Corbit Redden

String structures, 3-forms and

Notation

M^n a smooth closed n-manifold

g a Riemannian metric

$$\begin{array}{ccc} \mathrm{Spin} & \longrightarrow & P \\ & & \downarrow \\ & & M \end{array}$$ principal Spin(n)-bundle

A a connection on P

$\mathcal{S}$ string class

Outline

I. $\{\text{String structures}\}/\text{htpy}$.

II. Harmonic 3-forms on P

III. Geometry and tmf?

I.

Defn

- $$BString(n) \xrightarrow[\text{fiber}]{\text{htpy}} BSpin(n) \xrightarrow{\frac{p_1}{2}} K(\mathbb{Z}, 4)$$

definition of $BString(n)$ as a top. space

- A <u>string structure</u> on $P \xrightarrow{\pi} M$ is a lift of the classifying map:

$$\begin{tikzcd} P \arrow[d] & BString \arrow[d] \\ M \arrow[r] \arrow[ur, dashed] & BSpin \end{tikzcd}$$

Prop

- A string structure exists iff $\frac{p_1}{2}(P) = 0 \in H^4(M; \mathbb{Z})$.
- $$\{\text{String structures}\}/_{\text{htpy}} \overset{\text{canonical}}{\cong} \{\text{String classes}\}$$
$$:= \left\{ \begin{array}{l} \varsigma \in H^3(P, \mathbb{Z}) \text{ s.t.} \\ i^* \varsigma = 1 \in H^3(Spin, \mathbb{Z}) \end{array} \right\}$$
- $\{\text{string classes}\}$ is a torsor for $H^3(M, \mathbb{Z})$ under $\varsigma \to \varsigma + \pi^* H^3(M, \mathbb{Z})$.

<u>Proof</u> Universal example:

since something contractible

$$\begin{tikzcd} K(\mathbb{Z},3) \simeq \pi^* ESpin \arrow[r] \arrow[d] & ESpin \arrow[d] \\ BString \arrow[r, "\pi"'] & BSpin \end{tikzcd}$$

$\boxed{G = Spin}$

$\square$

This is at least a down-to-earth defn., though up to homotopy is not quite good.

<u>Why string structures?</u>

String structure on P $\overset{\text{transgresses}}{\rightsquigarrow}$ spin structure on

$$\begin{tikzcd} LSpin \arrow[r] & LP \arrow[d] \\ & LM \end{tikzcd}$$

<u>Need</u>: Central extension

$$S^1 \longrightarrow \widehat{LSpin} \longrightarrow LSpin$$

i.e. you want

$$\begin{tikzcd} \widehat{LSpin} \arrow[r] \arrow[d] & \widehat{LP} \arrow[d] \\ LSpin \arrow[r] & LP \end{tikzcd}$$

$$\mathcal{S} \in H^3(P, \mathbb{Z})$$

$$\xrightarrow{\text{transgress}} \quad (\pi_! \, ev^*)\mathcal{S} \in H^2(LP, \mathbb{Z})$$

$$\Big\downarrow \text{restricts}$$

universal extension $\quad H^2(LG, \mathbb{Z})$

So from that point of view, string structures give you what you'd want in the loop space case.

<u>String orientation</u> of tmf = Topological modular forms

$$MString^{-n}(pt) \xrightarrow{\sigma} tmf^{-n}(pt)$$

$\uparrow$

used to be known as $MO8$

A string class is exactly what you'd need to define <u>String bordism</u>.

Spin mfld, class on Spin(TM)

$$M, S \longrightarrow [M, S] \longrightarrow \sigma(M, S)$$

Relation to prev. picture:

$$\begin{tikzcd} & \mathrm{tmf} \arrow[dr, bend left] & \\ \mathrm{MString} \arrow[ur, "\sigma"] \arrow[rr, "\text{Witten genus}"'] & & \text{Mod. forms} \end{tikzcd}$$

σ is a lift of Witten genus.

$$\text{Witten' genus}(M) \overset{\text{heuristic}}{=} \text{“}\mathrm{index}^{S^1} \not{\partial}_{LM}\text{”}$$

So the real home for the index on loop space lives in tmf.

II. Harmonic representatives of S.

<u>Reminder</u> $\quad (M, g) \overset{\text{can form}}{\rightsquigarrow} \Delta = dd^* + d^*d$

Hodge Laplacian

$$H^k(M, \mathbb{R}) \cong \operatorname{Ker} \Delta^k_g \subset \Omega^k(M)$$

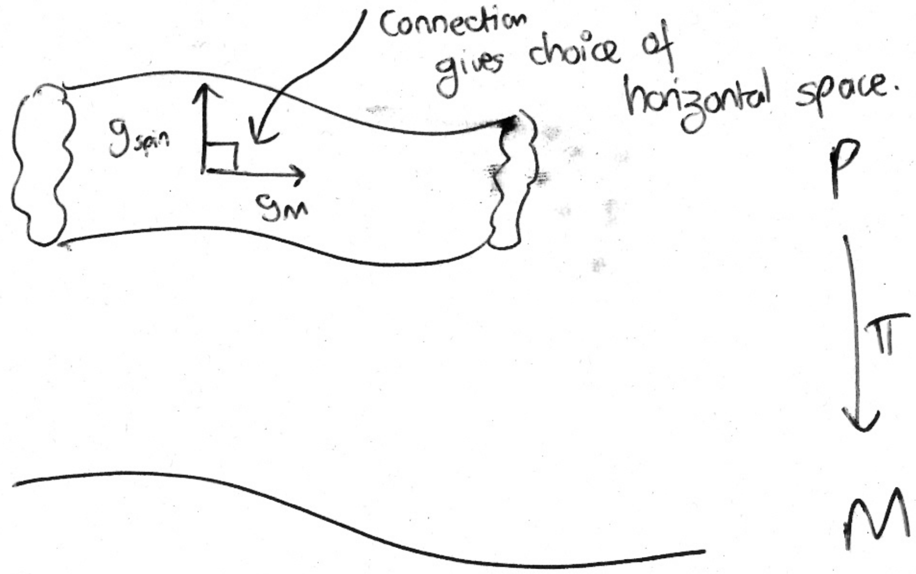
<u>Construction</u> : Start with $\left(P \xrightarrow{\pi} M, g_M, A\right)$

Choose a bi-invariant metric g_{spin} on ~~other~~ the fiber Spin(n).

$$g_P = \pi^* g_M \oplus \underset{\text{from connection}}{g_{spin}}$$

Connection gives choice of horizontal space.

wiggly means "S^5" not S^1.

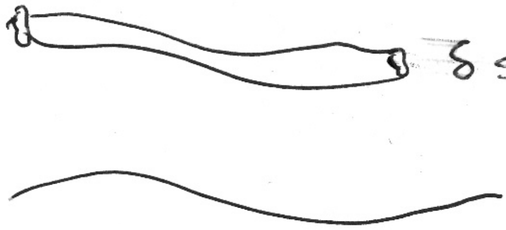


Let's scale away g_{spin}.

Introduce scaling factor $\delta > 0$

$$g_\delta := \pi^* g_M \oplus \delta^2 g_{spin}$$

and take $\lim_{\delta \to 0}$ i.e. adiabatic limit



δ small

the two defn's of spectral coincide!

So we have a family of metrics.

Thm (Mazzeo-Melrose, Dai, ~~[illegible]~~ Forman)

$\operatorname{Ker} \Delta_{g_\delta}$ extends smoothly to $\delta = 0$,

and comes from a filtration isomorphic to the Serre spectral sequence for $(Spin \to P \to M)$

i.e. path in a Grassmannian space

This means that still have Hodge iso

$$H^k(P, \mathbb{R}) \xrightarrow{\cong} \lim_{\delta \to 0} \operatorname{Ker} \Delta_{g_\delta} =: \mathcal{H}^k(P)$$

<u>Thm (R)</u> Given $(P \xrightarrow{\pi} M, g, A)$ and $\frac{p_1}{2}(P) = 0$,

then

$$H^3(P, \mathbb{Z}) \longrightarrow H^3(P, \mathbb{R}) \longrightarrow \mathcal{H}^3(P)$$

$$S \longmapsto \underbrace{CS_3(A)}_{\substack{\text{Chern-Simons} \\ \text{3-form}}} - \underbrace{\pi^* H}_{\pi^* \Omega^3(M)}$$

<u>not</u> in $\pi^*\Omega^3(M)$

<u>Remark</u>: In general,

$$[S]_{g_\delta} = CS_3(A) - \pi^* H + \underbrace{O(\delta)}_{\substack{\text{so you need} \\ \text{to take } \delta \to 0}}$$

What is H? (digression)

Thm (Cheeger-Simons, Chern, ...)

Given $(P \to M, A) \rightsquigarrow \frac{\check{p}_1}{2}(A) \in \check{H}^4(M)$

Remember, for differential theory

$$\begin{tikzcd} \Omega^3_{\mathbb{Z}}(M) \arrow[r] & \Omega^3(M) \arrow[r, "a"] & \check{H}^4(M) \arrow[r] & H^4(M,\mathbb{Z}) \arrow[r] & 0 \\ & H \arrow[r, mapsto] & \frac{\check{p}_1}{2}(A) \arrow[r, mapsto] & \frac{p_1}{2}(P) = 0 & \end{tikzcd}$$

Abstractly, the exact sequences of diff. cohomology would only determine the class and not H itself.

In particular,

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ Z_3(M) \arrow[ur, "\int H"] \arrow[r, "\frac{\check{p}_1}{2}(A)"'] & \mathbb{R}/\mathbb{Z} \end{tikzcd}$$

so I get something in the reals, as opposed to $\mathbb{R}/\mathbb{Z}$.

Also

$$\left.\begin{array}{l} d^{*}H = 0 \\[1em] \text{Together with} \\[0.5em] H \longmapsto \frac{\check{p}_1}{2}(A) \longmapsto \frac{p_1}{2}(P) = 0 \end{array}\right\}$$

determines H uniquely up to $\mathcal{H}^3_{\mathbb{Z}}(M) = \operatorname{Ker} \Delta_g$

this is where the metric comes in.

Equivariance

$$H_{S+\pi^{*}\psi} = H_S + \pi^{*} H_{\psi}$$

$$\begin{tikzcd} \text{where } \psi \arrow[d, mapsto] & \in & H^3(M,\mathbb{Z}) \arrow[d] \\ H_{\psi} & & H^3(M) \end{tikzcd}$$

take Harmonic representative

So pullback harmonic on the bundle (not true if you don't take limit)

$\cancel{M \times \{\text{String class}\} \to \Omega^3(P)}$

$$\begin{array}{ccc}
\mathrm{Metrics}(M) \times \mathrm{Connections}(P) \times \{\text{String classes}\} & & g, A, S \\
\downarrow & & \downarrow \\
\Omega^3(P) & & CS_3(A) - \pi^* H_{g,A,S} \\
\downarrow & & \downarrow \\
\Omega^3(M) & & H_{g,A,S}
\end{array}$$

Have map

$$\begin{tikzcd}
 & tmf^{-n} \arrow[d] \\
MString \arrow[ur, "\sigma"] \arrow[r] & MF
\end{tikzcd}$$

<u>Conj</u> (Stolz) If M is string, and admits a positive Ricci curvature metric, then

$$\mathrm{Witten}(M) = 0.$$

good shape

? How about also $\sigma(M, S) = 0$

as for K-theory ?

No way!!

<u>Hypothesis</u> If M admits (g, S),

$$\left[P = \mathrm{Spin}(TM), \quad A = \text{Levi-Civita} \right]$$

such that

$$\begin{cases} \mathrm{Ric}(g) > 0 \\ H_{g,S} = 0 \in \Omega^3(M) \end{cases}$$

this is really strong condition.

Then $\sigma(M, S) = 0 \in tmf^{-n}(pt)$

<u>Example</u> $M = S^3 = SU(2)$

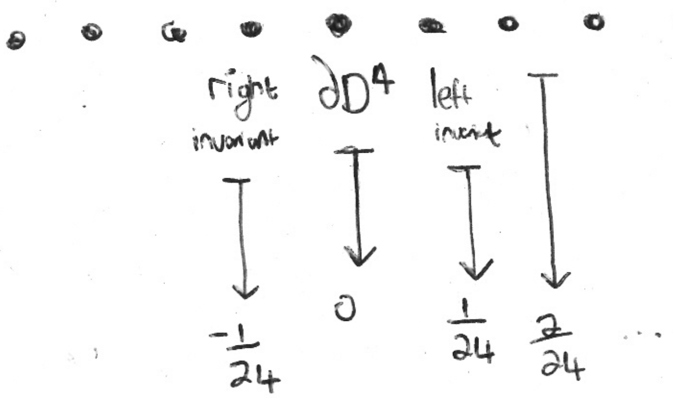
$p_1 \in H^4(S^3) = 0$

$H^3(S^3, \mathbb{Z}) = \mathbb{Z}$

$=$ # of string classes

$dH = d^* H = 0$

$\Rightarrow H \in H^3(S^3, \mathbb{R}) \cong \mathbb{R}$



String classes

$$\downarrow$$

$$MString^{-3} = \pi_3^s = tmf^{-3} = \mathbb{Z}/24$$

So string structures = framings.

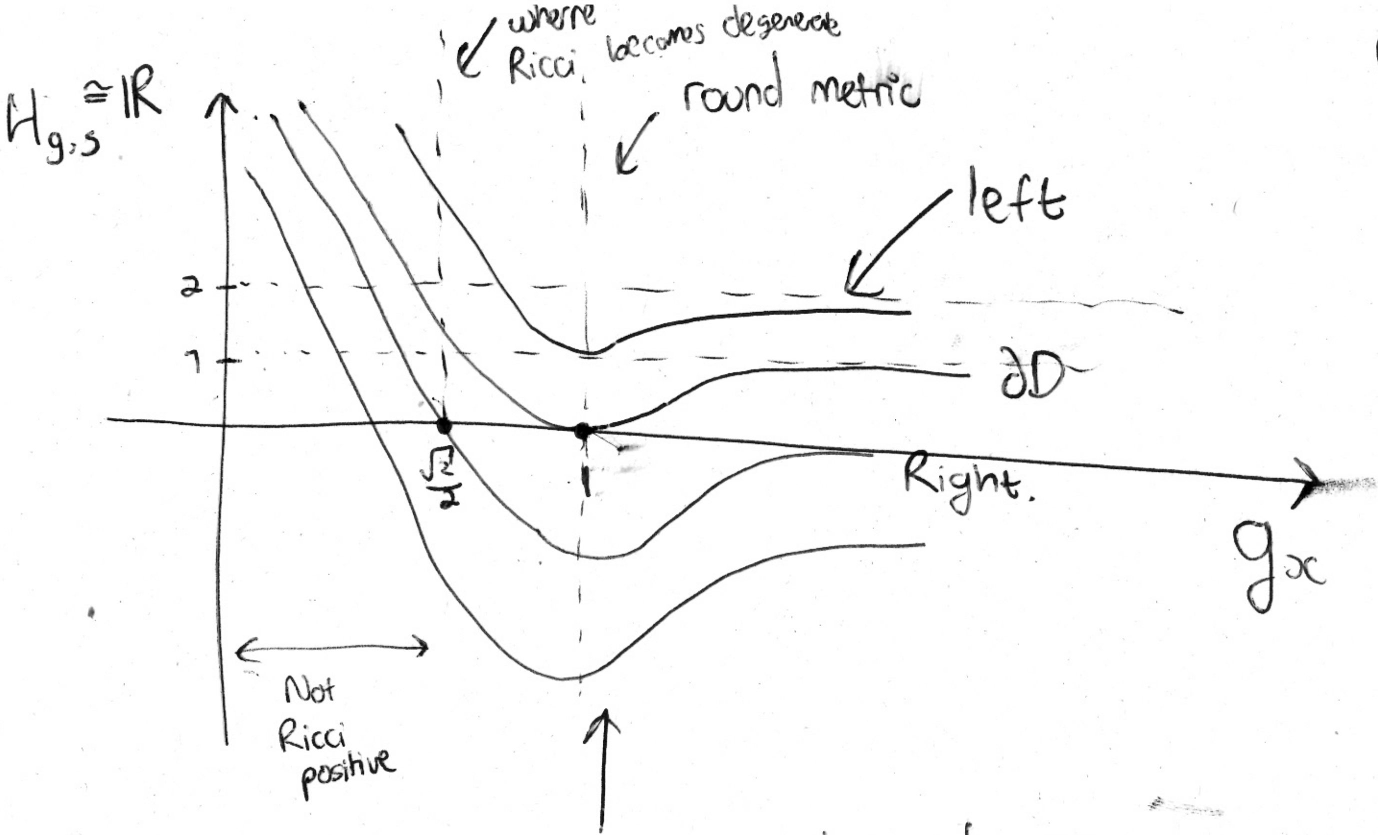
S^3 is as positively curved as you can get.

Also you have nonzero classes in tmf.

There's a 1-parameter family of "Berger metrics" on S^3.

Rescale fiber in Hopf fibration $S^1 \longrightarrow S^3 \longrightarrow S^2$

Call them g_x



equivariance = all graphs are vertical translates

So it just fits the hypothesis in each case!

Dan Freed You could use the metric and connection to define a torsion metric.

Corbit Yes ... (discussion)

this pair of conditions is equiv to scaling tensor being zero in one of the limits.

Stolz : The other limit is boring, Levi-Civita connection.