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; work with Owen Gwilliam

What is the analogue of deformation quantizan in QFT instead of quant. mechanics?

Deformation quantization

Classical mechanics.

A^{cl}, comm. algebra,

$\{\ ,\ \}$ Poisson bracket.

To quantize this, need to find an assoc. algebra A^q over the ring $\mathbb{R}[[\hbar]]$, such that

1. $A^q / \hbar A^q = A^{cl}$.

2. If $a, b \in A^{cl}$, $\tilde{a}, \tilde{b}$ are lifts to A^q,

then

$$\{\hat{a}, \hat{b}\} = \frac{1}{\hbar}[\tilde{a}, \tilde{b}] \mod \hbar.$$

These lectures: want to give an analog of this picture for QFT.

1) Need to explain what plays the role of commutative, Poisson and associative algebras.

2) ~~How to get~~ Explain how classical field theory is encoded in (analog of) commutative + Poisson.

3) Explain how to quantize the classical field theory.

Today: the structure that plays the role of assoc. alg is a <u>factorization algebra</u>.

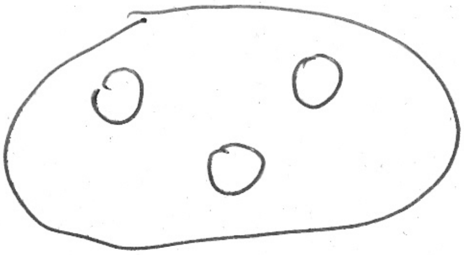
Not so terrifying as in book by Beilinson Drinfeld.

We work with a C^∞ (smooth) analogue of their stuff.

Let M be an manifold ("spacetime") on which we want to do QFT.

Let $B(M) = \{ \text{closed balls in } M \}$ ... an ∞-dim manifold if we look at Riemannian manifold

Let $B_n(M) = \{ n \text{ disjoint balls embedded in a larger ball} \}$

 $\in B_3(\mathbb{R}^2)$.

proj.

A <u>factorization algebra</u> is a (in practice, locally triv. bundles of ∞-dim top. vector spaces) vector bundle F on $B(M)$ (S_n-invariant) equipped with maps

$$p^*(F^{\boxtimes n}) \longrightarrow q^*(F)$$

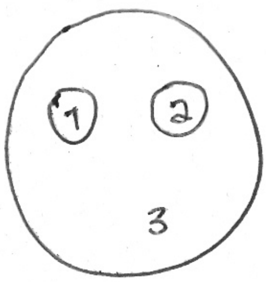
satisfying some compatibility.

projection maps

$$B(M) \xleftarrow[q]{\text{ball on outside}} B_n(M) \xrightarrow{p} B(M)^n$$

<u>Concretely</u>, F smoothly assigns a vector space to every ball $B \subseteq M$.

If we have some configuration of balls, like

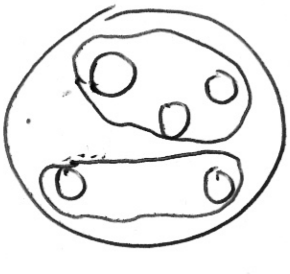


we get a map

$$F(B_1) \otimes F(B_2) \longrightarrow F(B_3)$$

These maps must vary smoothly as the configurations of the balls varies

Compatibility condition: like for little discs operad.



it doesn't matter if you apply map all in one go or twice.

i.e. the diagram

$$\begin{tikzcd}
\otimes F(\text{Yellow balls}) \arrow[r] \arrow[dr] & \otimes F(\text{green}) \arrow[d] \\
& F(\text{white})
\end{tikzcd}$$

commutes.

(i.e. its an algebra over the little discs operad (Urs)
Yes... absolutely. But it's a coulered operad (K. Costello)).

This is an algebra over coulered operad, colours are $B(M)$,
n-ary operations are $B_n(M)$. ~~Thesew~~
with extra condition saying the operations must depend
smoothly, i.e. the vector space we assign to
each color forms a smooth V. bundle and
the maps are compatible with this.

Explanation

Vector bundles come in 3 flavours:

1. C^∞ $\rightsquigarrow$ factorization algebra
2. holomorphic $\rightsquigarrow$ vertex algebra
3. Locally constant sheaves $\rightsquigarrow$ E_∞-algebra

Factorization algebras exist in settings 2+3 also.

Def^n (vague) A locally constant factorization algebra is like a factorization alg., except F is a locally constant sheaf on $B(M)$, and the structure maps are maps of l.c. sheaves.

Q: sheaves of what?

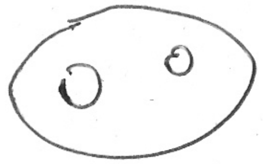
A: I've drunk the derived coolaid. Everything is a cochain complex.

Example Let F be a locally constant fact. alg. on $\mathbb{R}^n$.

$B(\mathbb{R}^n)$ is contractible,

F is quasi-iso to a trivial sheaf, with fibre V, a cochain complex.

If have



get a map

$V \otimes V \longrightarrow V$

As ~~[illegible]~~ vary, the conf. of discs vary,
the products change by homotopies.

i.e. a locally constant f. algebra on $\mathbb{R}^n$ is the same thing as an E_∞-algebra.

i.e. the set of colours is contractible.
the set of operations $\simeq$ little discs operad.

<u>Next</u> : Holomorphic factorization algebras.
(Here, For Rieman surfaces).

Let Σ be a Riemann surface.

We know what it means for a map from a complex manifold to $B(\Sigma)$ to be holomorphic.

(this is why go to 2d, can use ~~Riemann~~ open mapping theorem)

So, we can talk about hol. objects on $B(\Sigma)$.

(What is a map $U \longrightarrow B(\Sigma)$? Its a complex man.

$$\begin{tikzcd} M \arrow[r] \arrow[d] & \Sigma \\ U & \end{tikzcd}$$

where fibers of M are open balls)

(i.e. in the sense of a sheaf on the site of complex manifold)

André : problem with defn. Everything is holomorphic!

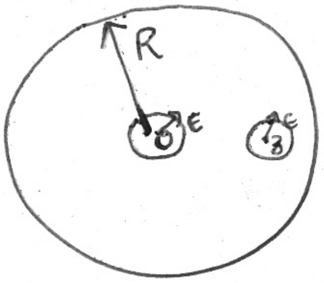
Kevin : you're prob. right. I apologize. Still nontrivial for maps to be holomorphic.

I think about smooth examples normally.

Let's consider a holomorphic fact. algebra on $\mathbb{C}$ which is translation invariant, and dilation invariant.

Let $V = F(\text{any round disc})$.

If we have a configuration of discs like



We get a map

$$m_z : V \otimes V \longrightarrow V$$

As z varies, with fixed radius, this map must vary holomorphically

m_z is a holomorphic map

$$\text{Annulus} \longrightarrow \text{Hom}(V \otimes V, V)$$

So it has a Laurent expansion:

$$m_z \sim \sum_{k \in \mathbb{Z}} z^k a_k ,$$

$a_k \in$ some completion of $\mathrm{Hom}(V \otimes V, V)$

If I had got defn right, we'd have seen this is reminiscent of a VOA.

In Beilonson + Drinfeld's book, they make this defn in the algebraic setting, call it a <u>chiral algebra</u>.

They show axioms for a chiral algebra on $\mathbb{C}$ are "essentially equivalent" to those of a vertex algebra.

Question: What about finite Laurent condition? You've described field-state.

Answer: I haven't worked this through. But in B-D book, they do get a correspondence.

Urs: What if you vary radius?

Answer: Get a filtered object, more along the lines of Huang.

Two situations in QFT where observables well understood

1. holomorphic / conformal field theory (hol. observables
qn eg. Segal)

2. quantum mechanics (obervables on $\mathbb{R}$).

In general, want factorization algebra on M.

<u>Note</u> Deformation quantization is a pert<u>urbativ</u>e thing.

We don't know what QFT is nonperturbatively.