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Background fields in twisted differential nonabelian cohomology

Plan

1. Motivation

Recently, it has become clearer:

A local QFT is an n-functor.

But so is the classical field theory!

$$\left(\text{local QFT as an } n\text{-functor}\right) \xleftarrow[\text{quantize}]{\text{systematically}} \left(\begin{array}{c}\text{classical} \\ \text{background fields also} \\ \text{as } n\text{-functors}\end{array}\right)$$

this is the point of my talk.

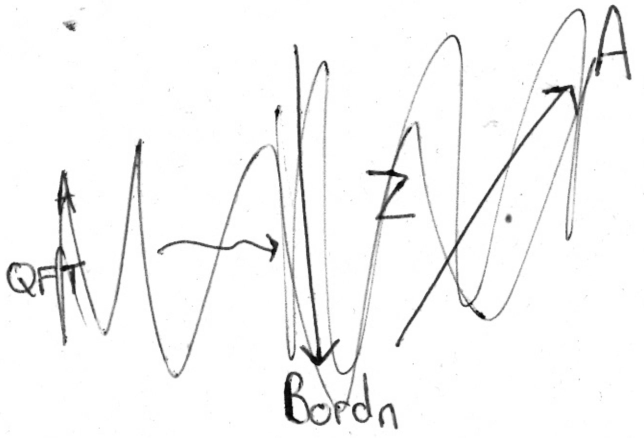
2. Understand smooth nonabelian cohomology.

3. differential nonabelian cohomology

4. twisted diff. nonabelian cohomology

5. Examples

1. Motivation



σ-model

A QFT determined by a target space X, and differential data ∇ on X.

want to understand these in this talk.

$$\begin{tikzcd}
 & & \nabla \\
\Pi_n(X) \arrow[d] \arrow[drr] & & \\
\text{field space} \rightsquigarrow \mathrm{Bord}_n(X) \arrow[rr, dashed, "\text{action}"] \arrow[d] & & A \\
\mathrm{Bord}_n \arrow[urr, "\text{quantum field theory}"'] & &
\end{tikzcd}$$

B. Richter: Is it

2. Smooth nonabelian cohomology

2.1. Toy example: Dijkgraaf-Witten theory

Physicists mean a σ-model has

- "target space" is $BG = \{ \cdot \circlearrowleft G \}$ groupoid
- background field is an n-function $BG \longrightarrow B^n U(1)$

n-groupoid with $U(1)$ in nth degree, zero elsewhere

How do we model this?

$$\infty \text{ Groupoids} = \text{Kan-complex}$$

the collection of ∞ Groupoids forms

a "category" enriched over ∞-Groupoids.

This is known as a <u>model</u> for $(\infty,1)$-Cat.

So

$$\text{Top} \in (\infty,1)\text{-Topos}$$

we have a notion of homotopy, cohomology here

let's generalize this to more general $(\infty,1)$-toposes.

$$H(X, A) \overset{\text{in Top}}{=} \pi_0 \, \text{Maps}(X, A)$$

$$\mathbb{H}(X, A) = \text{Hom}_{\infty\text{Groupoids}}(X, A)$$

objects are cocycles

morphisms are coboundisms

objects in an $(\infty,1)$-topos.

Stephan Stolz : ~~Is~~ Why call this cohomology ?

... discussion ensues.

$$H^1(X, G) = \mathbb{H}(X, \mathbb{B}G)$$

$$\cancel{H^n(X, G) = \mathbb{H}(X, B^n G)}$$

$H^n_{group}(G, U(1))$

$\cong \mathrm{Hom}(BG, B^n U(1))$.

weak functors from $\Pi_\infty(X)$ to $?^\alpha$ larrow

⋮

ΩG.

So in DW theory a background field is a cocycle in

$$\mathbb{H}(\mathbb{B}G, \mathbb{B}^n(U(1))) \cong H^n_{group}(G, U(1)).$$

2.2. Generalize to smooth cocycles

$\Rightarrow$ $\mathbb{H}$ on ∞-topos

So $\mathbb{H}$ is a collection of ∞-stacks

on some site S.

So $A \in \mathbb{H}$ $S^{op} \to \infty$ Grpds.

take $S = \mathrm{DIFF}$, then

$$A : \mathrm{Diff}^{op} \longrightarrow \infty\mathrm{Grpds}$$

$$M \longmapsto \mathrm{Maps}(M, A)$$

$$\in \infty\,\mathrm{Grpd}$$

So the upshot is :

pass from ∞ Grpds to Smooth ∞ Grpds

$\simeq \infty\,\mathrm{Stacks}(\mathrm{Diff})$

2.3 Model by ∞Grpd-valued sheaves

Let's look at:

$$A : \mathrm{Diff}^{op} \longrightarrow \infty\,\mathrm{Grpds} \left(\begin{array}{l} \cong \text{Kan complexes} \\ \subset \mathrm{SSet} \end{array} \right)$$

and impose on them the sheaf condition. Then you get

$$\mathrm{Sh}(\mathrm{Diff}, \mathrm{SSet}) = \mathrm{SSh}(\mathrm{Diff})$$

equipped with extra information that remembers which ∞-functors

$$f : A \longrightarrow B$$

would have weak inverses in the ~~following~~ full $(\infty,1)$-Cat ∞Stacks.

This is standard:

- model category structure on $[S^{op}, \mathrm{SSet}]$ (Joyal-Jardine)

simpler! →

- Brown-category structure ("category of fibrant objects", Kenneth Brown (1973))

Essential: remember class

$$\underbrace{\left\{ p : \begin{array}{c} Y \\ \downarrow \\ X \end{array} \;\middle|\; \begin{array}{l} p \text{ is locally} \\ \text{a surjective} \\ \text{equivalence of } \infty\text{-Grpds} \end{array} \right\}}_{\text{hypercovers}}.$$

$$\Rightarrow \boxed{\begin{aligned} \mathbb{H}(X,A) &:= \operatorname*{colim}_{\substack{\text{hypercovers} \\ Y \twoheadrightarrow X}} \quad Sh(Y,A) \\ &= \operatorname{colim} \left\{ \begin{array}{ccc} Y & \longrightarrow & A \\ \downarrow{\scriptstyle \simeq} & & \\ X & & \end{array} \right\} \end{aligned}}$$

So a <u>cocycle</u> on X with values in A is a span

$$\begin{array}{ccc} Y & \longrightarrow & A \\ \downarrow{\scriptstyle \simeq} & & \\ X & & \end{array}$$

<u>In particular</u>:

Observe the following chain of inclusions.

[illegible] eg. Crossed ~~[illegible]~~ complexes of groupoids $\simeq$ Strict ∞Grpd $\hookrightarrow$ ∞ Grpds

$\leftarrow$ nonabelian generalization of chain complexes of abelian groups

Thm (Brown '73) Let $S = Op(Z)$ (open subsets)

Suppose that $F \in Sh(Z, Ch_+(Ab))$.

$$\longmapsto A_F \in Sh(Z, \infty Grpds)$$

Claim: They coincide. That is,

$$\underbrace{H^n(X, F)}_{\substack{\text{sheaf cohomology,} \\ \text{i.e. right derived functor} \\ \text{of global sections}}} \simeq H(X, \mathbb{B}^n A_F)$$

homotopy classes of maps.

3. Differential cohomology

There is a functor

$$P_2(-) : \mathrm{Diff} \longrightarrow \text{Smooth } \infty \text{ Grpd}$$
$$M \longmapsto P_2(M)$$

a smooth model
of path groupoid
of M.

objects are points in M

morphisms are thin homotopy classes of paths in M

2-morphisms are surfaces.

Using this, let A be any smooth ∞-groupoid.

Produce

$$A^{P_2} = \mathrm{Sh}\big(P_2(-), A\big).$$

<u>Examples</u> (theorem) Let G be a Lie group, we get a smooth ∞-groupoid $\mathbb{B}G$. [actually a 1-groupoid].

Then
$$\mathbb{H}(X, \mathbb{B}G^{P_1}) \cong G\mathrm{Bun}_\nabla(X)$$
$$\mathbb{B}^2 U(1) \Rightarrow \mathbb{B}^2 U(1)^{P_2} \rightsquigarrow \mathbb{H}($$

$\rightsquigarrow \mathbb{H}(X, \mathbf{B}^2(U(1))^{P_2})$

$\cong$ $U(1)$-bundle gerbes on X with smooth connection.

Q: Where is the smoothness built in?

Urs: We are working in a smooth topos, everything is now smooth, we can't help it!

In fact,

$$(\mathbf{B}^2 U(1))^{P_2} \simeq \mathbb{Z}(3)_D^\infty$$

smooth Deligne cohomology

4. Twisted cohomology

Recall: a sequence

$$A \to \hat{B} \to B$$

of pointed smooth ∞ Groupoids is a fibration sequence

iff this diagram is a homotopy pullback:

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\ \hat{B} \arrow[r] & B \end{tikzcd}$$

This means that for all X, we can form

$$\begin{tikzcd} \mathbf{H}(X,A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "* \mapsto \mathrm{triv}"] \\ \mathbf{H}(X,B) \arrow[r] & \mathbf{H}(X,B) \end{tikzcd}$$

For some cocycle $c \in H(X,\hat{B})$,

the obstruction for lifting it to a A-cocycle is

a class $[\delta c]$.

<u>Defn</u> (Twisted nonabelian coh.)

For $A \to \hat{B} \to B$ a fibration sequence, and $c \in \mathbb{H}(X, B)$ a cocycle, define <u>c-twisted A-cohomology</u> on X as

$\mathbb{H}^c(X, A)$ ~~given~~

given by the homotopy pullback.

$$\begin{tikzcd} \mathbb{H}^c(X,A) \arrow[r] \arrow[d] & * \arrow[d, "* \mapsto c"] \\ \mathbb{H}(X,\hat{B}) \arrow[r] & \mathbb{H}(X,B) \end{tikzcd}$$

We get :

- This produces for suitable fibration sequences the twists by <u>magnetic higher charges</u> appearing in "nature"
- it allows you to describe non-flat differential cohomology

Non-flat diff. cohomology $:=$ $\begin{pmatrix}\text{curvature} \\ \text{char.} \\ \text{classes}\end{pmatrix}$ - twisted flat cohomology

This produces the <u>twisted</u> <u>Bianchi</u> <u>identities</u> that physicists run into.

<u>Q</u> Why not Lie groupoids?

<u>Q</u> Another example of nonabelian cohomology

Let G be a 2-group

$\mathbb{B}G$ is a one-object 2-groupoid.

$$H(X, \mathbb{B}G) \cong H\text{ - nonabelian gerbes} \quad \text{for } G \cong \mathrm{Aut}(H)$$

~~then~~

Twisting examples: Nonabelian cohomology on X

$\Updownarrow$

twisted abelian cohomology on X by

$$\begin{array}{c} \mathbb{B}^2 U(1) \\ \downarrow \\ \mathbb{B} \end{array}$$