Flux quantization of general higher gauge theories. We explain ([FSS23-Char][SS23a]) how Dirac's flux quantization generalizes to any higher gauge theory:

Higher Maxwell-type equations have a **characteristic L_∞-algebra $\mathfrak{a}$**: The flux densities are equivalently $\mathfrak{a}$-valued differential forms, and the Gauß law (17) is equivalently the condition that these be *closed* (i.e.: flat, aka "Maurer-Cartan element"; in Italian SuGra literature: "satisfying an FDA").

$$\mathrm{SolSpace}(X^d) \simeq \left\{ \underset{\text{flux densities on Cauchy surface}}{\vec{B} \equiv \Big(B^{(i)} \in \Omega^{\deg_i}_{\mathrm{dR}}\big(X^d\big)\Big)_{i \in I}} \;\middle|\; \underset{\text{satisfying Gauß's law}}{\mathrm{d}\,\vec{B} = \vec{P}(\vec{B})} \right\}$$
$$\simeq \Omega_{\mathrm{dR}}\big(X^d;\, \mathfrak{a}\big)_{\mathrm{clsd}} \quad \text{flat differential forms valued in characteristic } L_\infty\text{-algebra}.$$

Also every topological space $\mathcal{A}$ (under mild conditions) has a characteristic L_∞-algebra: Its $\mathbb{R}$-rational **Whitehead bracket L_∞-algebra $\mathfrak{l}\mathcal{A}$**.

$$\underset{\text{(homotopy type of) a topological space}}{\mathcal{A}} \;\underset{\mathbb{R}\text{-rationalization}}{\rightsquigarrow}\; \underset{\text{Whitehead } L_\infty\text{-algebra}}{\mathfrak{l}\mathcal{A}}$$

The **nonabelian** Chern-Dold **character map** turns $\mathcal{A}$-valued maps into closed $\mathfrak{l}\mathcal{A}$-valued differential forms, generalizing the Chern character for $\mathcal{A} = \mathrm{KU}_0$.

$$\text{charge } \big(\chi : X^d \to \mathcal{A}\big) \;\underset{\text{character map in } \mathcal{A}\text{-cohomology}}{\longmapsto}\; \mathrm{ch}(\chi) \in \Omega_{\mathrm{dR}}\big(X^d; \mathfrak{l}\mathcal{A}\big)_{\mathrm{clsd}}$$

The **possible flux quantization laws** for a given higher gauge field are those spaces $\mathcal{A}$ whose Whitehead L_∞-algebra is the characteristic one.

$$\mathrm{FluxQuantLaws} = \left\{ \underset{\text{classifying spaces}}{\mathcal{A}} \;\middle|\; \underset{\text{whose rational homotopy encodes the Gauß law}}{\mathfrak{l}\mathcal{A} \simeq \mathfrak{a}} \right\}$$

Given a flux quantization law $\mathcal{A}$, the corresponding **higher gauge potentials** are deformations of the flux densities into characters of a $\mathcal{A}$-valued map, witnessing the flux densities as reflecting discrete charges quantized in $\mathcal{A}$-cohomology.
(It is not obvious that this reduces to the usual notion of gauge potentials, but it does.)

$$\begin{array}{ccl} & & \chi \quad \text{charge} \\ & & \big\downarrow \text{ character} \\ & & \mathrm{ch}(\chi) \\ & \swarrow\!\!\!\!\swarrow & \widehat{A} \quad \text{gauge potential} \\ \text{flux density } \vec{F} \;\underset{\text{shape}}{\longmapsto}\; \vec{F} & & \end{array}$$

These non-perturbatively completed higher gauge fields form a *smooth higher groupoid*: the "canonical **differential $\mathcal{A}$-cohomology** moduli stack". Since these are now the flux-quantized on-shell fields, this is the **phase space** of the flux-quantized higher gauge theory (p. 13).

flux-quantized phase space stack is

$$\widehat{\mathcal{A}}(X^d) := \left\{ \left(\begin{array}{lcll} \vec{F} & \in & \Omega_{\mathrm{dR}}(X^d; \mathfrak{l}\mathcal{A})_{\mathrm{clsd}} & \text{flux} \\ \chi & \in & \mathrm{Map}(X; \mathcal{A}) & \text{charge} \\ \widehat{A} & : & \mathrm{ch}(\chi) \Rightarrow \vec{F} & \text{gauge} \end{array} \right) \right\}$$

differential $\mathcal{A}$-cohomology moduli stack

The topological sector of the phase space. The flux-quantized phase space hence subsumes the "solitonic" fields with non-trivial charge sectors χ, and as such is a non-perturbative completion of the traditional phase spaces (which correspond to a fixed charge sector only, typically to $\chi = 0$).

The shape (topological realization) of this phase space stack is the **space of topological fields**,

$$\textstyle\int \widehat{\mathcal{A}}(X^d) \simeq \mathcal{A}(X^d) = \mathrm{Map}\big(X^d, \mathcal{A}\big).$$

which implies that the ordinary homology of the phase space stack constitutes the **topological observables** on the higher gauge theory.

$$\begin{aligned} & H_\bullet\Big(\widehat{\mathcal{A}}(X^d); \mathbb{C}\Big) \\ \simeq\; & H_\bullet\Big(\mathcal{A}(X^d); \mathbb{C}\Big) \end{aligned}$$

Hence if we focus only on the solitonic or *topological field*-content of the phase space, then we see plain $\mathcal{A}$-cohomology moduli of the Cauchy surface. and the full phase space stack only serves to justify this object.

flux-quantized topological phase space

$$\mathcal{A}(X^d) := \Big\{ \chi \in \mathrm{Map}(X, \mathcal{A}) \Big\}$$

non-abelian $\mathcal{A}$-cohomology moduli space

Therefore the reader need not be further concernd with higher stack theory for the present purpose.