Deformation theory of commutative algebras

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HU Berlin, Summer semester 2013

Contents

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Lecture 1: 2013-04-17

1 Motivation

Consider the functor $A \to \Omega^1_A$, assigning to a commutative algebra A its module of Kähler differentials. It only captures infinitesimal deformations, therefore it is too coarse to capture all the deformation theory of A.

Idea: Left-derive this functor to get

$$A \to \mathbb{L}_A$$

where $\mathbb{L}_A$ is the cotangent complex of A, which is a complex of A-modules. It captures all the deformation theory.

For N a projective module, $N \otimes_A -$ is exact!

Recipe:

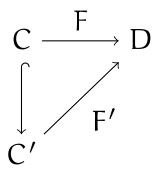
i) Embed $i\colon \mathrm{Mod}_A \hookrightarrow \mathrm{Ch}_A$, where Ch_A is the category of non-negatively graded chain complexes. For $N \in \mathrm{Mod}_A$, the complex $i(N)$ is concentrated at degree 0.

ii) Fact: For every $N \in \mathrm{Mod}_A$, there is a complex $P_\bullet \in \mathrm{Ch}_A$ of projectives and a quasi-isomorphism $\varphi\colon P_\bullet \to i(N)$, that is φ induces isomorphisms on the homology groups.

iii) We can now define $N \otimes^{\mathbb{L}}_A - := P_\bullet \otimes_A -$. This is a much richer object.

The abstract key features of this construction are as follows. For a functor $F\colon C \to D$, we would like to derive

- An embedding $C \hookrightarrow C'$, where C' is a suitable category that has some notion of qis (quasi-isomorphism).
- An extension F' of F:



- Some collection of objects $P \subseteq \mathrm{Ob}(C')$ well adapted to F.
- For all $c \in C$ there should be a $p \in P$ and a qis $\varphi\colon p \mapsto i(c)$.

We can then define $\mathbb{L}F(c) := F'(p)$ and finally apply this to Ω^1.

Recall that if

$$A \to B \to C$$

is a sequence of algebras and algebra morphisms, then

$$\Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0$$

is an exact sequence of C-modules. If $A \to B \to C$ are smooth morphisms, then the sequence is also left exact. There seems to be some kind of analogy between projective (modules) and smooth (morphisms).

What should C' be for Alg_A? Quillen gave the answer: It should be $C' = \mathrm{sAlg}_A$, the category of simplicial A-algebras.

Theorem 1.1 (Quillen)**.** *The category* sAlg_B *is a model category.*

2 Model categories

Model categories are abstract categories where homotopy theory works.

Definition 2.1 (Lift)**.** Let

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

be a commutative diagram in some category C. A lift is a morphism $h\colon B \to X$ such that the resulting triangles commute:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\big\downarrow & \overset{h}{\nearrow} & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

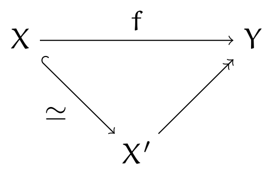
Definition 2.2 (Retract)**.** A morphism $f\colon X \to X'$ is a retract of $g\colon Y \to Y'$ if there exists a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{r} & X \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ X' & \xrightarrow[i']{} & Y' & \xrightarrow[r']{} & X' \end{array}$$

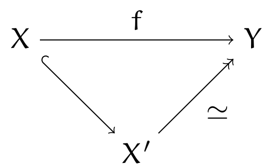
such that $r \circ i = \mathrm{id}$ and $r' \circ i' = \mathrm{id}$.

Definition 2.3 (Model category)**.** A model category structure on a category C consists of three classes of morphisms, weak equivalences ($\xrightarrow{\simeq}$), fibrations ($\twoheadrightarrow$) and cofibrations ($\hookrightarrow$), such that the following axioms hold.

1. Each class contains the identity and is closed under composition.
2. C has all limits and colimits.
3. Let f, g be composable morphisms. If two out of $\{g, f, g \circ f\}$ are weak equivalences, so is the third.
4. Let f be a retract of g. If g is a cofibration, fibration or weak equivalence, then f is a cofibration, fibration or weak equivalence, respectively.
5. In the diagram in Definition 2.1, a lift exists if
   i) i is a cofibration, p a fibration and a weak equivalence
   ii) i is a cofibration and a weak equivalence, p a fibration
6. Any morphism $f\colon X \to Y$ can be factored as both



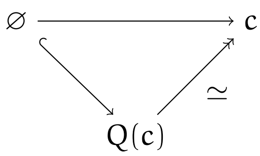
and

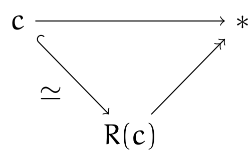


where both diagrams are commutative.

Definition 2.4 (Cofibrant and fibrant)**.** Let C be a model category. Since the (empty) limit and colimit exist, we get an initial object $\varnothing$ and a terminal object $*$. An object $c \in C$ is cofibrant if $\varnothing \to c$ is a cofibration and fibrant if $c \to *$ is a fibration.

Definition 2.5 ((Co)fibrant replacement)**.** For an object c, factor the morphism $\varnothing \to c$ and the morphism $c \to *$ as





The object $Q(c)$ is called a cofibrant replacement of c and $R(c)$ is called a fibrant replacement of c.

Example 2.6. The category Top. The three groups of morphisms are:

- Weak equivalences: weak homotopy equivalences, that is $f: X \to Y$ inducing for all points $x \in X$ bijections $\pi_0(X, x) \to \pi_0(Y, f(x))$ and group isomorphisms $\pi_i(X, x) \to \pi_i(Y, f(x))$ on the homotopy groups.
- Cofibrations: A continuous map $i: A \to B$ is a cofibration if it is a retract of a cell attachment.
- Fibrations: A continuous map $p: X \to Y$ is a fibration if it is a Serre fibration.

Example 2.7. Let A be a ring and consider the category Ch_A of non-negatively graded chain complexes.

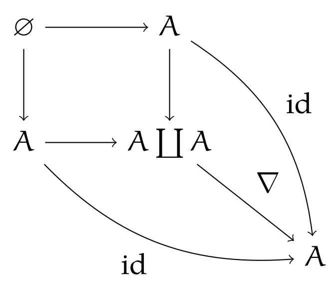
- $f: M_\bullet \to N_\bullet$ is a weak equivalence iff it is a qis.
- $i: M_\bullet \to N_\bullet$ is a cofibration iff $\forall n \geqslant 0$ the map $M_n \to N_n$ is injective with projective cokernel.
- $p: M_\bullet \to N_\bullet$ is a fibration iff $M_n \to N_n$ is surjective $\forall n \geqslant 1$.

In particular:

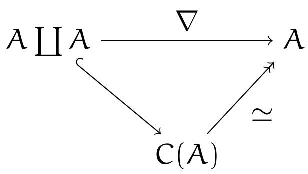
- Cofibrant objects are exactly the complexes of projectives.
- Cofibrant replacements are exactly the projective resolutions.
- Every object is fibrant.

2.1 The homotopy category

Definition 2.8 (Cylinder object)**.** Let ∇ be defined by the commutative diagram

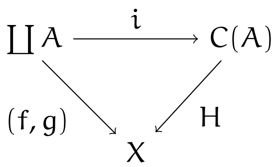


A cylinder object $C(A)$ for $A \in Ob(C)$ is a factorization



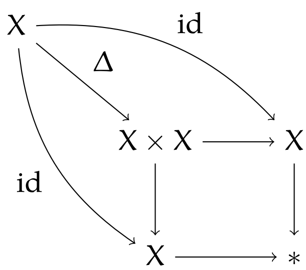
Roughly this corresponds to $C(A) = A \times [0, 1]$ in “normal” homotopy.

Definition 2.9 (Left homotopy)**.** Let $f\colon A \to X$ and $g\colon A \to X$. A left homotopy from f to g with respect to a cylinder object $C(A)$ for A is a map $H\colon C(A) \to X$ that fits in the commutative diagram

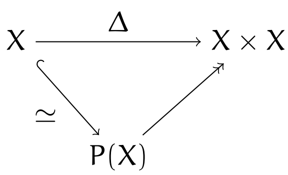


The dual notion is of course a right homotopy, which is defined in terms of a path object.

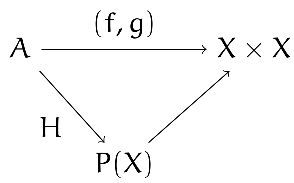
Definition 2.10 (Path object)**.** Let Δ be defined by the commutative diagram



A path object $P(X)$ for X is a factorization



Definition 2.11 (Right homotopy)**.** Let $f\colon A \to X$ and $g\colon A \to X$. A right homotopy from f to g with respect to a path object $P(X)$ for X is a map $H\colon A \to P(X)$ that fits into the commutative diagram



Definition 2.12 (Homotopy category)**.** The homotopy category $\mathrm{Ho}(C)$ of a category C with respect to some class of morphisms $W \subseteq \mathrm{Hom}(C)$ is $C[W^{-1}]$, satisfying the following universal property: There is a functor $\phi\colon C \to \mathrm{Ho}(C)$ such that for all functors $F\colon C \to D$ satisfying $\forall f \in W\colon F(f)$ is an isomorphism, there exists a unique functor $F'\colon \mathrm{Ho}(C) \to D$ with $F = F' \circ \phi$.

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Remark 2.13. For morphisms between cofibrant fibrant objects the notions of left and right homotopy coincide. Hence we get an equivalence relation on morphisms and can define the equivalence classes to be the homotopy classes of morphisms.

Definition 2.14. Let C be a model category. Then let C_{C_f} be the category with

- Objects the objects in C that are both cofibrant and fibrant.
- Morphisms the homotopy classes of morphisms in C.

Theorem 2.15 (Fundamental theorem of model categories)**.** *Let C be a model category, W the set of weak equivalences and $\mathrm{Ho}(C)$ the homotopy category of C with respect to W. Then $F'\colon \mathrm{Ho}(C) \to C_{C_f}$ is an equivalence of categories.*

2.2 Quillen functors

Definition 2.16. Let C, D be model categories. A Quillen functor is an adjoint pair (F, G) of functors $F\colon C \to D$ and $G\colon D \to C$ such that

i) F preserves cofibrations and weak equivalences between cofibrant objects.

ii) G preserves fibrations and weak equivalences between fibrant objects.

This means roughly that F is right exact and G is left exact.

Definition 2.17. Let (F, G) be a Quillen functor. The left derived functor of F is given by $LF(X) = F(Q(X))$ where $Q(X)$ is a cofibrant replacement for X. The right derived functor of G is given by $RG(Y) = G(R(Y))$ where R is a fibrant replacement for Y.

Theorem 2.18. *Let (F, G) be a Quillen functor. Then $LF\colon \mathrm{Ho}(C) \to \mathrm{Ho}(D)$ and $RG\colon \mathrm{Ho}(D) \to \mathrm{Ho}(C)$ is an adjoint pair.*

Remark 2.19. You should think of the Hom and $\otimes$ functors as an example of these.

3 Simplicial algebras

3.1 Simplicial sets – sSet

Definition 3.1. Let Δ be the category with objects $[n] = \{0, \dots, n\}$ and the usual ordering on this set and morphisms $\phi\colon [m] \to [n]$ that are order-preserving.

Example 3.2. We define two special examples of morphisms. Let $d^i\colon [n-1] \to [n]$ for $0 \leqslant i \leqslant n$ be defined by

$$d^i(j) = \begin{cases} j, & j < i \\ j+1, & j \geqslant i \end{cases}$$

This morphism "skips" i. Additionally, let $s^j\colon [n+1] \to [n]$ for $0 \leqslant j < n+1$ be defined by

$$s^j(i) = \begin{cases} i, & i \leqslant j \\ i-1, & i > j \end{cases}$$

This morphism "doubles" j.

Remark 3.3. Every $\phi \in \mathrm{Hom}(\Delta)$ is a composition of s^j and d^i.

Definition 3.4. Let C be a category. Its category of simplicial objects sC is the functor category $\mathrm{Fun}(\Delta^{\mathrm{op}}, C)$.

Remark 3.5. There is a geometric realization functor $|\ |\colon$ sSet $\to$ Top giving CW complexes.

Definition 3.6. We define $\Delta^n = \Delta(-, [n]) = \mathrm{Hom}(-, [n]) \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$

Remark 3.7. $|\Delta^n|$ is the n-simplex.

Definition 3.8. We set

$$\partial\Delta^n = \bigcup_{0 \leqslant i \leqslant n} d^i \Delta^{n-1} \subset \Delta^n$$

the boundary of Δ^n (so $|\partial\Delta^n|$ is the boundary of the n-simplex) and

$$\Delta^n_k = \bigcup_{i \neq k} d^i \Delta^{n-1} \subset \Delta^n$$

the k-th horn of Δ^n.

Theorem 3.9. *The category* sSet *is a model category with the following three classes of morphisms:*

1. *The weak equivalences are morphisms* $f\colon X \to Y$ *such that* $|f|\colon |X| \to |Y|$ *is a weak homotopy equivalence.*

2. *The cofibrations are* $i\colon X \to Y$ *such that* $i_n\colon X_n \to Y_n$ *are monomorphisms* $\forall n \geqslant 1$.

3. *The fibrations are* $p\colon X \to Y$ *such that* p *has the left lifting property with respect to all* $i\colon \Delta^n_k \to \Delta^n$ *for* $n \geqslant 1$ *and* $0 \leqslant k \leqslant n$.

The condition for fibrations is called the Kan condition.

3.2 The model category of simplicial algebras sAlg_A

Theorem 3.10 (Stealing along a right adjoint). *Let $(F, G)\colon C \to D$ be an adjoint pair and let C have a model structure generated by cofibrations. Let I be the set of generating cofibrations and J be the set of generating cofibrations and weak equivalences. Then $f\colon X \to Y \in \mathrm{Hom}(D)$ is a weak equivalence if and only if $G(f)$ is a weak equivalence and $p\colon X \to Y \in \mathrm{Hom}(D)$ is a fibration if and only if $G(p)$ is a fibration.*

Assume further that G commutes with directed colimits and every cofibration with the left lifting property with respect to all fibrations is a weak equivalence. Then D is a model category.

We now apply this to

$$\mathrm{sSet} \underset{\text{Free Mod}}{\overset{\text{for}}{\rightleftarrows}} \mathrm{sMod} \underset{\text{Sym}}{\overset{\text{for}}{\rightleftarrows}} \mathrm{sAlg}_A$$

Here "for" is short for the forgetful functor.

Theorem 3.11. *The categories sMod_A and sAlg_A are model categories with $f\colon X \to Y$ a weak equivalence (fibration) if and only if $\mathrm{for}(f)$ is a weak equivalence (fibration).*

Theorem 3.12 (Dold-Kan or Dold-Puppe). *The functor $N\colon \mathrm{sMod}_A \to \mathrm{Ch}_A$ given by*

$$N(M)_n = \frac{M_n}{s_0 M_{n-1} + \cdots + s_{n-1} M_{n-1}}$$

is an equivalence of categories.

4 The cotangent complex

Remark 4.1. Let $A \to B \in \mathrm{sAlg}_A$. Then Ω^1 applied levelwise gives $\Omega^1_{B/A} \in \mathrm{sMod}_B$:

$$\Omega^1_{B_0/A_0} \overset{\longleftarrow}{\underset{\longrightarrow}{\longleftarrow}} \Omega^1_{B_1/A_1} \quad \cdots$$

Definition 4.2. Let $A \to B \in \mathrm{sAlg}_A$ and

$$A \hookrightarrow X \xrightarrow{\simeq} B$$

be a cofibrant replacement (A is the initial object in sAlg_A). Define $\mathbb{L}_{B/A} = \Omega^1_{X/A} \otimes_X B \in \mathrm{sMod}_B$. Via the theorem we can regard it as a complex, called the cotangent complex of $A \to B$.

4.1 Quillen's construction

Recall the following. Let $A \to B \in \mathrm{Alg}_A$ and $M \in \mathrm{Mod}_B$. Then $B \oplus M \in \mathrm{Alg}_A/B$ is the trivial square-zero extension of B by M with multiplication

$$(b_1, m_1) \cdot (b_2, m_2) = (b_1 \cdot b_2, b_1 \cdot m_2 + b_2 \cdot m_1)$$

Definition 4.3. Let $A \to C \to B \in \mathrm{Alg}_A/B$ and $M \in \mathrm{Mod}_B$. We set

$$\mathrm{Der}_A(C, M) = \{D \in \mathrm{Hom}_A(C, M) \mid D(c1c_2) = c_1 D(c_2) + D(c_1)c_2\}$$

Definition 4.4. An object X in a category C is abelian if $C(\cdot, X)$ is naturally an abelian group.

Lemma 4.5. $\mathrm{Der}_A(C, M) \cong \mathrm{Hom}_{\mathrm{Alg}_A/B}(C, B \oplus M)$

Exercise 4.6. In Alg_A/B, $B \oplus M$ is an abelian group object.

Lemma 4.7. The functor $\Phi \colon \mathrm{Mod}_B \to (\mathrm{Alg}_A/B)_{ab}$ is an equivalence of categories. It sends M to $A \to B \oplus M \to B$. Hence we have a fancy way to describe $\mathbb{L}_{B/A}$ as adjoint of inclusion in the following diagram:

$$\begin{array}{ccccc} & & (\mathrm{Alg}_A/B)_{ab} & \qquad & A \to B \oplus M \to B \\ & \overset{ab}{\nearrow} \underset{incl.}{\swarrow} & \downarrow \uparrow \simeq & & \uparrow \\ \mathrm{Alg}_A/B & \longrightarrow & \mathrm{Mod}_B & & M \end{array}$$

$$A \to C \to B \longmapsto \Omega^1_{C/A} \otimes B$$

Hence

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Alg}_A/B}(C, B \oplus M) &\cong \mathrm{Der}_A(C, M) \\ &\cong \mathrm{Hom}_{\mathrm{Mod}_C}(\Omega^1_{C/A}, M) \\ &\cong \mathrm{Hom}_{\mathrm{Mod}_B}(\mathbb{L}_{B/A}, M) \end{aligned}$$

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Definition 4.8. Let $B \in \mathrm{sAlg}_A$. The category Mod^s_B is the category of modules over B i.e. $M \in \mathrm{Mod}^s_B$ is a simplicial abelian group plus a composition $B \times M \to M$ such that each M_i is a B_i-module and everything is compatible with the simplicial structure.

Remark 4.9. We have an inclusion $\mathrm{Alg}_A \hookrightarrow \mathrm{sAlg}_A$, given by $B \mapsto i(B)$, where $i(B)$ is the following simplicial algebra:

$$B \rightleftarrows B \;\substack{\to \\ \leftarrow \\ \to \\ \leftarrow \\ \to}\; B \quad \cdots$$

Here all arrows are the identity.

Definition 4.10. Let $B \in \mathrm{sAlg}_A$ and $M \in \mathrm{Mod}_B^s$. Define $B \oplus M$ by applying trivial square-zero extensions levelwise, i.e.

$$B \oplus M = B_0 \oplus M_0 \rightleftarrows B_1 \oplus M_1 \, \substack{\rightarrow \\ \leftarrow \\ \rightarrow \\ \leftarrow \\ \rightarrow} \, B_2 \oplus M_2 \quad \dots$$

Definition 4.11. Let $C \in \mathrm{sAlg}_A/B$, $M \in \mathrm{Mod}_B^s$. We saw that

$$\mathrm{Der}_A(C, M) = \mathrm{Hom}_{\mathrm{sAlg}_A/B}(C, B \oplus M)$$

are the derivations. We now define

$$\mathbb{R}\mathrm{Der}_A(C, M) = \mathrm{Hom}_{\mathrm{Ho}(\mathrm{sAlg}_A/B)}(C, B \oplus M) = \mathrm{Hom}_{\mathrm{sAlg}_A/B}(Q(C), R(B \oplus M))$$

Lemma 4.12. The pair of functors

$$\mathrm{sAlg}_A/B \underset{B\oplus -}{\overset{\Omega^1_{-/A} \otimes_{-} B}{\rightleftarrows}} \mathrm{Mod}_B^s$$

is a Quillen pair.

Proof. Adjointness follows from levelwise adjointness. As for the Quillen pair property, it is enough to check that $B \oplus -$ preserves fibrations and weak equivalences between fibrant objects (as we are treating adjoint functors between model categories). Therefore we can take the derived functors and obtain adjunction on the homotopy categories. □

Theorem 4.13. *The cotangent complex represents derived derivations, i.e.*

$$\mathbb{R}\mathrm{Der}_A(C, M) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Mod}_B^s)}(\mathbb{L}_{C/A} \oplus_C B, M)$$

Corollary 4.14. $\mathbb{L}_{B/A}$ is well-defined in $\mathrm{Ho}(\mathrm{Mod}_B^s)$ i.e. independent of the choice of cofibrant replacement.

Proof. Take $B = C \to B$ the identity and apply the theorem. We see that $\mathbb{L}_{B/A}$ represents $\mathbb{R}\mathrm{Der}_A(B, M)$ on $\mathrm{Ho}(\mathrm{Mod}_B^s)$. By Yoneda it is well-defined. □

4.2 Fundamental properties

We first recall on exact triangles (cofiber sequences). In topology we have the following mapping cone construction:

Why the cone? It is meaningful in the homotopy category on the level of chain complexes.

Definition 4.15. Let $f\colon M_\bullet \to N_\bullet$ be a map of chain complexes of A-modules. Remember that they are non-negatively graded and the differential goes down. Then we define

$$\mathrm{cyl}(f)_n = M_{n-1} \oplus M_n \oplus N_n$$

with differential

$$d_{\mathrm{cyl}(f)} = \begin{pmatrix} d_M & \mathrm{id}_M & -f \\ 0 & -d_M & 0 \\ 0 & 0 & d_N \end{pmatrix}$$

We also define the cone

$$\mathrm{cone}(f)_n = M_{n-1} \oplus N_n$$

with differential

$$d_{\mathrm{cone}(f)} = \begin{pmatrix} -d_M & -f \\ 0 & d_N \end{pmatrix}$$

Definition 4.16. A null-homotopy for $f\colon M_\bullet \to N_\bullet$ is a map $S\colon M \to N[-1]$ such that $(d_n \circ S_{n-1}) - (S_{n-2} \circ d_M) = f$, i.e.

$$\begin{array}{ccccc} M_n & \xrightarrow{d_M} & M_{n-1} & \xrightarrow{d_M} & M_{n-2} \\ \downarrow f_n & \swarrow s_{n-1} & \downarrow f_{n-1} & \swarrow s_{n-2} & \downarrow f_{n-2} \\ N_n & \xrightarrow[d_N]{} & N_{n-1} & \xrightarrow[d_N]{} & N_{n-2} \end{array}$$

We say that $f, g\colon M_\bullet \to N_\bullet$ are chain homotopic if $f - g$ is null-homotopic.

Definition 4.17. A map $f\colon M_\bullet \to N_\bullet$ is a chain homotopy equivalence if there is $g\colon N_\bullet \to M_\bullet$ such that $g \circ f$ is chain homotopic to id_M and $f \circ g$ is chain homotopic to id_N.

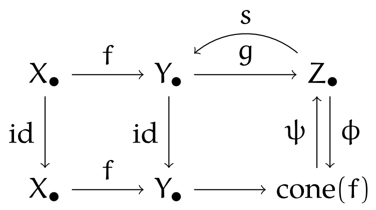
Definition 4.18. A sequence $X_\bullet \xrightarrow{g} Y_\bullet \xrightarrow{h} Z_\bullet$ in $\mathrm{Ch}(\mathrm{Mod}_A)$ is a cofiber sequence if there exists a diagram

$$\begin{array}{ccccc} X_\bullet & \xrightarrow{g} & Y_\bullet & \xrightarrow{h} & Z_\bullet \\ a\downarrow & & b\downarrow & & c\downarrow \\ M_\bullet & \xrightarrow{f} & N_\bullet & \longrightarrow & \mathrm{cone}(f) \end{array}$$

such that a, b, c are chain homotopy equivalences and both of the small squares commute up to homotopy.

Lemma 4.19. Every split exact sequence is a cofiber sequence.

Proof. Consider



where $\phi(z) = (sd_Z - d_Z s, s)$ and $\psi(x, y) = g(y)$. □

Remark 4.20. A cofiber sequence $X_\bullet \to Y_\bullet \to Z_\bullet$ gives rise to a long exact sequence in homology

$$\cdots \to H_1(X_\bullet) \to H_1(Y_\bullet) \to H_1(Z_\bullet) \to H_0(X_\bullet) \to H_0(Y_\bullet) \to H_0(Z_\bullet) \to 0$$

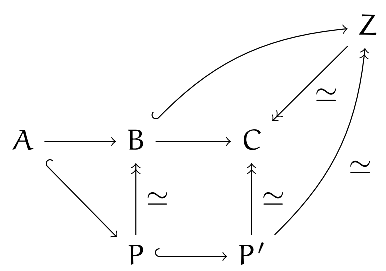
Remark 4.21. If $X_\bullet \to Y_\bullet \to Z_\bullet$ is a cofiber sequence, then $X_\bullet \to Z_\bullet \to X_\bullet[1]$ is a cofiber sequence as chain homotopy to $X_\bullet \to \mathrm{cyl}(f) \to \mathrm{cone}(f)$.

Proposition 4.22. *Let* $A \to B \to C$ *be a sequence of morphisms of algebras. Then we have a cofiber sequence*

$$\mathbb{L}_{B/A} \otimes_B C \to \mathbb{L}_{C/A} \to \mathbb{L}_{C/B}$$

in $\mathrm{Mod}_C^s \simeq \mathrm{Ch}(\mathrm{Mod}_C)$.

Proof. Choose cofibrant replacements P of B and P' of C and set $Z = P' \otimes_P B$. Then we get the following diagram (use the 2 out of 3 property):



Observe that the pushout of a weak equivalence in sAlg_A is again a weak equivalence. This is called left-properness of sAlg_A.

From the diagram we get a split short exact sequence

$$\Omega^1_{P/A} \otimes_P P' \to \Omega^1_{P'/A} \to \Omega^1_{P'/P} \tag{1}$$

Applying $- \otimes_{P'} C$ yields

$$\Omega^1_{P/A} \otimes_P C \to \Omega^1_{P'/A} \otimes_{P'} C \to \Omega^1_{P'/P} \otimes_{P'} C$$

On the other hand,

$$\Omega^1_{P'/P} \otimes_{P'} C \xrightarrow{\cong} \Omega^1_{Z/B}$$

and applying $- \otimes_Z C$ here yields

$$\Omega^1_{P'/P} \otimes_{P'} C \xrightarrow{\cong} \Omega^1_{Z/B} \otimes_B C$$

Thus (1) is

$$\mathbb{L}_{B/A} \otimes_B C \to \mathbb{L}_{C/A} \to \mathbb{L}_{C/B} \qquad \square$$

Lecture 4: 2013-05-15

Lemma 4.23. Let $A \to B$ be a morphism in Alg_A. Then $H_0(\mathbb{L}_{B/A}) \cong \Omega^1_{B/A}$.

Proof. Consider the factorization

$$A \longrightarrow P \xrightarrow{\simeq} B$$

Ω^1 is a left adjoint, therefore it preserves colimits (remember RAPL). Therefore

$$\Omega^1_{B/A} = \Omega^1_{-/B}(\varinjlim P_1 \rightrightarrows P_0) = \varinjlim(\Omega^1_{P_1/B} \rightrightarrows \Omega^1_{P_0/B}) = H_0(\mathbb{L}_{B/A}) \qquad \square$$

Corollary 4.24. Let $A \to B \to C$ be morphisms of algebras. There is a long exact sequence

$$\cdots \to H_1(\mathbb{L}_{B/A} \otimes_B C) \to H_1(\mathbb{L}_{C/A}) \to H_1(\mathbb{L}_{C/B}) \to$$
$$\Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0$$

4.2.1 Base change

Recall the base change property for Kähler differentials. For a cartesian diagram

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' = B \otimes_A A' \end{array}$$

we get an isomorphism

$$\Omega^1_{B/A} \otimes_B B' \xrightarrow{\cong} \Omega^1_{B'/A'}$$

Even without the cartesian property, a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

yields a morphism

$$\mathbb{L}_{B/A} \otimes_B B' \to \mathbb{L}_{B'/A'}$$

which is constructed as follows. First we factor $A \to B$ over P and get the diagram

$$\begin{array}{ccccc} A & \longrightarrow & A' & & \\ \big\downarrow & & \big\downarrow & & \\ P & \longrightarrow & A' \otimes_A P & \hookrightarrow & P' \\ \simeq \big\downarrow & & \big\downarrow & & \big\downarrow \simeq \\ B & \longrightarrow & B' & = & B' \end{array}$$

This gives us

$$\begin{array}{ccc} \Omega^1_{A' \otimes_A P/A} \otimes_{A \otimes_A P} P' & \longrightarrow & \Omega^1_{P'/A} \otimes_P B \\ \| & & \| \\ \mathbb{L}_{B/A} \otimes_B B' & \longrightarrow & \mathbb{L}_{B'/A'} \end{array}$$

Theorem 4.25 (Base change theorem for flat morphisms)**.** *Let*

$$\begin{array}{ccc} A & \xrightarrow{\;g\;} & A' \\ f \big\downarrow & & \big\downarrow f' \\ B & \xrightarrow[g']{} & B' = B \otimes_A A' \end{array}$$

be a cocartesian diagram of algebras. Assume that either f *or* g *are flat. Then*

$$\mathbb{L}_{B/A} \otimes_B B' \to \mathbb{L}_{B'/A'}$$

is an equivalence.

Proof. Without loss of generality, we may assume g to be flat. Then we get a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\;g\;} & A' \\ \big\downarrow & & \big\downarrow \\ P & \longrightarrow & P \otimes_A A' \\ \big\downarrow & & \big\downarrow \simeq \\ B & \longrightarrow & B' \end{array}$$

where the weak equivalence comes from the flatness property of g. By base change for Ω^1 we get

$$\Omega^1_{P/A} \otimes_P (P \otimes_A A') \xrightarrow{\simeq} \Omega^1_{P \otimes_A A'/A'}$$

Now apply $-\otimes_{P\otimes_A A'} B'$ which yields the equivalence

$$\mathbb{L}_{B/A} \otimes_B B' \xrightarrow{\simeq} \mathbb{L}_{B'/A} \qquad \square$$

Remark 4.26.

- The $\Omega^1_{P/A}$ are simplicial P-modules and a complex, not just modules.
- Let $P \in \mathrm{sAlg}_A$, $M \in \mathrm{Mod}^s_P$. Then M_i is a P_i-module, so in particular an A-module. Therefore $M \in \mathrm{sMod}_A \cong \mathrm{Ch}(\mathrm{Mod}_A)$.
- Underlying the simplicial stuff is always a chain complex.

4.2.2 Localization

Geometrically, if $U \hookrightarrow X$ is an open embedding, then $\Omega^1_{U/X} = 0$ because the tangents to U are the same as those to X. We get an algebraic analogue for the cotangent complex:

Proposition 4.27. *Let $S \subseteq A$ be a multiplicatively closed set, $B = S^{-1}A$. Then $\mathbb{L}_{B/A} \simeq 0$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} B = B \otimes_A B & \longleftarrow & B \\ \uparrow & & \uparrow \\ B & \longleftarrow & A \end{array}$$

By base change, we have

$$\mathbb{L}_{B/A} = \mathbb{L}_{B/A} \otimes_B B \xrightarrow{\simeq} \mathbb{L}_{B/B} = 0 \qquad \square$$

4.2.3 Mayer-Vietoris sequence

If

$$\begin{array}{ccc} B & \longrightarrow & X \\ \downarrow & & \downarrow \\ C & \longrightarrow & Y \end{array}$$

is a cocartesian diagram of A-algebras (so $Y = X \otimes_B C$), then we have the sequence

$$\Omega^1_{B/A} \otimes_B Y \to (\Omega^1_{X/A} \otimes_C Y) \oplus (\Omega^1_{C/A} \otimes_C Y) \to \Omega^1_{Y/A} \to 0$$

The geometrical interpretation is simply that the (relative) tangent space of a product (over a point) is the direct sum of the tangent spaces, in formulas

$$T_{X\times X} = T_X \oplus T_X, \quad T_{X/Y\times X/Y} = T_{X/Y} \oplus T_{X/Y}$$

We will now derive the same formula for the cotangent complex.

Proposition 4.28. *Let*

$$\begin{array}{ccc} B & \xrightarrow{g} & X \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ C & \longrightarrow & Y \end{array}$$

be a cocartesian diagram of A*-algebras (i.e.* $Y = X \otimes_B C$*) with either* f *or* g *flat. Then we have a cofiber sequence (which is the same as an exact triangle)*

$$\mathbb{L}_{B/A} \otimes_B Y \to (\mathbb{L}_{C/A} \otimes_C Y) \oplus (\mathbb{L}_{X/A} \otimes_X Y) \to \mathbb{L}_{Y/A}$$

Proof. As usual (take a cofiber replacement ...). □

Remark 4.29. If we replace

$$\begin{array}{ccc} A & \longrightarrow & B' \\ \big\downarrow & & \big\downarrow \\ B & \longrightarrow & B \otimes_A B' \end{array}$$

by the derived tensor product

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \big\downarrow & & \big\downarrow \\ P & \hookrightarrow & P \otimes_A A' \\ {\scriptstyle \simeq}\big\downarrow & & \\ B & & \end{array}$$

with $P \otimes_A A' := A' \otimes^{\mathbb{L}}_A B$, then base change and Mayer-Vietoris hold without flatness assumptions.

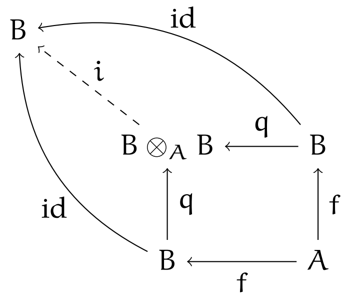
Recall that $f\colon A \to B$ is formally étale if for every square zero extension $T' \to T$ and all diagrams

$$\begin{array}{ccc} T & \longleftarrow & B \\ \big\uparrow & \swarrow_{h} & \big\uparrow{\scriptstyle f} \\ T' & \longleftarrow & A \end{array}$$

h exists and is unique.

Theorem 4.30. *If* $f\colon A \to B$ *is formally étale, then* $\mathbb{L}_{B/A} \simeq 0$.

Proof. Consider the following diagram:



Since f is étale, f is flat and i is a localization morphism (geometrically an open embedding). Therefore we can use flat base change:

$$\mathbb{L}_{B/A} \otimes_B (B \otimes_A B) \xrightarrow{\simeq} \mathbb{L}_{B \otimes_A B/B}$$

Now we use the transitivity triangle / cofiber sequence for the composition $i \circ q$:

$$\mathbb{L}_{B\otimes_A B/B} \otimes_{B \otimes_A B} B \to \underbrace{\mathbb{L}_{B/B}}_{=0} \to \mathbb{L}_{B/B\otimes_A B}$$

Remember that if $X \to Y \to Z$ is a cofiber sequence, then $Z[-1] \to X \to Y$ is, too. Also, if $X \to Y \to 0$ is a cofiber sequence, then $X \simeq Y$, i.e. X and Y are quasi-isomorphic. Therefore we get

$$\mathbb{L}_{B/B\otimes_A B}[-1] \xrightarrow{\simeq} \mathbb{L}_{B\otimes_A B/B} \otimes_{B\otimes_A B} B$$

But the first term is $\simeq 0$ since i is a localization morphism and the second term is

$$\mathbb{L}_{B\otimes_A B/B} \otimes_{B\otimes_A B} B \simeq \mathbb{L}_{B/A} \otimes_B (B \otimes_A B) \otimes_{B\otimes_A B} B \simeq \mathbb{L}_{B/A}$$

We used in the proof that you can just pass back and forth between simplicial B-modules and (non-negatively graded) chain complexes. □

Remark 4.31. If f is formally étale, then $\Omega^1_{B/A} = 0$. The converse is not true! But we will see later that f is formally étale if and only if $\mathbb{L}_{B/A} \simeq 0$ and f is finitely presented.

We see yet again that the cotangent complex is a much more powerful invariant. Here it can detect étaleness.

Lecture 5: 2013-05-22

Here are some references.

- Introduction to model categories: Dwyer-Spalinsky – Homotopy theory and model categories. It is contained in the handbook of algebraic topology or available from the homepage of William Dwyer.
- Cotangent complex and simplicial algebras: Goerss-Schemmerhorn – Model categories and simplicial methods. Available on the arXiv.
- Also good: The original article by Quillen – On the (co)homology of commutative rings. It appeared in some book.

5 Deformation theory

Today we start with real deformation theory. A (not very readable) reference is "Complex cotangent et déformations" by Illusie. It appeared as a Springer Lecture Notes book, available at the library.

5.1 Motivation

We give some motivational thoughts first.

The key thing for deformation theory is the extension of the Ω^1-sequence to the left. We will see this by considering the 2 central problems in deformation theory.

5.1.1 First problem: Kodaira-Spencer theory

Geometric situation Let X be a smooth projective variety over $k = \mathbb{C}$, $X \to \operatorname{Spec}(k) = *$. We deform X over a base S. Let $\mathfrak{X}$ be the deformation, that is we have a cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \text{flat} \\ * & \longrightarrow & S \end{array}$$

Question: If we take a slightly bigger base S', can we extend the deformation? Given a square-zero extension $S \hookrightarrow S'$ (given by an ideal sheaf of S in S' with $I^2 = 0$) and the diagram

$$\begin{array}{ccc} \mathfrak{X} & & \\ \downarrow & & \\ S & \hookrightarrow & S' \end{array}$$

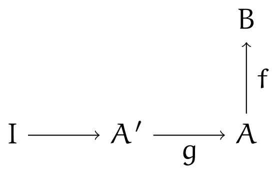
is it possible to find $\mathfrak{X}'$ such that

$$\begin{array}{ccc} \mathfrak{X} & \overset{}{\dashrightarrow} & \mathfrak{X}' \\ \downarrow & & \downarrow \\ S & \hookrightarrow & S' \end{array}$$

is cartesian?

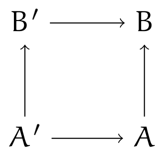
Answer (in the 60s, before Quillen): There exists a class $\alpha \in H^2(\mathfrak{X}, T_{\mathfrak{X}/S} \otimes I)$ such that $\alpha = 0$ if and only if $\mathfrak{X}' \to S'$ exists and makes the diagram cartesian. Here $T_{\mathfrak{X}/S}$ is the relative tangent sheaf. The set of solutions is then a torsor over $H^1(\mathfrak{X}, T_{\mathfrak{X}/S} \otimes I)$ if this set is not empty.

Algebraic situation We can instead do the same without all assumptions, purely algebraically. Here we start from a diagram in the category of algebras



with f flat and g a square-zero extension.

Question: Does there exist a commutative diagram



such that $B' \otimes_{A'} A \cong B$?

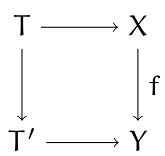
Answer: There exists a class $\alpha \in \mathrm{Ext}^2(\mathbb{L}_{B/A}, I \otimes_A B)$ such that $\alpha = 0$ if and only if such a B' exists and makes the diagram cartesian. The set of isomorphism classes of solutions is then a torsor over $\mathrm{Ext}^1(\mathbb{L}_{B/A}, I \otimes_A B)$.

Remark 5.1. In practice (with real examples) it can be very complicated to get a handle on this α. Therefore the theory is most useful if we know $\mathrm{Ext}^2 = 0$, for instance if $\mathbb{L}_{B/A}$ is small, e.g. concentrated in degree 0.

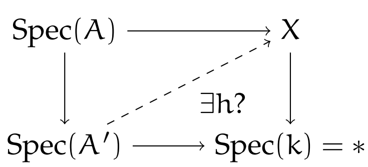
Remark 5.2. Illusie does everything with topoi, so that globally he doesn't need to concern himself with gluing problems.

5.1.2 Second problem: Lifting morphisms

Geometric situation Here we pose the following **Question**: Given a commutative diagram



where $T \hookrightarrow T'$ is a square-zero extension, is there a lift $h\colon T' \to X$, making both triangles commutative? The most important special case (e.g. for moduli problems) is



where A, A' are Artinian local algebras. Often we want to know how coherent sheaves deform under thickening of points. Are we sitting at a smooth point of our moduli space? The connection with the diagram is the following:

Definition 5.3 (Smoothness according to Grothendieck)**.** By definition, $f\colon X \to Y$ is smooth, if for all test diagrams

$$\begin{array}{ccc} T & \longrightarrow & X \\ \big\downarrow & \overset{h}{\nearrow} & \big\downarrow \\ T' & \longrightarrow & Y \end{array}$$

with $T \to T'$ a square-zero extension, the lifting h exists.

Remark 5.4. In general, this concept of smoothness differs from regularity.

Picture it like this: For a square-zero extension $T \to T'$, both T and T' have the same underlying topological space. We may think of square-zero extensions as a thickening up, adding a little bit more nilpotent stuff. For points, you can reduce to the case of Artinian local algebras.

Answer: We will later see that there exists a class $\alpha \in \mathrm{Ext}^1(p^*\mathbb{L}_{X/Y}, I)$ such that $\alpha = 0$ if and only if h exists. The set of solutions is a torsor over $\mathrm{Ext}^0(p^*\mathbb{L}_{X/Y}, I)$. Here I is the ideal of the square-zero extension.

Algebraic situation Once again we have the **Question**: For a commutative diagram

$$\begin{array}{ccc} T & \longleftarrow & B \\ \big\uparrow & \overset{\exists h?}{\swarrow} & \big\uparrow f \\ T' & \longleftarrow & A \end{array}$$

with $T' \to T$ a square-zero extension, is there a lift h making the diagram commutative?

Answer: There exists a class $\alpha \in \mathrm{Ext}^1(\mathbb{L}_{B/A} \otimes_B T, I)$ such that $\alpha = 0$ if and only if h exists and the set of solutions is a torsor over $\mathrm{Ext}^0(\mathbb{L}_{B/A} \otimes_B T, I)$.

5.1.3 Discussion

So we have this wonderful machine ($\mathbb{L}$) that takes difficult algebraic problems and transforms them into simple homological algebra. Why does it work? Why don't the Kähler differentials suffice?

This is the key point of the lecture. The reason is that the cotangent complex classifies square-zero extensions, i.e. to give a square-zero extension of A-algebras

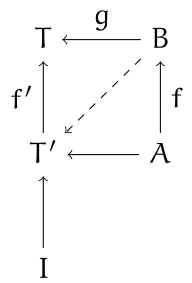
$$I \to B' \xrightarrow{\varphi} B$$

with kernel I is the same as to give a morphism

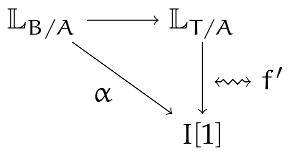
$$\mathbb{L}_{B/A} \to I[1]$$

where I[1] is the embedding of the kernel in Ch_A, degree-shifted by 1.

So there are two things to remember from this lecture: First, right adjoints preserve limits (RAPL) and second, deformation theory is all about square-zero extensions and $\mathbb{L}$ classifies them. An example for this, in the case of the second problem (lifting): To the diagram



of algebras corresponds the diagram of complexes



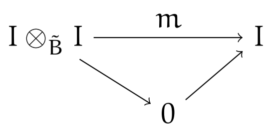
where $\alpha \in \mathrm{Ext}^1(\mathbb{L}_{B/A}, I)$. The map $\mathbb{L}_{B/A} \to \mathbb{L}_{T/A}$ is the derivative of g.

We will now go on to prove all this. There we have our program for the next 2–3 weeks.

5.2 Square-zero extensions give derivations

Definition 5.5 (Square-zero extension)**.** Let $\varphi\colon \tilde{B} \to B$ be a surjective morphism of A-algebras, $I = \ker\varphi$. We say φ is a square-zero extension, if $i \cdot i' = 0$ for every $i, i' \in I$.

This is equivalent to $I^2 = 0$ or to the commutativity of the diagram



where the multiplication map m is given by $i \otimes i' \mapsto i \cdot i'$.

Lemma 5.6. Let $\varphi\colon \tilde{B} \to B$ be a square-zero extension, $I = \ker\varphi$. By $b \cdot i := \tilde{b} \cdot i$, where $\varphi(\tilde{b}) = b$, we get a well-defined B-module structure on I.

Proof. By calculation. □

Definition 5.7 (Trivial square-zero extension)**.** A square-zero extension is called trivial, if there exists $s\colon B \to \tilde{B}$

$$\tilde{B} \underset{\varphi}{\overset{s}{\rightleftarrows}} B$$

such that $\varphi \circ s = \mathrm{id}_B$. By the splitting lemma, it follows that $\tilde{B} \cong B \oplus I$.

Lemma 5.8. Let $I \to \tilde{B} \xrightarrow{\varphi} B$ be a square-zero extension and $d \in \mathrm{Der}_A(B, I)$. Then $f\colon \tilde{B} \to \tilde{B}$, defined by $\tilde{b} \mapsto \tilde{b} + d\varphi(\tilde{b})$, is an automorphism of $\tilde{B}$.

Proof. We only check that it is an algebra homomorphism.

$$\begin{aligned} f(\tilde{b}) \cdot f(b') &= (\tilde{b} + d\varphi(\tilde{b})) \cdot (b' + d\varphi(b')) \\ &= \tilde{b}b' + \tilde{b}\, d\varphi(b') + b'\, d\varphi(\tilde{b}) + 0 \\ &= \tilde{b}b' + \varphi(\tilde{b})\, d\varphi(b') + \varphi(b')\, d\varphi(\tilde{b}) \\ &= \tilde{b}b' + d(\varphi(\tilde{b})\varphi(b')) \\ &= f(\tilde{b}b') \end{aligned}$$

□

Lemma 5.9. Let

$$I \longrightarrow \tilde{B} \underset{s'}{\overset{s}{\rightleftarrows}} B$$

be a trivial square-zero extension with two sections s and s'. Then $s - s' \in \mathrm{Der}_A(B, I)$.

Proof. Exercise. □

In summary, we have proved the following:

Proposition 5.10. *Let* $I \to \tilde{B} \xrightarrow{\varphi} B$ *a trivial square-zero extension. Then* $\mathrm{Aut}(\varphi) \cong \mathrm{Der}_A(B, I)$. *Here an element of* $\mathrm{Aut}(\varphi)$ *is an automorphism* α *of* $\tilde{B}$ *with* $\varphi = \varphi \circ \alpha$.

Remark 5.11. Here are just some other stupid ways of formulating the preceding lemma:

$$\begin{aligned} \mathrm{Aut}(\varphi) &\cong \mathrm{Der}_A(B, I) \\ &\cong \mathrm{Hom}_{\mathrm{Mod}_B}(\Omega^1_{B/A}, I) \\ &\cong \mathrm{Hom}_{\mathrm{Ch}_A}(\Omega^1_{B/A}, I) \\ &\cong \mathrm{Ext}^0_{\mathrm{Mod}_B}(\Omega^1_{B/A}, I) \end{aligned}$$

Remark 5.12. Since the automorphism group of a trivial square zero extension is isomorphic to $\mathrm{Ext}^0(\Omega_{B/A}, I)$, it is very tempting to think that isomorphism classes of general square-zero extensions should correspond to $\mathrm{Ext}^1(\Omega_{B/A}, I)$. However, this is not true! $\Omega_{B/A}$ again fails to do the job.

It only becomes true if we replace $\Omega^1_{B/A}$ by $\mathbb{L}_{B/A}$.

5.3 Derivations give square-zero extensions

Let $\alpha \in \mathrm{Ext}^1(\Omega^1_{B/A}, I)$. It represents a morphism $\eta\colon \Omega^1_{B/A} \to I[1]$ (caution: We are being sloppy about the categories we're working in and probably have to choose resolutions and so on). In the earlier lectures we proved

$$\mathrm{Hom}_{\mathrm{Mod}^s_B}(\Omega^1_{B/A}, I[1]) \cong \mathrm{Hom}_{\mathrm{sAlg}_{A/B}}(B, B \oplus I[1])$$

We can form the fiber product (cartesian diagram)

$$\begin{array}{ccc} B^\eta & \xrightarrow{f} & B \\ \downarrow & & \downarrow \eta \\ B & \xrightarrow[d_0]{} & B \oplus I[1] \end{array}$$

where d_0 is the trivial derivation, given as the section $B \to B \oplus M$, $b \mapsto (b, 0)$ of the map $B \oplus M \to B$. We now claim: f is a square-zero extension.

Lemma 5.13. B^η is a discrete algebra, i.e. $H_i(B^\eta) = 0$ for all $i \geqslant 1$.

Proof. The diagram

$$\begin{array}{ccc} B^\eta & \xrightarrow{f} & B \\ \downarrow & & \downarrow \eta \\ B & \xrightarrow[d_0]{} & B \oplus I[1] \end{array}$$

is a fiber product of chain complexes of Mod_B, which is an abelian category, therefore we get an exact triangle

$$B^\eta \to B \oplus B \to B \oplus I[1]$$

Here is a part of the corresponding long exact sequence in homology:

$$H_2(B \oplus I[1]) \to H_1(B\eta) \to H_1(B \oplus B) \to H_1(B \oplus I[1]) \to \cdots \to H_0(B \oplus I[1])$$

Since B is concentrated in degree 0 and $I[1]$ is concentrated in degree 1, we have $H_2(B \oplus I[1]) = 0$ and $H_1(B \oplus B) = 0$ and it follows $H_1(B^\eta) = 0$. The higher homology groups are 0 by a similar argument. □

Remark 5.14. If $I \xrightarrow{f} B \xrightarrow{\varphi} C$ is a morphism in sAlg_A, $I = \ker \varphi$ (taken level-wise), then there is a multiplication map $m\colon I \otimes_B I \to I$, given by

$$I \otimes_B I \xrightarrow{\mathrm{id} \otimes f} I \otimes_B B \cong I$$

Definition 5.15 (Square-zero extension)**.** Let $\varphi\colon B \to C$ a morphism in $sAlg_A$, levelwise surjective. We call φ a square-zero extension, if the diagram

$$\begin{array}{ccc} \ker\varphi \otimes_B \ker\varphi & \xrightarrow{m} & \ker\varphi \\ & \searrow \quad 0 \quad \nearrow & \end{array}$$

is commutative.

Lecture 6: 2013-06-05

Last time we made the statement that (infinitesimal) deformation theory is equivalent to understanding square-zero extensions. We had the problem of lifting morphisms, whether there is

$$\begin{array}{ccc} \mathrm{Spec}(A) & \longrightarrow & X \\ \downarrow & \overset{\exists h?}{\nearrow} & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(k) = * \end{array}$$

for all local artinian k-algebras A with residue field k and square-zero extensions $A' \to A$. We say again that this can be very hard to check. $\mathrm{Spec}(A)$ is never a variety, except when it is a field. These objects are not in itself very interesting. Basic examples are $\mathrm{Spec}(k[x]/x^2)$, which describes first-order deformations of the point in one direction and $\mathrm{Spec}(k[x]/x^3)$ which describes second-order deformations. Sometimes it is enough to check all $\mathrm{Spec}(k[x]/x^n)$ (called curvilinear square-zero extensions), but the assumptions non X etc. for this to work are horrible. The condition on X is called T^1.

Our goal is therefore to show that $\mathbb{L}_{B/A}$ classifies square-zero extensions, i.e.

$$\mathrm{Exalcomm}_A(B, I) \simeq \mathrm{Ext}^1(\mathbb{L}_{B/A}, I) = \mathrm{Hom}_{D(\mathrm{Mod}_B)}(\mathbb{L}_{B/A}, I[1])$$

The first objects has its name from french "extensions algèbres commutatifs". Isomorphisms of square-zero extensions are similar to group extensions:

$$\begin{array}{ccccc} I & \longrightarrow & B' & \longrightarrow & B \\ \| & & \uparrow \cong & & \| \\ I & \longrightarrow & B'' & \longrightarrow & B \end{array}$$

How does this equivalence work? Given $B \in sAlg_A$ and a morphism $\eta\colon \Omega^1_{B/A} \to M[1]$ of B-modules this corresponds to $\eta\colon B \to B \oplus M[1]$ and we know

$$\mathrm{Hom}_{\mathrm{Mod}^s_B}(\Omega^1_{B/A}, M[1]) \cong \mathrm{Hom}_{sAlg_A/B}(B, B \oplus M[1]) =: \mathrm{Der}_A(B, M[1])$$

Then we can form the fiber product (what kind of fiber product?):

$$\begin{array}{ccc} B^\eta & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \eta \\ B & \xrightarrow[s_0]{} & B \oplus M[1] \end{array}$$

where s_0 is the zero derivation $b \mapsto (b, 0)$. We now claim that $B^\eta \to B$ is a square-zero extension and $M = \ker \phi$.

Remark 5.16. Consider morphisms $I \xrightarrow{\psi} B \xrightarrow{\phi} C$ in sAlg_A where $I = \ker \phi$. Then we have a multiplication map

$$m \colon I \otimes_B I \xrightarrow{\mathrm{id} \otimes \phi} I \otimes_B B \xrightarrow{\cong} I$$

Definition 5.17. A morphism $B \xrightarrow{\phi} C$ in sAlg_A is called square-zero extension if the multiplication map $m \colon I \otimes_B I \to I$ factors through 0.

Proposition 5.18. *Let* $B \in \mathrm{sAlg}_A$ *and* $\eta \colon \Omega^1_{B/A} \to M[1]$ *be an element of* $\mathrm{Der}_A(B, M[1])$. *Then* $B^\eta \xrightarrow{\phi} B$ *is a square-zero extension.*

Proof. The proof was messed up and complete nonsense. We will try again next time. □

The upshot is thus: The functor

$$\Phi \colon \Omega^1_{B/A} \backslash \mathrm{Mod}^s_B \to \mathrm{sAlg}_A / B$$

factors over the square-zero extensions. By $A \backslash \mathcal{C}$ we denote the under category $\mathcal{C}$ under A with objects $A \to B$, the dual notion to the more familiar over (or comma) category $\mathcal{C}/A$. We want an adjoint functor in the other direction.

Definition 5.19. Define a functor $\Psi \colon \mathrm{sAlg}_A / B \to \Omega^1_{B/A} \backslash \mathrm{Mod}^s_B$ by sending $A \to B \to C$ to $\Omega^1_{B/A} \to \Omega^1_{B/C}$.

Remark 5.20. Technically the category $\Omega^1_{B/A} \backslash \mathrm{Mod}^s_B$ starts in degree 1. It is a pain to show this is a model category.

Lemma 5.21. The functors Ψ and Φ are an adjoint pair, i.e. Ψ is left adjoint to Φ.

Proof. Only a sketch, the proof is very boring. We have to show

$$\mathrm{Hom}_{\Omega^1_{B/A} \backslash \mathrm{Mod}^s_B}(\Omega^1_{B/A}, M[1]) \cong \mathrm{Hom}_{\mathrm{sAlg}_A / B}(C, B^\eta)$$

Consider the diagram

$$\begin{array}{ccccc} C & & & & \\ & \overset{f}{\searrow} & & & \\ & & B^{\eta} & \xrightarrow{\phi} & B \\ & & \downarrow & & \downarrow \\ & & B & \xrightarrow[s_0]{} & B \oplus M[1] \end{array} \qquad (h\colon C \to B,\ g\colon C \to B)$$

Giving f is the same as giving g, h. Since Ω^1 represents derivations, this is the same as

$$\begin{array}{ccc} \Omega^1_{C/A} \otimes_C B & \longrightarrow & \Omega^1_{B/A} \\ & \overset{0}{\searrow} & \downarrow \\ & & M[1] \end{array}$$

The Ω give an exact sequence so this is the same as

$$\begin{array}{ccc} \Omega^1_{B/A} & \longrightarrow & \Omega^1_{B/C} \\ \downarrow & \overset{!}{\swarrow} & \\ M[1] & & \end{array}$$

Here we have the morphism that we wanted. □

Lemma 5.22. Even more is true: The functors Ψ and Φ are equivalences of categories and form a quillen pair.

Proof. Since cofibrations are hard objects to deal with, we show something about weak equivalences and fibrations. It suffices to show that Φ preserves fibrations and weak equivalences. Then automatically it preserves cofibrations. We didn't prove this (it follows from stealing), but it's true.

Let $U\colon \mathrm{sAlg}_A/B \to s(\mathrm{Mod}_A)$ be the forgetful functor. It is a right adjoint, so it preserves limits. So f is a fibration (respectively a weak equivalence) if and only if $U(f)$ is. Now we know that f is a fibration if and only if it is surjective. Let

$$\begin{array}{ccc} & \Omega^1_{B/A} & \\ {}^{\eta}\swarrow & & \searrow^{\theta} \\ M[1] & \longrightarrow & N[1] \end{array}$$

Since

$$\begin{array}{ccc} B^\eta & \longrightarrow & B \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \oplus M[1] \end{array}$$

is a pullback,

$$\begin{array}{ccc} U(B^\eta) & \longrightarrow & U(B) \\ \downarrow & & \downarrow \\ U(B) & \longrightarrow & U(B \oplus M[1]) \end{array}$$

is a pullback in sMod_B (since RAPL) which is an abelian category. Therefore

$$0 \to U(B^\eta) \to U(B) \oplus U(B) \to U(B \oplus M[1]) \to 0$$

is an exact sequence. We now drop the U from the notation. We get the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B^\eta & \longrightarrow & B \oplus B & \longrightarrow & B \oplus M[1] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Phi(f)} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id} \oplus f} & & \\ 0 & \longrightarrow & B^\theta & \longrightarrow & B \oplus B & \longrightarrow & B \oplus N[1] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & 0 & \longrightarrow & 0 & & \end{array}$$

where the cokernels are exact too, therefore 0. This shows $\Phi(f)$ is surjective, hence Φ preserves fibrations.

For weak equivalences, we consider the same diagram with f a w.e., then we have to show $\Phi(f)$ is a quasi-isomorphism. This corresponds to the cone being zero. Instead of the cokernel, we therefore take the cones, they are $\simeq 0$ in the third and fourth column. Every exact sequence is an exact triangle, hence the cones also form an exact tirangle. It follows

$$\mathrm{cone}(\Phi(f)) \simeq 0$$

and therefore $\Phi(f)$ is a weak equivalence. □

Lecture 7: 2013-06-19

Let's try again with the proposition of last time.

Proposition 5.23. *If* $B \in \mathrm{sAlg}_A$ *and* $\eta\colon \Omega^1_{B/A} \to M[1]$*, then* $B^\eta \to B$ *is a square-zero extension.*

Proof. Let $\tilde{M} = \ker \eta$. We have

$$\begin{array}{ccccc} B & \longrightarrow & B \oplus \Omega^1_{B/A} & \xrightarrow{\eta} & B \oplus M[1] \\ b & \longmapsto & (b, db) & \longmapsto & (b, \eta\, db) \end{array}$$

and

$$\begin{array}{ccccc} B & \longrightarrow & B \oplus \Omega^1_{B/A} & \xrightarrow{s_0} & B \oplus M[1] \\ b & \longmapsto & (b, db) & \longmapsto & (b, 0) \end{array}$$

Hence B^η as a set is $\{(b, b') \mid (b, \eta\, db) = (b', 0)\}$.

Let's consider

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \downarrow & & \downarrow d \\ \tilde{M} & \xrightarrow{f} & \Omega^1_{B/A} \end{array}$$

hence B' as a set is

$$B' = \{(b, \tilde{m}) \mid db = f(m)\}$$

Multiplication is $(b, \tilde{m}) \cdot (b', \tilde{m}') = (b \cdot b', b \cdot \tilde{m}' + b' \cdot \tilde{m})$ and the morphism $B' \to B$ is given by $(b, \tilde{m}) \mapsto b$, hence B' is a square-zero extension. But $B^\eta \simeq B'$ by $(b, b') \mapsto (b, db)$. □

So now we can say that Φ factors over the square-zero extensions. As (Ψ, Φ) is a Quillen pair, we have adjunctions

$$\mathrm{Ho}(\mathrm{sAlg}_A /B) \underset{\mathrm{Ho}(\Phi)}{\overset{\mathrm{Ho}(\Psi)}{\rightleftarrows}} \mathrm{Ho}((\Omega^1_{B/A} \backslash \mathrm{Mod}^s_B)_1)$$

A problem is, that (Φ, Ψ) is not a Quillen equivalence, even on square-zero extensions.

5.4 Some further homological algebra

Let A be a ring, M a complex of A-modules (non-negatively graded).

Definition 5.24. M is

- n-truncated if $H_i(M) = 0$ for all $i > n$.
- n-connective if $H_i(M) = 0$ for all $i < n$.
- discrete if M is zero-truncated.

Remark 5.25. Let $M \xrightarrow{f} N$ be a morphism of complexes. Then $M \xrightarrow{f} N \to \mathrm{cone}(f)$ is an exact triangle. We get that $\mathrm{cone}(f)[-1] \to M \to N$ is again an exact triangle. Caveat: This makes sense i.e. stays exact in our category if $\mathrm{Ho}(M) \to \mathrm{Ho}(N)$.

Remark 5.26. Other terminology for $\mathrm{cone}(f)$ is the cofiber of f and $\mathrm{cone}(f)[-1]$ is called the cocone of f or the fiber of f.

Definition 5.27. $M \xrightarrow{f} N$ is called

- n-connective if $\mathrm{fib}(f)$ is n-connective.
- n-truncated if $\mathrm{fib}(f)$ is n-truncated.

Lemma 5.28. A morphism $f\colon M \to N$ is n-connective if and only if $H_i(f)$ is an isomorphism for $i < n$ and surjective for $i = n$.

Proof. Check the long exact sequence coming from $\mathrm{fib}(f) \to M \to N$. □

Remark 5.29. There is an internal Hom on complexes of non-negatively graded A-modules:

$$(\underline{\mathrm{Hom}}(M, N))_n = \prod_q \mathrm{Hom}(M_q, N_{q+n})$$

The differential df is defined by

$$df(m) = d_N fm - (-1)^{|f|} f(d_M(m))$$

where $|f| = n$ if $f \in (\underline{\mathrm{Hom}}(M, N))_n$.

Check: $H_i(\underline{\mathrm{Hom}}(M, N))$ are the chain homotopy classes of maps $M \to N[i]$.

Definition 5.30. Let $P \to M$ be a projective resolution (cofibrant replacement). Then

$$\mathbb{R}\underline{\mathrm{Hom}}(M, N) := \underline{\mathrm{Hom}}(P, N)$$

and

$$H_i(\mathbb{R}\underline{\mathrm{Hom}}(M, N)) := \mathrm{Ext}^i(M, N) = \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Ch}_A))}(M, N[i])$$

Recall that $\underline{\mathrm{Hom}}(-, N)\colon \mathrm{Ch}_A \to \mathrm{Ch}_A$ and $\mathbb{R}\underline{\mathrm{Hom}}(-, N)\colon \mathrm{Ho}(\mathrm{Ch}_A) \to \mathrm{Ho}(\mathrm{Ch}_A)$ where $\mathrm{Ho}(\mathrm{Ch}_A) = D(A)$.

5.5 n-concentrated square-zero extensions

Definition 5.31. A derivation $\Omega^1_{B/A} \to M[1]$ is n-concentrated if M is n-truncated and n-connective, i.e $M \cong N[n]$ for N discrete.

Definition 5.32. We call $A \to C \xrightarrow{f} B$ in sAlg_A/B an n-concentrated square-zero extension if

- if $n > 0$ then $\mathrm{fib}(f)$ is n-truncated and n-connective
- if $n = 0$ then $\mathrm{fib}(f$ is 0-truncated and 0-connective and $\mathrm{fib}(f) \otimes_C \mathrm{fib}(f) \to \mathrm{fib}(f)$ factors over the 0 morphism.

Lecture 8: 2013-06-26

Assume B is cofibrant. Then our functors do the following:

$$\mathrm{Ho}(\Psi)\colon (A \to C \to B) \mapsto (\mathbb{L}_{B/A} \to \mathbb{L}_{B/C})$$

and

$$\mathrm{Ho}(\Phi)\colon (\mathbb{L}_{B/A} \xrightarrow{\eta} M) \longmapsto \begin{array}{ccc} B^\eta & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle s_0} \\ B & \longrightarrow & B \oplus M \end{array} \longmapsto (A \to B^\eta \to B)$$

Sadly, this is not an equivalence even for square-zero extensions.

Remark 5.33. Let A, B be discrete algebras. Then $A \to B' \to B$ is 0-concentrated if and only if $B' \to B$ is a square-zero extension of B in the usual sense.

Definition 5.34. $\mathrm{SqZ}_n^{A/B} \subset \mathrm{Ho}(\mathrm{sAlg}_A/B)$ is the full subcategory of n-concentrated square-zero extensions.

Definition 5.35. $\mathrm{Der}_n^{A/B} \subset \mathrm{Ho}(\Omega^1_{B/A} \backslash \mathrm{Mod}_B)$ is the full subcategory of n-concentrated derivations.

Lemma 5.36. Given $\Omega^1_{B/A} \xrightarrow{\eta} M$, let $B^\eta \xrightarrow{f} B$ be the image of η under $\mathrm{Ho}(\Phi)$. Then $\mathrm{fib}(f) \simeq M[-1]$.

Proof. We have the homotopy fiber sequence

$$\begin{array}{ccc} B^\eta & \xrightarrow{f} & B \\ \downarrow & & \downarrow{\scriptstyle \eta} \\ B & \xrightarrow[s_0]{} & B \oplus M \end{array}$$

Hence $\mathrm{fib}(s_0) \simeq \mathrm{fib}(f)$. Also $B \to B \oplus M \to M$ is an exact triangle, so $M[-1] \to B \to B \oplus M$ is an exact triangle. Therefore $\mathrm{fib}(s_0) \simeq M[-1]$. □

Remark 5.37. Let $A \to B$ be a morphism of algebras, $m\colon B \oplus_A B \to B$ the multiplication map, $I = \ker m$, $I/I^2 = \Omega^1_{B/A}$. If $A \to B$ is a square-zero extension, then $I^2 = 0$, therefore $\Omega^1_{B/A} = I = \ker m$.

Now let $A \to B$ be a morphism of simplicial algebras and $I' = \mathrm{fib}(m\colon B \otimes_A B \to B)$. We get the homotopy pushout

$$\begin{array}{ccc} I' \otimes I' & \xrightarrow{m} & I' \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{L}_{B/A} \end{array}$$

If $A \to B$ is a square-zero extension, then $\mathbb{L}_{B/A} = \mathrm{fib}(m)$.

Theorem 5.38 (Main theorem of the lecture)**.** *Assume $A \to B$ is cofibrant. Then*

$$\Phi_n \colon \mathrm{Der}_n^{A/B} \to \mathrm{SqZ}_n^{A/B}$$

is an equivalence of categories.

Proof. We split the proof in several parts. It will be quite long.

- Let $\eta \in \mathrm{Der}_n^{A/B}$. Then $\Phi_n(\eta) \in \mathrm{SqZ}_n^{A/B}$ since $\mathbb{L}_{B/A} \xrightarrow{\eta} M$ is concentrated in degree $n+1$ and therefore $\mathrm{Ho}(\Phi)(\eta) \colon B^\eta \xrightarrow{f} B$, $\mathrm{fib}(f) \simeq M[-1]$ which is concentrated in degree n. Moreover, $\mathrm{fib}(f) \otimes \mathrm{fib}(f) \to \mathrm{fib}(f)$ factors over the 0 morphism.

- Adjointness problem:

  $$\mathrm{Ho}(\Psi) \colon (A \to C \to B) \mapsto (\mathbb{L}_{B/A} \to \mathbb{L}_{B/C})$$

  does not end up only in degree $n+1$. But Φ_n admits a left adjoint by $\Psi_n \coloneqq \tau_{\leqslant n+1} \circ \mathrm{Ho}(\Psi)$ where $\tau_{\leqslant n+1}$ simply truncates at $n+1$.
  We have to show that

  $$\mathbb{L}_{B/A} \to \mathbb{L}_{B/C} \to \tau_{\leqslant n+1}\mathbb{L}_{B/C} \in \mathrm{Der}_n^{A/B}$$

  Identify

  $$\mathbb{L}_{B/C} \rightleftarrows B \otimes_C B \underset{\mathrm{id}\otimes f}{\overset{m}{\rightleftarrows}} B \cong B \otimes_C C \rightleftarrows B \otimes_C \mathrm{fib}(f)$$

  The first and the last three terms are each an exact triangle with section. So

  $$\mathbb{L}_{B/C} \simeq (\mathrm{fib}(f) \otimes_C B)[1]$$

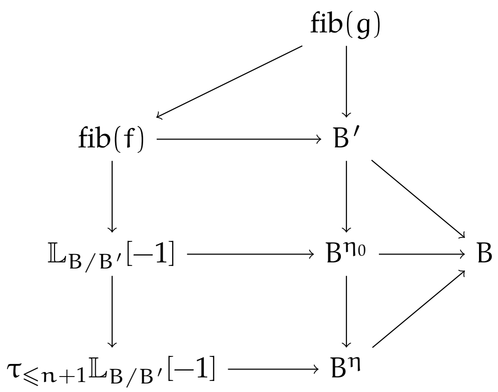
  and therefore Ψ_n has target $\mathrm{Der}_n^{A/B}$.

- We saw that $\mathrm{Ho}(\Phi)$ is conservative, i.e. f is an isomorphism if and only if $\mathrm{Ho}(\Phi)(f)$ is an isomorphism. Therefore Φ_n is conservative.

- It suffices to check that the unit transformation $u \colon \mathrm{id} \to \Phi_n \circ \Psi_n$ is an isomorphism.

$$\begin{array}{ccccc}
B' \xrightarrow{f} B \in \mathrm{SqZ}_n^{A/B} & & B' & & \\
\downarrow \mathrm{Ho}(\Psi) & & \downarrow g & \searrow f & \\
(\mathbb{L}_{B/A} \xrightarrow{\eta_0} \mathbb{L}_{B/B'}) & \overset{\Phi}{\longmapsto} & B^{\eta_0} & \xrightarrow{f'} & B \\
\downarrow \tau_{\leqslant n+1} & & \downarrow g' & & \| \\
(\mathbb{L}_{B/A} \xrightarrow{\eta} \tau_{\leqslant n+1}\mathbb{L}_{B/B'}) & \overset{\Phi}{\longmapsto} & B^{\eta} & \xrightarrow{f''} & B
\end{array}$$

We have to show that $g' \circ g$ is an equivalence of simplicial algebras. Consider the kernels $\ker f = \mathrm{fib}(f)$, $\ker f' = \mathbb{L}_{B/B'}[-1]$, $\ker f'' = \tau_{\leqslant n+1}\mathbb{L}_{B/B'}[-1]$. This results in the following diagram:



- It suffices to show that $\mathrm{fib}(f) \to \mathrm{fib}(f')$ is an isomorphism in degree n. We have an exact triangle
$$\mathrm{fib}(g) \to \mathrm{fib}(f) \to \mathrm{fib}(f') = \mathbb{L}_{B/B'}[-1]$$
We have just seen
$$\mathbb{L}_{B/B'} \simeq (\mathrm{fib}(f) \otimes_{B'} B)[1]$$
and therefore $\mathbb{L}_{B/B'}[-1] \simeq \mathrm{fib}(f) \otimes_{B'} B$. This gives us
$$\mathrm{fib}(g) \to \mathrm{fib}(f) \ \to \mathrm{fib}(f) \otimes_{B'} B$$
and
$$\mathrm{fib}(f) \otimes_{B'} \mathrm{fib}(f) \xrightarrow{m} \mathrm{fib}(f) \otimes_{B'} B \xrightarrow{\mathrm{id} \otimes f} \mathrm{fib}(f) \otimes_{B'} B$$
and m is homotopic to 0. Then
$$\mathrm{fib}(f) \otimes_{B'} B \to \mathrm{fib}(f) \otimes_{B'} B \to (\mathrm{fib}(f) \otimes_{B'} \mathrm{fib}(f))[1]$$
and so
$$\mathrm{fib}(f') \simeq \mathrm{fib}(f) \otimes_{B'} B \simeq \mathrm{fib}(f) \otimes \mathrm{fib}(g)[1]$$
The degrees are $\geqslant n$, n and $2n+1$. Then $\mathrm{fib}(f')$ and $\mathrm{fib}(f)$ must be isomorphic in degree n. □

Remark 5.39. For $n = 0$ and ordinary algebras, the "cofibrant" can be omitted.

Lecture 9: 2013-07-03

Remember: We had a correspondence between sAlg_A / B and $\Omega^1_{B/A} \backslash \mathrm{Mod}^s_B$. Assume B is cofibrant as A-algebra. Then $\Omega^1_{B/A} = \mathbb{L}_{B/A}$. Here $\Omega^1_{B/A}$ does *not* denote the complex concentrated in degree 0, but Ω is applied level-wise. Passing to the homotopy category we get

$$\mathrm{Ho}(\mathrm{sAlg}_A/B) \underset{\mathrm{Ho}(\Phi)}{\overset{\mathrm{Ho}(\Psi)}{\rightleftarrows}} \mathrm{Ho}(\mathbb{L}_{B/A} \backslash \mathrm{Mod}_B^s)$$

We also had the following theorem:

Theorem 5.40 (The fundamental theorem). *There is an equivalence of categories*

$$\mathrm{SqZ}_n^{A/B} \simeq \mathrm{Der}_n^{A/B}$$

induced by Ψ_n *and* Φ_n.

Remark 5.41. For ordinary algebras A and B we can drop the cofibrancy condition on B. The theorem then still holds for $n = 0$ with $\mathrm{Der}_n^{A/B}$ replaced by

$$\mathrm{d\,Der}_n^{A/B} = \mathrm{Ho}(\mathbb{L}_{B/A} \backslash \mathrm{sMod}_B)$$

6 Solution of deformation theory problems

6.1 Lifting morphisms

Let the following commutative diagram be given:

$$\begin{array}{ccc} T & \longleftarrow & B \\ \uparrow & & \uparrow{\scriptstyle f} \\ T' & \longleftarrow & A \end{array}$$

Here $A \to T' \to T$ is an element in $\mathrm{SqZ}_n^{A/T}$. By the correspondence we get $\eta \colon \mathbb{L}_{T/A} \to M[1]$. Then we can look at

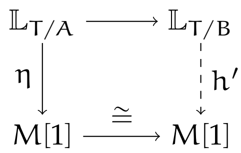
$$\alpha \colon \mathbb{L}_{B/A} \otimes_B T \to \mathbb{L}_{T/A} \to M[1]$$

We get the first morphism from the transitivity sequence. This gives us a class $[\alpha] \in \mathrm{Ext}_T^1(\mathbb{L}_{B/A} \otimes_B T, M)$.

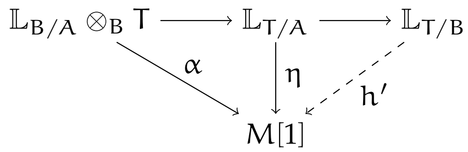
As promised: Let $f \colon A \to B$ be either a morphism of simplicial algebras with B cofibrant or an ordinary morphism of algebras with $n = 0$. Then:

Theorem 6.1. *The class* $[\alpha]$ *is* 0 *if and only if a lifting* $h \colon B \to T'$ *exists, making the resulting diagram commutative.*

Proof. The question is equivalent to: Is $T' \to T$ a square-zero extension of B-algebras? Also, the existence of h is equivalent to the existence of h' such that the following diagram commutes up to homotopy:

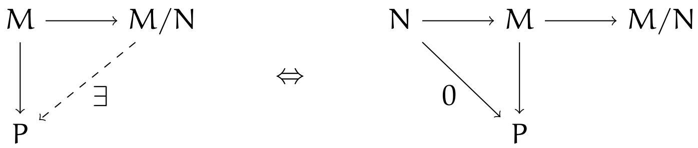


This again is equivalent to α being homotopic to 0 in the following diagram where the top row is given by the exact triangle cofiber sequence:



And this of course is equivalent to $[\alpha] = 0$. □

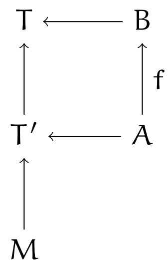
Remark 6.2. The analogue of the method used in the proof for modules is



Remark 6.3. Isomorphisms classes of liftings are given by $\mathrm{Ext}^0_T(\mathbb{L}_{B/A} \otimes_B T, M)$. As usually, they are parametrized by Ext in one degree less.

Theorem 6.4. *Let $A \to B$ be a morphism of rings, B finitely presented as an A-algebra. Then f is étale if and only if $\mathbb{L}_{B/A} \simeq 0$. Remember: For Ω in $\Leftarrow$ you only get unramified.*

Proof. We did $\Rightarrow$ a long time ago. Therefore, let $\mathbb{L}_{B/A} \simeq 0$. Since B is finitely presented, formally étale implies étale. We check a test diagram



The obstruction to the existence of a lifting h is

$$[\alpha] \in \mathrm{Ext}^1_T(\underbrace{\mathbb{L}_{B/A}}_{=0} \otimes_B T, M[1]) = 0$$

therefore we get the lift. Similarly, $\mathrm{Ext}^0_T(\mathbb{L}_{B/A} \otimes BT, M) = 0$ and hence h is unique up to isomorphism. □

Remark 6.5. Let $f\colon A \to B$ where B is finitely presented.

- f smooth follows from $\mathbb{L}_{B/A} \simeq \Omega^1_{B/A}$ with $\Omega^1_{B/A}$ projective by essentially the same proof (Ext^1 vanishes, therefore we have existence, not necessarily unique, therefore smooth).
- f unramified follows from $H_0(\mathbb{L}_{B/A}) \simeq \Omega^1_{B/A} \simeq 0$.

6.2 Kodaira-Spencer theory

Recall the problem. We start with

$$\begin{array}{ccccc} B & & & & \\ \uparrow{\scriptstyle \text{flat}} & & & & \\ A & \xleftarrow[f]{} & A' & \longleftarrow & I \end{array}$$

We ask whether there exists an A'-algebra B' with $B' \otimes^{\mathbb{L}}_{A'} A \cong B$. A weaker condition to satisfy would be $B' \otimes_{A'} A \cong B$ and

$$\begin{array}{ccccc} B & \longleftarrow & B' & \longleftarrow & J \\ \uparrow & & \uparrow & & \uparrow{\scriptstyle \exists u} \\ A & \longleftarrow & A' & \longleftarrow & I \end{array}$$

This is weaker because u doesn't have to be an isomorphism. We have u is an iso if and only if it is flat.

Lemma 6.6. Assume B' exists. Then $\mathrm{fib}(B' \to B) = I \otimes_{A'} B'$.

Proof. We have the cofiber sequences

$$I \to A' \to A \to I[1]$$

Therefore

$$\begin{array}{ccccc} A' & \longrightarrow & A & \longrightarrow & I[1] \\ \downarrow & & \downarrow & & \downarrow \\ B' & \longrightarrow & B & \longrightarrow & I[1] \otimes_A B \end{array}$$

where both squares are homotopy pushouts. Hence

$$\mathrm{cofib}(B' \to B) = I \otimes_A B[1]$$

and therefore

$$\mathrm{fib}(B' \to B) = I \otimes_A B \cong I \otimes_{A'} B' \qquad \square$$

Lemma 6.7. Assume B' exists. Then $A' \to B'$ is flat.

Proof. Let $M \in \mathrm{Mod}_A$. We have to show $M \otimes^{\mathbb{L}}_{A'} B'$ is discrete. We have an exact sequence

$$0 \to IM \to M \to M/IM \to 0$$

and applying $- \otimes_{A'} B'$ gives

$$IM \otimes^{\mathbb{L}}_{A'} B' \to M \otimes^{\mathbb{L}}_{A'} B' \to M/IM \otimes^{\mathbb{L}}_{A'} B'$$

is a cofiber sequence (an exact triangle). To show the discreteness (i.e. concentrated in degree 0) it therefore suffices to show $IM \otimes^{\mathbb{L}}_{A'} B'$ and $M/IM \otimes^{\mathbb{L}}_{A'} B'$ are discrete. Let N be one of these two. In both cases $IN = 0$, therefore N is an A'-module. It follow

$$N \otimes^{\mathbb{L}}_{A'} B' \cong N \otimes^{\mathbb{L}}_{A} (A \otimes_{A'} B') \cong N \otimes^{\mathbb{L}}_{A} B$$

which is discrete because B is flat over A. $\square$

Remark 6.8. If $J \to B' \to B$ and $I \to A' \to A$ are square-zero extensions, then

$$\begin{array}{ccccc} B & \longleftarrow & B' & \longleftarrow & J \\ \uparrow & & \uparrow & & \uparrow \\ A & \longleftarrow & A' & \longleftarrow & I \end{array}$$

is a morphism of square-zero extensions if and only if

$$\begin{array}{ccc} \mathbb{L}_B & \xrightarrow{\eta_J} & J[1] \\ \uparrow{\scriptstyle df} & & \uparrow \\ \mathbb{L}_A \otimes_A B & \xrightarrow{\eta_I} & I[1] \end{array}$$

commutes.

As before, take

$$\begin{array}{ccccc} B & & & & \\ \uparrow & & & & \\ A & \longleftarrow & A' & \longleftarrow & I \end{array}$$

We have the following diagram:

$$\begin{array}{ccc} \mathbb{L}_{B/A}[-1] & \longrightarrow & \mathbb{L}_A \otimes_A B \\ & \searrow^{\alpha} & \downarrow \\ & & I[1] \otimes_A B \end{array}$$

By shifting around

$$\mathbb{L}_A \otimes_A B \to \mathbb{L}_B \to \mathbb{L}_{B/A}$$

we can introduce $\mathbb{L}_{B/A}[-1] \to \mathbb{L}_A \otimes_A B$ on the left. So $\alpha\colon \mathbb{L}_{B/A}[-1] \to I \otimes_A B[1]$ induces $\alpha\colon \mathbb{L}_{B/A} \to I \otimes_A B[2]$ and therefore a class $[\alpha] \in \mathrm{Ext}^2_B(\mathbb{L}_{B/A}, I \otimes_A B)$.

Theorem 6.9. *We have* $[\alpha] = 0$ *if and only if there exists an* A'*-algebra* B' *such that* $B' \otimes^{\mathbb{L}}_{A'} A \cong B$.

Proof. For the necessary condition we need:

1. $B' \to B$ is a square-zero extension by $I \otimes_A B$
2. For commutativity:

$$\begin{array}{ccc} \mathbb{L}_A \otimes_A B & \longrightarrow & I \otimes_A B[1] \\ \downarrow & & \downarrow \cong \\ \mathbb{L}_b & \overset{\beta}{\dashrightarrow} & I \otimes_A B[1] \end{array}$$

Therefore B' exists if and only if there is a map β as in the above diagram which is the case if and only if there is an $\alpha \sim 0$ as in the following diagram making the left triangle commutative:

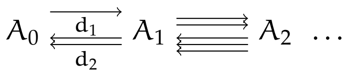
$$\begin{array}{ccccc} \mathbb{L}_{B/A}[-1] & \longrightarrow & \mathbb{L}_A \otimes_A B & \longrightarrow & \mathbb{L}_B \\ & \searrow^{\alpha} & \downarrow & \swarrow_{\beta} & \\ & & I \otimes_A B[1] & & \end{array}$$

This is equivalent to $[\alpha] = 0$ in $\mathrm{Ext}^2_B(\mathbb{L}_{B/A}, I \otimes_A B)$. □

The isomorphism classes are again given by Ext^1 and the automorphisms are given by Ext^0.

7 The conormal sequence and Postnikov decomposition

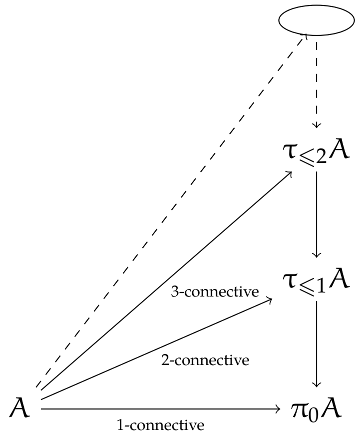
Let A be a simplicial algebra. A priori this seems to be a very complicated object. A is given by



We now define

$$\pi_0(A) = A_0/(d_1(A_1) - d_2(A_1))$$

and start constructing a sequence $\tau_{\leqslant i}A$ as in the following schematic:



Here $\tau_{\leqslant 0}A = \pi_0 A$ is 0-truncated, $\tau_{\leqslant 1}A$ is 1-truncated, i.e. $\pi_i(\tau_{\leqslant 1}A) = 0$ for all $i \geqslant 2$ and so on. Remember: i-connective means isomorphism on $k < i$ and surjective on i. The fascinating thing is: All $\tau_{\leqslant k}A \to \tau_{\leqslant k-1}A$ are square-zero extensions.

That means: Simplicial algebras are not *that* complicated. Instead of an infinite amount of algebras you start with one (i.e. $\pi_0 A$) and add square-zero extensions. Especially it is easier to find morphisms than one could think.

Geometrically this means: We start with a derived scheme $X = \mathrm{Spec}(A)$ and get a decomposition into a "usual" scheme $\mathrm{Spec}(\pi_0(A))$ and schemes $\tau_{\leqslant k}X$ with additional nilpotents in every degree $\leqslant k$. X will then be equivalent to the homotopy colimit of the $\tau_{\leqslant k}X$.

Lecture 10: 2013-07-10

Okay, let's do this in greater detail. As an introduction, consider the manifold S^2 which is embedded in $\mathbb{R}^3$ by $i\colon S^2 \hookrightarrow \mathbb{R}^3$. At a point $p \in S^2$ we have the 2-dimensional tangent space $T_p S^2$ and the fiber N_p of the normal bundle. They fit into the exact sequence

$$0 \to T_p S^2 \hookrightarrow T_{i(p)}\mathbb{R}^3 \to N_p \to 0$$

The dual sequence is

$$0 \to N_p^* \to (T_{i(p)}\mathbb{R}^3)^* \to (T_p S^2)^* \to 0$$

Algebraically, we start with a sequence

$$0 \to I \to A \to B \to 0$$

and the conormal exact sequence is

$$I/I^2 \to \Omega^1_A \otimes_A B \to \Omega^1_B \to \Omega^1_{B/A} = 0$$

On the level of the cotangent complex this becomes

$$\mathbb{L}_A \otimes_A B \to \mathbb{L}_B \to \mathbb{L}_{B/A}$$

which gives us the long exact sequence

$$\begin{array}{ccccccccccc}
H_1(\mathbb{L}_A \otimes_A B) & \to & H_1(\mathbb{L}_B) & \to & H_1(\mathbb{L}_{B/A}) & \to & H_0(\mathbb{L}_A \otimes_A B) & \to & H_0(\mathbb{L}_B) & \to & H_0(\mathbb{L}_{B/A}) \\
 & & & & \| & & \| & & \| & & \| \\
 & & & & I/I_2 & \longrightarrow & \Omega^1_A \otimes_A B & \longrightarrow & \Omega^1_B & \longrightarrow & \Omega^1_{B/A} = 0
\end{array}$$

We can rephrase this is a fancy way. Let $A \to B$ be surjective. Then

$$(I \otimes_A B)[1] = I/I^2[1] \to \mathbb{L}_{B/A}$$

is an isomorphism in degrees 0 and 1.

Theorem 7.1. *Let* $f\colon A \to B$ *be a morphism of simplicial rings with* n*-connective cofiber* K*. This means* $A \xrightarrow{f} B \to \operatorname{cofib}(f)$ *is an exact triangle such that* $\pi_i(\operatorname{cofib}(f)) = 0$ *for* $i < n$*. Then there exists a canonical* $(2n)$*-connective morphism* $\epsilon_f\colon K \otimes_A B \to \mathbb{L}_{B/A}$*.*

Proof. This is pretty hard, we are not going to do it! □

In the preceding theorem, we can also have ∞-connective which means a weak equivalence. We're going to have a look at the construction of ϵ_f: The morphism

$$\mathbb{L}_A \otimes_A B \xrightarrow{\alpha} \mathbb{L}_B \xrightarrow{\eta} \mathbb{L}_{B/A}$$

corresponds to a square-zero extension $A \to B \leftarrow B^\eta$. Does it lift? The obstruction to lifting is $\eta \circ \alpha$. But since we have an exact triangle, $[\eta \circ \alpha] = 0$, so it indeed lifts:

$$\begin{array}{ccccc}
A & \longrightarrow & B^\eta & & \\
\| & & \downarrow & & \\
A & \longrightarrow & B & \longrightarrow & K \\
\downarrow & & \downarrow & & \downarrow \epsilon'_f \\
0 & \longrightarrow & \mathbb{L}_{B/A} & \xrightarrow{=} & \mathbb{L}_{B/A}
\end{array}$$

The second and third row are cofiber sequences. By adjunction, we get $\epsilon_f\colon K \otimes_A B \to \mathbb{L}_{B/A}$.

Corollary 7.2. Let $f\colon A \to B$ be a morphism of simplicial rings such that $\pi_0(f)\colon \pi_0(A) \to \pi_0(B)$ is an isomorphism. Then $\mathbb{L}_{B/A}$ is n-connective if and only if $\operatorname{cofib}(f)$ is n-connective.

Proof. No proof given. Here $\Rightarrow$ is the hard direction to prove. Checking $\mathbb{L}_{B/A}$ is just homological algebra, while checking how surjective a ring morphism is is hard. □

7.1 Postnikov decomposition

The first step is the natural map $A \to \pi_0(A)$. Remember that A is given in nonnegative degrees and

$$\pi_0(A) = A_0/(d_1 A_1 - d_2 A_1)$$

Lemma 7.3. $A \xrightarrow{f} \pi_0(A)$ has 2-connective cofiber (which is the same as saying f is 1-connective).

Proof. We know that f is an isomorphism on π_0 and surjective on π_1. Check the long exact sequence belonging to $A \to \pi_0(A) \to K$, this immediately gives $\pi_0(K) = 0$ and $\pi_1(K) = 0$, i.e. K is 2-connective. □

Corollary 7.4. $\mathbb{L}_{\pi_0(A)/A}$ is 2-connective.

Remark 7.5. Let M be any n-connective module. Then $M[-n]$ is 0-connective, therefore $M[-n] \to \pi_0(M[-n]) = \pi_n(M)$, therefore $M \to \pi_n(M)[n]$ is n-connective.

Now define A_1 by

$$\eta\colon \mathbb{L}_{\pi_0(A)} \to \mathbb{L}_{\pi_0(A)/A} \to \pi_2(\mathbb{L}_{\pi_0(A)/A})[2]$$

We know that η classifies a square-zero extension

$$\begin{array}{ccc} & & A_1 \coloneqq \pi_0(A)^{\eta} \\ & & \downarrow \\ A & \longrightarrow & \pi_0(A) \end{array}$$

Lemma 7.6. $A \to \pi_0(A)$ lifts to $A \to A_1$.

Proof. The obstruction is given by

$$\mathbb{L}_A \otimes_A \pi_0(A) \xrightarrow{\alpha} \mathbb{L}_{\pi_0(A)} \xrightarrow{\eta} \pi_2(\mathbb{L}_{\pi_0(A)/A})[2]$$

By definition η factors as

$$\eta\colon \mathbb{L}_{\pi_0(A)} \xrightarrow{\phi} \mathbb{L}_{\pi_0(A)/A} \to \pi_2(\mathbb{L}_{\pi_0(A)/A})[2]$$

and $[\phi \circ \alpha]$ is 0 (exact triangle). □

We have thus constructed

$$\begin{array}{ccc} & & A_1 \\ & \overset{f_1}{\nearrow} & \downarrow \scriptstyle\text{square-zero} \\ A & \xrightarrow[f]{} & \pi_0(A) \end{array}$$

and f is 1-connective. We now show that A_1 is a better approximation to A than $\pi_0(A)$: It sees both π_0 and π_1 of A.

Lemma 7.7. $A \xrightarrow{f_1} A_1$ is 2-connective.

Proof. It suffices to show that $\mathbb{L}_{A_1/A}$ is 3-connective, since

$$\begin{aligned} & A \xrightarrow{f_1} A_1 \text{ 2-connective} \\ \Leftrightarrow & \operatorname{fib}(f_1) \text{ 2-connective} \\ \Leftrightarrow & \operatorname{cofib}(f_1) = \operatorname{fib}(f_1)[1] \text{ 3-connective} \\ \Leftrightarrow & \mathbb{L}_{A_1/A} \text{ 3-connective} \end{aligned}$$

Here

$$\mathbb{L}_{A_1/A} \otimes_{A_1} \pi_0(A) \to \mathbb{L}_{\pi_0(A)/A} \to \mathbb{L}_{\pi_0(A)/A_1}$$

But we know what $\mathbb{L}_{\pi_0(A)/A_1}$ is. Since $A_1 \to \pi_0(A)$ is a square-zero extension, we have a cofiber sequence

$$A_1 \to \pi_0(A) \to \pi_2(\mathbb{L}_{\pi_0(A)/A})[2]$$

using $\epsilon_f \colon \pi_2(\mathbb{L}_{\pi_0(A)/A_1}) \cong \pi_2(\mathbb{L}_{\pi_0(A)/A})$. Now look at the corresponding long exact sequence in homology. Everything in degree $\leqslant 2$ vanishes, as does $H_2(\mathbb{L}_{A_1/A} \otimes_{A_1} \pi_0(A))$. The other degree-2-parts are then isomorphic and we get that $\mathbb{L}_{A_1/A}$ is 3-connective. □

Now proceed inductively. Assume $f_n \colon A \to A_n$ is n-connective. Define A_{n+1} via

$$\mathbb{L}_{A_n} \to \mathbb{L}_{A_n/A} \to \pi_{n+2}(\mathbb{L}_{A_n/A})[n+2]$$

Using exactly the same arguments

$$\begin{array}{ccc} & & A_{n+1} \\ & \overset{f_{n+1}}{\nearrow} & \downarrow \\ A & \xrightarrow[f_n]{} & A_n \end{array}$$

lifts (obstruction is 0 by exact triangle) and $f_{n+1} \colon A \to A_{n+1}$ is $(n+1)$-connective. We have proved the following theorem.

Theorem 7.8. *Let A be a simplicial ring. Then*

$$A \xrightarrow{\simeq} \operatorname{holim}\{\cdots \to A_1 \to \pi_0(A)\}$$

where each $A_{n+1} \to A_n$ is a square-zero extension and the A_n are n-truncated.

8 Final remarks

There are a lot of open questions about $\mathbb{L}_X$. For example, Quillen conjectured the following. Let $A \to B$ a homomorphism of Noetherian rings and assume that $\mathbb{L}_{B/A}$ is of finite projective dimension. Then:

1. The projective dimension of $\mathbb{L}_{B/A}$ is $\leqslant 2$.

2. Assume B is of finite Tor dimension as A-module. Then $\mathbb{L}_{B/A}$ has projective dimension $\leqslant 1$.

The first one is *very* open. The second is now a theorem of Abramovich. So there are effectively 3 possibilities for $\mathbb{L}_{B/A}$. It is smooth, LCI or it explodes.