Twisted, Equivariant, Differential (TED) K-Theory via Diffeological Stacks

Urs Schreiber on joint work with Hisham Sati

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& Center for Quantum and Topological Systems

New York University, Abu Dhabi

Center for Quantum & Topological Systems

talk via:

Global Diffeology Seminar @ DIFFEOLOGY.NET 04 Nov 2022

slides and pointers at: `ncatlab.org/schreiber/show/TED+cohomology`

Motivation:

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start simple: understand quantum physics

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Tool: Mathematical foundations.

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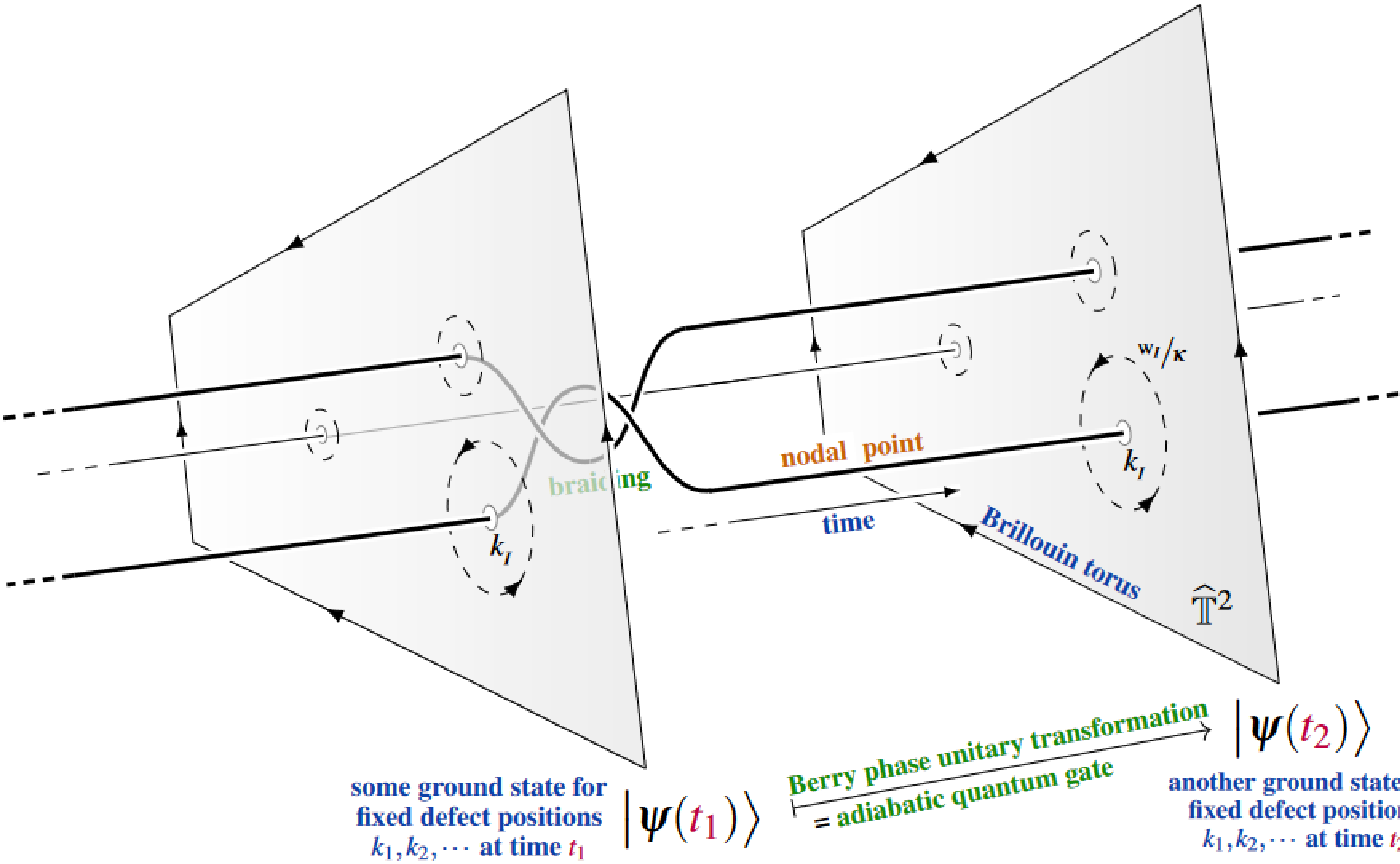

Figure 1 – Adiabatic braid quantum gate. Schematically indicated is the unitary transformation induced on the topologically ordered ground state (as discussed below in §3.3) of an effectively 2-dimensional topological semi-metal (as in §3.1) under adiabatic braiding (Rem. 1.1) of nodal points in the Brillouin torus (Rem. 3.9).

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$$\begin{array}{c}\textbf{TED}\text{ K-cohomology}\\ \text{of configuration spaces}\end{array} \xleftrightarrow[\text{[Sati.-S. (2022)]}]{\text{Hypothesis H}} \begin{array}{c}\text{anyonic}\\ \text{defect branes}\end{array}$$

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[Sati.-S. (2022)]

TED K-cohomology of configuration spaces

AdS/CMT @ $N\sim 1$ (M-theor. holography)

assorted lore from quantum field theory & solid state physics

[Sati-S., RMP (2022)]

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realistic topological quantum logic gates

TED Gauss-Manin connection

[Sati-S., PlanQC (2022)]

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0 – Cohesive ∞-Topoi

I – Equivariant ∞-Bundles

II – TED-K-Theory

III – Anyonic Defect Branes

IV – Quantum Computation

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This part is
a lightning indication of
how diffeological spaces fit
into the scheme of things,

following:

Equivariant Principal ∞-bundles [arXiv:2112.13654]
Proper Orbifold Cohomology [arXiv:2008.01101]
Diff. Cohomology in a Cohesive ∞-Topos [arXiv:1310.7930]

Diffeological Spaces in Perspective.

Souriau (1979), Iglesias-Zemmour (1985) – paraphrased:

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\mathbb{R}^{n'} & \mapsto & \mathrm{Plots}\left(\mathbb{R}^{n'}, X\right) \arrow[u, "f^*"', "\text{precompos.}"] & \subset & \mathrm{Hom}_{\mathrm{Set}}\left(\mathbb{R}^{n'}, X\right) \arrow[u, "\mathrm{Hom}_{\mathrm{Set}}\left(f, X\right)"]
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convenience

category of
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$$\mathrm{TopSpc} \xrightarrow{\;\mathrm{Pth}\;} \mathrm{Grpd}_\infty$$

form path ∞-groupoids
(sing. simpl. compl.)

base ∞-topos of
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convenience

category of **topological spaces**

cartesian closed category of **compactly generated topological spaces**

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\begin{tikzcd}
\mathrm{TopSpc} \arrow[r, shift right, "k"'] \arrow[r, phantom, "\perp"] \arrow[drr, "\mathrm{Pth}"'] & \mathrm{kTopSpc} \arrow[l, hook', shift right] \arrow[dr, "\mathrm{Pth}"] & \\
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Hurewicz (1948)

Gale (1950)

Steenrod (1967)

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cart. clsd. & locl. pres. category of **Δ-generated topological spaces**

$$\begin{tikzcd}
\mathrm{TopSpc} \arrow[r, shift right=2, "k"'] \arrow[r, phantom, "\perp"] \arrow[drrr, rounded corners] & \mathrm{kTopSpc} \arrow[l, hook', shift right=2] \arrow[r, shift right=2, "\mathrm{Cdfflg}"'] \arrow[r, phantom, "\perp"] \arrow[drr, rounded corners] & \mathrm{DTopSpc} \arrow[l, hook', shift right=2] \arrow[dr, rounded corners, "\mathrm{Pth}"] & \\
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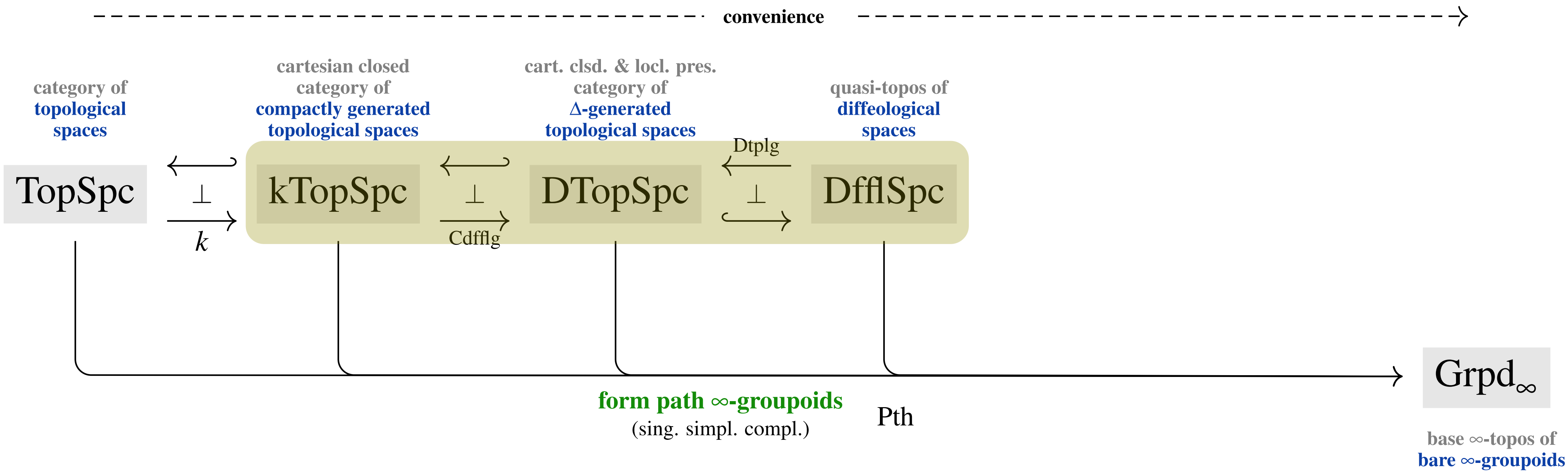

Shimakawa, Yoshida & Haraguchi (2010)

Christensen, Sinnamon & Wu (2014)

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cohesive topos of **smooth sets**

$$
\begin{tikzcd}[column sep=large]
\mathrm{TopSpc} \arrow[r, shift right=2, "k"'] \arrow[drrrrr, bend right=10] & \mathrm{kTopSpc} \arrow[l, hook', shift right=2] \arrow[l, phantom, "\perp"] \arrow[r, shift right=2, "\mathrm{Cdfflg}"'] \arrow[drrrr, bend right=10] & \mathrm{DTopSpc} \arrow[l, hook', shift right=2] \arrow[l, phantom, "\perp"] \arrow[r, hook, shift right=2] \arrow[drrr, bend right=10] & \mathrm{DfflSpc} \arrow[l, shift right=2, "\mathrm{Dtplg}"'] \arrow[l, phantom, "\perp"] \arrow[r, hook, "i_{\sharp 1}"'] \arrow[drr, bend right=10, "\mathrm{Pth}"'] & \mathrm{SmthGrpd}_0 & \\
 & & & & & \mathrm{Grpd}_\infty
\end{tikzcd}
$$

form path ∞-groupoids
(sing. simpl. compl.)

base ∞-topos of **bare ∞-groupoids**

Diffeological Spaces in Perspective.

convenience $\dashrightarrow$

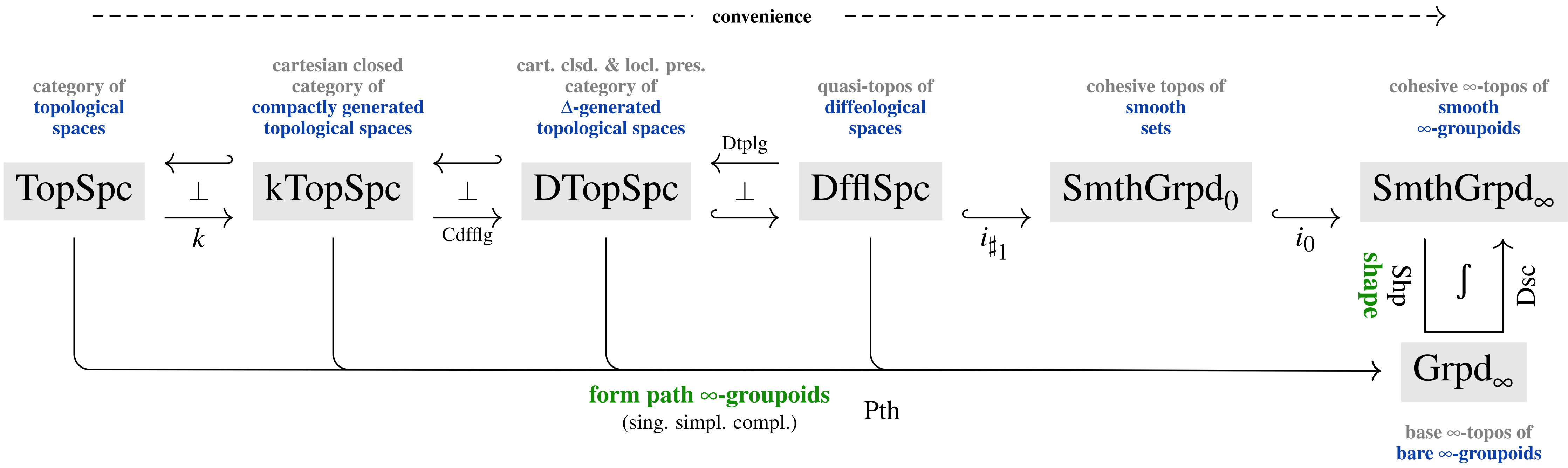
category of **topological spaces**	cartesian closed category of **compactly generated topological spaces**	cart. clsd. & locl. pres. category of **Δ-generated topological spaces**	quasi-topos of **diffeological spaces**	cohesive topos of **smooth sets**	cohesive ∞-topos of **smooth ∞-groupoids**

$$
\begin{tikzcd}
\mathrm{TopSpc}
  \arrow[r, shift right=1.2ex, "k"']
  \arrow[r, phantom, "\perp"]
  \arrow[drrrrr, rounded corners]
&
\mathrm{kTopSpc}
  \arrow[l, hook', shift right=1.2ex]
  \arrow[r, shift right=1.2ex, "\mathrm{Cdfflg}"']
  \arrow[r, phantom, "\perp"]
  \arrow[drrrr, rounded corners]
&
\mathrm{DTopSpc}
  \arrow[l, hook', shift right=1.2ex]
  \arrow[r, hook, shift right=1.2ex]
  \arrow[r, phantom, "\perp"]
  \arrow[drrr, rounded corners]
&
\mathrm{DfflSpc}
  \arrow[l, shift right=1.2ex, "\mathrm{Dtplg}"']
  \arrow[r, hook, "i_{\sharp_1}"']
  \arrow[drr, rounded corners, "\mathrm{Pth}"']
&
\mathrm{SmthGrpd}_0
  \arrow[r, hook, "i_0"']
&
\mathrm{SmthGrpd}_\infty
\\
& & & & & \mathrm{Grpd}_\infty
\end{tikzcd}
$$

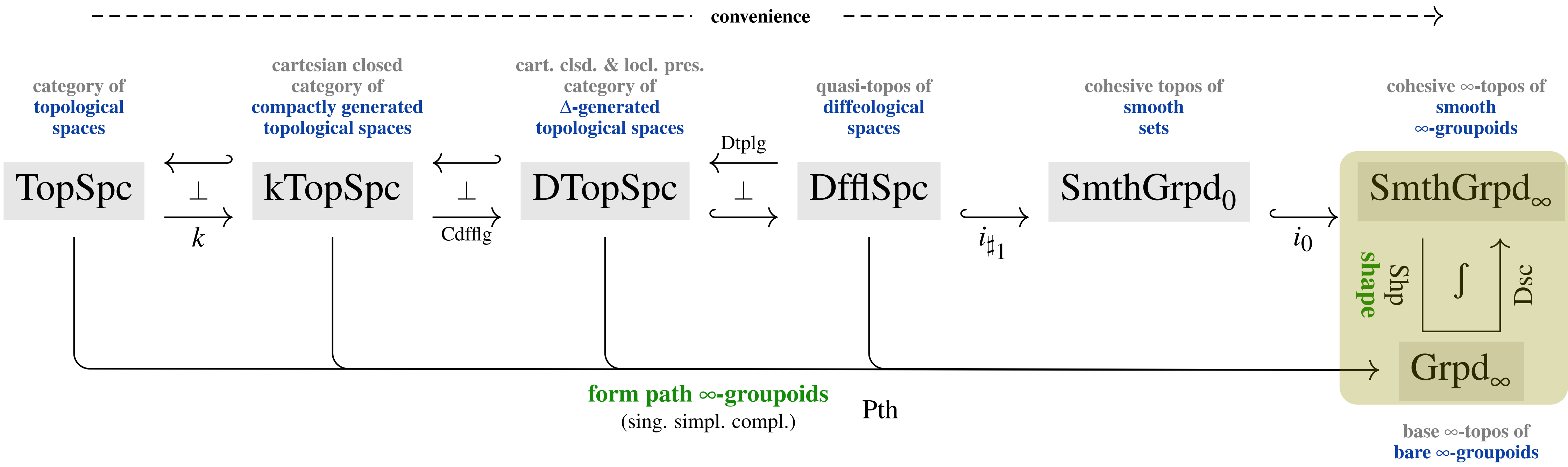
form path ∞-groupoids
(sing. simpl. compl.)

base ∞-topos of
bare ∞-groupoids

Diffeological Spaces in Perspective.



Diffeological Spaces in Perspective.



Dugger (1998)

Cisinski (2003)

Schreiber (2013) – cohesive terminology borrowed from Lawvere (1994)

Pavlov (2014)

Pavlov et al. (2019)

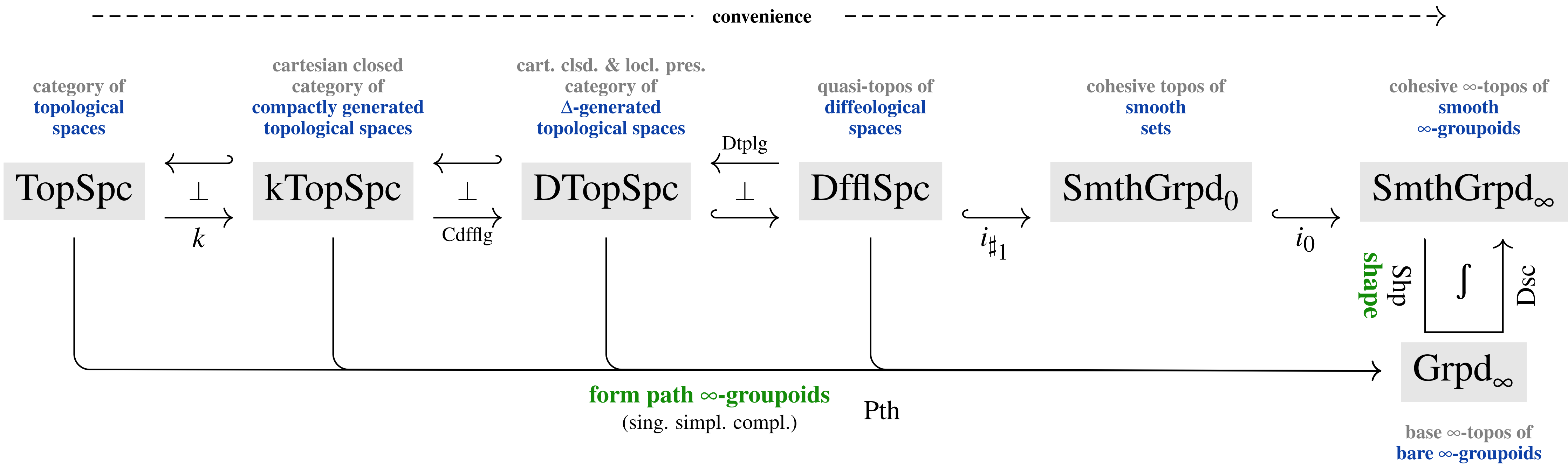
Bunk (2020)

Sati & Schreiber (2020)

Sati & Schreiber (2021)

Clough (2021)

Diffeological Spaces in Perspective.



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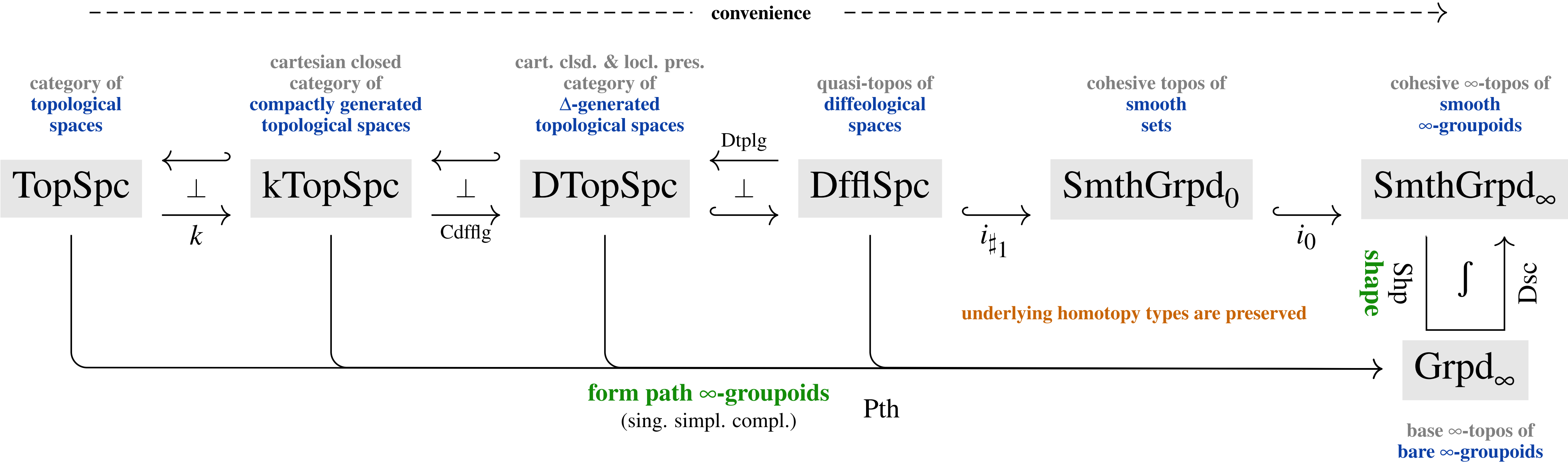
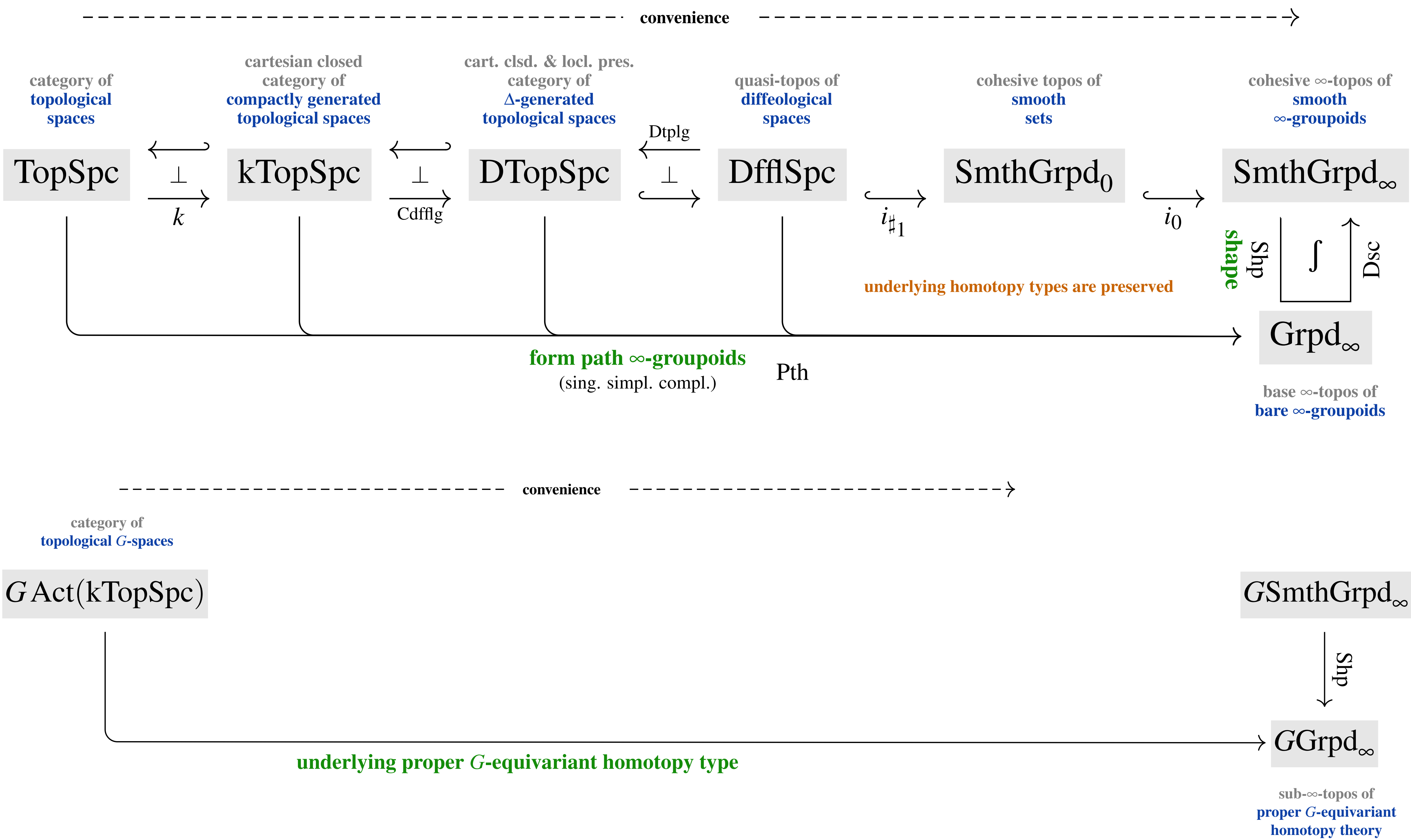
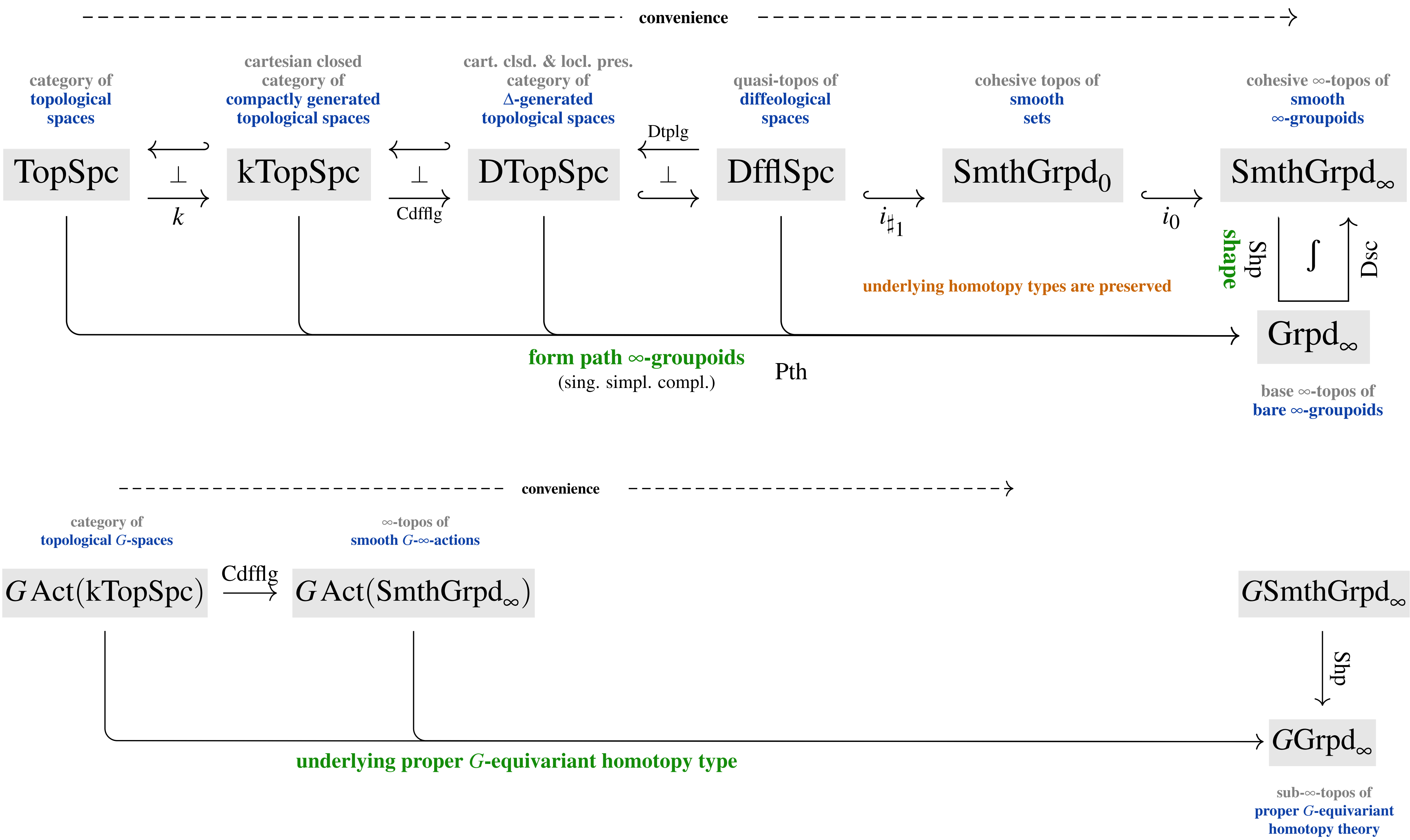


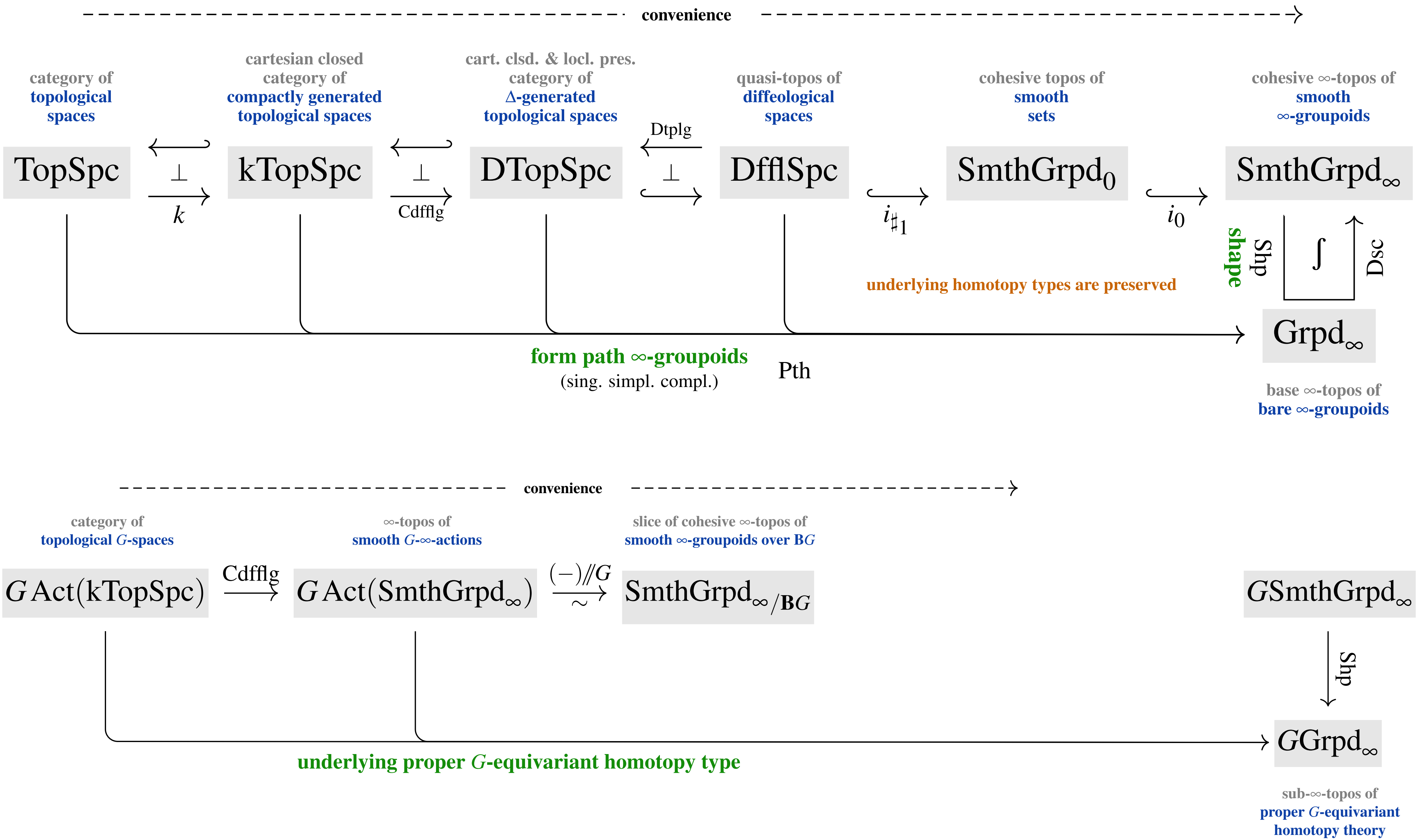
Diffeological Spaces in Perspective.



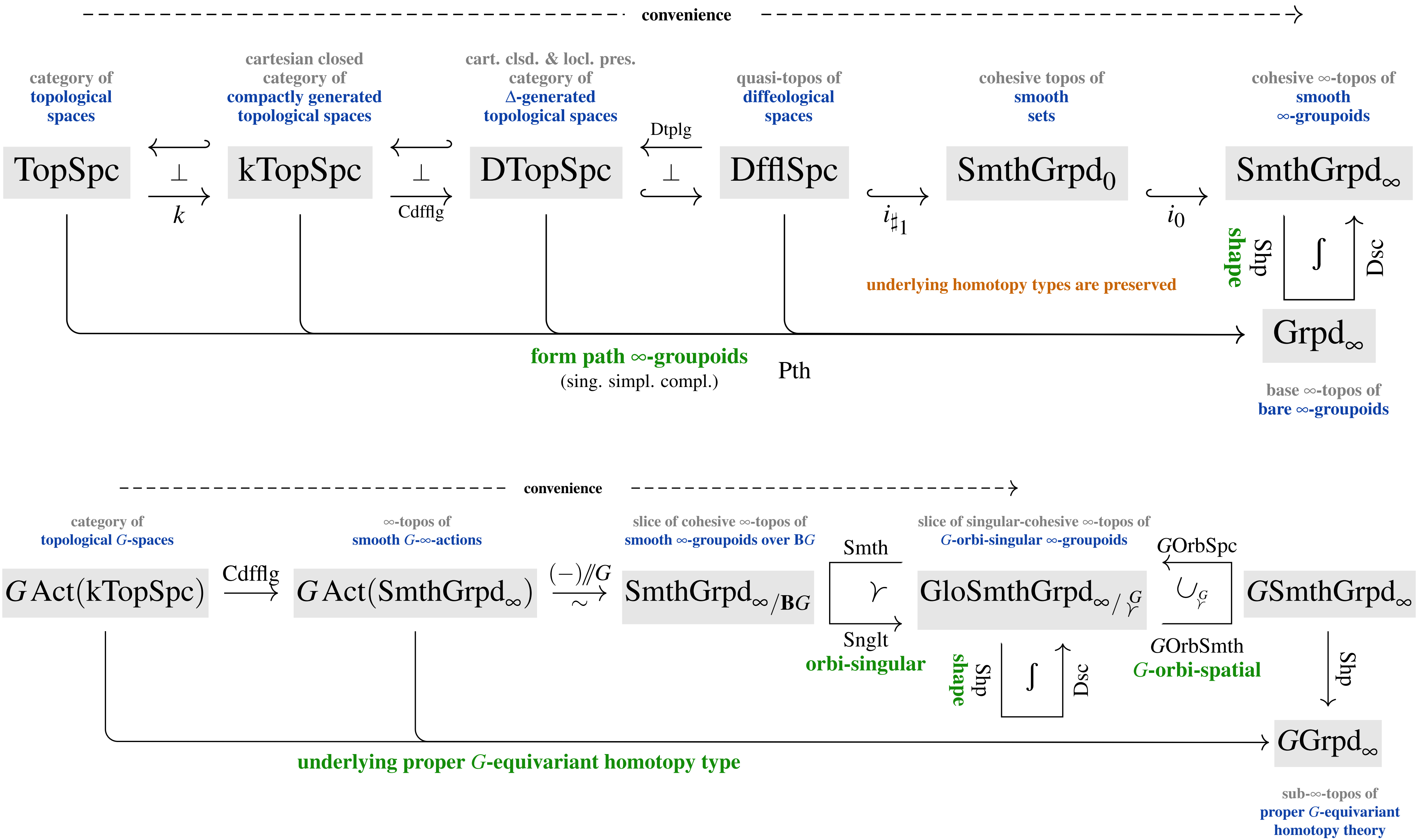
Diffeological Spaces in Perspective.



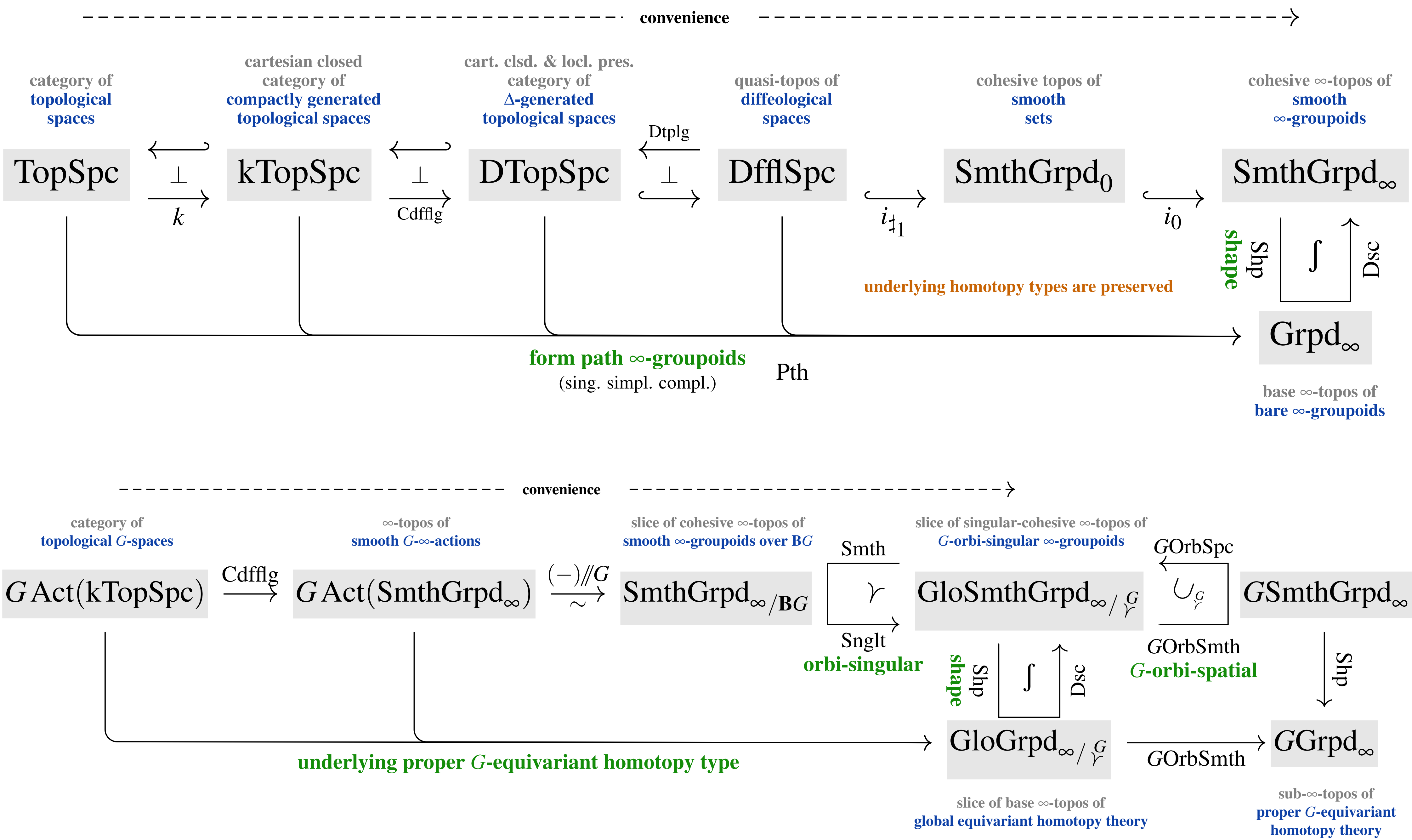
Diffeological Spaces in Perspective.



Diffeological Spaces in Perspective.



Diffeological Spaces in Perspective.



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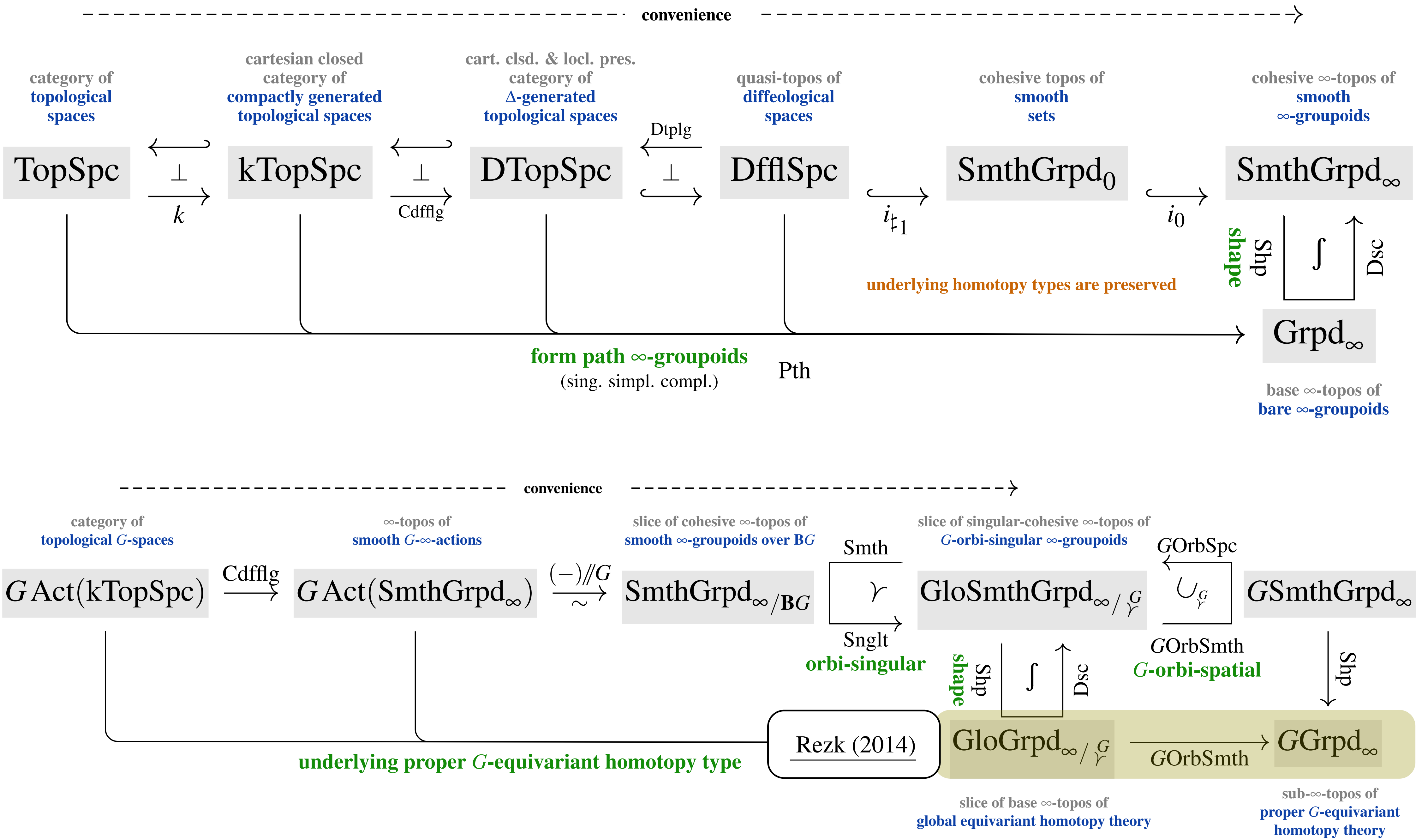

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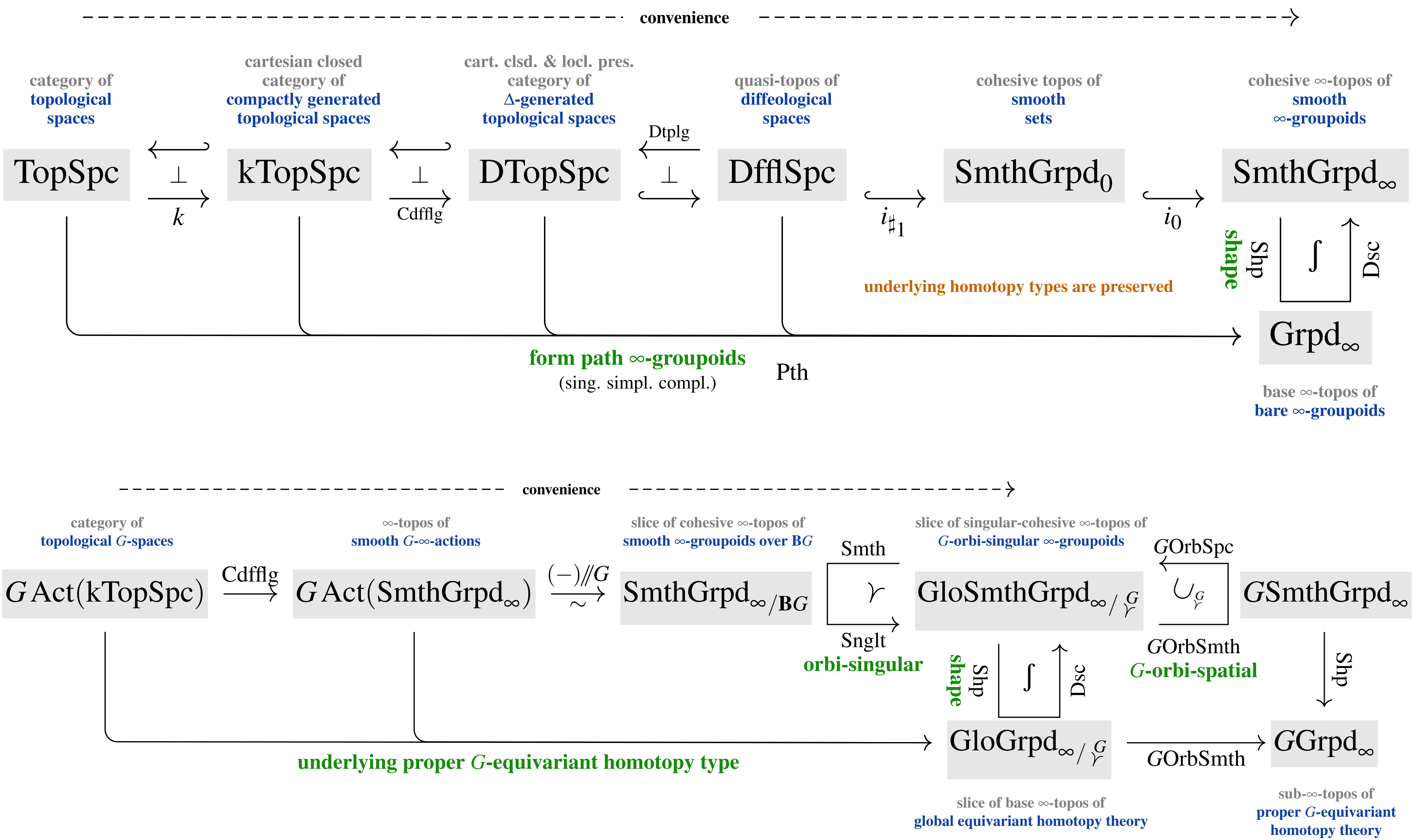

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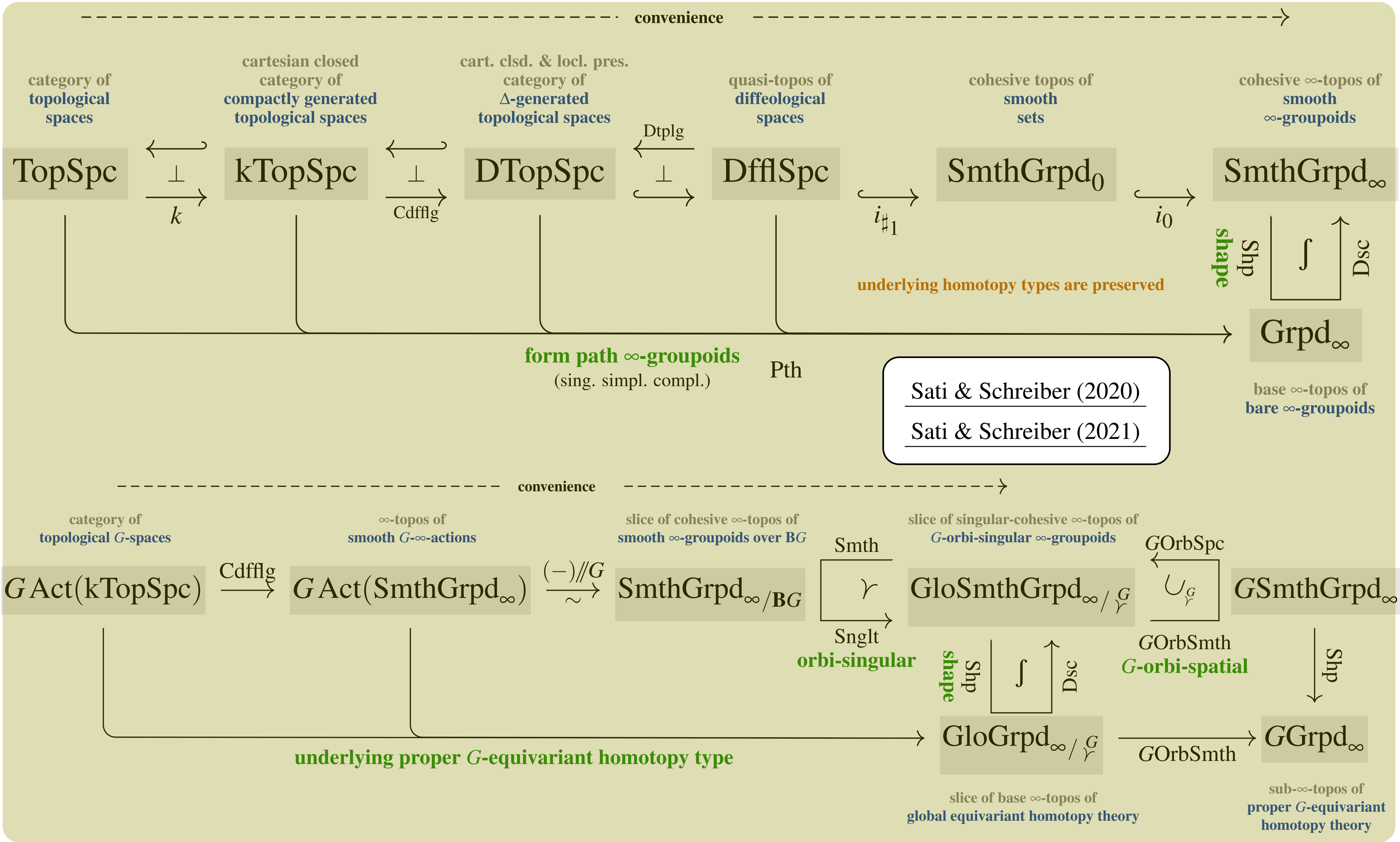

0 – Cohesive ∞-Topoi

I – Equivariant ∞-Bundles

II – TED-K-Theory

III – Anyonic Defect Branes

IV – Quantum Computation

This part is
a gentle exposition of
the most basic concept
underlying these articles:

Principal ∞-bundles [arXiv:1207.0248/49]
Equivariant Principal ∞-bundles [arXiv:2112.13654]
Proper Orbifold Cohomology [arXiv:2008.01101]

following

Diff. Cohomology in a Cohesive ∞-Topos [arXiv:1310.7930]

Motivation, Overview, Summary and Outlook – in one single slide:

Generalized Cohomology Theories $\longleftrightarrow$ **Cohesive Higher Fiber Bundles**

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Here "generalized" subsumes "Whitehead-generalized cohomology" ($\leftrightarrow$ spectra) but goes further:

Cohomology	$\leftrightarrow$	**Higher Bundles**
non-abelian	$\leftrightarrow$	general fibers
twisted	$\leftrightarrow$	associated
differential	$\leftrightarrow$	cohesive
G-equivariant	$\leftrightarrow$	sliced over $\mathbf{B}G$

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A major phenomenon/subtlety is that the last two aspects go hand-in-hand:

Proper G-equivariance corresponds to the **cohesive slice** over $\mathbf{B}G$,

while

Borel equivariance corresponds just to the **slice of shapes**.

2-Groupoids

2-Groupoids are the algebra of 2-dimensional pasting,
such that all composition is associative and invertible:

In general we need n-groupoids for $n \in \{1, 2, 3, \cdots, \infty\}$
but for sake of exposition we may focus on $n = 2$.

2-Groupoids

2-Groupoids are the algebra of 2-dimensional pasting,
such that all composition is associative and invertible:

Accurate intuition:
homotopy classes of surfaces Σ relative boundary paths γ
in a topological space:

$$\begin{tikzcd}
x_1 \arrow[r, "\gamma"] \arrow[d] \arrow[dr, phantom, "\Swarrow\Sigma"] & y_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & z_1 \arrow[d] \\
x_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & y_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & z_2 \arrow[d] \\
x_3 \arrow[r] & y_3 \arrow[r] & z_3
\end{tikzcd}$$

2-Groupoids – Examples.

For G a discrete group, there is its *delooping 1-groupoid* – $\boxed{\mathbf{B}G}$:

$$\begin{tikzcd} & \bullet \arrow[dr, "g_2"] \arrow[d, phantom, "\Vert"] & \\ \bullet \arrow[ur, "g_1"] \arrow[rr, "g_1 \cdot g_2"'] & {} & \bullet \end{tikzcd} \qquad g_i \in G$$

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$$
\begin{tikzcd}
 & \bullet \arrow[dr, "g_2"] & \\
\bullet \arrow[ur, "g_1"] \arrow[rr, "g_1 \cdot g_2"'] \arrow[rr, phantom, shift left=4, "\Vert"] & & \bullet
\end{tikzcd}
\qquad g_i \in G
$$

For $X \circlearrowleft G$ a G-action on a set X there is
its *action groupoid* or *homotopy quotient* – $\boxed{X /\!\!/ S}$:

$$
\begin{tikzcd}
 & x \cdot g_1 \arrow[dr, "g_2"] & \\
x \arrow[ur, "g_1"] \arrow[rr, "g_1 \cdot g_2"'] \arrow[rr, phantom, shift left=4, "\Vert"] & & x \cdot g_1 \cdot g_2
\end{tikzcd}
\qquad
\begin{array}{c} g_i \in G \\ x \in X \end{array}
$$

Hence: $\mathbf{B}G \simeq * /\!\!/ G$.

2-Groupoids – Examples.

For A an *abelian* group there is the *double delooping 2-groupoid*

$$\boxed{\mathbf{B}^2 A = \mathbf{B}\big(\overbrace{\mathbf{B}A}^{\text{“2-group”}}\big)}$$

$$
\begin{tikzcd}
\bullet \arrow[r, "\bullet"] \arrow[d, "\bullet"'] \arrow[dr, phantom, "\Downarrow a_1" near start] & \bullet \arrow[d, "\bullet"] \\
\bullet \arrow[r, "\bullet"'] \arrow[ur, "\bullet" description] \arrow[ur, phantom, "\Downarrow a_2" near end] & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, "\bullet"] \arrow[d, "\bullet"'] \arrow[dr, phantom, "\Downarrow a_1+a_2"] & \bullet \arrow[d, "\bullet"] \\
\bullet \arrow[r, "\bullet"'] & \bullet
\end{tikzcd}
\qquad a_i \in A
$$

2-Groupoids – Examples.

For $A \curvearrowleft G$ a *linear* action, i.e. by group automorphisms,

there is the delooping 2-groupoid $\mathbf{B}\big(\underbrace{(\mathbf{B}A) \rtimes G}\big) \;\simeq\; \big(\mathbf{B}^2 A\big) /\!\!/ G$

of the *semidirect product 2-group*:

$$
\begin{tikzcd}[column sep=huge]
\bullet \arrow[r, bend left=70, "g"{name=U}] \arrow[r, "g"{description, name=M}] \arrow[r, bend right=70, "g"'{name=D}] & \bullet
\arrow[Rightarrow, from=U, to=M, "a_1"] \arrow[Rightarrow, from=M, to=D, "a_2"]
\end{tikzcd}
\;=\;
\begin{tikzcd}[column sep=huge]
\bullet \arrow[r, bend left=70, "g"{name=U}] \arrow[r, bend right=70, "g"'{name=D}] & \bullet
\arrow[Rightarrow, from=U, to=D, "a_1 + a_2"]
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=huge]
\bullet \arrow[r, bend left=50, "{\color{purple}g_1}"{name=U1}] \arrow[r, bend right=50, "g_1"'{name=D1}] & \bullet \arrow[r, bend left=50, "g_2"{name=U2}] \arrow[r, bend right=50, "g_2"'{name=D2}] & \bullet
\arrow[Rightarrow, from=U1, to=D1, "a_1"] \arrow[Rightarrow, from=U2, to=D2, "a_2"]
\end{tikzcd}
\;=\;
\begin{tikzcd}[column sep=huge]
\bullet \arrow[r, bend left=70, "g_1 \cdot g_2"{name=U}] \arrow[r, bend right=70, "g_1 \cdot g_2"'{name=D}] & \bullet
\arrow[Rightarrow, from=U, to=D, "a_1 + {\color{purple}g_1}(a_2)"]
\end{tikzcd}
$$

2-Groupoids – Examples.

This is a special case of the delooping of the *automorphism 2-group* of a group Γ:

$$\mathbf{B}\Big(\mathrm{Aut}\big(\mathbf{B}\Gamma\big)\Big) \;=\; \mathbf{B}\big(\overbrace{\mathrm{Aut}(\Gamma)/\!\!/\Gamma}\big)$$

$$\begin{tikzcd}[column sep=huge]
(\mathbf{B}\Gamma) & & (\mathbf{B}\Gamma) \arrow[ll, bend right, "\alpha"'] \arrow[ll, bend left, "\mathrm{Ad}_\gamma\circ\alpha"] \arrow[ll, phantom, "\Downarrow\gamma"]
\end{tikzcd}$$

$\alpha \in \mathrm{Aut}(\Gamma)$

$\gamma \in \Gamma$

$\Gamma \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\Gamma)$

$$\begin{tikzcd}[column sep=large]
\bullet & & \bullet \arrow[ll, bend right, "\alpha_1"'] \arrow[ll, bend left, "\mathrm{Ad}_\gamma\circ\alpha_1"] \arrow[ll, phantom, "\Downarrow\gamma_1"] & & \bullet \arrow[ll, bend right, "\alpha_2"'] \arrow[ll, bend left, "\mathrm{Ad}_{\gamma_2}\circ\alpha_2"] \arrow[ll, phantom, "\Downarrow\gamma_2"] & = & \bullet & & \bullet \arrow[ll, bend right, "\alpha_1\circ\alpha_2"'] \arrow[ll, bend left, "\mathrm{Ad}_{\gamma_1}\circ\alpha_1\circ\mathrm{Ad}_{\gamma_2}\circ\alpha_2"] \arrow[ll, phantom, "\Downarrow\gamma_1\cdot\alpha_1(\gamma_2)"]
\end{tikzcd}$$

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$$\begin{tikzcd}[column sep=huge]
(\mathbf{B}\Gamma) & (\mathbf{B}\Gamma) \arrow[l, bend right=40, "\alpha"'] \arrow[l, bend left=40, "\mathrm{Ad}_\gamma \circ \alpha"] \arrow[l, phantom, "\Downarrow \gamma"]
\end{tikzcd}
\qquad
\begin{aligned}
&\alpha \in \mathrm{Aut}(\Gamma) \\
&\gamma \in \Gamma \\
&\Gamma \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\Gamma)
\end{aligned}$$

$$\begin{tikzcd}[column sep=huge]
\bullet & \bullet \arrow[l, bend right=50, "\color{purple}\alpha_1"'] \arrow[l, bend left=50, "\mathrm{Ad}_\gamma \circ \alpha_1"] \arrow[l, phantom, "\Downarrow \gamma_1"] & \bullet \arrow[l, bend right=50, "\alpha_2"'] \arrow[l, bend left=50, "\mathrm{Ad}_{\gamma_2} \circ \alpha_2"] \arrow[l, phantom, "\Downarrow \gamma_2"]
\end{tikzcd}
\;=\;
\begin{tikzcd}[column sep=huge]
\bullet & \bullet \arrow[l, bend right=50, "\alpha_1 \circ \alpha_2"'] \arrow[l, bend left=50, "\mathrm{Ad}_{\gamma_1} \circ \alpha_1 \circ \mathrm{Ad}_{\gamma_2} \circ \alpha_2"] \arrow[l, phantom, "\Vert\;\; \gamma_1 \cdot {\color{purple}\alpha_1}(\gamma_2) \;\;\Downarrow"]
\end{tikzcd}$$

NB: Always need to choose whether actions are right- or left-actions,
hence whether group multiplication is opposite or aligned to arrow composition.
Before long we want *structure groups* to act *from the left*
and *equivariance groups* to act *from the right*.

2-Groupoids – Examples.

Notice:

(1) $(\mathbf{B}A) \rtimes G$ is a non-abelian 2-group iff G is a non-abelian group;

2-Groupoids – Examples.

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(1) $(\mathbf{B}A) \rtimes G$ is a non-abelian 2-group iff G is a non-abelian group;

(2) its delooping sits in this fiber sequence:

$$
\begin{tikzcd}
 & \mathbf{B}\big((\mathbf{B}A) \rtimes G\big) \arrow[d, phantom, "\parallel"] & \\
\mathbf{B}^2 A \arrow[r, "\mathrm{fib}(p)"] & (\mathbf{B}^2 A)/\!\!/G \arrow[r, "p", "\in \mathrm{KanFib}"'] & \mathbf{B}G
\end{tikzcd}
$$

$$
\begin{tikzcd}
\bullet \arrow[r, bend left=50, "\bullet"{name=U}] \arrow[r, bend right=50, "\bullet"'{name=D}] \arrow[Rightarrow, from=U, to=D, "a"] & \bullet
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
\bullet \arrow[r, bend left=50, "\mathrm{e}"{name=U}] \arrow[r, bend right=50, "\mathrm{e}"'{name=D}] \arrow[Rightarrow, from=U, to=D, "a"] & \bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
\bullet \arrow[r, bend left=50, "g"{name=U}] \arrow[r, bend right=50, "g"'{name=D}] \arrow[Rightarrow, from=U, to=D, "a"] & \bullet
\end{tikzcd}
\quad \longmapsto \quad
\bullet \xrightarrow{\;g\;} \bullet
$$

2-Groupoids – 2-Functors.

A *2-functor* is a map between 2-groupoids respecting identities and composition.

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E.g.: if $\mathbb{Z} \curvearrowleft \mathbb{Z}_2$ by sign inversion, and $G \xrightarrow{\sigma} \mathbb{Z}_2$ a homomorphism then
2nd group cohomology of G with coefficients in $G \underset{\sigma}{\curvearrowright} \mathbb{Z}$ is 2-functors:

$$
\begin{tikzcd}
\mathbf{B}G \arrow[rr, dashed, "\text{2-functor = 2-cocycle}"] \arrow[dr, "\mathbf{B}\sigma"'] & & (\mathbf{B}^2 A) /\!\!/ \mathbb{Z}_2 \arrow[dl] \\
& \mathbf{B}\mathbb{Z}_2 &
\end{tikzcd}
$$

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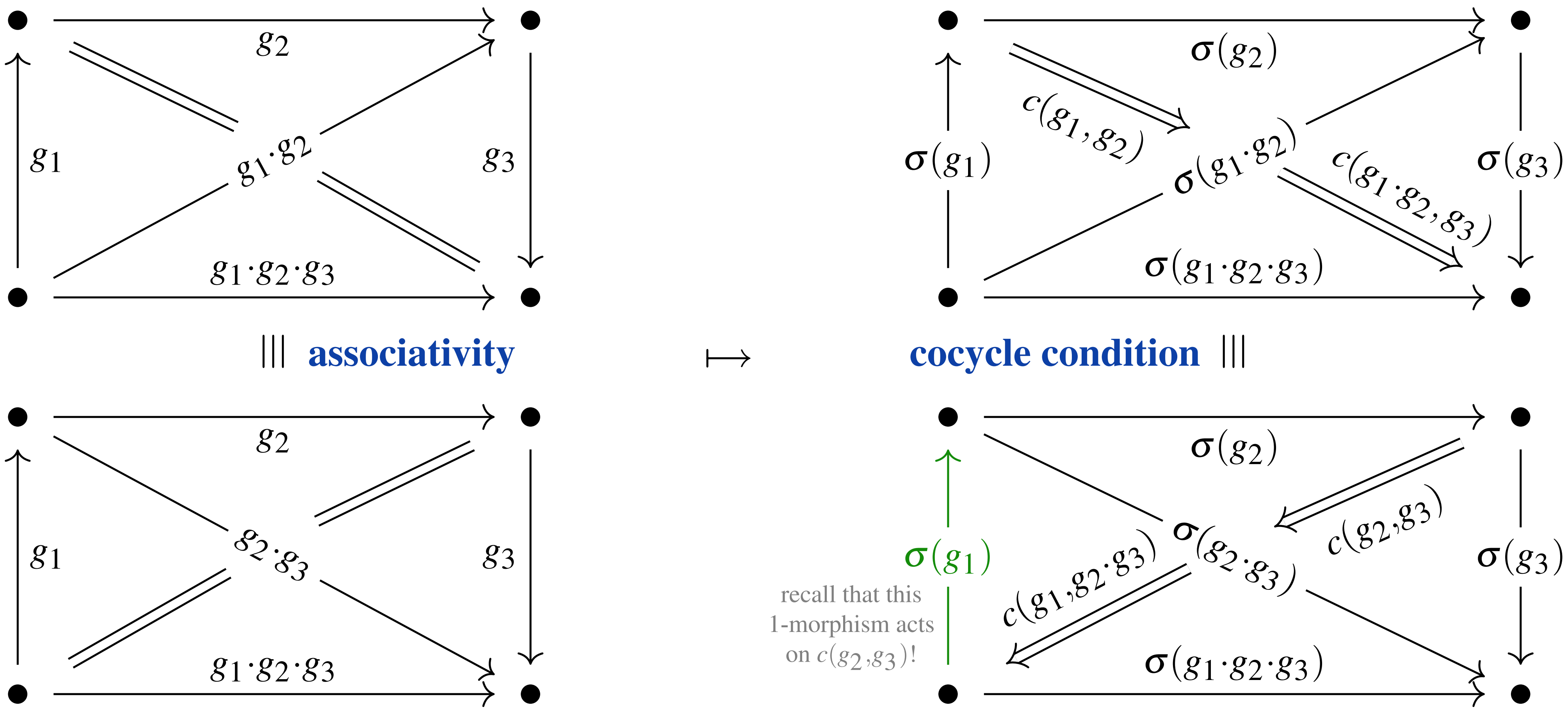
& \mathbf{B}\mathbb{Z}_2 &
\end{tikzcd}
$$



2-Groupoids with smooth structure.

A *smooth 2-groupoid* $\mathscr{X}$ is given by a rule

which to each chart $\mathbb{R}^n$, $n \in \mathbb{N}$, assigns the plain 2-groupoid $\mathrm{Probe}(\mathbb{R}^n, \mathscr{X})$

of ways of smoothly mapping $\mathbb{R}^n$ into the would-be $\mathscr{X}$

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and to each smooth map of charts a 2-functor of pre-composing such *probes*

$$
\begin{tikzcd}
\text{a chart} & \mathbb{R}^{n_1} \arrow[d, "f", "\text{smooth}"'] \arrow[r, mapsto] & \mathrm{Probe}(\mathbb{R}^{n_1}, \mathscr{X}) & \text{set of probes by this chart} \\
\text{another chart} & \mathbb{R}^{n_2} \arrow[r, mapsto] & \mathrm{Probe}(\mathbb{R}^{n_2}, \mathscr{X}) \arrow[u, "f^*", "\text{pullback}"']
\end{tikzcd}
$$

such that this respects composition and identities of smooth functions f.

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$$\begin{tikzcd}
\text{a chart} & \mathbb{R}^{n_1} \arrow[d, "f", "\text{smooth}"'] & \longmapsto & \mathrm{Probe}(\mathbb{R}^{n_1}, \mathscr{X}) & \text{set of probes by this chart} \\
\text{another chart} & \mathbb{R}^{n_2} & \longmapsto & \mathrm{Probe}(\mathbb{R}^{n_2}, \mathscr{X}) \arrow[u, "f^*", "\text{pullback}"']
\end{tikzcd}$$

such that this respects composition and identities of smooth functions f.

So $\mathrm{Probe}(*, \mathscr{X}) = \mathrm{Probe}(\mathbb{R}^0, \mathscr{X})$ is the underlying 2-groupoid

and the system of $\mathrm{Probe}(\mathbb{R}^{\bullet > 0}, \mathscr{X})$ is *smooth structure* on it.

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Grothendieck (1965): "functorial geometry"

common jargon: "pre-2-stacks on the site of Cartesian spaces"

2-Groupoids with smooth structure – Examples.

If X is a topological space, then as a smooth 2-groupoid it's this assignment:

$$\mathrm{X} \quad : \quad \mathbb{R}^n \quad \mapsto \quad \mathrm{Probe}(\mathbb{R}^n, \mathrm{X}) := C^0(\mathbb{R}^n, \mathrm{X})$$

$$\begin{tikzcd} & \mathbf{x} \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\|"] & \\ \mathbf{x} \arrow[rr, equal] & & \mathbf{x} \end{tikzcd} \qquad \mathbf{x} \in C^0(\mathbb{R}^n, \mathrm{X})$$

If X is a smooth manifold, then as a smooth 2-groupoid it's this assignment:

$$\mathrm{X} \quad : \quad \mathbb{R}^n \quad \mapsto \quad \mathrm{Probe}(\mathbb{R}^n, \mathrm{X}) := C^\infty(\mathbb{R}^n, \mathrm{X})$$

$$\begin{tikzcd} & \mathbf{x} \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\|"] & \\ \mathbf{x} \arrow[rr, equal] & & \mathbf{x} \end{tikzcd} \qquad \mathbf{x} \in C^\infty(\mathbb{R}^n, \mathrm{X})$$

(also known as X in its incarnation as a *diffeological space*).

2-Groupoids with smooth structure – Examples.

If Γ a *Lie* group, then the sets of smooth functions $C^\infty(\mathbb{R}^n, \Gamma)$ are plain groups, and the *smooth delooping groupoid* $\mathbf{B}\Gamma$ is:

$$\mathbf{B}\Gamma \quad : \quad \mathbb{R}^n \quad \mapsto \quad \mathrm{Probe}(\mathbb{R}^n, \mathbf{B}\Gamma) := \mathbf{B}\big(C^\infty(\mathbb{R}^n, \Gamma)\big)$$

$$\begin{tikzcd} & \bullet \arrow[dr, "\gamma_2"] \arrow[d, phantom, "\|"] & \\ \bullet \arrow[ur, "\gamma_1"] \arrow[rr, "\gamma_1 \cdot \gamma_2"'] & {} & \bullet \end{tikzcd} \qquad \gamma_i \in C^\infty(\mathbb{R}^n, \Gamma)$$

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If V is a Γ-representation, then the *smooth moduli space of V-valued differential forms* is

$$\Omega^d_{\mathrm{dR}}(-; V) /\!\!/ \Gamma \quad : \quad \mathbb{R}^n \quad \mapsto \quad \Omega^d_{\mathrm{dR}}(\mathbb{R}^n; V) /\!\!/ \Gamma$$

$$\begin{tikzcd} & \omega_d \cdot \gamma_1 \arrow[dr, "\gamma_2"] \arrow[d, equal] & \\ \omega_d \arrow[ur, "\gamma_1"] \arrow[rr, "\gamma_1 \cdot \gamma_2"'] & {} & \omega_d \cdot \gamma_1 \cdot \gamma_2 \end{tikzcd} \qquad \begin{array}{l} \gamma_i \in C^\infty(\mathbb{R}^n, \Gamma) \\ \omega_d \in \Omega^d_{\mathrm{dR}}(\mathbb{R}^n; V) \end{array}$$

2-Groupoids with smooth structure – As smooth homotopy types.

Smooth 2-groupoids are *models* for *smooth 2-stacks* aka **smooth homotopy 2-types**

2-Groupoids with smooth structure – As smooth homotopy types.

Smooth 2-groupoids are *models* for *smooth 2-stacks* aka **smooth homotopy 2-types**

A smooth 2-functor $\mathscr{X} \xrightarrow{f} \mathscr{Y}$ is called:

PrjFib	*projective* *fibration*	iff for each $\mathbb{R}^n$, every $k+1$-morphism in $\mathrm{Probe}(\mathbb{R}^n, \mathscr{Y})$ that starts at k-morphisms which come from $\mathrm{Probe}(\mathbb{R}^n, \mathscr{X})$ lifts compatibly to a $k+1$-morphism in $\mathrm{Probe}(\mathbb{R}^n, \mathscr{X})$
LWEq	*local* *weak equivalence*	iff for every $\mathbb{R}^n$ there exists an open ball $0 \in \mathbb{D}^n_\varepsilon \overset{i}{\hookrightarrow} \mathbb{R}^n$ such that $\mathrm{Probe}(\mathbb{R}^n, f)_{\vert i}$ is a weak homotopy equivalence namely an iso on the evident homotopy groups
PrjCof	*projective* *cofibration*	if (Dugger's sufficient condition): for all k, the spaces of k-morphisms are disjoint unions of charts $\mathbb{R}^n$ (for any n-s)

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Fact/Def.: *Maps of 2-stacks* are *modeled* by

$$\varnothing \xrightarrow[\in \mathrm{PrjCof}]{\text{cof. domain}} \widehat{\mathscr{X}} \xrightarrow[\text{smooth 2-functor}]{\phi} \mathscr{Y} \xrightarrow[\in \mathrm{PrjFib}]{\substack{\text{fib.} \\ \text{co-domain}}} *$$

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Fact/Def.: *Maps of 2-stacks* and their *homotopy fibers* are *modeled* by pullbacks of this form:

(because 2-stackification is an ∞-lex reflection)

$$
\begin{tikzcd}
 & \mathrm{HoFib}_y(\phi) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \widehat{\ast} \arrow[d, "\in \mathrm{PrjFib}"] & \ast \arrow[l, "\text{fib. resolution}"', "\in \mathrm{LWEq}"] \arrow[dl, "y"] \\
\varnothing \arrow[r, "\text{cof. domain}", "\in \mathrm{PrjCof}"'] & \widehat{\mathscr{X}} \arrow[r, "\phi", "\text{smooth 2-functor}"'] & \mathscr{Y} \arrow[r, "\text{fib. co-domain}", "\in \mathrm{PrjFib}"'] & \ast
\end{tikzcd}
$$

2-Groupoids with smooth structure – As smooth homotopy types.

Smooth 2-groupoids are *models* for *smooth 2-stacks* aka **smooth homotopy 2-types**

A smooth 2-functor $\mathscr{X} \xrightarrow{f} \mathscr{Y}$ is called:

PrjFib	*projec* *fibrati*	iff for each $\mathbb{R}^n$, $\mathscr{X}$)
LWEq	*local* *weak*	alence s
PrjCof	*projec* *cofibration*	disjoint unions of charts $\mathbb{R}^n$ (for any n-s)

Definition/Example:
For $\widehat{\mathscr{X}}$ projectively cofibrant
and $\mathscr{Y}$ projectively fibrant
the smooth *mapping 2-stack* is modeled by

$$\mathrm{Map}\big(\widehat{\mathscr{X}}, \mathscr{Y}\big) \;:\; \mathbb{R}^n \mapsto \underbrace{\mathrm{Hom}\big(\widehat{\mathscr{X}} \times \mathbb{R}^n, \mathscr{Y}\big)}_{\text{2-groupoid of smooth 2-functors}}$$

Fact/Def.: *Maps of 2-stacks* and their *homotopy fibers* are *modeled* by pullbacks of this form:

(because 2-stackification is an ∞-lex reflection)

$$\begin{tikzcd}
 & \mathrm{HoFib}_y(\phi) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \widehat{*} \arrow[d, "\in\mathrm{PrjFib}"] & * \arrow[l, "\text{fib. resolution}"', "\in\mathrm{LWEq}"] \arrow[dl, "y"] \\
\varnothing \arrow[r, "\text{cof. domain}", "\in\mathrm{PrjCof}"'] & \widehat{\mathscr{X}} \arrow[r, "\phi", "\text{smooth 2-functor}"'] & \mathscr{Y} \arrow[r, "\text{fib. co-domain}", "\in\mathrm{PrjFib}"'] & *
\end{tikzcd}$$

2-Groupoids with smooth structure – Homotopy fiber sequences.

$$
\begin{tikzcd}
U(1) \arrow[r, hook, "\text{locally trivial circle-extension}"'] & \Gamma \arrow[r, two heads] & \Gamma/U(1) \; \text{smooth group} & & & & \\
& & \Gamma /\!/ U(1) \arrow[u, "\in \mathrm{LWEq}"'] \arrow[r, "\in \mathrm{PrjFib}"'] & \mathbf{B}U(1) \arrow[r] & \mathbf{B}\Gamma \arrow[r] & \mathbf{B}(\Gamma/U(1)) & \\
& & & & & \mathbf{B}\Gamma /\!/ \mathbf{B}U(1) \arrow[u, "\in \mathrm{LWEq}"'] \arrow[r, "\text{2-cocycle classifying the extension}"'] & \mathbf{B}^2 U(1)
\end{tikzcd}
$$

2-Groupoids with smooth structure – Homotopy fiber sequences.

$$\begin{tikzcd}
\mathrm{U}(1) \arrow[r, hook, "\text{locally trivial circle-extension}"'] & \Gamma \arrow[r, two heads] & \Gamma/\mathrm{U}(1)
\end{tikzcd}$$

smooth group

$$\begin{tikzcd}
 & & \Gamma/\mathrm{U}(1) \\
 & & \Gamma/\!/\mathrm{U}(1) \arrow[u, "\in \mathrm{LWEq}"'] \arrow[r, "\in \mathrm{PrjFib}"'] & \mathbf{B}\mathrm{U}(1) \arrow[r] & \mathbf{B}\Gamma \arrow[r] & \mathbf{B}(\Gamma/\mathrm{U}(1)) \\
 & & & & & \mathbf{B}\Gamma/\!/\mathbf{B}\mathrm{U}(1) \arrow[u, "\in \mathrm{LWEq}"'] \arrow[r, "\text{2-cocyle classifying the extension}"'] & \mathbf{B}^2\mathrm{U}(1)
\end{tikzcd}$$

$$\begin{tikzcd}
c \arrow[d, equal] & \mapsto & \gamma \arrow[d, equal] & \mapsto & {[\gamma]} \arrow[d, equal] \\
c & & \gamma & & {[\gamma]}
\end{tikzcd}$$

$$\begin{tikzcd}
\gamma \arrow[d, "c"'] & \mapsto & \bullet \arrow[d, "c"'] & \mapsto & \bullet \arrow[d, "\gamma"'] & \mapsto & \bullet \arrow[d, "{[\gamma]}"'] \\
c \cdot \gamma & & \bullet & & \bullet & & \bullet
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[d, bend right, "\gamma"'] \arrow[d, bend left, "c\cdot\gamma"] \arrow[d, phantom, "\overset{c}{\Longrightarrow}"] & \mapsto & \bullet \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\overset{c}{\Longrightarrow}"] \\
\bullet & & \bullet
\end{tikzcd}$$

For example, write $\mathrm{U}(n),\ n \in \mathbb{N} \sqcup \{\omega\}$
for the unitary group on a countably-dimensional complex Hilbert space
and regard this as a smooth group by its "continuous diffeology":

$$\mathrm{U}(n) \ : \ \mathbb{R}^k \ \mapsto \ \mathrm{Probe}(\mathbb{R}^k, \mathrm{U}(n)) \ := \ C^0(\mathbb{R}^k, \mathrm{U}(n))$$

2-Groupoids with smooth structure – Dixmier-Douady class.

For example, write $\mathrm{U}(n)$, $n \in \mathbb{N} \sqcup \{\omega\}$
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Then we have the following long fiber sequence of smooth 2-groupoids:

$$\begin{tikzcd}
\mathrm{U}(1) \arrow[r, hook] & \mathrm{U}(n) \arrow[r, two heads] & \mathrm{PU}(n) & & & \\
& & \mathrm{U}(n) /\!/ \mathrm{U}(1) \arrow[u, "\in \mathrm{LWEq}"'] \arrow[r] & \mathbf{B}\mathrm{U}(1) \arrow[r] & \mathbf{B}\mathrm{U}(n) \arrow[r] & \mathbf{B}\mathrm{PU}(n) & \\
& & & & & \mathbf{B}\mathrm{U}(n) /\!/ \mathbf{B}\mathrm{U}(1) \arrow[u, "\in \mathrm{LWEq}"'] \arrow[r] & \mathbf{B}^2\mathrm{U}(1)
\end{tikzcd}$$

For example, write $\mathrm{U}(n),\ n \in \mathbb{N} \sqcup \{\omega\}$
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& & \mathrm{U}(n)/\!\!/\mathrm{U}(1) \arrow[u, "\in\mathrm{LWEq}"'] \arrow[r] & \mathbf{B}\mathrm{U}(1) \arrow[r] & \mathbf{B}\mathrm{U}(n) \arrow[r] & \mathbf{B}\mathrm{PU}(n) \\
& & & & & \mathbf{B}\mathrm{U}(n)/\!\!/\mathbf{B}\mathrm{U}(1) \arrow[u, "\in\mathrm{LWEq}"'] \arrow[r] & \mathbf{B}^2\mathrm{U}(1)
\end{tikzcd}$$

This is compatible with complex conjugation, so we have a map of 2-stacks like this:

universal Dixmier-Douady class

$$\mathbf{B}\mathrm{PU}(n)/\!\!/\mathbb{Z}_2 \xleftarrow[\in\mathrm{LWEq}]{} \xrightarrow{\mathrm{DD}} \mathbf{B}^2\mathrm{U}(1)/\!\!/\mathbb{Z}_2$$

2-Groupoids with smooth structure – Čech groupoids.

For X a smooth manifold with $\left\{ \mathrm{U}_i \hookrightarrow \mathrm{X} \right\}_{i \in I}$ a good open cover, in that

$$\big(\mathbf{x}, (i_1, \cdots, i_n)\big) \in C^\infty\big(\mathbb{R}^m, U_{i_1} \cap \cdots \cap U_{i_n}\big) \qquad \Rightarrow \qquad U_{i_1} \cap \cdots \cap U_{i_n} \underset{\text{diff}}{\simeq} \mathbb{R}^{\dim(\mathrm{X})},$$

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we have the smooth *Čech 2-groupoid*:

$$\varnothing \xrightarrow[\in \mathrm{PrjCof}]{} \underset{\substack{\text{Čech 2-groupoid of} \\ \text{good open cover}}}{\widehat{\mathrm{X}}} \xrightarrow[\in \mathrm{LWEq}]{\text{cofibrant resolution}} \mathrm{X} \;\; \substack{\text{smooth manifold regarded} \\ \text{as smooth 2-groupoid}}$$

$$\begin{tikzcd}
 & (\mathbf{x}, j) \arrow[dr, "{(\mathbf{x},(j,k))}"] \arrow[d, phantom, "\Downarrow {(\mathbf{x},(i,j,k))}"] & \\
(\mathbf{x}, i) \arrow[ur, "{(\mathbf{x},(i,j))}"] \arrow[rr, "{(\mathbf{x},(i,k))}"'] & {} & (\mathbf{x}, k)
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
 & \mathbf{x} \arrow[dr, equal] \arrow[d, phantom, "\Vert"] & \\
\mathbf{x} \arrow[ur, equal] \arrow[rr, equal] & {} & \mathbf{x}
\end{tikzcd}$$

which is a *projectively cofibrant resolution* of X.

2-Groupoids with smooth structure – Čech cocycles.

Smooth 2-functors from such a Čech resolution $\widehat{X} \to X$

to the delooping $\mathbf{B}\Gamma$ of a Lie group

are *cocycles* in the *Čech cohomology* of X with coefficients in Γ:

$$\widehat{X} \xrightarrow{\text{smooth functor = Čech cocycle}} \mathbf{B}\Gamma$$

$$\begin{tikzcd}
& (x,j) \arrow[dr, "{(x,(j,k))}"] & \\
(x,i) \arrow[ur, "{(x,(i,j))}"] \arrow[rr, "{(x,(i,k))}"'] \arrow[rr, phantom, bend left=40, "{\overset{(x,(i,j,k))}{\Downarrow}}"] & & (x,k)
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
& \bullet \arrow[dr, "\gamma_{jk}(x)"] & \\
\bullet \arrow[ur, "\gamma_{ij}(x)"] \arrow[rr, "\gamma_{ik}(x)"'] \arrow[rr, phantom, bend left=40, "\Vert"] & & \bullet
\end{tikzcd}$$

Čech relations **cocycle condition**

Principal bundles via smooth groupoids – Universal principal bundles.

The inclusion of the unique base point into $\mathbf{B}\Gamma$ has the following *fibrant resolution*:

$$\begin{tikzcd} * \arrow[d, "\in \mathrm{LWEq}"] \\ \mathbf{E}\Gamma \arrow[d, "\in \mathrm{PrjFib}"] \\ \mathbf{B}\Gamma \end{tikzcd}$$

universal
Γ-principal bundle

in its stacky incarnation

$$\begin{tikzcd} & \gamma \cdot \gamma_1 \arrow[dr, "\gamma_2"] \arrow[d, phantom, "\Vert"] & \\ \gamma \arrow[ur, "\gamma_1"] \arrow[rr, "\gamma_1 \cdot \gamma_2"'] & {} \arrow[d, mapsto] & \gamma \cdot \gamma_1 \cdot \gamma_2 \\ & \bullet \arrow[dr, "\gamma_2"] \arrow[d, phantom, "\Vert"] & \\ \bullet \arrow[ur, "\gamma_1"] \arrow[rr, "\gamma_1 \cdot \gamma_2"'] & {} & \bullet \end{tikzcd}$$

Principal bundles via smooth groupoids.

The *homotopy fiber* of a 2-functor = Čech cocycle is equivalently
the principal bundle P *it classifies*:

$$
\begin{tikzcd}
\mathrm{P} \arrow[d] & \overset{\textbf{principal bundle}}{\widehat{\mathrm{P}}} \arrow[l, "\in \mathrm{LWEq}"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathbf{E}\Gamma \arrow[d, "\in \mathrm{PrjFib}"] & \ast \arrow[l, "\in \mathrm{LWEq}"'] \arrow[dl, bend left] \\
\mathrm{X} & \widehat{\mathrm{X}} \arrow[l, "\in \mathrm{LWEq}"] \arrow[r, "\textbf{Čech cocycle}"'] & \mathbf{B}\Gamma &
\end{tikzcd}
$$

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$$
\begin{tikzcd}
 & \gamma_x \cdot \gamma_{ij}(x) \arrow[dr, "\gamma_{jk}(x)"] \arrow[d, phantom, "\Vert"] & \\
\gamma_x \arrow[ur, "\gamma_{ij}(x)"] \arrow[rr, "\gamma_{ik}(x)"'] & & \gamma_x \cdot \gamma_{ik}(x)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{P} \arrow[d] & \widehat{\mathrm{P}} \arrow[l, "\in \mathrm{LWEq}"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] \arrow[r, phantom, "\textbf{principal bundle}", yshift=4ex] & \mathbf{E}\Gamma \arrow[d, "\in \mathrm{PrjFib}"] & \ast \arrow[l, "\in \mathrm{LWEq}"'] \arrow[dl, bend left] \\
\mathrm{X} & \widehat{\mathrm{X}} \arrow[l, "\in \mathrm{LWEq}"] \arrow[r, "\textbf{Čech cocycle}"'] & \mathbf{B}\Gamma &
\end{tikzcd}
\qquad \longmapsto
$$

$$
\begin{tikzcd}
 & (x,j) \arrow[dr] \arrow[d, phantom, "\Vert"] & \\
(x,i) \arrow[ur] \arrow[rr] & & (x,k)
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
 & \bullet \arrow[dr, "\gamma_{jk}(x)"] \arrow[d, phantom, "\Vert"] & \\
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$$
\begin{tikzcd}
 & \big((x,j),\gamma_x\cdot\gamma_{ij}(x)\big) \arrow[dr] & \\
\big((x,i),\gamma_x\big) \arrow[ur] \arrow[rr] \arrow[urr, phantom, "\overset{||}{\underset{||}{\text{gluing prescr.}}}"] & & \big((x,k),\gamma_x\cdot\gamma_{ik}(x)\big)
\end{tikzcd}
\longmapsto
\begin{tikzcd}
 & \gamma_x\cdot\gamma_{ij}(x) \arrow[dr, "\gamma_{jk}(x)"] & \\
\gamma_x \arrow[ur, "\gamma_{ij}(x)"] \arrow[rr, "\gamma_{ik}(x)"'] \arrow[urr, phantom, "\|"] & & \gamma_x\cdot\gamma_{ik}(x)
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & \text{principal bundle} & & \\
\mathrm{P} \arrow[d] & \widehat{\mathrm{P}} \arrow[l, "\in\mathrm{LWEq}"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathbf{E}\Gamma \arrow[d, "\in\mathrm{PrjFib}"] & \ast \arrow[l, "\in\mathrm{LWEq}"'] \arrow[dl, bend left] \\
\mathrm{X} & \widehat{\mathrm{X}} \arrow[l, "\in\mathrm{LWEq}"'] \arrow[r, "\text{Čech cocycle}"'] & \mathbf{B}\Gamma &
\end{tikzcd}
$$

$\big\downarrow$ $\qquad\qquad$ $\big\downarrow$

$$
\begin{tikzcd}
 & (x,j) \arrow[dr] & \\
(x,i) \arrow[ur] \arrow[rr] \arrow[urr, phantom, "\|"] & & (x,k)
\end{tikzcd}
\longmapsto
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 & \bullet \arrow[dr, "\gamma_{jk}(x)"] & \\
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\end{tikzcd}
$$

Principal 2-bundles via smooth 2-groupoids.

This neat formulation of ordinary principal bundles immediatly generalizes to give principal 2-bundles:

Principal 2-bundles via smooth 2-groupoids.

This neat formulation of ordinary principal bundles

immediatly generalizes to give principal 2-bundles:

E.g. for the structure 2-group $\mathrm{Aut}(\mathbf{B}\Gamma)$

these are equivalently Giraud's *non-abelian gerbes*:

$$
\begin{tikzcd}
\mathrm{P} \arrow[d] & \overset{\textbf{principal 2-bundle}}{\widehat{\mathrm{P}}} \arrow[l, "\in \mathrm{LWEq}"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathbf{E}\mathrm{Aut}(\mathbf{B}\Gamma) \arrow[d, "\in \mathrm{PrjFib}"] & \ast \arrow[l, "\in \mathrm{LWEq}"'] \arrow[dl, bend left] \\
\mathrm{X} & \widehat{\mathrm{X}} \arrow[l, "\in \mathrm{LWEq}"'] \arrow[r, "\textbf{Čech cocycle}"'] & \mathbf{B}\mathrm{Aut}(\mathbf{B}\Gamma) &
\end{tikzcd}
$$

While it's tradition to be esoteric about this simple affair,

here to highlight that this is really about *twisted cohomology*: $\longrightarrow$

Principal 2-bundles via smooth 2-groupoids – Example: Twisted cohomology.

For structure 2-group $\mathrm{Aut}(\mathbf{B}\mathbb{Z}) \simeq (\mathbf{B}\mathbb{Z}) \rtimes \mathbb{Z}_2$,
with $\mathbf{B}\mathrm{Aut}(\mathbf{B}\mathbb{Z}) \simeq (\mathbf{B}^2\mathbb{Z})/\!/\mathbb{Z}_2$ and $\widehat{\mathrm{X}} \xrightarrow{\sigma} \mathbf{B}\mathbb{Z}_2$ a double covering, then
2nd integral cohomology of X with local coefficients is smooth 2-functors:

$$\begin{tikzcd}
\widehat{\mathrm{X}} \arrow[rr, dashed, "\text{smooth 2-functor = Čech 2-cocycle}"] \arrow[dr, "\sigma"'] & & (\mathbf{B}^2\mathbb{Z})/\!/\mathbb{Z}_2 \arrow[dl] \\
& \mathbf{B}\mathbb{Z}_2 &
\end{tikzcd}$$

Principal 2-bundles via smooth 2-groupoids – Example: Twisted cohomology.

For structure 2-group $\mathrm{Aut}(\mathbf{B}\mathbb{Z}) \simeq (\mathbf{B}\mathbb{Z}) \rtimes \mathbb{Z}_2$,
with $\mathbf{B}\mathrm{Aut}(\mathbf{B}\mathbb{Z}) \simeq (\mathbf{B}^2\mathbb{Z}) /\!\!/ \mathbb{Z}_2$ and $\widehat{\mathrm{X}} \xrightarrow{\sigma} \mathbf{B}\mathbb{Z}_2$ a double covering, then
2nd integral cohomology of X with local coefficients is smooth 2-functors:

$$
\begin{tikzcd}
\widehat{\mathrm{X}} \arrow[rr, dashed, "\text{smooth 2-functor = Čech 2-cocycle}"] \arrow[dr, "\sigma"'] & & (\mathbf{B}^2\mathbb{Z}) /\!\!/ \mathbb{Z}_2 \arrow[dl] \\
& \mathbf{B}\mathbb{Z}_2 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(x,j) \arrow[r] \arrow[dr, phantom, "\Rightarrow"] & (x,k) \arrow[d] \\
(x,i) \arrow[u] \arrow[ur] \arrow[r] \arrow[ur, phantom, "\Rightarrow"] & (x,l)
\end{tikzcd}
\qquad\longmapsto\qquad
\begin{tikzcd}
\bullet \arrow[rr, "\sigma_{jk}(x)"'] \arrow[dr, phantom, "\Downarrow c_{ijk}(x)"] & & \bullet \arrow[dd, "\sigma_{kl}(x)"] \\
& \sigma_{ik}(x) \arrow[dr, phantom, "\Downarrow c_{ikl}(x)"] & \\
\bullet \arrow[uu, "\sigma_{ij}(x)"] \arrow[uurr, "\sigma_{ik}(x)"] \arrow[rr, "\sigma_{il}(x)"] & & \bullet
\end{tikzcd}
$$

||| **Čech relations** $\qquad\mapsto\qquad$ **cocycle condition** |||

$$
\begin{tikzcd}
(x,j) \arrow[r] \arrow[dr] \arrow[dr, phantom, "\Leftarrow"] & (x,k) \arrow[d] \\
(x,i) \arrow[u] \arrow[r] & (x,l)
\end{tikzcd}
\qquad\longmapsto\qquad
\begin{tikzcd}
\bullet \arrow[rr, "\sigma_{jk}(x)"'] \arrow[ddrr, "\sigma_{jl}(x)"] & & \bullet \arrow[dd, "\sigma_{kl}(x)"] \arrow[dl, phantom, "\Leftarrow c_{jkl}(x)"] \\
& \sigma_{jl}(x) \arrow[dl, phantom, "\Leftarrow c_{ijl}(x)"] & \\
\bullet \arrow[uu, "\sigma_{ij}(x)"] \arrow[rr, "c_{il}(x)"] & & \bullet
\end{tikzcd}
$$

recall that this 1-morphism acts on $c_{jkl}(x)$!

Principal 2-bundles via smooth 2-groupoids – Example: Jandl bundle gerbes.

For structure 2-group $\mathrm{Aut}(\mathbf{B}\mathrm{U}(1)) \simeq (\mathbf{B}\mathrm{U}(1)) \rtimes \mathbb{Z}_2$,
with $\mathbf{B}\mathrm{Aut}(\mathbf{B}\mathrm{U}(1)) \simeq (\mathbf{B}^2\mathrm{U}(1))/\!\!/\mathbb{Z}_2$ and $\widehat{\mathrm{X}} \xrightarrow{\sigma} \mathbf{B}\mathbb{Z}_2$ a double covering, then
2nd $\mathrm{U}(1)$**-valued cohomology** of X with local coefficients is smooth 2-functors:

$$\begin{tikzcd}
\widehat{\mathrm{X}} \arrow[rr, dashed, "\text{smooth 2-functor = Čech 2-cocycle}", "\text{“real bundle gerbe”}"'] \arrow[dr, "\sigma"'] & & (\mathbf{B}^2\mathrm{U}(1))/\!\!/\mathbb{Z}_2 \arrow[dl] \\
& \mathbf{B}\mathbb{Z}_2 &
\end{tikzcd}$$

$$\begin{tikzcd}
(x,j) \arrow[rr] & \arrow[d, phantom, "\Searrow"] & (x,k) \arrow[dd] \\
& \arrow[dr, phantom, "\Searrow"] & \\
(x,i) \arrow[uu] \arrow[uurr] \arrow[rr] & & (x,l)
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[rr, "\sigma_{jk}(x)"'] & \arrow[d, phantom, "\Searrow c_{ijk}(x)"] & \bullet \arrow[dd, "\sigma_{kl}(x)"] \\
& \sigma_{ik}(x) \arrow[dr, phantom, "\Searrow c_{ikl}(x)"] & \\
\bullet \arrow[uu, "\sigma_{ij}(x)"] \arrow[uurr] \arrow[rr, "\sigma_{il}(x)"] & & \bullet
\end{tikzcd}$$

$\|\|$ **Čech relations** $\qquad\qquad \mapsto \qquad\qquad$ **cocycle condition** $\|\|$

$$\begin{tikzcd}
(x,j) \arrow[rr] \arrow[ddrr] & \arrow[dl, phantom, "\Swarrow"] & (x,k) \arrow[dd] \\
\arrow[d, phantom, "\Swarrow"] & & \\
(x,i) \arrow[uu] \arrow[rr] & & (x,l)
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[rr, "\sigma_{jk}(x)"'] \arrow[ddrr] & \sigma_{jl}(x) \arrow[dr, phantom, "\Swarrow c_{jkl}(x)"] & \bullet \arrow[dd, "\sigma_{kl}(x)"] \\
\arrow[d, phantom, "\Swarrow c_{ijl}(x)"] & & \\
\bullet \arrow[uu, "\sigma_{ij}(x)"] \arrow[rr, "c_{il}(x)"] & & \bullet
\end{tikzcd}$$

recall that this
1-morphism acts
on $c_{jkl}(x)$!

Principal 2-bundles via smooth 2-groupoids – Example: Twisted cohomology.

For equivariant de Rham coefficients with $G \circlearrowright V$ a representation of a finite group:

$$
\begin{tikzcd}
\widehat{\mathrm{X}} \arrow[rr, dashed, "\text{smooth functor = Čech 1-cocycle}"] \arrow[dr, "\sigma", "\substack{\text{flat } G\text{-bundle} \\ \text{local system}}"'] & & \Omega^d_{\mathrm{dR}}\big(-;V\big) /\!\!/ G \arrow[dl] \\
& \mathbf{B}G &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(x,j) \arrow[rr] \arrow[drr, phantom, "\Rightarrow"] & & (x,k) \arrow[dd] \\
& & \\
(x,i) \arrow[uu] \arrow[uurr] \arrow[rr] \arrow[uurr, phantom, "\Rightarrow"'] & & (x,l)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\omega_j(x) \arrow[rr, "\sigma_{jk}"'] \arrow[dr, equal] & & \omega_k(x) \arrow[dd, "\sigma_{kl}"] \\
& \sigma_{ik} \arrow[ur] \arrow[dr, equal] & \\
\omega_l(x) \arrow[uu, "\sigma_{ij}"] \arrow[ur, dash] \arrow[rr, "\sigma_{il}"] & & \omega_k(x)
\end{tikzcd}
$$

$|||$ **Čech relations** $\qquad\mapsto\qquad$ **cocycle condition** $|||$

$$
\begin{tikzcd}
(x,j) \arrow[rr] \arrow[ddrr] & & (x,k) \arrow[dd] \arrow[dl, phantom, "\Leftarrow"] \\
& & \\
(x,i) \arrow[uu] \arrow[rr] \arrow[ur, phantom, "\Leftarrow"] & & (x,l)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\omega_j(x) \arrow[rr, "\sigma_{jk}"'] \arrow[dr, dash] & & \omega_k(x) \arrow[dd, "\sigma_{kl}"] \arrow[dl, equal] \\
& \sigma_{jl} \arrow[dr] \arrow[dl, equal] & \\
\omega_i(x) \arrow[uu, "\sigma_{ij}"] \arrow[rr, "\sigma_{il}"] & & \omega_l(x)
\end{tikzcd}
$$

Principal 2-bundles via smooth 2-groupoids – Punchline.

So:

Non-abelian 1-cohomology is modulated by 1-stacks $\mathbf{B}\Gamma$,
abelian 2-cohomology is modulated by 2-stacks $\mathbf{B}^2 A$, etc.

Higher fiber/principal bundles are *bundles of such moduli stacks*,
hence are families of moduli stacks that vary over the base space,
hence locally modulate cohomology as before,
but now subject to global twists.

Principal 2-bundles via smooth 2-groupoids – Equivariance.

Finally, the "higher topos" of smooth 2-groupoids
has *equivariance* natively built into it: just let domain spaces be groupoids, too.

Principal 2-bundles via smooth 2-groupoids – Equivariance.

Finally, the "higher topos" of smooth 2-groupoids
has *equivariance* natively built into it: just let domain spaces be groupoids, too:

For $\mathrm{X} \circlearrowleft G$ a smooth action of a finite group on a smooth manifold.

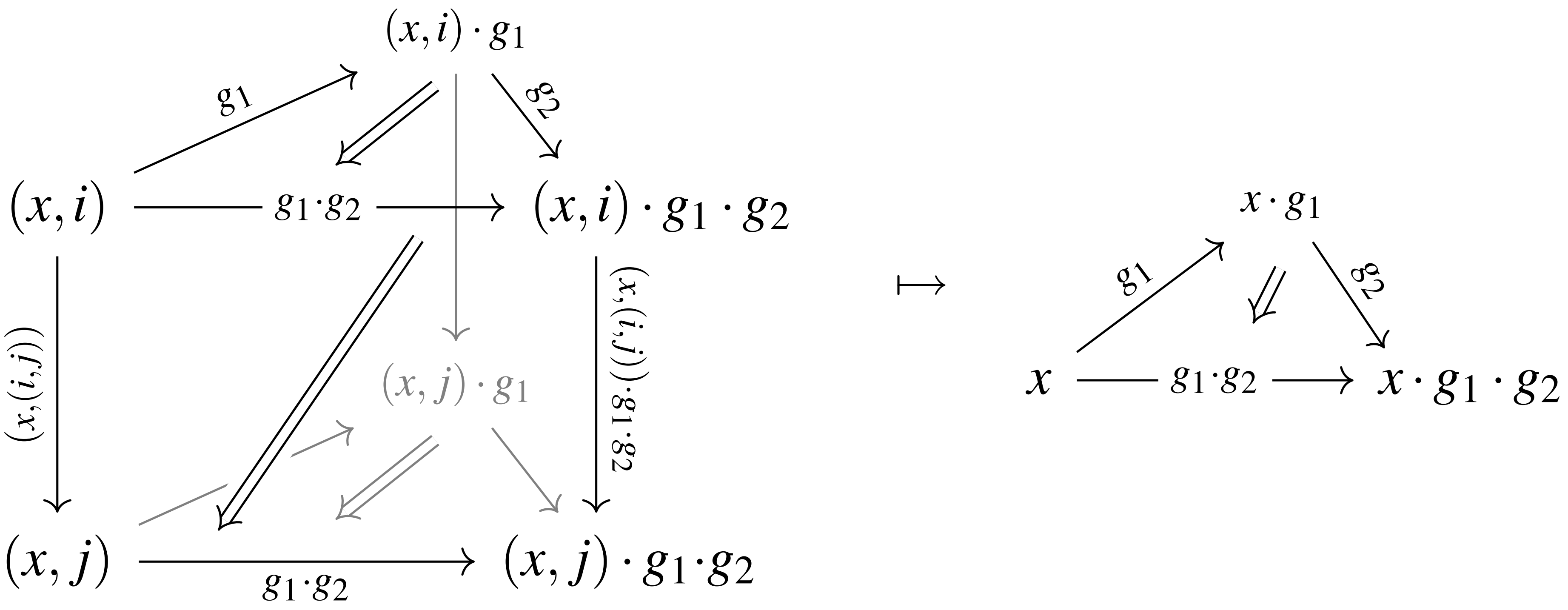
there exists an *equivariant good open cover*

$$\overset{\circlearrowleft G}{\coprod_{i \in I} \mathrm{U}_i} \longrightarrow\!\!\!\!\!\rightarrow \overset{\circlearrowleft G}{\mathrm{X}}$$

and its *equivariant Čech groupoid*:

$$\varnothing \xrightarrow[\in \mathrm{PrjCof}]{} \widehat{\mathrm{X} /\!\!/ G} \xrightarrow[\in \mathrm{LWEq}]{\text{cofibrant resolution}} \underbrace{\mathrm{X} /\!\!/ G}_{}$$

action groupoid



Principal 2-bundles via smooth 2-groupoids – Equivariance.

For $\mathrm{X} \circlearrowleft G$ a smooth manifold and $\Gamma \circlearrowleft G$ a smooth 2-group both equipped with smooth G-action, a G-equivariant Γ-principal 2-bundle on X is modulated by a smooth 2-functor like this:

$$
\begin{tikzcd}
\widehat{\mathrm{X} /\!\!/ G} \arrow[rr, dashed, "\text{modulating map}"'] \arrow[dr] & & \mathbf{B}\Gamma /\!\!/ G \arrow[dl] \\
& \mathbf{B}G &
\end{tikzcd}
$$

Principal 2-bundles via smooth 2-groupoids – Equivariance.

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$$
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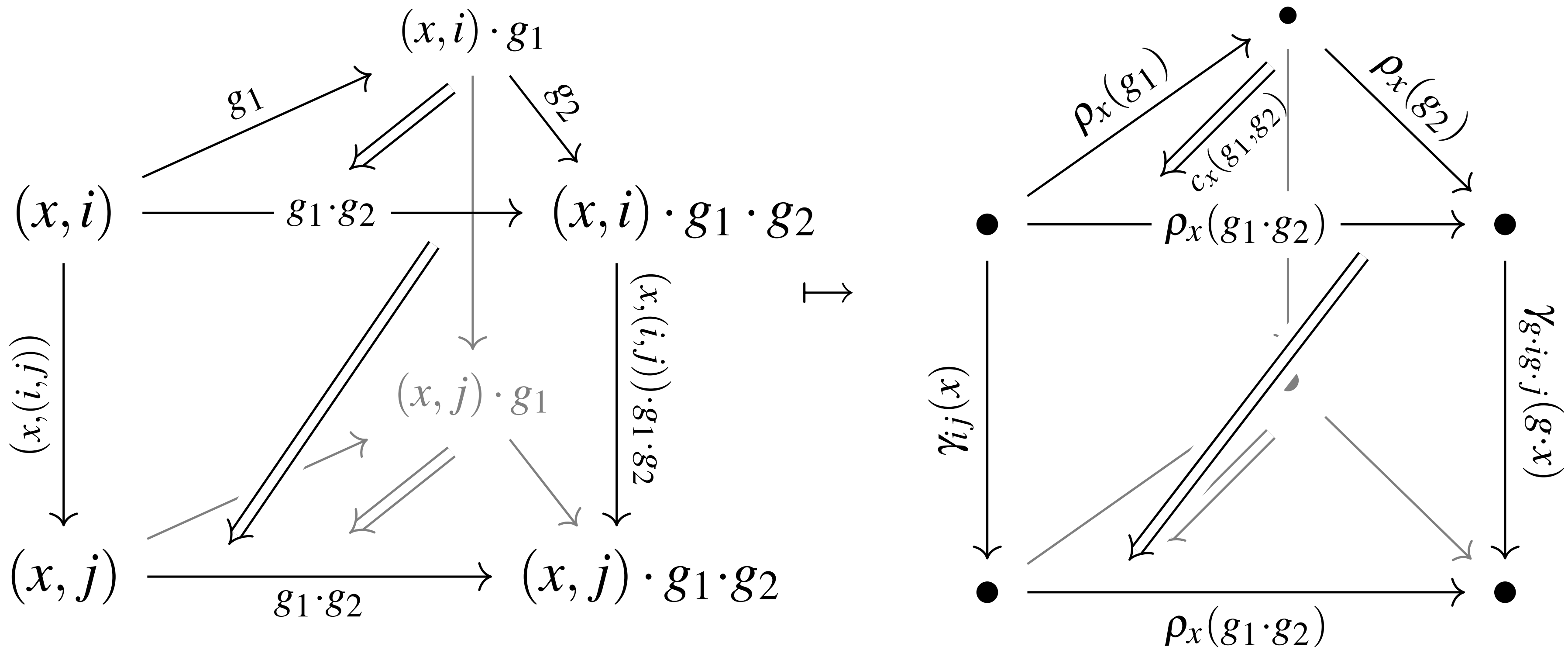
& \mathbf{B}G &
\end{tikzcd}
$$



equivariant Čech relations

equivariant 2-cocyle condition

Principal 2-bundles via smooth 2-groupoids – Equivariance.

For $\mathrm{X} \circlearrowleft G$ a smooth manifold and $\Gamma \circlearrowleft G$ a smooth 2-group
both equipped with smooth G-action, a
G-equivariant Γ-principal 2-bundle on X is modulated by a smooth 2-functor like this:

$$\begin{tikzcd}
\widehat{\mathrm{X} /\!\!/ G} \arrow[rr, dashed, "\text{modulating map}"'] \arrow[dr] & & \mathbf{B}\Gamma /\!\!/ G \simeq \mathbf{B}\left(\Gamma \rtimes G\right) \arrow[dl] \\
& \mathbf{B}G &
\end{tikzcd}$$

on the right we have equivalently the semidirect product 2-group.

Principal 2-bundles via smooth 2-groupoids – Equivariance.

For $\mathrm{X} \circlearrowleft G$ a smooth manifold and $\Gamma \circlearrowleft G$ a smooth 2-group both equipped with smooth G-action, a G-equivariant Γ-principal 2-bundle on X is modulated by a smooth 2-functor like this:

$$\begin{tikzcd}
\widehat{\mathrm{X} /\!/ G} \arrow[rr, "\text{eg. equivariant real bundle gerbe}", "\text{modulating map}"'] \arrow[dr] & & \mathbf{B}^2\mathrm{U}(1) /\!/ \mathbb{Z}_2 \simeq \mathbf{B}\big(\mathbf{B}\mathrm{U}(1) \rtimes \mathbb{Z}_2\big) \arrow[dl] \\
& \mathbf{B}\mathbb{Z}_2 &
\end{tikzcd}$$

on the right we have equivalently the semidirect product 2-group.

Principal 2-bundles via smooth 2-groupoids – Associated 2-bundles.

$$
\begin{tikzcd}
& & A /\!\!/ \big(\Gamma \rtimes G\big) \arrow[d, ->>, "\substack{\text{universal equivariant} \\ \text{associated } A\text{-fiber bundle}}"] & \substack{\text{homotopy quotient of} \\ \text{smooth 2-groupoid } A} \\
\widehat{\mathrm{X} /\!\!/ G} \arrow[rr, "\tau", "\text{modulating map}"'] \arrow[dr] & & \mathbf{B}\big(\Gamma \rtimes G\big) \arrow[dl] & \\
& \mathbf{B}G & &
\end{tikzcd}
$$

A fibration over that is equivalently
an equivariant ∞-action $(\Gamma \rtimes G) \circlearrowright A$
embodied by its universal *associated 2-bundle*.

Principal 2-bundles via smooth 2-groupoids – Associated 2-bundles.

$$
\begin{tikzcd}
\widehat{E /\!\!/ G} \arrow[rr] \arrow[d, "\substack{\textbf{associated equivariant}\\ A\textbf{-fiber 2-bundle}}"'] \arrow[drr, phantom, "\text{(pb)}"] & & A /\!\!/ \left(\Gamma \rtimes G\right) \arrow[d, ->>, "\substack{\textbf{universal equivariant}\\ \textbf{associated } A\textbf{-fiber bundle}}"] & \substack{\textbf{homotopy quotient of}\\ \textbf{smooth 2-groupoid } A} \\
\widehat{\mathrm{X} /\!\!/ G} \arrow[rr, "\tau", "\textbf{modulating map}"'] \arrow[dr] & & \mathbf{B}\left(\Gamma \rtimes G\right) \arrow[dl] & \\
& \mathbf{B}G & &
\end{tikzcd}
$$

Its pullback is
the equivariant A-fiber 2-bundle
which is associated to
the given equivariant principal 2-bundle.

Principal 2-bundles via smooth 2-groupoids – Associated 2-bundles.

$$
\begin{tikzcd}
\widehat{E /\!\!/ G} \arrow[rr] \arrow[d] & & A /\!\!/ \left(\Gamma \rtimes G\right) \arrow[d, ->>, "\text{universal equivariant associated } A\text{-fiber bundle}"] & \text{homotopy quotient of smooth 2-groupoid } A \\
\widehat{X /\!\!/ G} \arrow[u, bend left, dashed, "\text{section}"] \arrow[urr, dashed, "\text{lift of the modulating map}"] \arrow[rr, "\tau", "\text{modulating map}"'] \arrow[dr] & & \mathbf{B}\left(\Gamma \rtimes G\right) \arrow[dl] & \\
& \mathbf{B}G & &
\end{tikzcd}
$$

The equivariant sections are equivalently
the lifts of the modulating map.

Twisted equivariant non-abelian cohomology.

$$
\begin{tikzcd}
 & & A /\!\!/ \left(\Gamma \rtimes G\right) \arrow[d, two heads, "\text{universal equivariant associated } A\text{-fiber bundle}"] & \text{homotopy quotient of smooth 2-groupoid } A \\
\widehat{\mathrm{X} /\!\!/ G} \arrow[urr, dashed, "\text{cocycle in } \tau\text{-twisted } G\text{-equivariant } A\text{-cohomology of } \mathrm{X}"] \arrow[rr, "\tau", "\text{twist}"'] \arrow[dr] & & \mathbf{B}\left(\Gamma \rtimes G\right) \arrow[dl] \\
 & \mathbf{B}G &
\end{tikzcd}
$$

Equivalently, these are the *cocycles* of
τ-twisted G-equivariant A-cohomology.

Twisted equivariant non-abelian cohomology.

twisted equivariant
cohomology set

homotopy quotient of
smooth 2-groupoid A

$$\mathrm{A}^{\tau}_{G}(\mathrm{X}) \;:=\; \left\{ \begin{tikzcd} & & A /\!\!/ (\Gamma \rtimes G) \arrow[d, ->>, "\text{universal equivariant associated } A\text{-fiber bundle}"] \\ \widehat{\mathrm{X} /\!\!/ G} \arrow[urr, dashed, "\text{cocycle in } \tau\text{-twisted } G\text{-equivariant } A\text{-cohomology of } \mathrm{X}"] \arrow[rr, "\tau", "\text{twist}"'] \arrow[dr] & & \mathbf{B}(\Gamma \rtimes G) \arrow[dl] \\ & \mathbf{B}G & \end{tikzcd} \right\} \Big/ \sim_{\mathrm{htpy}}$$

Equivalently, these are the *cocycles* of
τ-twisted G-equivariant A-cohomology.

0 – Cohesive ∞-Topoi

I – Equivariant ∞-Bundles

II – TED-K-Theory

III – Anyonic Defect Branes

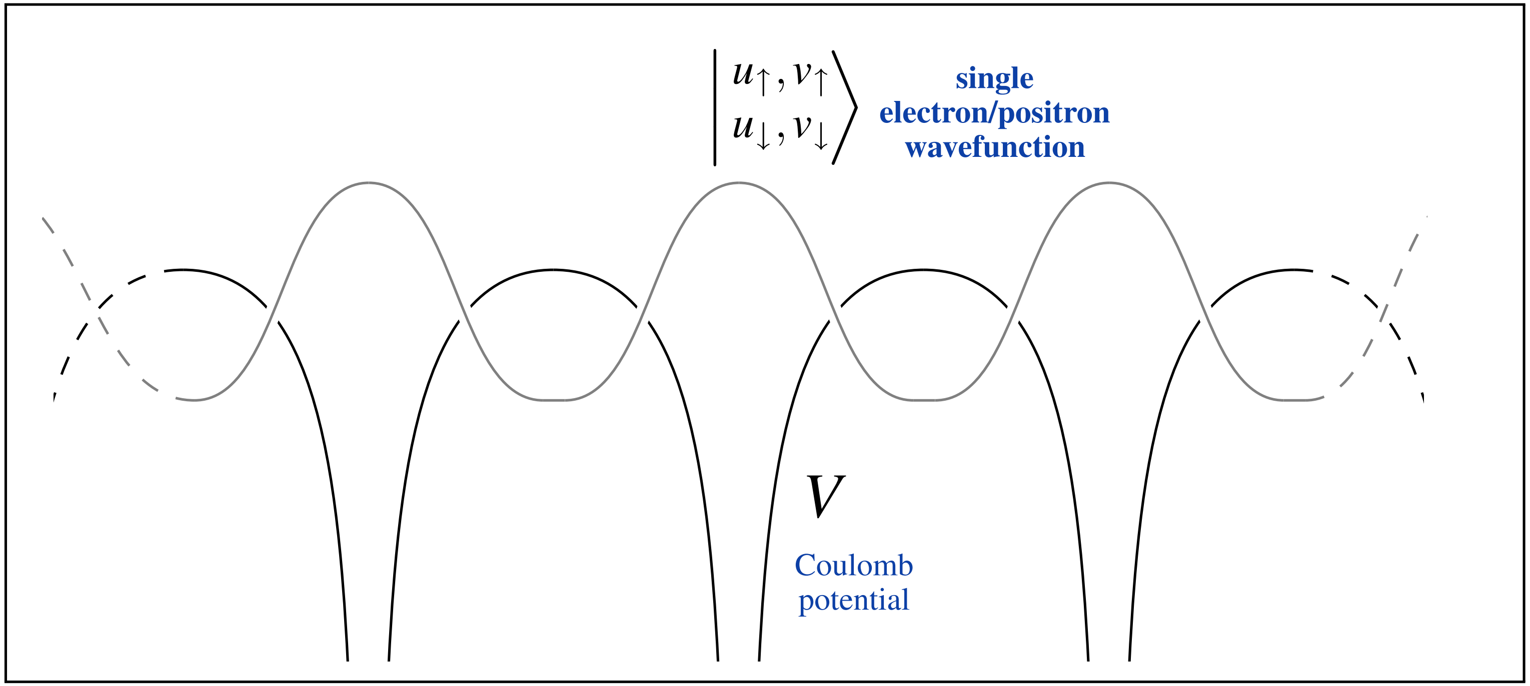
IV – Quantum Computation

This part is a
quick motivation and exposition of
twisted equivariant KR-theory
following these articles:

Equivariant Principal ∞-bundles [arXiv:2112.13654]
Anyonic Defect Branes in TED-K-Theory [arXiv:2203.11838]
The TED character map (in preparation)

Vacua of electron/positron field in Coulomb background.

Fact ([KS77][CHO82]). The vacua of the free Dirac quantum field in a classical Coulomb background...



Vacua of electron/positron field in Coulomb background.

Fact ([KS77][CHO82]). The vacua of the free Dirac quantum field in a classical Coulomb background are characterized by Fredholm operators...

$$
\begin{array}{ccccccc}
\textbf{finite-dimensional kernel} & & & \textbf{Fredholm operator} & & & \textbf{finite-dimensional cokernel}\\
\ker(F) & \hookrightarrow & \mathcal{H} & \xrightarrow[\text{bounded linear}]{F} & \mathcal{H} & \twoheadrightarrow & \mathrm{coker}(F)\\
\overbrace{\psi \in \mathcal{H} \,\Big|\, \forall_\phi \; \langle \phi | F | \psi \rangle = 0} & & & & & & \overbrace{\psi \in \mathcal{H} \,\big|\, \forall_\phi \; \langle \psi | F | \phi \rangle = 0}
\end{array}
$$

Vacua of electron/positron field in Coulomb background.

Fact ([KS77][CHO82]). The vacua of the free Dirac field in a classical Coulomb background are characterized by Fredholm operators

finite-dimensional kernel — Fredholm operator — finite-dimensional cokernel

$$\underbrace{\ker(F)}_{\left\{\psi \in \mathcal{H} \,\middle|\, \forall_\phi \, \langle \phi | F | \psi \rangle = 0\right\}} \hookrightarrow \mathcal{H} \xrightarrow[\text{bounded linear}]{F} \mathcal{H} \twoheadrightarrow \underbrace{\mathrm{coker}(F)}_{\left\{\psi \in \mathcal{H} \,\middle|\, \forall_\phi \, \langle \psi | F | \phi \rangle = 0\right\}}$$

on the single-electron/positron Hilbert space:

single electron Hilbert space

$$\begin{tikzcd}
\text{electron states in dressed vacuum} \quad \ker(F) \arrow[r, hook] & \mathcal{H} \arrow[drr, "F"] & & \mathcal{H} \\
& \oplus & & \oplus \\
& \mathcal{H} \arrow[urr, "F^*"'] & & \mathcal{H} \arrow[r, two heads] & \mathrm{coker}(F) \quad \text{positron states in dressed vacuum}
\end{tikzcd}$$

dressed vacuum Fredholm operator

single positron Hilbert space

total charge in dressed vacuum		number of electrons in dressed vacuum state		number of positrons in dressed vacuum state
$\mathrm{ind}(F)$	$=$	$\dim\big(\ker(F)\big)$	$-$	$\dim\big(\mathrm{coker}(F)\big)$
	$=$	$\dim\big(\mathrm{coker}(F^*)\big)$	$-$	$\dim\big(\ker(F^*)\big)$

Quantum symmetries.

On these dressed vacua of electron/positron states
the following *CPT-twisted projective group*

even projective unitary group

grading involution

complex conjugation

$$\frac{\mathrm{U}(\mathcal{H}) \times \mathrm{U}(\mathcal{H})}{\mathrm{U}(1)} \rtimes \Big(\underbrace{\mathbb{Z}_2}_{\{\mathrm{e}, P\}} \times \underbrace{\mathbb{Z}_2}_{\{\mathrm{e}, T\}} \Big)$$

group of quantum symmetries

$$C := PT\,, \quad P \cdot \big[U_+, U_- \big] := \big[U_-, U_+ \big] \cdot P\,, \qquad T \cdot \big[U_+, U_- \big] := \big[\overline{U}_+, \overline{U}_- \big] \cdot T$$

Quantum symmetries.

On these dressed vacua of electron/positron states
the following *CPT-twisted projective group*

$$\underbrace{\overbrace{\frac{\mathrm{U}(\mathcal{H})\times\mathrm{U}(\mathcal{H})}{\mathrm{U}(1)}}^{\text{even projective unitary group}} \rtimes \Big(\underbrace{\overset{\text{grading involution}}{\mathbb{Z}_2}}_{\{\mathrm{e},P\}} \times \underbrace{\overset{\text{complex conjugation}}{\mathbb{Z}_2}}_{\{\mathrm{e},T\}} \Big)}_{\text{group of quantum symmetries}}$$

$$C := PT\,, \quad P\cdot\big[U_+\,,U_-\big] := \big[U_-\,,U_+\big]\cdot P\,, \qquad T\cdot\big[U_+\,,U_-\big] := \big[\overline{U}_+\,,\overline{U}_-\big]\cdot T$$

naturally acts by conjugation:

$$\begin{array}{rcccl} [U_+,U_-] & : & F & \longmapsto & U_+^{-1}\circ F\circ U_- \\ C\cdot[U_+,U_-] & : & F & \longmapsto & U_-^{-1}\circ F^t\circ U_+ \\ P\cdot[U_+,U_-] & : & F & \longmapsto & U_-^{-1}\circ F^*\circ U_+ \\ T\cdot[U_+,U_-] & : & F & \longmapsto & U_+^{-1}\circ \overline{F}\circ U_- \end{array}$$

Twisted equivariant KR-theory – As a single diagram of smooth groupoids.

Homotopy classes of quantum-symmetry equivariant families
of such self-adjoint odd Fredholm operators
constitute *twisted equivariant* KR-*cohomology*:

$$
\mathrm{KR}^{\tau}_{G}(\mathrm{X}) \;:=\; \left\{
\begin{tikzcd}
 & \overset{\text{space of self-adjoint odd Fredholm operators}}{\mathrm{Fred}^0_{\mathbb{C}}} /\!\!/ \overset{\text{group of quantum symmetries}}{\left(\frac{\mathrm{U}(\mathcal{H}) \times \mathrm{U}(\mathcal{H})}{\mathrm{U}(1)} \rtimes \{\mathrm{e}, P\} \times \{\mathrm{e}, T\} \right)} \arrow[d] \\
\overset{\text{orbi-orientifold}}{\mathrm{X} /\!\!/ G} \arrow[ur, dashed, "\text{equivariant family of Fredholm operators}", "\text{cocycle in TE-K-theory}"'] \arrow[r, "\text{twist } \tau"'] & {}
\end{tikzcd}
\right\} \Big/ \sim_{\mathrm{htpy}}
$$

Twisted equivariant KR-theory – As a single diagram of smooth groupoids.

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\mathrm{KR}^{\tau}_{G}(\mathrm{X}) \;:=\; \left\{ \begin{tikzcd}[column sep=large, row sep=huge]
 & \overset{\text{space of self-adjoint odd Fredholm operators}}{\mathrm{Fred}^{0}_{\mathbb{C}}} /\!\!/ \overset{\text{group of quantum symmetries}}{\left(\frac{\mathrm{U}(\mathcal{H})\times\mathrm{U}(\mathcal{H})}{\mathrm{U}(1)} \rtimes \{\mathrm{e}, P\} \times \{\mathrm{e}, T\} \right)} \arrow[d, "\substack{\text{universal bundle of} \\ \text{self-adjoint odd Fredholm operators} \\ \text{over moduli stack of quantum symmetries}}"] \\
\underset{\substack{\text{orbi-} \\ \text{orientifold}}}{\mathrm{X} /\!\!/ G} \arrow[ur, dashed, "\substack{\text{equivariant family of} \\ \text{Fredholm operators}}", "\text{cocycle in TE-K-theory}"'] \arrow[r, "\text{twist } \tau"'] & \mathbf{B}\left(\frac{\mathrm{U}(\mathcal{H})\times\mathrm{U}(\mathcal{H})}{\mathrm{U}(1)} \rtimes \{\mathrm{e}, P\} \times \{\mathrm{e}, T\} \right)
\end{tikzcd} \right\} \Big/ \sim_{\mathrm{htpy}}
$$

Twisted equivariant KR-theory – As a single diagram of smooth groupoids.

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$$
\mathrm{KR}^{\tau}_{G}(\mathrm{X}) \;:=\; \left\{
\begin{tikzcd}[column sep=large, row sep=huge]
 & & \overset{\text{space of self-adjoint odd Fredholm operators}}{\mathrm{Fred}^{0}_{\mathbb{C}}} \sslash \Big(\overset{\text{group of quantum symmetries}}{\frac{\mathrm{U}(\mathcal{H}) \times \mathrm{U}(\mathcal{H})}{\mathrm{U}(1)} \rtimes \{\mathrm{e}, P\} \times \{\mathrm{e}, T\}} \Big) \arrow[d, "\substack{\text{universal bundle of}\\ \text{self-adjoint odd Fredholm operators}\\ \text{over moduli stack of quantum symmetries}}"] \\
\overset{\substack{\text{orbi-}\\ \text{orientifold}}}{\mathrm{X} \sslash G} \arrow[urr, dashed, "\substack{\text{equivariant family of}\\ \text{Fredholm operators}}", "\text{cocycle in TE-K-theory}"'] \arrow[rr, "\text{twist } \tau"'] \arrow[dr, "\substack{\text{underlying}\\ \text{CPT symmetry}}"'] & & \mathbf{B}\Big(\frac{\mathrm{U}(\mathcal{H}) \times \mathrm{U}(\mathcal{H})}{\mathrm{U}(1)} \rtimes \{\mathrm{e}, P\} \times \{\mathrm{e}, T\} \Big) \arrow[dl, "C = PT"] \\
 & \mathbf{B}\big(\{\mathrm{e}, C\} \times \{\mathrm{e}, T\} \big) &
\end{tikzcd}
\right\} \Big/ \sim_{\mathrm{htpy}}
$$

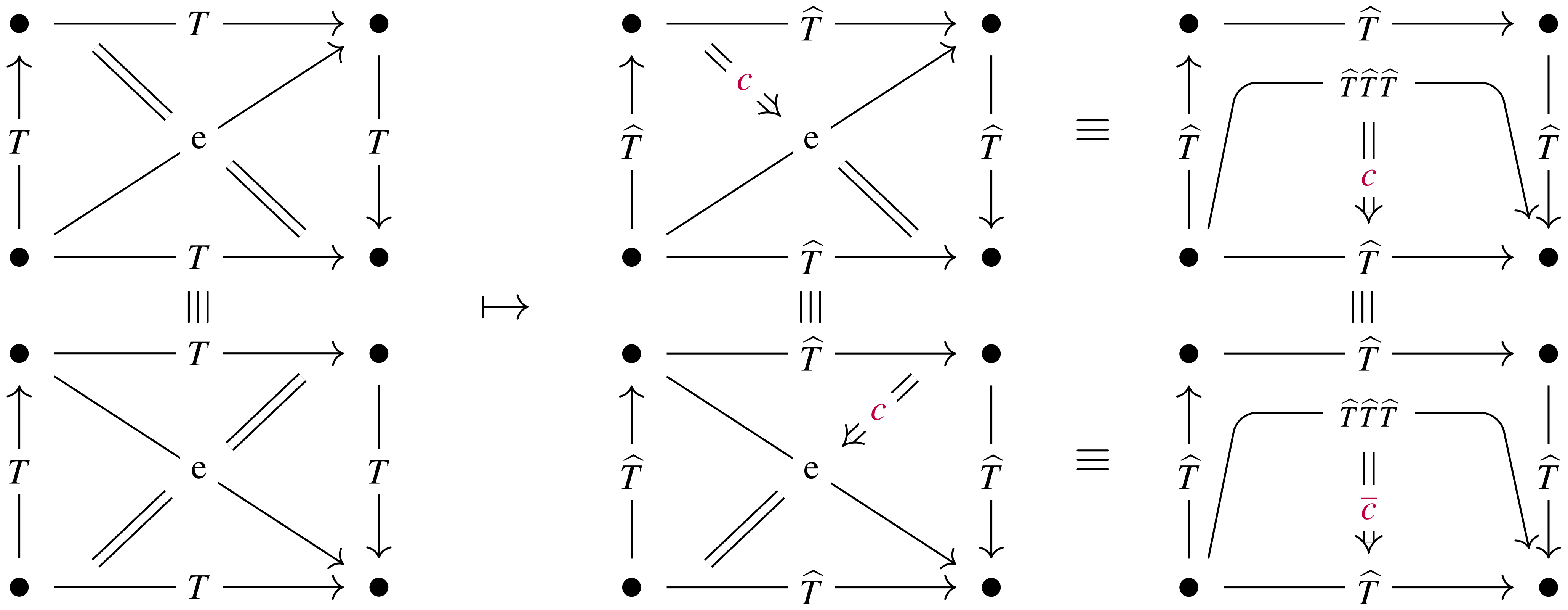
CPT Quantum symmetries.

$$
\begin{tikzcd}
\mathbf{B}\big(\{\mathrm{e}, T\}\big) \arrow[rr, "{\substack{\text{pure quantum T-symmetry} \\ T \longmapsto \widehat{T}}}"] \arrow[dr, hook] & & \mathbf{B}\left(\dfrac{\mathrm{U}(\mathcal{H}) \times \mathrm{U}(\mathcal{H})}{\mathrm{U}(1)} \rtimes \{\mathrm{e}, T\}\right) \arrow[r] \arrow[dl] & \mathbf{B}\big(\mathbf{B}\mathrm{U}(1) \rtimes \{\mathrm{e}, T\}\big) \\
& \mathbf{B}\big(\{\mathrm{e}, P\} \times \{\mathrm{e}, T\}\big) & &
\end{tikzcd}
$$

Let’s use the previous machinery to compute the possible quantum T-symmetries...

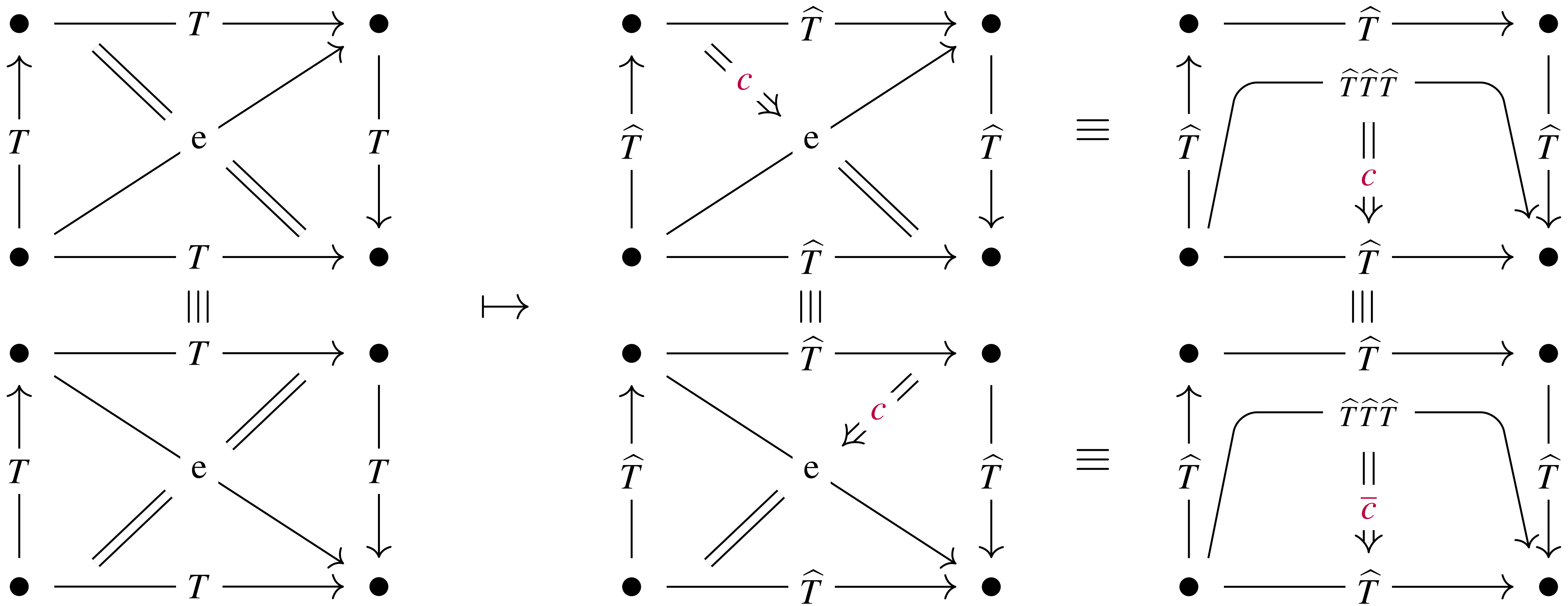
CPT Quantum symmetries.

$$
\begin{tikzcd}
\mathbf{B}\big(\{\mathrm{e}, T\}\big) \arrow[rr, "\substack{\text{pure quantum T-symmetry} \\ T \longmapsto \widehat{T}}"] \arrow[dr, hook] & & \mathbf{B}\left(\dfrac{\mathrm{U}(\mathcal{H}) \times \mathrm{U}(\mathcal{H})}{\mathrm{U}(1)} \rtimes \{\mathrm{e}, T\} \right) \arrow[r] \arrow[dl] & \mathbf{B}\big(\mathbf{B}\mathrm{U}(1) \rtimes \{\mathrm{e}, T\}\big) \\
& \mathbf{B}\big(\{\mathrm{e}, P\} \times \{\mathrm{e}, T\}\big) & &
\end{tikzcd}
$$



CPT Quantum symmetries.

$$
\begin{tikzcd}
\mathbf{B}\big(\{\mathrm{e}, T\}\big) \arrow[rr, "T \longmapsto \widehat{T}", "\text{pure quantum T-symmetry}"'] \arrow[dr, hook] & & \mathbf{B}\left(\dfrac{\mathrm{U}(\mathcal{H}) \times \mathrm{U}(\mathcal{H})}{\mathrm{U}(1)} \rtimes \{\mathrm{e}, T\} \right) \arrow[r] \arrow[dl] & \mathbf{B}\big(\mathbf{B}\mathrm{U}(1) \rtimes \{\mathrm{e}, T\}\big) \\
& \mathbf{B}\big(\{\mathrm{e}, P\} \times \{\mathrm{e}, T\}\big) & &
\end{tikzcd}
$$



So $\overline{c} = c$ and hence there are two choices for quantum T-symmetry, up to homotopy:

$$\widehat{T}^2 = \pm 1 \quad \text{and similarly} \quad \widehat{C}^2 = \pm 1\,.$$

Example – Orientifold KR-theory

Let I be *I*nversion action on 2-torus $\widetilde{\mathbb{T}}^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$ and trivial action on observables

$$
\begin{array}{ccc}
\mathbb{T}^2 & \xrightarrow{\quad I \quad} & \mathbb{T}^2 \\
k & \longmapsto & -k\,,
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathrm{Fred}^0_{\mathbb{C}} & \xrightarrow{\quad I \quad} & \mathrm{Fred}^0_{\mathbb{C}} \\
F & \longmapsto & F\,.
\end{array}
$$

If T acts as I on $\mathbb{T}^2$, then $\mathrm{KR}^{\widehat{T}^2 = +1}$ is *Atiyah's Real K-theory* aka *orienti-fold* K-theory:

$$
\mathrm{KR}\left(\widetilde{\mathbb{T}}^{0,2}\right) \;\simeq\; \left\{
\begin{tikzcd}
 & & & & \mathrm{Fred}^0_{\mathbb{C}} /\!\!/ \left(\mathrm{U}(\mathcal{H}) \rtimes \{\mathrm{e}, T\}\right) \arrow[d] \\
\mathbb{T}^2 /\!\!/ \{\mathrm{e}, I\} \arrow[urrrr, dashed] \arrow[rrrr, "\widehat{T}^2 = +1"] \arrow[drr, "\text{inversion of space}", "I \mapsto T"'] & & & & \mathbf{B}\left(\mathrm{U}(\mathcal{H}) \rtimes \{\mathrm{e}, T\}\right) \arrow[dll, "\text{complex conj. of observables}"'] \\
 & & \mathbf{B}\{\mathrm{e}, T\} \arrow[u, phantom, "\text{combined with}"] & &
\end{tikzcd}
\right\}_{/\sim_{\mathrm{htpy}}}
$$

Example – Orientifold KR-theory

Let I be *I*nversion action on 2-torus $\widetilde{\mathbb{T}}^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$ and trivial action on observables

$$
\begin{array}{ccc}
\mathbb{T}^2 & \xrightarrow{\;\;I\;\;} & \mathbb{T}^2 \\
k & \longmapsto & -k\,,
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathrm{Fred}^0_{\mathbb{C}} & \xrightarrow{\;\;I\;\;} & \mathrm{Fred}^0_{\mathbb{C}} \\
F & \longmapsto & F\,.
\end{array}
$$

If T acts as I on $\mathbb{T}^2$, then $\mathrm{KR}^{\widehat{T}^2 = +1}$ is *Atiyah's Real K-theory* aka *orienti-fold* K-theory:

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 & & \mathrm{Fred}^0_{\mathbb{C}} /\!\!/ \big(\mathrm{U}(\mathcal{H}) \rtimes \{\mathrm{e}, T\}\big) \arrow[d] \\
\mathbb{T}^2 /\!\!/ \{\mathrm{e}, I\} \arrow[urr, dashed] \arrow[rr, "\widehat{T}^2 = +1"] \arrow[dr, "I \mapsto T"', "\text{inversion of space}"] & & \mathbf{B}\big(\mathrm{U}(\mathcal{H}) \rtimes \{\mathrm{e}, T\}\big) \arrow[dl, "\text{complex conj. of observables}"'] \\
 & \mathbf{B}\{\mathrm{e}, T\} \arrow[u, phantom, "\text{combined with}"] &
\end{tikzcd}
\right\}\Big/\!\sim_{\mathrm{htpy}}
$$

But what happens on I-fixed loci i.e. on "orientifolds" ? $\longrightarrow$

CPT Quantum symmetries – 10 global choices. (following [FM12, Prop. 6.4])

Equivariance group	$G =$	$\{\mathrm{e}\}$	$\{\mathrm{e}, P\}$	$\{\mathrm{e}, T\}$		$\{\mathrm{e}, C\}$		$\{\mathrm{e}, T\} \times \{\mathrm{e}, C\}$			
Realization as quantum symmetry $\tau :$	$\widehat{T}^2 =$			$+1$	-1			$+1$	-1	-1	$+1$
	$\widehat{C}^2 =$					$+1$	-1	$+1$	$+1$	-1	-1
Maximal induced Clifford action anticommuting with all G-invariant odd Fredholm operators	$E_{-3} =$								$\mathrm{i}\widehat{T}\widehat{C}\beta$		
	$E_{-2} =$					$\mathrm{i}\widehat{C}\beta$			$\mathrm{i}\widehat{C}\beta$		
	$E_{-1} =$		$\widehat{P}\beta$			$\widehat{C}\beta$		$\widehat{C}\beta$	$\widehat{C}\beta$		
	$E_{+0} =$	β	β	β	$\begin{pmatrix} \beta & 0 \\ 0 & -\beta \end{pmatrix}$	β	β	β	β	β	β
	$E_{+1} =$				$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$		$\widehat{C}\beta$			$\widehat{C}\beta$	$\widehat{C}\beta$
	$E_{+2} =$				$\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$		$\mathrm{i}\widehat{C}\beta$			$\mathrm{i}\widehat{C}\beta$	
	$E_{+3} =$				$\begin{pmatrix} 0 & -\widehat{T} \\ \widehat{T} & 0 \end{pmatrix}$					$\mathrm{i}\widehat{T}\widehat{C}\beta$	
	$E_{+4} =$				$\begin{pmatrix} 0 & \mathrm{i}\widehat{T} \\ \mathrm{i}\widehat{T} & 0 \end{pmatrix}$						
τ-twisted G-equivariant KR-theory of fixed loci	$\mathrm{KR}^\tau =$	KU^0	KU^1	KO^0	KO^4	KO^2	KO^6	KO^1	KO^3	KO^5	KO^7

$$
\left\{
\begin{array}{lll}
\text{bounded opers.} & \widehat{F} : \mathcal{H}^2 \xrightarrow[\mathbb{K}-\text{linear}]{\text{bounded}} \mathcal{H}^2 \\
\text{self-adjoint} & \widehat{F}^* = \widehat{F} := F + F^* \\
\text{Fredholm} & \dim\big(\ker(\widehat{F})\big) < \infty
\end{array}
\;\middle|\;
\begin{array}{l}
\text{graded comm.} \\
E_i \circ \widehat{F} = -\widehat{F} \circ E_i
\end{array}
\;\text{with}\;
\begin{array}{ll}
\text{bounded oper.} & E_0, \cdots, E_p : \mathcal{H}^2 \xrightarrow[\mathbb{K}-\text{linear}]{\text{bounded}} \mathcal{H}^2 \\
\text{(anti-)self-adjoint} & (E_i)^* = \mathrm{sgn}_i \cdot E_i \\
\text{Clifford gen.} & E_i \circ E_j + E_j \circ E_i = 2\mathrm{sgn}_i \cdot \delta_{ij}
\end{array}
\right\}
$$

$$
=: \mathrm{Fred}_{\mathbb{C}}^{-p}
$$

[Karoubi 70]: $\left\{ X \xrightarrow[\text{cnts}]{} \mathrm{Fred}^p_{\mathbb{K}} \right\} \big/_{\sim \text{htpy}} = \left\{ \begin{array}{lll} \mathrm{KU}^p(X) = \mathrm{KU}^{p+2}(X) & | & \mathbb{K} = \mathbb{C} \\ \mathrm{KO}^p(X) = \mathrm{KO}^{p+8}(X) & | & \mathbb{K} = \mathbb{R} \end{array} \right.$

Maximal induced Clifford action anticommuting with all G-invariant odd Fredholm operators	$E_{-3} =$								$\mathrm{i}\widehat{T}\widehat{C}\beta$		
	$E_{-2} =$					$\mathrm{i}\widehat{C}\beta$			$\mathrm{i}\widehat{C}\beta$		
	$E_{-1} =$		$\widehat{P}\beta$			$\widehat{C}\beta$		$\widehat{C}\beta$	$\widehat{C}\beta$		
	$E_{+0} =$	β	β	β	$\begin{pmatrix} \beta & 0 \\ 0 & -\beta \end{pmatrix}$	β	β	β	β	β	β
	$E_{+1} =$				$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$		$\widehat{C}\beta$			$\widehat{C}\beta$	$\widehat{C}\beta$
	$E_{+2} =$				$\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$		$\mathrm{i}\widehat{C}\beta$			$\mathrm{i}\widehat{C}\beta$	
	$E_{+3} =$				$\begin{pmatrix} 0 & -\widehat{T} \\ \widehat{T} & 0 \end{pmatrix}$					$\mathrm{i}\widehat{T}\widehat{C}\beta$	
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τ-twisted G-equivariant KR-theory of fixed loci	$\mathrm{KR}^\tau =$	KU^0	KU^1	KO^0	KO^4	KO^2	KO^6	KO^1	KO^3	KO^5	KO^7

Example – TI-equivariant KR-theory is KO^0-theory.

The combination $T \cdot I$ acts trivially on the domain space and by complex conjugation on observables.

Hence $(T \cdot I)$-equivariant $(\widehat{T}^2 = +1)$-twisted KR-theory is KO^0-theory:

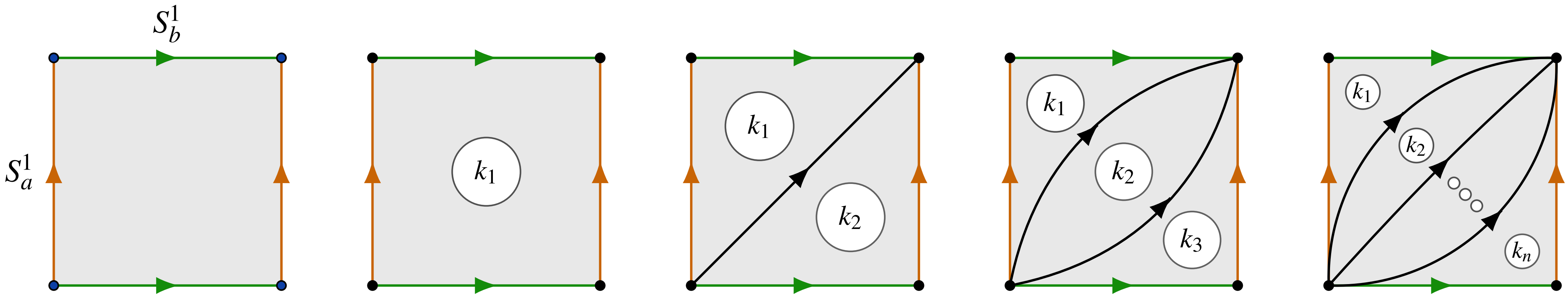
$$
\mathrm{KO}^0(X) \;\simeq\; \left\{
\begin{tikzcd}
 & & \mathrm{Fred}^0_{\mathbb{C}} /\!\!/ \big(\mathrm{U}(\mathcal{H}) \rtimes \{\mathrm{e}, T\}\big) \arrow[d] \\
X \times * /\!\!/ \{\mathrm{e}, TI\} \arrow[urr, dashed] \arrow[rr, "\widehat{T}^2 = +1"] \arrow[dr, "TI \mapsto T"', "\substack{\text{no action}\\ \text{on space}}"] & & \mathbf{B}\big(\mathrm{U}(\mathcal{H}) \rtimes \{\mathrm{e}, T\}\big) \arrow[dl, "\substack{\text{complex conj.}\\ \text{of observables}}"'] \\
 & \mathbf{B}\{\mathrm{e}, T\} \arrow[u, phantom, "\substack{\text{combined}\\ \text{with}}"] &
\end{tikzcd}
\right\}_{\big/ \sim_{\mathrm{htpy}}}
$$

$n =$	0	1	2	3	4	5	6	7	8	9	$\cdots$
$\mathrm{KO}^0(S^n_*) =$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\cdots$

Example – TI-equivariant KR-theory of punctured torus.

So the TI-equivariant $(\widehat{T}^2 = +1)$-twisted KR-theory of the N-punctured torus is

$$
\begin{aligned}
&\mathrm{KR}^{(\widehat{T}^2 = +1)}\left(\widetilde{\mathbb{T}}^2 \setminus \{k_1, \cdots, k_N\}\right) \\
&\simeq \mathrm{KO}^0\left(\widetilde{\mathbb{T}}^2 \setminus \{k_1, \cdots, k_N\}\right) \\
&\simeq \mathrm{KO}^0\left(\bigvee_{1+N} S^1_*\right) \qquad (N \geq 1) \\
&\simeq \bigoplus_{1+N} \mathbb{Z}_2
\end{aligned}
$$



$\widetilde{\mathbb{T}}^2 \underset{\text{stable}}{\simeq} S^1_a \vee S^1_b \vee S^2_{\text{bulk}}$

$\widetilde{\mathbb{T}}^2 \setminus \{k_1\} \underset{\text{htpy}}{\simeq} S^1_a \vee S^1_b$

$\widetilde{\mathbb{T}}^2 \setminus \{k_1, k_2\} \underset{\text{htpy}}{\simeq} S^1_a \vee S^1_b \vee S^1$

$\widetilde{\mathbb{T}}^2 \setminus \{k_1, k_2, k_3\} \underset{\text{htpy}}{\simeq} S^1_a \vee S^1_b \vee S^1 \vee S^1$

$\widetilde{\mathbb{T}}^2 \setminus \{k_1, \cdots, k_n\} \underset{\text{htpy}}{\simeq} \bigvee_{1+n} S^1$

The B-field twist.

Besides these CPT-quantum symmetries,
K-theory generically admits the famous *twisting by a B-field*:

The homotopy fiber sequence of 2-stacks discussed before

$$\mathbf{B}\mathrm{U}(\mathcal{H}) \longrightarrow \mathbf{B}\big(\mathrm{U}(\mathcal{H})/\mathrm{U}(1)\big) \xrightarrow[\text{}]{\mathrm{DD}} \mathbf{B}^2\mathrm{U}(1)$$

universal Dixmier-Douady class

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induces a surjection of equivalence classes of equivariant higher bundles

$$\underbrace{\pi_0 \; \mathrm{Map}\Big(\widehat{\mathrm{X}/\!\!/G}, \mathbf{B}\big(\mathrm{U}(\mathcal{H})/\mathrm{U}(1)\big)\Big)}_{\text{equivariant projective bundles}} \xrightarrow{\;\mathrm{DD}_*\;} \!\!\!\!\twoheadrightarrow \underbrace{\pi_0 \,\mathrm{Map}\Big(\widehat{\mathrm{X}/\!\!/G}, \mathbf{B}^2\mathrm{U}(1)\Big)}_{\text{equivariant bundle gerbes}}$$

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which has a natural section:

$$\overbrace{\underbrace{\pi_0\mathrm{Map}\big(\widehat{\mathrm{X}/\!\!/G},\,\mathbf{B}^2\mathrm{U}(1)\big)}_{\text{equivariant bundle gerbes}}}^{\text{“stable twists”}} \hookrightarrow \underbrace{\pi_0\,\mathrm{Map}\bigg(\widehat{\mathrm{X}/\!\!/G},\,\mathbf{B}\Big(\frac{\mathrm{U}(\mathcal{H})\times\mathrm{U}(\mathcal{H})}{\mathrm{U}(1)} \rtimes \big(\{\mathrm{e},C\}\times\{\mathrm{e},P\}\big)\Big)\bigg)}_{\text{full quantum-symmetry twists}}.$$

The B-field twist – Inner local systems.

On fixed loci (orbi-singularities)

$$\mathrm{X}/\!\!/G \;\simeq\; \mathrm{X} \times * /\!\!/ G \;=\; \mathrm{X} \times \mathbf{B}G$$

the B-field twist induces *secondary* twists by “inner local systems”:

$$\underbrace{\mathrm{Map}\big(\mathrm{X} \times * /\!\!/ G, \mathbf{B}^2\mathrm{U}(1)\big)}^{\text{stable twists over fixed locus}} \simeq \mathrm{Map}\big(\mathrm{X} \times \mathbf{B}G, \mathbf{B}^2\mathrm{U}(1)\big)$$

$$\simeq \; \mathrm{Map}\big(\mathrm{X}, \mathrm{Map}(\mathbf{B}G, \mathbf{B}^2\mathrm{U}(1))\big)$$

The B-field twist – Inner local systems.

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the B-field twist induces *secondary* twists by “inner local systems”:

$$\begin{aligned}
\overbrace{\mathrm{Map}\big(\mathrm{X} \times * /\!\!/ G,\, \mathbf{B}^2\mathrm{U}(1)\big)}^{\text{stable twists over fixed locus}} &\simeq \mathrm{Map}\big(\mathrm{X} \times \mathbf{B}G,\, \mathbf{B}^2\mathrm{U}(1)\big) \\
&\simeq \mathrm{Map}\big(\mathrm{X},\, \mathrm{Map}(\mathbf{B}G,\, \mathbf{B}^2\mathrm{U}(1))\big) \\
&\simeq \mathrm{Map}\big(\mathrm{X},\, \mathbf{B}G^* \times \mathbf{B}^2\mathrm{U}(1)\big)
\end{aligned}$$

Here we are assuming $G \underset{\mathrm{fin}}{\subset} \mathrm{SU}(2)$ so that $H^2_{\mathrm{Grp}}\big(G,\, \mathrm{U}(1)\big) = 0$.

And $G^* := \mathrm{Hom}\big(G, \mathrm{U}(1)\big)$ denotes the Pontrjagin-dual group.

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&\simeq \mathrm{Map}\big(\mathrm{X},\, \mathrm{Map}(\mathbf{B}G,\, \mathbf{B}^2\mathrm{U}(1))\big) \\
&\simeq \mathrm{Map}\big(\mathrm{X},\, \mathbf{B}G^* \times \mathbf{B}^2\mathrm{U}(1)\big) \\
&\simeq \underbrace{\mathrm{Map}\big(\mathrm{X},\, \mathbf{B}G^*\big)}_{\text{inner local systems}} \times \underbrace{\mathrm{Map}\big(\mathrm{X},\, \mathbf{B}^2\mathrm{U}(1)\big)}_{\text{bundle gerbes}}
\end{aligned}$$

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The B-field twist – Inner local systems – The diagrammatics.

Hence the

inner local system-twisted KU-*cohomology*
of a G-orbi-singularity of shape X

arises as follows:

$$
\mathrm{KU}_G^{n+[\omega_1]}(X) \;=\; \left\{
\begin{tikzcd}
 & \mathrm{Fred}^n_{\mathbb{C}} /\!\!/ \mathrm{PU}(\mathcal{H}) \arrow[d] \\
X \times * /\!\!/ G \arrow[ur, dashed, "\text{cocycle}"] \arrow[r, "\tau", "\text{inner local system twist}"'] & \mathbf{B}\mathrm{PU}(\mathcal{H})
\end{tikzcd}
\right\}_{\Big/ \sim_{\mathrm{htpy}}}
$$

The B-field twist – Inner local systems – The diagrammatics.

Hence the

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$$\mathrm{KU}_G^{n+[\omega_1]}\big(X\big) \;=\; \left\{ \begin{tikzcd} & \mathrm{Map}\big(\mathbf{B}G, \mathrm{Fred}^n_{\mathbb{C}} /\!\!/ \mathrm{PU}(\mathcal{H})\big) \arrow[d] \\ X \arrow[ur, dashed, "\text{cocycle}"] \arrow[r, "\widetilde{\tau}", "\text{adjunct twist}"'] & \mathrm{Map}\big(\mathbf{B}G, \mathbf{B}\mathrm{PU}(\mathcal{H})\big) \end{tikzcd} \right\}\Big/\sim_{\mathrm{htpy}}$$

The B-field twist – Inner local systems – The diagrammatics.

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 & & \mathrm{Map}\big(\mathbf{B}G,\, \mathrm{Fred}^n_{\mathbb{C}} /\!\!/ \mathrm{PU}(\mathcal{H})\big) \arrow[d] \\
X \arrow[urr, dashed, "\text{cocycle}"] \arrow[r, "\omega_1"', "\text{inner local system}"'] & \mathbf{B}G^* \arrow[r, "\substack{\text{automorphisms of} \\ \text{univ. stable equiv. twist}}"'] & \mathrm{Map}\big(\mathbf{B}G,\, \mathbf{B}\mathrm{PU}(\mathcal{H})\big)
\end{tikzcd} \right\}_{\big/ \sim_{\mathrm{htpy}}}
$$

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arises as follows:

$$
\mathrm{KU}_G^{n+[\omega_1]}(X) \;=\; \left\{
\begin{tikzcd}
 & \big(\mathrm{Fred}^0_{\mathbb{C}}\big)^G /\!\!/ G^* \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & \mathrm{Map}\big(\mathbf{B}G, \mathrm{Fred}^n_{\mathbb{C}} /\!\!/ \mathrm{PU}(\mathcal{H})\big) \arrow[d] \\
X \arrow[r, "\omega_1", "\text{inner local system}"'] \arrow[ur, dashed, "\text{cocycle}"] & \mathbf{B}G^* \arrow[r, "\substack{\text{automorphisms of}\\ \text{univ. stable equiv. twist}}"'] & \mathrm{Map}\big(\mathbf{B}G, \mathbf{B}\mathrm{PU}(\mathcal{H})\big)
\end{tikzcd}
\right\} \Big/ {\sim_{\mathrm{htpy}}}
$$

The B-field twist – Inner local systems – The proof.

For the proof we consider the following diagram [SS22-Bun, Ex. 4.1.56][SS22, §3]:

$$
\begin{tikzcd}
\mathbf{B}G \arrow[rr, bend left, "\mathrm{stable}_0"] \arrow[rr, bend right, "\mathrm{stable}_0"'] \arrow[rr, phantom, "\Downarrow \rho \in G^*"] & & \mathbf{B}\mathrm{PU}(\mathcal{H})
\end{tikzcd}
$$

$$
\begin{tikzcd}
\bullet \arrow[d, "g"] & & \underset{\substack{[\rho_i] \in \\ \mathrm{Irr}(G)}}{\bigoplus} \rho_i \otimes \ell^2(\mathbb{C}) \arrow[rr, "v \mapsto 1_\rho \otimes v"] \arrow[d, "\underset{[\rho_i]}{\oplus}(\rho_i(g)\otimes \mathrm{id})"] & & \underset{\substack{[\rho_i] \in \\ \mathrm{Irr}(G)}}{\bigoplus} \rho_i \otimes \ell^2(\mathbb{C}) \arrow[d, "\underset{[\rho_i]}{\oplus}(\rho_i(g)\otimes \mathrm{id})"] \\
\bullet & \longmapsto & \underset{\substack{[\rho_i] \in \\ \mathrm{Irr}(G)}}{\bigoplus} \rho_i \otimes \ell^2(\mathbb{C}) \arrow[rr, "v \mapsto 1_\rho \otimes v"'] \arrow[urr, Rightarrow, "\rho(g)(1_\rho)"] & & \underset{\substack{[\rho_i] \in \\ \mathrm{Irr}(G)}}{\bigoplus} \rho_i \otimes \ell^2(\mathbb{C})
\end{tikzcd}
$$

The B-field twist – Inner local systems – The proof.

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$$
\begin{tikzcd}
\mathbf{B}G \arrow[rr, bend left, "\mathrm{stable}_0"] \arrow[rr, bend right, "\mathrm{stable}_0"'] \arrow[rr, phantom, "\Downarrow \rho \in G^*"] & & \mathbf{B}\mathrm{PU}(\mathcal{H})
\end{tikzcd}
$$

stable G-representation

$$
\begin{tikzcd}[column sep=small]
\bigoplus_i \rho_i \otimes \ell^2(\mathbb{C}) \arrow[rr, "v \mapsto 1_\rho \otimes v"] \arrow[dr, "F", "\text{Fredholm operator}"'] \arrow[dd, "\bigoplus_i \rho_i \otimes \ell^2(\mathbb{C})"] & & \bigoplus_i \rho_i \otimes \ell^2(\mathbb{C}) \arrow[dr, "{[\rho] \cdot F}"] \arrow[d, phantom, "\text{action of group character on equivariant Fredholm operator}"] & \\
& \bigoplus_i \rho_i \otimes \ell^2(\mathbb{C}) \arrow[rr, "v \mapsto 1_\rho \otimes v", "\text{tensoring with unit of group character}"'] \arrow[dd, "\bigoplus_s \rho_i(g) \otimes \mathrm{id}"] \arrow[ddrr, phantom, "\text{projective intertwining action of group character}"] & & \bigoplus_i \rho_i \otimes \ell^2(\mathbb{C}) \arrow[dd, "\bigoplus_s \rho_i(g) \otimes \mathrm{id}"] \\
\bigoplus_i \rho_i \otimes \ell^2(\mathbb{C}) \arrow[dr, "F"'] \arrow[ur, phantom, "\text{equivariance of Fredholm operator}"] & & & \\
& \bigoplus_i \rho_i \otimes \ell^2(\mathbb{C}) \arrow[rr, "v \mapsto 1_\rho \otimes v"'] & & \bigoplus_i \rho_i \otimes \ell^2(\mathbb{C})
\end{tikzcd}
$$

The B-field twist – Inner local systems – Chern character.

One aspect of these twistings becomes transparent under the *Chern character*:

$$\underbrace{\mathrm{KU}^0(\mathrm{X})}_{\text{complex K-theory}} \xrightarrow[\text{Chern character}]{} \mathrm{KU}^0(\mathrm{X};\mathbb{C}) \;\simeq\; \underbrace{\bigoplus_{d\in\mathbb{N}} H^{2d}\Big(\Omega^{\bullet}_{\mathrm{dR}}(\mathrm{X};\mathbb{C}),\mathrm{d}\Big)}_{\text{periodic de Rham cohomology}}$$

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For twist by B-field $\widehat{B}_2$ there is a closed differential 3-form H_3 such that:

$$\underset{\text{plain B-field -twisted K-theory}}{\mathrm{KU}^{n+\widehat{B}_2}(\mathrm{X})} \xrightarrow[\text{Chern character}]{\text{twisted}} \mathrm{KU}^{\widehat{B}_2}(\mathrm{X};\mathbb{C}) \;\simeq\; \underbrace{\bigoplus_{d\in\mathbb{Z}} H^{n+2d}\Big(\Omega^{\bullet}_{\mathrm{dR}}\big(\mathrm{X};\mathbb{C}\big), \mathrm{d}+H_3\wedge\Big)}_{\text{3-twisted periodic de Rham cohomology}}$$

The B-field twist – Inner local systems – Chern character.

One aspect of these twistings becomes transparent under the Chern character:

$$\underset{\text{complex K-theory}}{\mathrm{KU}^0(\mathrm{X})} \xrightarrow[\text{Chern character}]{} \mathrm{KU}^0(\mathrm{X};\mathbb{C}) \simeq \overbrace{\bigoplus_{d\in\mathbb{N}} H^{2d}\Big(\Omega^\bullet_{\mathrm{dR}}(\mathrm{X};\mathbb{C}),\mathrm{d}\Big)}^{\text{periodic de Rham cohomology}}$$

For twist by B-field $\widehat{B}_2$ there is a closed differential 3-form H_3 such that:

plain B-field -twisted K-theory

3-twisted periodic de Rham cohomology

$$\mathrm{KU}^{n+\widehat{B}_2}(\mathrm{X}) \xrightarrow[\text{Chern character}]{\text{twisted}} \mathrm{KU}^{\widehat{B}_2}(\mathrm{X};\mathbb{C}) \simeq \bigoplus_{d\in\mathbb{Z}} H^{n+2d}\Big(\Omega^\bullet_{\mathrm{dR}}(\mathrm{X};\mathbb{C}),\mathrm{d}+H_3\wedge\Big)$$

For twist by inner C_κ-local system, there is closed 1-form ω_1 with holon. in $C_\kappa \subset \mathrm{U}(1)$ such that:

inner local system -twisted K-theory of A-type singularity

1-twisted periodic de Rham cohomology

$$\mathrm{KU}^{n+[\omega_1]}_{C_\kappa}(\mathrm{X}) \xrightarrow[\text{Chern character}]{\text{twisted equivariant}} \bigoplus_{\substack{d\in\mathbb{Z}\\ 1\leq r\leq\kappa}} H^{n+2d}\Big(\Omega^\bullet_{\mathrm{dR}}(\mathrm{X};\mathbb{C}),\mathrm{d}+r\cdot\omega_1\wedge\Big)$$

The B-field twist – Inner local systems – Chern character.

One aspect of these twistings becomes transparent under the Chern character:

This is the hidden 1-twisting in TED-K – that we will next relate to anyons. $\longrightarrow$

$$
\underset{\substack{\text{of A-type singularity}}}{\overset{\substack{\text{inner local system}\\ \text{-twisted K-theory}}}{\mathrm{KU}^{n+[\omega_1]}_{C_\kappa}(\mathrm{X})}}
\xrightarrow[\text{Chern character}]{\text{twisted equivariant}}
\overset{\text{1-twisted periodic de Rham cohomology}}{\bigoplus_{\substack{d \in \mathbb{Z} \\ 1 \leq r \leq \kappa}} H^{n+2d}\Big(\Omega^\bullet_{\mathrm{dR}}\big(\mathrm{X};\mathbb{C}\big), \mathrm{d} + r \cdot \omega_1 \wedge\Big)}
$$

0 – Cohesive ∞-Topoi

I – Equivariant ∞-Bundles

II – TED-K-Theory

III – Anyonic Defect Branes

IV – Quantum Computation

This part is a brief indication
of a few aspects discussed in:

Anyonic Defect Branes in TED-K-Theory [arXiv:2203.11838]

K-Theory classifies non-perturbative vacua.

Solid state physics	K-theory	String theory

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Solid state physics	**K-theory**	**String theory**
Single electron state	Line bundle	Single D-brane
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Valence bundle of electron/positron states	Virtual bundle of their kernels and cokernels	D-brane Sen vacuum after tachyon condensation
Topological phase	K-theory class	Stable D-brane charge

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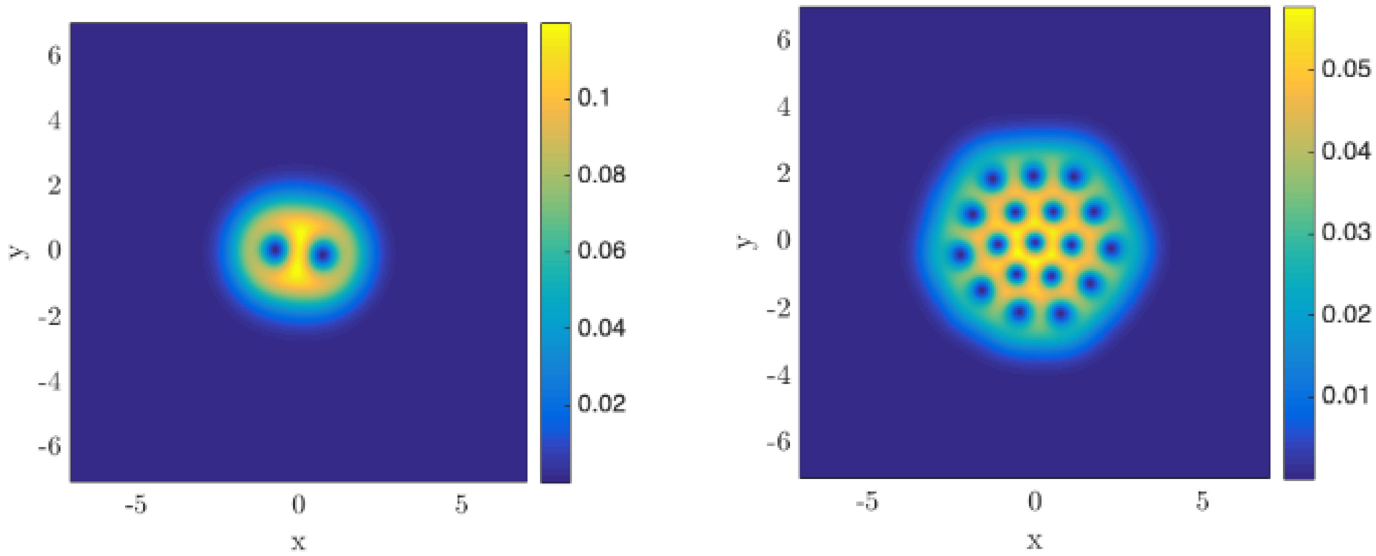
Anyons	Punctures	Defect branes

Anyons in condensed matter & string theory.

In solid state physics

anyons are presumed pointlike defects in gapped topological phases of effectively 2-dimensional materials whose adiabatic dynamics is that of Wilson lines in $\mathfrak{su}(2)$-CS theory.



(numerical simulation from `arXiv:1901.10739`)

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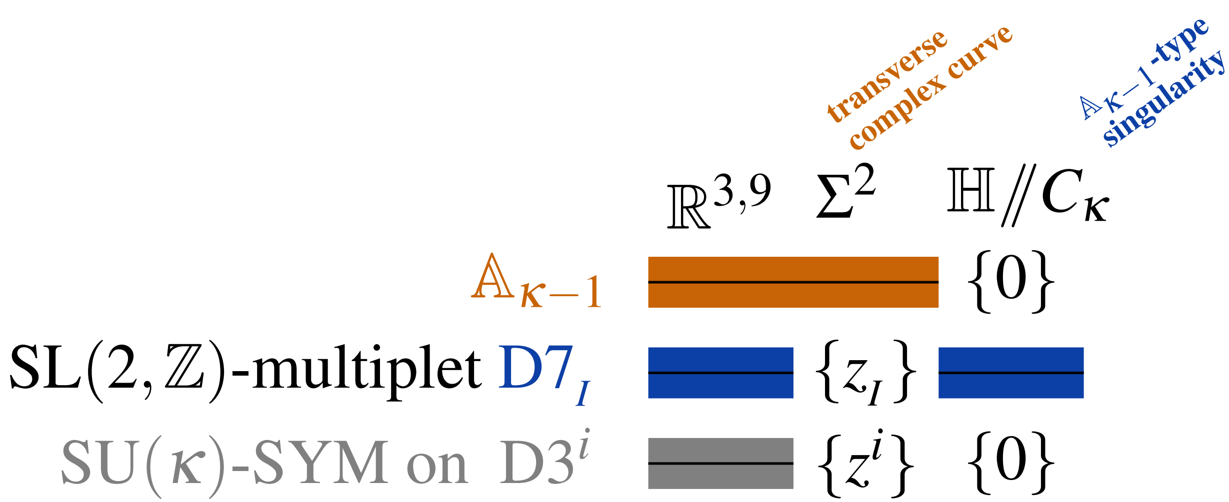

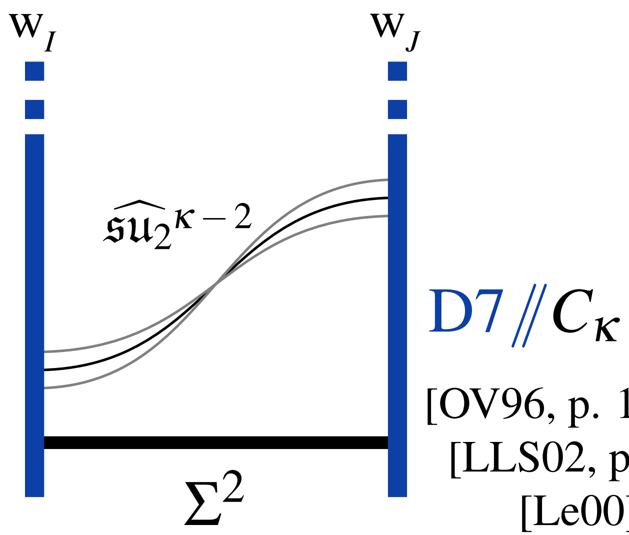

[OV96, p. 10-12]
[LLS02, p. 4s]
[Le00]

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Concretely, it is expected that:

$$\left\{ \begin{array}{c} \text{ground state wave functions of} \\ \text{spin=w}_I \;\; \widehat{\mathfrak{su}_2}^k\text{-anyons at} \\ \text{positions } z_I \text{ in transverse plane} \end{array} \right\} \simeq \overset{\text{space of “conformal blocks”}}{\mathrm{ConfBlck}^{\bullet}_{\widehat{\mathfrak{sl}_2}^k}\big(\vec{\mathrm{w}}, \vec{z}\big)}$$

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Yes! $\longrightarrow$

TED-Cohomological incarnation of Conformal blocks.

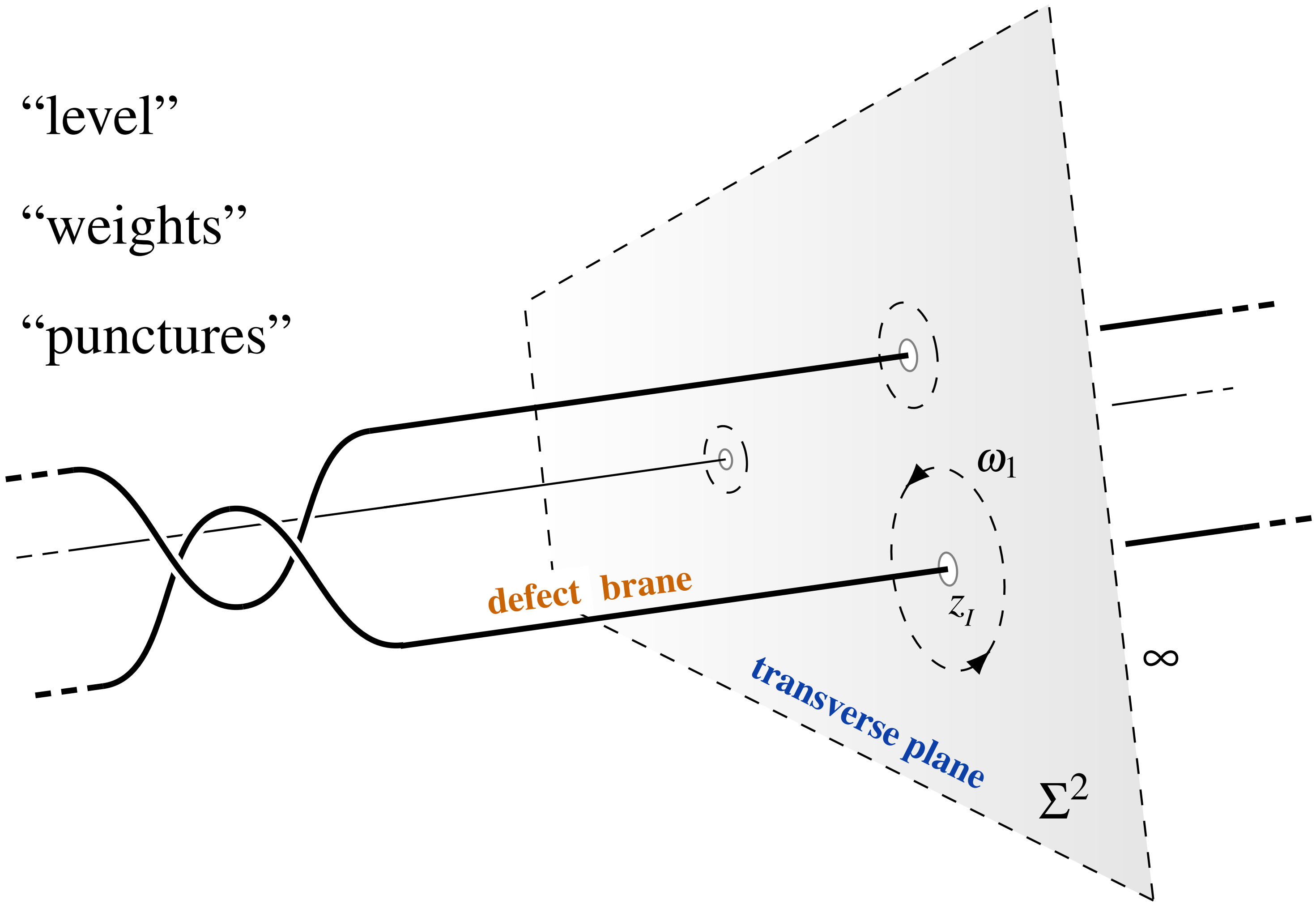
Consider

$$\kappa \quad := \quad k+2 \qquad \text{“level”}$$

$$\mathrm{w}_I \quad \in \quad \{0, \cdots, k\} \qquad \text{“weights”}$$

$$z_I \quad \in \quad \{z_1, \cdots, z_N\} \qquad \text{“punctures”}$$

$$\omega_1 \quad := \quad \sum_I -\frac{\mathrm{w}_I}{\kappa}\frac{\mathrm{d}z}{z-z_I}$$



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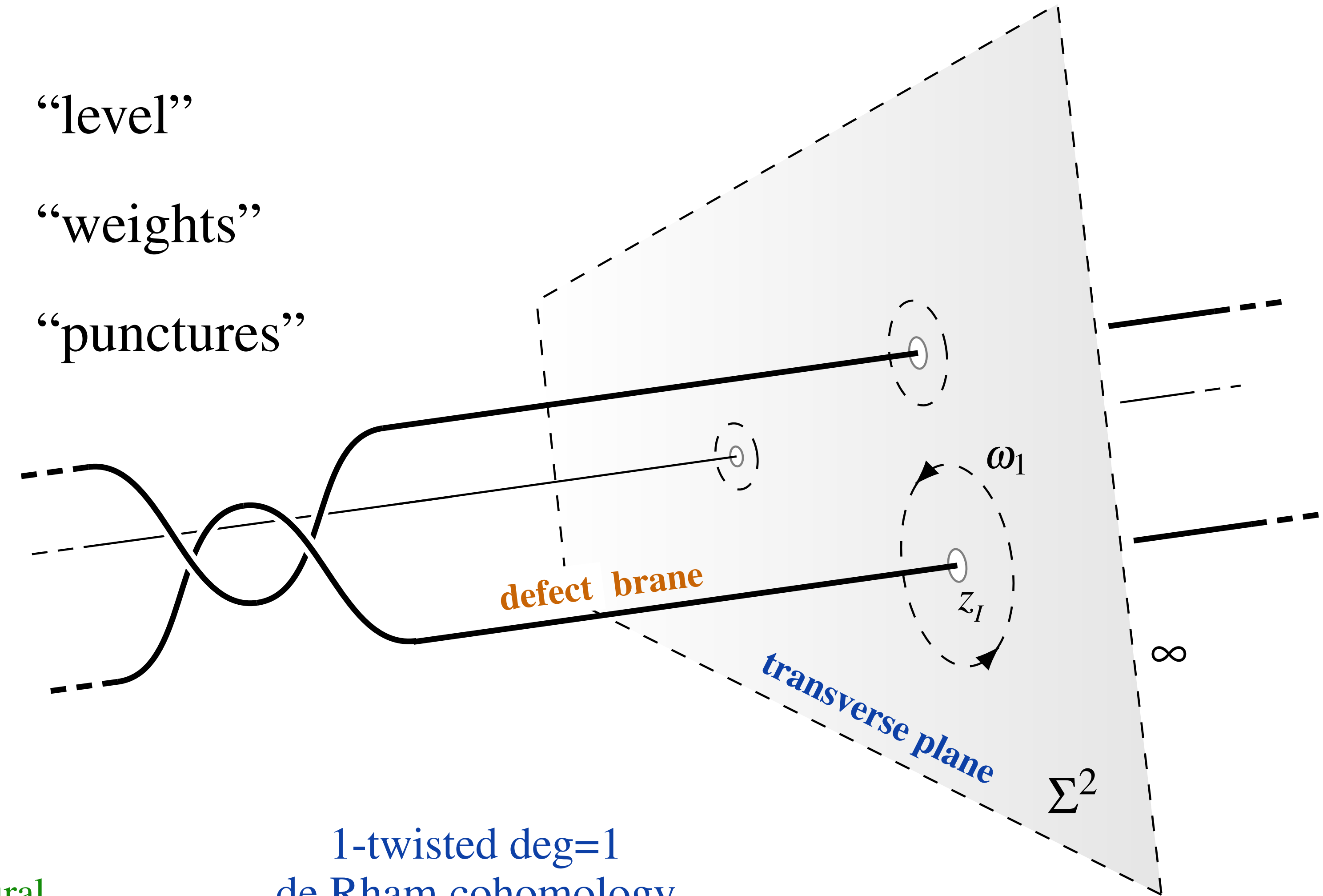

$$\underset{\text{conformal blocks}}{\overset{\mathfrak{su}(2)\text{-affine deg=1}}{\mathrm{CnfBlck}^1_{\widehat{\mathfrak{sl}_2}^k}(\vec{\mathrm{w}}, \vec{z})}} \xhookrightarrow{\text{natural inclusion}} \underset{\text{de Rham cohomology}}{\overset{\text{1-twisted deg=1}}{H^1\Big(\Omega^\bullet_{\mathrm{dR}}\big(\mathbb{C}\setminus\{\vec{z}\}\big), \mathrm{d} + \omega_1 \wedge\Big)}} \qquad \text{[FSV94, Cor. 3.4.2]}$$

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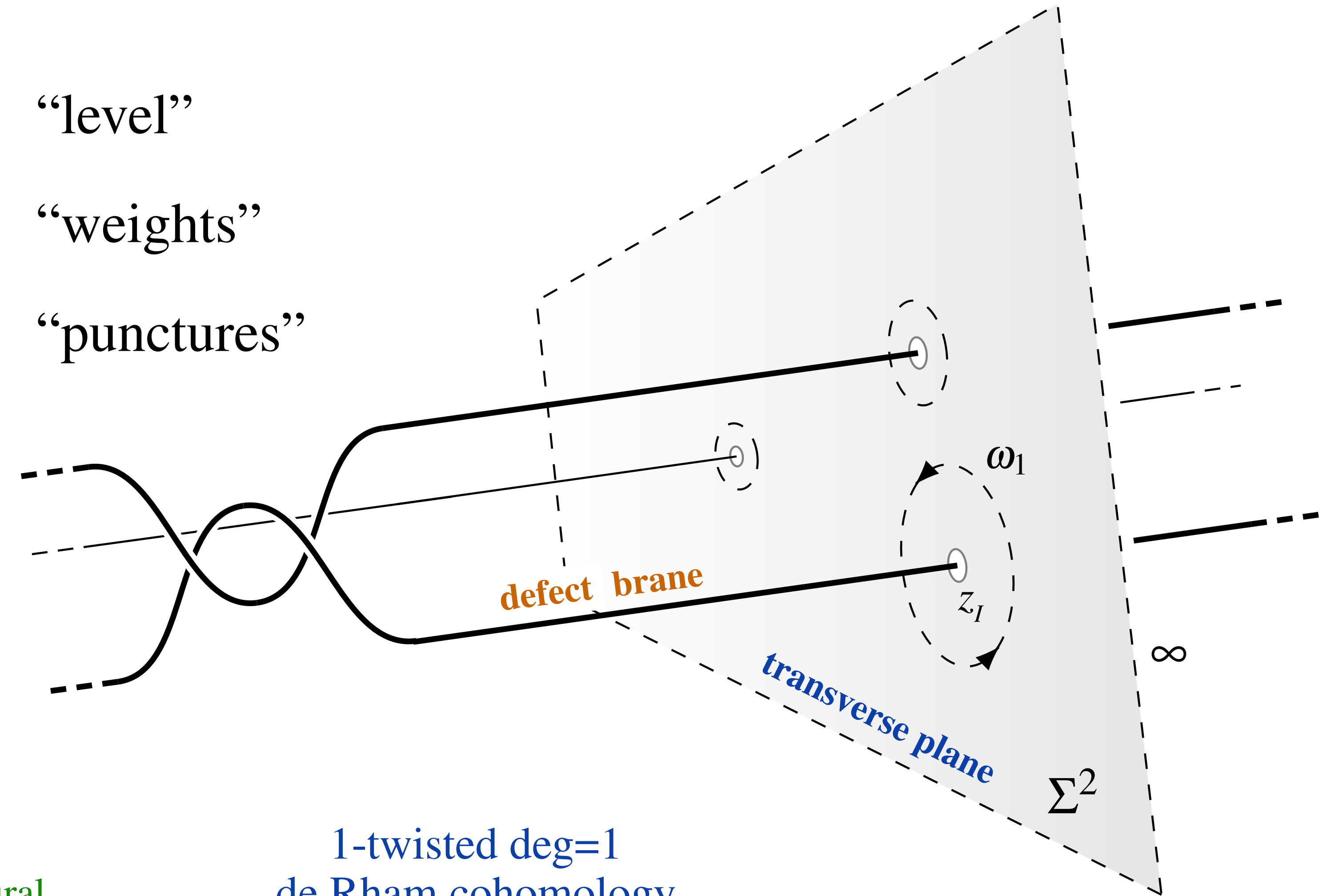

$$
\underset{\substack{\mathfrak{su}(2)\text{-affine deg=1} \\ \text{conformal blocks}}}{\mathrm{CnfBlck}^1_{\widehat{\mathfrak{sl}_2}^k}(\vec{\mathrm{w}}, \vec{z})} \xhookrightarrow{\substack{\text{natural} \\ \text{inclusion}}} \underset{\substack{\text{1-twisted deg=1} \\ \text{de Rham cohomology}}}{H^1\Big(\Omega^\bullet_{\mathrm{dR}}\big(\mathbb{C} \setminus \{\vec{z}\}\big), \mathrm{d} + \omega_1 \wedge\Big)} \quad \text{[FSV94, Cor. 3.4.2]}
$$

$$
\xhookrightarrow{\substack{\text{natural} \\ \text{inclusion}}} \underset{\substack{\text{inner local system-twisted deg=1} \\ \text{K-theory of } \mathbb{A}_{\kappa-1}\text{-singularity}}}{\mathrm{KU}^{1+\omega_1}\Big(\big(\mathbb{C} \setminus \{\vec{z}\}\big) \times * /\!\!/ C_\kappa; \mathbb{C}\Big)} \quad \text{[SS22, Prop. 2.16]}
$$

(as explained above)

TED-Cohomological incarnation of Conformal blocks.

Generally, consider *configuration spaces of points* (e.g. [SS19, §2.2])

$$\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathrm{X}\big) \;:=\; \Big\{ z^1, \cdots, z^n \in \mathrm{X} \;\Big|\; \underset{i<j}{\forall}\; z^i \neq z^j \Big\}.$$

with $\quad \omega_1 := \sum_{1 \leq i \leq n} \sum_{I} -\frac{\mathrm{w}_I}{\kappa} \frac{\mathrm{d}z}{z - z_I} + \sum_{1 \leq i < j \leq n} \frac{2}{\kappa} \frac{\mathrm{d}z}{z^i - z^j} \quad$ on $\quad \underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C} \setminus \{\vec{z}\}\big)$

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$\mathfrak{su}(2)$-affine deg=n conformal blocks

1-twisted deg=n de Rham cohomology of configuration space of n points

$$\mathrm{CnfBlck}^n_{\widehat{\mathfrak{sl}_2}^k}(\vec{\mathrm{w}}, \vec{z}) \hookrightarrow H^n\Big(\Omega^\bullet_{\mathrm{dR}}\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{\vec{z}\}\big)\Big), \mathrm{d} + \omega_1 \wedge\Big)$$

[FSV94, Cor. 3.4.2]

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inner local system-twisted deg=n K-theory
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$$\hookrightarrow \underbrace{\mathrm{KU}^{n+\omega_1}\bigg(\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{\vec{z}\}\big)\Big) \times * /\!\!/ C_\kappa; \mathbb{C}\bigg)}_{\substack{\text{inner local system-twisted deg}=n \text{ K-theory} \\ \text{of configurations in } \mathbb{A}_{\kappa-1}\text{-singularity}}} \qquad \text{[SS22, Thm. 2.18]}$$

The previous statement is subsumed since $\underset{\{1\}}{\mathrm{Conf}}(\mathrm{X}) = \mathrm{X}$.

Conclusion.

The commonly expected $\widehat{\mathfrak{su}_2}^k$-charges of anyons and defect branes *are* reflected in the TED-K-theory of *configuration spaces of points* in 2-dimensional transverse spaces *inside* $\mathbb{A}_{k+1}$*-orbi-singularities*.

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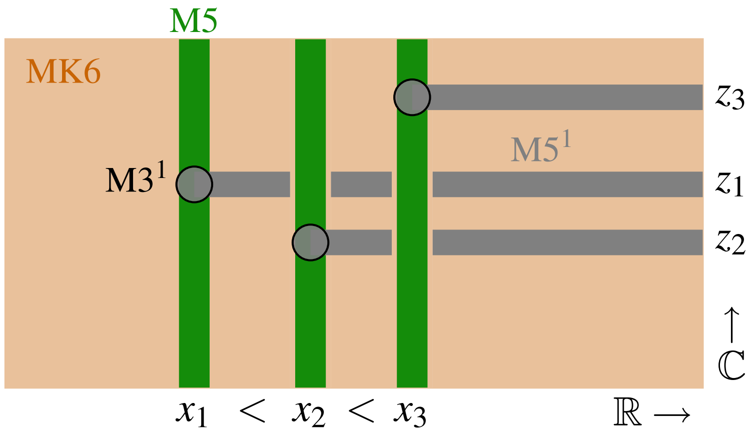
Configuration space of ordered points in the plane — **3-Cohomotopy cocycle space for codim=1 branes** — **3-Cohomotopy cocycle space for codim-2 branes**

$$\coprod_n \underset{\{1,\cdots,n\}}{\mathrm{Conf}}(\mathbb{C}) \;\simeq\; \overbrace{\coprod_n \mathrm{Conf}_n(\mathbb{C};\mathbb{R}_{\mathrm{cpt}})}^{\mathrm{Map}^*(\mathbb{R}_+\wedge\mathbb{C}_{\mathrm{cpt}},\,S^3)\,\simeq} \underset{\coprod_n \mathrm{Conf}_n(*;\,(\mathbb{R}\times\mathbb{C})_{\mathrm{cpt}})}{\times} \overbrace{\coprod_n \mathrm{Conf}_n(\mathbb{R};\mathbb{C}_{\mathrm{cpt}})}^{\mathrm{Map}^*(\mathbb{R}_{\mathrm{cpt}}\wedge\mathbb{C}_+,\,S^3)\,\simeq}$$

Fiber product of respective configuration spaces (of un-ordered points escaping to transverse infinity) reflecting the brane intersections

e.g.: $\underset{\{1,\cdots,3\}}{\mathrm{Conf}}(\mathbb{C}) \simeq \Bigg\{$



$\Bigg\}$

The moduli space of flat M3-branes according to *Hypothesis H* is the configuration space of ordered points in their transverse plane.

0 – Cohesive ∞-Topoi

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For expository outlook on

Topological Quantum Programming in `TED-K` [arXiv:2209.08331]

see:

ncatlab.org/nlab/files/CQTS-InitialResearcherMeeting-Schreiber-220914.pdf

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Twisted, Equivariant, Differential (TED) K-Theory via Diffeological Stacks

Urs Schreiber on joint work with Hisham Sati

NYU AD Science Division, Program of Mathematics

Thanks!

$\longleftarrow$ back to contents

Global Diffeology Seminar @ DIFFEOLOGY.NET 04 Nov 2022

slides and pointers at: ncatlab.org/schreiber/show/TED+cohomology