Equivariant Stable Cohomotopy and Branes

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Part II

joint work with H. Sati

and S. Burton

Recall conclusion of Part I:

A compelling candidate for M-brane charge cohomology theory is...

by discussion in Part I	non-linear brane/anti-brane bound states	fundamental & black M-branes	by [HSS18] (Huerta's talk)
absolute K-theory	$K\mathbb{F}_1$	$\overset{G_{\mathrm{ADE}}}{\curvearrowright} S^4$	ADE-equivariant Cohomotopy
stable Cohomotopy	$\mathbb{S}$	$S^{\mathbb{H}}$	in RO-degree $\mathbb{H}$ (quaternions)

$$\begin{tikzcd}
K\mathbb{F}_1 \arrow[d, phantom, "\wr"] & & \overset{G_{\mathrm{ADE}}}{\curvearrowright} S^4 \arrow[d, phantom, "\wr"] \\
\mathbb{S} \arrow[dr, mapsto, "\text{equivariantize}"] & & S^{\mathbb{H}} \arrow[dl, mapsto, "\text{stabilize}"'] \\
& \mathbb{S}^{\mathbb{H}}_{G_{\mathrm{ADE}}} \simeq \Sigma^\infty_{G_{\mathrm{ADE}}} S^{\mathbb{H}} &
\end{tikzcd}$$

a

ADE-equivariant stable Cohomotopy in RO-degree $\mathbb{H}$

Hypothesis H:

The **generalized cohomology theory** for **M-brane charge**	is	**ADE-equivariant stable Cohomotopy in RO-degree $\mathbb{H}$**

Hypothesis **H** predicts M-brane charge groups:

$$\mathbb{S}^{\mathbb{H}}_{G_{\mathrm{ADE}}}\Big(\underbrace{\overset{G_{\mathrm{ADE}}}{\circlearrowright} X}_{\substack{\text{11d spacetime} \\ \text{orbifold}}} \Big)$$

How does this compare to / clarify folklore of perturbative string theory:

- intersecting MK6-branes $\rightsquigarrow$ fractional D-branes ?
- M-theoretic "discrete torsion" of fractional M5-branes ?
- GUT at E-type singularities ?
- $\cdots$

This we discuss now $\longrightarrow$

Strategy for testing Hypothesis **H**

1. **Identify** suitable comparison homomorphism

$$\boxed{\begin{array}{c}\text{putative}\\ \text{M-brane charge}\\ \text{in}\\ \text{equivariant stable Cohomotopy}\end{array}} \xrightarrow[\substack{\text{putative}\\ \text{perturbative}\\ \text{approximation}}]{\text{“}\beta\circ\alpha\text{”}} \boxed{\begin{array}{c}\text{putative}\\ \text{D-brane charge}\\ \text{in}\\ \text{equivariant K-theory}\end{array}}$$

2. **Compute**:

$$\frac{\text{the } \textbf{co-kernel} \text{ of } \beta\circ\alpha; \quad \text{reflects} \quad \begin{array}{c}\text{D-brane configurations}\\ \text{that do not lift}\\ \text{to M-theory}\end{array}}{\text{the } \textbf{kernel} \text{ of } \beta\circ\alpha; \quad \text{reflects} \quad \begin{array}{c}\text{M-brane degrees of freedom}\\ \text{invisible in}\\ \text{perturbative string theory}\end{array}}$$

Hypothesis H finds support if the **cokernel of** $\beta \circ \alpha$ is

1. **small** $\Leftrightarrow$ putative M-brane charge mostly reproduces string theory folklore,
2. **plausible** $\Leftrightarrow$ the putative D-brane states in the co-kernel are dubious.

If so, Hypothsis **H** predicts the **kernel of** $\beta \circ \alpha$ as hidden M-theoretic DOFs.

Outline

Since the sphere spectrum $\mathbb{S}$
is the *initial* commutative ring spectrum,
there is a unique multiplicative comparison morphism
from stable cohomotopy
to *every* other multiplicative cohomology theory $\mathcal{A}$,
called the
equivariant generalized Boardman homomorphism

$$\mathbb{S}^{\alpha}_{G}(X) \xrightarrow{G_A} \mathcal{A}^{\alpha}_{G}(X)$$

Here we present two cases:

1. **Comparison map A** to
K-theory and RR-charge of fractional D-branes
2. **Comparison map B** to
ordinary cohomology and "discrete torsion" of fractional M5-branes

Comparison A to

K-theory

and

fractional RR-charge of D-branes

Finite subgroups $G_{\mathrm{ADE}} \subset \mathrm{SU}(2)$ – Classification

Dynkin Label	**Finite subgroup of** $\mathrm{SU}(2)$	**Name of group**
$\mathbb{A}_{n\geq 1}$	$\mathbb{Z}_{n+1}$	Cyclic
$\mathbb{D}_{n\geq 4}$	$2\mathrm{D}_{2(n-2)}$	Binary dihedral
$\mathbb{E}_6$	$2\mathrm{T}$	Binary tetrahedral
$\mathbb{E}_7$	$2\mathrm{O}$	Binary octahedral
$\mathbb{E}_8$	$2\mathrm{I}$	Binary icosahedral

Assumption: In the following, consider finite groups

$G = G_{\mathrm{DE}} \subset \mathbb{E} \subset \mathrm{SU}(2)$

in the D- or E-series

and

in the exceptional subgroup lattice. $\xrightarrow{\text{next slide}}$

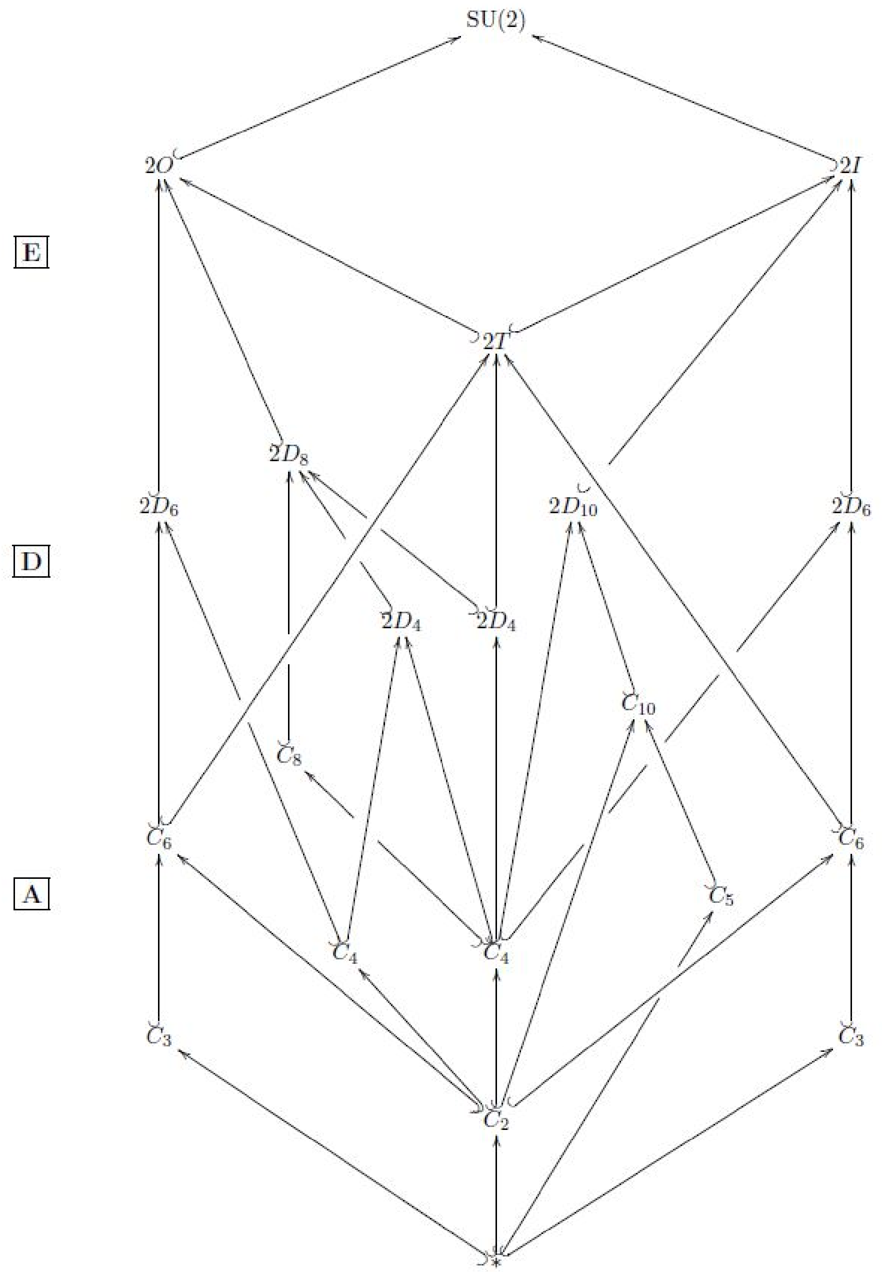
Implies in particular:

G-orbi-folds are G-*orienti-folds*,

the relevant K-theory for fractional D-brane charge

at G-fixed points is is KO-theory

Finite subgroups $G_{\mathrm{ADE}} \subset \mathrm{SU}(2)$ – Exceptional subgroup lattice



The comparison homomorphism A

charge group of M-branes at two intersecting MK6 ADE-singularities visisible in M-theory (?)	charge lattice of fractional M-branes at an MK6-singularity visisible in M-theory (?)	charge lattice of fractional D-branes at orientifold singularity expected in perturbative string theory

equivariant stable cohomotopy in $\mathrm{RO}(G \times G')$-degree $\mathbb{H}$

equivarian KO-theory in degree 0

$$
\begin{tikzcd}
\mathbb{S}^{\mathbb{H}}_{(G\times G')}(\mathbb{R}^{10,1}) \arrow[r, "\boxed{\text{zoom in onto one MK6}}"] \arrow[d, phantom, "\wr\mid"] & \mathbb{S}_G^{\left(\mathbb{H}^{G'}\right)}\left(\mathbb{R}^{6,1}\right) \arrow[r, "\boxed{\text{perturbative approximation}}"] \arrow[d, phantom, "\wr\mid"] & \mathrm{KO}^0_G(\mathbb{R}^{6,1}) \arrow[d, phantom, "\wr\mid"] \\
\mathbb{S}^{\mathbb{H}}_{(G\times G')}(*) \arrow[r, "\alpha", "\boxed{\text{form geometric } G'\text{-fixed spectrum}}"'] & \mathbb{S}^0_G(*) \arrow[r, "\boxed{\text{Boardman homomorphism}}"] \arrow[d, phantom, "\wr\mid"] & \mathrm{KO}^0_G(*) \arrow[d, phantom, "\wr\mid"] \\
 & A\,(G) \arrow[r, "\beta", "\boxed{\text{linearize } G\text{-actions}}"'] & \mathrm{RO}\,(G)
\end{tikzcd}
$$

module over $(G \times G')$-Burnside ring **Thm. 1** G-Burnside ring Thm. 2 G-Representation ring

Theorem 1

charge group
of M-branes
at two intersecting
MK6 ADE-singularities

visible in
M-theory
(?)

charge lattice
of fractional M-branes
at an MK6-singularity

visible in
M-theory
(?)

Proof.
Use Prop. II 9.13 in [LewisMaySteinberger86].
□

equivariant stable cohomotopy in $\mathrm{RO}(G \times G')$-degree $\mathbb{H}$

$$
\begin{tikzcd}
\mathbb{S}^{\mathbb{H}}_{(G \times G')}\left(\mathbb{R}^{10,1}\right) \arrow[rr, "\boxed{\text{zoom in onto one MK6}}"] \arrow[d, "\wr" description, phantom] & & \mathbb{S}_G^{\left(\mathbb{H}^{G'}\right)}\left(\mathbb{R}^{6,1}\right) \arrow[d, "\wr" description, phantom] \\
\mathbb{S}^{\mathbb{H}}_{(G \times G')}(*) \arrow[rr, "\alpha", "\boxed{\text{form geometric } G'\text{-fixed spectrum}}"'] & & \mathbb{S}^0_G(*) \arrow[d, "\wr" description, phantom] \\
& & A(G)
\end{tikzcd}
$$

module over
$(G \times G')$-Burnside ring

Is surjective.

G-Burnside ring

hence: $\mathrm{coker}(\beta \circ \alpha) \simeq \mathrm{coker}(\beta)$

The comparison homomorphism A

charge group
of M-branes
at two intersecting
MK6 ADE-singularities

visisible in
M-theory
(?)

charge lattice
of fractional M-branes
at an MK6-singularity

visisible in
M-theory
(?)

charge lattice
of fractional D-branes
at orientifold singularity

expected in
perturbative
string theory

equivariant stable cohomotopy in $\mathrm{RO}(G \times G')$-degree $\mathbb{H}$

equivarian KO-theory
in degree 0

$$
\begin{tikzcd}
\mathbb{S}^{\mathbb{H}}_{(G\times G')}(\mathbb{R}^{10,1}) \arrow[r, "\boxed{\text{zoom in onto one MK6}}"] \arrow[d, phantom, "\wr|"] & \mathbb{S}_G^{\left(\mathbb{H}^{G'}\right)}\left(\mathbb{R}^{6,1}\right) \arrow[r, "\boxed{\text{perturbative approximation}}"] \arrow[d, phantom, "\wr|"] & \mathrm{KO}^0_G(\mathbb{R}^{6,1}) \arrow[d, phantom, "\wr|"] \\
\mathbb{S}^{\mathbb{H}}_{(G\times G')}(\ast) \arrow[r, "\alpha", "\boxed{\text{form geometric } G'\text{-fixed spectrum}}"'] & \mathbb{S}^0_G(\ast) \arrow[r, "\boxed{\text{Bordman homomorphism}}"] \arrow[d, phantom, "\wr|"] & \mathrm{KO}^0_G(\ast) \arrow[d, phantom, "\wr|"] \\
 & A\left(G\right) \arrow[r, "\beta", "\boxed{\text{linearize } G\text{-actions}}"'] & \mathrm{RO}\left(G\right)
\end{tikzcd}
$$

module over
$(G \times G')$-Burnside ring

Thm. 1

G-Burnside ring

Thm. 2

G-Representation ring

Theorem 2

$$\underbrace{\mathrm{RO}^{\mathrm{irrational}}(G)}_{\text{irrational characters}} \simeq \mathrm{RO}(G) \;/\; \underbrace{\mathrm{RO}^{\mathrm{int}}(G)}_{\text{integral characters}}$$

charge lattice of fractional M-branes at an MK6-singularity visible in M-theory (?)	charge lattice of fractional D-branes at orientifold singularity expected in perturbative string theory
equivarian stable cohomotopy in degree 0	equivarian KO-theory in degree 0

$$\begin{array}{ccc}
\mathbb{S}_G^{\left(\mathbb{H}^{G'}\right)}\left(\mathbb{R}^{6,1}\right) & \xrightarrow{\boxed{\text{perturbative approximation}}} & \mathrm{KO}_G^0\left(\mathbb{R}^{6,1}\right) \\
\wr\| & & \wr\| \\
\mathbb{S}_G^0(*) & \xrightarrow{\boxed{\text{Bordman homomorphism}}} & \mathrm{KO}_G^0(*) \\
\wr\| & & \wr\| \\
A\,(G) & \underset{\boxed{\text{linearize } G\text{-actions}}}{\xrightarrow{\;\;\beta\;\;}} & \mathrm{RO}\,(G)
\end{array}$$

$\boxed{G\text{-Burnside ring}}$ $\qquad \mathrm{coker}(\beta) \simeq \mathrm{RO}^{\mathrm{irrational}}(G) \qquad$ $\boxed{G\text{-Representation ring}}$

Theorem 2 – Ingredients

G-actions on finite sets (not-linear)

G-representations on $\mathbb{F}$-vector spaces (linear)

$$\begin{tikzcd}
(G\mathrm{Set}_{\mathrm{fin}}, \sqcup) \arrow[r, "\substack{\text{linearize} \\ \mathbb{F}[-]}"] \arrow[d, "K"'] & (G\mathrm{Rep}_{\mathbb{F}}, \oplus) \arrow[d, "K"] \\
A(G) \arrow[r, "\beta"] & R_{\mathbb{F}}(G)
\end{tikzcd}$$

Burnside ring of virtual G-sets

Representation ring of virtual representations

$\boxed{G\text{-set}}$

$$G \curvearrowright S \longmapsto G \curvearrowright \mathbb{F}[S]$$

"permutation representation"

the **cokernel** of β	reflects	linear algebra invisible to pure combinatorics
the **kernel** of β	reflects	pure combinatorics invisible to linear algebras

Compute:

1. set of conjugacy classes $\{[H_i]\}$ of subgroups $H \subset G$
2. the Burnside product $[G/H_i] \times [G/H_j] \;=\; \coprod_\ell \underbrace{n^\ell_{ij}}_{\substack{\text{structure}\\ \text{constants}}} \cdot [G/H_\ell]$
3. its matrix of total multiplicities $\;\mathrm{mult}_{ij} \;\coloneqq\; \sum_\ell n^\ell_{ij}$
4. its integral row reduction $\;\underbrace{H}_{\substack{\text{upper}\\ \text{triangular}}} \;\coloneqq\; \underbrace{U}_{\in \mathrm{GL}(N,\mathbb{Z})} \cdot \mathrm{mult}$

Lemma. *The rows of H span* $\mathrm{im}(\beta) \subset R_{\mathbb{F}}(G)$.

This yields an effective algorithm computing coker(β) $\;=\; R_{\mathbb{F}}(G)/\mathrm{im}(\beta)$

Simon Burton has implemented this algorithm in `Python`.

$\Rightarrow$ **Proof of Theorem 2:** By brute force automatized computation. $\square$

Theorem 2 – Proof

coker	$A(G) \overset{\beta_{\mathbb{F}}}{\to} R_{\mathbb{F}}(G)$			$A(G) \overset{\beta^{\text{int}}_{\mathbb{F}}}{\to} R^{\text{int}}_{\mathbb{F}}(G)$		
	ground field $\mathbb{F}$			ground field $\mathbb{F}$		
finite group G	$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{C}$
$2D_4$	0	0	$\frac{\mathbb{Z}[\rho_5]}{\mathbb{Z}[2\rho_5]}$	0	0	$\frac{\mathbb{Z}[\rho_5]}{\mathbb{Z}[2\rho_5]}$
$2D_6$	0	0	$\frac{\mathbb{Z}[\rho_3,\rho_4,\rho_6]}{\mathbb{Z}[\rho_3+\rho_4,2\rho_6]}$	0	0	$\frac{\mathbb{Z}[\rho_6]}{\mathbb{Z}[2\rho_6]}$
$2D_8$	0	$\frac{\mathbb{Z}[2\rho_6,2\rho_7]}{\mathbb{Z}[2\rho_6+2\rho_7]}$	$\frac{\mathbb{Z}[\rho_6,\rho_7]}{\mathbb{Z}[2\rho_6+2\rho_7]}$	0	0	$\frac{\mathbb{Z}[\rho_6+\rho_7]}{\mathbb{Z}[2\rho_6+2\rho_7]}$
$2D_{10}$	0	$\frac{\mathbb{Z}[\rho_3,\rho_4,\rho_5,\rho_6,\rho_7,\rho_8]}{\mathbb{Z}[\rho_3+\rho_4,\rho_5+\rho_6,2\rho_7+2\rho_8]}$	$\frac{\mathbb{Z}[\rho_3,\rho_4,\rho_5,\rho_6,2\rho_7,2\rho_8]}{\mathbb{Z}[\rho_3+\rho_4,\rho_5+\rho_6,2\rho_7+2\rho_8]}$	0	0	$\frac{\mathbb{Z}[\rho_7+\rho_8]}{\mathbb{Z}[2\rho_7+2\rho_8]}$
$2D_{12}$	0	$\frac{\mathbb{Z}[\rho_7,\rho_8,\rho_9]}{\mathbb{Z}[2\rho_7,2\rho_8+2\rho_9]}$	$\frac{\mathbb{Z}[2\rho_8,2\rho_9]}{\mathbb{Z}[2\rho_8+2\rho_9]}$	0	0	$\frac{\mathbb{Z}[\rho_7]}{\mathbb{Z}[2\rho_7]}$
$2T$	0	0	$\frac{\mathbb{Z}[\rho_2,\rho_2^*,\rho_4,\rho_4^*,\rho_5]}{\mathbb{Z}[\rho_2+\rho_2^*,\rho_4+\rho_4^*,2\rho_5]}$	0	0	$\frac{\mathbb{Z}[\rho_5]}{\mathbb{Z}[2\rho_5]}$
$2O$	0	$\frac{\mathbb{Z}[2\rho_6,2\rho_7]}{\mathbb{Z}[2\rho_6+2\rho_7]}$	$\frac{\mathbb{Z}[\rho_6,\rho_7,\rho_8]}{\mathbb{Z}[2\rho_6+2\rho_7,2\rho_8]}$	0	0	$\frac{\mathbb{Z}[\rho_8]}{\mathbb{Z}[2\rho_8]}$
$2I$	0	$\frac{\mathbb{Z}[2\rho_2,2\rho_3,\rho_4,\rho_5]}{\mathbb{Z}[2\rho_2+2\rho_3,\rho_4+\rho_5]}$	$\frac{\mathbb{Z}[\rho_2,\rho_3,\rho_4,\rho_5,\rho_7,\rho_9]}{\mathbb{Z}[2\rho_2+2\rho_3,\rho_4+\rho_5,2\rho_7,2\rho_9]}$	0	0	$\frac{\mathbb{Z}[\rho_2+\rho_3,\rho_4+\rho_5,\rho_7,\rho_9]}{\mathbb{Z}[2\rho_2+2\rho_3,\rho_4+\rho_5,2\rho_7,2\rho_9]}$

Theorem 2 – Physics interpretation

Let $V \in K_G(*) \simeq R(G)$ a fractional D-brane $\leftrightarrow$ G-representation,

RR-charge in the g-twisted closed string sector

is the value of its character at g:

$$Q_V^{\mathrm{RR}}(g) \;=\; \frac{1}{|G|}\chi_V(g)$$

([DouglasGreeneMorrisson97, (3.8)], [DiGo00, (2.4)], [BCR00, (4.65) with (4.41)], [EGJ05, (4.5)], [ReSc13, 4.102])

Theorem 2 – Physics reformulation:

Hypothesis **H** *implies*
that fractional D-branes with irrational RR-charge are spurious.

Physically plausible?

Some $V \in K_G(*)$ *must* be spurious [BDHKMMS02, 4.5.2].

Irrational RR-charge called a *paradox* in [BachasDouglasSchweigert00, (2.8)], also [Taylor00, Zho01, Rajan02], apparently unresolved.

If this is indeed a *paradox*,
then hypothesis **H** exactly resolves it.

Theorem 2 – Physics interpretation

Regard coker(β) under **McKay correspondence** :

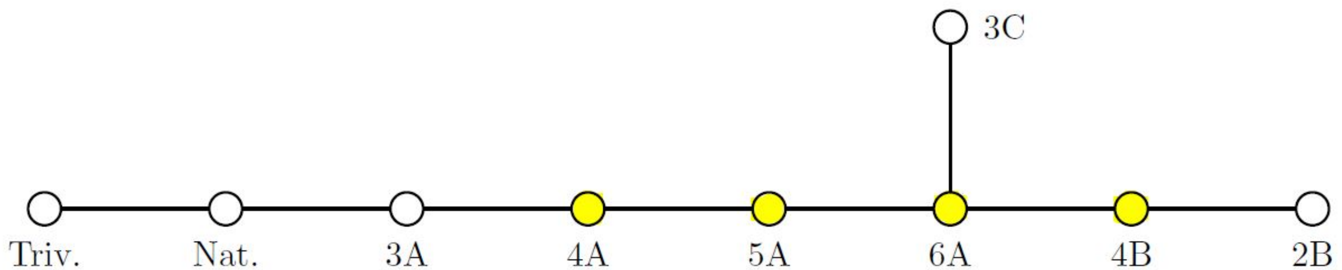
$$\left\{ \text{ irreps } \rho \in R_{\mathbb{C}}(G_{\mathrm{ADE}}) \right\} \simeq \left\{ \begin{array}{c} \text{vertices of corresponding} \\ \text{ADE-type Dynkin diagram} \end{array} \right\}$$

Most exceptional Example: $G = 2I$:

integral/
non-irrational
characters

	e	a	a^2	a^3	a^2b	a^4	a^3b	a^5	a^4b
Triv.	1	1	1	1	1	1	1	1	1
Nat.	2	$\frac{1}{2}(1+\sqrt{5})$	$-\frac{1}{2}(1-\sqrt{5})$	$\frac{1}{2}(1-\sqrt{5})$	1	$-\frac{1}{2}(1+\sqrt{5})$	0	-2	-1
3A	3	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1-\sqrt{5})$	$\frac{1}{2}(1-\sqrt{5})$	0	$\frac{1}{2}(1+\sqrt{5})$	-1	3	0
4A	4	1	-1	1	-1	-1	0	-4	1
5A	5	0	0	0	-1	0	1	5	-1
6A	6	-1	1	-1	0	1	0	-6	0
2B	2	$\frac{1}{2}(1-\sqrt{5})$	$-\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1+\sqrt{5})$	1	$-\frac{1}{2}(1-\sqrt{5})$	0	-2	-1
4B	4	-1	-1	-1	1	-1	0	4	1
3C	3	$\frac{1}{2}(1-\sqrt{5})$	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1+\sqrt{5})$	0	$\frac{1}{2}(1-\sqrt{5})$	-1	3	0

Dynkin
diagram



hence:

$$\mathrm{im}(\beta)_{|\mathrm{irred}} \subset \mathrm{RO}(2I)_{|\mathrm{irred}} \leftrightarrow \underbrace{SU(5)}_{\text{actual GUT group}} \subset \underbrace{E_8}_{\text{stringy GUT group}}$$

Comparison B to

ordinary Cohomology

and

"discrete torsion" of fractional M5-branes

The comparison homomorphism B

Away from the singular locus
of a black M2-brane

$$
\begin{array}{ccccc}
 & & \cdots & & \\
 & \overset{\ell_P \gg 1}{\swarrow} & & \overset{\ell_P \ll 1}{\searrow} & \\
\mathrm{AdS}_4 \times \underbrace{S^7/G_{\mathrm{A}}}_{\text{spherical space form}} & & & & \mathbb{R}^{2,1} \times \underbrace{\mathbb{R}^8 /\!\!/ G_{\mathrm{A}}}_{\text{orbifold du Val singularity}}
\end{array}
$$

the orbifold is smooth and, for A-type singularities, so is the RO-degree:

$$
\begin{array}{cccc}
\begin{array}{c}\text{coefficient}\\ \text{bundle}\end{array} & \big((\mathbb{R}^{10,1} - \mathbb{R}^{2,1}) \times \mathbb{H}\big) /\!\!/ G_A & & \big(\mathbb{R}^{10,1} - \mathbb{R}^{2,1}\big)/G_A \times \mathbb{R}^4 \\
 & \downarrow & \simeq & \downarrow \\
\begin{array}{c}\text{spacetime}\\ \text{orbifold}\end{array} & \big(\mathbb{R}^{10,1} - \mathbb{R}^{2,1}\big) /\!\!/ G_A & & \big(\mathbb{R}^{10,1} - \mathbb{R}^{2,1}\big)/G_A
\end{array}
$$

$\Rightarrow$ relevant comparison morphism is equivariant Boardman homomorphism

$$
\underbrace{\mathbb{S}^4_{G_A}(S^7)}_{\substack{\text{equivariant}\\ \text{stable Cohomotopy}}} \xrightarrow{\quad \beta_{G_A} \quad} \underbrace{H\mathbb{Z}^4_{G_A}(S^7)}_{\substack{\text{equivariant}\\ \text{ordinary cohomology}}} \simeq \underbrace{H^4(S^7/G_A, \mathbb{Z})}_{\substack{\text{Borel}\\ \text{equivariance}}}
$$

The comparison homomorphism B

Theorem 3 i): factors through plain Boardman homomorphism:

$$\boxed{\begin{array}{c}\text{putative}\\ \text{C-field flux /}\\ \text{M5-brane charge}\\ \text{visible in M-theory}\\ \text{(?)}\end{array}} \xrightarrow[\substack{\text{putative}\\ \text{perturbative}\\ \text{approximation}}]{\text{“}\beta \circ \alpha\text{”}} \boxed{\begin{array}{c}\text{putative}\\ \text{M-theoretic discrete torsion}\\ \text{of fractional M5-brane charge}\\ \text{[AharonyBergmanJafferis08, Sharpe03]}\end{array}}$$

equivariant Boardmamn homomorphism

$$\begin{tikzcd}
\mathbb{S}^4_{G_A}\left(S^7\right) \arrow[rr, "\beta_{G_A}"] \arrow[dr, "\alpha"'] & & H^4_{G_A}\left(S^7/G_{\mathrm{A}}, \mathbb{Z}\right) \\
& \mathbb{S}^4\left(S^7/G_{\mathrm{A}}\right) \arrow[ur, "\beta"'] &
\end{tikzcd}$$

Carlsson-Segal completion

plain Boardman homomorphism

Proof. Use [Schwede18, Example 4.5.19]. $\square$.

Theorem 3 ii): $4\,\mathrm{coker}(\beta) = 0$

Proof. By [Arlettaz04, Theorem 1.2 b)]. $\square$

The comparison homomorphism B

Theorem 3 iii): factors isomorphically through ko^4:

$$\begin{tikzcd}[column sep=huge] \boxed{\begin{array}{c}\text{putative}\\ \text{C-field flux /}\\ \text{M5-brane charge}\\ \text{visible in M-theory}\\ \text{(?)}\end{array}} \arrow[r, "\text{“}\beta \circ \alpha\text{”}", "\begin{array}{c}\text{putative}\\ \text{perturbative}\\ \text{approximation}\end{array}"'] & \boxed{\begin{array}{c}\text{putative}\\ \text{M-theoretic discrete torsion}\\ \text{of fractional M5-brane charge}\\ \text{[AharonyBergmanJafferis08, Sharpe03]}\end{array}} \end{tikzcd}$$

$$\boxed{\text{equivariant Boardmamn homomorphism}}$$

$$\begin{tikzcd} & \mathbb{S}^4_{G_A}\left(S^7\right) \arrow[rrr, "\beta_{G_A}"] \arrow[dr, "\alpha"'] & & & H^4\left(S^7/G_{\mathrm{A}}, \mathbb{Z}\right) \\ \boxed{\text{Carlsson-Segal completion}} & & \mathbb{S}^4\left(S^7/G_{\mathrm{A}}\right) \arrow[urr, "\beta"] \arrow[rr, "\simeq", "\boxed{\begin{array}{c}\text{Boardman homomorphism}\\ \text{for ko}\end{array}}"'] & & \mathrm{ko}^4(S^7/G_A) \arrow[u, "p_1", "\boxed{\begin{array}{c}\text{first}\\ \text{Pontryagin class}\end{array}}"'] \end{tikzcd}$$

Proof. By the AHSS and using $\pi_{\bullet\leq 2}(\mathbb{S}) = \pi_{\bullet\leq 2}(\mathrm{ko})$ $\square$.

Physically reasonable?

This coker(β) is KO-version of what was argued for KU in [DiaconescuMooreWitten00].

Conclusion

1. Part I – Motivation of equivariant stable Cohomotopy:

   (a) Derivation of equivariant cohomotopy$_{/\mathbb{Q}}$ from first principles via super homotopy theory of Green-Schwarz sigma-models for M2/M5 [FSS13, FSS16a, FSS16b, BSS18, HSS18]

   (b) Inclusion of anti-branes *should* mean homotopy-theoretic stabilization " Hypothesis **H**"

2. Part II – Consistency checks of Hypothesis **H**:

   (a) reproduces fractional D-brane charge in equivariant K-theory

      i. excluding exactly the spurious irrational RR-charges,

      ii. which may correspond, via McKay, to breaking E_8 to SU(5) GUT

   (b) reproduces discrete torsion of fractional M5-branes with DMW-correction.

   In particular, equivariant stable cohomotopy somehow unifies ordinary cohomology of the C-field with K-theory of D-branes.

Outlook:

Bring in super-gravity EOMs $\Rightarrow$ "self-duality" of M-brane flux $G_7 = \star G_4$.

Idea: Require cocycle in equivariant stable Cohomotopy to be super-torsion free in higher super Cartan geometry ([HSS18, 1. ii)]).

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