Don Freed

w J. Distler, G. Moore

$$\Sigma^2 \xrightarrow{\phi} X^{10}$$

compact
2-manifold
"worldsheet"

smooth ~~manifold~~ orbifold
"spacetime"

2dim theory $\rightsquigarrow$ 10dim theory

Today: given double cover

$$\begin{array}{c} \circlearrowleft \sigma \\ X_w \\ \downarrow \\ X_{10} \end{array}$$

"spacetime with involution"

This will be an orientifold.

Two important cases:

I) σ trivial (Type I)

II) section $X_W \underset{\text{section}}{\overset{\pi}{\rightleftarrows}} X$ (Type II).

Jacques showed me a formula.

$i : F \hookrightarrow X_W$ fixed point set

RR charge in 2d theory [Raravos, Same, Same ?]

$$= \pm\, 2^{\#}\, i_*\left(\sqrt{\frac{L'(F)}{L'(\nu)}}\right)$$

the normal bundle of $F \hookrightarrow X_W$

$$L' = \prod \frac{x/4u}{\tanh x/4u}$$

the Bott element in K-theory

looks similar to Hirzebruch L-genus

1. Definition of fields / theory
2. Derive RR charge formula over $\mathbb{Z}[\frac{1}{2}]$ from 10d.
3. Anomaly cancellation in 2d.

Some background

Differential cohomology

h cohomology theory.

$$h^\bullet(- ; \mathbb{Q}) = H(- ; \underbrace{h(pt; \mathbb{Q})}_{h_{\mathbb{Q}}})^\bullet$$

Will tensor with reals.

$$h^\cdot(\quad) \longrightarrow H(\quad ; \mathbb{R})$$

$$\begin{tikzcd}
\check{h}^\cdot(M) \arrow[r, dashed] \arrow[d, dash, dotted] & \Omega(M; h_{\mathbb{R}})^\bullet_{\text{closed}} \arrow[d] \\
h^\cdot(M) \arrow[r] & H(M; h_{\mathbb{R}})
\end{tikzcd}$$

differential cohomology is the "homotopy fiber product".

i.e. a class is $a \in h(M)$, b in $\Omega(M; h_{\mathbb{R}})^\cdot_{\text{closed}}$, and a homotopy from a to b in $H(M; h_{\mathbb{R}})$.

Exact sequences:

$$0 \to h(M; h_{\mathbb{R}} \otimes \mathbb{R}/\mathbb{Z})^{q-1} \longrightarrow \check{h}^q(M) \xrightarrow{\text{curv}} \Omega(M; h_{\mathbb{R}})^q_{\mathbb{Z}} \longrightarrow 0$$

$\leftarrow$ integral forms

$$0 \longrightarrow \text{forms} \longrightarrow \check{h}^q(M) \longrightarrow h^q(M) \longrightarrow 0$$

For ordinary diff cohomology: Cheeger + Simons, and Deligne.

Also Hopkins + Singer.

Functorial: given h, get differential cohomology.

$$h^q(M) = \pi_0 \left(\mathrm{Map}(M, h_q) \right).$$

so really objects in higher groupoids.
(Need $\pi_1, \ldots$)

Also for $\check{h}^q(M)$.

We will freely use equivariant version, though not entirely worked out.

Eg. $\check{H}^1(M) = \mathrm{Maps}(M, \mathbb{T})$. $\quad H^1(M) = \pi_0 \mathrm{Map}(M; \mathbb{T})$

$$\check{H}^2(M)_{ob} = \left\{ \begin{array}{c} P \\ \downarrow \mathbb{T} \\ M \end{array} \text{ with connection} \right\} \qquad H^2(M) =$$

$\check{H}^3(M) =$ gerbes, bundle gerbes etc.

In Dirac quantization of monopoles ... good example.

These enter in defining topological terms (eg WZW) or gauge theory (Dirac charge quant.)

Twistings of $KR(X_w)$

$$\begin{array}{c} X_w \circlearrowleft \sigma \\ \downarrow \pi \\ X \end{array}$$

double cover. (in world of groupoids)

could be groupoid approach.

object in $KR^0(X_w)$ = $\begin{array}{c} E \\ \downarrow \\ X_w \end{array}$ $\mathbb{Z}_2$-graded complex vector bundle

+ twisted lift of involution

$$\tilde{\sigma} : \sigma^* E \longrightarrow E$$

$$\tilde{\sigma}^2 = \pm 1.$$

special cases : I) σ acts trivially : $KO^0(X_w)$.

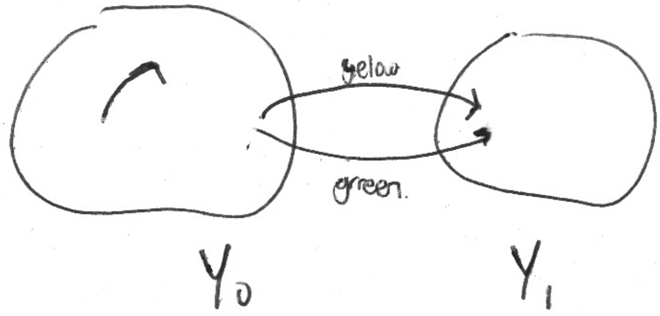
II) $X_w \rightleftarrows X$ $\quad K^0(X)$.

<u>Twisting</u> May have to Pass to a locally equivalent groupoid

$$Y_w \longrightarrow Y$$

Y is a groupoid

$$Y_0 \underset{p_1}{\overset{p_0}{\leftleftarrows}} Y_1 \Lleftarrow \cdots$$



<u>Notation:</u> $${}^{\phi}V = \begin{cases} V, & \phi = 0 \\ \overline{V}, & \phi = 1. \end{cases}$$

<u>Defn</u> A twisting of $KR(X_w)$ is a locally equivalent $Y_w \to Y$ and a triple $\tau = (d, L, \theta)$ where:

$d : Y_0 \longrightarrow \mathbb{Z}$ continuous

$L \longrightarrow Y_1$ a $\mathbb{Z}_2$-graded hermition line bundle over arrows.

$\theta : {}^{\phi(f)}L_g \otimes L_f \longrightarrow L_{gf} \quad (a \xrightarrow{f} b \xrightarrow{g} c)$

\+ cocycle conditions:

$d(b) = d(a)$ for $(a \xrightarrow{f} b)$.

\+ cocycle data for θ.

Cohomology group:

For $K(X)$,

$$\pi_{\{0,1,2,3\}} h \cong \{\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, 0, \mathbb{Z}\}$$

So without d, $\mathbb{Z}_2$-grading, get cohomology class only in 3.

Now also in 0 and 1.

For $KO(X_w)$ a Postnikov section of connected KO.

$KR(X_w)$: the iso classes are

$$H^0(X;\mathbb{Z}) \times H^1(X;\mathbb{Z}/2\mathbb{Z}) \times H^{w+3}(X;\mathbb{Z}).$$

as a set.

For differential theory, add connections. Get out twisted B-fields, etc.

$$u \in K^2(pt).$$

lifts to $KR^{\tau_1+2}(pt)$, $4\tau_1 \cong 0$

concrete model for u

(PTO)

$$\mathbb{C}^{1|1} = \mathbb{C} \oplus \Pi\mathbb{C}.$$

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

$$\tilde{\sigma}(z_+, z_-) = (-\bar{z}_-, \bar{z}_+).$$

an <u>odd</u> map which squares to <u>minus</u> the identity.

In particular,

$$\tau \longrightarrow \tau + (\tau_1, +2)$$

ie. An action of Bott periodicity on these twistings

Defn An $\underbrace{\text{NS}}_{\text{Neveu Schwarz}}A\times 2$ <u>superstring background</u> is

(i) X a smooth 10d orbifold w/ metric and real function (dilaton data.)

(ii) $\pi : X_w \longrightarrow X$ orientifold double cover.

(iii) $\check{\beta}$ a differential twisting of $KR(X_w)$.
<u>B-field</u>

(iv) $K : R(\beta) \longrightarrow \tau^{KO}(TX - 2)$

iso of twistings of $KO(X)$

$\therefore$ (a twisted spin structure)

$\Longrightarrow \begin{cases} w_1(X) = tW \\ w_2(X) = tW^2 + aW \end{cases}$, t and a come from Bott twists.

W: class of double cover

$t = 0 \rightarrow$ type B
$t = 1 \rightarrow$ type A.

There's a Bott shift.

$$\check{\beta} \longmapsto \check{\beta} + (\check{\tau}_{+3})$$
$$K \longmapsto (u\bar{u})^{-1} K.$$

2d theory

A worldsheet Σ^2 metric

α a spin structure

on $\hat{\Sigma} \longrightarrow \Sigma$

orientation double cover.

the worldsheet is a map

$$\phi : \Sigma \longrightarrow X$$

plus pullback

$$\phi^* w \cong \hat{w}.$$

i.e.

$$\begin{tikzcd} \hat{\Sigma} \arrow[r] \arrow[d] & X_w \arrow[d] \\ \Sigma \arrow[r] & X \end{tikzcd}$$

spinor fields ψ, χ on $\hat{\Sigma}$.

in fact, not a function but a section of Pfaffian line bundle.

$$e^{-\text{Eff action}} = \left\{ \text{Pfaffian } D_{\hat{\Sigma}, \alpha} \left(\phi^* TX - 2\right) \right\}$$

$$\times \exp 2\pi i \int_{\Sigma} \phi^* \check{\beta} \;.\; + \ldots$$

Parameter space S.

$$\underbrace{\text{Section of } \begin{array}{c} L_\psi \\ \downarrow \\ S \end{array}}_{\text{1st term in action}} \qquad \underbrace{\text{Section of } \begin{array}{c} L_B \\ \downarrow \\ S \end{array}}_{\text{Second term.}}$$

<u>Thm</u> There is a (canonical) trivialization of

L

That's good, because we need a <u>function</u> to integrate.

<u>Anomaly</u> (sources of)

1. Integrals over fermions ... eg. L_ψ above. (has to do with Pfaffian line bundles) ... geometric index theory of Dirac operators. } quantu

2. Simultaneous electric and magnetic current (or self-dual current).

3. Boundaries of topological terms, eg WZW, Chern-Simons

} Classical

4 NEW : exotic orientation (L_B).

Roughly: to integrate B-field,

(use Schreiber-Waldorf for bosonic case)

need spin structure.

Insert extra classes: Differential KO of a point is not trivial!

$$e^{-\text{Eff action}} = \text{pfaffian of } D_{\hat{\Sigma}_\alpha} (\phi^* TX - 2)$$

$$\cdot \exp 2\pi i \int_{\Sigma/S} \underbrace{\hat{\xi}\, \hat{\epsilon}}_{} \, \phi^* \check{\beta}$$

extra classes.