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Equivariant Cohomotopy and Branes
—
Towards microscopic M-Theory

talk at
String and M-Theory: The New Geometry of the 21st Century
Singapore 2018

joint work with

Hisham Sati & V. Braunack-Mayer

based on [FSS13, FSS15, FSS16a, FSS16b, HS17, HSS18, BSS18]

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ncatlab.org/schreiber/print/Equivariant+Cohomotopy+and+Branes

Motivation.

Nonperturbative QFT and an old Prophecy

Part I.

Some M-Theory from Super homotopy theory

Part II.

Some corners of M-theory

Motivation

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Glaring open problem of contemporary quantum field theory:

All non-perturbative physics.

Such as:

-quark confinement in hadrons	(existence of ordinary matter!)
-quark-gluon plasma & nucleosynthesis	(becoming of ordinary matter!)
-Higgs field metastability	(existence of vacuum spacetime!)
-QCD cosmology	(becoming of vacuum spacetime!)

Important non-answers:

lattice QFT numerics is (great but) *not the answer*:
like Bohr-Sommerfeld's "old quantization"
it allows to compute some numbers
but without conceptual understanding

string theory is (great but) *not the answer*:
string scattering series just as perturbative as Feynman series
(vanishing radius of convergence, both)

But string theory is the vehicle with which to glimpse **M-Branes**...

The emerging answer: Intersecting M-branes

Web of plausibility arguments and consistency checks suggests:

Non-perturbative standard model of particle physics & cosmology
arises on intersecting M-branes
at asymptotic boundary of approximately AdS spacetime.

In particular

Witten-Sakai-Sugimoto model for QCD:

N_c M5-branes intersecting N_f M9-branes
KK-compactified, breaking all supersymmetry, to
N_c D4-branes intersecting N_f D8-branes

yields QFT at least close to non-perturbative QCD
with transparent interpretation of non-perturbative effects

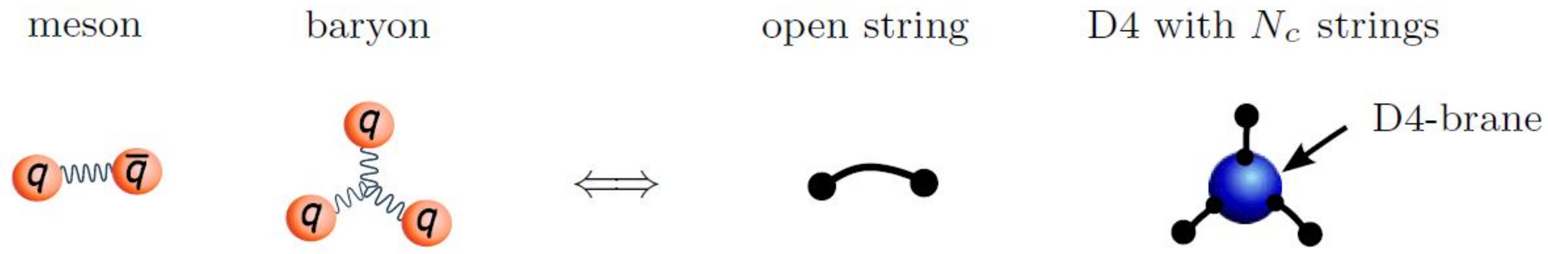

Fig. 15.1. mesons and baryons in quark model and string theory

graphics from Sugimoto 16

However...

Glaring open problem of contemporary M-brane theory:

What is it, really?

> *We still have no fundamental formulation of "M-theory"* -
> *Work on formulating the fundamental principles underlying M-theory has noticeably waned. [...]. If history is a good guide, then we should expect that anything as profound and far-reaching as a fully satisfactory formulation of M-theory is surely going to lead to new and novel mathematics. Regrettably, it is a problem the community seems to have put aside - temporarily. But, ultimately,*
> *Physical Mathematics must return to this grand issue.*

G. Moore, *Physical Mathematics and the Future*, at Strings 2014

Important non-answer:

BFSS/IKKT matrix model is (great but) *not the answer*:

like lattice QFT numerics
it allows to compute some numbers
but without conceptual understanding

What is missing?..

An old prophecy

> *Back in the '70s, the Italian physicist, D, Amati reportedly said that string theory was part of 21st-century physics that fell by chance into the 20th century. I think it was a very wise remark. How wise it was is so clear from the fact that 30 years later we're still trying to understand what string theory really is.*

E. Witten, Nova Interview 2003

New development brought by the 21st century:

Homotopy theory & higher topos theory ("higher structures")

	physics	**mathematics**
	gauge principle	homotopy theory
&	Pauli exclusion	super-geometry
=		**super homotopy theory**

Part I.

Some M-theory from Super homotopy theory

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Super homotopy theory

and the Atom of Superspace

back to Part I

Global equivariant Super homotopy theory

Definition. Consider the 2-site

$$\text{SuperSingularities} := \left\{ \underbrace{\mathbb{R}^{d|N}}_{\begin{array}{c}\text{super-}\\ \text{space}\end{array}} \times \underbrace{\mathbb{D}}_{\begin{array}{c}\text{infinitesimal}\\ \text{disk}\end{array}} \times \underbrace{\mathbb{B}G}_{\begin{array}{c}\text{orbifold}\\ \text{singularity}\end{array}} \right\}$$

Global equivariant
super homotopy theory
is the ∞-stack ∞-topos
over SuperSingularities:

$$\boxed{\mathbf{H} := \underbrace{\mathrm{Sh}_\infty}_{\begin{array}{c}\text{generalized}\\ \text{spaces}\end{array}}(\text{SuperSingularities})}$$

probe-able
by these
local model spaces

[Sch13]
& [Rezk14]

[BMSS19]

$$\begin{array}{c}\text{space-time}\\ \hline \left.\begin{array}{l}\text{SuperManifolds,}\\ \text{SuperOrbifolds}\\ \text{with } G\text{-structure}\end{array}\right\}\end{array} \hookrightarrow \mathbf{H} \hookleftarrow \begin{array}{c}\text{higher gauge fields}\\ \hline \left\{\begin{array}{c}\text{classifying spaces for}\\ \text{equivariant, differential}\\ \text{generalized cohomology theories}\end{array}\right\}\end{array}$$

(e.g.
super-Riemannian-, super-conformal-, ...
Spin-, String-, Fivebrane-, ... ([SSS09])
structure)

The Modalities of Super homotopy theory

The terminal functor factors into a system of dualities = adjunctions.

$$\overbrace{\mathbf{H}}^{\substack{\text{supergeometric} \\ \infty\text{-groupoids}}} \xrightarrow{\qquad\qquad\qquad\qquad \Gamma \qquad\qquad\qquad\qquad} *$$

The Modalities of Super homotopy theory

$\varnothing$

nothing

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}[column sep=huge]
\overbrace{\mathbf{H}}^{\substack{\text{supergeometric}\\ \infty\text{-groupoids}}} \arrow[r, shift right, "\Gamma" description] & \ast \arrow[l, hook', shift right, "\Delta" description]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$\varnothing \dashv *$

pure being

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}[column sep=huge]
\overbrace{\mathbf{H}}^{\substack{\text{supergeometric}\\ \infty\text{-groupoids}}} \arrow[r, shift left=0, "\Gamma" description] & * \arrow[l, hook', shift right=3] \arrow[l, hook', shift left=3, "\nabla" description]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\begin{array}{ccc} \flat & & \\ \vee & & \\ \varnothing & \dashv & \ast \end{array}$$

discrete

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}
\overbrace{\mathbf{H}}^{\substack{\text{supergeometric}\\ \infty\text{-groupoids}}} \arrow[rr, "\Gamma"] & & \overbrace{\mathbf{H}_\flat}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}} \arrow[ll, hook', shift right=6, "\Delta"'] \arrow[ll, hook, shift left=6] \arrow[r] & \ast \arrow[l, hook', shift right=6] \arrow[l, hook, shift left=6]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\boxed{\begin{array}{ccc} \flat & \dashv & \sharp \\ \vee & & \vee \\ \varnothing & \dashv & \ast \end{array}}$$

continuous

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}[column sep=huge]
\overbrace{\mathbf{H}}^{\substack{\text{supergeometric} \\ \infty\text{-groupoids}}} \arrow[rr, "\Gamma" description] & & \overbrace{\mathbf{H}_\flat}^{\substack{\text{geometrically discrete} \\ \infty\text{-groupoids}}} \arrow[ll, hook', shift right=8] \arrow[ll, hook, shift left=8, "\nabla" description] \arrow[r] & \ast \arrow[l, hook', shift right=8] \arrow[l, hook, shift left=8]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$
\boxed{
\begin{array}{ccccc}
\int & \dashv & \flat & \dashv & \sharp \\
 & & \vee & & \vee \\
 & & \varnothing & \dashv & \ast
\end{array}
}
$$

shaped

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}
\overbrace{\mathbf{H}}^{\substack{\text{supergeometric}\\ \infty\text{-groupoids}}}
\arrow[r, shift left=4.5, "\Pi" description]
\arrow[r, hookleftarrow, shift left=1.5, "\Delta" description]
\arrow[r, shift right=1.5]
\arrow[r, hookleftarrow, shift right=4.5]
& \overbrace{\mathbf{H}_\flat}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}
\arrow[r, hookleftarrow, shift left=3]
\arrow[r]
\arrow[r, hookleftarrow, shift right=3]
& \ast
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\begin{array}{ccccc} \Im & & & & \\ \vee & & & & \\ \int & \dashv & \flat & \dashv & \sharp \\ & & \vee & & \vee \\ & & \varnothing & \dashv & \ast \end{array}$$

infinitesimally shaped

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}[column sep=huge]
\overbrace{\mathbf{H}}^{\substack{\text{supergeometric}\\ \infty\text{-groupoids}}}
\arrow[r, shift left=3, "\Pi" description]
\arrow[r, shift right=1]
& \overbrace{\mathbf{H}_{\Re}}^{\substack{\text{reduced differential-geometric}\\ \infty\text{-groupoids}}}
\arrow[l, hook', shift right=1, "\Delta" description]
\arrow[r, shift left=3]
\arrow[r, shift right=1]
& \overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}
\arrow[l, hook', shift right=1]
\arrow[ll, hook', shift left=5]
\arrow[r, shift right=1]
& \ast
\arrow[l, hook', shift right=1]
\arrow[l, hook', shift left=5]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\boxed{\begin{array}{ccccc} \Im & \dashv & \& & & \\ \vee & & \vee & & \\ \int & \dashv & \flat & \dashv & \sharp \\ & & \vee & & \vee \\ & & \varnothing & \dashv & * \end{array}}$$

infinitesimally discrete

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}[column sep=huge]
\overbrace{\mathbf{H}}^{\substack{\text{supergeometric} \\ \infty\text{-groupoids}}}
\arrow[r, shift left=3]
\arrow[r, shift right=1, "\Gamma"']
&
\overbrace{\mathbf{H}_{\Re}}^{\substack{\text{reduced differential-geometric} \\ \infty\text{-groupoids}}}
\arrow[l, hook', shift right=1, "\Delta"']
\arrow[r, shift left=3]
\arrow[r, shift right=1]
&
\overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete} \\ \infty\text{-groupoids}}}
\arrow[l, hook', shift right=1]
\arrow[ll, hook', shift left=5, bend left=10]
\arrow[r, shift right=1]
&
*
\arrow[l, hook', shift right=1]
\arrow[l, hook', shift left=3]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\begin{array}{ccccccccc}
\Re & \dashv & \Im & \dashv & \& & & \\
 & & \vee & & \vee & & \\
 & & \int & \dashv & \flat & \dashv & \sharp \\
 & & & & \vee & & \vee \\
 & & & & \varnothing & \dashv & *
\end{array}$$

reduced

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}[column sep=huge]
\overbrace{\mathbf{H}}^{\text{supergeometric}} \arrow[r, shift left=1, "\Pi" description] \arrow[r, shift right=3] & \overbrace{\mathbf{H}_{\Re}}^{\text{reduced differential-geometric}} \arrow[l, hook', shift right=3, "\iota" description] \arrow[l, hook', shift left=1] \arrow[r, shift left=2] \arrow[r, shift right=2] & \overbrace{\mathbf{H}_{\flat}}^{\text{geometrically discrete } \infty\text{-groupoids}} \arrow[l, hook'] \arrow[ll, hook', bend left=20] \arrow[r] & * \arrow[l, hook', shift right=2] \arrow[l, hook', shift left=2]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\boxed{\begin{array}{ccccccc}
{\color{blue}\rightsquigarrow} & & & & & & \\
\vee & & & & & & \\
\Re & \dashv & \Im & \dashv & \& & & \\
 & & \vee & & \vee & & \\
 & & \int & \dashv & \flat & \dashv & \sharp \\
 & & & & \vee & & \vee \\
 & & & & \emptyset & \dashv & *
\end{array}}$$

bosonic

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}[column sep=huge]
\overbrace{\mathbf{H}}^{\text{supergeometric}}
\arrow[r, "{\color{blue}\Pi}" description]
\arrow[rr, shift right=8]
& \overbrace{\mathbf{H}_{\rightsquigarrow}}^{\color{blue}\text{bosonic}}
\arrow[l, hook', shift right=4, "{\color{blue}\iota}" description]
\arrow[l, hook, shift left=4]
\arrow[r]
& \overbrace{\mathbf{H}_{\Re}}^{\text{reduced}}
\arrow[l, hook', shift right=4]
\arrow[l, hook, shift left=4]
\arrow[r]
\arrow[r, shift right=8]
& \overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}
\arrow[l, hook, shift left=4]
\arrow[lll, hook, shift left=12]
\arrow[r, shift right=8]
& *
\arrow[l, hook, shift left=4]
\arrow[l, hook, shift left=12]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\boxed{\begin{array}{ccccccc}
\rightsquigarrow & \dashv & \mathrm{Rh} & & & & \\
\vee & & \vee & & & & \\
\Re & \dashv & \Im & \dashv & \& & & \\
 & & \vee & & \vee & & \\
 & & \int & \dashv & \flat & \dashv & \sharp \\
 & & & & \vee & & \vee \\
 & & & & \emptyset & \dashv & \ast
\end{array}}$$

rheonomic

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}[column sep=huge]
\overbrace{\mathbf{H}}^{\text{supergeometric}} \arrow[r, shift left=1, "\Pi"] \arrow[rr, bend right=25] & \overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}} \arrow[l, hook', shift right=4] \arrow[l, hook, shift left=-1, "\Delta"] \arrow[r, shift left=1] & \overbrace{\mathbf{H}_{\Re}}^{\text{reduced}} \arrow[l, hook', shift right=4] \arrow[l, hook, shift left=-1] \arrow[r, shift left=3] \arrow[r, shift right=3] & \overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete} \\ \infty\text{-groupoids}}} \arrow[l, hook'] \arrow[lll, hook, bend left=30] \arrow[r, shift right=3] & \ast \arrow[l, hook'] \arrow[l, hook, shift left=6]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\boxed{\begin{array}{ccccccccc}
\rightrightarrows & \dashv & \rightsquigarrow & \dashv & \mathrm{Rh} & & & & \\
 & & \vee & & \vee & & & & \\
 & & \Re & \dashv & \Im & \dashv & \& & & \\
 & & & & \vee & & \vee & & \\
 & & & & \int & \dashv & \flat & \dashv & \sharp \\
 & & & & & & \vee & & \vee \\
 & & & & & & \varnothing & \dashv & \ast
\end{array}}$$

even

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}
\overbrace{\mathbf{H}}^{\substack{\text{super-}\\\text{geometric}}}
\arrow[r, shift left=6, "\epsilon"]
\arrow[r, shift left=2]
\arrow[rr, bend right=20, shift right=2]
& \overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}}
\arrow[l, hook, shift left=2, "\iota"]
\arrow[l, hook, shift right=2]
\arrow[r, shift left=2]
& \overbrace{\mathbf{H}_{\Re}}^{\text{reduced}}
\arrow[l, hook, shift left=6]
\arrow[l, hook, shift right=2]
\arrow[r, shift left=2]
\arrow[r, shift right=4]
& \overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete}\\\infty\text{-groupoids}}}
\arrow[l, hook, shift right=2]
\arrow[lll, hook, bend left=20, shift left=4]
\arrow[r, shift right=4]
& \ast
\arrow[l, hook, shift right=2]
\arrow[l, hook, shift left=6]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\boxed{\begin{array}{ccccccccc}
\mathrm{id} & \dashv & \mathrm{id} & & & & & & \\
\vee & & \vee & & & & & & \\
\rightrightarrows & \dashv & \rightsquigarrow & \dashv & \mathrm{Rh} & & & & \\
 & & \vee & & \vee & & & & \\
 & & \Re & \dashv & \Im & \dashv & \& & & \\
 & & & & \vee & & \vee & & \\
 & & & & \int & \dashv & \flat & \dashv & \sharp \\
 & & & & & & \vee & & \vee \\
 & & & & & & \varnothing & \dashv & \ast
\end{array}}$$

super-geometric

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}
\overbrace{\mathbf{H}}^{\text{super-geometric}} \arrow[r, shift left=9] \arrow[r, shift left=3] \arrow[rr, shift right=9] & \overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}} \arrow[l, hook, shift left=6] \arrow[l, hook] \arrow[r, shift left=3] & \overbrace{\mathbf{H}_{\Re}}^{\text{reduced}} \arrow[l, hook, shift right=6] \arrow[l, hook] \arrow[r, shift left=3] \arrow[r, shift right=9] & \overbrace{\mathbf{H}_{\flat}}^{\text{geometrically discrete } \infty\text{-groupoids}} \arrow[l, hook] \arrow[lll, hook, shift left=12] \arrow[r, shift right=9] & \ast \arrow[l, hook] \arrow[l, hook, shift left=12]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$
\boxed{
\begin{array}{ccccccccc}
\mathrm{id} & \dashv & \mathrm{id} & & & & & & \\
\vee & & \vee & & & & & & \\
\rightrightarrows & \dashv & \rightsquigarrow & \dashv & \mathrm{Rh} & & & & \\
 & & \vee & & \vee & & & & \\
 & & \Re & \dashv & \Im & \dashv & \& & & \\
 & & & & \vee & & \vee & & \\
 & & & & \int & \dashv & \flat & \dashv & \sharp \\
 & & & & & & \vee & & \vee \\
 & & & & & & \emptyset & \dashv & \ast
\end{array}
}
$$

$\mathbb{A}^1$-*local*

The central modalities are motivic $\mathbb{A}^1$-localizations.

$$\begin{tikzcd}
\overbrace{\mathbf{H}}^{\text{super-geometric}} \arrow[r, shift left=6] \arrow[r, shift left=1.5, "\Pi_{\mathrm{sup}}"] \arrow[rr, bend right, shift right=2] & \overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}} \arrow[l, hook', shift left=3] \arrow[l, hook', shift left=1.5, "\Delta_{\mathrm{sup}}"] \arrow[r, shift left=1.5, "\Pi_{\mathrm{inf}}"] & \overbrace{\mathbf{H}_{\Re}}^{\text{reduced}} \arrow[l, hook', shift right=3] \arrow[l, hook', shift left=1.5, "\Delta_{\mathrm{inf}}"] \arrow[r, shift left=1.5, "\Pi"] \arrow[r, shift right=3] & \overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}} \arrow[l, hook', shift left=1.5, "\Delta"] \arrow[lll, hook', bend left, shift left=2] \arrow[r, shift right=3] & \ast \arrow[l, hook', shift right=1.5] \arrow[l, hook', shift left=6]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\boxed{\begin{array}{ccccccccc}
\mathrm{id} & \dashv & \mathrm{id} & & & & & & \\
\vee & & \vee & & & & & & \\
\rightrightarrows & \dashv & \rightsquigarrow & \dashv & \mathrm{Rh} & & & & \\
 & & \vee & & \vee & & & & \\
 & & \Re & \dashv & \Im & \dashv & \& & & \\
 & & & & \vee & & \vee & & \\
 & & & & \color{blue}\boxed{\mathbb{R}^1} & \dashv & \flat & \dashv & \sharp \\
 & & & & & & \vee & & \vee \\
 & & & & & & \varnothing & \dashv & \ast
\end{array}}$$

continuum-local

The central modalities are motivic $\mathbb{A}^1$-localizations.

$$\begin{tikzcd}
\overbrace{\mathbf{H}}^{\substack{\text{super-}\\\text{geometric}}} \arrow[r, bend left=40] \arrow[r, shift left, "\color{blue}\Pi_{\mathrm{sup}}"] \arrow[rr, bend right=30] & \overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}} \arrow[l, hook, bend right=20] \arrow[l, hook, shift left, "\color{blue}\Delta_{\mathrm{sup}}"] \arrow[r, shift left, "\color{blue}\Pi_{\mathrm{inf}}"] & \overbrace{\mathbf{H}_{\Re}}^{\text{reduced}} \arrow[l, hook, bend right=20] \arrow[l, hook, shift left, "\color{blue}\Delta_{\mathrm{inf}}"] \arrow[r, shift left, "\color{blue}\Pi"] \arrow[r, bend right=30] & \overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete}\\\infty\text{-groupoids}}} \arrow[l, hook, shift left, "\color{blue}\Delta"] \arrow[lll, hook, bend left=30] \arrow[r, bend right=30] & \ast \arrow[l, hook] \arrow[l, hook, bend left=30]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\boxed{\begin{array}{ccccccccc}
\mathrm{id} & \dashv & \mathrm{id} & & & & & & \\
\vee & & \vee & & & & & & \\
\rightrightarrows & \dashv & \rightsquigarrow & \dashv & \mathrm{R\!\!h} & & & & \\
 & & \vee & & \vee & & & & \\
 & & \Re & \dashv & \boxed{\mathbb{D}} & \dashv & \& & & \\
 & & & & \vee & & \vee & & \\
 & & & & \boxed{\mathbb{R}^1} & \dashv & \flat & \dashv & \sharp \\
 & & & & & & \vee & & \vee \\
 & & & & & & \varnothing & \dashv & \ast
\end{array}}$$

infinitum-local

The central modalities are motivic $\mathbb{A}^1$-localizations.

$$\begin{tikzcd}[column sep=large]
\overbrace{\mathbf{H}}^{\substack{\text{super-}\\ \text{geometric}}} \arrow[r, bend left=50] \arrow[r, shift left, "\Pi_{\mathrm{sup}}"] \arrow[rr, bend right=40] & \overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}} \arrow[l, hook, bend left=25] \arrow[l, hook, shift left, "\Delta_{\mathrm{sup}}"] \arrow[r, shift left, "\Pi_{\mathrm{inf}}"] & \overbrace{\mathbf{H}_{\Re}}^{\text{reduced}} \arrow[l, hook, bend right=30] \arrow[l, hook, shift left, "\Delta_{\mathrm{inf}}"] \arrow[r, shift left=2] \arrow[r, shift right=2] & \overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}} \arrow[l, hook] \arrow[lll, hook, bend left=40] \arrow[r, shift right] & \ast \arrow[l, hook, shift right=2] \arrow[l, hook, bend left=30]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\boxed{\begin{array}{ccccccccc}
\mathrm{id} & \dashv & \mathrm{id} & & & & & & \\
\vee & & \vee & & & & & & \\
\rightrightarrows & \dashv & \rightsquigarrow & \dashv & \boxed{\mathbb{R}^{0|1}} & & & & \\
 & & \vee & & \vee & & & & \\
 & & \Re & \dashv & \boxed{\mathbb{D}} & \dashv & \& & & \\
 & & & & \vee & & \vee & & \\
 & & & & \boxed{\mathbb{R}^{1}} & \dashv & \flat & \dashv & \sharp \\
 & & & & & & \vee & & \vee \\
 & & & & & & \varnothing & \dashv & \ast
\end{array}}$$

superpoint-local

The central modalities are motivic $\mathbb{A}^1$-localizations.

$$\begin{tikzcd}[column sep=huge]
\overbrace{\mathbf{H}}^{\text{super-geometric}} \arrow[r, shift left=6] \arrow[r, shift left=2, "\Pi_{\mathrm{sup}}"] \arrow[rr, shift right=4]
& \overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}} \arrow[l, hook', shift right=4] \arrow[l, hook', "\Delta_{\mathrm{sup}}"] \arrow[r, shift left=2]
& \overbrace{\mathbf{H}_{\Re}}^{\text{reduced}} \arrow[l, hook', shift right=4] \arrow[l, hook'] \arrow[r, shift left=2] \arrow[r, shift right=4]
& \overbrace{\mathbf{H}_{\flat}}^{\text{geometrically discrete }\infty\text{-groupoids}} \arrow[l, hook'] \arrow[lll, hook', shift left=6] \arrow[r, shift right=4]
& \ast \arrow[l, hook'] \arrow[l, hook', shift left=6]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\begin{tikzcd}
\mathrm{id} \arrow[r, phantom, "\dashv"] \arrow[d, phantom, "\vee"] & \mathrm{id} \arrow[d, phantom, "\vee"] & & & \\
\rightrightarrows \arrow[r, phantom, "\dashv"] & \rightsquigarrow \arrow[r, phantom, "\dashv"] \arrow[d, phantom, "\vee"] & \boxed{\mathbb{R}^{0|1}} \arrow[d, phantom, "\vee"] & & \\
 & \Re \arrow[r, phantom, "\dashv"] & \boxed{\mathbb{D}} \arrow[r, phantom, "\dashv"] \arrow[d, phantom, "\vee"] & \& \arrow[d, phantom, "\vee"] & \\
 & & \boxed{\mathbb{R}^{1}} \arrow[r, phantom, "\dashv"] & \flat \arrow[r, phantom, "\dashv"] \arrow[d, phantom, "\vee"] & \sharp \arrow[d, phantom, "\vee"] \\
 & & & \varnothing \arrow[r, phantom, "\dashv"] \arrow[uuul, rightsquigarrow, bend right] & \ast
\end{tikzcd}$$

⇒ emergence of Atom of Superspace from `\nothing`

The Modalities of Super homotopy theory

$$\begin{tikzcd}
\mathrm{id} \arrow[r, phantom, "\dashv"] \arrow[d, phantom, "\vee"] & \mathrm{id} \arrow[d, phantom, "\vee"] & & & \\
\rightrightarrows \arrow[r, phantom, "\dashv"] & \rightsquigarrow \arrow[r, phantom, "\dashv"] \arrow[d, phantom, "\vee"] & \boxed{\mathbb{R}^{0|1}} \arrow[d, phantom, "\vee"] & & \\
& \Re \arrow[r, phantom, "\dashv"] & \boxed{\mathbb{D}} \arrow[r, phantom, "\dashv"] \arrow[d, phantom, "\vee"] & \& \arrow[d, phantom, "\vee"] & \\
& & \boxed{\mathbb{R}^1} \arrow[r, phantom, "\dashv"] & \flat \arrow[r, phantom, "\dashv"] \arrow[d, phantom, "\vee"] & \sharp \arrow[d, phantom, "\vee"] \\
& & & \emptyset \arrow[r, phantom, "\dashv"] \arrow[uuul, squiggly, bend right] & \ast
\end{tikzcd}$$

$\Rightarrow$ emergence of Atom of Superspace from \\`nothing`

now apply the microscope of homotopy theory
to discover what emerges, in turn, out of the superpoint...

Rational
Super homotopy theory

and the fundamental super p-Branes

back to Part I

Higher super Lie theory and Rational homotopy

$$\left.\begin{array}{c}\text{infinitesimal}\\ \text{rational}\end{array}\right\}\ \text{approximation of super-homotopy by}\ \left\{\begin{array}{c}\text{higher Lie integration}\\ \text{Sullivan construction}\end{array}\right.$$

Definition.

$$\underbrace{\mathrm{FDAs}}_{\substack{\text{terminology}\\ \text{common in}\\ \text{supergravity}\\ \text{([vanNieuwenhuizen82])}}} \;:=\; \underbrace{\mathrm{dgcSuperAlg}_{\mathbb{R},\mathrm{cn}}}_{\substack{\infty\text{-category of}\\ \text{differential}\\ \text{graded-commutative}\\ \text{superalgebras}}} \xleftarrow[\simeq]{\ \mathrm{CE}\ } \Big(\underbrace{\mathrm{Super}L_\infty\mathrm{Algbd}^{\mathrm{nil}}_{\mathbb{R},\mathrm{cn}}}_{\substack{\infty\text{-category of}\\ \text{nilpotent}\\ \text{super } L_\infty\text{-algebroids}}}\Big)^{\mathrm{op}}$$

Theorem. ([BM18], see Sch18)

Sullivan rational homotopy theory

$$\begin{tikzcd} \mathrm{Super}L_\infty\mathrm{Algbd}^{\mathrm{nil}}_{\mathbb{R},\mathrm{cn}} \arrow[r, phantom, "\simeq"] & \overbrace{\mathrm{FDAs}^{\mathrm{op}}}^{\substack{\text{rational super-}\\ \infty\text{-groupoids}}} \arrow[r, shift right, "\mathrm{Spec}"'] & \overbrace{\mathbf{H}}^{\substack{\text{supergeometric}\\ \infty\text{-groupoids}}} \arrow[l, shift right, "\mathcal{O}"'] \arrow[r, shift right, "\Gamma"'] & \overbrace{\mathbf{H}_\flat}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}} \arrow[l, hook', shift right, "\mathrm{Disc}"'] \end{tikzcd}$$

higher super Lie integration

$$\begin{tikzcd} \underbrace{\mathbb{R}^{0|1}}_{\substack{D=0,\ \mathcal{N}=1\\ \text{supersymmetry}\\ \text{super Lie algebra}}} \arrow[rr, mapsto] & & \underbrace{\mathbb{R}^{0|1}}_{\text{superpoint}} \end{tikzcd}$$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

the Atom of Superspace

$\mathbb{R}^{0|\mathbf{1}}$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$\mathbb{R}^{0|\mathbf{1}}$

$\boxed{\text{Type I}}$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift right] \arrow[l, shift left] \\
\boxed{\text{Type IIB}} & \boxed{\text{Type I}}
\end{tikzcd}$$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

[HS17]

$$\begin{tikzcd}
 & \mathbb{R}^{2,1|\mathbf{2}} \arrow[dl] \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] \\
\boxed{\text{Type IIB}} & \boxed{\text{Type I}}
\end{tikzcd}$$

universal central extension: 3d super-Minkowski spacetime

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

[HS17]

$$\begin{tikzcd}
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right]
\end{tikzcd}$$

Type IIB

Type I

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & \mathbb{R}^{3,1|\mathbf{4}} \arrow[dl] \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] \\
\boxed{\text{Type IIB}} & \boxed{\text{Type I}}
\end{tikzcd}$$

[HS17]

universal invariant central extension: 4d super-Minkowski spacetime

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift right] \arrow[l, shift left] \arrow[dl] \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift right] \arrow[l, shift left] \arrow[dl] \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift right] \arrow[l, shift left]
\end{tikzcd}$$

[HS17]

Type IIB

Type I

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & {\color{blue}\mathbb{R}^{5,1|\mathbf{8}}} \arrow[dl] \\
\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift right] \arrow[l, shift left] \arrow[dl] \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift right] \arrow[l, shift left] \arrow[dl] \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift right] \arrow[l, shift left] \\
\boxed{\text{Type IIB}} & \boxed{\text{Type I}}
\end{tikzcd}$$

[HS17]

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & \mathbb{R}^{5,1|\mathbf{8}} \arrow[dl] \arrow[r, shift left] \arrow[r, shift right] & {\color{blue}\mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}}} \\
\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & \\
\boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}}
\end{tikzcd}$$

[HS17]

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & \mathbb{R}^{9,1|\mathbf{16}} \arrow[dr] & \\
 & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & \\
\boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}}
\end{tikzcd}$$

[HS17]

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
\mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \\
& \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & \\
\boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}}
\end{tikzcd}$$

[HS17]

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & \mathbb{R}^{10,1|\mathbf{32}} \arrow[dr] & \\
\mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift right] \arrow[l, shift left] \arrow[r, shift right] \arrow[r, shift left] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \\
 & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift right] \arrow[r, shift left] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift right] \arrow[l, shift left] \arrow[dl] & \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift right] \arrow[l, shift left] \arrow[dl] & \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift right] \arrow[l, shift left] & \\
\boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}}
\end{tikzcd}$$

[HS17]

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & \mathbb{R}^{10,1|\mathbf{32}} \arrow[dr] & \\
\mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \\
 & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \text{emergent spacetime} \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & \\
\boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}}
\end{tikzcd}$$

[HS17]

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] & & \\
\mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & \text{[FSS13]} & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] \\
 & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & \\
 & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & \\
 & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \begin{array}{c}\text{emergent}\\ \text{spacetime}\end{array} & \text{[HS17]} \\
 & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & \\
 & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & \\
 & \boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}} &
\end{tikzcd}$$

universal *higher* central invariant extension: stringy extended super-spacetimes

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
\mathfrak{d5brane} \arrow[ddr] & \mathfrak{d3brane} \arrow[dd] & \mathfrak{d1brane} \arrow[ddl] & & \mathfrak{d0brane} \arrow[ddr] & \mathfrak{d2brane} \arrow[dd] & \mathfrak{d4brane} \arrow[ddl] \\
\mathfrak{d7brane} \arrow[dr] & & & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] & & & \mathfrak{d6brane} \arrow[dl] \\
\mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & \text{[FSS13]} & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] \\
& & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift right] \arrow[l, shift left] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & \\
& & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & \\
& & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift right] \arrow[l, shift left] \arrow[dl] & \text{emergent spacetime} & & \text{[HS17]} \\
& & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift right] \arrow[l, shift left] \arrow[dl] & & & \\
& & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift right] \arrow[l, shift left] & & &
\end{tikzcd}$$

$\boxed{\text{Type IIB}}$ $\boxed{\text{Type I}}$ $\boxed{\text{Type IIA}}$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & & & \mathfrak{m2brane} \arrow[dd] & & & & \\
\mathfrak{d5brane} \arrow[ddr] & \mathfrak{d3brane} \arrow[dd] & \mathfrak{d1brane} \arrow[ddl] & & \mathfrak{d0brane} \arrow[ddr] & \mathfrak{d2brane} \arrow[dd] & \mathfrak{d4brane} \arrow[ddl] & \\
\mathfrak{d7brane} \arrow[dr] & & & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] & & & \mathfrak{d6brane} \arrow[dl] & \\
\mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & \text{[FSS13]} & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] & \\
 & & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & & \\
 & & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & & \\
 & & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \text{emergent spacetime} & & & \text{[HS17]} \\
 & & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & & \\
 & & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & & &
\end{tikzcd}$$

$$\boxed{\text{Type IIB}} \quad \boxed{\text{Type I}} \quad \boxed{\text{Type IIA}}$$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & & & {\color{blue}\mathfrak{m5brane}} \arrow[d] & & & & \text{[FSS15]} \\
 & & & \mathfrak{m2brane} \arrow[dd] & & & & \\
\mathfrak{d5brane} \arrow[ddr] & \mathfrak{d3brane} \arrow[dd] & \mathfrak{d1brane} \arrow[ddl] & & \mathfrak{d0brane} \arrow[ddr] & \mathfrak{d2brane} \arrow[dd] & \mathfrak{d4brane} \arrow[ddl] & \\
\mathfrak{d7brane} \arrow[dr] & & & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] & & & \mathfrak{d6brane} \arrow[dl] & \\
\mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & \text{[FSS13]} & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] & \\
 & & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift right] \arrow[l, shift left] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & & \\
 & & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & & \\
 & & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift right] \arrow[l, shift left] \arrow[dl] & \text{emergent spacetime} & & & \text{[HS17]} \\
 & & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift right] \arrow[l, shift left] \arrow[dl] & & & & \\
 & & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift right] \arrow[l, shift left] & & & & \\
 & & \boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}} & & &
\end{tikzcd}$$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & & & \mathfrak{m5brane} \arrow[d] & & & & \text{[FSS15]} \\
 & & & \mathfrak{m2brane} \arrow[dd] & & & & \\
\mathfrak{d5brane} \arrow[ddr] & \mathfrak{d3brane} \arrow[dd] & \mathfrak{d1brane} \arrow[ddl] & & \mathfrak{d0brane} \arrow[ddr] & \mathfrak{d2brane} \arrow[dd] & \mathfrak{d4brane} \arrow[ddl] & \\
\mathfrak{d7brane} \arrow[dr] & & \text{brane bouquet} & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] & & & \mathfrak{d6brane} \arrow[dl] & \\
\mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & \text{[FSS13]} & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] & \\
 & & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & & \\
 & & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & & \\
 & & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \text{emergent spacetime} & & & \text{[HS17]} \\
 & & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & & \\
 & & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & & & \\
 & & \boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}} & & &
\end{tikzcd}$$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & & & \mathfrak{m5brane} \arrow[dd] & & & & \text{[FSS15]} \\
 & & & & \text{M/IIA Duality} \arrow[l, dotted] \arrow[dr, dotted] & & & \text{[FSS16a]} \\
 & & & \mathfrak{m2brane} \arrow[dd] & & & & \\
\mathfrak{d5brane} \arrow[ddr] & \mathfrak{d3brane} \arrow[dd] & \mathfrak{d1brane} \arrow[ddl] & & \mathfrak{d0brane} \arrow[dl, dotted] \arrow[ddr] & \mathfrak{d2brane} \arrow[dd] & \mathfrak{d4brane} \arrow[ddl] & \\
\mathfrak{d7brane} \arrow[dr] & & \text{brane bouquet} & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] & \text{(pb)} & & \mathfrak{d6brane} \arrow[dl] & \\
\mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & \text{[FSS13]} & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] & \\
 & & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & & \\
 & & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & & \\
 & & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \text{emergent spacetime} & & & \text{[HS17]} \\
 & & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & & \\
 & & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & & & \\
 & & \boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}} & & &
\end{tikzcd}$$

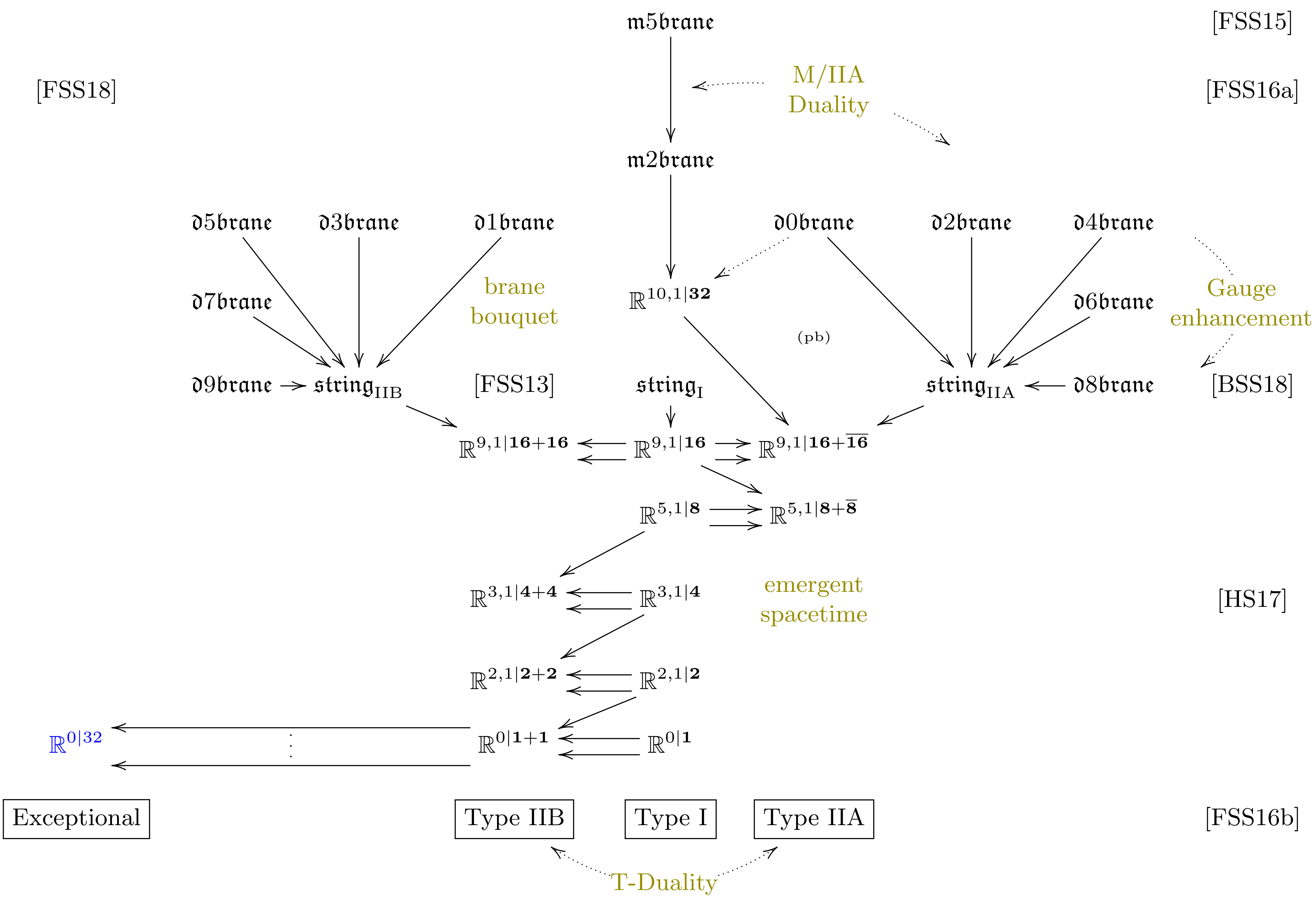
Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & & & \mathfrak{m5brane} \arrow[dd] & & & & \text{[FSS15]} \\
 & & & {} & \text{M/IIA Duality} \arrow[l, dotted] \arrow[dr, dotted] & & & \text{[FSS16a]} \\
 & & & \mathfrak{m2brane} \arrow[dd] & & {} & & \\
\mathfrak{d5brane} \arrow[ddr] & \mathfrak{d3brane} \arrow[dd] & \mathfrak{d1brane} \arrow[ddl] & & \mathfrak{d0brane} \arrow[dl, dotted] \arrow[ddr] & \mathfrak{d2brane} \arrow[dd] & \mathfrak{d4brane} \arrow[ddl] & {} \arrow[d, dash, dotted] \\
\mathfrak{d7brane} \arrow[dr] & & \text{brane bouquet} & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] & \text{(pb)} & & \mathfrak{d6brane} \arrow[dl] & \text{Gauge enhancement} \arrow[d, dotted] \\
\mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & \text{[FSS13]} & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] & \text{[BSS18]} \\
 & & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & & \\
 & & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & & \\
 & & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \text{emergent spacetime} & & & \text{[HS17]} \\
 & & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & & \\
 & & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & & & \\
 & & \boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}} & & &
\end{tikzcd}$$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & & & \mathfrak{m5brane} \arrow[dd] & & & & \text{[FSS15]} \\
 & & & & \text{M/IIA Duality} \arrow[l, dotted] \arrow[dr, dotted] & & & \text{[FSS16a]} \\
 & & & \mathfrak{m2brane} \arrow[dd] & & & & \\
\mathfrak{d5brane} \arrow[ddr] & \mathfrak{d3brane} \arrow[dd] & \mathfrak{d1brane} \arrow[ddl] & & \mathfrak{d0brane} \arrow[dl, dotted] \arrow[ddr] & \mathfrak{d2brane} \arrow[dd] & \mathfrak{d4brane} \arrow[ddl] \arrow[dd, dotted, bend left] & \\
\mathfrak{d7brane} \arrow[dr] & & \text{brane bouquet} & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] & \text{(pb)} & & \mathfrak{d6brane} \arrow[dl] & \text{Gauge enhancement} \\
\mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & \text{[FSS13]} & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] & \text{[BSS18]} \\
 & & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & & \\
 & & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & & \\
 & & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \text{emergent spacetime} & & & \text{[HS17]} \\
 & & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & & \\
 & & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & & & \\
 & & \boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}} & & & \text{[FSS16b]} \\
 & & & \text{T-Duality} \arrow[ul, dotted] \arrow[ur, dotted] & & & &
\end{tikzcd}$$

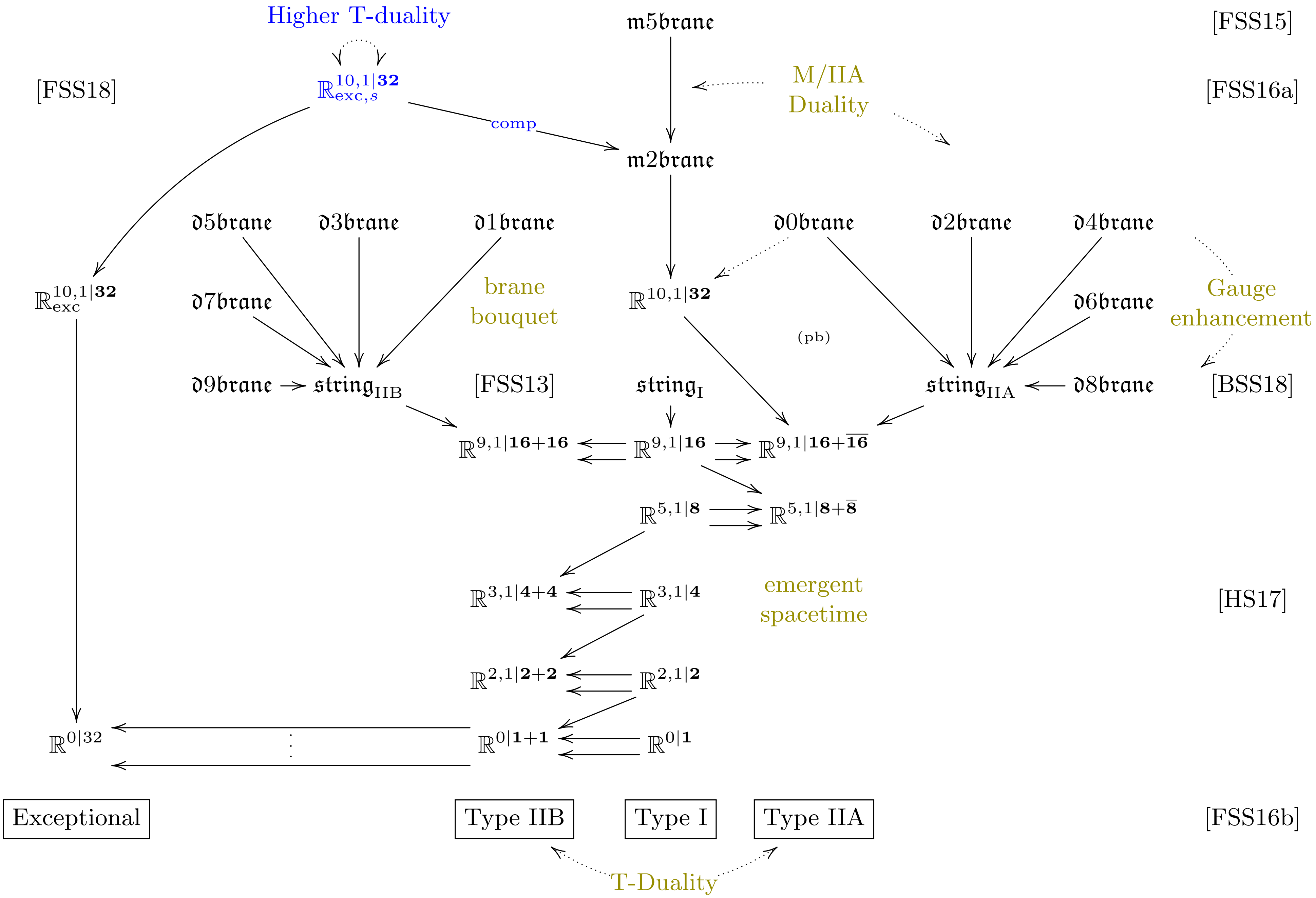
Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet



Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & & & & \mathfrak{m5brane} \arrow[dd] & & & & \text{[FSS15]} \\
\text{[FSS18]} & & & & & \text{M/IIA Duality} \arrow[l, dotted] \arrow[dr, dotted] & & & \text{[FSS16a]} \\
 & & & & \mathfrak{m2brane} \arrow[dd] & & & & \\
 & \mathfrak{d5brane} \arrow[ddr] & \mathfrak{d3brane} \arrow[dd] & \mathfrak{d1brane} \arrow[ddl] & & \mathfrak{d0brane} \arrow[dl, dotted] \arrow[ddr] & \mathfrak{d2brane} \arrow[dd] & \mathfrak{d4brane} \arrow[ddl] \arrow[dr, dotted, no head] & \\
\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}} \arrow[dddddd] & \mathfrak{d7brane} \arrow[dr] & & \text{brane bouquet} & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] & \text{(pb)} & & \mathfrak{d6brane} \arrow[dl] & \text{Gauge enhancement} \arrow[dl, dotted] \\
 & \mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & \text{[FSS13]} & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] & \text{[BSS18]} \\
 & & & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & & \\
 & & & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & & \\
 & & & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \text{emergent spacetime} & & & \text{[HS17]} \\
 & & & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & & \\
\mathbb{R}^{0|32} & & & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} \arrow[lll, shift left=2] \arrow[lll, shift right=2] \arrow[lll, phantom, "\vdots"] & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & & & \\
\boxed{\text{Exceptional}} & & & \boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}} & & & \text{[FSS16b]} \\
 & & & & \text{T-Duality} \arrow[ul, dotted] \arrow[ur, dotted] & & & &
\end{tikzcd}$$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet



Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & & \text{Higher T-duality} & & \mathfrak{m5brane} \arrow[dd] & & & & \text{[FSS15]} \\
\text{[FSS18]} & & \mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s} \arrow[loop above, dotted] \arrow[rrd, "\mathrm{comp}"] \arrow[dddll, bend right] & & & \text{M/IIA Duality} \arrow[l, dotted] \arrow[rd, dotted] & & & \text{[FSS16a]} \\
 & & & & \mathfrak{m2brane} \arrow[dd] & & & & \\
 & \mathfrak{d5brane} \arrow[rdd] & \mathfrak{d3brane} \arrow[dd] & \mathfrak{d1brane} \arrow[ldd] & & \mathfrak{d0brane} \arrow[ld, dotted] \arrow[rdd] & \mathfrak{d2brane} \arrow[dd] & \mathfrak{d4brane} \arrow[ldd] & \\
\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}} \arrow[dddddd] & \mathfrak{d7brane} \arrow[rd] & & \text{brane bouquet} & \mathbb{R}^{10,1|\mathbf{32}} \arrow[rdd] & {\scriptstyle (\mathrm{pb})} & & \mathfrak{d6brane} \arrow[ld] & \text{Gauge enhancement} \arrow[d, dotted] \\
 & \mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[rd] & \text{[FSS13]} & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[ld] & \mathfrak{d8brane} \arrow[l] & \text{[BSS18]} \\
 & & & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[rd] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & & \\
 & & & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[ld] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & & \\
 & & & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[ld] & \text{emergent spacetime} & & & \text{[HS17]} \\
 & & & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[ld] & & & & \\
\mathbb{R}^{0|32} & & & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} \arrow[lll, shift left=2] \arrow[lll, shift right=2] \arrow[lll, phantom, "\vdots"] & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & & & \\
\boxed{\text{Exceptional}} & & & \boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}} & & & \text{[FSS16b]} \\
 & & & & \text{T-Duality} \arrow[lu, dotted] \arrow[ru, dotted] & & & &
\end{tikzcd}$$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & \mathfrak{m5brane} \arrow[d] & & \text{[FSS15]} \\
 & \mathfrak{m2brane} \arrow[d] & & \\
 & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] & & \\
\text{[FSS13]} & & & \\
 & \mathbb{R}^{9,1|\mathbf{16}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & \\
 & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & \\
\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & \text{[HS17]} \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] \arrow[uuuuuuu, rightsquigarrow, bend right=60] & &
\end{tikzcd}$$

emergence of fundamental M-branes from the Atom of Superspace

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

[FSS15]

$$\begin{tikzcd}
\mathfrak{m5brane} \arrow[d] \\
\mathfrak{m2brane} \arrow[d] \\
\mathbb{R}^{10,1|\mathbf{32}}
\end{tikzcd}$$

zoom in on the fundamental M-brane super-extensions

The fundamental M2/M5-brane cocycle

[FSS15]

$$\begin{tikzcd}
\mathfrak{m5brane} \arrow[d] & \\
\mathfrak{m2brane} \arrow[d, "\mathrm{hofib}(\mu_{\mathrm{M2}})"'] & \\
\mathbb{R}^{10,1|\mathbf{32}} \arrow[dr, "\mu_{\mathrm{M2}}"'] & \\
 & B^4\mathbb{R}
\end{tikzcd}$$

$$\mu_{\mathrm{M2}} = dL^{\mathrm{WZW}}_{\mathrm{M2}} = \tfrac{i}{2}\left(\overline{\psi}\Gamma_{a_1 a_2}\psi\right) \wedge e^{a_1} \wedge e^{a_2}$$

the WZW-curvature of the Green-Schwarz-type sigma-model super-membrane

The fundamental M2/M5-brane cocycle

[FSS15]

$$\begin{tikzcd}
\mathfrak{m5brane} \arrow[d, "\mathrm{hofib}(\mu_{\mathrm{M5}})"'] & \\
\mathfrak{m2brane} \arrow[r, "\mu_{\mathrm{M5}}"] \arrow[d, "\mathrm{hofib}\big(\mu_{\mathrm{M2}}\big)"'] & S^7_{\mathbb{R}} \\
\mathbb{R}^{10,1|\mathbf{32}} \arrow[dr, "\mu_{\mathrm{M2}}"'] & \\
& B^4\mathbb{R}
\end{tikzcd}$$

$$\mu_{\mathrm{M5}} \; = dL^{\mathrm{WZW}}_{\mathrm{M5}} \;=\; \tfrac{1}{5!}\left(\overline{\psi}\Gamma_{a_1\cdots a_5}\psi\right)\wedge e^{a_1}\wedge\cdots\wedge e^{a_5} \;+\; c_3\wedge\tfrac{i}{2}\left(\overline{\psi}\Gamma_{a_1a_2}\psi\right)\wedge e^{a_1}\wedge e^{a_2}$$

the WZW-curvature of the Green-Schwarz-type sigma-model super-fivebrane

The fundamental M2/M5-brane cocycle

[FSS15]

$$\begin{tikzcd}
\mathfrak{m5brane} \arrow[d, "\mathrm{hofib}(\mu_{\mathrm{M5}})"'] & & \\
\mathfrak{m2brane} \arrow[rr, "\mu_{\mathrm{M5}}"] \arrow[d, "\mathrm{hofib}(\mu_{\mathrm{M2}})"'] & & S^7_{\mathbb{R}} \arrow[d, "\mathrm{hofib}(c_2)"] \\
\mathbb{R}^{10,1\vert\mathbf{32}} \arrow[dr, "\mu_{\mathrm{M2}}"'] & & S^4_{\mathbb{R}} \arrow[dl, "c_2"] \\
& B^4\mathbb{R} &
\end{tikzcd}$$

the quaternionic Hopf fibration (in rational homotopy theory)

The fundamental M2/M5-brane cocycle

[FSS15]

$$\begin{tikzcd}
\mathfrak{m5brane} \arrow[d, "\mathrm{hofib}(\mu_{\mathrm{M5}})"'] & & \\
\mathfrak{m2brane} \arrow[rr, "\mu_{\mathrm{M5}}"] \arrow[d, "\mathrm{hofib}\big(\mu_{\mathrm{M2}}\big)"'] & & S^7_{\mathbb{R}} \arrow[d, "\mathrm{hofib}(c_2)"] \arrow[dll, Rightarrow, shorten <=6em, shorten >=6em] \\
\mathbb{R}^{10,1|\mathbf{32}} \arrow[rr, dashed, "\mu_{\mathrm{M2/M5}}"] \arrow[dr, "\mu_{\mathrm{M2}}"'] & & S^4_{\mathbb{R}} \arrow[dl, "c_2"] \\
& B^4\mathbb{R} &
\end{tikzcd}$$

the unified M2/M5-cocycle

The fundamental M2/M5-brane cocycle

[FSS15]

$$\begin{tikzcd}
\mathbb{R}^{10,1|\mathbf{32}} \arrow[r, "\mu_{\mathrm{M2/M5}}"] & S^4_{\mathbb{R}}
\end{tikzcd}$$

the unified M2/M5-cocycle is in rational Cohomotopy in degree 4

The fundamental M2/M5-brane cocycle

[FSS15]

$$\begin{tikzcd}
\mathbb{R}^{10,1|\mathbf{32}} \arrow[r, "\mu_{\mathrm{M2/M5}}"] & S^4_{\mathbb{R}} \\
\tfrac{i}{2}\left(\overline{\psi}\Gamma_{a_1 a_2}\psi\right) \wedge e^{a_1} \wedge e^{a_2} & G_4 \arrow[l, mapsto] \\
\tfrac{1}{5!}\left(\overline{\psi}\Gamma_{a_1 \cdots a_5}\psi\right) \wedge e^{a_1} \cdots e^{a_5} & G_7 \arrow[l, mapsto]
\end{tikzcd}$$

Sullivan model: $\mathcal{O}\left(S^4_{\mathbb{R}}\right) \;\simeq\; \mathbb{R}[G_4, G_7]/\left(\begin{smallmatrix} dG_4 = 0 \\ dG_7 = -\frac{1}{2} G_4 \wedge G_4 \end{smallmatrix}\right)$

= 11d supergravity equations of motion of the C-field ([Sati13, Sect. 2.5])

The fundamental M2/M5-brane cocycle

[FSS15]

$$\begin{tikzcd}\mathbb{R}^{10,1|\mathbf{32}} \arrow[rr, "\mu_{\mathrm{M2/M5}}"] & & S^4_{\mathbb{R}}\end{tikzcd}$$

the unified M2/M5-cocycle

The fundamental M2/M5-brane cocycle

[FSS15]

$$\begin{tikzcd}
\mathbb{R}^{10,1|\mathbf{32}} \arrow[rr, "\mu_{\mathrm{M2/M5}}"] & {} \arrow[d, "\text{double dimensional reduction \& gauge enhancement}" description] & S^4_{\mathbb{R}} \\
\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \arrow[rr, "\mu_{F1/D_{2p}}"'] & {} & \mathrm{ku}_{/\!\!/ B^2\mathbb{R}}
\end{tikzcd}$$

D-brane charge in twisted K-theory, rationally
[BSS18]

The rational conclusion.

$$\text{In} \left\{ \begin{array}{c} \text{infinitesimal} \\ \text{rational} \end{array} \right\} \text{approximation}$$

brane charge quantization follows from first principles

and reveals this situation:

brane species	**cohomology theory of charge quantization**
D-branes	twisted K-theory
M-branes	Cohomotopy in degree 4

The rational conclusion.

In $\left\{\begin{array}{c}\text{infinitesimal}\\ \text{rational}\end{array}\right\}$ approximation

brane charge quantization follows from first principles

and reveals this situation:

brane species	**cohomology theory of charge quantization**
D-branes	twisted K-theory
M-branes	Cohomotopy in degree 4

Lift beyond $\left\{\begin{array}{c}\text{infinitesimal}\\ \text{rational}\end{array}\right\}$ approximation is not unique

but one lift of rational Cohomotopy is *minimal* (in number of cells):

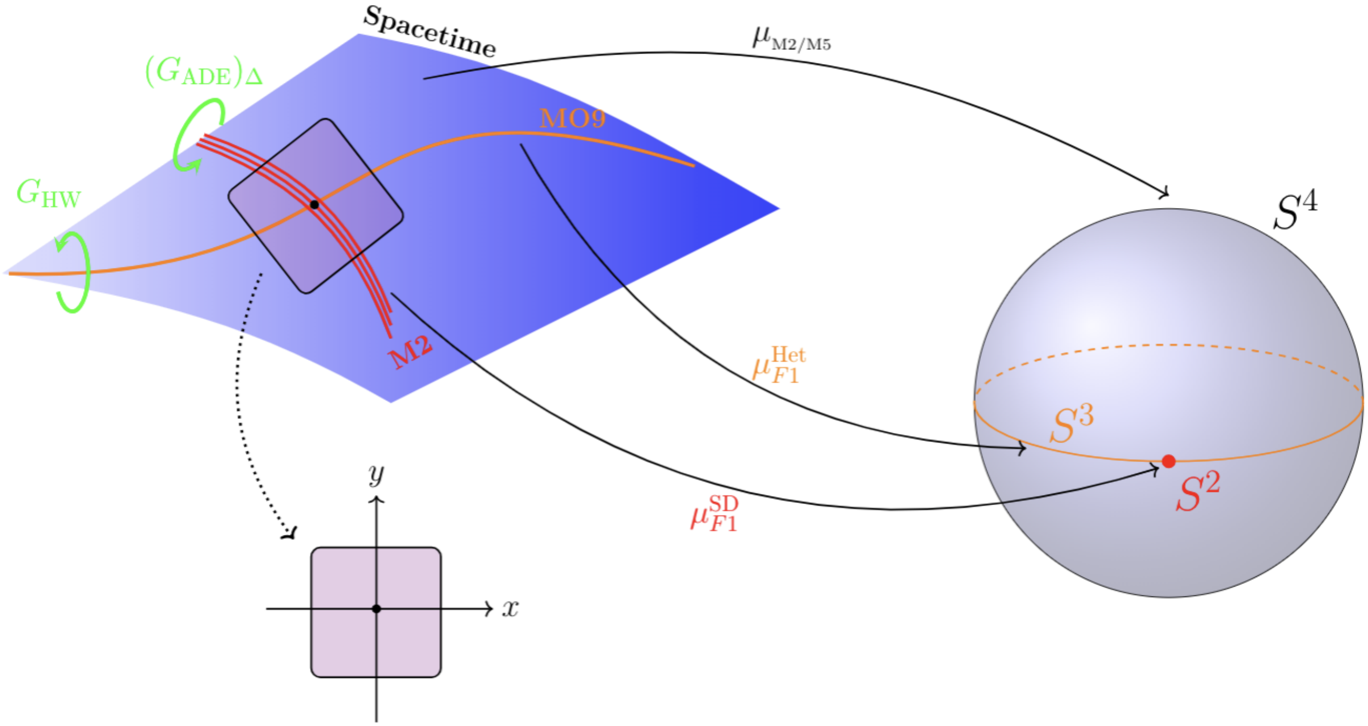
actual Cohomotopy represented by the actual 4-sphere

$$\begin{tikzcd}
 & S^4 \arrow[d, "\text{rationalization}"] \\
X \arrow[ur, "{\substack{\text{cocycle in}\\ \text{actual Cohomotopy}}}"] \arrow[r, "{\substack{\text{cocycle in}\\ \text{rational cohomotopy}}}"'] & S^4_{\mathbb{R}}
\end{tikzcd}$$

Towards microscopic M-theory

1. Construct
differential equivariant Cohomotopy $\widehat{S^4_\gamma}$
of 11d super-orbifold spacetimes $\mathcal{X}$

2. lifting super-tangent-space-wise
the fundamental M2/M5-brane cocycle.



3. Compare the resulting observables on
M-brane charge quantized supergravity field moduli
with expected limiting corners of M-theory

Global equivariant Super homotopy theory

and the C-field at singularities

back to Part I

orbifolded

$\rightsquigarrow$

Global equivariant
Super homotopy theory

and the C-field at singularities

back to Part I

The modalities of global equivariant homotopy theory

$\vee$	$\dashv$	$\cup$	$\dashv$	$\prec$
singular		smooth		orbi-singular

possibly singular/orbifolded

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\overbrace{\mathbf{H}}^{\text{super-geometric}}
 \arrow[r, shift left=4.5] \arrow[r, shift right=1.5]
 \arrow[rr, bend right=25]
 \arrow[d, shift right=4.5] \arrow[d, shift left=1.5]
&
\overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}}
 \arrow[l, hook, shift right=1.5] \arrow[l, hook, shift left=4.5]
 \arrow[r, shift right=1.5]
 \arrow[d, shift right=4.5] \arrow[d, shift left=1.5]
&
\overbrace{\mathbf{H}_{\Re}}^{\text{reduced}}
 \arrow[l, hook, shift right=4.5] \arrow[l, hook, shift left=1.5]
 \arrow[r, shift left=4.5] \arrow[r, shift right=1.5]
 \arrow[d, shift right=4.5] \arrow[d, shift left=1.5]
&
\overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}
 \arrow[l, hook, shift left=1.5]
 \arrow[lll, hook, bend left=30]
 \arrow[r, shift right=1.5]
 \arrow[d, shift right=4.5] \arrow[d, shift left=1.5]
&
\ast
 \arrow[l, hook, shift right=1.5] \arrow[l, hook, shift left=4.5]
 \arrow[d, equal]
\\
\overbrace{\mathbf{H}_{\cup}}^{\text{super-geometric}}
 \arrow[u, hook, shift left=1.5] \arrow[u, hook, shift right=4.5]
 \arrow[r, shift left=4.5] \arrow[r, shift right=1.5]
 \arrow[rr, bend right=25]
&
\overbrace{\mathbf{H}_{\underset{\rightsquigarrow}{\cup}}}^{\text{bosonic}}
 \arrow[u, hook, shift left=1.5] \arrow[u, hook, shift right=4.5]
 \arrow[l, hook, shift right=1.5] \arrow[l, hook, shift left=4.5]
 \arrow[r, shift right=1.5]
&
\overbrace{\mathbf{H}_{\underset{\Re}{\cup}}}^{\text{reduced}}
 \arrow[u, hook, shift left=1.5] \arrow[u, hook, shift right=4.5]
 \arrow[l, hook, shift right=4.5] \arrow[l, hook, shift left=1.5]
 \arrow[r, shift left=4.5] \arrow[r, shift right=1.5]
&
\overbrace{\mathbf{H}_{\underset{\flat}{\cup}}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}
 \arrow[u, hook, shift left=1.5] \arrow[u, hook, shift right=4.5]
 \arrow[l, hook, shift left=1.5]
 \arrow[lll, hook, bend left=30]
 \arrow[r, shift right=1.5]
&
\ast
 \arrow[l, hook, shift right=1.5] \arrow[l, hook, shift left=4.5]
\end{tikzcd}$$

smooth = non-singular

The modalities of global equivariant homotopy theory

$\textcolor{blue}{\veebar}$	$\dashv$	$\cup$	$\dashv$	$\curlyvee$
singular		smooth		orbi- singular

possibly singular/orbifolded

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\overbrace{\mathbf{H}}^{\substack{\text{super-}\\ \text{geometric}}}
  \arrow[r, shift left=6]
  \arrow[r, shift left=2]
  \arrow[rr, bend left=0, shift right=4]
  \arrow[d, shift right=6, "\textcolor{blue}{\Pi}"']
  \arrow[d, shift left=2]
&
\overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}}
  \arrow[l, hook', shift left=4]
  \arrow[l, hook', shift left=0]
  \arrow[r, shift left=2]
  \arrow[d, shift right=6]
  \arrow[d, shift left=2]
&
\overbrace{\mathbf{H}_{\Re}}^{\text{reduced}}
  \arrow[l, hook', shift left=4]
  \arrow[l, hook', shift left=0]
  \arrow[r, shift left=2]
  \arrow[r, shift right=4]
  \arrow[d, shift right=6]
  \arrow[d, shift left=2]
&
\overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}
  \arrow[l, hook', shift left=0]
  \arrow[lll, hook', bend left=20]
  \arrow[r, shift right=4]
  \arrow[d, shift right=6]
  \arrow[d, shift left=2]
&
\ast
  \arrow[l, hook', shift left=0]
  \arrow[l, hook', shift left=8]
  \arrow[d, equal]
\\
\overbrace{\mathbf{H}_{\cup}}^{\substack{\text{super-}\\ \text{geometric}}}
  \arrow[u, hook, shift right=2, "\textcolor{blue}{\Delta}"']
  \arrow[u, hook, shift left=6]
  \arrow[r, shift left=6]
  \arrow[r, shift left=2]
  \arrow[rr, shift right=4]
&
\overbrace{\mathbf{H}_{\overset{\cup}{\rightsquigarrow}}}^{\text{bosonic}}
  \arrow[u, hook, shift right=2]
  \arrow[u, hook, shift left=6]
  \arrow[l, hook', shift left=4]
  \arrow[l, hook', shift left=0]
  \arrow[r, shift left=2]
&
\overbrace{\mathbf{H}_{\overset{\cup}{\Re}}}^{\text{reduced}}
  \arrow[u, hook, shift right=2]
  \arrow[u, hook, shift left=6]
  \arrow[l, hook', shift left=4]
  \arrow[l, hook', shift left=0]
  \arrow[r, shift left=2]
  \arrow[r, shift right=4]
&
\overbrace{\mathbf{H}_{\overset{\cup}{\flat}}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}
  \arrow[u, hook, shift right=2]
  \arrow[u, hook, shift left=6]
  \arrow[l, hook', shift left=0]
  \arrow[lll, hook', bend right=20]
  \arrow[r, shift right=4]
&
\ast
  \arrow[l, hook', shift left=0]
  \arrow[l, hook', shift right=8]
\end{tikzcd}$$

smooth = non-singular

The modalities of global equivariant homotopy theory

$$\boxed{\begin{array}{ccccc} \vee & \dashv & \cup & \dashv & \curlyvee \\ \text{singular} & & \text{smooth} & & \substack{\text{orbi-}\\ \text{singular}} \end{array}}$$

possibly singular/orbifolded

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\overbrace{\mathbf{H}}^{\substack{\text{super-}\\ \text{geometric}}}
\arrow[r, shift left=6]
\arrow[r, shift left=2]
\arrow[rr, shift right=2]
\arrow[d, shift right=6]
\arrow[d, shift left=2, "\Gamma"]
&
\overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}}
\arrow[l, hook', shift right=4]
\arrow[l, hook', shift left=0]
\arrow[r, shift left=2]
\arrow[d, shift right=6]
\arrow[d, shift left=2]
&
\overbrace{\mathbf{H}_{\Re}}^{\text{reduced}}
\arrow[l, hook', shift right=4]
\arrow[l, hook', shift left=0]
\arrow[r, shift left=2]
\arrow[r, shift right=2]
\arrow[d, shift right=6]
\arrow[d, shift left=2]
&
\overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}
\arrow[l, hook', shift left=0]
\arrow[lll, hook', shift left=4]
\arrow[r, shift right=2]
\arrow[d, shift right=6]
\arrow[d, shift left=2]
&
\ast
\arrow[l, hook', shift left=0]
\arrow[l, hook', shift left=4]
\arrow[d, equal]
\\
\overbrace{\mathbf{H}_{\cup}}^{\substack{\text{super-}\\ \text{geometric}}}
\arrow[u, hook, shift right=2, "\Delta"]
\arrow[u, hook, shift left=6]
\arrow[r, shift left=6]
\arrow[r, shift left=2]
\arrow[rr, shift right=2]
&
\overbrace{\mathbf{H}_{\underset{\rightsquigarrow}{\cup}}}^{\text{bosonic}}
\arrow[u, hook, shift right=2]
\arrow[u, hook, shift left=6]
\arrow[l, hook', shift right=4]
\arrow[l, hook', shift left=0]
\arrow[r, shift left=2]
&
\overbrace{\mathbf{H}_{\underset{\Re}{\cup}}}^{\text{reduced}}
\arrow[u, hook, shift right=2]
\arrow[u, hook, shift left=6]
\arrow[l, hook', shift right=4]
\arrow[l, hook', shift left=0]
\arrow[r, shift left=2]
\arrow[r, shift right=2]
&
\overbrace{\mathbf{H}_{\underset{\flat}{\cup}}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}
\arrow[u, hook, shift right=2]
\arrow[u, hook, shift left=6]
\arrow[l, hook', shift left=0]
\arrow[lll, hook', shift left=4]
\arrow[r, shift right=2]
&
\ast
\arrow[l, hook', shift left=0]
\arrow[l, hook', shift left=4]
\end{tikzcd}$$

smooth = non-singular

The modalities of global equivariant homotopy theory

$\rotatebox[origin=c]{70}{$<$}$	$\dashv$	$\cup$	$\dashv$	$\rotatebox[origin=c]{70}{$\prec$}$
singular		smooth		orbi- singular

possibly singular/orbifolded

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\overbrace{\mathbf{H}}^{\substack{\text{super-}\\ \text{geometric}}}
\arrow[r, shift left=3ex] \arrow[r, shift right=1ex] \arrow[rr, shift right=5ex]
\arrow[d, shift right=4.5ex] \arrow[d, shift left=1.5ex, "\Gamma"]
&
\overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}}
\arrow[l, hook', shift right=1ex] \arrow[l, hook', shift left=3ex]
\arrow[r, shift right=1ex]
\arrow[d, shift right=4.5ex] \arrow[d, shift left=1.5ex]
&
\overbrace{\mathbf{H}_{\Re}}^{\text{reduced}}
\arrow[l, hook', shift right=1ex] \arrow[l, hook', shift left=3ex]
\arrow[r, shift left=1ex] \arrow[r, shift right=3ex]
\arrow[d, shift right=4.5ex] \arrow[d, shift left=1.5ex]
&
\overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}
\arrow[l, hook', shift right=1ex] \arrow[lll, hook', shift right=7ex]
\arrow[r, shift right=3ex]
\arrow[d, shift right=4.5ex] \arrow[d, shift left=1.5ex]
&
\ast
\arrow[l, hook', shift right=1ex] \arrow[l, hook', shift right=5ex]
\arrow[d, equal]
\\
\overbrace{\mathbf{H}_{\cup}}^{\substack{\text{super-}\\ \text{geometric}}}
\arrow[u, hook, shift right=1.5ex] \arrow[u, hook, shift left=4.5ex, "\nabla"']
\arrow[r, shift left=3ex] \arrow[r, shift right=1ex] \arrow[rr, shift right=5ex]
&
\overbrace{\mathbf{H}_{\underset{\rightsquigarrow}{\cup}}}^{\text{bosonic}}
\arrow[u, hook, shift right=1.5ex] \arrow[u, hook, shift left=4.5ex]
\arrow[l, hook', shift right=1ex] \arrow[l, hook', shift left=3ex]
\arrow[r, shift right=1ex]
&
\overbrace{\mathbf{H}_{\underset{\Re}{\cup}}}^{\text{reduced}}
\arrow[u, hook, shift right=1.5ex] \arrow[u, hook, shift left=4.5ex]
\arrow[l, hook', shift right=1ex] \arrow[l, hook', shift left=3ex]
\arrow[r, shift left=1ex] \arrow[r, shift right=3ex]
&
\overbrace{\mathbf{H}_{\underset{\flat}{\cup}}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}
\arrow[u, hook, shift right=1.5ex] \arrow[u, hook, shift left=4.5ex]
\arrow[l, hook', shift right=1ex] \arrow[lll, hook', shift right=7ex]
\arrow[r, shift right=3ex]
&
\ast
\arrow[l, hook', shift right=1ex] \arrow[l, hook', shift right=5ex]
\end{tikzcd}$$

smooth = non-singular

Super-Orbifolds – Abstract definition

Let $\underbrace{V}_{\substack{\text{tangent}\\ \text{space}\\ \text{model}}}$, $\underbrace{G}_{\substack{\text{generic}\\ \text{singularity}\\ \text{type}}}$ $\in \mathrm{Grp}(\mathbf{H})$ be group objects.

Definition. A **G-orbi V-fold** is

- an object $\mathcal{X} \in \mathbf{H}_{/\mathbf{B}G_{\prec}}$

which is

1. 0-truncated: $\tau_0(\mathcal{X}) \simeq \mathcal{X}$
2. orbi-singular: $\prec(\mathcal{X}) \simeq \mathcal{X}$
3. a V-fold: there exists a V-*atlas*

$$\begin{tikzcd} & U \arrow[dl, "p_V"'] \arrow[dr, "p_X"] & \\ V & & \mathcal{X}_{\cup} \end{tikzcd}$$

   (a) p_X is a covering: $(\tau_{-1})_{/X}(p_X) \simeq *$

   (b) p_X is a local diffeomorphism: $\Im_{/X}(p_X) \simeq p_X$

   (c) p_V is a local diffeomorphism: $\Im_{/V}(p_V) \simeq p_V$

The global equivariant 4-sphere

In the following $G := \mathrm{Pin}(5)^\flat$
the unoriented spin group in 5d, regarded as geometrically discrete.

This unifies
ADE-singularities
with
O-plane singularities

$$\begin{tikzcd}
 & \text{[HSS18]} & & \\
\overbrace{(G_{\mathrm{ADE}} \times_Z G'_{\mathrm{ADE}})}^{\substack{\text{ADE-singularity} \\ \text{[MFFGME09, MFF10]}}} \arrow[d, hook] & \times & \overbrace{\mathbb{Z}_2}^{\text{O-plane}} \arrow[d, equal] & \\
\mathrm{Spin}(4) & \times & O(1) \arrow[r, hook] & \mathrm{Pin}(5)
\end{tikzcd}$$

Write $\mathbf{S}^4 \in \mathrm{SmoothManifolds}$ for the smooth 4-sphere.

$\hookrightarrow \mathbf{H}$

with $S^4 := \int\left(\mathbf{S}^4\right) \in \infty\mathrm{Groupoids}$ its shape.

Then

$$\mathbf{S}^4_{\curlyvee} := \left(\mathbf{S}^4 /\!\!/ \flat\mathrm{Pin}(5)\right)_{\curlyvee} \in \mathbf{H}_{/\mathbf{B}\mathrm{Pin}(5)^\flat_{\curlyvee}}$$ is a $\mathrm{Pin}(5)^\flat$-orbi $\mathbb{R}^4$-fold

$$S^4_{\curlyvee} := \int\left(\mathbf{S}^4 /\!\!/ \mathrm{Pin}(5)^\flat\right)_{\curlyvee}$$ is its shape orbi-space

Equivariant Cohomotpy of Super-orbifolds

Let

$\mathbb{R}^{10,1|\mathbf{32}} \in \mathrm{Grp}(\mathbf{H})$ $D = 1, \mathcal{N} = 1$ translational supersymmetry

$\mathcal{X} \in \mathbf{H}_{/\mathbf{B}\mathrm{Pin}(5)^\flat_{\rotatebox[origin=c]{70}{$\prec$}}}$ a $\mathrm{Pin}(5)^\flat$-orbi $\mathbb{R}^{10,1|\mathbf{32}}$-fold.

Definition.

The cocycle space of *equivariant Cohomotopy* of $\mathcal{X}$ is

$$\mathbf{H}_{/\mathbf{B}\mathrm{Pin}(5)^\flat_{\rotatebox[origin=c]{70}{$\prec$}}}\left(\mathcal{X}\,,\, S^4_{\rotatebox[origin=c]{70}{$\prec$}}\right) \;=\; \left\{ \begin{tikzcd} \mathcal{X} \arrow[rr, dashed, "\substack{\text{cocycle in}\\ \text{equivariant Cohomotopy}}"] \arrow[dr] & & S^4_{\rotatebox[origin=c]{70}{$\prec$}} \arrow[dl] \\ & \mathbf{B}\mathrm{Pin}(5)^\flat_{\rotatebox[origin=c]{70}{$\prec$}} \arrow[from=1-2, to=2-2, phantom, "\Swarrow"] & \end{tikzcd} \right\}$$

and so the cohomology set is

$$H\left(\mathcal{X}, S^4_{\rotatebox[origin=c]{70}{$\prec$}}\right) \;:=\; \pi_0 \mathbf{H}_{/\mathbf{B}\mathrm{Pin}(5)^\flat_{\rotatebox[origin=c]{70}{$\prec$}}}\left(\mathcal{X}, S^4_{\rotatebox[origin=c]{70}{$\prec$}}\right)$$

Differential Equivariant Cohomotopy of Super-Orbifolds

Definition. $\mathbf{\Omega}_{\mathrm{flat}}\left(-, \mathfrak{l}S^4_{\curlyvee}\right) \;\in\; \mathbf{H}_{/\mathbb{B}\flat\mathrm{Pin}(5)}$

is the universal moduli space of

$$\mathbb{R}^{d|N} \times \mathbb{D} \times \left(\begin{array}{c} \mathbb{B}K \\ \downarrow \\ \mathbb{B}\flat\mathrm{Pin}(5) \end{array} \right) \;\mapsto\; \overbrace{\Omega_{\mathrm{flat}}\Big(\mathbb{R}^{d|N} \times \mathbb{D},\, \mathfrak{l}\, \underbrace{\underbrace{\left(S^4\right)^K}_{\text{fixed point sphere}}}_{\substack{L_\infty\text{-algebra dual to its} \\ \text{minimal Sullivan model}}} \Big)}^{\substack{\text{flat } L_\infty\text{-algebra valued} \\ \text{super-differential forms}}}$$

Claim: $\quad \int\!\Big(\mathbf{\Omega}_{\mathrm{flat}}\left(-, \mathfrak{l}S^4_{\curlyvee}\right)\Big) \simeq \left(S^4_{\curlyvee}\right)_{\mathbb{R}}$

Definition. The *differential equivariant 4-sphere* is

$$\widehat{S^4_{\curlyvee}} \;:=\; S^4_{\curlyvee} \underset{\left(S^4_{\curlyvee}\right)_{\mathbb{R}}}{\times} \mathbf{\Omega}_{\mathrm{flat}}\left(-, \mathfrak{l}S^4_{\curlyvee}\right)$$

Hence *differential equivariant Cohomotopy in degree 4* is

$$H\left(\mathcal{X}, \widehat{S^4_{\curlyvee}}\right) \;:=\; \pi_0 \mathbf{H}_{/\mathbf{B}\mathrm{Pin}(5)^{\flat}_{\curlyvee}}\left(\mathcal{X}\,,\, \widehat{S^4_{\curlyvee}}\right)$$

M-brane charge quantized C-field

C-field	**electromagnetic field (“A-field”)**
flux forms $\mathcal{X} \xrightarrow{(G_4,G_7)} \mathbf{\Omega}_{\mathrm{flat}}\big(-, \mathfrak{l} S^4_{\curlyvee}\big)$	Faraday tensor $X \xrightarrow{F_2} \underbrace{\mathbf{\Omega}_{\mathrm{flat}}\big(-\, \mathfrak{l} BU(1)\big)}_{=\mathbf{\Omega}^2_{\mathrm{clsd}}(-)}$
M-brane charge quantized C-field $\begin{tikzcd} & \widehat{S^4_{\curlyvee}} \arrow[d] \\ \mathcal{X} \arrow[ur, dashed, "\hat{C}"] \arrow[r, "(G_4,G_7)"'] & \mathbf{\Omega}_{\mathrm{flat}}\big(-, \mathfrak{l} S^4_{\curlyvee}\big) \end{tikzcd}$	Dirac charge quantized electromagnetic field $\begin{tikzcd} & \mathbf{B}U(1)_{\mathrm{conn}} \arrow[d] \\ X \arrow[ur, dashed, "\hat{A}"] \arrow[r, "F_2"'] & \mathbf{\Omega}^2_{\mathrm{clsd}}\big(-\big) \end{tikzcd}$

Super Cartan geometry

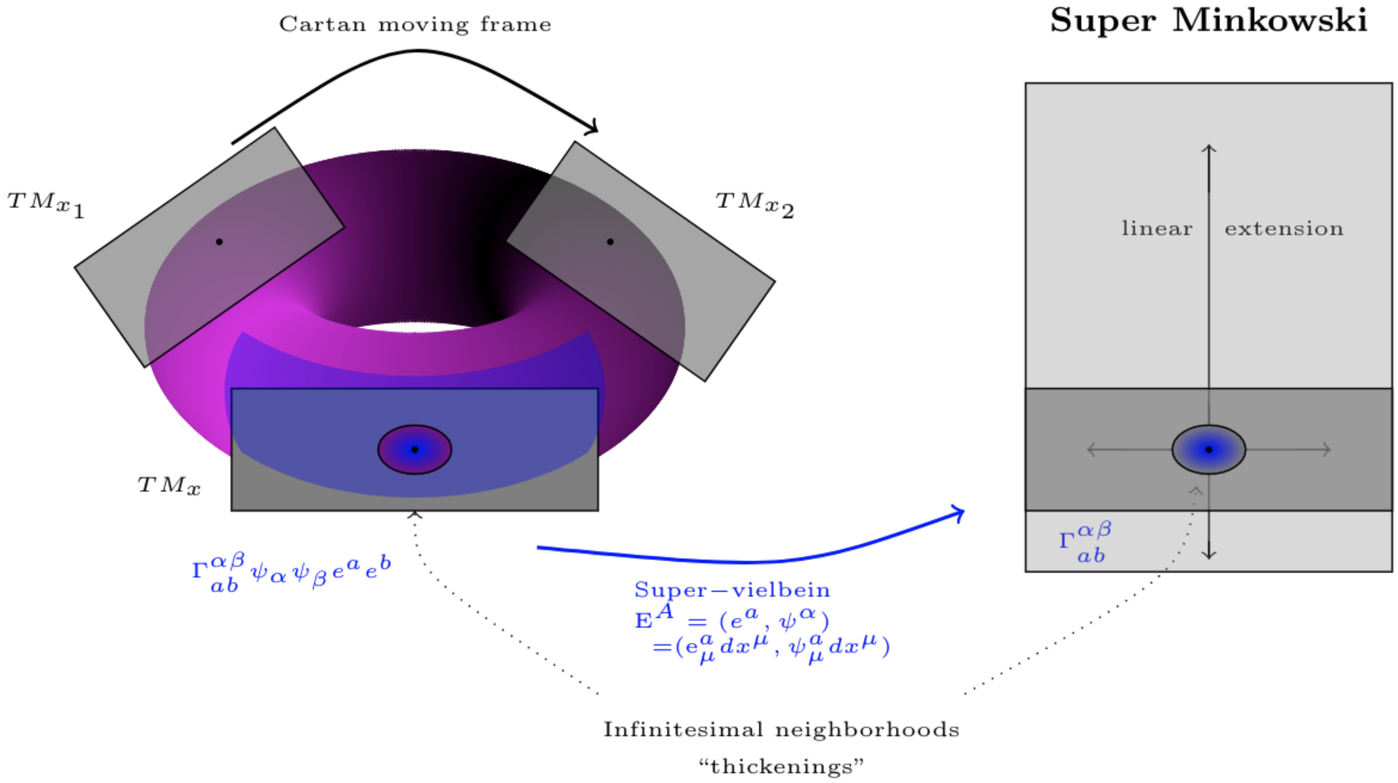
and 11d orbifold supergravity

back to Part I

Cartan geometry formalizes Einstein principle of equivalence

Spacetime is locally equivalent to Minkowski spacetime,
namely in the infinitesimal neighbourhood of every point

We now generalize this
from manifolds to super-orbifolds...



G-Structures on orbi V-folds ([Wellen17, Sch13])

Def.: infinitesimal disk around origin:

$$\mathbb{D}^V \;:=\; V \underset{\Im(V)}{\times} \{e\} \hookrightarrow V$$

Prop.: every orbi V-fold $\mathcal{X}$ carries
its canonical V*-frame bundle*

$$\mathcal{X}_{\cup} \xrightarrow{\text{frame}} \mathbf{BAut}\left(\mathbb{D}^V\right)$$

Def.: for $G \xrightarrow{\text{homom.}} \mathbf{Aut}\left(\mathbb{D}^V\right)$
a G-structure is a lift
(E is the *vielbein*)

$$\begin{tikzcd}
\mathcal{X}_{\cup} \arrow[rr, dashed, ""{name=U, below}] \arrow[dr, "\text{frame}"', ""{name=D}] & & \mathbf{B}G \arrow[dl] \\
& \mathbf{BAut}\left(\mathbb{D}^V\right) &
\arrow[Rightarrow, from=U, to=D, "E"]
\end{tikzcd}$$

Prop.: V itself carries
canonical G-structure
given by left translation

$$\begin{tikzcd}
V \arrow[rr, dashed, ""{name=U, below}] \arrow[dr, "\text{frame}"', ""{name=D}] & & \mathbf{B}G \arrow[dl] \\
& \mathbf{BAut}\left(\mathbb{D}^V\right) &
\arrow[Rightarrow, from=U, to=D, "E_{\text{li}}"]
\end{tikzcd}$$

Def.: a G-structure is *torsion-free and flat*
if it coincides with this canonical one
on each infinitesimal disk

$$E_{|\mathbb{D}^V_x} \;\simeq\; \left(E_{\text{li}}\right)_{|\mathbb{D}^V_e}$$

11d Supergravity from Super homotopy theory

Consider now $V = \mathbb{R}^{10,1|\mathbf{32}}$ and $\mathcal{X}$ an orbi $\mathbb{R}^{10,1|\mathbf{32}}$-fold.

Claim:

$G := \mathbf{Aut}^{\rightsquigarrow}_{\mathrm{Grp}}\big(\mathbb{R}^{10,1|\mathbf{32}}\big)$	$\simeq$	$\mathrm{Spin}(10,1)$
G-structure on $\mathcal{X}$	$\simeq$ $\simeq$	super-vielbein on $\mathcal{X}$ metric/field of gravity
G-structure is torsion-free:	$\Leftrightarrow$ $\underset{\substack{\text{[CaLe93]}\\ \text{[How97]}}}{\Leftrightarrow}$	super-torsion on $\mathcal{X}$ vanishes $\mathcal{X}$ is solution to 11d supergravity with vanishing bosonic flux
G-structure is flat:	$\Leftrightarrow$	$\mathcal{X}$ is a “flat” super-orbifold solution to 11d supergravity

11d Supergravity from Super homotopy theory

Consider now $V = \mathbb{R}^{10,1|\mathbf{32}}$ and $\mathcal{X}$ an orbi $\mathbb{R}^{10,1|\mathbf{32}}$-fold.

Claim:

$G := \mathbf{Aut}^{\rightsquigarrow}_{\mathrm{Grp}}\big(\mathbb{R}^{10,1|\mathbf{32}}\big) \quad \simeq \quad \mathrm{Spin}(10,1)$

G-structure on $\mathcal{X}$ $\quad \simeq \quad$ super-vielbein on $\mathcal{X}$

$\simeq \quad$ metric/field of gravity

G-structure is torsion-free: $\quad \Leftrightarrow \quad$ super-torsion on $\mathcal{X}$ vanishes

$\underset{\text{[CaLe93] [How97]}}{\Leftrightarrow} \quad$ $\mathcal{X}$ is solution to 11d supergravity with vanishing bosonic flux

---

G-structure is flat: $\quad \Leftrightarrow \quad$ $\mathcal{X}$ is a “flat” super-orbifold solution to 11d supergravity

$$\Rightarrow \quad \text{all} \left\{ \begin{array}{c} \text{curvature} \\ \&\ G_4\text{-flux} \end{array} \right\} \text{hence all} \left\{ \begin{array}{c} \text{higher curvature corrections} \\ \&\ \text{flux quantization} \end{array} \right\}$$

crammed into orbifold singularities
and thus taken care of by the *equivariance*
of charge quantization in differential equivariant Cohomotopy

Flat & fluxless except at curvature- & flux- singularities

Plausibility check:

Black M2/M5-brane solutions to SuGra interpolate ([AFFHS98]) between:

$$\text{Planck-scale curved throat} \xleftrightarrow{\text{near/far horizon geometry}} \begin{array}{c}\text{orbifold-singularity}\\ \text{in flat \& fluxless space}\end{array}$$

$$\begin{tikzcd} & \text{black M2} \arrow[dl, "\ell_P \gg 1"'] \arrow[dr, "\ell_P \ll 1"] & \\ \mathrm{AdS}_4 \times \underbrace{S^7/G_{\mathrm{A}}}_{\text{spherical space form}} & & \mathbb{R}^{2,1} \times \underbrace{\mathbb{R}^8 /\!\!/ G_{\mathrm{A}}}_{\text{orbifold du Val singularity}} \end{tikzcd}$$

inconsistent:
Planck-scale throat ($\ell_P \gg 1$)
spurious in SuGra ($\ell_P \ll 1$)
(evaded only by
macroscopic $N \gg 1$)

consistent:
all Planck-scale geometry
crammed into orbi-singularity
(necessary for
microscopic $N = 1$)

Hence, indeed, a consistent & complete picture:

1. is flat & fluxless away from singularities,
2. has hidden degrees of freedom inside the singularities.

Flat & fluxless except at curvature- & flux- singularities

Plausibility check:

Black M2/M5-brane solutions to SuGra interpolate ([AFFHS98]) between:

$$\text{Planck-scale curved throat} \xleftrightarrow{\text{near/far horizon geometry}} \begin{array}{c}\text{orbifold-singularity}\\ \text{in flat \& fluxless space}\end{array}$$

$$\begin{tikzcd} & \text{black M5} \arrow[dl, "\ell_P \gg 1"'] \arrow[dr, "\ell_P \ll 1"] & \\ \mathrm{AdS}_7 \times \underbrace{S^4 /\!\!/ G}_{\text{[MFF10]}} & & \mathbb{R}^{6,1} \times \underbrace{\mathbb{R}^4 /\!\!/ G}_{\text{orbifold du Val singularity}} \end{tikzcd}$$

inconsistent:
Planck-scale throat ($\ell_P \gg 1$)
spurious in SuGra ($\ell_P \ll 1$)
(evaded only by
macroscopic $N \gg 1$)

consistent:
all Planck-scale geometry
crammed into orbi-singularity
(necessary for
microscopic $N = 1$)

Hence, indeed, a consistent & complete picture:

1. is flat & fluxless away from singularities,
2. has hidden degrees of freedom inside the singularities.

In conclusion

the following enhancement of 11d supergravity
naturally emerges out of super homotopy theory

The full covariant phase space is...

$$\mathrm{CovariantPhaseSpace} := \coprod_{\substack{[\mathcal{X}] \in \\ \mathrm{Pin}(5)^\flat\mathrm{Orbi-} \\ \mathbb{R}^{10,1|\mathbf{32}}\mathrm{Folds}(\mathbf{H})_\sim}} \left(\begin{tikzcd} & \mathrm{CovariantPhaseSpace}_{\mathcal{X}} \arrow[dl] \arrow[dr] & \\ \underline{\mathbf{H}}\Big(\mathcal{X}_{\cup}, \mathbf{B}\mathrm{Spin}(10,1)\Big)^{\tau=0}_{/\mathbf{BAut}(\mathbb{D}^{10,1|\mathbf{32}})} \arrow[dr] & {\scriptstyle (\mathrm{pb})} & \underline{\mathbf{H}}\Big(\mathcal{X}_{\curlyvee}, \widehat{S^4_{\curlyvee}}\Big)_{/\mathbf{B}\mathrm{Pin}(5)^\flat_{\curlyvee}} \arrow[dl] \\ & \underline{\mathbf{H}}\Big(\mathcal{X}_{\cup}, \mathbf{\Omega}(-, \text{ʃ} S^4)\Big) & \end{tikzcd} \right)$$

$$\begin{aligned} \mathrm{Observables} &:= \widehat{H\mathbb{R}}_{\curlyvee}\Big(\mathrm{CovariantPhaseSpace}\Big) \\ &= \widehat{H\mathbb{R}}_{\curlyvee}\Big(\Sigma^{\infty}_{\mathrm{Pin}(5)^\flat}\mathrm{CovariantPhaseSpace}\Big) \end{aligned}$$

In conclusion

the following enhancement of 11d supergravity
naturally emerges out of super homotopy theory

... for each class of super-orbifolds $\mathcal{X}$...

$$\mathrm{CovariantPhaseSpace} := \coprod_{\substack{[\mathcal{X}] \in \\ \mathrm{Pin}(5)^\flat\mathrm{Orbi}- \\ \mathbb{R}^{10,1|\mathbf{32}}\mathrm{Folds}(\mathbf{H})_\sim}} \left(\begin{tikzcd} & \mathrm{CovariantPhaseSpace}_{\mathcal{X}} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\text{(pb)}"] & \\ \underline{\mathbf{H}}\Big(\mathcal{X}_{\cup}, \mathbf{B}\mathrm{Spin}(10,1)\Big)^{\tau=0}_{/\mathbf{BAut}(\mathbb{D}^{10,1|\mathbf{32}})} \arrow[dr] & & \underline{\mathbf{H}}\Big(\mathcal{X}_{\rotatebox[origin=c]{70}{$\prec$}}, \widehat{S^4_{\rotatebox[origin=c]{70}{$\prec$}}}\Big)_{/\mathbf{B}\mathrm{Pin}(5)^\flat_{\rotatebox[origin=c]{70}{$\prec$}}} \arrow[dl] \\ & \underline{\mathbf{H}}\Big(\mathcal{X}_{\cup}, \mathbf{\Omega}(-, ʃ S^4)\Big) & \end{tikzcd} \right)$$

$$\begin{aligned} \mathrm{Observables} &:= \widehat{H\mathbb{R}}_{\rotatebox[origin=c]{70}{$\prec$}}\Big(\mathrm{CovariantPhaseSpace}\Big) \\ &= \widehat{H\mathbb{R}}_{\rotatebox[origin=c]{70}{$\prec$}}\Big(\Sigma^\infty_{\mathrm{Pin}(5)^\flat}\mathrm{CovariantPhaseSpace}\Big) \end{aligned}$$

In conclusion

the following enhancement of 11d supergravity
naturally emerges out of super homotopy theory

... a super-torsion-free Spin-structure
encoding the fields of supergravity...

$$\mathrm{CovariantPhaseSpace} :=$$

$$\coprod_{\substack{[\mathcal{X}] \in \\ \mathrm{Pin}(5)^\flat\mathrm{Orbi}- \\ \mathbb{R}^{10,1|\mathbf{32}}\mathrm{Folds}(\mathbf{H})_\sim}} \left(\begin{tikzcd} & \mathrm{CovariantPhaseSpace}_{\mathcal{X}} \arrow[dl] \arrow[dr] & \\ \underline{\mathbf{H}}\Big(\mathcal{X}_{\cup}, \mathbf{B}\mathrm{Spin}(10,1)\Big)^{\tau=0}_{/\mathbf{BAut}(\mathbb{D}^{10,1|\mathbf{32}})} \arrow[dr] & {\scriptstyle (\mathrm{pb})} & \underline{\mathbf{H}}\Big(\mathcal{X}_{\rotatebox[origin=c]{70}{$\prec$}}, \widehat{S^4_{\rotatebox[origin=c]{70}{$\prec$}}}\Big)_{/\mathbf{B}\mathrm{Pin}(5)^\flat_{\rotatebox[origin=c]{70}{$\prec$}}} \arrow[dl] \\ & \underline{\mathbf{H}}\Big(\mathcal{X}_{\cup}, \mathbf{\Omega}\big(-, ʃ S^4\big)\Big) & \end{tikzcd} \right)$$

$$\mathrm{Observables} := \widehat{H\mathbb{R}}_{\rotatebox[origin=c]{70}{$\prec$}}\Big(\mathrm{CovariantPhaseSpace}\Big)$$

$$= \widehat{H\mathbb{R}}_{\rotatebox[origin=c]{70}{$\prec$}}\Big(\Sigma^\infty_{\mathrm{Pin}(5)^\flat}\mathrm{CovariantPhaseSpace}\Big)$$

In conclusion

the following enhancement of 11d supergravity
naturally emerges out of super homotopy theory

... equipped with a compatible lift of the flux forms
to a cocycle in differential equivariant Cohomotopy (charge quantization).

$$
\mathrm{CovariantPhaseSpace} :=
$$

$$
\underset{\substack{[\mathcal{X}] \in \\ \mathrm{Pin}(5)^\flat\mathrm{Orbi}- \\ \mathbb{R}^{10,1|\mathbf{32}}\mathrm{Folds}(\mathbf{H})_\sim}}{\coprod}
\left(
\begin{tikzcd}
 & \mathrm{CovariantPhaseSpace}_{\mathcal{X}} \arrow[dl] \arrow[dr] & \\
\underline{\mathbf{H}}\Big(\mathcal{X}_{\cup}, \mathbf{B}\mathrm{Spin}(10,1)\Big)^{\tau=0}_{/\mathbf{BAut}(\mathbb{D}^{10,1|\mathbf{32}})} \arrow[dr] \arrow[rr, phantom, "{\scriptstyle (\mathrm{pb})}"] & & \underline{\mathbf{H}}\Big(\mathcal{X}_{\rotatebox[origin=c]{70}{$\prec$}}, \widehat{S^4_{\rotatebox[origin=c]{70}{$\prec$}}}\Big)_{/\mathbf{B}\mathrm{Pin}(5)^\flat_{\rotatebox[origin=c]{70}{$\prec$}}} \arrow[dl] \\
 & \underline{\mathbf{H}}\Big(\mathcal{X}_{\cup}, \mathbf{\Omega}\big(-, \mathfrak{l}S^4\big)\Big) &
\end{tikzcd}
\right)
$$

$$
\begin{aligned}
\mathrm{Observables} &:= \widehat{H\mathbb{R}}_{\rotatebox[origin=c]{70}{$\prec$}}\Big(\mathrm{CovariantPhaseSpace}\Big) \\
&= \widehat{H\mathbb{R}}_{\rotatebox[origin=c]{70}{$\prec$}}\Big(\Sigma^\infty_{\mathrm{Pin}(5)^\flat}\mathrm{CovariantPhaseSpace}\Big)
\end{aligned}
$$

Part II.

Some corners of M-theory

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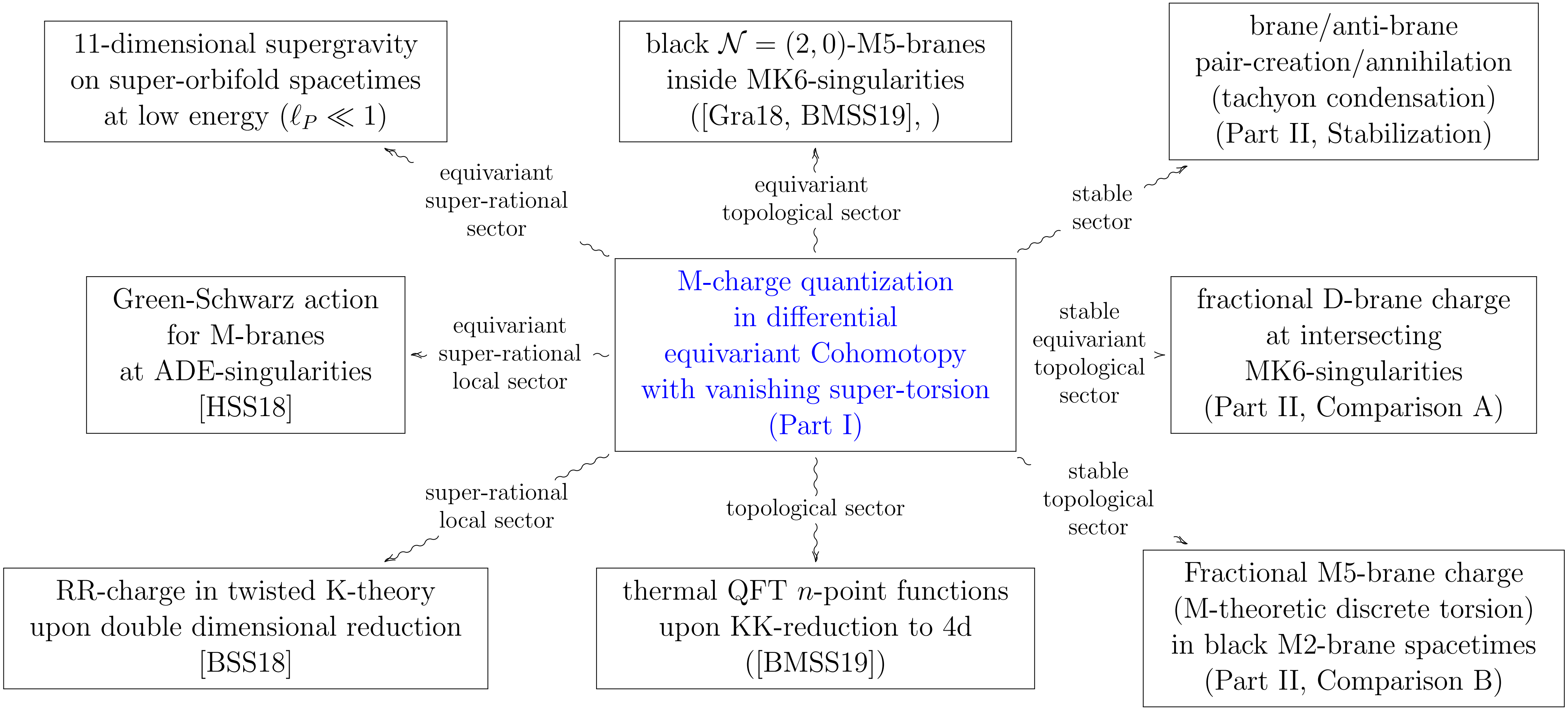
Some corners of M-theory



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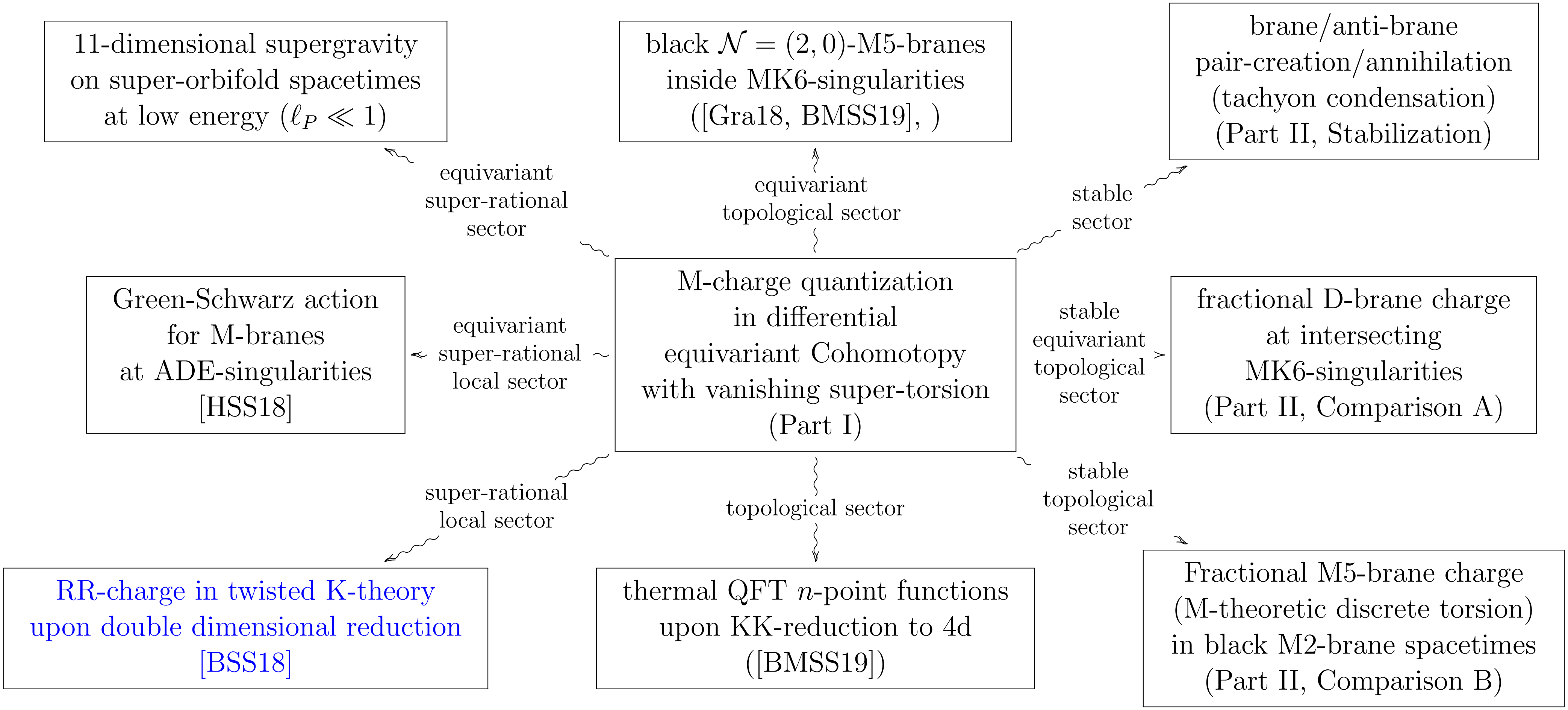
Part II.

Some corners of M-theory



11-dimensional supergravity
on super-orbifold spacetimes
at low energy ($\ell_P \ll 1$)

black $\mathcal{N} = (2,0)$-M5-branes
inside MK6-singularities
([Gra18, BMSS19],)

brane/anti-brane
pair-creation/annihilation
(tachyon condensation)
(Part II, Stabilization)

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equivariant
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M-charge quantization
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Fractional M5-brane charge
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(Part II, Comparison B)

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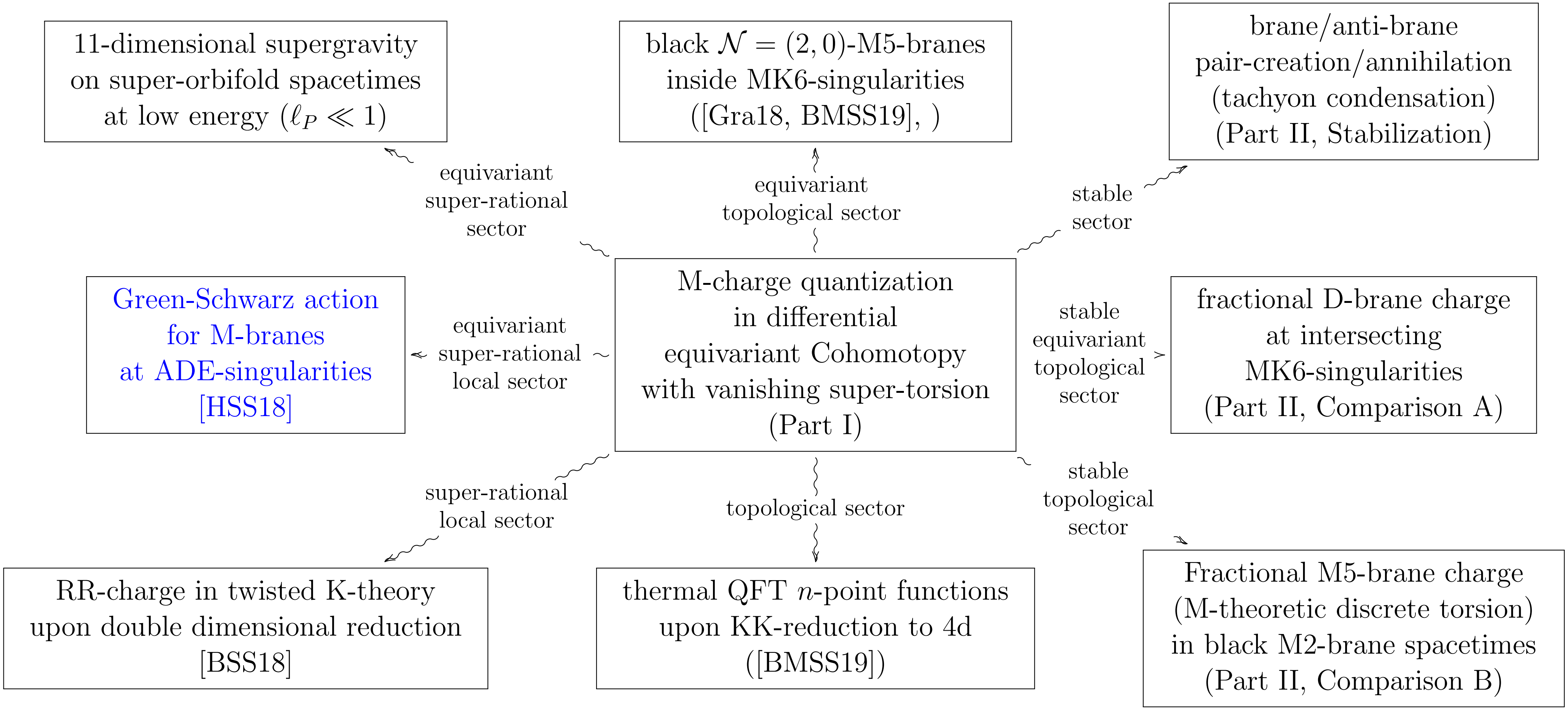
Some corners of M-theory



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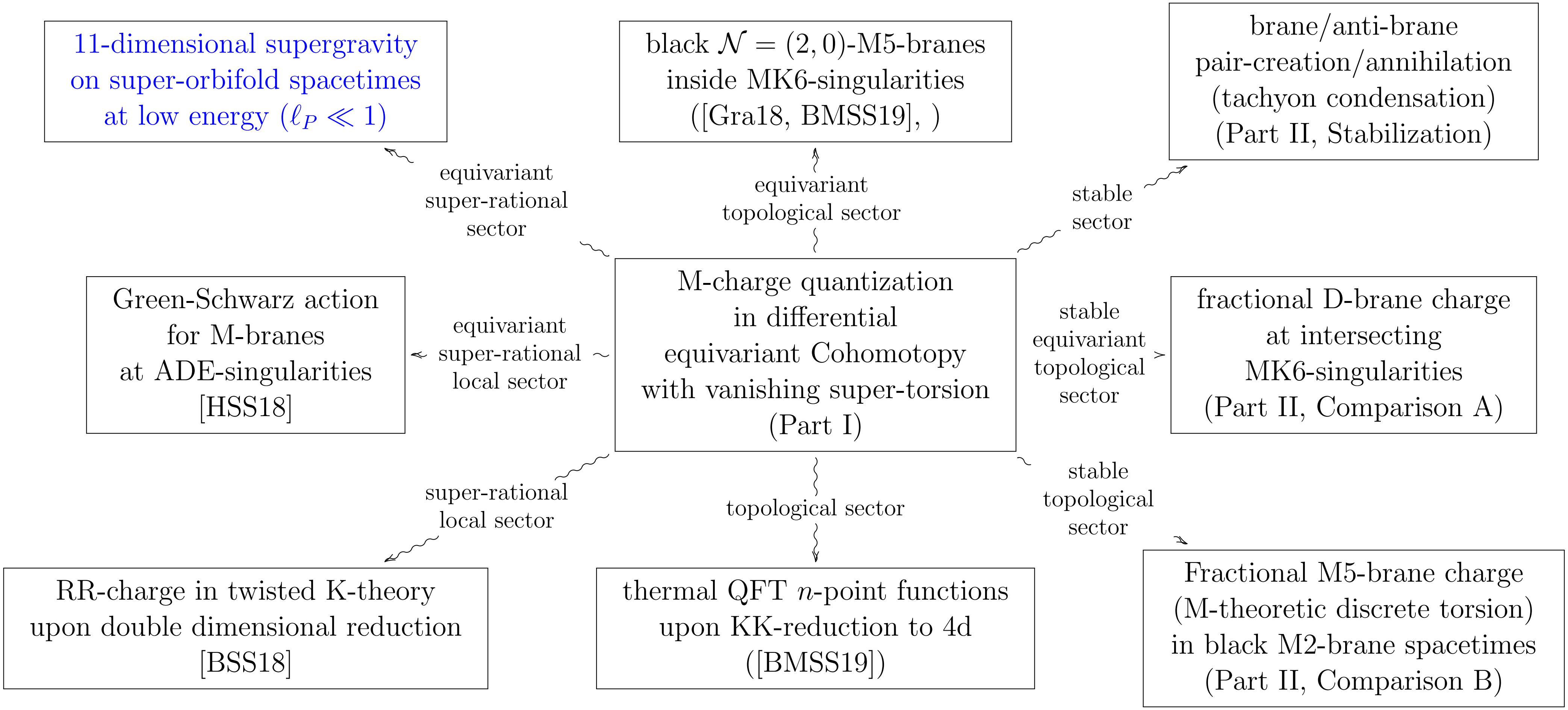

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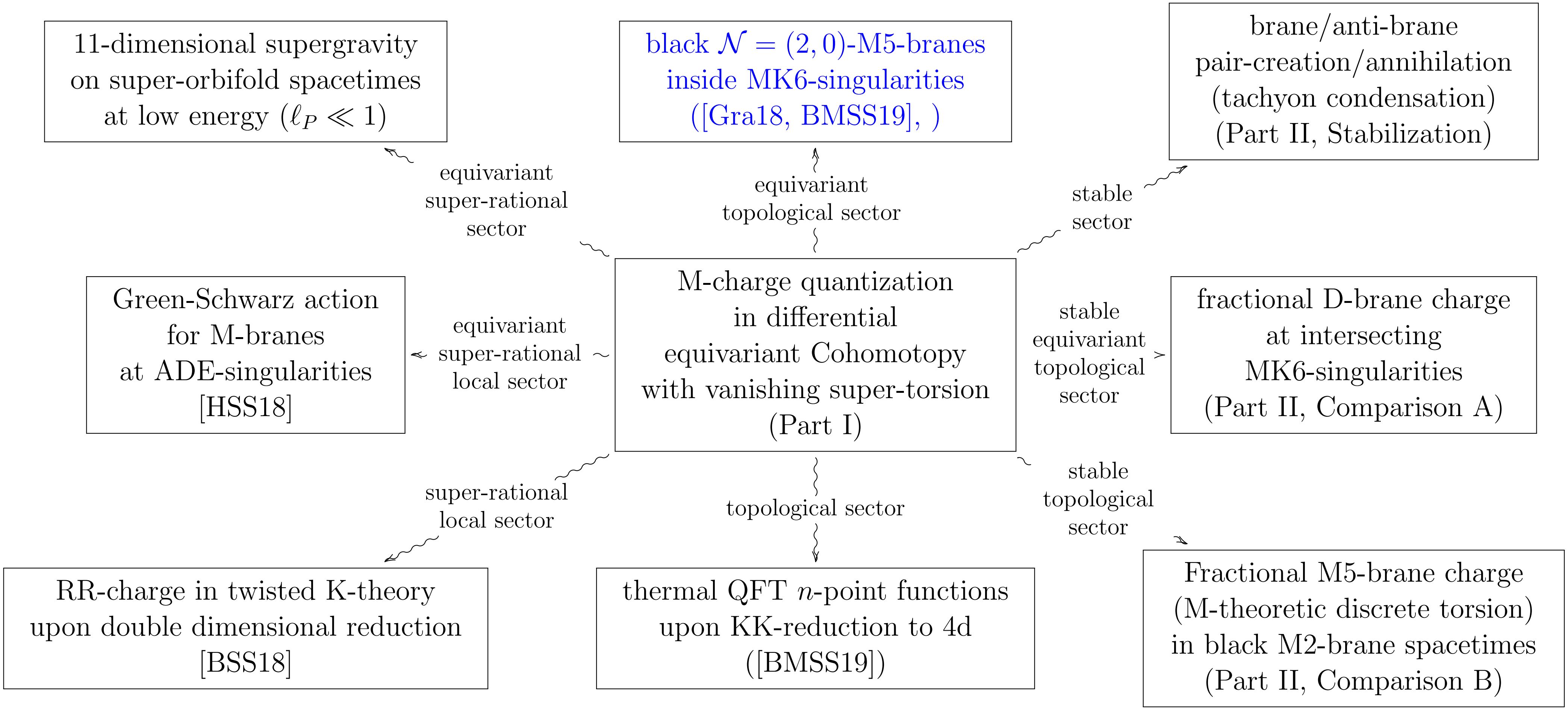

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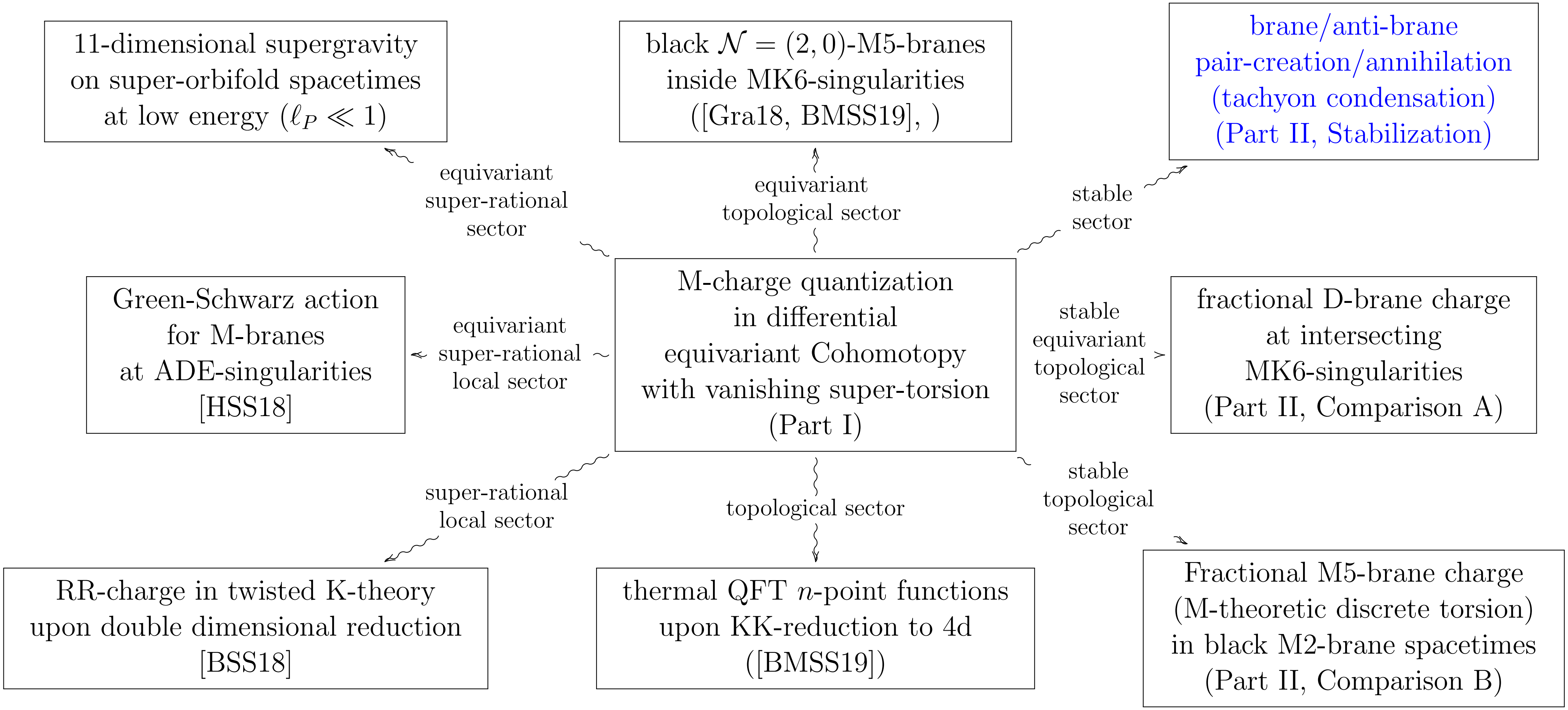

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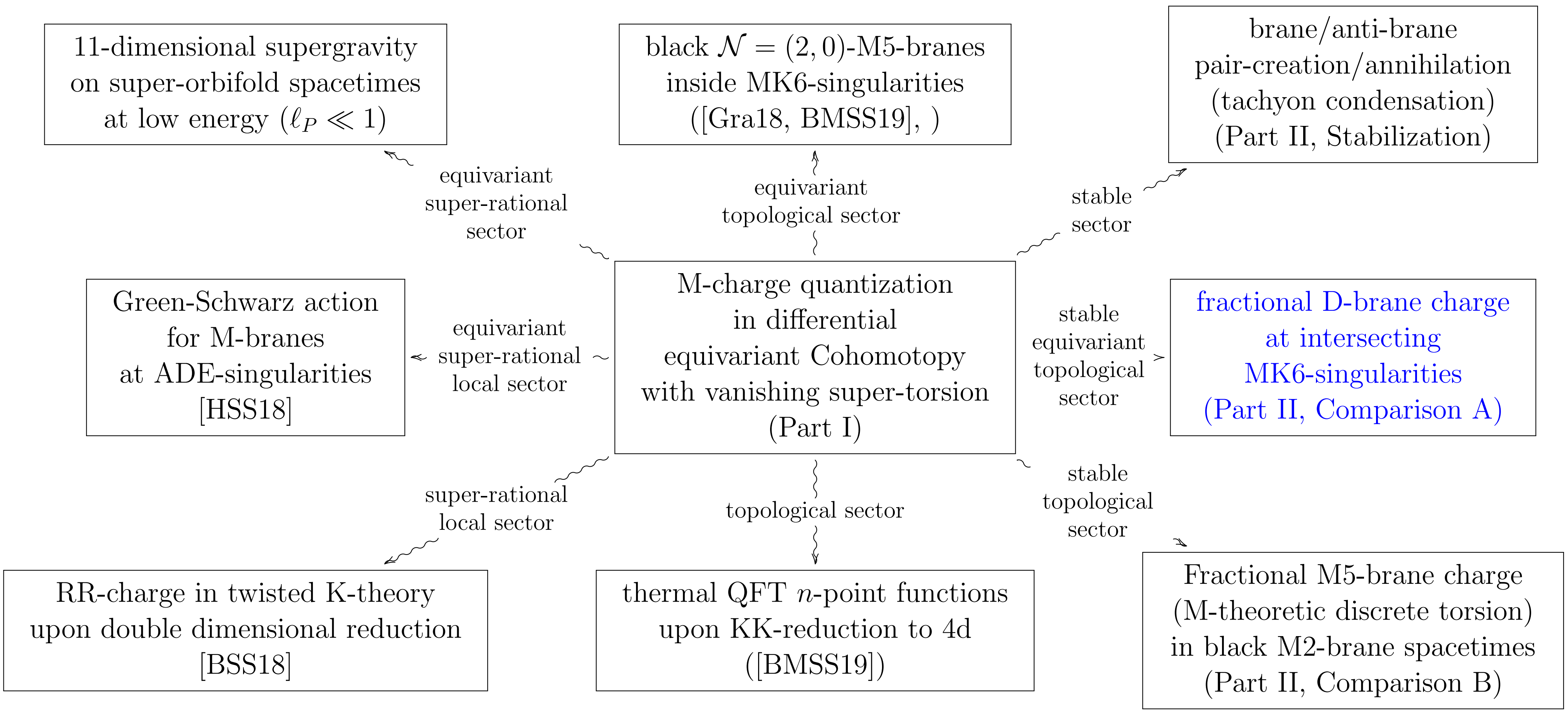

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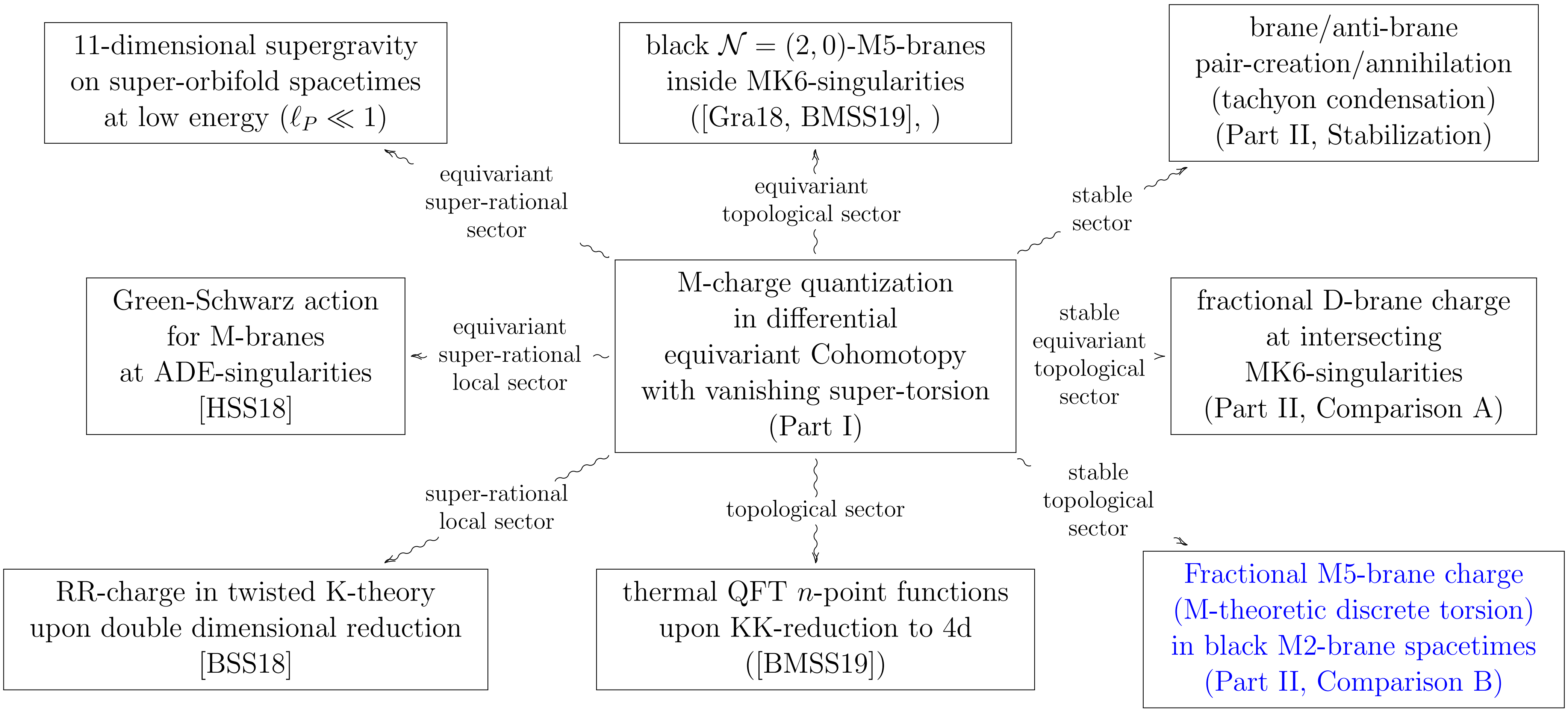

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Stable homotopy theory

and anti-branes

back to Part II

Stable Cohomotopy from observing Cohomotopy

Any space of observables is *linear*:
Observables may be added and subtracted.

Linear + Homotopy theory = *Stable homotopy theory*

$$\underbrace{\widehat{H\mathbb{R}}_G\Big(\mathrm{CovariantPhaseSpace}\Big)}_{\substack{\text{homotopy-linear}\\ \text{space of observables}}} = \widehat{H\mathbb{R}}_G\Big(\underbrace{\Sigma^\infty_G \mathrm{CovariantPhaseSpace}}_{\substack{\text{free homotopy-linearized}\\ \text{covariant phase space}}}\Big)$$

There is canonical comparison map

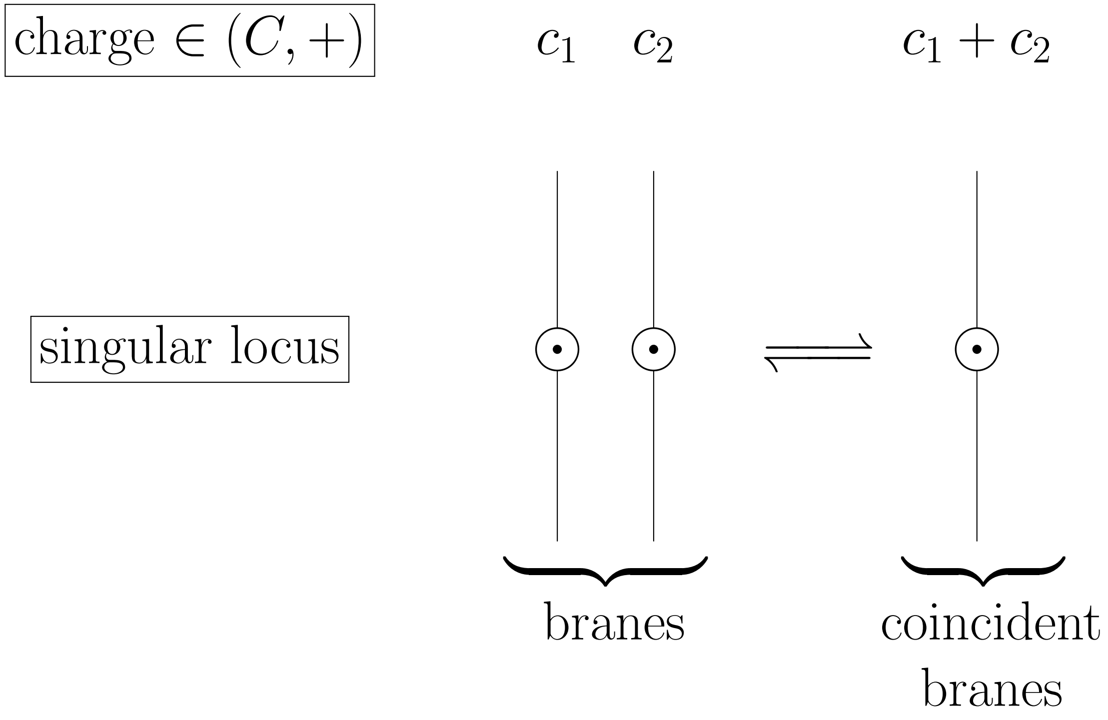
$$\Sigma^\infty \underbrace{\mathbf{H}_{/\mathbf{B}\mathrm{Pin}(5)^\flat_{\curlyvee}}(\mathcal{X}, S^4_{\curlyvee})}_{\text{Cohomotopy}} \xrightarrow[\text{approximation}]{\text{homotopy-linear}} \underbrace{\mathrm{Stab}(\mathbf{H})_{\mathrm{Pin}^\flat_{\curlyvee}}\Big(\Sigma^\infty_G \mathcal{X},\ \overbrace{\mathbb{S}^4_{\curlyvee}}^{\substack{\text{4-shifted}\\ \text{equivariant}\\ \text{sphere spectrum}}}\Big)}_{\text{stable Cohomotopy}}$$

to fields in *stable equivariant Cohomotopy*

We now discuss what this means...

Brane charge – 1st order approximation

Let $(C, +)$ be
an abelian semigroup
a “commutative monoid”
of charges.



charge $\in (C, +)$

$c_1 \quad c_2$

$c_1 + c_2$

singular locus

$\rightleftharpoons$

branes

coincident branes

Fundamental example: the natural numbers

$$(C, +) \; = \; (\mathbb{N}, +)$$

charge c = number of coincident branes $\Leftrightarrow$ each brane carries unit charge

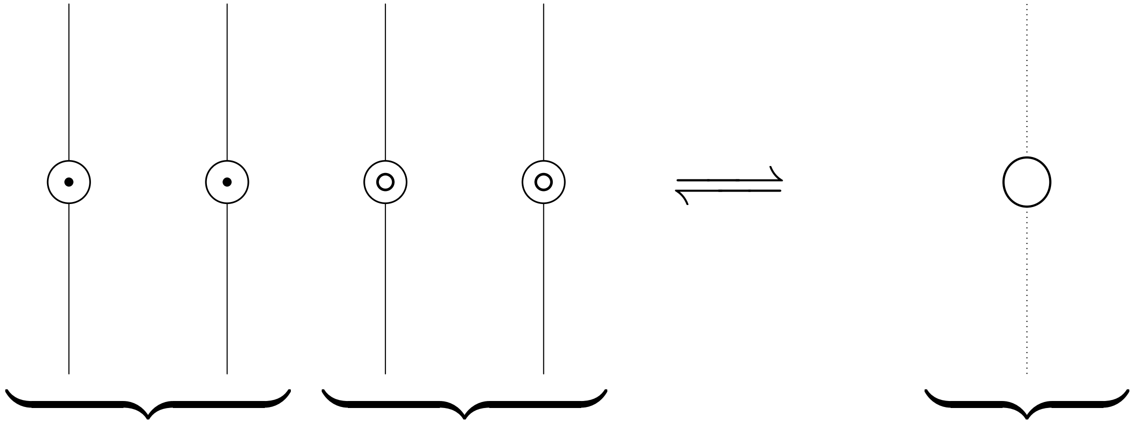
Brane charge – 2nd order approximation

Including anti-brane charges, hence negative brane charges,
means to pass to the abelian group completion of the charge monoid:

$$K(C,+) \;\coloneqq\; \Big\{ \, (c^+, c^-) \mid c^\pm \in C \, \Big\} \Big/ \Big(\, (c,c) \sim 0 \, \Big)$$

$\boxed{\text{charge} \in K(C,+)}$ $(c_1, 0)$ $(c_2, 0)$ $(0, c_2)$ $(0, c_1)$ $(c_1 + c_2, c_1 + c_2) \sim 0$

$\boxed{\text{singular locus}}$



branes

anti-branes

coincident branes

Fundamental example: the integers: $K(\mathbb{N}, +) \;=\; (\mathbb{Z}, +)$

$(c^+, c^-) =$ number of coincident branes minus number of anti-branes $\Leftrightarrow$ each brane carries unit charge
each anti-brane carries negative unit charge

Brane charge – 3rd order approximation

The categorification of *commutative monoid* $(C, +)$ is *symmetric monoidal category* $(\mathcal{C}, \oplus)$

$$(C, +) \;=\; \pi_0 \, (\mathcal{C}, \oplus)$$

- **Fundamental non-linear example**

  Finite pointed sets with disjoint unition $(\mathcal{C}, \oplus) \;=\; \left(\mathrm{Set}^{*/}_{\mathrm{fin}}, \sqcup\right)$

  this categorifies the previous example: $\pi_0 \left(\mathrm{Set}^{*/}_{\mathrm{fin}}, \sqcup\right) \;\simeq\; (\mathbb{N}, +)$

- **Fundamental linear example** for $\mathbb{F}$ a field

  finite-dim vector spaces with direct sum $(\mathcal{C}, \oplus) = \left(\mathbb{F}\mathrm{Vect}_{\mathrm{fin}}, \oplus\right)$

  this *also* categorifies the previous example: $\pi_0 \left(\mathbb{F}\mathrm{Vect}_{\mathrm{fin}}, \oplus\right) \;\simeq\; (\mathbb{N}, +)$

Unified on a deeper level:

pointed sets may be regarded as the vector spaces over the "absolute ground-field with one element" $\mathbb{F}_1$

$$\left(\mathrm{Set}^{*/}_{\mathrm{fin}}, \sqcup\right) \;\simeq\; \left(\mathbb{F}_1\mathrm{Vect}_{\mathrm{fin}}, \oplus\right)$$

$$\begin{tikzcd}
\mathrm{Spec}(\mathbb{F}) \arrow[dd] \arrow[dr] & \\
 & \mathrm{Spec}(\mathbb{Z}) \arrow[dl] \\
\mathrm{Spec}(\mathbb{F}_1) &
\end{tikzcd}$$

algebraic base

combinatorial absolute base

Brane charge in generalized cohomology

Brane/anti-brane annihilation may be varying over spacetime X

⇝ enhance discrete abelian group of charges to a *space* of charges

The homotopification of *abelian group* $(A,+)$ is *∞-loop space / spectrum* $\mathcal{A}$

$$(A,+) \;=\; \pi_0\,(\mathcal{A})$$

brane charge

locally constant	**locally varying**
$X \longrightarrow \underbrace{(A,+)}_{\text{discrete abelian group}}$	$X \longrightarrow \underbrace{\mathcal{A}}_{\infty\text{-loop space or spectrum}}$

Hence brane charge group on spacetime X is generalized cohomology group:

$$\mathcal{A}(X) \;\coloneqq\; \pi_0 \mathrm{Maps}\,(\; X \;,\; \mathcal{A} \;)$$

Example: D-brane/anti-D-brane bound states

open string tachyon condensation profile:

$$X \longrightarrow \mathrm{KU} \qquad\qquad \text{(conjecturally, or similar)}$$

↖ K-theory spectrum

Algebraic K-Theory – locally varying brane/anti-brane annihilation

in conclusion:

brane/anti-brane annihilation

$\rightsquigarrow$

abelian group completion
of charge monoid

locally varying brane charge

$\rightsquigarrow$

spectral enhancement
of charge group

combine:

The categorification / homotopification of *abelian group completion* $K\big(\pi_0(\mathcal{C},\oplus)\big)$ is *algebraic K-theory spectrum* $\mathbb{K}(\mathcal{C},\oplus) \coloneqq \Omega B_\oplus \mathcal{C}$

$$K\Big(\pi_0(\mathcal{C},\oplus)\Big) \;=\; \pi_0\Big(\mathbb{K}(\mathcal{C},\oplus)\Big)$$

Algebraic K-Theory – Examples

- **algebraic K-theory spectrum** of a field $\mathbb{F}$

$$K\mathbb{F} \;\simeq\; \mathbb{K}\big(\mathbb{F}\mathrm{Vect}_{\mathrm{fin}}\big)$$

- **complex algebraic K-theory** $K\mathbb{C}$

  is *flat* K-theory:

$$\begin{tikzcd}
 & \overbrace{\widehat{\mathrm{ku}}}^{\substack{\text{differential}\\ \text{K-theory}}} \arrow[dr, "\substack{\text{project out}\\ \text{topological class}}"] & \\
\underbrace{K\mathbb{C}}_{\substack{\text{algebraic}\\ \text{K-theory}}} \arrow[ur, "\substack{\text{include}\\ \text{flat cocycles}}"] \arrow[rr] & & \mathrm{KU}
\end{tikzcd}$$

- **absolute algebraic K-theory** $K\mathbb{F}_1 \;\coloneqq\; \mathbb{K}\big(\mathbb{F}_1\mathrm{Vect}_{\mathrm{fin}}\big) \;\simeq\; \mathbb{K}\big(\mathrm{Set}^{*/}_{\mathrm{fin}}\big)$

  is stable Cohomotopy theory (Barrat-Priddy-Quillen theorem):

$$\boxed{\underbrace{K\mathbb{F}_1}_{\substack{\text{absolute algebraic}\\ \text{K-theory spectrum}}} \;\simeq\; \underbrace{\mathbb{S}}_{\substack{\text{sphere}\\ \text{spectrum}}}}$$

Brane charge on Orbifolds – Equivariant generalized cohomology

A *representation sphere* $\overset{\overset{G}{\curvearrowright}}{S^V}$:= one-point compactification of linear representation $\overset{\overset{G}{\curvearrowright}}{V}$

A G-equivariant spectrum $\mathcal{A}$ is
a spectrum of G-spaces indexed by representation spheres, hence

1. a system of pointed G-spaces $\left\{ \overset{\overset{G}{\curvearrowright}}{\mathcal{A}_V} \;\middle|\; \overset{\overset{G}{\curvearrowright}}{V} \text{ a linear } G\text{-representation} \right\}$
2. with equivariant suspension morphisms $S^V \wedge \mathcal{A}_W \xrightarrow{\sigma_{V,W}} \mathcal{A}_{V \oplus W}$

Examples

- The *equivariant suspension spectrum* of a G-space $\overset{\overset{G}{\curvearrowright}}{S^V}$ is

$$\Sigma^\infty_G X \;:\; V \;\mapsto\; S^V \wedge X\,.$$

- The equivariant K-theory of a contractible space is the representation ring

$$\mathrm{KU}_G\big(\overset{\overset{G}{\curvearrowright}}{\mathbb{R}^{d,1}}\big) \;\simeq\; \mathrm{KU}_G\big(\overset{\overset{G}{\curvearrowright}}{\ast}\big) \;\simeq\; R_{\mathbb{C}}(G) \;\simeq\; \mathbb{Z}[\overbrace{\rho_1, \cdots, \rho_n}^{\text{irreps}}] \qquad \text{“fractional D-branes”}$$

In conclusion, from Part I:

A compelling candidate for M-brane charge cohomology theory is...

by discussion
in Part I

non-linear
brane/anti-brane
bound states

fundamental
& black
M-branes

by [HSS18]

absolute
K-theory

ADE-equivariant
Cohomotopy

stable
Cohomotopy

in RO-degree
$\mathbb{H}$
↖ quaternions

$$\begin{tikzcd}
K\mathbb{F}_1 \arrow[d, phantom, "\wr"] & & \underset{S^4}{\overset{G_{\mathrm{ADE}}}{\curvearrowright}} \arrow[d, phantom, "\wr"] \\
\mathbb{S} \arrow[dr, mapsto, "\text{equivariantize}"] & & S^{\mathbb{H}} \arrow[dl, mapsto, "\text{stabilize}"'] \\
& \mathbb{S}^{\mathbb{H}}_{G_{\mathrm{ADE}}} \simeq \Sigma^{\infty}_{G_{\mathrm{ADE}}} S^{\mathbb{H}} &
\end{tikzcd}$$

a

ADE-equivariant
stable
Cohomotopy
in RO-degree $\mathbb{H}$

Hypothesis H:

The
generalized cohomology theory
for
M-brane charge

is

ADE-equivariant
stable
Cohomotopy
in RO-degree $\mathbb{H}$

Hypothesis **H** predicts M-brane charge groups:

$$\mathbb{S}^{\mathbb{H}}_{G_{\mathrm{ADE}}}\Big(\underbrace{\overset{\overset{G_{\mathrm{ADE}}}{\curvearrowright}}{X}}_{\substack{\text{11d spacetime}\\ \text{orbifold}}} \Big)$$

How does this compare to / clarify folklore of perturbative string theory:

- intersecting MK6-branes ⇝ fractional D-branes ?
- M-theoretic “discrete torsion” of fractional M5-branes ?
- GUT at E-type singularities ?
- $\cdots$

This we discuss now ⟶

Strategy for testing Hypothesis **H**

1. **Identify** suitable comparison homomorphism

$$\boxed{\begin{array}{c}\text{putative}\\ \text{M-brane charge}\\ \text{in}\\ \text{equivariant stable Cohomotopy}\end{array}} \xrightarrow[\begin{array}{c}\text{putative}\\ \text{perturbative}\\ \text{approximation}\end{array}]{\text{“}\beta \circ \alpha\text{”}} \boxed{\begin{array}{c}\text{putative}\\ \text{D-brane charge}\\ \text{in}\\ \text{equivariant K-theory}\end{array}}$$

2. **Compute**: $$\frac{\text{the } \textbf{co-kernel} \text{ of } \beta \circ \alpha; \;\; \text{reflects} \;\; \begin{array}{c}\text{D-brane configurations}\\ \text{that do not lift}\\ \text{to M-theory}\end{array}}{\text{the } \textbf{kernel} \text{ of } \beta \circ \alpha; \;\; \text{reflects} \;\; \begin{array}{c}\text{M-brane degrees of freedom}\\ \text{invisible in}\\ \text{perturbative string theory}\end{array}}$$

Hypothesis H finds support if the **cokernel of** $\beta \circ \alpha$ is

1. **small** ⇔ putative M-brane charge mostly reproduces string theory folklore,
2. **plausible** ⇔ the putative D-brane states in the co-kernel are dubious.

If so, Hypothsis **H** predicts the **kernel of** $\beta \circ \alpha$ as hidden M-theoretic DOFs.

Outline

Since the sphere spectrum $\mathbb{S}$
is the *initial* commutative ring spectrum,
there is a unique multiplicative comparison morphism
from stable cohomotopy
to *every* other multiplicative cohomology theory $\mathcal{A}$,
called the
equivariant generalized Boardman homomorphism

$$\mathbb{S}^{\alpha}_{G}(X) \xrightarrow{G} \mathcal{A}^{\alpha}_{G}(X)$$

Here we present two cases:

1. **Comparison map A** to
   K-theory and RR-charge of fractional D-branes
2. **Comparison map B** to
   ordinary cohomology and “discrete torsion” of fractional M5-branes

Comparison A to

K-theory
and
fractional RR-charge of D-branes

back to Part II

Finite subgroups $G_{\mathrm{ADE}} \subset \mathrm{SU}(2)$ – Classification

Dynkin Label	**Finite subgroup of** $\mathrm{SU}(2)$	**Name of group**
$\mathbb{A}_{n\geq 1}$	$\mathbb{Z}_{n+1}$	Cyclic
$\mathbb{D}_{n\geq 4}$	$2\mathrm{D}_{2(n-2)}$	Binary dihedral
$\mathbb{E}_6$	$2\mathrm{T}$	Binary tetrahedral
$\mathbb{E}_7$	$2\mathrm{O}$	Binary octahedral
$\mathbb{E}_8$	$2\mathrm{I}$	Binary icosahedral

Assumption: In the following, consider finite groups

$G = G_{\mathrm{DE}} \subset \mathbb{E} \subset \mathrm{SU}(2)$

in the D- or E-series

and

in the exceptional subgroup lattice. $\xrightarrow{\text{next slide}}$

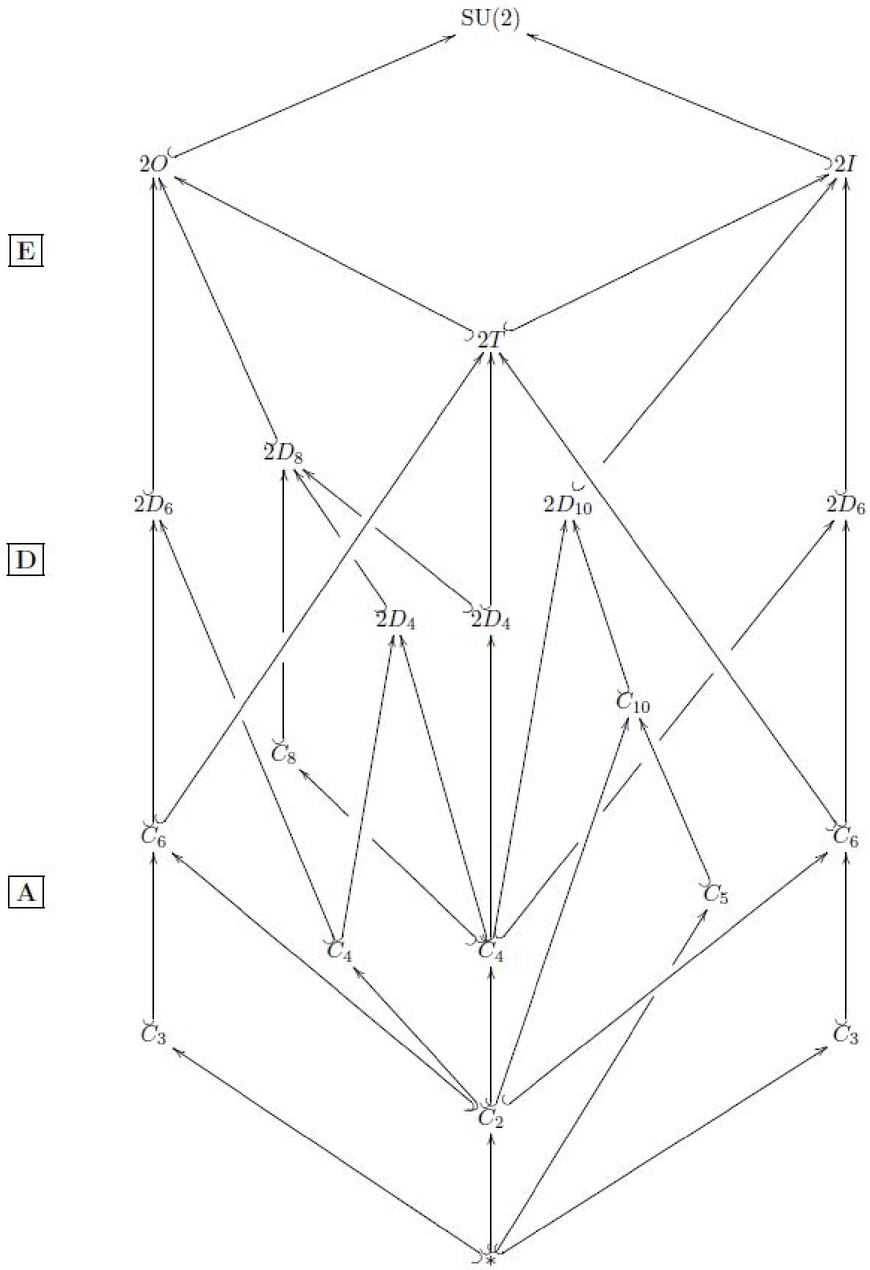
Implies in particular:

G-orbi-folds are G-*orienti-folds*,

the relevant K-theory for fractional D-brane charge

at G-fixed points is is KO-theory

Finite subgroups $G_{\mathrm{ADE}} \subset \mathrm{SU}(2)$ – Exceptional subgroup lattice



The comparison homomorphism A

charge group
of M-branes
at two intersecting
MK6 ADE-singularities

visible in
M-theory
(?)

charge lattice
of fractional M-branes
at an MK6-singularity

visible in
M-theory
(?)

charge lattice
of fractional D-branes
at orientifold singularity

expected in
perturbative
string theory

equivariant stable cohomotopy in RO$(G \times G')$-degree $\mathbb{H}$

equivariant KO-theory
in degree 0

$$\begin{tikzcd}
\mathbb{S}^{\mathbb{H}}_{(G\times G')}(\mathbb{R}^{10,1}) \arrow[r, "\boxed{\text{zoom in onto one MK6}}"] \arrow[d, phantom, sloped, "\simeq"] & \mathbb{S}_G^{\left(\mathbb{H}^{G'}\right)}\left(\mathbb{R}^{6,1}\right) \arrow[r, "\boxed{\text{perturbative approximation}}"] \arrow[d, phantom, sloped, "\simeq"] & \mathrm{KO}^0_G(\mathbb{R}^{6,1}) \arrow[d, phantom, sloped, "\simeq"] \\
\mathbb{S}^{\mathbb{H}}_{(G\times G')}(\ast) \arrow[r, "\alpha", "\boxed{\text{form geometric } G'\text{-fixed spectrum}}"'] & \mathbb{S}^0_G(\ast) \arrow[r, "\boxed{\text{Boardman homomorphism}}"] \arrow[d, phantom, sloped, "\simeq"] & \mathrm{KO}^0_G(\ast) \arrow[d, phantom, sloped, "\simeq"] \\
 & A\,(G) \arrow[r, "\beta", "\boxed{\text{linearize } G\text{-actions}}"'] & \mathrm{RO}\,(G)
\end{tikzcd}$$

module over
$(G \times G')$-Burnside ring

Thm. 1

G-Burnside ring

Thm. 2

G-Representation ring

Theorem 1

charge group
of M-branes
at two intersecting
MK6 ADE-singularities

visible in
M-theory
(?)

charge lattice
of fractional M-branes
at an MK6-singularity

visible in
M-theory
(?)

Proof.
Use Prop. II 9.13 in
[LewisMaySteinberger86].
□

equivariant stable cohomotopy in RO$(G \times G')$-degree $\mathbb{H}$

$$\begin{tikzcd}
\mathbb{S}^{\mathbb{H}}_{(G\times G')}\left(\mathbb{R}^{10,1}\right) \arrow[r, "\boxed{\text{zoom in onto one MK6}}"] \arrow[d, phantom, sloped, "\simeq"] & \mathbb{S}_G^{\left(\mathbb{H}^{G'}\right)}\left(\mathbb{R}^{6,1}\right) \arrow[d, phantom, sloped, "\simeq"] \\
\mathbb{S}^{\mathbb{H}}_{(G\times G')}(*) \arrow[r, "\alpha", "\boxed{\text{form geometric } G'\text{-fixed spectrum}}"'] & \mathbb{S}^0_G(*) \arrow[d, phantom, sloped, "\simeq"] \\
 & A(G)
\end{tikzcd}$$

module over
$(G \times G')$-Burnside ring

Is surjective.

G-Burnside ring

hence: $\mathrm{coker}(\beta \circ \alpha) \simeq \mathrm{coker}(\beta)$

The comparison homomorphism A

charge group
of M-branes
at two intersecting
MK6 ADE-singularities

visisible in
M-theory
(?)

charge lattice
of fractional M-branes
at an MK6-singularity

visisible in
M-theory
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charge lattice
of fractional D-branes
at orientifold singularity

expected in
perturbative
string theory

equivariant stable cohomotopy in RO($G \times G'$)-degree $\mathbb{H}$

equivariant KO-theory
in degree 0

$$\begin{tikzcd}
\mathbb{S}^{\mathbb{H}}_{(G\times G')}(\mathbb{R}^{10,1}) \arrow[r, "\boxed{\text{zoom in onto one MK6}}"] \arrow[d, phantom, "\vert\wr"] & \mathbb{S}_G^{\left(\mathbb{H}^{G'}\right)}\left(\mathbb{R}^{6,1}\right) \arrow[r, "\boxed{\text{perturbative approximation}}"] \arrow[d, phantom, "\vert\wr"] & \mathrm{KO}^0_G(\mathbb{R}^{6,1}) \arrow[d, phantom, "\vert\wr"] \\
\mathbb{S}^{\mathbb{H}}_{(G\times G')}(\ast) \arrow[r, "\alpha", "\boxed{\text{form geometric } G'\text{-fixed spectrum}}"'] & \mathbb{S}^0_G(\ast) \arrow[r, "\boxed{\text{Bordman homomorphism}}"] \arrow[d, phantom, "\vert\wr"] & \mathrm{KO}^0_G(\ast) \arrow[d, phantom, "\vert\wr"] \\
 & A\,(G) \arrow[r, "\beta", "\boxed{\text{linearize } G\text{-actions}}"'] & \mathrm{RO}\,(G)
\end{tikzcd}$$

module over
($G \times G'$)-Burnside ring

Thm. 1

G-Burnside ring

Thm. 2

G-Representation ring

Theorem 2

$$\overbrace{\mathrm{RO}^{\mathrm{irrational}}(G)}^{\text{irrational characters}} \simeq \mathrm{RO}(G) \;/\; \underbrace{\mathrm{RO}^{\mathrm{int}}(G)}_{\text{integral characters}}$$

charge lattice
of fractional M-branes
at an MK6-singularity

visible in
M-theory
(?)

charge lattice
of fractional D-branes
at orientifold singularity

expected in
perturbative
string theory

equivariant stable cohomotopy
in degree 0

equivariant KO-theory
in degree 0

$$\begin{tikzcd}
\mathbb{S}_G^{\left(\mathbb{H}^{G'}\right)}\left(\mathbb{R}^{6,1}\right) \arrow[r, "\boxed{\text{perturbative approximation}}"] & \mathrm{KO}_G^0(\mathbb{R}^{6,1}) \\
\mathbb{S}_G^0(\ast) \arrow[u, phantom, sloped, "\simeq"] \arrow[r, "\boxed{\text{Bordman homomorphism}}"] & \mathrm{KO}_G^0(\ast) \arrow[u, phantom, sloped, "\simeq"] \\
A\,(G) \arrow[u, phantom, sloped, "\simeq"] \arrow[r, "\beta", "\boxed{\text{linearize } G\text{-actions}}"'] & \mathrm{RO}\,(G) \arrow[u, phantom, sloped, "\simeq"]
\end{tikzcd}$$

G-Burnside ring

$\mathrm{coker}(\beta) \simeq \mathrm{RO}^{\mathrm{irrational}}(G)$

G-Representation ring

Theorem 2 – Ingredients

G-actions
on finite sets
(not-linear)

G-representations
on $\mathbb{F}$-vector spaces
(linear)

$$\begin{tikzcd}
(G\mathrm{Set}_{\mathrm{fin}}, \sqcup) \arrow[r, "\substack{\text{linearize}\\ \mathbb{F}[-]}"] \arrow[d, "K"'] & (G\mathrm{Rep}_{\mathbb{F}}, \oplus) \arrow[d, "K"] \\
A(G) \arrow[r, "\beta"] & R_{\mathbb{F}}(G) \\
\overset{G}{\overset{\curvearrowright}{S}} \arrow[r, mapsto] & \overset{G}{\overset{\curvearrowright}{\mathbb{F}[S]}}
\end{tikzcd}$$

Burnside ring of
virtual G-sets

Representation ring
of virtual representations

G-set

"permutation
representation"

$$\frac{\text{the \textbf{cokernel} of } \beta \quad \text{reflects} \quad \begin{array}{c}\text{linear algebra}\\ \text{invisible to}\\ \text{pure combinatorics}\end{array}}{\text{the \textbf{kernel} of } \beta \quad \text{reflects} \quad \begin{array}{c}\text{pure combinatorics}\\ \text{invisible to}\\ \text{linear algebras}\end{array}}$$

Theorem 2 – Proof

Compute:

1. set of conjugacy classes $\{[H_i]\}$ of subgroups $H \subset G$
2. the Burnside product $[G/H_i] \times [G/H_j] = \coprod_{\ell} \underbrace{n_{ij}^{\ell}}_{\text{structure constants}} \cdot [G/H_\ell]$
3. its matrix of total multiplicities $\mathrm{mult}_{ij} \coloneqq \sum_{\ell} n_{ij}^{\ell}$
4. its integral row reduction $\underbrace{H}_{\text{upper triangular}} \coloneqq \underbrace{U}_{\in \mathrm{GL}(N,\mathbb{Z})} \cdot \mathrm{mult}$

Lemma. *The rows of H span* $\mathrm{im}(\beta) \subset R_{\mathbb{F}}(G)$.

This yields an effective algorithm computing $\mathrm{coker}(\beta) = R_{\mathbb{F}}(G)/\mathrm{im}(\beta)$

Simon Burton has implemented this algorithm in `Python`.

$\Rightarrow$ **Proof of Theorem 2:** By brute force automatized computation. $\square$

Theorem 2 – Proof

	coker	$A(G) \xrightarrow{\beta_{\mathbb{F}}} R_{\mathbb{F}}(G)$			$A(G) \xrightarrow{\beta^{\text{int}}_{\mathbb{F}}} R^{\text{int}}_{\mathbb{F}}(G)$		
		ground field $\mathbb{F}$			ground field $\mathbb{F}$		
		$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{C}$
finite group G	$2D_4$	0	0	$\frac{\mathbb{Z}[\rho_5]}{\mathbb{Z}[2\rho_5]}$	0	0	$\frac{\mathbb{Z}[\rho_5]}{\mathbb{Z}[2\rho_5]}$
	$2D_6$	0	0	$\frac{\mathbb{Z}[\rho_3,\rho_4,\rho_6]}{\mathbb{Z}[\rho_3+\rho_4,2\rho_6]}$	0	0	$\frac{\mathbb{Z}[\rho_6]}{\mathbb{Z}[2\rho_6]}$
	$2D_8$	0	$\frac{\mathbb{Z}[2\rho_6,2\rho_7]}{\mathbb{Z}[2\rho_6+2\rho_7]}$	$\frac{\mathbb{Z}[\rho_6,\rho_7]}{\mathbb{Z}[2\rho_6+2\rho_7]}$	0	0	$\frac{\mathbb{Z}[\rho_6+\rho_7]}{\mathbb{Z}[2\rho_6+2\rho_7]}$
	$2D_{10}$	0	$\frac{\mathbb{Z}[\rho_3,\rho_4,\rho_5,\rho_6,\rho_7,\rho_8]}{\mathbb{Z}[\rho_3+\rho_4,\rho_5+\rho_6,2\rho_7+2\rho_8]}$	$\frac{\mathbb{Z}[\rho_3,\rho_4,\rho_5,\rho_6,2\rho_7,2\rho_8]}{\mathbb{Z}[\rho_3+\rho_4,\rho_5+\rho_6,2\rho_7+2\rho_8]}$	0	0	$\frac{\mathbb{Z}[\rho_7+\rho_8]}{\mathbb{Z}[2\rho_7+2\rho_8]}$
	$2D_{12}$	0	$\frac{\mathbb{Z}[\rho_7,\rho_8,\rho_9]}{\mathbb{Z}[2\rho_7,2\rho_8+2\rho_9]}$	$\frac{\mathbb{Z}[2\rho_8,2\rho_9]}{\mathbb{Z}[2\rho_8+2\rho_9]}$	0	0	$\frac{\mathbb{Z}[\rho_7]}{\mathbb{Z}[2\rho_7]}$
	$2T$	0	0	$\frac{\mathbb{Z}[\rho_2,\rho_2^*,\rho_4,\rho_4^*,\rho_5]}{\mathbb{Z}[\rho_2+\rho_2^*,\rho_4+\rho_4^*,2\rho_5]}$	0	0	$\frac{\mathbb{Z}[\rho_5]}{\mathbb{Z}[2\rho_5]}$
	$2O$	0	$\frac{\mathbb{Z}[2\rho_6,2\rho_7]}{\mathbb{Z}[2\rho_6+2\rho_7]}$	$\frac{\mathbb{Z}[\rho_6,\rho_7,\rho_8]}{\mathbb{Z}[2\rho_6+2\rho_7,2\rho_8]}$	0	0	$\frac{\mathbb{Z}[\rho_8]}{\mathbb{Z}[2\rho_8]}$
	$2I$	0	$\frac{\mathbb{Z}[2\rho_2,2\rho_3,\rho_4,\rho_5]}{\mathbb{Z}[2\rho_2+2\rho_3,\rho_4+\rho_5]}$	$\frac{\mathbb{Z}[\rho_2,\rho_3,\rho_4,\rho_5,\rho_7,\rho_9]}{\mathbb{Z}[2\rho_2+2\rho_3,\rho_4+\rho_5,2\rho_7,2\rho_9]}$	0	0	$\frac{\mathbb{Z}[\rho_2+\rho_3,\rho_4+\rho_5,\rho_7,\rho_9]}{\mathbb{Z}[2\rho_2+2\rho_3,\rho_4+\rho_5,2\rho_7,2\rho_9]}$

Theorem 2 – Physics interpretation

Let $V \in K_G(*) \simeq R(G)$ a fractional D-brane ↔ G-representation,

RR-charge in the g-twisted closed string sector

is the value of its character at g:

$$Q_V^{\mathrm{RR}}(g) \;=\; \frac{1}{|G|}\chi_V(g)$$

([DouglasGreeneMorrisson97, (3.8)], [DiGo00, (2.4)], [BCR00, (4.65) with (4.41)], [EGJ05, (4.5)], [ReSc13, 4.102])

Theorem 2 – Physics reformulation:

Hypothesis **H** *implies*
that fractional D-branes with irrational RR-charge are spurious.

Physically plausible?

Some $V \in K_G(*)$ *must* be spurious [BDHKMMS02, 4.5.2].

Irrational RR-charge called a *paradox* in [BachasDouglasSchweigert00, (2.8)], also [Taylor00, Zho01, Rajan02], apparently unresolved.

If this is indeed a *paradox*,
then hypothesis **H** exactly resolves it.

Theorem 2 – Physics interpretation

Regard coker(β) under **McKay correspondence** :

$$\left\{ \text{ irreps } \rho \in R_{\mathbb{C}}\left(G_{\mathrm{ADE}}\right) \right\} \simeq \left\{ \begin{array}{c} \text{vertices of corresponding} \\ \text{ADE-type Dynkin diagram} \end{array} \right\}$$

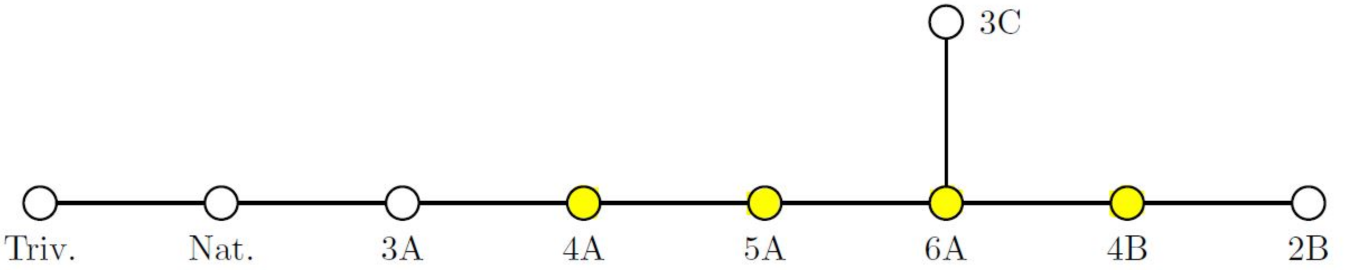
Most exceptional Example: $G = 2I$:

integral/
non-irrational

characters

	e	a	a^2	a^3	a^2b	a^4	a^3b	a^5	a^4b
Triv.	1	1	1	1	1	1	1	1	1
Nat.	2	$\frac{1}{2}(1+\sqrt{5})$	$-\frac{1}{2}(1-\sqrt{5})$	$\frac{1}{2}(1-\sqrt{5})$	1	$-\frac{1}{2}(1+\sqrt{5})$	0	-2	-1
3A	3	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1-\sqrt{5})$	$\frac{1}{2}(1-\sqrt{5})$	0	$\frac{1}{2}(1+\sqrt{5})$	-1	3	0
4A	4	1	-1	1	-1	-1	0	-4	1
5A	5	0	0	0	-1	0	1	5	-1
6A	6	-1	1	-1	0	1	0	-6	0
2B	2	$\frac{1}{2}(1-\sqrt{5})$	$-\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1+\sqrt{5})$	1	$-\frac{1}{2}(1-\sqrt{5})$	0	-2	-1
4B	4	-1	-1	-1	1	-1	0	4	1
3C	3	$\frac{1}{2}(1-\sqrt{5})$	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1+\sqrt{5})$	0	$\frac{1}{2}(1-\sqrt{5})$	-1	3	0

Dynkin
diagram



hence:

$$\mathrm{im}(\beta)_{|\mathrm{irred}} \subset \mathrm{RO}(2I)_{|\mathrm{irred}} \quad \leftrightarrow \quad \underbrace{SU(5)}_{\substack{\text{actual} \\ \text{GUT group}}} \quad \subset \quad \underbrace{E_8}_{\substack{\text{stringy} \\ \text{GUT group}}}$$

Comparison B to

ordinary Cohomology
and
"discrete torsion" of fractional M5-branes

back to Part II

The comparison homomorphism B

Away from the singular locus
of a black M2-brane

$$\begin{tikzcd} {} \arrow[d, squiggly] & & \cdots \arrow[dl, "\ell_P \gg 1"'] \arrow[dr, "\ell_P \ll 1"] & \\ \mathrm{AdS}_4 \times \underbrace{S^7/G_{\mathrm{A}}}_{\text{spherical space form}} & {} & & \mathbb{R}^{2,1} \times \underbrace{\mathbb{R}^8 /\!\!/ G_{\mathrm{A}}}_{\text{orbifold du Val singularity}} \end{tikzcd}$$

the orbifold is smooth and, for A-type singularities, so is the RO-degree:

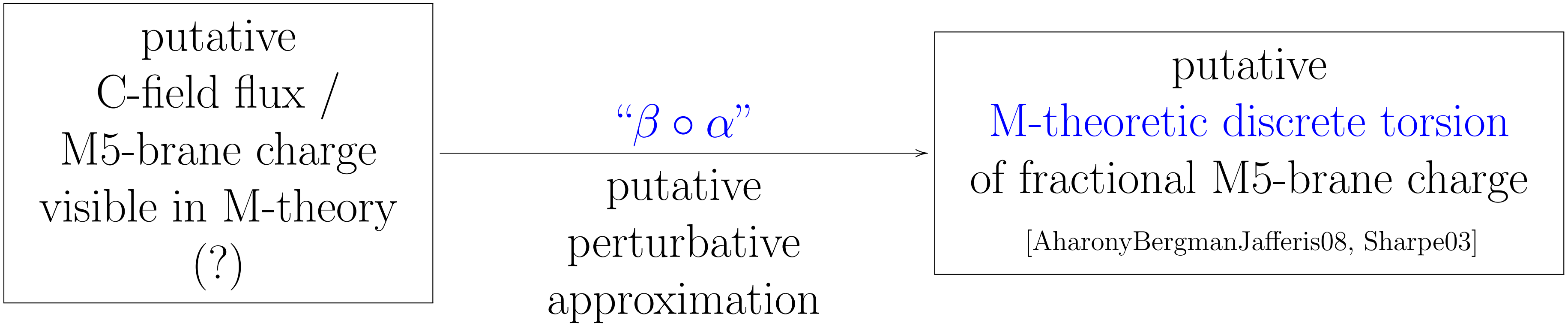
$$\begin{tikzcd} \substack{\text{coefficient}\\ \text{bundle}} & \big(\big(\mathbb{R}^{10,1} - \mathbb{R}^{2,1} \big) \times \mathbb{H} \big) /\!\!/ G \arrow[d] & \simeq & \big(\mathbb{R}^{10,1} - \mathbb{R}^{2,1} \big) / G \times \mathbb{R}^4 \arrow[d] \\ \substack{\text{spacetime}\\ \text{orbifold}} & \big(\mathbb{R}^{10,1} - \mathbb{R}^{2,1} \big) /\!\!/ G_A & & \big(\mathbb{R}^{10,1} - \mathbb{R}^{2,1} \big) / G_A \end{tikzcd}$$

⇒ relevant comparison morphism is equivariant Boardman homomorphism

$$\underbrace{\mathbb{S}^4_{G_A}\big(S^7\big)}_{\substack{\text{equivariant}\\ \text{stable Cohomotopy}}} \xrightarrow{\beta_{G_A}} \underbrace{H\mathbb{Z}^4_{G_A}\big(S^7\big)}_{\substack{\text{equivariant}\\ \text{ordinary cohomology}}} \simeq \underbrace{H^4\big(S^7/G_A, \mathbb{Z}\big)}_{\substack{\text{Borel}\\ \text{equivariance}}}$$

The comparison homomorphism B

Theorem 3 i): factors through plain Boardman homomorphism:



putative
C-field flux /
M5-brane charge
visible in M-theory
(?)

“$\beta \circ \alpha$”

putative
perturbative
approximation

putative
M-theoretic discrete torsion
of fractional M5-brane charge
[AharonyBergmanJafferis08, Sharpe03]

equivariant Boardmamn homomorphism

$$\begin{tikzcd}
\mathbb{S}^4_{G_A}\left(S^7\right) \arrow[rr, "\beta_{G_A}"] \arrow[dr, "\alpha"'] & & H^4_{G_A}\left(S^7/G_{\mathrm{A}}, \mathbb{Z}\right) \\
& \mathbb{S}^4\left(S^7/G_{\mathrm{A}}\right) \arrow[ur, "\beta"'] &
\end{tikzcd}$$

Carlsson-Segal completion

plain Boardman homomorphism

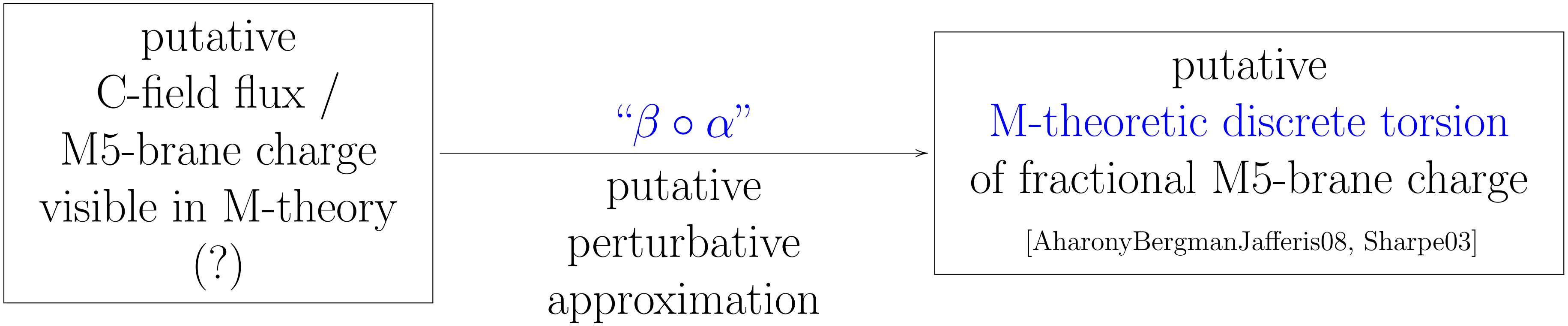
Proof. Use [Schwede18, Example 4.5.19]. □.

Theorem 3 ii): $4\,\mathrm{coker}(\beta) = 0$

Proof. By [Arlettaz04, Theorem 1.2 b)]. □

The comparison homomorphism B

Theorem 3 iii): factors isomorphically through ko^4:



putative
C-field flux /
M5-brane charge
visible in M-theory
(?)

“$\beta \circ \alpha$”

putative
perturbative
approximation

putative
M-theoretic discrete torsion
of fractional M5-brane charge
[AharonyBergmanJafferis08, Sharpe03]

equivariant Boardmamn homomorphism

$$\begin{tikzcd}
\mathbb{S}^4_{G_A}\left(S^7\right) \arrow[rr, "\beta_{G_A}"] \arrow[dr, "\alpha"'] & & H^4\left(S^7/G_{\mathrm{A}}, \mathbb{Z}\right) \\
& \mathbb{S}^4\left(S^7/G_{\mathrm{A}}\right) \arrow[ur, "\beta"] \arrow[r, "\simeq"] & \mathrm{ko}^4(S^7/G_A) \arrow[u, "p_1"']
\end{tikzcd}$$

Carlsson-Segal completion

Boardman homomorphism
for ko

first
Pontryagin class

Proof. By the AHSS and using $\pi_{\bullet\leq 2}(\mathbb{S}) = \pi_{\bullet \leq 2}(\mathrm{ko})$ $\square$.

Physically reasonable?

This coker(β) is KO-version of what was argued for KU in [DiaconescuMooreWitten00].

Conclusion

1. Part I – Motivation of differential equivariant Cohomotopy:
   (a) Derivation of equivariant cohomotopy$_{/\mathbb{Q}}$ from first principles via rational super homotopy theory [FSS13, FSS16a, FSS16b, BSS18, HSS18]
   (b) actual Cohomotopy is the minimal non-rational lift – differential equivariant Cohomotopy of super-orbifolds exists in global equivariant super homotopy theory
   (c) Hypothesis **H**:
   the observables of M-theory are the differential equivariant real cohomology of the moduli stack of supertorsion-free differential equivariant Cohomotopy of spacetime $\mathrm{Pin}(5)^\flat$-orbi $\mathbb{R}^{10,1|\mathbf{32}}$-folds
2. Part II – Consistency checks of Hypothesis **H**:
   (a) reproduces fractional D-brane charge in equivariant K-theory
      i. excluding exactly the spurious irrational RR-charges,
      ii. which may correspond, via McKay, to breaking E_8 to SU(5) GUT
   (b) reproduces discrete torsion of fractional M5-branes with DMW-correction.

In particular, equivariant stable cohomotopy somehow unifies ordinary cohomology of the C-field with K-theory of D-branes.

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