Lie algebroids and homological vector fields

A. Yu. Vaintrob

The notion of a Lie algebroid, introduced by J. Pradines, is an analogue of the algebra of a Lie group for differentiable groupoids. Lie algebroids combine the properties of Lie algebras and manifolds and are used in differential geometry, symplectic geometry, and representation theory (see [1]–[3]). The goal of this paper is to demonstrate that the theory of Lie algebroids is a special case of the theory of homological vector fields on supermanifolds [4], [5], and to indicate the possible applications of this approach.

1. Definitions. A vector bundle $A \to M$ with Lie bracket $[\ ,\ ]$ on the cross-section space $\Gamma(A)$ and with fibre homomorphism $a\colon A \to TM$, called an *anchor*, is called a *Lie algebroid* if a is a Lie algebra homomorphism and $[x, fy] = (a(x)f)y + f \cdot [x, y]$ for $x, y \in A$, $f \in C^\infty(M)$.

The standard examples of Lie algebroids are bundles of Lie algebras ($a = 0$), the tangent bundle of a manifold, and the cotangent bundle of a Poisson manifold.

An odd vector field V on a supermanifold is called a *homological vector field* if $[V, V] = 2V^2 = 0$. Many objects of various branches of mathematics can be described and studied in terms of homological fields (see [4], [5]).

2. Theorem. *We denote by $\mathcal{M} = (M, \Lambda^\bullet A^*)$ and $\mathcal{M}' = (M, \Lambda^\bullet A)$ two supermanifolds associated with the bundle $A \to M$. The following three classes of objects are in a natural one-to-one correspondence*:

(i) *Lie algebroid structures on $A \to M$;*
(ii) *homological vector fields of degree 1 on $\mathcal{M}$;*
(iii) *odd linear Poisson structures on $\mathcal{M}'$.*

Proof. In a local coordinate system (x, ξ) on $\mathcal{M}$ (where (x^α) are coordinates on M and (ξ^i) is a local basis of A^*) any vector field of degree 1 has the form $V = \sum c^k_{ij}\xi^i\xi^j\partial_{\xi^k} + \sum a^\alpha_i \xi^i \partial_{x^\alpha}$, where c and a are functions depending on x^α. We denote by (ε_i) a local basis of A, dual to (ξ^i). We put $a\colon A \to TM\colon a(X) = \sum f^i a^\alpha_i \partial_{x^\alpha}$ and $[X, Y] = \sum f^i g^j c^k_{ij}\varepsilon_k + \sum a(X)(g^j)\varepsilon_j - \sum a(Y)(f^i)\varepsilon_i$, where $X = \sum f^i(x)\varepsilon_i$ and $Y = \sum g^i(x)\varepsilon_i \in \Gamma(A)$. Conversely, for a bracket $[\ ,\ ]$ and for a map $a\colon A \to TM$ we can find the functions c^k_{ij} and a^α_i and we can define the vector field V. It can be verified directly that the pair $([\ ,\ ], a)$ yields a Lie algebroid structure if and only if V is a homological vector field.

The canonical duality between A and A^* transforms ξ^i into ∂_{ε_i} and ∂_{ξ^i} into ε_i, and V becomes the odd bivector field $\pi = \sum c^k_{ij}\varepsilon_k \partial_{\varepsilon_i} \wedge \partial_{\varepsilon_j} + \sum a^\alpha_i \partial_{\varepsilon_i} \wedge \partial_{x^\alpha}$ on $\mathcal{M}'$. It follows from the homological interpretation of Poisson structures [4] that π defines a Poisson structure on $\mathcal{M}'$ if and only if V is a homological field. This can be verified also by direct computation. The bracket defined by π is the odd variant of the linear Poisson structure on A^*, introduced by Courant [6].

3. Morphisms. It is clear what must be called a morphism of Lie algebroids over the same base M, whereas the general definition given by Alameida and Kumpeira [7] (see also [8]) is non-trivial. It is difficult to apply this definition; it is not even obvious that the composition of two morphisms is a morphism. As we change to the language of homological vector fields, the situation changes completely. We recall that a morphism of two vector fields V on P and W on Q is a map $h\colon P \to Q$ such that $h_*(V_p) = W_{h(p)}$ for any $p \in P$.

Theorem. *Let V and W be homological vector fields on $\mathcal{M}$ and $\mathcal{N}$, corresponding to the Lie algebroids $A \to M$ and $B \to N$. A morphism $\varphi\colon A \to B$, $f\colon M \to N$ of the bundles is a Lie algebroid morphism if and only if the induced supermanifold map $\Phi\colon \mathcal{M} \to \mathcal{N}$ is a morphism of the homological fields V and W.*

4. Modules over Lie algebroids. We recall that a module over a homological vector field V on a supermanifold P is a fibering $E \to P$ with a planar V-connection ∇ (that is, with a linear map $\nabla\colon \Gamma(E) \to \Gamma(E)$ such that $\nabla(fe) = V(f)e + (-1)^{|f|} f \cdot \nabla e$ and $\nabla^2 = 0$). This leads to the following definition. A *module over a Lie algebroid* $A \to M$ is a homogeneous fibering E over $\mathcal{M}$ (that is, a $\mathbb{Z}$-graded $C^\infty(\mathcal{M})$-module) with a planar V-connection ∇ of degree 1. The homology space $H^\bullet(A; E) := H(\nabla) = \operatorname{Ker}(\nabla)/\operatorname{Im}(\nabla)$ is called the *space of cohomologies of A with coefficients in E.*

The representations of an algebroid $A \to M$ in the sense of Mackenzie [1] correspond to the special type of modules for which $E = \pi^* F$, where F is a fibering over M and $\pi\colon \mathcal{M} \to M$ is the canonical projection.

Proposition. *A fibering $F \to M$ with linear map $m\colon \Gamma(A) \otimes \Gamma(F) \to \Gamma(F)$ is a representation of the Lie algebroid $A \to M$ if and only if the induced map $m^*\colon \Gamma(F) \to \Gamma(F) \otimes \Gamma(A^*)$ extended to a V-connection ∇ on $E = \pi^* F \to \mathcal{M}$ is an A-module (that is, $\nabla^2 = 0$).*

5. Example: tensor modules. For a homological vector field V on $\mathcal{M}$ the space of tensors on $\mathcal{M}$ of a definite type is a V-module such that $\nabla = L_V$, the Lie derivative. In particular, if V is the homological vector field corresponding to the Lie algebroid $A \to M$, we obtain a *tensor A-module.* With the exception of the trivial case where $a = 0$ (that is, when A is a fibering of Lie algebras), an arbitrary tensor module is not a representation in the sense of [1]. Nevertheless, this type contains many important modules like the adjoint $Ad = T\mathcal{M}$, the coadjoint $Ad^* = \Omega^1\mathcal{M}$, and the dualiser $\operatorname{Ber} = \operatorname{Vol}(\mathcal{M})$. If $\mathcal{M}$ is compact, then integration yields a non-degenerate pairing between $\operatorname{Vol}(\mathcal{M})$ and $C^\infty(\mathcal{M})$, compatible with the $\mathbb{Z}$-grading. This pairing induces a canonical duality between $H^\bullet(A)$ and $H^\bullet(A; \operatorname{Ber})$ that generalizes both the classical Poincaré duality and its analogue for Lie algebras.

6. Deformations. As in the case of Lie algebras, cohomologies of a Lie algebroid with coefficients in the adjoint module arise in the study of deformations. Results on deformations of homological vector fields [5] provide the following theorem.

Theorem. *Let $Ad = T\mathcal{M}$ be the adjoint module of the Lie algebroid $A \to M$. Then*

(i) *the Lie algebra of the automorphism group of A is isomorphic to $H^0(A, Ad)$;*
(ii) *the space of infinitesimal deformations of A is isomorphic to $H^1(A, Ad)$;*
(iii) *the obstructions to the extension of deformation belong to $H^2(A, Ad)$;*
(iv) *if $\dim H^1(A, Ad) = d < \infty$ and $H^2(A, Ad) = 0$, then A has a versal deformation with a smooth base of dimension d.*

Bibliography

[1] K. Mackenzie, *Lie groupoids and Lie algebroids in differential geometry*, London Math. Soc. Lect. Ser. Vol. 124, Cambridge University Press, Cambridge 1987.
[2] A.Costé, P. Dazord, and A. D. Weinstein, *Publ. Dép. Math. Univ. Claude-Bernard, Lyon. Nouvelle Sér. A*, Vol. 2 (1987), 1–62.
[3] A. Beilinson and J. Bernstein, *Advances in Soviet Math.* **16**:1 (1993), 1–50.
[4] A. Vaintrob, "On homological vector fields", Preprint.
[5] A. Vaintrob, "Normal forms of homological vector fields", *J. Math. Sciences* (to appear).
[6] T. Courant, *Trans. Amer. Math. Soc.* **319** (1990), 631–661.
[7] R. Alameida and A. Kumpeira, *Ann. Acad. Brasil. Ciênc.* **53** (1981), 247–250.
[8] P. Higgins and K. Mackenzie, *J. Algebra* **129** (1990), 194–230.

New Mexico State University

Received 18/JAN/97