)_x \pitchfork V$ for all $x \in X$. Let E_x denote the orthogonal projection of E'_x to V and define $F \to X$ by $F_x = T_1^{-1}E_x$. Note that orthogonal projection is an isomorphism $E' \xrightarrow{\cong} E$. Compute

$$\begin{aligned} i(T_2T_1) &= [(T_2T_1)^*K_W] - [\underline{W}] \\ &= [(T_2T_1)^*K_W] - [T_2^*K_W] + i(T_2) \\ &= [F] - [E] + i(T_2) \\ &= [T_1^*K_V] - [\underline{V}] + i(T_2) \\ &= i(T_1) + i(T_2). \end{aligned} \tag{9.29}$$

In the penultimate step we use the exact sequence $0 \to F \to T_1^*K_V \to \underline{V}/E \to 0$, analogous to (9.28).

To prove that i is surjective observe from Proposition 3.15 that any element of $K(X)$ has the form $[E] - N$ for some vector bundle $E \to X$ and $N \in \mathbb{Z}^{\geqslant 0}$. By Example 9.9 there is a constant family of Fredholm operators whose index is N. Embed $E \hookrightarrow \underline{\mathbb{E}}$ in a trivial bundle (Proposition 3.15 again) and define $p_x \in \operatorname{End}\mathbb{E}$ as orthogonal projection with kernel E_x. Finally, embed $\mathbb{E} \hookrightarrow H$ and extend p_x to be the identity on $\mathbb{E}^\perp$.

To prove that i is injective, if $i(T) = 0$ for $T\colon X \to \operatorname{Fred}(H)$, then for some finite dimensional vector space $\mathbb{E}$ there exists an isomorphism

$$T^*K_W \oplus \underline{\mathbb{E}} \xrightarrow{\varphi} \underline{W} \oplus \underline{\mathbb{E}} \tag{9.30}$$

of vector bundles over X. The fiber of T^*K_W at $x \in X$ is $T_x^{-1}W$. Add to (9.30) the isomorphism $T_x\colon (T_x^{-1}W)^\perp \to H \twoheadrightarrow W^\perp$ to obtain the family of isomorphisms

$$H \oplus \mathbb{E} \xrightarrow{\Phi_x = T_x + \varphi_x} H \oplus \mathbb{E}. \tag{9.31}$$

Then $t \mapsto T + t\varphi$ is a homotopy from T to this family of invertibles. By Kuiper's Theorem 9.5(iii) the latter is homotopically trivial. (To obtain operators on H rather than $H \oplus \mathbb{E}$ conjugate by an isomorphism $H \to H \oplus \mathbb{E}$.)

Since i is a bijective homomorphism of monoids, and $K(X)$ is an abelian group, it follows that $[X, \operatorname{Fred}(H)]$ is also an abelian group and i is an isomorphism of abelian groups. ☐

Further remarks

We will have more to say about Fredholm operators in future lectures. For now a few comments will suffice.

(9.32) *Invertibles as a fat basepoint.* In homotopy theory we work with pointed topological spaces, that is, topological spaces with a distinguished basepoint. For sure $\operatorname{Fred}(H)$ has one—the identity operator—though if $H^0 \neq H^1$ then $\operatorname{Fred}(H^0, H^1)$ does not have a distinguished basepoint. In both cases there is a natural contractible subspace, the subspace of invertible operators. So we can work with the pair $(\operatorname{Fred}(H), \operatorname{Fred}(H)^\times)$ in lieu of a pointed space.

(9.33) *Relative K-theory.* In this spirit if (X, A) is a pair of spaces and $T\colon X \to \operatorname{Fred}(H)$ such that T_a is invertible for all $a \in A$, then T defines an element in the *relative K-theory* group $K(X, A)$. (Take this as the definition of relative K-theory.) The *support* of a Fredholm family is the set of points at which the Fredholm operator fails to be invertible. The family of linear operators in Example 1.11 is trivially Fredholm, since they act on a finite dimensional space, and the support is the set of eigenvalues of the given operator.

(9.34) *Topology of* $\mathrm{Fred}(H)$. By Corollary 9.26 we can write

$$\mathrm{Fred}(H) = \coprod_n \mathrm{Fred}^{(n)}(H) \tag{9.35}$$

as the disjoint union of connected spaces of Fredholms of a fixed index. The components are all homeomorphic, and the underlying homotopy type is that of BU, the classifying space of the group described in Remark 3.32.

Half of a cohomology theory (Bonus material)

In lecture we covered these ideas following [A1, Ha] in the context of compact Hausdorff spaces. At this point we have defined a map

$$X \longmapsto K(X) = [X, \mathrm{Fred}(H)] \tag{9.36}$$

which attaches an abelian group to every space X. For compact Hausdorff spaces Theorem 9.24 asserts that this is the same as Definition 3.10. It is conventional to restrict to a category of "nice" spaces, such as CW complexes or compactly generated spaces. We will see that for free we obtain half of a cohomology theory through suspensions and loopings, but we need new ideas to recover the other half. For K-theory that idea is Bott periodicity (Theorem 3.33, Theorem 3.35) In the next several lectures we give a proof of Bott periodicity in the context of Fredholm operators.

A reference for this section is [A1].

(9.37) *Pointed spaces.* Let $\mathcal{S}$ denote a convenient category of spaces (CW complexes, compactly generated spaces), $\mathcal{S}_*$ the category of pointed spaces—spaces with a (nondegenerate) basepoint—and $\mathcal{S}_2$ the category of (excisive) pairs. There are functors

$$\begin{tikzcd} \mathcal{S}_2 \arrow[r] & \mathcal{S}_* \\ (X, A) \arrow[r, mapsto] & X/A \\ (X, \{x_0\}) & X \arrow[l, mapsto] \end{tikzcd} \tag{9.38}$$

where in the last formula X has basepoint x_0. Note that unpointed spaces $\mathcal{S}$ map to $\mathcal{S}_2$ via $X \mapsto (X, \varnothing)$, and

$$X/\varnothing = X_+ = X \amalg \{*\} \tag{9.39}$$

is the union of X with a disjoint basepoint. Recall that the *suspension* ΣX of a pointed space is the smash product $S^1 \wedge X$.

Definition 9.40. A *cohomology theory* is a sequence of abelian group-valued functors

$$\widetilde{K}^n\colon \mathcal{S}_*^{\mathrm{op}} \longrightarrow \mathrm{AbGp}, \tag{9.41}$$

one for each $n \in \mathbb{Z}$, and a sequence of natural transformations

$$\widetilde{K}^n(X) \longrightarrow \widetilde{K}^{n+1}(\Sigma X), \qquad X \in \mathcal{S}_*, \tag{9.42}$$

such that

(i) if $f_t\colon X \to Y$ is a homotopy, then

$$f_0^* = f_1^*\colon \widetilde{K}^n(Y) \longrightarrow \widetilde{K}^n(X); \tag{9.43}$$

(ii) for $f\colon X \to Y$ with mapping cone C_f and $j\colon Y \hookrightarrow C_f$ the inclusion, the sequence

$$\widetilde{K}^n(C_f) \xrightarrow{j^*} \widetilde{K}^n(Y) \xrightarrow{f^*} \widetilde{K}^n(X) \tag{9.44}$$

is exact;

(iii) the suspension homomorphisms (9.42) are isomorphisms; and

(iv) if $X = \bigvee\limits_{\alpha\in A} X_\alpha$, then the natural map

$$\widetilde{K}^n(X) \longrightarrow \prod_{\alpha\in A} \widetilde{K}^n(X_\alpha) \tag{9.45}$$

is an isomorphism.

(9.46) *Nonpositive degree K-theory.* From the definition

$$K^0(X) = [X, \mathrm{Fred}(H)] \tag{9.47}$$

developed earlier in this lecture we extract the reduced cohomology

$$\widetilde{K}^0(X) = \bigl[(X, x_0), (\mathrm{Fred}(H), \mathrm{id}_H)\bigr] \tag{9.48}$$

of a pointed space, where we use the identity operator as the basepoint in the space of Fredholms. Then for *nonnegative integers* $n \in \mathbb{Z}^{\geqslant 0}$ define

$$\widetilde{K}^{-n}(X) = \widetilde{K}^0(\Sigma^n X) = [\Sigma^n X, \mathrm{Fred}(H)] = [X, \Omega^n \,\mathrm{Fred}(H)], \tag{9.49}$$

where X is pointed and maps and homotopies preserve basepoints. Just as the space of Fredholms is the classifying space (9.47) for K^0, its n^{th} *loop space* is the classifying space for K^{-n}. To define K^n for $n > 0$ it is clear that we need *deloopings* of $\mathrm{Fred}(H)$. That is the challenge for any cohomology theory.

Remark 9.50. For example, we can define $H^0(X;\mathbb{Z}) = [X, \mathbb{Z}]$, where $\mathbb{Z}$ is a discrete space with basepoint $0 \in \mathbb{Z}$. Its loop spaces are all trivial—they consist of one point—and so $H^{-n}(X;\mathbb{Z}) = 0$ for $n > 0$. But this gives no clue how to define $H^n(X;\mathbb{Z})$ for $n > 0$.

(9.51) *Bott periodicity.* For K-theory the groups $\widetilde{K}^{-n}(X)$, $n > 0$, are periodic and so it is easy to extend to positive degree. We sketch the proofs in the next few lectures for both the real and complex cases.

Theorem 9.52 (Bott)**.** *There are homotopy equivalences* $\Omega^2 \operatorname{Fred}(H) \simeq \operatorname{Fred}(H)$ *and* $\Omega^8 \operatorname{Fred}(H_{\mathbb{R}}) \simeq \operatorname{Fred}(H_{\mathbb{R}})$*.*

(9.53) *Suspensions as Thom complexes.* We give an alternative picture of suspension and a twisted version. Let X be an *unpointed* space. Its n^{th} suspension is

$$\Sigma^n X_+ = X \times S^n \,/\, X \times \{*\} \simeq X \times \mathbb{R}^n \,/\, X \times \bigl(\mathbb{R}^n \backslash B_r(0)\bigr), \tag{9.54}$$

where $B_r(0) \subset \mathbb{R}^n$ is the open ball of radius $r > 0$.

The construction generalizes to twisted suspensions, replacing $X \times \mathbb{R}^n$ by a real vector bundle $V \to X$. Fix an inner product on the bundle and also fix a real number $r > 0$. Define $B_r(0) \subset V$ as the open subspace of vectors of norm strictly less than r. The quotient space

$$X^V = V/\bigl(V\backslash B_r(0)\bigr) \tag{9.55}$$

is the *Thom complex* of V, and up to homeomorphism it is independent of the inner product and choice of $r > 0$. Note that X^V has a natural basepoint.

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Lecture 11: Clifford algebras

In this lecture we introduce Clifford algebras, which will play an important role in the rest of the class. The link with K-theory is the *Atiyah-Bott-Shapiro construction* [ABS], which implements the K-theory of suspensions via Clifford modules. We will begin the next lecture with this ABS construction.

An algebra from the orthogonal group

The orthogonal group O_n is a subset of an algebra: the algebra $M_n\mathbb{R}$ of $n \times n$ matrices. The *Clifford algebra* plays a similar role for a double cover group of the orthogonal group.

(11.1) *Heuristic motivation.* Orthogonal transformations are products of reflections. For a unit norm vector $\xi \in \mathbb{R}^n$ define

$$\rho_\xi(\eta) = \eta - 2\langle \eta, \xi\rangle \xi, \tag{11.2}$$

where $\langle -, -\rangle$ is the standard inner product.

Theorem 11.3 (Sylvester)**.** *Any $g \in O_n$ is the composition of $\leqslant n$ reflections.*

Proof. The statement is trivial for $n = 1$. Proceed by induction: if $g \in O_n$ fixes a unit norm vector ξ then it fixes the orthogonal complement $(\mathbb{R}\cdot\xi)^\perp$, and we are reduced to the theorem for O_{n-1}. If there are no fixed unit norm vectors, then for any unit norm vector ζ set $\xi = \frac{g(\zeta)-\zeta}{|g(\zeta)-\zeta|}$. The composition $\rho_\xi \circ g$ fixes ζ and again we reduce to the $(n-1)$-dimensional orthogonal complement. ☐

Now generate an algebra from the unit norm vectors, with relations inspired by those of reflections. Note immediately that the vectors $\pm\xi$ both correspond to the same reflection $\rho_\xi = \rho_{-\xi}$. Therefore, we expect from the beginning that the Clifford algebra "double counts" orthogonal transformations. Now since the square of a reflection is the identity, we impose the relation

$$\xi^2 = \pm 1, \qquad |\xi| = 1. \tag{11.4}$$

The sign ambiguity is that described above, and we choose a sign independent of ξ. It follows that

$$\xi^2 = \pm|\xi|^2 \tag{11.5}$$

for any $\xi \in \mathbb{R}^n$. Now if $\langle \xi_1, \xi_2\rangle = 0$, then $(\xi_1 + \xi_2)/\sqrt{2}$ has unit norm and from

$$\pm 1 = \left(\frac{\xi_1 + \xi_2}{\sqrt{2}}\right)^2 = \frac{\xi_1^2 + \xi_2^2 + (\xi_1\xi_2 + \xi_2\xi_1)}{2} = \frac{\pm 2 + (\xi_1\xi_2 + \xi_2\xi_1)}{2} \tag{11.6}$$

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we deduce

$$\xi_1\xi_2 + \xi_2\xi_1 = 0, \qquad \langle \xi_1, \xi_2 \rangle = 0. \tag{11.7}$$

Equations (11.4) and (11.7) are the defining relations for the Clifford algebra. Check that the reflection (11.2) is given by

$$\rho_\xi(\eta) = -\xi\eta\xi^{-1} \tag{11.8}$$

in the Clifford algebra. By composition using Theorem 11.3 we obtain the action of any orthogonal transformation on $\eta \in \mathbb{R}^n$.

Definition 11.9. For $n \in \mathbb{Z}$ define the *real Clifford algebra* $\mathrm{C}\ell_n$ as the unital associative real algebra generated by $e_1, \dots, e_{|n|}$ subject to the relations

$$\begin{aligned} e_i^2 &= \pm 1, \qquad & i &= 1, \dots, n \\ e_ie_j + e_je_i &= 0, \qquad & i &\neq j. \end{aligned} \tag{11.10}$$

The complex Clifford algebra $\mathrm{C}\ell_n^{\mathbb{C}}$ is the complex algebra with the same generators and same relations.

Note $\mathrm{C}\ell_0 = \mathbb{R}$ and $\mathrm{C}\ell_0^{\mathbb{C}} = \mathbb{C}$.

Example 11.11. There is an isomorphism $\mathrm{C}\ell_{-n}^{\mathbb{C}} \cong \mathrm{C}\ell_n^{\mathbb{C}}$ obtained by multiplying each generator e_i by $\sqrt{-1}$.

Example 11.12. $\mathrm{C}\ell_{-1}$ can be embedded in the matrix algebra $M_2\mathbb{R}$ by setting

$$e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{11.13}$$

The same equation embeds $\mathrm{C}\ell_{-1}^{\mathbb{C}}$ in $M_2\mathbb{C}$.

Example 11.14. We identify $\mathrm{C}\ell_{-2}^{\mathbb{C}}$ with $\mathrm{End}(\mathbb{C}^2)$ by setting

$$e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{11.15}$$

where $i = \sqrt{-1}$. This does *not* work over the reals. The product

$$e_1e_2 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{11.16}$$

is $-i$ times a *grading operator* on $\mathbb{C}^2$.

Example 11.17. The real Clifford algebras $\mathrm{C}\ell_1$ and $\mathrm{C}\ell_{-1}$ are not isomorphic. We embed in $M_2\mathbb{R}$ in the former case by setting

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{11.18}$$

and in the latter using (11.13). Note that the doubled orthogonal group $\{\pm 1, \pm e_1\}$ is different in the two cases: in $\mathrm{C}\ell_1$ it is isomorphic to the Klein group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ whereas in $\mathrm{C}\ell_{-1}$ it is cyclic of order four.

(11.19) *Spin and Pin.* For $n > 0$ let $S(\mathbb{R}^n) \subset \mathbb{R}^n$ denote the sphere of unit norm vectors. Since $\mathbb{R}^n$ embeds in $\mathrm{C}\ell_{\pm n}$, so too does $S(\mathbb{R}^n)$. We assert without proof that the group it generates is a Lie group $\mathrm{Pin}_{\pm n} \subset \mathrm{C}\ell_{\pm n}$. It follows from Theorem 11.3 that there is a surjection $\mathrm{Pin}_{\pm n} \to O_n$ defined by composing the reflections (11.8). The inverse image $\mathrm{Spin}_{\pm n}$ of the special orthogonal group SO_n consists of products of an even number of elements in $S(\mathbb{R}^n)$. There is an isomorphism $\mathrm{Spin}_n \cong \mathrm{Spin}_{-n}$, but as we saw in Example 11.14 this is not true in general for Pin.

(11.20) *The Dirac operator.* The Clifford algebra arises from the following question, posed by Dirac: Find a square root of the Laplace operator. We work on flat Euclidean space $\mathbb{E}^n$, which is the affine space $\mathbb{A}^n$ endowed with the translation-invariant metric constructed from the standard inner product on the underlying vector space $\mathbb{R}^n$ of translations. Let $x^1, \dots, x^n$ be the standard affine coordinates on $\mathbb{E}^n$. The Laplace operator is

$$\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{(\partial x^i)^2}. \tag{11.21}$$

A first-order operator

$$D = \gamma^i \frac{\partial}{\partial x^i} \tag{11.22}$$

satisfies $D^2 = \Delta$ if and only if γ^i satisfy the Clifford relation

$$\gamma^i\gamma^j + \gamma^j\gamma^i = -2\delta^{ij}, \qquad 1 \leqslant i, j \leqslant n, \tag{11.23}$$

as in (11.10). If we let (11.21), (11.22) act on the space $C^\infty(\mathbb{E}^n; \mathbb{S})$ of functions with values in a vector space $\mathbb{S}$, then we conclude that $\mathbb{S}$ is a $\mathrm{C}\ell_{-n}$*-module*.

(11.24) *$\mathbb{Z}/2\mathbb{Z}$-gradings.* So far we have not emphasized the $\mathbb{Z}/2\mathbb{Z}$-grading evident in the examples: odd products of generators such as (11.13), (11.15), (11.18) are represented by block off-diagonal matrices whereas even products of generators (11.16) are represented by block diagonal matrices.

Superalgebra

For a more systematic treatment, see [DM, §1]. We use ‘super’ synonymously with ‘$\mathbb{Z}/2\mathbb{Z}$-graded’.

(11.25) *Super vector spaces.* Let k be a field, which in our application will always be $\mathbb{R}$ or $\mathbb{C}$. A super vector space $\mathbb{S} = \mathbb{S}^0 \oplus \mathbb{S}^1$ is a pair $(\mathbb{S}, \epsilon)$ of a vector space over k and an operator ϵ with $\epsilon^2 = \mathrm{id}_{\mathbb{S}}$. The subspaces $\mathbb{S}^0, \mathbb{S}^1$ are the $+1, -1$-eigenspaces, respectively. Eigenvectors are called even, odd. The tensor product $\mathbb{S}' \otimes \mathbb{S}''$ of super vector spaces carries the grading $\epsilon' \otimes \epsilon''$. The main new point is the *Koszul sign rule*, which is the symmetry of the tensor product:

$$\begin{aligned} \mathbb{S}' \otimes \mathbb{S}'' &\longrightarrow \mathbb{S}'' \otimes \mathbb{S}' \\ s' \otimes s'' &\longmapsto (-1)^{|s'||s''|} s'' \otimes s', \end{aligned} \tag{11.26}$$

(11.27) *Superalgebras.* Let $A = A^0 \oplus A^1$ be a super algebra, an algebra with a compatible grading: $A^i \cdot A^j \subset A^{i+j}$, where the degree is taken in $\mathbb{Z}/2\mathbb{Z}$. A homogeneous element z in its *center* satisfies $za = (-1)^{|z||a|}az$ for all homogeneous $a \in A$. The center is itself a super algebra, which is of course commutative (in the $\mathbb{Z}/2\mathbb{Z}$-graded sense). The *opposite* super algebra A^{op} to a super algebra A is the same underlying vector space with product $a_1 \cdot a_2 = (-1)^{|a_1||a_2|}a_2a_1$ on homogeneous elements. All algebras are assumed unital. Tensor products of super algebras are taken in the graded sense: the multiplication in $A' \otimes A''$ is

$$(a_1' \otimes a_1'')(a_2' \otimes a_2'') = (-1)^{|a_1''||a_2'|}a_1'a_2' \otimes a_1''a_2''. \tag{11.28}$$

Undecorated tensor products are over the ground field. Unless otherwise stated a module is a left module. An ideal $I \subset A$ in a super algebra is *graded* if $I = (I \cap A^0) \oplus (I \cap A^1)$.

(11.29) *Super matrix algebras.* Let $S = S^0 \oplus S^1$ be a finite dimensional super vector space over k. Then $\operatorname{End} S$ is a central simple super algebra. Endomorphisms which preserve the grading on S are even, those which reverse it are odd. A super algebra isomorphic to $\operatorname{End} S$ is called a *super matrix algebra.*

Clifford algebras

For more details see [ABS, Part I], [De1, §2].

A quadratic form on a vector space V is a function $Q\colon V \to k$ such that

$$B(\xi_1, \xi_2) = Q(\xi_1 + \xi_2) - Q(\xi_1) - Q(\xi_2), \qquad \xi_1, \xi_2 \in V, \tag{11.30}$$

is bilinear and $Q(n\xi) = n^2Q(\xi)$.

Definition 11.31. The *Clifford algebra* $C\ell(V, Q) = C\ell(V)$ of a quadratic vector space is an algebra equipped with a linear map $i\colon V \to C\ell(V, Q)$ which satisfies the following universal property: If $\varphi\colon V \to A$ is a linear map to an algebra A such that

$$\varphi(\xi)^2 = Q(\xi) \cdot 1_A, \qquad \xi \in V, \tag{11.32}$$

then there exists a unique algebra homomorphism $\tilde{\varphi}\colon C\ell(V, Q) \to A$ such that $\varphi = \tilde{\varphi} \circ i$.

We leave the reader to prove that i is injective and that $C\ell(V, Q)$ is unique up to unique isomorphism. Furthermore, there is a surjection

$$\otimes V \longrightarrow C\ell(V, Q) \tag{11.33}$$

from the tensor algebra, as follows from its universal property. This gives an explicit construction of $C\ell(V, Q)$ as the quotient of $\otimes V$ by the 2-sided ideal generated by $\xi^2 - Q(\xi) \cdot 1_{\otimes V}$, $\xi \in V$. The tensor algebra is $\mathbb{Z}$-graded, and since the ideal sits in even degree the quotient Clifford algebra is

$\mathbb{Z}/2\mathbb{Z}$-graded. The increasing filtration $\otimes^0 V \subset \otimes^{\leqslant 1} V \subset \otimes^{\leqslant 2} V \subset \cdots$ induces an increasing filtration on $\mathrm{C}\ell(V,Q)$ whose associated graded is isomorphic to the ($\mathbb{Z}$-graded) exterior algebra $\bigwedge^\bullet V$. There is a canonical isomorphism

$$\mathrm{C}\ell(V' \oplus V'', Q' \oplus Q'') \cong \mathrm{C}\ell(V',Q') \otimes \mathrm{C}\ell(V'',Q''), \tag{11.34}$$

deduced from the universal property. The standard Clifford algebras in Definition 11.9 have the form $\mathrm{C}\ell(V,Q)$ for $V = \mathbb{R}^n, \mathbb{C}^n$ and Q the positive or negative definite standard quadratic form on V.

The Clifford algebras are *central simple* as $\mathbb{Z}/2\mathbb{Z}$-graded algebras. I will leave the simplicity (there are no nontrivial 2-sided homogeneous ideals) as an exercise and here prove the centrality.

Proposition 11.35. *$\mathrm{C}\ell(V,Q)$ has center k.*

Proof. Suppose $x = x^0 + x^1$ is a central element. Fix an orthonormal basis $e_1, \dots, e_n$ of V. Then for every $i = 1, \dots, n$ we have

$$\begin{aligned} x^0 e_i &= e_i x^0 \\ x^1 e_i &= -e_i x^1 \end{aligned} \tag{11.36}$$

There is a unique decomposition $x^0 = a^0 + e_i b^1$ where a^0, b^1 belong to the Clifford algebra generated by the basis elements excluding e_i. Then

$$\begin{aligned} x^0 e_i &= a^0 e_i + e_i b^1 e_i = e_i a^0 - (e_i)^2 b^1 \\ e_i x^0 &= e_i a^0 + (e_i)^2 b^1. \end{aligned} \tag{11.37}$$

Since x^0 is central we have $x^0 e_i = e_i x^0$, and so (11.37) implies that $b^1 = 0$. Since this holds for every i, we conclude that x^0 is a scalar. Similarly, write $x^1 = a^1 + e_i b^0$ so that

$$\begin{aligned} x^1 e_i &= a^1 e_i + e_i b^0 e_i = -e_i a^1 + (e_i)^2 b^0 \\ -e_i x^1 &= -e_i a^1 - (e_i)^2 b^0 \end{aligned} \tag{11.38}$$

from which $x^1 = 0$. ☐

For a vector space L and $\theta \in L^*$ let ϵ_θ denote exterior multiplication by θ, which is an endomorphism of the exterior algebra $\bigwedge^\bullet L^*$. For $\ell \in L$ the adjoint of exterior multiplication by ℓ is contraction ι_ℓ, an endomorphism of $\bigwedge^\bullet L^*$ of degree -1.

Proposition 11.39. *Suppose $V = L \oplus L^*$ with the split quadratic form $Q(\ell + \theta) = \theta(\ell)$, $\ell \in L$, $\theta \in L^*$. Set $\mathbb{S} = \bigwedge^\bullet L^*$ with its $\mathbb{Z}/2\mathbb{Z}$-grading by the parity of the degree. Then the map $V \to \operatorname{End} \mathbb{S}$*

$$\begin{aligned} \ell &\longmapsto \iota_\ell \\ \theta &\longmapsto \epsilon_\theta \end{aligned} \tag{11.40}$$

extends to an isomorphism $\mathrm{C}\ell(V) \xrightarrow{\cong} \operatorname{End} \mathbb{S}$ of the Clifford algebra with a super matrix algebra.

Proof. Using (11.34) we reduce to the case $\dim L = 1$ which can be checked by hand; it is essentially Example 11.14. ☐

(11.41) *Algebraic Bott periodicity.* We may in the future discuss basic *Morita theory*, in which we will see that super matrix algebras are in some sense trivial. That is the spirit of the following theorem. We say the dimension of a finite dimensional super vector space $\mathbb{S} = \mathbb{S}^0 \oplus \mathbb{S}^1$ is $d^0|d^1$ if $\dim \mathbb{S}^i = d^i$.

Theorem 11.42. *There are isomorphisms of superalgebras*

$$\begin{aligned} \mathrm{C}\ell_{-2}^{\mathbb{C}} &\xrightarrow{\cong} \operatorname{End}(\mathbb{S}), &\qquad \dim \mathbb{S} \ &= 1|1, \\ \mathrm{C}\ell_{-8} &\xrightarrow{\cong} \operatorname{End}(\mathbb{S}_{\mathbb{R}}), &\qquad \dim \mathbb{S}_{\mathbb{R}} &= 8|8. \end{aligned} \tag{11.43}$$

Proof. The complex case is Example 11.14. For the real case we let $\mathrm{C}\ell_{-2}$ act on $\mathbb{W} = \mathbb{C}^{1|1}$ via the formulas in (11.15). This action commutes (in the graded sense) with the *odd* real structure

$$J(z^0, z^1) = (\overline{z^1}, \overline{z^0}). \tag{11.44}$$

That is, $J\colon \mathbb{W} \to \mathbb{W}$ is antilinear, odd, and squares to $-\mathrm{id}_{\mathbb{W}}$. Set $\mathbb{S} = W^{\otimes 4}$. It carries an action of $\mathrm{C}\ell_{-2}^{\otimes 4} \cong \mathrm{C}\ell_{-8}$ which commutes with $J^{\otimes 4}$. The latter is antilinear, even, and squares to $\mathrm{id}_{\mathbb{S}}$, so is a real structure. ☐

As stated in the proof, $\mathbb{W}^{\otimes 2}$ carries a *quaternionic* structure $J^{\otimes 2}$: the Koszul sign rule (11.26) implies that $J^{\otimes 2}$ squares to *minus* the identity. (Check that sign! It will test your understanding of the sign rule.)

(11.45) *Spin and Pin redux.* Sitting inside the Clifford algebra $\mathrm{C}\ell(V,Q)$ is the pin group $\mathrm{Pin}(V,Q)$ generated by $S(V)$ and its even subgroup $\mathrm{Spin}(V,Q) = \mathrm{Pin}(V,Q) \cap \mathrm{C}\ell(V,Q)^0$. When V is real and Q is definite these are compact Lie groups. In that case we can average a metric over a real or complex Clifford module $\mathbb{S} = \mathbb{S}^0 \oplus \mathbb{S}^1$ so that $\mathrm{Pin}(V,Q)$ acts orthogonally (unitarily in the complex case). It follows that $e \in S(V)$ is self- or skew-adjoint, according as Q is positive or negative definite.

Remark 11.46. There is a tricky sign in the proper definition of 'self-adjoint' and 'skew-adjoint' in the super world. There is a way around that sign to a more standard convention, which is the one we use; see [DM, §4.4], [De2, §4].

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Lecture 12: Kuiper's theorem, classifying spaces, Atiyah-Singer loop map, Atiyah-Bott-Shapiro construction

We begin this lecture by giving a proof of Kuiper's theorem on the contractibility of the general linear group of an infinite dimensional Hilbert space. We follow [ASe1, Appendix C] closely; see also the original paper [Ku]. One consequence is the contractibility of infinite dimensional Stiefel manifolds, which we use to construct geometric classifying spaces for compact Lie groups. Then we introduce spaces of Fredholm operators which incorporate Clifford algebras. We state the Atiyah-Singer looping construction **(12.59)**, which is the main theorem in [AS3], and show how together with algebraic Bott periodicity **(11.41)** it implies Bott periodicity. In the next two lectures we sketch the main points in the proof of that theorem, which relies on Kuiper's contractibility result. We conclude with some important material not covered in class: the Atiyah-Bott-Shapiro construction.

Kuiper's Theorem

Theorem 12.1 ([Ku]). *Let H be an infinite dimensional real or complex separable Hilbert space. Then the general linear group $\mathrm{Aut}(H)$ is contractible in the norm topology.*

The general linear group deformation retracts onto the unitary group of automorphisms which preserve the inner product. Namely, to any operator P is associated a nonnegative self-adjoint operator $|P|$ such that $|P|^2 = P^*P$. The operator $|P|$ is positive if P is invertible. Then the retraction is

$$P_t = P\big((1-t)\,\mathrm{id}_H + t|P|^{-1}\big). \tag{12.2}$$

Corollary 12.3. *The unitary group $U(H)$ is contractible in the norm topology*

In the real case 'unitary' is usually called 'orthogonal' and the group is denoted $O(H)$.

Definition 12.4. A continuous map $f\colon X \to Y$ of topological spaces is a *weak homotopy equivalence* if $f_*\colon \pi_0X \to \pi_0Y$ is a bijection and for every $x \in X$ the map $f_*\colon \pi_q(X,x) \to \pi_q(Y,f(x))$ is an isomorphism for all $q > 0$.

Whitehead proved that a weak homotopy equivalence of CW complexes is a homotopy equivalence, and the same is true for spaces with the homotopy type of a CW complex [Mi]. This applies in particular to open subsets of a Banach space,[1] so to the general linear group in the norm topology. Therefore, to prove Theorem 12.1 it suffices to show all homotopy groups of $\mathrm{Aut}(H)$ vanish.

Let X be a compact simplicial complex and $f\colon X \to \mathrm{Aut}(H)$ a continuous map. We prove by a series of deformations that $f = f_0$ is homotopic to the constant map with value id_H.

Lemma 12.5. *There exists a homotopy $f_0 \simeq f_1$ and a finite dimensional subspace $V \subset \mathrm{End}(H)$ such that $f_1(x) \in V$ for all $x \in X$.*

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[1] Milnor [Mi] proves that an absolute neighborhood retract (ANR) has the homotopy type of a CW complex. A (paracompact) Banach manifold is an ANR [Pa2].

Of course, we make a choice of the homotopy whose existence is proved! (Similar comment for the lemmas which follow.)

Proof. Since $Aut(H) \subset \mathrm{End}(H)$ is open it has a cover by balls, and since X is compact it is covered by a finite number of their inverse images. Subdivide X so that every simplex lies in such a ball. Define f_1 to agree with f_0 on vertices and be affine on each simplex. Then $f_t = (1-t)f_0 + tf_1$ is the desired homotopy and V is the span of $f(x)$ in $\mathrm{End}(H)$ over the vertices x in X. □

Lemma 12.6. *There exists an orthogonal decomposition* $H = H_1 \oplus H_2 \oplus H_3$ *such that (i)* $\alpha(H_1) \perp H_3$ *for all* $\alpha \in V$*, (ii)* H_1 *is infinite dimensional, and (iii) there is an isomorphism* $T\colon H_1 \to H_3$*.*

Proof. Let $P_1 \subset H$ be a line and $P_2 \subset H$ a finite dimensional orthogonal subspace so that $\alpha(P_1) \subset P_1 \oplus P_2$ for all $\alpha \in V$. Let P_3 be a line orthogonal to $P_1 \oplus P_2$. We begin an iterative process. Choose a line Q_1 orthogonal to $P_1 \oplus P_2 \oplus P_3$ and a finite dimensional subspace Q_2 such that the sum $P_1 \oplus Q_1 \oplus P_2 \oplus Q_2$ is orthogonal and contains $\alpha(Q_1)$ for all $\alpha \in V$. Let Q_3 be a line orthogonal to $P_1 \oplus Q_1 \oplus P_2 \oplus Q_2 \oplus P_3$. Set $P_i^{(1)} = P_i \oplus Q_i$, $i = 1, 2, 3$. Then $\alpha(P_1^{(1)}) \subset P_1^{(1)} \oplus P_2^{(1)}$, $\dim P_1^{(1)} > \dim P_1$, and $P_1^{(1)} \cong P_3^{(1)}$. Iterate to find $P_i^{(1)} \subset P_i^{(2)} \subset \cdots$ with these properties. Set

$$H_i = \overline{\bigcup_{k=1}^{\infty} P_i^{(k)}}, \qquad i = 1, 3, \tag{12.7}$$

and choose $H_2 = (H_1 \oplus H_3)^\perp$. □

Lemma 12.8. *There exists a homotopy* $f_1 \simeq f_3$ *so that* $f_3\big|_{H_1} = \mathrm{id}_{H_1}$*.*

Proof. For $x \in X$ let $H_x = \big(f_1(x)H_1 \oplus H_3\big)^\perp$. The identity transformation is connected to the rotation

$$\begin{aligned} H_1 \oplus H_x \oplus H_1 &\longrightarrow H_1 \oplus H_x \oplus H_1 \\ \xi \oplus \eta \oplus \zeta &\longmapsto (-\zeta) \oplus \eta \oplus \xi \end{aligned} \tag{12.9}$$

by a path (of unitaries). Conjugate this path by

$$f_1(x) \oplus \mathrm{id}_{H_x} \oplus T\colon H_1 \oplus H_x \oplus H_1 \longrightarrow f_1(x)H_1 \oplus H_x \oplus H_3 \tag{12.10}$$

to obtain a path from id_H to

$$\begin{aligned} \varphi_x\colon f_1(x)H_1 \oplus H_x \oplus H_3 &\longrightarrow f_1(x)H_1 \oplus H_x \oplus H_3 \\ f_1(x)\xi \oplus \eta \oplus T\zeta &\longmapsto -f_1(x)\zeta \oplus \eta \oplus T\xi \end{aligned} \tag{12.11}$$

Set $f_2(x) = \varphi_x^{-1} f_1(x)$. Then f_2 is continuous, $f_1 \simeq f_2$, and $f_2(x)\big|_{H_1} = -T\colon H_1 \to H_3$ for all $x \in X$. Now compose with the rotation

$$\begin{aligned} H_1 \oplus H_2 \oplus H_3 &\longrightarrow H_1 \oplus H_2 \oplus H_3 \\ \xi \oplus \lambda \oplus T\zeta &\longmapsto -\zeta + \lambda \oplus T\xi \end{aligned} \tag{12.12}$$

to obtain f_3 homotopic to f_2 with $f_3\big|_{H_1} = \mathrm{id}_{H_1}$. □

Proof of Theorem 12.1. We execute the *Eilenberg swindle*. First, relative to the orthogonal decomposition $H = H_1^\perp \oplus H_1$ we have

$$f_3(x) = \begin{pmatrix} u(x) & 0 \\ * & 1 \end{pmatrix}, \tag{12.13}$$

where '1' denotes the identity operator. By a simple homotopy multiplying the operator '$*$' by t we move to a homotopic family

$$f_4(x) = \begin{pmatrix} u(x) & 0 \\ 0 & 1 \end{pmatrix}. \tag{12.14}$$

Since H_1 is infinite dimensional and separable we can write it as a countable orthogonal direct sum

$$H_1 = K_1 \oplus K_2 \oplus K_3 \oplus \cdots \tag{12.15}$$

of infinite dimensional subspaces, each equipped with an isomorphism to $H_1^\perp$. Now the path of operators

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} u & \\ & 1 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} u^{-1} & \\ & 1 \end{pmatrix} \tag{12.16}$$

on $H_1^\perp \oplus H_1^\perp$ begins at $t = 0$ with the identity operator and concludes at $t = 1$ with $\left(\begin{smallmatrix} u^{-1} & \\ & u \end{smallmatrix}\right)$. Exchanging the roles of u, u^{-1} we obtain a path of operators from $\left(\begin{smallmatrix} u & \\ & u^{-1} \end{smallmatrix}\right)$ to the identity. Therefore, we obtain a homotopy

$$f_4 = \begin{pmatrix} u &&&& \\ & 1 &&& \\ && 1 && \\ &&& 1 & \\ &&&& \ddots \end{pmatrix} \simeq \begin{pmatrix} u &&&& \\ & u^{-1} &&& \\ && u && \\ &&& u^{-1} & \\ &&&& \ddots \end{pmatrix} \simeq \begin{pmatrix} 1 &&&& \\ & 1 &&& \\ && 1 && \\ &&& 1 & \\ &&&& \ddots \end{pmatrix} \tag{12.17}$$

of operators on $H = H_1^\perp \oplus K_1 \oplus K_2 \oplus K_3 \oplus \cdots$. □

Stiefel manifolds and classifying spaces for principal bundles

Recall first the definition.

Definition 12.18. Let G be a Lie group. A *principal G bundle* is a fiber bundle $\pi\colon P \to M$ over a smooth manifold M equipped with a right G-action $P \times G \to P$ which is simply transitive on each fiber.

The hypothesis that π is a fiber bundle means it admits local trivializations. For a *principal* bundle a local trivialization is equivalent to a local section. In one direction, if $U \subset M$ and $s\colon U \to P$ is a section of $\pi\big|_U\colon P\big|_U \to U$, then there is an induced local trivialization

$$\begin{aligned} \varphi\colon U \times G &\longrightarrow P \\ x, g &\longmapsto s(x)\cdot g \end{aligned} \tag{12.19}$$

where '$\cdot$' denotes the G-action on P.

(12.20) *From vector bundles to principal bundles and back.* Let $\pi\colon E \to M$ be a vector bundle of rank k. Assume for definiteness that π is a *real* vector bundle. There is an associated principal $GL_k(\mathbb{R})$-bundle $\mathcal{B}(E) \to M$ whose fiber at $x \in M$ is the spaces of bases $b\colon \mathbb{R}^k \xrightarrow{\cong} E_x$. These fit together into a principal bundle which admits local sections: a local section of the principal bundle $\mathcal{B}(E) \to M$ is a local trivialization of the vector bundle $E \to M$. Conversely, if $P \to M$ is a principal $G = GL_k(\mathbb{R})$-bundle, then there is an *associated* rank k vector bundle $E \to M$ defined as

$$E = P \times \mathbb{R}^k/G, \tag{12.21}$$

where the right G-action on $P \times \mathbb{R}^k$ is

$$(p,\xi)\cdot g = (p\cdot g, g^{-1}\xi), \qquad p \in P, \quad \xi \in \mathbb{R}^k, \quad g \in G, \tag{12.22}$$

and we use the standard action of $GL_k(\mathbb{R})$ on $\mathbb{R}^k$ to define $g^{-1}\xi$.

A section of the associated bundle E is a G-equivariant map $s\colon P \to \mathbb{R}^k$. Note that G acts on the *right* on both P and $\mathbb{R}^k$. A special case of this construction is the frame bundle of a smooth k-dimensional manifold M, a case the reader should thing through carefully if this is new.

(12.23) *More general associated bundles.* Let $P \to M$ be a principal G-bundle and F a space equipped with a *left* G-action. Then there is an associated fiber bundle with fiber F; the total space is the quotient

$$F_P = (P \times F)/G = P \times_G F \tag{12.24}$$

where $g \in G$ acts on the right of $(p, f) \in P \times F$ to give $(p\cdot g, g^{-1}\cdot f)$. A section of the associated bundle is a G-equivariant function $P \to F$. The fibers of $F_P \to M$ are identified with F only up to the action of G. The principal bundle controls this uncertainty. More precisely, each point $p \in P_x$ gives an identification of the fiber $(F_P)_x$ with F. In that sense points of a principal bundle are generalized bases for all associated fiber bundles, and it is the principal bundle which controls the geometry and topology.

This geometric viewpoint on fiber bundles was advocated by Steenrod [St].

(12.25) *Fiber bundles with contractible fiber.* We quote the following general proposition in the theory of fiber bundles.

Proposition 12.26. *Let $\pi\colon \mathcal{E}\to M$ be a fiber bundle whose fiber F is contractible and a metrizable topological manifold, possibly infinite dimensional. Assume that the base M is metrizable. Then π admits a section. Furthermore, if $\mathcal{E}, M, F$ all have the homotopy type of a CW complex, then π is a homotopy equivalence.*

See [Pa2] for a proof of the first assertion. The last assertion follows from the long exact sequence of homotopy groups and Whitehead's theorem, stated after Definition 12.4. The takeaway, after stripping off the technical hypotheses, is that a fiber bundle with contractible fibers is a homotopy equivalence. But don't forget that there are technical hypotheses!

(12.27) *Classifying maps for principal bundles.* Now we characterize universal principal bundles.

Theorem 12.28. *Let G be a Lie group. Suppose $\pi^{\text{univ}}\colon P^{\text{univ}}\to B$ is a principal G-bundle and P^{univ} is a contractible metrizable topological manifold.*[2] *Then for any continuous principal G-bundle $P\to M$ with M metrizable, there is a classifying diagram*

(12.29)
$$\begin{tikzcd} P \arrow[r, "\tilde{\varphi}"] \arrow[d] & P^{\text{univ}} \arrow[d] \\ M \arrow[r, "\varphi"] & B \end{tikzcd}$$

In the commutative diagram (12.29) the map $\tilde{\varphi}$ commutes with the G-actions on P, P^{univ}, i.e., it is a map of principal G-bundles.

Proof. A G-map $\tilde{\varphi}$ is equivalently a section of the associated fiber bundle

$$(P\times P^{\text{univ}})/G\to M \tag{12.30}$$

formed by taking the quotient by the diagonal right G-action. The fiber of the bundle (12.30) is P^{univ}. Sections exist by Proposition 12.26, since P^{univ} is contractible. $\square$

(12.31) *Stiefel manifolds.* Let H be a separable (complex) Hilbert space. Introduce the infinite dimensional *Stiefel manifold*

$$St_k(H)=\{b\colon \mathbb{C}^k\to H : b \text{ is an isometry}\}. \tag{12.32}$$

It is an open subset of the linear space $\mathrm{Hom}(\mathbb{C}^k, H)\cong H\oplus\cdots\oplus H$, which we give the topology of a Hilbert space. Then the open subset $St_k(H)$ is a Hilbert manifold. There is an obvious projection

$$\pi\colon St_k(H)\longrightarrow Gr_k(H) \tag{12.33}$$

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[2] We allow an infinite dimensional manifold modeled on a Hilbert space, say.

to the Grassmannian

$$Gr_k(H) = \{W \subset H : \dim W = k\}. \tag{12.34}$$

which maps b to its image $b(\mathbb{C}^k) \subset H$. We leave the reader to check that π is smooth. In fact, π is a principal bundle with structure group the unitary group U_k.

Theorem 12.35. *$St_k(H)$ is contractible.*

Proof. The unitary group $U(H)$ acts transitively on $St_k(H)$ by left composition. The stabilizer of a k-frame $b_0\colon \mathbb{C}^k \to H$ is the unitary group of the orthogonal complement $b(\mathbb{C}^k)^\perp$. Both unitary groups are contractible, by Kuiper (Corollary 12.3). So too is the quotient homogeneous space, which is diffeomorphic to $St_k(H)$. ☐

Corollary 12.36. *The bundle (12.33) is a universal U_k-bundle.*

Remark 12.37. There is a simpler proof that $St_k(H)$ is contractible based on the contractibility of the unit sphere in H.

(12.38) *Other Lie groups.* Let G be a *compact* Lie group. (Note G need not be connected.) The *Peter-Weyl theorem* asserts that there is an embedding $G \subset U_k$ for some $k > 0$. Let $EG = St_k(H)$ be the Stiefel manifold for a *complex* separable Hilbert space H. Then the restriction of the free $GL_k(\mathbb{C})$-action to G is also free; let BG be the quotient. It is a Hilbert manifold, and

$$EG \longrightarrow BG \tag{12.39}$$

is a universal principal G-bundle, by Theorem 12.28.

This gives Hilbert manifold models for the classifying space of any compact Lie group.

Fredholms and Clifford algebras

Fix a $\mathbb{Z}/2\mathbb{Z}$-graded complex Hilbert space $H = H^0 \oplus H^1$ for which both H^0 and H^1 are infinite dimensional. With few modifications what we do applies to real Hilbert spaces, but we defer that to the next lecture.

Remark 12.40. Following **(11.25)** strictly, *which I strongly recommend* [DF], leads to some awkward conventions [DM, §4.4]. First, H should have an inner product $\langle -, - \rangle$, which we assume is even so that $H^0 \perp H^1$. But then the sign rule implies that $\langle \xi, \xi \rangle \in i\mathbb{R}$ for $\xi \in H^1$. Worse, an odd skew-adjoint operator on H has eigenvalues which are multiples of a primitive eighth root of unity; they are neither purely real nor purely imaginary. We will take the easy way out to conform with the (old) literature, which does not follow the Koszul sign rule. So each of H^0, H^1 is a Hilbert space with a usual inner product, and a continuous odd skew-adjoint operator A on $H = H^0 \oplus H^1$ has the form

$$A = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix} \tag{12.41}$$

where $T\colon H^0 \to H^1$ is continuous and $T^*\colon H^1 \to H^0$ is its usual adjoint relative to the inner products on H^0, H^1.

Definition 12.42. Let $H = H^0 \oplus H^1$ be a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space. The following spaces of operators are endowed with the norm topology.

(i) $\mathrm{Fred}_0(H)$ is the space of odd skew-adjoint Fredholm operators on H.

(ii) For $n > 0$, $\mathrm{Fred}_{-n}(H) \subset \mathrm{Fred}_0(\mathrm{C}\ell^{\mathbb{C}}_{-n} \otimes H)$ is the subspace[3] of operators which (graded) commute with the left action of $\mathrm{C}\ell^{\mathbb{C}}_{-n}$.

We endow $\mathrm{C}\ell^{\mathbb{C}}_{-n}$ with the Hermitian inner product which renders products $e_{i_1} \cdots e_{i_q}$ of basis elements orthonormal; then the pin group Pin_{-n} acts unitarily and the generators e_i act by odd skew-adjoint unitary isomorphisms. An odd operator A on $\mathrm{C}\ell^{\mathbb{C}}_{-n} \otimes H$ commutes with $\mathrm{C}\ell^{\mathbb{C}}_{-n}$ if and only if $Ae_i = -e_i A$ for $i = 1, \dots, n$.

Remark 12.43. The matrix (12.41) makes clear that $\mathrm{Fred}_0(H)$ is canonically identified with $\mathrm{Fred}(H^0, H^1)$, the space studied in Lecture 1. There is also an ungraded interpretation of $\mathrm{Fred}_{-n}(H)$. First observe that the $n-1$ elements $f_i = e_i e_n,\ i = 1, \dots, n-1$ generate a Clifford algebra isomorphic to $\mathrm{C}\ell^{\mathbb{C}}_{-(n-1)}$. Then if $A \in \mathrm{Fred}_{-n}(H)$ the restriction of $e_n A$ to the even subspace of $\mathrm{C}\ell^{\mathbb{C}}_{-n} \otimes H$ is a skew-adjoint Fredholm operator whcih anticommutes with the f_i. This is the form of $\mathrm{Fred}_{-n}(H)$ which is most studied in [AS3].

(12.44) *Periodicity of* $\mathrm{Fred}_{-n}(H)$. Recall from Theorem 11.42 that there is an isomorphism

$$\mathrm{C}\ell^{\mathbb{C}}_{-2} \cong \mathrm{End}(\mathbb{S}) \cong \mathbb{S} \otimes \mathbb{S}^* \tag{12.45}$$

for a complex super vector space of dimension $1|1$.

Proposition 12.46. *The map*

$$\begin{aligned} \mathrm{Fred}_0(\mathbb{S}^* \otimes H) &\longrightarrow \mathrm{Fred}_{-2}(H) \subset \mathrm{Fred}_0(\mathbb{S} \otimes \mathbb{S}^* \otimes H) \\ A &\longmapsto \mathrm{id}_{\mathbb{S}} \otimes A \end{aligned} \tag{12.47}$$

is a homeomorphism.

This follows since the only central endomorphisms of $\mathbb{S}$ are multiples of $\mathrm{id}_{\mathbb{S}}$.

Kuiper's Theorem 12.1 implies that there is a contractible space of isomorphisms $H \xrightarrow{\cong} \mathbb{S}^* \otimes H$.

Corollary 12.48. *Up to a contractible choice the isomorphism* (12.45) *leads to a homeomorphism*

$$\mathrm{Fred}_0(H) \xrightarrow{\cong} \mathrm{Fred}_{-2}(H). \tag{12.49}$$

The same argument leads to contractible spaces of isomorphisms

$$\begin{aligned} \mathrm{Fred}_0(H) \cong \mathrm{Fred}_{-2}(H) \cong \mathrm{Fred}_{-4}(H) \cong \cdots \\ \mathrm{Fred}_{-1}(H) \cong \mathrm{Fred}_{-3}(H) \cong \mathrm{Fred}_{-5}(H) \cong \cdots \end{aligned} \tag{12.50}$$

---

[3] We add another condition in the next lecture to get rid of contractible components.

It is natural to extend the definition

$$\mathrm{Fred}_n(H) \subset \mathrm{Fred}_0(\mathrm{C}\ell_n^{\mathbb{C}} \otimes H), \qquad n \in \mathbb{Z}, \tag{12.51}$$

to allow positive integers as well. Note that the generators e_i of the Clifford algebra are odd *self*-adjoint if $n > 0$. Then (12.50) extends in both directions:

$$\begin{gathered} \cdots \cong \mathrm{Fred}_2(H) \cong \mathrm{Fred}_0(H) \cong \mathrm{Fred}_{-2}(H) \cong \cdots \\ \cdots \cong \mathrm{Fred}_1(H) \cong \mathrm{Fred}_{-1}(H) \cong \cdots \end{gathered} \tag{12.52}$$

(12.53) *Atiyah-Singer loop map.* For each $n > 0$ define

$$\begin{aligned} \alpha\colon\ \mathrm{Fred}_{-n}(H) &\longrightarrow \Omega\,\mathrm{Fred}_{-(n-1)}(\mathrm{C}\ell_{-1}^{\mathbb{C}} \otimes H) \\ A &\longmapsto \bigl(t \mapsto e_n \cos \pi t + A \sin \pi t\bigr), \qquad 0 \leqslant t \leqslant 1. \end{aligned} \tag{12.54}$$

The Clifford algebra $\mathrm{C}\ell_{-1}^{\mathbb{C}}$ in the codomain has generator e_n. Note that the operators in the domain and codomain both act on the same Hilbert space $\mathrm{C}\ell_{-n}^{\mathbb{C}} \otimes H$. The codomain consists of paths from e_n to $-e_n$, the endpoints fixed independent of A. The space of such paths is homotopy equivalent to the based loop space with any basepoint. (Recall **(9.32)**.) The reader should check that $e_n \cos \pi t + A \sin \pi t$ is indeed Fredholm, and in fact is invertible if $t \neq 1/2$.

Theorem 12.55 ([AS3])**.** *α is a homotopy equivalence.*

Then algebraic Bott periodicity in the form Corollary 12.48 combines with Theorem 12.55 to prove Bott periodicity (Theorem 9.52).

Corollary 12.56. *There is a homotopy equivalence*

$$\Omega^2\,\mathrm{Fred}_0(H) \simeq \mathrm{Fred}_0(H) \tag{12.57}$$

We sketch a proof of Theorem 12.55 in the next few lectures.

Atiyah-Bott-Shapiro construction (Bonus material)

We did not have time in lecture for this important construction, which we explain in the next section is "adjoint" to the Atiyah-Singer loop map in some sense. Clifford modules implement the suspension in the definition (9.49) of the negative K-groups. The constructions work equally over $\mathbb{R}$ and $\mathbb{C}$, though our notation assumes the latter. The picture of suspension is (9.54), and the construction applies as well to the twisted suspension, or Thom complex, in (9.55).

(12.58) *A family of operators parametrized by a real vector space.* Let (V, Q) be a real quadratic vector space with Q negative definite and $\mathbb{S} = \mathbb{S}^0 \oplus \mathbb{S}^1$ a $\mathrm{C}\ell(V,Q)$-module. Then $\xi \in V$ determines an odd endomorphism $c(\xi) \in (\mathrm{End}\,\mathbb{S})^1$. Since Q is negative definite we can choose a compatible inner product on $\mathbb{S}$; then $c(\xi)$ is skew-adjoint. The family of operators $\xi \mapsto c(\xi)$ is supported at $0 \in V$, i.e., the operator $c(\xi)$ is invertible if $\xi \neq 0$. This defines an element in the relative K-theory group $K^0(V, V\backslash 0) \cong K^0(V, V\backslash B_r(0))$, as in **(9.33)**.

(12.59) *A vector bundle over the sphere.* Let $V \oplus \mathbb{R}$ have the direct sum inner product which is negative definite. The sphere $S(V \oplus \mathbb{R})$ is the 1-point compactification of V and is naturally decomposed as

$$S(V \oplus \mathbb{R}) = D^+ \cup_{S(V)} D^-, \qquad D^{\pm} = \{(\xi, t) \in S(V \oplus \mathbb{R}) : \pm t > 0\}. \tag{12.60}$$

We identify D^+ as the closed unit ball in V. Glue the trivial bundles $D^+ \times \mathbb{S}^0$ and $D^- \times \mathbb{S}^1$ using the isomorphisms $c(\xi)\colon \mathbb{S}^0 \to \mathbb{S}^1$ for $\xi \in S(V)$. This gives a vector bundle over $S(V \oplus \mathbb{R})$ whose K-theory class agrees with the class constructed in **(12.58)**. Note that the vector bundle comes trivialized on D^-, which is the 1-point compactification of the complement of the open unit ball in V.

(12.61) *The clutching function.* By Proposition 3.26 the K-theory class is determined by the homotopy class of the clutching function

$$\begin{aligned} S(V) &\longrightarrow \operatorname{Aut}(\mathbb{S}^0) \\ \xi &\longmapsto c(\xi_0)^{-1}c(\xi), \end{aligned} \tag{12.62}$$

where $\xi_0 \in S(V)$ is a basepoint.

Note that **(12.58)**, **(12.59)**, and **(12.61)** describe three geometric objects which represent the same information, all defined from a Clifford module.

Example 12.63. Take $V = \mathbb{R}^2$ and $\mathbb{S} = \mathbb{C}^{1|1}$ the complex Clifford module with

$$c(e_1) = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \qquad c(e_2) = \begin{pmatrix} & i \\ i & \end{pmatrix}. \tag{12.64}$$

Then

$$c(\cos\theta\, e_1 + \sin\theta\, e_2) = \begin{pmatrix} & -e^{-i\theta} \\ e^{i\theta} & \end{pmatrix} \tag{12.65}$$

and the product with $c(e_1)^{-1}$ on the left is $\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$. Restricted to $\mathbb{S}^0$ we get the clutching function for the hyperplane bundle over S^2.

(12.66) *Modding out by modules which extend.* Suppose $\mathbb{S}$ extends to a $C\ell(V \oplus \mathbb{R})$-module. Then we use the extra generator e_{n+1} to make a homotopy of clutching functions (12.62):

$$c(\xi_0)^{-1}\left[\cos \pi t/2\, c(\xi) + \sin \pi t/2\, c(e_{n+1})\right], \qquad 0 \leqslant t \leqslant 1. \tag{12.67}$$

This shows that the bundle we obtain over $S(V \oplus \mathbb{R})$ is trivializable. ABS [ABS, §11] use this to define a map from the graded ring of Clifford modules modulo those which extend to the reduced K-theory of the sphere—working both over the reals and the complexes—and they prove that this map is an isomorphism of graded rings. The Atiyah-Singer theorem [AS3] we are proving is a generalization.

(12.68) *Parametrized version; twisted K-theory.* Let $V \to X$ be a real vector bundle equipped with a family Q of negative definite quadratic forms. Then we obtain a bundle $\mathrm{C}\ell(V,Q) \to X$ of Clifford algebras. Let $S \to X$ be a $\mathrm{C}\ell(V,Q)$-module, that is, a super vector bundle with an action of $\mathrm{C}\ell(V,Q)$ fiberwise. The previous constructions can be carried out fiber by fiber to construct a class in the Thom space $K^0(X^V)$. This can be identified with a K-theory class of degree $-n$ in the base, but in general it lives in *twisted* K-theory, a topic we will return to shortly.

Comparison of ABS and AS suspension maps (Bonus material)

Consider the ABS construction **(12.58)** in a special case. Suppose $E \to X$ is a complex vector bundle. The ABS construction yields a complex vector bundle $\underline{\mathrm{C}\ell^{\mathbb{C}}_{-1}} \otimes pr_1^*E \to X \times \mathbb{R}$ equipped with the family of odd skew-adjoint operators which at $(x,s) \in X \times \mathbb{R}$ is Clifford multiplication multiplication by

$$se_1 \otimes \mathrm{id}_E\,. \tag{12.69}$$

The operator is invertible except at $s=0$, when it is the zero operator. This family of operators is clutching data for a vector bundle over ΣX_+.

The AS map (12.54) is a sort of adjoint in the world of Fredholm operator representatives for K-theory. The relevant special case of (12.54) is

$$\begin{aligned} \alpha\colon\ \mathrm{Fred}_{-1}(H) &\longrightarrow \Omega\,\mathrm{Fred}_0(\mathrm{C}\ell^{\mathbb{C}}_{-1}\otimes H) \\ A &\longmapsto \bigl(e_1\cos\pi t + A\sin\pi t,\ 0\leqslant t\leqslant 1\bigr). \end{aligned} \tag{12.70}$$

Write

$$e_1\cos\pi t + A\sin\pi t = e_1(\cos\pi t - e_1A\sin\pi t) \tag{12.71}$$

and note that $(e_1A)^* = A^*e_1^* = (-A)(-e_1) = Ae_1 = -e_1A$ is skew-adjoint. Since $\cos\pi t$ times the identity operator is invertible self-adjoint except at $t=1/2$, it follows that the operator in (12.70) is invertible except at $t=1/2$. At $t=1/2$ it is the Fredholm operator A. Recalling that invertible operators are a “fat basepoint” in Fredholms, we can homotop the family (12.70) to a family of Fredholm operators

$$s \longmapsto se_1\otimes\mathrm{id} + \mathrm{id}\otimes A, \qquad s\in\mathbb{R}, \tag{12.72}$$

which is more parallel to the ABS construction. It is in this sense that the AS loop map is adjoint to the ABS suspension.

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Lecture 13: Topology of skew-adjoint Fredholm operators

We present background and many of the ideas in the proof of Theorem 12.55, the key result in the proof of Bott periodicity given by Atiyah and Singer [AS3]; the last part of the proof is deferred to the next lecture. In this lecture we get as far as explaining the contractible components of skew-adjoint Fredholms and so making complete Definition 12.42 for n odd. We emphasize the two key deformations in the proof: the deformation retraction to unitary operators (12.2) and modding out by the contractible space of compact operators. For the latter we need to know that a fiber bundle $\mathcal{E} \to M$ with contractible fibers is a homotopy equivalence, conditions for which are set out in Proposition 12.26. The real case is parallel to the complex case. We will not present complete proofs, but highlight most of the main ideas as a reader's guide to [AS3]. We work in an ungraded (non-super) situation which contains most of the key ideas. Along the way we review basic facts about compact operators and the relation to Fredholm operators. We also introduce Banach Lie groups which have the homotopy type of BGL_∞; see [F3] for more along these lines. In the first part of this lecture we give some context; see the end of Lecture 9 for additional relevant material. In particular, we describe the geometric model of K-theory that we are developing, something very important for the rest of the course. (Some of the technical background, especially for the equivariant case that we will use later, may be found in the appendix to [FHT1].)

We continue with some of the notation from previous lectures.

The periodic K-theory spectra

We present the definition of a spectrum and its antecedents: prespectra and Ω-prespectra. These definitions and terms vary in the literature. Spectra are the basic objects of *stable homotopy theory*.

Definition 13.1.

(i) A *prespectrum* $T_\bullet$ is a sequence $\{T_n\}_{n\in\mathbb{Z}^{>0}}$ of pointed spaces and maps $s_n\colon \Sigma T_n \to T_{n+1}$.

(ii) An Ω*-prespectrum* is a prespectrum $T_\bullet$ such that the adjoints $t_n\colon T_n \to \Omega T_{n+1}$ of the structure maps are weak homotopy equivalences.

(iii) A *spectrum* is a prespectrum $T_\bullet$ such that the adjoints $t_n\colon T_n \to \Omega T_{n+1}$ of the structure maps are homeomorphisms.

Obviously a spectrum is an Ω-prespectrum is a prespectrum. We can take the sequence of pointed spaces $T_{n_0}, T_{n_0+1}, T_{n_0+2}, \dots$ to begin at any integer $n_0 \in \mathbb{Z}$. If $T_\bullet$ is a *spectrum* which begins at n_0, then we can extend to a sequence of pointed spaces T_n defined for *all* integers n by setting

$$T_n = \Omega^{n_0-n}T_{n_0}, \qquad n < n_0. \tag{13.2}$$

Note that each T_n, in particular T_0, is an *infinite* loop space:

$$T_0 \simeq \Omega T_1 \simeq \Omega^2 T_2 \simeq \cdots \tag{13.3}$$

Example 13.4. Let X be a pointed space. The *suspension prespectrum* of X is defined by setting $T_n = \Sigma^n X$ for $n \geqslant 0$ and letting the structure maps s_n be the identity maps. In particular, for $X = S^0$ we obtain the sphere prespectrum with $T_n = S^n$.

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(13.5) *Spectra from prespectra.* Associated to each prespectrum $T_\bullet$ is a spectrum[1] $LT_\bullet$ called its *spectrification.* It is easiest to construct in case the adjoint structure maps $t_n\colon T_n \to \Omega T_{n+1}$ are inclusions. Then set $(LT)_n$ to be the colimit of

$$T_n \xrightarrow{t_n} \Omega T_{n+1} \xrightarrow{\Omega t_{n+1}} \Omega^2 T_{n+2} \longrightarrow \cdots \tag{13.6}$$

which is computed as an union. For the suspension spectrum of a pointed space X the 0-space is

$$(LT)_0 = \operatorname*{colim}_{\ell\to\infty} \Omega^\ell \Sigma^\ell X, \tag{13.7}$$

which is usually denoted QX.

(13.8) *Homotopy and homology of prespectra.* Let $T_\bullet$ be a prespectrum. Define its homotopy groups by

$$\pi_n(T) = \operatorname*{colim}_{\ell\to\infty} \pi_{n+\ell} T_\ell, \tag{13.9}$$

where the colimit is over the sequence of maps

$$\pi_{n+\ell} T_\ell \xrightarrow{\pi_{n+\ell} t_\ell} \pi_{n+\ell} \Omega T_{\ell+1} \xrightarrow{\text{adjunction}} \pi_{n+\ell+1} T_{\ell+1} \tag{13.10}$$

For an Ω-prespectrum the composition (13.10) is an isomorphism and there is no need for the colimit. Similarly, define the homology groups as the colimit

$$H_n(T) = \operatorname*{colim}_{\ell\to\infty} \widetilde{H}_{n+\ell} T_\ell, \tag{13.11}$$

where $\widetilde{H}$ denotes the reduced homology of a pointed space. We might be tempted to define the cohomology similarly, but that does not work.[2]

(13.12) *Cohomology theory of a spectrum.* A prespectrum $T_\bullet$ determines a cohomology theory h_T on CW complexes and other nice categories of spaces. Assume for simplicity that $T_\bullet$ is an Ω-prespectrum. Then the reduced cohomology of a pointed space X is

$$\tilde{h}^n_T(X) = [X, T_n], \tag{13.13}$$

where we take homotopy classes of pointed maps. All the computational tools (long exact sequences, spectral sequences, etc.) work for generalized cohomology theories. One account is [DaKi].

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[1] The notation 'L' indicates 'left adjoint'.

[2] Homotopy and homology commute with colimits, but cohomology does not: there is a derived functor $\lim^1$ which measures the deviation.

(13.14) *Periodic K-theory spectra.* Theorem 12.55 tells that Fredholm operators give an Ω-prespectrum whose n^{th} space is $\text{Fred}_n(H)$ and whose structure maps are the adjoint of α in (12.54). Bott periodicity (Corollary 12.56) tells that this spectrum is 2-periodic. It is the periodic complex topological K-theory spectrum; the corresponding cohomology groups of a space X are denoted $K^n(X)$. The real version of the theorems gives an Ω-prespectrum whose n^{th} space is $\text{Fred}_n(H_{\mathbb{R}})$; the structure maps are the adjoint of α. Now the spectrum is 8-periodic and the corresponding cohomology groups are $KO^n(X)$, the real K-theory groups.

The geometric model of K-theory

Our point of view in this course is to develop a *geometric model* of K-theory and to see it arise in geometry and physics. Were we to discuss real singular cohomology in place of K-theory the geometric model of interest is restricted to smooth manifolds: the de Rham complex. A closed differential form on a smooth manifold determines a real cohomology class, and this brings topological methods into differential geometry. Absent the geometric model we would not be able to recognize and use the topology underlying Chern-Weil forms, a symplectic form, and many other closed forms which occur naturally in geometry.

The paper [FHT1], especially the appendix, contains much more about this model of K-theory, including the equivariant and twisted cases which we need later in the course.

(13.15) *Untwisted classes.* The model so far consists of a *fixed* super Hilbert space $H = H^0 \oplus H^1$ with the left action of a *fixed* superalgebra $\text{C}\ell_n^{\mathbb{C}}$, the complex Clifford algebra. (It is important that every irreducible Clifford module appear infinitely often in the Hilbert space, which is why in Definition 12.42 we write the Hilbert space as $\text{C}\ell_n^{\mathbb{C}} \otimes H'$ for an infinite dimensional separable Hilbert space with no Clifford action.) Then a K-theory class in $K^n(X)$ on a space X is represented geometrically by a family $X \to \text{Fred}_n(H)$ of odd skew-adjoint Fredholm operators on the fixed Hilbert space H which commute with the fixed algebra $\text{C}\ell_n^{\mathbb{C}}$.

(13.16) *Invertibles.* Kuiper's theorem asserts that the invertibles are a contractible subspace of $\text{Fred}_n(H)$; see **(9.32)** and **(9.33)**. Thus families of invertible operators determine the zero K-theory class, and more generally families of Fredholms $X \to \text{Fred}_n(H)$ determine a class relative to the subspace $A \subset X$ consisting of $x \in X$ such that $T(x)$ is invertible.

(13.17) *Twisted classes.* A more flexible model is obtained by allowing the Hilbert space H and the superalgebra $\text{C}\ell_n^{\mathbb{C}}$ to also depend on the point $x \in X$. As usual, we want them to vary in a locally trivial family, so form fiber bundles. We need to pay some point-set attention to define a locally trivial family of Hilbert spaces, though in fact Definition 1.12 goes over *verbatim.* We can generalize the standard Clifford algebras to central simple superalgebras and so consider fiber bundles of such equipped with a supermodule which is a Hilbert space bundle. Now the family of odd skew-adjoint Fredholms act on variable Hilbert spaces; they still commute with the superalgebra action. This is a geometric model for *twisted* K-theory which we will come to shortly.

Of course, there is a real version as well. This model extends nicely to groupoids, as we will discuss in a future lecture.

(13.18) *Finite rank vector bundles.* It is convenient to allow finite rank Hilbert bundles, i.e., ordinary finite rank vector bundles, via a simple construction. Let $E \to X$ be a finite rank complex vector bundle. Fix a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space $H = H^0 \oplus H^1$ whose homogeneous subspaces are infinite dimensional. Then $E \oplus \underline{H}$ is a Hilbert bundle and the constant family of Fredholms $0 \oplus \mathrm{id}_H$ has kernel the original vector bundle $E \to X$. In the sequel we use finite rank bundles as geometric representatives of K-theory with no further comment.

(13.19) *Warning about finite dimensional representatives of K^1.* Suppose $E = E^0 \oplus E^1 \to X$ is a finite rank complex super vector bundle with a $\mathrm{C}\ell^{\mathbb{C}}_{-1}$-module structure on each fiber. Let e_1 denote the action of the Clifford generator and ϵ the grading operator. Define $e_2 = ie_1\epsilon$. Then a simple computation shows that $e_2^2 = e_1^2 = -1$ and $e_1e_2 = -e_2e_1$. One interpretation is that $E \to X$ automatically extends to a bundle of $\mathrm{C}\ell^{\mathbb{C}}_{2}$-modules. Another is that e_2 is an *invertible* odd skew-adjoint endomorphism of finite rank $\mathrm{C}\ell^{\mathbb{C}}_{-1}$-modules, and furthermore the homotopy $t \mapsto te_2$ connects the zero operator on E to an invertible operator. Adding the identity on a fixed infinite dimensional $\mathrm{C}\ell^{\mathbb{C}}_{-1}$-module, as in **(13.18)**, we see that we get the zero element of $K^1(X)$ from this finite rank $\mathrm{C}\ell^{\mathbb{C}}_{-1}$-module over X.

A similar argument works for finite rank real Clifford modules except in degrees congruent to $0, 8 \pmod 8$.

Compact operators

Let H^0, H^1 be *ungraded* Hilbert spaces.

Definition 13.20.

(i) An operator $T\colon H^0 \to H^1$ is *finite rank* if the image $T(H^0) \subset H^1$ is a finite dimensional subspace.

(ii) An operator $T\colon H^0 \to H^1$ is *compact* if the closure $\overline{T\bigl(B(H^0)\bigr)} \subset H^1$ of the image of the unit ball is compact.

We topologize the set $\mathrm{cpt}(H^0, H^1) \subset \mathrm{Hom}(H^0, H^1)$ of compact operators by the norm topology. Some basic facts whose proof we leave to the reader: The space of compact operators is closed, and in fact is the closure of the set of finite rank operators. The composition of a bounded operator and a compact operator is compact. Hence the compact operators $\mathrm{cpt}(H) \subset \mathrm{End}(H)$ on a Hilbert space H form a closed 2-sided ideal in the space of bounded operators. A Hilbert space H is finite dimensional if and only if $\mathrm{id}_H\colon H \to H$ is compact.

We will prove a basic fact (Proposition 13.23) relating Fredholm and compact operators. It will be convenient to first prove that the closed range condition is superfluous in the definition (Definition 9.6) of a Fredholm operator.

Lemma 13.21. *Let H^0, H^1 be Hilbert spaces and $T\colon H^0 \to H^1$ a continuous linear map with finite dimensional kernel and finite dimensional cokernel. Then T is Fredholm.*

We use the fact that a finite dimensional subspace of a Hilbert space is closed.

Proof. The kernel $\ker T$ is a closed finite dimensional subspace of H^0 and the image of T equals that of T restricted to the closed subspace $(\ker T)^{\perp}$, so to prove that T is closed range we may assume that T is injective. Choose a finite dimensional complement V to $T(H^0) \subset H^1$; it exists since T has finite dimensional cokernel. Then

$$H^1 = T(H^0) \oplus V = V^\perp \oplus V, \tag{13.22}$$

where the latter is the direct sum of closed subspaces. Let π denote orthogonal projection onto $V^\perp$. Then πT is bijective and continuous, so by the open mapping theorem its inverse F is also continuous.

Now suppose $\{\xi_n\} \subset H^0$ is a sequence such that $T\xi_n \to \eta_\infty$ as $n \to \infty$. Then $\pi T \xi_n \to \pi\eta_\infty$. Apply TF to conclude that $T\xi_n \to TF\pi\eta_\infty$. It follows that $\eta_\infty = TF\pi\eta_\infty$, which shows that η_∞ lies in the image of T. This proves that T has closed range. □

Proposition 13.23. *A bounded operator $T\colon H^0 \to H^1$ is Fredholm if and only if there exist bounded operators $S, S'\colon H^1 \to H^0$ such that $\mathrm{id}_{H^0} - ST$ and $\mathrm{id}_{H^1} - TS'$ are compact.*

We can replace 'compact' by 'finite rank', as is clear from the proof, which also makes clear that we can choose $S' = S$. The operators S, S' are called *parametrices* for T.

Proof. If T is Fredholm, write $H^0 = \ker T \oplus (\ker T)^\perp$ and $H^1 = T(H^0)^\perp \oplus T(H^0)$ as orthogonal sums of closed subspaces. Since T restricted to $(\ker T)^\perp$ is an isomorphism onto $T(H^0)$, it has a continuous inverse on those spaces by the open mapping theorem. Define $S = S'$ to be the extension of this inverse by zero on $T(H^0)^\perp$.

Conversely, if the parametrices exist, restrict $\mathrm{id}_{H^0} - ST$ to $\ker T$ to deduce that $\mathrm{id}_{\ker T}$ is compact. Also, the operator $\mathrm{id}_{H^1} - TS'$ is compact and preserves $T(H^0)$, thus $\mathrm{id}_{\operatorname{coker} T}$ is compact. □

We turn now to groups—in fact, complex Banach Lie groups—and so switch notation to emphasize that the linear spaces of operators we have been using are Lie algebras:

$$\begin{aligned} \mathrm{Aut} &\longrightarrow GL \\ \mathrm{End} &\longrightarrow \mathfrak{gl} \\ \mathrm{cpt} &\longrightarrow \mathfrak{cpt} \end{aligned} \tag{13.24}$$

We often omit the Hilbert space from the notation for visual clarity.

Definition 13.25. $GL^{\mathrm{cpt}}(H) = \{P\colon H \to H$ such that P is invertible and $P - \mathrm{id}_H$ is compact$\}$.

GL^{cpt} is a Banach Lie group with Lie algebra $\mathfrak{cpt}$.

Fix a filtration $H_1 \subset H_2 \subset H_3 \subset \cdots$ of H by subspaces with $\dim H_n = n$ such that $\overline{\bigcup_{n=1}^{\infty} H_n} = H$. We can achieve this by choosing a countable basis (H is always assumed separable) and letting H_n be the span of the first n basis vectors. There is an induced increasing sequence of groups

$$GL(H_1) \subset GL(H_2) \subset GL(H_3) \subset \cdots \tag{13.26}$$

where the n^{th} group consists of invertible operators which are the identity on $H_n^\perp$.

Theorem 13.27 (Palais [Pa3]). *The inclusion* $\bigcup_{n=1}^{\infty} GL(H_n) \hookrightarrow GL^{\mathrm{cpt}}(H)$ *is a homotopy equivalence.*

The union of the groups (13.26), denoted GL_∞, has the colimit topology: a subset is open iff its intersection which each group in (13.26) is open. We encountered this group—rather its deformation retraction to the unitary subgroup (and with a different notation)—in Remark 3.32.

The Calkin algebra and its subgroups

Definition 13.28. The *Calkin algebra* of a Hilbert space H is the quotient algebra $\operatorname{End}(H)/\operatorname{cpt}(H)$.

Since the ideal of compact operators is closed, the Calkin algebra inherits a Banach space structure. (It is not only a Banach algebra but a C^*-algebra.) So we can talk about unitary elements, skew-adjoint elements, the spectrum of an element, etc. We usually use the notation '$\mathfrak{gl}/\mathfrak{cpt}$' to emphasize that the Calkin algebra is the Lie algebra of a Banach Lie group.

That group is the quotient GL/GL^{cpt}. The quotient map $GL \to GL/GL^{\mathrm{cpt}}$ is a principal bundle. (To prove that we need the existence of local sections, which follows from a theorem of Bartle and Graves, for example; see also [Pa2].) Kuiper's Theorem 12.1 asserts that GL is contractible. It follows from Theorem 12.28 that $GL \to GL/GL^{\mathrm{cpt}}$ is a universal bundle in the sense that it classifies principal GL^{cpt}-bundles (over metrizable bases). In particular, we have proved

Proposition 13.29. *The group* GL/GL^{cpt} *has the homotopy type* BGL_∞.

We now have two homotopy types on the table: GL_∞ and BGL_∞. There are also two homotopy types in (12.52). The main result implies a match if we replace BGL_∞ with $\mathbb{Z} \times BGL_\infty$.

Now we bring in the deformation retraction to unitaries (12.2). GL retracts to the contractible group U, as in Corollary 12.3, and GL^{cpt} retracts onto U^{cpt}, which has the same homotopy type. Denote

$$G = U/U^{\mathrm{cpt}}, \tag{13.30}$$

which is a deformation retract of GL/GL^{cpt} and thus has the homotopy type BGL_∞. We summarize the groups defined so far in the diagram

$$\begin{tikzcd}
U \arrow[r, hook, "\text{d.r.}"] \arrow[d, "U^{\mathrm{cpt}}"'] & GL \arrow[r, hook] \arrow[d, "GL^{\mathrm{cpt}}"'] & \mathfrak{gl} \arrow[d, "\pi"] \\
G = U/U^{\mathrm{cpt}} \arrow[r, hook, "\text{d.r.}"] & GL/GL^{\mathrm{cpt}} \arrow[r, hook] & \mathfrak{gl}/\mathfrak{cpt}
\end{tikzcd} \tag{13.31}$$

The labeled horizontal arrows are deformation retractions and the first two vertical arrows are principal bundles.

Let $(\mathfrak{gl}/\mathfrak{cpt})^\times \subset \mathfrak{gl}/\mathfrak{cpt}$ denote the Banach Lie group of invertible elements.

Lemma 13.32. GL/GL^{cpt} *is the identity component of* $(\mathfrak{gl}/\mathfrak{cpt})^\times$.

We leave the proof to the reader; it can be found in [F3]. The intuition is that GL/GL^{cpt} is a Banach Lie group with Lie algebra the Calkin algebra $\mathfrak{gl}/\mathfrak{cpt}$. So too is $(\mathfrak{gl}/\mathfrak{cpt})^{\times}$.

The following is a restatement of Proposition 13.23.

Proposition 13.33. $\pi^{-1}\bigl((\mathfrak{gl}/\mathfrak{cpt})^{\times}\bigr) = \mathrm{Fred} \subset \mathfrak{gl}$. *Also,* $\pi^{-1}\bigl(GL/GL^{\mathrm{cpt}}\bigr) = \mathrm{Fred}^{(0)}$ *is the space of Fredholm operators of numerical index zero.*

For the latter statement we use Corollary 9.26. We summarize in an expanded version of (13.31):

(13.34)
$$\begin{tikzcd}
U \arrow[r, hook, "\text{d.r.}"] \arrow[d, "U^{\mathrm{cpt}}"'] & GL \arrow[r, hook] \arrow[d, "GL^{\mathrm{cpt}}"'] & \mathfrak{gl} \arrow[d, "\pi"] & \mathrm{Fred} \arrow[l, hook'] \arrow[d, "\pi"] & \mathrm{Fred}^{(0)} \arrow[l, hook'] \arrow[d, "\pi"] \\
G = U/U^{\mathrm{cpt}} \arrow[r, hook, "\text{d.r.}"] & GL/GL^{\mathrm{cpt}} \arrow[r, hook] & \mathfrak{gl}/\mathfrak{cpt} & (\mathfrak{gl}/\mathfrak{cpt})^{\times} \arrow[l, hook'] & GL/GL^{\mathrm{cpt}} \arrow[l, hook']
\end{tikzcd}$$

The central vertical map is the quotient by $\mathfrak{cpt}$ which defines the Calkin algebra. To the right are restrictions of that quotient to Fredholm operators. To the left are principal bundles with total space a group.

Corollary 13.35. *The space* $\mathrm{Fred}^{(0)}$ *of Fredholm operators of index zero has the homotopy type* BGL_{∞} *and the space* Fred *of all Fredholm operators has the homotopy type* $\mathbb{Z} \times BGL_{\infty}$.

Since $\mathrm{Fred}_0(H)$ in (12.52) is isomorphic to the space of Fredholm operators on an ungraded Hilbert space, this determines the homotopy type of the spaces in the first line of (12.52).

Remark 13.36. It follows that G also has the homotopy type BGL_{∞}. We caution that [AS3] uses the symbol 'G' for the unitary retraction of $(\mathfrak{gl}/\mathfrak{cpt})^{\times}$, a group whose identity component is our 'G'.

Spaces of skew-adjoint Fredholm operators

Recall that the Lie algebra of the group of unitary operators is the space of skew-adjoint operators. (This is true in finite dimensions.) For any space of operators, or operator algebra, we use a "hat" to denote the subspace of skew-adjoint elements. Thus if we use $\mathcal{F} = \mathrm{Fred}$ for all Fredholm operators, then $\widehat{\mathcal{F}}$ is the notation for skew-adjoint Fredholms.

(13.37) *An ungraded version of* Fred_1. This is essentially a reprise of the text following (12.70). Let $H = H^0 \oplus H^1$ be a super Hilbert space and suppose $A \subset \mathrm{Fred}_1(H)$. Let e_1 denote the action of the Clifford generator. Then e_1A is even and skew adjoint: $(e_1A)^* = A^*e_1^* = (-A)(-e_1) = Ae_1 = -e_1A$. Let T denote its restriction to the even part of $C\ell_1^{\mathbb{C}} \otimes H$. The loop map (13.41) which appears below is essentially the Atiyah-Singer map (12.70); see (12.71).

(13.38) *Main theorem.* We now state the ungraded version of Theorem 12.55 whose proof we sketch in the next lecture.

Theorem 13.39 ([AS3])**.** *The space* $\widehat{\mathcal{F}}$ *has three components*

$$\widehat{\mathcal{F}} = \widehat{\mathcal{F}}_{+} \amalg \widehat{\mathcal{F}}_{-} \amalg \widehat{\mathcal{F}}_{*}. \qquad (13.40)$$

The components $\widehat{\mathcal{F}}_\pm$ *are contractible, and the map*

$$\begin{aligned} \alpha\colon \widehat{\mathcal{F}}_* &\longrightarrow \Omega\mathcal{F} \\ T &\longmapsto (\cos\pi t + T\sin\pi t,\ 0 \leqslant t \leqslant 1) \end{aligned} \tag{13.41}$$

is a homotopy equivalence.

Notice that the Atiyah-Singer loop map (13.41) has domain a space of skew-adjoint operators and codomain loops in a space related to a Lie group (Theorem 13.33). So we might expect—after retracting to unitaries which *do* have skew-adjoints as the Lie algebra—that (13.41) is closely related to exponentiation from a Lie algebra to a Lie group. It is.

(13.42) *The contractible components of skew-adjoint Fredholms.* In the diagram

$$\begin{tikzcd} & \widehat{\mathcal{F}} \arrow[d, "\mathfrak{cpt}"] \\ \widehat{G} \arrow[r, hook, "\mathrm{d.r.}"] & \widehat{GL/GL}^{\mathrm{cpt}} \end{tikzcd} \tag{13.43}$$

the vertical arrow is a fiber bundle with contractible fibers and the horizontal arrow is a deformation retraction. So $\widehat{G}$ is homotopy equivalent to $\widehat{\mathcal{F}}$. Now an element $x \in \widehat{G}$ is unitary skew-adjoint, so $xx^* = 1$ and $x = -x^*$, which implies $x^2 = -1$. It follows that $\operatorname{spec} x \subset \{+i -i\}$. Since the spectrum is nonempty, there are three possibilities. This decomposes $\widehat{G}$ into three disjoint subspaces which one can prove are components:

$$\widehat{G} = \widehat{G}_+ \amalg \widehat{G}_- \amalg \widehat{G}_* \tag{13.44}$$

Furthermore, the spaces $\widehat{G}_\pm$ each contain a single element $\pm i$. The decomposition (13.40) of $\widehat{\mathcal{F}}$ follows as does the contractibility of the two components consisting of skew-adjoint Fredholms whose *essential spectrum* is $\{+i\}$ or $\{-i\}$.

Remark 13.45. There are contractible components of $\operatorname{Fred}_n(H_{\mathbb{R}})$ in the real case if $n \equiv 1 \pmod 4$.

(13.46) *The noncontractible component of skew-adjoint Fredholms.* Replacing the spaces in (13.41) by homotopy equivalent spaces, we reduce the remaining part of Theorem 13.39 to the following.

Theorem 13.47. *The exponential map*

$$\begin{aligned} \epsilon\colon \widehat{G}_* &\longrightarrow \Omega G \\ x &\longmapsto (\exp\pi t x,\ 0 \leqslant t \leqslant 1) \end{aligned} \tag{13.48}$$

is a homotopy equivalence.

Now, as promised, the loop map is exponentiation, since $x^2 = -1$ implies

$$\cos\pi t + x\sin\pi t = \exp\pi t x. \tag{13.49}$$

We sketch a proof of Theorem 13.47 in the next lecture.

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Lecture 14: Proof of Bott periodicity (con't)

There are many spaces of operators in the proof, and it is confusing to follow at first. So we'll first try to sort things out a bit.

For a *super* Hilbert space $H_s = H^0 \oplus H^1$ we have a sequence of spaces of skew-adjoint odd Fredholm operators which exhibit just two homeomorphism types, as in (12.52). Letting H denote a (non-super) Hilbert space, and identifying $H = H^0 = H^1$, we can identify $\mathrm{Fred}_0(H_s)$ with $\mathcal{F} = \mathrm{Fred}(H)$ by identifying $\left(\begin{smallmatrix} 0 & -T^* \\ T & 0 \end{smallmatrix}\right)$ with T; see (12.41). Also, the argument after (12.71) shows that we can identify $\mathrm{Fred}_{-1}(H_s)$ with the space $\widehat{\mathcal{F}}$ of skew-adjoint Fredholms (on $(C\ell^{\mathbb{C}}_{-1} \otimes H_s)^0$) by identifying A with $e_1 A$ restricted to the even subspace. We proved in Corollary 13.35 that $\mathcal{F}$ has the homotopy type $\mathbb{Z} \times BGL_\infty$, which is then the homotopy type of all spaces in the first line of (12.52). Its loop space $\Omega\mathcal{F}$ has the homotopy type GL_∞. Theorem 13.39, whose proof we complete in this lecture, says that that is also the homotopy type of the nontrivial component $\widehat{\mathcal{F}_*}$ of $\widehat{\mathcal{F}}$. The identification with $\mathrm{Fred}_{-1}(H_s)$, which we re-define to denote only this nontrivial component, then determines the homotopy type of the spaces in the second line of (12.52) as GL_∞. This completes the proof of Bott periodicity, which in this form is Corollary 12.56.

Fiber bundles, fibrations, and quasifibrations

If $p\colon E \to B$ is a continuous map with contractible fibers we might like to conclude that p is a homotopy equivalence, but that is not always true. (Counterexample: Take $E = B = \mathbb{R}$ and p the identity map, but topologize E as a discrete set and B with the usual topology.) Not surprisingly, we need control over the fibers. The three classes of maps in the title are successively more general yet retain just such control. Namely, assuming the base is path connected, the fibers are respectively (i) homeomorphic, (ii) homotopy equivalent, (iii) weakly homotopy equivalent.

For convenience assume B is path connected, and always assume that E, B are metrizable.

(14.1) *Fiber bundles.* We already discussed these in **(12.23)**.

Definition 14.2. $p\colon E \to B$ is a *fiber bundle* if for every $b \in B$ there exists an open neighborhood U and a local trivialization

(14.3)
$$\begin{tikzcd} U \times p^{-1}(b) \arrow[rr] \arrow[dr] & & p^{-1}(U) \arrow[dl] \\ & B & \end{tikzcd}$$

Many important maps in geometry are fiber bundles.

(14.4) *Fibrations.* Now assume that E, B are pointed spaces with basepoints $e, \pi(e) = b$. A fibration is characterized by the *homotopy lifting property.*

Definition 14.5. $p\colon E \to B$ is a *fibration* if for every pointed space X, continuous map $f\colon [0,1] \times X \to B$ and lift $\tilde{f}_0\colon X \to E$ of f_0 there exists an extension $\tilde{f}\colon [0,1] \times X \to E$ lifting f.

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The lift is encoded in the diagram

(14.6)
$$\begin{tikzcd} \{0\}\times X \arrow[r, "\tilde f_0"] \arrow[d] & E \arrow[d, "p"] \\ {[0,1]}\times X \arrow[r, "f"] \arrow[ur, dashed, "\tilde f"] & B \end{tikzcd}$$

Theorem 14.7. *Suppose $p\colon E\to B$ is a fibration.*

(i) $p_*\colon \pi_n(E,p^{-1}(b),e)\to\pi_n(B,b)$ *is an isomorphism for all* $n\in\mathbb{Z}^{\geqslant 0}$.

(ii) There is a long exact sequence

(14.8)
$$\cdots\longrightarrow\pi_n(F,e)\longrightarrow\pi_n(E,e)\longrightarrow\pi_n(B,b)\longrightarrow\pi_{n-1}(F,e)\longrightarrow\cdots$$

in which $F=p^{-1}(b)$.

Proposition 14.9. *Let $p\colon (E,e)\to(B,b)$ be a fibration, $b'\in B$, and $P_e\bigl(E;p^{-1}(b')\bigr)$ the space of paths in E which begin at e and terminate on the subspace $p^{-1}(b')$. Then p induces a fibration*

(14.10)
$$P_e\bigl(E;p^{-1}(b')\bigr)\longrightarrow P_b(B;b')$$

with contractible fibers, so is a weak homotopy equivalence.

The last conclusion follows from the long exact sequence (14.8). We leave the reader to provide a proof of Proposition 14.9 using the homotopy lifting property.

(14.11) *Quasifibrations.* A quasifibration is a map for which the statements in Theorem 14.7 hold, but the homotopy lifting property does not necessarily hold. See Figure 4 for a typical example.

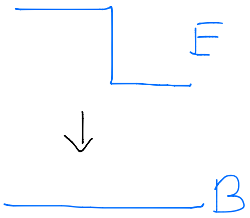

FIGURE 4. A quasifibration which is not a quasifibration

Definition 14.12. A map $p\colon E\to B$ (of unpointed spaces) is a *quasifibration* if $p_*\colon\pi_n(E,p^{-1}(b),e)\to\pi_n(B,b)$ is an isomorphism for all $b\in B$, $e\in p^{-1}(b)$, and $n\in\mathbb{Z}^{\geqslant 0}$.

The long exact sequence (14.8) follows.

An equivalent condition is that the natural map from each fiber to the *homotopy fiber* is a weak homotopy equivalence.

Quasifibrations are useful in part because of the following criterion to recognize them. This was proved by Dold-Thom [DT], who introduced quasifibrations.

Theorem 14.13. *Suppose $q\colon E\to B$ is a surjective map with B path connected. Let*

$$F_0B\subset F_1B\subset F_2B\subset\cdots \tag{14.14}$$

be an increasing filtration of B with $\bigcup\limits_{n=0}^{\infty}F_nB=B$ such that

(*i*) *$q\big|_U$ is a quasifibration for all open $U\subset F_nB\backslash F_{n-1}B$, and*

(*ii*) *For $n\geqslant 1$ there exists an open neighborhood $U_n\subset F_nB$ of $F_{n-1}B$ and deformation retractions*

$$\begin{aligned}U_n&\xrightarrow{h_t}F_{n-1}B\\ q^{-1}U_n&\xrightarrow{H_t}q^{-1}F_{n-1}B\end{aligned} \tag{14.15}$$

such that $H_1\colon q^{-1}(b)\to q^{-1}(h_1b)$ is a weak homotopy equivalence.

Then q is a quasifibration.

There is a nice exposition of quasifibrations in [Ha2, pp. 476–481] based on [Ma]. You will find the proofs of the theorems and much more there.

The basic diagram

We continue where we left off in Lecture 13. Introduce

$$\widehat{F}_*=\{T\in\pi^{-1}(\widehat{G}_*):\|T\|=1\}. \tag{14.16}$$

Thus an operator $T\colon H\to H$ in $\widehat{F}_*$ satisfies:

$$\begin{aligned}&T\text{ is Fredholm,}\\ &T^*=-T,\\ &\|T\|=1,\\ &\operatorname{ess\,spec}T=\{+i,-i\}.\end{aligned} \tag{14.17}$$

Lemma 14.18. *$\widehat{F}_*$ is a deformation retract of $\widehat{\mathcal{F}_*}$.*

Proof. First use the deformation retraction $\big((1-t)+t\|\pi(T)^{-1}\|\big)T$ onto the subspace of $S\in\widehat{\mathcal{F}_*}$ with $\|\pi(S)^{-1}\|=1$. Then deformation retract $i\mathbb{R}$ symmetrically onto $[-i,+i]$ and use the spectral theorem. (The symmetry ensures we stay in the space of skew-adjoint operators.) ☐

Corollary 14.19. *$\hat{\pi}\colon\widehat{F}_*\to\widehat{G}_*$ is a homotopy equivalence*

Now we have the basic diagram

$$\begin{tikzcd}\widehat{F}_* \arrow[r, "\delta"] \arrow[d, "\hat{\pi}"'] & P_1(U,-U^{\mathrm{cpt}}) \arrow[d, "\rho"] \\ \widehat{G}_* \arrow[r, "\epsilon"] & P_1(G,-1)\end{tikzcd} \tag{14.20}$$

Both δ and ϵ are given by the formula

$$x \longmapsto \exp \pi t x, \qquad 0 \leqslant t \leqslant 1. \tag{14.21}$$

In (14.20) we know that $\hat{\pi}$ is a homotopy equivalence and we need to prove that ϵ is a homotopy equivalence (Theorem 13.47). We will do so by proving that δ, ρ are homotopy equivalences.

That ρ is a weak homotopy equivalence follows directly from Proposition 14.9 once we observe (see (13.31)) that $U \to G$ is a principal fiber bundle (hence fibration) with fiber U^{cpt}. All spaces in the game have the homotopy type of CW complexes, so weak homotopy equivalences are homotopy equivalences.

Proposition 14.22. *Evaluation at the endpoint is a homotopy equivalence*

$$P_1(U, -U^{\mathrm{cpt}}) \longrightarrow -U^{\mathrm{cpt}} \tag{14.23}$$

Proof. The map (14.23) is a fibration with fiber ΩU, and the latter is contractible by Kuiper's Theorem 12.1. ☐

From the basic diagram (14.20) we are reduced to proving the following.

Theorem 14.24. *The map*

$$\begin{aligned} q\colon \widehat{F}_* &\longrightarrow -U^{\mathrm{cpt}} \\ T &\longmapsto \exp(\pi T) \end{aligned} \tag{14.25}$$

is a homotopy equivalence.

A dense quasifibration

To gain some intuition, let's look at a few fibers of the map q in (14.25). We write $P \in -U^{\mathrm{cpt}}$ as $P = -\operatorname{id}_H + \ell$ where $\ell \in \operatorname{cpt}(H)$ is a compact operator.

Example 14.26. Suppose that ℓ has finite rank. Then $K = \ker(\ell)$ is a closed subspace of finite codimension and $H = K \oplus K^{\perp}$; the dimension of $K^{\perp}$ is finite. Suppose $T \in q^{-1}(P)$ so $\exp \pi T = P$ and T satisfies the conditions in (14.17). The first observation is that $T\big|_{K^{\perp}}$ is determined by $P\big|_{K^{\perp}}$. For on this finite dimensional space we can diagonalize the operators and we are studying the map $\exp(\pi -)\colon [-i, +i] \to \mathbb{T}$, which is an isomorphism except at the endpoints, both of which map to $-i \in \mathbb{T}$. On $K^{\perp}$ the operator P does not have eigenvalue $-i$ so the logarithm (inverse image under q) is unique. On the other hand, the operator $T\big|_K$ has spectrum contained in $\{+i, -i\}$, and by the last condition in (14.17) both $+i$ and $-i$ are in the spectrum with "infinite multiplicity". It follows that there is a decomposition

$$K = K_+ \oplus K_- \tag{14.27}$$

with $T\big|_{K_\pm} = \pm i$ and $\dim K_+ = \dim K_- = \infty$. The fiber $q^{-1}(P)$ is then identified with the Grassmannian of all splittings (14.27). This Grassmannian is diffeomorphic to the homogeneous space $U(K) \,/\, U(K_+) \times U(K_-)$. All three unitary groups are contractible by Kuiper, hence so is the fiber.

Example 14.28. Suppose $e_1, e_2, \dots$ is an orthonormal basis of the Hilbert space H. Consider the following two operators in $-U^{\mathrm{cpt}}$:

$$\begin{aligned} P_1(e_n) &= \exp\bigl(\pi i(1-\frac{1}{n})\bigr) \\ P_2(e_n) &= \exp\bigl(\pi i(1+\frac{(-1)^n}{n})\bigr) \end{aligned} \tag{14.29}$$

There is a unique operator $T_i\colon H \to H$ which exponentiates to P_i under $\exp(\pi -)$, but as the essential spectrum of T_1 is $\{+i\}$ it is not an element of $\widehat{F}_*$. Thus $q^{-1}(P_1)$ is empty whereas $q^{-1}(P_2)$ has a single point.

Since not all fibers of q are weakly homotopy equivalent, q is not a quasifibration. However, it is still a homotopy equivalence. Atiyah-Singer prove this by proving that q is a quasifibration over the dense subspace of operators of the form treated in Example 14.26, and in turn the inclusion of the subspaces of both base and total space are homotopy equivalences.

Definition 14.30. Let $n \in \mathbb{Z}^{>0}$. Define

(i) $-U^{\mathrm{cpt}}(n) \subset -U^{\mathrm{cpt}}$ as the subset $\{P = -\mathrm{id}_H + \ell : \operatorname{rank} \ell \leqslant n\}$,
(ii) $\widehat{F}_*(n) \subset \widehat{F}_*$ as the subset $q^{-1}\bigl(-U^{\mathrm{cpt}}(n)\bigr)$.

In each case we have an increasing filtration of the unions

$$\begin{aligned} -U^{\mathrm{cpt}}(\infty) &= \bigcup_{n=1}^{\infty} -U^{\mathrm{cpt}}(n) \\ \widehat{F}_*(\infty) &= \bigcup_{n=1}^{\infty} \widehat{F}_*(n) \end{aligned} \tag{14.31}$$

The first union is the space of all unitaries which differ from $-\mathrm{id}_H$ by a finite rank operator. That resembles the union of the groups (13.26) in which we fix the subspaces on which the operator deviates from $-\mathrm{id}_H$. In any case the homotopy type of the unions are the same.

Proposition 14.32. *The inclusion maps*

$$\begin{aligned} i\colon\ -U^{\mathrm{cpt}}(\infty) &\longrightarrow -U^{\mathrm{cpt}} \\ i\colon \widehat{F}_*(\infty) &\longrightarrow \widehat{F}_* \end{aligned} \tag{14.33}$$

are homotopy equivalences.

Proposition 14.34. $q\big|_{\widehat{F}_*(\infty)}$ *is a quasifibration with contractible fibers.*

Theorem 14.24 follows immediately from these propositions.

We sketch the proofs (literally) and defer to [AS3] for details. For Proposition 14.32 we must show that any compact $X \subset \widehat{F}_*$ can be deformed to a subset of $\widehat{F}_*(n)$ for some n. We do that by a spectral deformation, illustrated in Figure 5 in which $0 < \alpha < 1$. The key observation is that

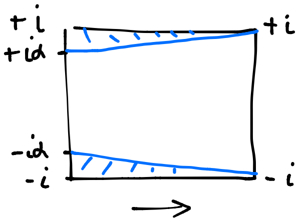

Figure 5. Spectral deformation for Proposition 14.32

any operator in $\widehat{F}_*$ has only a finite spectrum in the interval $[-i\alpha, i\alpha]$ as the essential spectrum is $\{-i, +i\}$. The argument for U^{cpt} is similar.

For Proposition 14.34 we use the Dold-Thom criterion Theorem 14.13. To verify condition (i) we sup up the argument in Example 14.26 to show that over the subspace where rank $\ell = n$ is constant the kernels K form a vector bundle, as do their orthogonal complements. Thus the restriction of q over this subspace is a fiber bundle with contractible fibers, so in particular is a quasifibration on any open subset. For (ii) we observe that an operator in $-U^{\mathrm{cpt}}(n)$ has at most n eigenvalues not equal to -1. We need to deform a neighborhood of $-U^{\mathrm{cpt}}(n-1)$ in $-U^{\mathrm{cpt}}(n)$ to $-U^{\mathrm{cpt}}(n-1)$. Let U_n be the subset where there is such an eigenvalue with negative real part. Make a spectral deformation as illustrated in Figure 6. It is easy to check that the induced map on fibers is a homotopy equivalence.

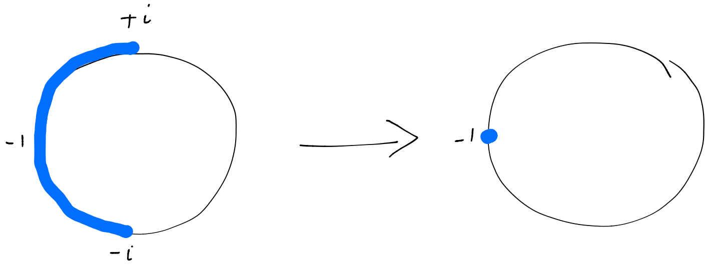

Figure 6. Spectral deformation for Proposition 14.34

McDuff's proof of Bott periodicity (Bonus material)

We begin with the observation of an exchange between finite and infinite dimensions. Let V be a finite dimensional complex vector space and H an infinite dimensional Hilbert space. Then whereas $GL(H)$ is contractible (Kuiper), $GL(V)$ definitely has interesting topology: the colimit for $\dim V$ large is the homotopy type GL_∞. On the other hand, the space $\mathrm{Fred}(H)$ of Fredholm operators is interesting—it has the homotopy type $\mathbb{Z} \times BGL_\infty$—whereas the space $\mathrm{End}(V)$ is contractible.

McDuff [McD] gave a proof of Bott periodicity by constructing a variation of (14.25) from finite dimensional spaces which exchanges the spaces of interest: the total space in her quasifibration is contractible, whereas it is the fibers of (14.25) which are contractible. Thus her quasifibration has the form

(14.35)
$$\begin{tikzcd}
\mathbb{Z}\times BGL_\infty \arrow[r] & \mathrm{pt} \arrow[d] \\
 & GL_\infty
\end{tikzcd}$$

This shows that $\Omega GL_\infty \simeq \mathbb{Z}\times BGL_\infty$. It is trivial that $\Omega(\mathbb{Z}\times BGL_\infty)\simeq GL_\infty$, and the two statements together immediately imply Bott periodicity.

For each finite dimensional vector space V we let $\mathfrak{u}(V)_{\leqslant 1}$ be the subspace of skew-adjoint operators with operator norm $\leqslant 1$, and as in (14.25) consider the map

(14.36)
$$\begin{aligned} q\colon \mathfrak{u}(V)_{\leqslant 1} &\longrightarrow U(V) \\ T &\longmapsto \exp(\pi T) \end{aligned}$$

This is a quasifibrations with fibers the Grassmannian of the $(-i)$-eigenspace, exactly as in the analysis of Example 14.26. The idea is to replace V by the colimit $\mathbb{C}^\infty$ of finite dimensional vector spaces and work with a "restricted Grassmannian" and "restricted general linear group". Details of the argument are worked out in the series of papers [AP], [Beh1], [Beh2]; the latter also works out real Bott periodicity.

We want not only Bott periodicity but also a geometric model of K-theory. As we meet Fredholm operators in geometry this alternative proof, while very beautiful and elegant, does not suffice for our purposes.

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Lecture 15: Groupoids and vector bundles

So far we have two notions of 'space': a topological space and a smooth manifold. But in many situations this is not adequate: the objects represented by the points of a "space" have internal structure, or symmetries. For example, consider the moduli "space" of triangles in the plane, where two triangles represent the same point if there is an isometry of the plane which carries one to the other. Then some triangles, for example an isosceles triangle, admit self-symmetries. We need a mathematical structure which tracks these internal symmetries. In physics too we meet the same phenomenon. For example, some fields in field theory, such as gauge fields (connections), admit internal symmetries. In fact, in both geometry and physics there are geometric objects with more than one layer of internal structure, but in this course we restrict ourselves to a single layer. The intrinsic geometric object we need is called a *stack*. Stacks are presented by *groupoids*, which are more concrete, and we focus on them. Again we consider a topological version (*topological groupoids*) and a smooth version (*Lie groupoids*). In both cases we need to restrict to *local quotient groupoids* to develop K-theory. A key idea in this lecture is *local equivalence* of groupoids: groupoids which are locally equivalent represent the same underlying stack. We prove that a local equivalence of groupoids induces an equivalence of the categories of vector bundles. We also prove homotopy invariance for vector bundles over local quotient groupoids. This enables us to define K-theory for local quotient groupoids, which we will pursue in subsequent lectures. A reference for this material is [FHT1, Appendix].

The important special case of a global quotient groupoid leads to the K-theory of equivariant bundles [Se2].

We begin these notes with background material about categories and simplicial sets, topics which will not be covered in lecture. That makes these notes very definition-heavy, a burden surmountable by careful consideration of examples on the part of the reader.

Categories, functors, and natural transformations

Definition 15.1. A *category* C consists of a collection of objects, for each pair of objects y_0, y_1 a set of morphisms $C(y_0, y_1)$, for each object y a distinguished morphism $\mathrm{id}_y \in C(y, y)$, and for each triple of objects y_0, y_1, y_2 a composition law

$$\circ\colon C(y_1, y_2) \times C(y_0, y_1) \longrightarrow C(y_0, y_2) \tag{15.2}$$

such that $\circ$ is associative and id_y is an identity for $\circ$.

The last phrase indicates two conditions: for all $f \in C(y_0, y_1)$ we have

$$\mathrm{id}_{y_1} \circ f = f \circ \mathrm{id}_{y_0} = f \tag{15.3}$$

and for all $f_1 \in C(y_0, y_1)$, $f_2 \in C(y_1, y_2)$, and $f_3 \in C(y_2, y_3)$ we have

$$(f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1). \tag{15.4}$$

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We use the notation $y \in C$ for an object of C and $f\colon y_0 \to y_1$ for a morphism $f \in C(y_0, y_1)$.

Remark 15.5 (set theory)*.* The words 'collection' and 'set' are used deliberately. Russell pointed out that the collection of all sets is not a set, yet we still want to consider a category whose objects are sets. For many categories the objects do form a set. In that case the moniker 'small category' is often used. In these lecture we will be sloppy about the underlying set theory and simply talk about a set of objects.

Definition 15.6. Let C be a category.

(i) A morphism $f \in C(y_0, y_1)$ is *invertible* (or an *isomorphism*) if there exists $g \in C(y_1, y_0)$ such that $g \circ f = \mathrm{id}_{y_0}$ and $f \circ g = \mathrm{id}_{y_1}$.

(ii) If every morphism in C is invertible, then we call C a *groupoid*.

(15.7) *Reformulation.* To emphasize that a category is an algebraic structure like any other, we indicate how to formulate the definition in terms of sets[1] and functions. Then a category $C = (C_0, C_1)$ consists of a set C_0 of objects, a set C_1 of morphisms, and structure maps

$$\begin{aligned} i\colon C_0 &\longrightarrow C_1 \\ s,t\colon C_1 &\longrightarrow C_0 \\ c\colon C_1 \times_{C_0} C_1 &\longrightarrow C_1 \end{aligned} \tag{15.8}$$

which satisfy certain conditions. The map i attaches to each object y the identity morphism id_y, the maps s,t assign to a morphism $(f\colon y_0 \to y_1) \in C_1$ the source $s(f) = y_0$ and target $t(f) = y_1$, and c is the composition law. The fiber product $C_1 \times_{C_0} C_1$ is the set of pairs $(f_2, f_1) \in C_1 \times C_1$ such that $t(f_1) = s(f_2)$. The conditions (15.3) and (15.4) can be expressed as equations for these maps. If C is a groupoid, then there is another structure map

$$\iota\colon C_1 \longrightarrow C_1 \tag{15.9}$$

which attaches to every arrow its inverse.

Definition 15.10. Let C, D be categories.

(i) A *functor* or *homomorphism* $F\colon C \to D$ is a pair of maps $F_0\colon C_0 \to D_0$, $F_1\colon C_1 \to D_1$ which commute with the structure maps (15.8).

(ii) Suppose $F, G\colon C \to D$ are functors. A *natural transformation* η from F to G is a map of sets $\eta\colon C_0 \to D_1$ such that for all morphisms $(f\colon y_0 \to y_1) \in C_1$ the diagram

$$\begin{tikzcd} Fy_0 \arrow[r, "Ff"] \arrow[d, "\eta(y_0)"'] & Fy_1 \arrow[d, "\eta(y_1)"] \\ Gy_0 \arrow[r, "Gf"] & Gy_1 \end{tikzcd} \tag{15.11}$$

commutes. We write $\eta\colon F \to G$.

---

[1] ignoring set-theoretic complications, as in Remark 15.5

(iii) A natural transformation $\eta\colon F \to G$ is an *isomorphism* if $\eta(y)\colon Fy \to Gy$ is an isomorphism for all $y \in C$.

(iv) A functor $F\colon C \to D$ is an *equivalence of categories* if there exist a functor $F'\colon D \to C$, a natural equivalence $F' \circ F \to \mathrm{id}_C$, and a natural equivalence $F \circ F' \to \mathrm{id}_{C'}$.

In (i) the commutation with the structure maps means that F is a homomorphism in the usual sense of algebra: it preserves compositions and takes identities to identities. A natural transformation is often depicted in a diagram

$$\begin{tikzcd} C \arrow[rr, bend left, "G"] \arrow[rr, bend right, "F"'] \arrow[rr, phantom, "\Uparrow \eta"] & & D \end{tikzcd} \tag{15.12}$$

with a double arrow.

Definition 15.13. Let $F\colon C \to D$ be a functor. F is *essentially surjective* if for each $z \in D_0$ there exists $y \in C_0$ such that Fy is isomorphic to z. It is *faithful* if for every $y_0, y_1 \in C_0$ the map

$$F\colon C(y_0, y_1) \to D(Fy_0, Fy_1) \tag{15.14}$$

is injective, and it is *full* if (15.14) is surjective.

The following lemma characterizes equivalences of categories; the proof, which we leave to the reader, invokes the axiom of choice to construct an inverse equivalence.

Lemma 15.15. *A functor $F\colon C \to D$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

Example 15.16. Let Vect denote the category of vector spaces over a fixed field with linear maps as morphisms. There is a functor $**\colon \mathrm{Vect} \to \mathrm{Vect}$ which maps a vector space V to its double dual V^{**}. But this is not enough to define it—we must also specify the map on morphisms. Thus if $f\colon V_0 \to V_1$ is a linear map, there is an induced linear map $f^{**}\colon V_0^{**} \to V_1^{**}$. (Recall that $f^*\colon V_1^* \to V_0^*$ is defined by $\langle f^*(v_1^*), v_0\rangle = \langle v_1^*, f(v_0)\rangle$ for all $v_0 \in V_0$, $V_1^* \in V_1^*$. Then define $f^{**} = (f^*)^*$.) Now there is a natural transformation $\eta\colon \mathrm{id}_{\mathrm{Vect}} \to **$ defined on a vector space V as

$$\begin{aligned} \eta(V)\colon V &\longrightarrow V^{**} \\ v &\longmapsto \bigl(v^* \mapsto \langle v^*, v\rangle\bigr) \end{aligned} \tag{15.17}$$

for all $v^* \in V^*$. I encourage you to check (15.11) carefully.

Simplices, simplicial sets, and the nerve

Let S be a nonempty finite ordered set. For example, we have the set

$$[n] = \{0, 1, 2, \dots, n\} \tag{15.18}$$

with the given total order. Any S is uniquely isomorphic to $[n]$, where the cardinality of S is $n+1$. Let $A(S)$ be the affine space generated by S and $\Sigma(S) \subset A(S)$ the simplex with vertex set S. So if $S = \{s_1, s_1, \dots, s_n\}$, then $A(S)$ consists of formal sums

$$p = t^0 s_0 + t^1 s_1 + \cdots + t^n s_n, \qquad t^i \in \mathbb{R}, \quad t^0 + t^1 + \cdots + t^n = 1, \tag{15.19}$$

and $\Sigma(S)$ consists of those sums with $t^i \geqslant 0$. We write $\mathbb{A}^n = A([n])$ and $\Delta^n = \Sigma([n])$. For these standard spaces the point $i \in [n]$ is $(\dots, 0, 1, 0, \dots)$ with 1 in the i^{th} position.

Let Δ be the category whose objects are nonempty finite ordered sets and whose morphisms are order-preserving maps (which may be neither injective nor surjective). The category Δ is generated by the morphisms

$$\begin{tikzcd} {[0]} \arrow[r, shift left=2] \arrow[r, shift right=2] & {[1]} \arrow[l, dashed] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & {[2]} \arrow[l, dashed, shift left=2] \arrow[l, dashed, shift right=2] & \cdots \end{tikzcd} \tag{15.20}$$

where the right-pointing maps are injective and the left-pointing maps are surjective. For example, the map $d_i\colon [1] \to [2]$, $i = 0, 1, 2$ is the unique injective order-preserving map which does not contain $i \in [2]$ in its image. The map $s_i\colon [2] \to [1]$, $i = 0, 1$, is the unique surjective order-preserving map for which $s_i^{-1}(i)$ has two elements. Any morphism in Δ is a composition of the maps d_i, s_i and identity maps.

Each object $S \in \Delta$ determines a simplex $\Sigma(S)$, as defined above. This assignment extends to a functor

$$\Sigma\colon S \longrightarrow \mathrm{Top} \tag{15.21}$$

to the category of topological spaces and continuous maps. A morphism $\theta\colon S_0 \to S_1$ maps to the affine extension $\theta_*\colon \Sigma(S_0) \to \Sigma(S_1)$ of the map θ on vertices.

Recall the definition **(15.7)** of a category.

Definition 15.22. Let C be a category. The *opposite category* C^{op} is defined by

$$C_0^{\mathrm{op}} = C_0, \quad C_1^{\mathrm{op}} = C_1, \quad s^{\mathrm{op}} = t, \quad t^{\mathrm{op}} = s, \quad i^{\mathrm{op}} = i, \tag{15.23}$$

and the composition law is reversed: $g^{\mathrm{op}} \circ f^{\mathrm{op}} = (f \circ g)^{\mathrm{op}}$.

Here recall that C_0 is the set of objects, C_1 the set of morphisms, and $s, t\colon C_1 \to C_0$ the source and target maps. The opposite category has the same objects and morphisms but with the direction of the morphisms reversed.

The following definition is slick, and at first encounter needs unpacking (see [Fr], for example).

Definition 15.24. A *simplicial set* is a functor

$$X\colon \Delta^{\mathrm{op}} \longrightarrow \mathrm{Set} \tag{15.25}$$

It suffices to specify the sets $X_n = X([n])$ and the basic maps (15.20) between them. Thus we obtain a diagram

$$\begin{tikzcd} X_0 \arrow[r, dashed, shift left=0] & X_1 \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[r, dashed, shift left=2] \arrow[r, dashed, shift right=2] & X_2 \arrow[l, shift right=4] \arrow[l] \arrow[l, shift left=4] & \cdots \end{tikzcd} \tag{15.26}$$

We label the maps d_i and s_i as before. The d_i are called *face maps* and the s_i *degeneracy maps*. The set X_n is a set of abstract simplices. An element of X_n is degenerate if it lies in the image of some s_i.

The morphisms in an abstract simplicial set are gluing instructions for concrete simplices.

Definition 15.27. Let $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ be a simplicial set. The *geometric realization* is the topological space $|X|$ obtained as the quotient of the disjoint union

$$\coprod_S X(S) \times \Sigma(S) \tag{15.28}$$

by the equivalence relation

$$(\sigma_1, \theta_* p_0) \sim (\theta^* \sigma_1, p_0), \qquad \theta\colon S_0 \to S_1, \quad \sigma_1 \in X(S_1), \quad p_0 \in \Sigma(S_0). \tag{15.29}$$

The map $\theta_* = \Sigma(\theta)$ is defined after (15.21) and $\theta^* = X(\theta)$ is part of the data of the simplicial set X. Alternatively, the geometric realization map be computed from (15.26) as

$$\coprod_n X_n \times \Delta^n \;/\!\sim, \tag{15.30}$$

where the equivalence relation is generated by the face and degeneracy maps.

Remark 15.31. The geometric realization can be given the structure of a CW complex.

Example 15.32. Let X be a simplicial set whose nondegenerate simplices are

$$X_0 = \{A, B, C, D\}, \qquad X_1 = \{a, b, c, d\}. \tag{15.33}$$

The face maps are as indicated in Figure 4. For example $d_0(a) = B$, $d_1(a) = A$, etc. (This requires a choice not depicted in Figure 4.) The level 0 and 1 subset of the disjoint union (15.30) is pictured in Figure 5. The 1-simplices a, b, c, d glue to the 0-simplices A, B, C, D to give the space depicted in Figure 4. The red 1-simplices labeled A, B, C, D are degenerate, and they collapse under the equivalence relation (15.29) applied to the degeneracy map s_0.

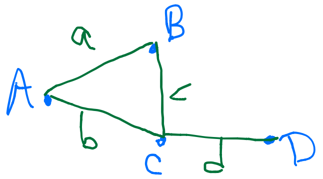

Figure 4. The geometric realization of a simplicial set

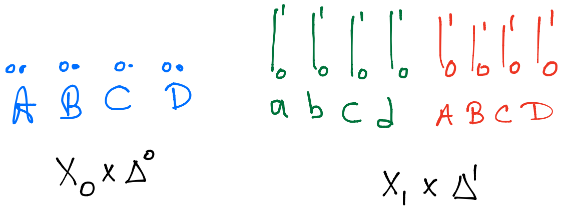

Figure 5. Gluing the simplicial set

(15.34) *The nerve of a category.* Let $C = (C_0, C_1)$ be a category, which in part is encoded in the diagram

$$\begin{tikzcd} C_0 \arrow[r, dashed] & C_1 \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd} \tag{15.35}$$

The solid left-pointing arrows are the source s and target t of a morphism; the dashed right-pointing arrow i assigns the identity map to each object. This looks like the start of a simplicial set, and indeed there is a simplicial set NC, the *nerve* of the category C, which begins precisely this way: $NC_0 = C_0$, $NC_1 = C_1$, $d_0 = t$, $d_1 = s$, and $s_0 = i$. A slick definition runs like this: a finite nonempty ordered set S determines a category with objects S and a unique arrow $s \to s'$ if $s \leqslant s'$ in the order. Then

$$NC(S) = \mathrm{Fun}(S, C) \tag{15.36}$$

where $\mathrm{Fun}(-,-)$ denotes the set of functors. As is clear from Figure 6, $NC([n])$ consists of sets of n composable arrows in C. The degeneracy maps in NC insert an identity morphism. The face map d_i omits the i^{th} vertex and composes the morphisms at that spot; if i is an endpoint $i = 0$ or $i = n$, then d_i omits one of the morphisms.

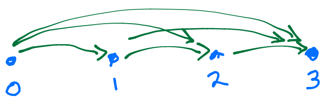

Figure 6. A totally ordered set as a category

Example 15.37. Let M be a monoid, regarded as a category with a single object. Then

$$NM_n = M^{\times n}. \tag{15.38}$$

It is a good exercise to write out the face maps.

Definition 15.39. Let C be a category. The *classifying space* BC of C is the geometric realization $|NC|$ of the nerve of C.

Example 15.40. Suppose $G = \mathbb{Z}/2\mathbb{Z}$ is the cyclic group of order two, viewed as a category with one object. Then NG_n has a single nondegenerate simplex $(g, \dots, g)$ for each n, where $g \in \mathbb{Z}/2\mathbb{Z}$ is the non-identity element. So BG is glued together with a single simplex in each dimension. We leave the reader to verify that in fact $BG \simeq \mathbb{RP}^\infty$.

Topological and Lie groupoids

From now on we use the formulation **(15.7)** of a category, recall (Definition 15.6) that a groupoid is a category in which all morphisms are invertible, and we identify a groupoid $X = (X_0, X_1)$ with its nerve $NX_\bullet = X_\bullet$, which is a simplicial set **(15.34)**.

Definition 15.41. Let $X = (X_0, X_1)$ be a groupoid.

(i) X is a *topological groupoid* if X_0, X_1 have the structure of topological spaces and if the structure maps i, s, t, c, ι in (15.8), (15.9) are continuous.[2]

(ii) X is a *Lie groupoid* if X_0, X_1 have the structure of smooth manifolds, the structure maps i, s, t, c, ι are smooth, and the source and target maps $s, t\colon X_1 \to X_0$ are submersions.

The submersion condition guarantees that the fiber product $X_1 \times_{X_0} X_1$ is a smooth manifold, which is necessary if the composition c in (15.8) is to be a smooth map.

We now give many examples to illustrate the pervasiveness and utility of topological and Lie groupoids.

Example 15.42 (groups)**.** A groupoid with a single object $X_0 = \{*\}$ is a group; that is, X_1 is a group. A topological groupoid with a single object is a topological group. A Lie groupoid with a single object is a Lie group. The groupoid attached to a (topological, Lie) group G is often denoted BG, but we reserve that notation for classifying spaces. Instead we use the notation 'pt $/\!/G$', explained below in Example 15.44.

Example 15.43 (spaces)**.** A groupoid with only identity arrows ($i\colon X_0 \to X_1$ is a bijection) is a set X_0. A topological groupoid with only identity arrows is a space. A Lie groupoid with only identity arrows is a smooth manifold.

Example 15.44 (group actions)**.** Let X be a set, G a group, and suppose G acts on X on the right.[3] Then we construct a groupoid[4] $Y = X/\!/G$ variously called the *quotient groupoid* or *action groupoid*. We have $Y_0 = X$ and $Y_1 = X \times G$. The source map is projection $X \times G \to X$ and the target is the action $X \times G \to X$. Composition is defined using the group action. If X is a space, G a topological group, and the action is continuous, then Y is a topological groupoid in a natural way. Similarly, if X is a smooth manifold, G a Lie group, and the action is smooth, then Y is a Lie groupoid in a natural way.

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[2] It is sometimes convenient to also ask that s, t be *open* maps.

[3] There is a similar construction for left actions.

[4] This notation is not universally admired as it conflicts with the notation for symplectic or Kähler or GIT quotients. Other possibilities include '$X\!:\!G$', '$G \ltimes X$', and '$X \rtimes G$'.

Example 15.45 (principal bundles)**.** As a special case of the previous example, suppose P is a space (or smooth manifold) with a continuous (smooth) right G-action, assume the action is free, and suppose furthermore that continuous (or smooth) local slices exist. That is, for every $p \in P$ there exists a set $U \subset P$ containing p such that the restriction of the projection $\pi \colon P \to P/G$ to U is a homeomorphism (diffeomorphism) onto an open subset of P/G. Then $\pi \colon P \to P/G$ is called a *principal bundle* with base P/G and structure group G. The action groupoid $P//G$ is equivalent to the space P/G (see Definition 15.10(iv)), and we will see below that π defines a *local equivalence*.

Example 15.46 ($G/\!/G$)**.** Let G be a topological or Lie group. Let G act on itself by conjugation, and denote the resulting quotient groupoid as $G/\!/G$. Even for finite groups this is an important groupoid, for example in proofs of the Sylow theorems. It will play a large role in our later study of loop groups and the Verlinde ring.

Example 15.47 (open covers)**.** Let X be a topological space and $\{U_i\}_{i\in I}$ an open cover. Define

$$Y_0 = \coprod_{i\in I} U_i \tag{15.48}$$

as the disjoint union of the sets in the cover, with the obvious topology. There is a projection $\pi \colon Y_0 \to X$ which is a continuous surjection. Given that we can construct a topological groupoid (Y_0, Y_1) by setting

$$Y_1 = Y_0 \times_X Y_0 \tag{15.49}$$

as the fiber product of $\pi \colon Y_0 \to X$ with itself. So a point of Y_0 is an ordered pair of points $x_0 \in U_{i_0}$, $x_1 \in U_{i_1}$ such that $x_0 = x_1$ as points of X. We can take higher fiber products to construct the simplicial set $Y_\bullet$ which is the nerve of the groupoid Y.

The next examples can be considered to be *moduli spaces*, except that they are groupoids rather than spaces.[5] They are parameter spaces for geometric objects with internal symmetries.

Example 15.50 (Galois coverings)**.** Fix a space X and a discrete group G. Then there is a groupoid $Y = \mathrm{Bun}_G(X)$ whose objects are Galois covers $P \to X$ with group (of deck transformations) G and whose morphisms $X(P, P')$ are homeomorphisms $\varphi \colon P \to P'$ which cover the identity map id_X and commute with the G-actions. For $G = \mathbb{Z}/2\mathbb{Z}$ we obtain the groupoid of double covers. Suppose $X = S^1 = \mathbb{R}/\mathbb{Z}$. Let Y be the groupoid whose objects are Galois covers $P \to S^1$ equipped with a basepoint in the fiber P_0; the morphisms need not fix the basepoint. Then we obtain a diagram

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ X & & G/\!/G \end{tikzcd} \tag{15.51}$$

in which the left arrow forgets the basepoint and the right arrow maps a cover to its *holonomy*: the path $[0,1] \hookrightarrow \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ has a lift to a Galois cover $P \to \mathbb{R}/\mathbb{Z}$ with initial point the basepoint $* \in P_0$, and the terminal point is $* \cdot h$, where $h \in G$ is the holonomy. The maps in (15.51) are equivalences of groupoids, in fact, local equivalences.

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[5] Thus the term 'moduli stack' is more accurately used.

Example 15.52 (connections on principal bundles)**.** We will discuss connections systematically in a later lecture. Here we want to observe that connections give an extension of Example 15.50 when G is not discrete. Namely, fix a smooth manifold M and a Lie group G. Let $\mathrm{Conn}_G(M)$ be the groupoid whose objects are pairs (P, Θ) consisting of a smooth principal G-bundle $P \to M$ and a connection $\Theta \in \Omega^1_P(\mathfrak{g})$. (Here $\mathfrak{g} = \mathrm{Lie}(G)$ is the Lie algebra of G.) A morphism $(P, \Theta) \to (P', \Theta')$ is a smooth map $\varphi\colon P \to P'$ such that $\varphi^*\Theta' = \Theta$. If G is discrete, then every principal bundle carries a unique connection and $\mathrm{Conn}_G(M) = \mathrm{Bun}_G(M)$. For $M = S^1$ the holonomy map gives an equivalence with the Lie groupoid $G/\!/G$.

Remark 15.53. $\mathrm{Conn}_G(M)$ is not a Lie groupoid as presented. It is (locally) equivalent to a groupoid which is the global quotient of an infinite dimensional manifold by an infinite dimensional Lie group; the details of what type of manifold and Lie group depend on whether we use smooth connections or complete to Banach spaces of connections.

Example 15.54 (Riemannian metrics; complex structures)**.** Connections are *extrinsic* to the geometry of the smooth manifold M. There are also natural groupoids of *intrinsic* geometric structures, which are often quotient groupoids of spaces by the action of the diffeomorphism group $\mathrm{Diff}(M)$. Examples include the space of Riemannian metrics and the space of complex structures. (The latter may be empty, for example if M has odd dimension.) If M is an oriented compact connected 2-manifold of genus g, then the groupoid of compatible complex structures is a model for the moduli stack of curves of genus g.

Example 15.55 (spin structures)**.** Again, we will discuss spin structures in detail later in the course. Here we remark that if M is a fixed smooth manifold, then there is a groupoid whose objects are spin structures and whose morphisms are maps of spin structures. This groupoid is empty if M is not spinable.

One particular type of groupoid is important in differential geometry as a mild generalization of a smooth manifold.

Definition 15.56. A Lie groupoid X is *étale* if the target and source maps $p_0, p_1\colon X_1 \to X_0$ are local diffeomorphisms.

In this case the underlying topological stack (defined below) is called an *orbifold* or *smooth Deligne-Mumford stack* and the representing groupoid an *orbifold groupoid*. We remark that smooth Deligne-Mumford stacks may be presented by Lie groupoids which are not étale—for example, if $P \to M$ is a principal G-bundle over a smooth manifold, then $P/\!/G$ is locally equivalent to M. Orbifolds have a more concrete differential-geometric description as "V-manifolds" in the work of Satake, Kawasaki, Thurston and others; see [ALR] and the references therein for a discussion of the various approaches.

Local equivalence of groupoids

An equivalence of groupoids (Definition 15.10(iv), Lemma 15.15) has an inverse equivalence, but an equivalence of *topological* groupoids does not necessarily have a *continuous* inverse.

Example 15.57. A principal G-bundle $\pi\colon P \to X$ induces an continuous equivalence $P/\!/X \to X$ which has a continuous inverse if and only if π admits a continuous global section (which only happens if the principal bundle is globally trivializable). As another example, an open cover $\{U_i\}_{i\in I}$ of a topological space X gives rise to an equivalence of groupoids $\pi\colon Y \to X$, where Y is the groupoid of Example 15.47. It admits a global continuous inverse only if each component of X appears as a set in the cover.

The following definition encodes the notion of continuous *local* inverses.

Definition 15.58. Let $f\colon X \to Y$ be a continuous equivalence of topological groupoids. Then f is a *local equivalence* if for each $y_0 \in Y_0$ there exists a neighborhood $i\colon U \hookrightarrow Y_0$ of y_0 and a lift $\tilde{i}$

(15.59)

$$\begin{tikzcd} & \widetilde{X}_0 \arrow[r, dashed] \arrow[d, dashed] & X_0 \arrow[d, "f"] \\ & Y_1 \arrow[r, "t"] \arrow[d, "s"] & Y_0 \\ U \arrow[r, "i"] \arrow[uur, dotted, "\tilde{i}"] & Y_0 & \end{tikzcd}$$

which makes the diagram commute.

In the diagram the upper square is a fiber product. Concretely, for $y \in U$ the lift $\tilde{i}(y)$ gives, in a continuous way, $x \in X_0$ and an arrow $\bigl(a\colon y \to f(x)\bigr)$.

Example 15.60. We list examples of local equivalences whose verification we leave to the reader.

1. A principal G-bundle $\pi\colon P \to X$ induces a local equivalence $P/\!/G \to X$.
2. An open cover of a space X induces a local equivalence $Y \to X$, where Y is defined in Example 15.47.
3. Let G be a Lie group. The holonomy map determines a local equivalence $\mathrm{Conn}_G(S^1) \to G/\!/G$.
4. The composition of local equivalences is a local equivalence. The pullback of a local equivalence is a local equivalence. The fiber product $P \times_X Q \to X$ of local equivalences $P \to X$, $Q \to X$ is a local equivalence.

Remark 15.61. Definition 15.58 fits into the theory of *presheaves of groupoids* on the category of topological spaces: it says that f induces a map of stalks. See [FHT1, Remark A.5] for more explanation.

Coarse moduli space

A topological groupoid X has an associated topological space $[X]$. For a groupoid quotient $X/\!/G$ as in Example 15.44, $[X/\!/G] = X/G$ is the quotient space.

Definition 15.62. Let $X = (X_0, X_1)$ be a topological groupoid. Define an equivalence relation $\sim$ on X_0 by $x \sim x'$ if there exists $f \in X_1$ such that $s(f) = x$ and $t(f) = x'$. Let $X_0 \to [X]$ be the quotient map of the equivalence relation, and topologize $[X]$ as a quotient. The space $[X]$ is the *coarse moduli space* of the groupoid X.

Recall (15.8) that s, t are the source and target maps, respectively. We write $f\colon x \to x'$. The coarse moduli space, or *orbit space*, can be bad, very bad. In particular, it may not be Hausdorff or paracompact.

Example 15.63. Consider an irrational rotation of S^1, which generates a $\mathbb{Z}$-action. The quotient space $S^1/\mathbb{Z}$, which is the coarse moduli space of the quotient groupoid $S^1/\!/\mathbb{Z}$, is not Hausdorff.

We will soon restrict to a class of groupoids (*local quotient groupoids*) whose coarse moduli space is paracompact Hausdorff.

If X is a groupoid, S a space, and $\phi\colon S \to [X]$ a continuous map, then there is a pullback groupoid $Y = \phi^*X$ with coarse moduli space $[\phi^*X] = S$. Namely, define Y_0 as the pullback

(15.64)
$$\begin{tikzcd} Y_0 \arrow[r, dashed] \arrow[d, dashed] & X_0 \arrow[d, "\pi"] \\ S \arrow[r, "\phi"] & {[X]} \end{tikzcd}$$

and Y_1 as the pullback

(15.65)
$$\begin{tikzcd} Y_1 \arrow[r, dashed] \arrow[d, dashed] & X_1 \arrow[d, "\pi\circ s=\pi\circ t"] \\ S \arrow[r, "\phi"] & {[X]} \end{tikzcd}$$

The structure maps pullback from the structure maps of X. In particular, we can take f to be an inclusion. For example, an open cover of $[X]$ induces an open cover of X by groupoids.

The course moduli space of a topological stack is invariant under local equivalences if we add the hypothesis that the source (hence target) maps be open.[6]

Lemma 15.66. *If the source map $s\colon Y_1 \to Y_0$ of a topological groupoid is open, then so too is the quotient map $q\colon Y_0 \to [Y]$.*

Proof. For $U \subset Y_0$ open we must show $q(U) \subset [Y]$ is open, which is equivalent to $q^{-1}q(U) \subset Y_0$ open. But $q^{-1}q(U) = ts^{-1}(U)$, and t is an open map if s is, since inversion is continuous in a topological groupoid. ☐

Proposition 15.67. *Let $F\colon X \to Y$ be a local equivalence of topological groupoids. Assume the source map $Y_1 \to Y_0$ in $Y = (Y_0, Y_1)$ is continuous. Then the induced map $[F]\colon [X] \to [Y]$ is a homeomorphism.*

Proof. In (iii) since F is an equivalence of discrete groupoids it induces a bijection $[F]$ on equivalence classes. For any continuous map F of groupoids the induced map $[F]$ is continuous. Given $[y_0] \in [Y]$ we choose an open neighborhood $U \subset Y_0$ of y_0 and a local inverse, as in (15.59). The composition

(15.68)
$$U \xrightarrow{\tilde{i}} \widetilde{X}_0 \longrightarrow X_0 \longrightarrow [X]$$

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[6] The version of Lecture 1 posted online omitted a hypothesis in the definition of a topological groupoid $Y = (Y_0, Y_1)$: the inversion map $\iota\colon Y_1 \to Y_1$ should also be assumed continuous.

is continuous and factors through the quotient map $U \to [U] \subset [X]$. The resulting map $[U] \to [X]$ is the inverse to $[F]$ restricted to $[U]$. Since the quotient map is open, by Lemma 15.66, $[U]$ is an open neighborhood of $[y_0]$. It follows that $[F]^{-1}$ is continuous at $[y_0]$. ☐

The homotopy category of groupoids: stacks

Let Top be the category of topological spaces. A *weak equivalence* is a continuous map $\phi\colon X \to Y$ of spaces which induces an isomorphism on π_0 and an isomorphism of all homotopy groups $\pi_q(X;x) \to \pi_q(Y;\phi(x))$ for all $x \in X$. The *homotopy category of spaces* is obtained from Top by formally inverting all weak equivalences. The resulting category can be considered to have the same objects as Top—topological spaces—but only the underlying homotopy type has invariant meaning. Similarly, let TGpd denote the category whose objects are topological groupoids and whose morphisms are continuous maps of groupoids. Now invert the weak equivalences to obtain the homotopy category of *stacks*. See [SiTe, MV] for a detailed development and [FH] for a gentle introduction to sheaves (on the category of smooth manifolds rather than Top, but the basic ideas are the same).

Local quotient groupoids

Even if we require the coarse moduli space $[X]$ of a topological groupoid X to be paracompact Hausdorff, we will not have homotopy invariance. For example, consider the groupoid $X = \mathrm{pt}/\!/\mathbb{R}$, the real line as a group under addition. As we shall see shortly, a vector bundle over X is simply a representation of $\mathbb{R}$. There is a continuous family of nonisomorphic 1-dimensional unitary representations

$$\begin{aligned}\rho_\xi\colon \mathbb{R} &\longrightarrow \mathbb{T}\\ x &\longmapsto e^{i\xi x}\end{aligned} \tag{15.69}$$

parametrized by $\xi \in \mathbb{R}$, where $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ is the multiplicative group of unit norm complex numbers. Thus there is a complex line bundle over $[0,1] \times X$ not isomorphic to its restriction to the endpoints. The lesson is that the representations of the automorphism groups of the groupoid must form a discrete set if homotopy invariance is to have a chance of working: representations must be discrete. This happens for compact Lie groups.

Definition 15.70. A topological groupoid X is a *local quotient groupoid* if $[X]$ admits a countable open cover $\{U_i\}_{i\in I}$ such that each groupoid X_{U_i} is locally equivalent to a groupoid of the form $S/\!/G$, where S is a paracompact Hausdorff locally contractible space and G is a compact Lie group.

Proposition 15.71.

(*i*) *Let X be a local quotient groupoid. Then $[X]$ is paracompact Hausdorff.*

(*ii*) *Let $F\colon X \to Y$ be a local equivalence of topological groupoids with open source maps. Then X is a local quotient groupoid if and only if Y is.*

Proof. For (i) it suffices, based on Proposition 15.67, to show that the quotient space S/G is paracompact Hausdorff. Paracompactness follows from [En, (5.1.33)] and Hausdorffness from [tD, (I.3.1)]. The second assertion (ii) is immediate from Proposition 15.67. ☐

Remark 15.72. The source map of a local quotient groupoid is open [tD, (I.3.1)].

Vector bundles over groupoids

A topological groupoid X can be viewed as a space X_0 of points together with gluing data X_1, and a composition law on gluing data. A vector bundle over a topological groupoid, then, is: a vector space for each point $x \in X$, gluing data for each arrow $(x_0 \xrightarrow{f} x_1) \in X_1$, and a consistency condition for each composable pair of arrows $(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2) \in X_2$. Continuity is ensured by specifying this data all at once. We use the nerve **(15.34)** of the groupoid and write in terms of the face maps d_i.

Remark 15.73. The discussion in this section applies to fiber bundles, not just vector bundles; see [FHT1, §A.3].

Definition 15.74. Let X be a topological groupoid.

(i) A *vector bundle* $E \to X$ is a pair $E = (E_0, \psi)$ consisting of a vector bundle $E_0 \to X_0$ and an isomorphism $\psi\colon d_1^*E_0 \to d_0^*E_0$ on X_1 which satisfies the cocycle constraint

$$\psi_{f_2 \circ f_1} = \psi_{f_2} \circ \psi_{f_1}. \tag{15.75}$$

for $(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2) \in X_2$.

(ii) A map $\phi\colon E \to E'$ of vector bundles over X is a map $\phi\colon E_0 \to E_0'$ of vector bundles over X_0 such that for every $(x_0 \xrightarrow{f} x_1) \in X_1$ the diagram

(15.76)
$$\begin{tikzcd} E_{x_0} \arrow[r, "\psi_f"] \arrow[d, "\phi_{x_0}"'] & E_{x_1} \arrow[d, "\phi_{x_1}"] \\ E'_{x_0} \arrow[r, "\psi'_f"] & E'_{x_1} \end{tikzcd}$$

commutes.

The notation is that the isomorphism ψ at $(x_0 \xrightarrow{f} x_1) \in X_1$ is $\psi_f\colon (E_0)_{x_0} \to (E_0)_{x_1}$. This data determines a groupoid $E = \bigl(E_0 \overset{\tilde{d}_1}{\underset{\tilde{d}_0}{\leftleftarrows}} E_1\bigr)$ where E_1 is the pullback $d_1^*E_0$ and $\tilde{d}_0\colon E_1 \to E_0$ is the composition $d_1^*E_0 \xrightarrow{\psi} d_0^*E_0 \to E_0$.

There is a category $\mathrm{Vect}(X)$ of vector bundles over a topological groupoid X. If $F\colon X \to Y$ is a continuous map of groupoids, there is a pullback functor

$$F^*\colon \mathrm{Vect}(Y) \longrightarrow \mathrm{Vect}(X) \tag{15.77}$$

Example 15.78. For a topological group G a vector bundle over $\mathrm{pt}/\!/G$ is a continuous representation of G; see Figure 2 in Lecture 1.

Example 15.79. Let a topological group G act on a topological space X. Then a vector bundle over $X/\!/G$ is a G-equivariant bundle over X.

Example 15.80. Let G be a finite group. A vector bundle over $G/\!/G$ has support over a union of conjugacy classes in $X_0 = G$. If the support is a single conjugacy class, the bundle is determined up to isomorphism by its restriction to any $g \in G$ in that conjugacy class, and that restriction is a representation of the centralizer subgroup $Z(g) \subset G$. So the simple objects in the category of vector bundles over X are parametrized by pairs $(\mathcal{O}, \rho)$ consisting of a conjugacy class in G and an irreducible representation of the centralizer of an element in that conjugacy class.

Example 15.81 (orbifolds)**.** If X is an orbifold groupoid its tangent bundle $TX \to X$ is the vector bundle $TX_0 \to X_0$ with the natural isomorphism $d_1^*TX_0 \to d_0^*TX_0$ from the fact that d_0, d_1 are local diffeomorphisms. A tensor field on an orbifold groupoid X is a tensor field t on X_0 which satisfies $d_0^*t = d_1^*t$. This includes functions, Riemannian metrics, etc.

Example 15.82 (open covers)**.** A vector bundle over the groupoid Y associated to an open cover $\{U_i\}_{i\in I}$ of a space (Example 15.47) is a vector bundle over each U_i together with gluing data on the overlaps $U_{i_0} \cap U_{i_1}$ which satisfies a cocycle condition on triple overlaps $U_{i_0} \cap U_{i_1} \cap U_{i_2}$. Thus Definition 15.74 includes the clutching construction of vector bundles; see **(1.16)**, **(1.18)**.

Proposition 15.83. *Let $F\colon X \to Y$ be a local equivalence of topological groupoids. Then the pullback functor* (15.77) *is an equivalence of categories.*

Proof. We sketch the construction of an inverse equivalence

$$F_*\colon \mathrm{Vect}(X) \longrightarrow \mathrm{Vect}(Y) \tag{15.84}$$

called *descent*; we leave the verification of details to the reader. Suppose $E \to X$ is a vector bundle. For each $y_0 \in Y_0$ consider the set of pairs $(x, g) \in X_0 \times Y_1$ where $g\colon y_0 \to Fx$. They form the objects of a groupoid $\mathcal{G}_{y_0}$ which is contractible in the sense that there is a unique arrow between any two objects. The restriction of the vector bundle $E_0 \to X_0$ to this groupoid has a limit which is a vector space isomorphic to the fiber over any object. (You may think of it as the vector space of invariant sections over the contractible groupoid $\mathcal{G}_{y_0}$.) Define the fiber of $(F_*E)_0$ at y_0 to be this vector space. To topologize and see we get a vector bundle $(F_*E)_0 \to Y_0$ use the local lifts $\tilde{i}$ in (15.59). You will need to check that the topology is independent of the local lift. The clutching data ψ also descends: if $(y_0 \xrightarrow{h} y_1) \in Y_1$ and we choose $x_0, x_1 \in X_0$ together with arrows $(y_0 \xrightarrow{g_0} Fx_0)$, $(y_1 \xrightarrow{g_1} Fx_1)$ in Y_1, then the composite $g_1 h g_0^{-1}$ has a unique lift to $(x_0 \xrightarrow{f} x_1) \in X_1$ and we use ψ_f to define an isomorphism between the fibers of $(F_*E)_0$ at y_0 and y_1. ☐

Homotopy invariance

The definition of local quotient groupoid is designed in part so that the following extension of Theorem 2.1 holds.

Theorem 15.85. *Let X be a local quotient groupoid. Suppose $E \to [0,1] \times X$ is a vector bundle, and denote by $j_t\colon X \to [0,1] \times X$ the inclusion $j_t(x) = (t,x)$. Then there exists an isomorphism*

$$j_0^*E \xrightarrow{\cong} j_1^*E. \tag{15.86}$$

Proof. First we prove the homotopy invariance for a global quotient. Thus suppose S is a paracompact Hausdorff space with the continuous action of a compact Lie group G, and let $E \to [0,1] \times S$ be a vector bundle. By Theorem 2.1 there exists a nonequivariant trivialization

$$[0,1] \times S \times \mathbb{E} \longrightarrow E \tag{15.87}$$

of vector bundles over $[0,1] \times S$. The G-action on $E \to [0,1] \times S$ transports to a continuous family

$$\psi_g(t,s) \in \operatorname{End}\mathbb{E}, \qquad g \in G, \quad t \in [0,1], \quad s \in S, \tag{15.88}$$

of endomorphisms, thought of as the lift of the arrow $(t,s) \xrightarrow{g} (t,gs)$. For $t \leqslant t'$ we average against Haar measure dg on the compact Lie group G to define

$$\varphi(t,t';s) = \int_G dg\, \psi_g(t', g^{-1}s)\psi_{g^{-1}}(t,s) \quad \in \operatorname{End}\mathbb{E}, \tag{15.89}$$

thought of as a homomorphism $\mathbb{E}_{(t,s)} \to \mathbb{E}_{(t',s)}$. A direct check shows it is G-invariant. Since isomorphisms are open in $\operatorname{End}\mathbb{E}$, and $\varphi(t,t;s) = \mathrm{id}_{\mathbb{E}}$, we see that $\varphi(t,t';s)$ is an isomorphism for t' sufficiently close to t. Now argue as in the topological proof of Theorem 2.1 in Lecture 1. First, for each $s \in S$ we see by compactness of $[0,1]$ that there exists $0 < t_1 < t_2 < \cdots < t_N < 1$ such that the composition

$$\varphi(t_N,1;s) \circ \cdots \circ \varphi(t_1,t_2;s) \circ \varphi(0,t_1;s) \tag{15.90}$$

is an isomorphism. Then by paracompactness cover S by open sets U for which we string together G-inveriant isomorphisms (15.90) for all $s \in U$. Use a partition of unity for a locally finite refinement to patch.

Returning to the general local quotient groupoid X, cover $[X]$ by open sets on which the restriction of X is locally equivalent to a global quotient S/G. Then apply the argument of Theorem 2.1 again to paste the trivializations of the previous paragraph. ☐

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