commutative for every i such that $\tau_S n(i) \in v_n^{-1}(A)$. Next for every $f \in S^e\mathbf{n}$ we define

$$\bar{\gamma}_f^n : J_n'(\bar{v}_n(f) \longrightarrow J_n^e(f)$$

using stalk-construction in the obvious way.

Clearly, $(\bar{v}, \bar{\gamma}) \cdot (w, \beta) = (v, \gamma)$

The fact that the functor associated to the morphism $(\bar{v}, \bar{\gamma})$ preserves ultra-products follows directly from Lemma 4.1.

The proof is complete.

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REMARKS ON REPRESENTATIONS OF UNIVERSAL ALGEBRAS BY SHEAVES OF QUOTIENT ALGEBRAS

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Abstract. In this paper, we give a sheaf representation for any universal algebra whose congruence lattice contains a particular type of subframe. This representation yields the following known representation theorems: Wolf-Maddana Swamy's representations for those algebras whose congruence lattices are distributive; Koh-Simmons' representation theorem for strongly harmonic rings which includes Mulvey's representations of Gelfand rings; and Georgescu and Voiculescu's representations of normal unital quantales which include Keimel's representations of F-rings and Cornish's representations of normal lattices.

1. Introduction

There are many sheaf representation theorems for a wide variety of algebras. One might hope to establish a unified proof of the sheaf representation theorems by searching for a sheaf representation of universal algebras. Non-trivial sheaf representation theorems are known for the class of algebras whose congruence lattices are distributive (see [16],[10]) and the class whose congruence lattices are normal commutative unital quantales (see [4]). Unfortunately such results do not include the sheaf representation theorems for rings with identity.

We present a sheaf representation for a class of universal algebras which specializes to give some of the known representation theorems including some representations of non-commutative rings. This solves the problem posed by Wolf [16], but it does not yield Grothendieck representations, nor some of the representations of Lambek [8]. This paper will not explore the applications of the individual representations, nor will we yet consider general categorical treatments.

For basic terminology the reader is referred to [5] and [1].

2. Preliminaries

Let L be a complete lattice. Then $\mathcal{L} \subset L$ is called a subframe of L if $\mathcal{L}$ is closed under infinite joins and finite meets and $\mathcal{L}$ is itself a locale. Equivalently,

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$\mathcal{L}$ is a subframe of L if it is a frame and the inclusion preserves finite meets and arbitrary joins (including the top and bottom elements).

For such an $\mathcal{L}$ let $Pt(\mathcal{L})$ denote the set of all prime elements of $\mathcal{L}$ endowed with the usual hull-kernel topology in which the open sets have the form $d(\theta) = \{\pi \in Pt(\mathcal{L}) \mid \theta \not\leq \pi\}$, for $\theta \in \mathcal{L}$.

Unless the context indicates otherwise, suppose henceforward that A is a universal algebra, that CA is its congruence lattice, and that $\mathcal{L}$ is a compact subframe of CA.

Let ω be the smallest (diagonal) congruence on A and write for each $\theta \in \mathcal{L}$

$$\theta^* = \bigvee\{\phi \in \mathcal{L} \mid \phi \cap \theta = \omega\}.$$

Then $\theta^* \in \mathcal{L}$ and $\theta^* \cap \theta = \omega$. For each $\pi \in Pt(\mathcal{L})$, let

$$r(\pi) = \bigcup\{\theta^* \mid \theta \not\leq \pi\}.$$

Notice that $\pi \supset r(\pi)$ and $\theta^* \subset \bigcap\{r(\pi) \mid \pi \in d(\theta)\}$.

Lemma 1. *For each $\theta \in \mathcal{L}$, $\theta = \bigcap\{\pi \mid \pi \in Pt(\mathcal{L})$ and $\theta \subset \pi\}$.*

Proof. See Cornish [3] Proposition 2.1. □

Lemma 2. *For each $\theta \in \mathcal{L}$, $\theta^* = \bigwedge_{\mathcal{L}}\{\pi \mid \pi \in d(\theta)\}$.*

Proof. Notice that

$$\begin{aligned}\theta \cap (\bigwedge_{\mathcal{L}}\{\pi \mid \pi \in d(\theta)\}) &= \theta \cap (\bigwedge_{\mathcal{L}}\{\pi \in Pt(\mathcal{L}) \mid \theta \not\leq \pi\}) \\ &= \bigwedge_{\mathcal{L}}\{\pi \in Pt(\mathcal{L}) \mid \theta \leq \pi\} \cap \bigwedge_{\mathcal{L}}\{\pi \in Pt(\mathcal{L}) \mid \theta \not\leq \pi\} \\ &= \omega\end{aligned}$$

(using Lemma 1). Thus we have $\theta^* \geq \bigwedge_{\mathcal{L}}\{\pi \mid \pi \in d(\theta)\}$. The reverse inequality is straightforward. □

Recall that an element x of a lattice L is said to be *compact* if for each family of elements of L whose join is above x there is a finite subfamily whose join is above x, and that a complete lattice L is called *algebraic* if each element x of L can be expressed as the join of a family of compact elements. For example any ideal lattice (two-sided or one-sided) of a ring is an algebraic complete lattice.

Lemma 3. *Let L be an algebraic complete lattice and $\mathcal{L}$ a subframe of L. Then $\mathcal{L}$ is a spatial locale.*

Proof. Let $a, b \in \mathcal{L}$ with $a \not\leq b$. Then there is a compact $c \leq a$ with $c \not\leq b$ since L is algebraic. Consider the set $Q(c) = \{x \in \mathcal{L} \mid c \not\leq x, b \leq x\}$. Then $Q(c)$ is non-empty since $b \in Q(c)$ and $Q(c)$ is closed under joins of chains in $Q(c)$ because c is compact. Hence by Zorn's Lemma there is a maximal element $m \in Q(c)$ with $b \leq m$. Of course, $c \not\leq m$.

It remains to show that m is a prime element in $\mathcal{L}$. Let $y_1, y_2 \in \mathcal{L}$ with $y_i \not\leq m$; then $c \leq y_i \vee m$ by the maximality of m. Furthermore, $c \leq (y_1 \vee m) \wedge (y_2 \vee m) = y_1 \wedge y_2 \vee m$, and hence $y_1 \wedge y_2 \not\leq m$. □

Remark 4. Let x and y be elements of a lattice L. We say that x is *way below* y if for any family of L whose join is above y there is a finite subfamily whose join is above x. A lattice L is called *continuous* if each element of L can be expressed as the join of those elements which are way below it. It is clear that every algebraic lattice is continuous.

The hypotheses of Lemma 3 are actually stronger than is required. Using our argument and a well-known characterization of continuous lattices, Lemma 3 holds for L a *continuous* complete lattice. We thank D.S. Zhao (Cambridge) for bringing this to our attention.

Corollary 5. *Every continuous locale is spatial.*

We will use the following Chinese Remainder Theorem [16].

Theorem 6. (CRT): *Let θ_i, $i = 1, 2, \ldots, n$ be mutually permutable congruences on A which generate a distributive sublattice $\mathcal{L}$ of CA. If a_i, $i = 1, 2, \ldots, n$ are elements of A such that $(a_i, a_k) \in \theta_i \vee \theta_k$ for all $i, k = 1, 2, \ldots, n$, then there is an element $a \in A$ such that $(a, a_i) \in \theta_i$ for all $i = 1, 2, \ldots, n$.*

Lemma 7. *Let $\theta_i \in \mathcal{L}$, $i = 1, 2, \ldots, n$ be mutually permutable with join $A \times A$; and $a_i \in A, i = 1, 2, \ldots, n$ such that $(a_i, a_j) \in (\theta_i \cap \theta_j)^*$ for all $i, j = 1, 2, \ldots, n$. Then there exists $a \in A$ such that $(a, a_i) \in \theta_i^*$ for all $i \leq n$.*

Proof. Since θ^* is the largest element in $\mathcal{L}$ with the property that $\theta \cap \theta^* = \omega$, we have $(\theta \cap \phi)^* \cap \theta \leq \phi^*$ for all $\theta, \phi \in \mathcal{L}$. Now let $i, j = 1, 2, \ldots, n$ and $\theta = \bigcap_{k=1}^{n}\{(\theta_i \cap \theta_k)^* \vee (\theta_k \cap \theta_j)^*\}$ and notice that $(a_i, a_j) \in \theta$. Then

$$\theta \cap \theta_k \subset ((\theta_i \cap \theta_k)^* \cap \theta_k) \vee ((\theta_k \cap \theta_j)^* \cap \theta_k) \subset \theta_i^* \vee \theta_j^*$$

for $1 \leq k \leq n$, which implies that $\theta = \theta \cap (\bigvee_{k=1}^{n} \theta_k) = \bigvee_{k=1}^{n}(\theta \cap \theta_k) \subset \theta_i^* \vee \theta_j^*$.

On the other hand, $\theta_i^* \vee \theta_j^* \subset (\theta_i \cap \theta_k)^* \vee (\theta_k \cap \theta_j)^*$ for $1 \leq k \leq n$ and hence $\theta_i^* \vee \theta_j^* \subset \theta$; so that $\theta = \theta_i^* \vee \theta_j^*$. Therefore, we have $(a_i, a_j) \in \theta_i^* \vee \theta_j^*$ for $1 \leq i, j \leq n$. Note that each $\theta_i^* \in \mathcal{L}$ and $\mathcal{L}$ is distributive. The lemma now follows from CRT (Theorem 6). □

3. A sheaf representation

We will construct the sheaf $\mathcal{F}$ in the usual way: for each $\pi \in Pt(\mathcal{L})$, we denote by A_π the quotient algebra $A/r(\pi)$.

Let E be the disjoint union of the algebras A_π, $\pi \in Pt(\mathcal{L})$. Each $a \in A$ determines a Gelfand function $\widehat{a}\colon Pt(\mathcal{L}) \to E$ by defining $\widehat{a}(\pi) = [a]r(\pi)$ for all $\pi \in Pt(\mathcal{L})$.

We endow E with the finest topology for which all the maps $\widehat{a}$ are continuous.

Proposition 8. *The sets of the form $\widehat{a}(d(\theta))$ for $a \in A$ and $\theta \in \mathcal{L}$ form a basis of the topology on E and $\eta : E \to Pt(\mathcal{L})$ is a sheaf $\mathcal{F}$ of universal algebras. For each $a \in A$ the function $\widehat{a}$ is a global section and $a \mapsto \widehat{a}$ is an algebra homomorphism.*

Proof. First we show that the set

$$V = \{\pi \in Pt(\mathcal{L}) \mid (a,b) \in r(\pi)\}$$

is open for all $a, b \in A$: If $\pi \in V$, then $(a,b) \in \theta^*$ for some $\theta \in \mathcal{L}$ with $\theta \not\leq \pi$. Now $\pi \subset d(\theta) \subset V$ is easily seen. Next we see that for each $a, b \in A$,

$$\widehat{b}^{-1}(\widehat{a}(d(\theta))) = \{\pi \in Pt(\mathcal{L}) \mid (a,b) \in r(\pi)\} \cap d(\theta)$$

since $\pi \in \widehat{b}^{-1}(\widehat{a}(d(\theta)))$ if and only if $\widehat{b}(\pi) = \widehat{a}(\pi')$ for some $\pi' \in d(\theta)$ but this occurs if and only if $(a,b) \in \pi$ and $\pi = \pi' \in d(\theta)$. The rest of the proof is straightforward. □

We will require $\mathcal{L}$ to satisfy the following property (*):
For each finite set $\{\theta_i\}_1^n$ of $\mathcal{L}$ with $\bigvee \theta_i = A \times A$, there exists a finite set $\{\phi_i\}_1^n$ of $\mathcal{L}$ such that $\bigvee \phi_i = A \times A$, $\phi_i \leq \theta_i$, $i \leq n$ and

$$(\phi_i \cap \phi_j)^* \supset \bigcap \{r(\pi) \in Pt(\mathcal{L}) \mid \pi \in d(\theta_i) \cap d(\theta_j)\}$$

for all $i, j \leq n$.

Theorem 9. *Let A be a universal algebra with congruence lattice CA and let $\mathcal{L}$ be a compact subframe of CA which satisfies property (*) and is permutable. Then $a \mapsto \widehat{a}$ is an isomorphism from A onto the algebra of all sections of the sheaf $\mathcal{F}$ over $Pt(\mathcal{L})$ constructed above.*

Proof. The injectivity follows from Lemma 1. Now we show the surjectivity: Let σ be a section of $\mathcal{F}$. For each $\pi \in Pt(\mathcal{L})$, there is an $a_\pi \in A$ with $\sigma(\pi) = \widehat{a}_\pi(\pi)$. Then $\sigma^{-1}(\widehat{a}_\pi(Pt(\mathcal{L}))$ is a neighbourhood of π which contains some basic neighbourhood $d(\theta_\pi)$ of π, where $\theta_\pi \in \mathcal{L}$, so that $\sigma|d(\theta_\pi) = \widehat{a}_\pi|d(\theta_\pi)$ since they are sections. Thus $\{d(\theta_\pi) \mid \pi \in Pt(\mathcal{L})\}$ is an open cover of $Pt(\mathcal{L})$ and hence, since $Pt(\mathcal{L})$ is compact, there are $\theta_1, \theta_2, \ldots, \theta_n \in \mathcal{L}$ whose join is $A \times A$, and elements $a_1, a_2, \ldots, a_n \in A$ such that

$$\sigma|d(\theta_i) = \widehat{a}_i|d(\theta_i)$$

for $i = 1, 2, \ldots, n$. Furthermore, there exist $\{\phi_i\}_1^n$ satisfying (*).

Now for $i, k = 1, 2, \ldots, n$ we claim that $(a_i, a_k) \in (\phi_i \cap \phi_k)^*$: indeed, for all $\pi \in d(\theta_i \cap \theta_k) = d(\theta_i) \cap d(\theta_k)$ we have $\widehat{a}_i(\pi) = \sigma(\pi) = \widehat{a}_k(\pi)$, whence $(a_i, a_k) \in r(\pi)$ since $a_k \in [a_i]r(\pi)$; and hence $(a_i, a_k) \in \bigcap\{r(\pi) \mid \pi \in d(\theta_i \cap \theta_k)\} \subset (\phi_i \cap \phi_k)^*$ by (*). Now by Lemma 7, there is $a \in A$ such that $(a, a_i) \in {\phi_i}^*$, and hence in $\bigcap\{r(\pi) \mid \pi \in d(\phi_i)\}$ for $i = 1, 2, \ldots, n$ (using the definition of $r(\pi)$). Furthermore, $\widehat{a}|d(\phi_i) = \widehat{a}_i|d(\phi_i) = \sigma|d(\phi_i)$. As $\bigcup_{i=1}^n d(\phi_i) = d(A \times A) = Pt(\mathcal{L})$, we conclude that $\widehat{a} = \sigma$. □

Theorem 9 can be generalized as follows:

Theorem 10. *Let A be a universal algebra with congruence lattice CA and let $\mathcal{L}$ be a sublattice of CA satisfying (0) $\mathcal{L}$ and CA have the same top and bottom elements; (1) $\mathcal{L}$ itself is a compact spatial locale; (2) $\omega = \bigcap\{\pi \mid \pi \in Pt(\mathcal{L})\}$; (3) property (*) holds; (4) $\mathcal{L}$ is permutable. Then $a \mapsto \widehat{a}$ is an isomorphism from A onto the algebra of all sections of the sheaf $\mathcal{F}$ over $Pt(\mathcal{L})$.*

4. Quantales

To apply our sheaf representation to ring theory, we need the following notions (see [12] and [11]). Recall that a quantale L is a complete lattice with an associative binary operator $\cdot$ satisfying the infinite distributive law:

$$a \cdot (\bigvee S) = \bigvee \{a \cdot s \mid s \in S\}; \ (\bigvee S) \cdot a = \bigvee \{s \cdot a \mid s \in S\}$$

for any $a \in L$ and $S \subset L$. We will often write as for $a \cdot s$.

A *unital quantale* is one in which the top element is also an identity with respect to $\cdot$ (a similar setting was called an *integral cl-groupoid* in [1]).

Let L be a unital quantale. An element $p \leq 1$ is called m*-prime* if for any $a, b \in L$ then $a \cdot b \leq p$ implies either $a \leq p$ or $b \leq p$, while $p \leq 1$ is called *maximal* if $p \leq a$ implies $a = p$ or $a = 1$.

Then for any element x of a unital quantale, x maximal implies x m-prime which implies that x is prime (in the lattice-theoretic sense).

Following Simmons([14]), a unital quantale which is an upper continuous lattice will be called a *carrier*. Examples of carriers include complete Heyting algebras, the ideal lattices of bounded distributive lattices and the 2-sided ideal lattices of rings, as well as the congruence lattice of any universal algebra with distributive congruences.

Note that subframes $\mathcal{L}$ of a carrier L correspond to order-preserving and top-preserving maps $g : L \to L$ by defining for any $a \in L$

$$g(a) = \bigvee \{x \in \mathcal{L} \mid x \leq a\}.$$

The following are basic examples of quantales with chosen subframes.

Example 11. Let $Id\,R$ be the set of 2-sided ideals of R. Following Borceux *et al* (see [2]), a 2-sided ideal I is called pure if $I \vee a^* = R$ for each $a \in I$, where a^* denotes the right annihilator of a. The set of all pure ideals of R is a subframe of $Id\,R$.

Example 12. Let $Id\,R$ be the set of 2-sided ideals of R. Following Simmons ([13]), a 2-sided ideal I is called *uniform virginal* if $I \vee (aR)^* = R$ where $(aR)^*$ is the right annihilator of the right ideal aR. Then the set of all uniform virginal ideals of R is a subframe of $Id\,R$.

Example 13. More generally, let L be any 2-sided quantale which is an upper continuous lattice. An element $a \in L$ is called right regular if $a = \bigvee\{x \in L \mid x^* \vee a = 1\}$, where $x^* = \bigvee\{y \in L \mid xy = 0\}$. Define $w : L \to L$ by sending each $a \in L$ to the join of all the right regular elements less than a. Then $w(L)$ is a subframe of L.

It is shown in [4] that, if L is a commutative continuous unital normal quantale, then w restricts to an isomorphism between $Max(L)$ and $Pt(w(L))$. Their proof does not appear to use commutativity but does make use of continuity. The scope of this result has been extended by Simmons [14] for 2-sided normal carriers with a compact top. It can also be extended to include 1-sided cases, see [15].

Example 14. Let R be a ring and $E(R)$ the set of all idempotent central elements of R; then $E(R)$ is boolean with respect to $a \wedge b = ab$, $a \vee b = a + b - ab$. Define $g : Id\,R \to Id\,R$ by $g(A) = (A \cap E(R))R$. Then $g(Id\,R)$ is a subframe of $Id\,R$.

Now we record some examples of subframes which satisfy property (*).

Example 15. Let A be a universal algebra with distributive congruence lattice CA then $\mathcal{L} = CA$ satisfies (*).

Example 16. Let A be an algebraic normal unital quantale with a compact top τ. For example, A could be the 2-sided ideal lattice of a strongly harmonic ring. Then $\mathcal{L} = w(A)$ as defined in Example 13 satisfies (*).

Proof. We will prove that for any finite open cover $\{d(a_i)\}_1^n$ of $Pt(\mathcal{L})$ there exists a finite open cover $\{d(b_i)\}_1^n$ such that each $d(b_i) \subset d(a_i)$ and $(b_i \wedge b_j)^* \supset \bigcap\{\pi \in Pt(\mathcal{L}) \mid \pi \in d(a_i \wedge a_j)\}$. To do this, define $\rho : A \to A$ by assigning to each $a \in A$ the join of those elements x below a such that $x^* \vee a = \tau$. Write $\tilde{\pi}$ for the unique maximal element containing π. It is known that $\rho(\tilde{\pi}) = \pi$, that $\rho = w$ and that $\rho(A)$ is a compact regular subframe of A (which implies $Pt(\mathcal{L})$ is a compact Haudorff space). So for any open cover of $Pt(\mathcal{L})$ there is a finite refinement $\{d(a_i)\}_1^n$. By regularity, there is a finite open cover $\{d(b_i)\}_1^n$ such that for each $i \leq n$, the closure $\overline{d(b_i)} \subset d(a_i)$, and hence

$$\overline{d(b_i \wedge b_j)} = \overline{d(b_i) \cap d(b_j)} \subset \overline{d(b_i)} \cap \overline{d(b_j)} \subset d(a_i) \cap d(a_j) = d(a_i \wedge a_j).$$

Now:

(a) $\pi \in d(a)$ if and only if $\tilde{\pi} \in \tilde{d}(a)$, where the $\tilde{d}(x)$, $x \in \mathcal{L}$, are the open sets for the usual topology on the space of maximal elements of A.

Proof of (a): If $a \not\leq \pi$, then $a \not\leq \tilde{\pi}$. Otherwise $a \leq \tilde{\pi}$ implies $a = \rho(a) \leq \rho(\tilde{\pi}) = \pi$ which is a contradiction. The converse is trivial.

(b) $\tilde{\pi} \in \overline{\tilde{d}(a)}$ implies $\rho(\tilde{\pi}) \in \overline{d(a)}$.

Proof of (b): Consider any neighbourhood $d(b)$ of $\pi = \rho(\tilde{\pi})$, i.e., $b \in \mathcal{L}$ with $b \not\leq \pi$. By (a), $\tilde{d}(b)$ is a neighbourhood of $\tilde{\pi}$. So there exists $m \in \tilde{d}(b) \cap \tilde{d}(a) = \tilde{d}(b \wedge a)$ which implies $\rho(m) \in d(b \wedge a)$ using (a) again. That is to say $d(a) \cap d(b) \neq \emptyset$. Thus $\pi \in \overline{d(a)}$.

Now write $b_{i,j}$ for $b_i \wedge b_j$ and $a_{i,j}$ for $a_i \wedge a_j$. We will first show that

$$\bigcap\{\pi \mid \pi \in \overline{d(b_{i,j})}\} \subset \rho(\bigcap\{\tilde{\pi} \mid \tilde{\pi} \in \overline{\tilde{d}(b_{i,j})}\}).$$

It suffices to show that for each compact element $c \leq \bigcap\{\pi \mid \pi \in \overline{d(b_{i,j})}\}$,

$$c^* \vee \bigcap\{\tilde{\pi} \mid \tilde{\pi} \in \overline{\tilde{d}(b_{i,j})}\} = \tau.$$

If not, there is a maximal element m_0 containing $c^* \vee \bigcap\{\tilde{\pi} \mid \tilde{\pi} \in \overline{\tilde{d}(b_{i,j})}\}$ since τ is compact. In particular, $m_0 \supset \bigcap\{\tilde{\pi} \mid \tilde{\pi} \in \overline{\tilde{d}(b_{i,j})}\}$ which implies that $m_0 \in \{\tilde{\pi} \mid \tilde{\pi} \in \overline{\tilde{d}(b_{i,j})}\}$ since $\overline{\tilde{d}(b_{i,j})}$ is closed. Hence $\rho(m_0) \in \overline{d(b_{i,j})}$ by (b) above. On the other hand, $c \leq \bigcap\{\pi \mid \pi \in \overline{d(b_{i,j})}\}$ and so $c \leq \rho(m_0)$ which implies $c^* \vee m_0 = \tau$ (since c is compact) and $m_0 = \tau$ which is a contradiction.

Now the proof is completed as follows:

$$\bigcap\{\pi \mid \pi \in \overline{d(b_{i,j})}\} \subset \rho(\bigcap\{\tilde{\pi} \mid \tilde{\pi} \in \overline{\tilde{d}(b_{i,j})}\})$$

$$= w(\bigcap\{\tilde{\pi} \mid \tilde{\pi} \in \overline{\tilde{d}(b_{i,j})}\}) \subset w(\bigcap\{w(\tilde{\pi}) \mid \tilde{\pi} \in \tilde{d}(b_{i,j})\})$$

$$= w(\bigcap\{\pi \mid \pi \in d(b_{i,j})\}) = (b_{i,j})^* \text{ by Lemma 2}.$$

Finally, we have

$$\bigcap\{\pi \mid \pi \in d(a_{i,j})\} \subset \bigcap\{\pi \mid \pi \in \overline{d(b_{i,j})}\} \subset (b_{i,j})^*. \quad \square$$

Remark 17. In this example, we can weaken the assumption that the lattice is algebraic to require just that the lattice is continuous. The proof of the more general case is similar to that given above. Using results from [15] we can also weaken the assumption that the quantale is unital to require only that it be 1-sided.

5. Applications

We now use Theorem 9 to derive several known sheaf representation theorems. We begin by applying Lemma 2 to obtain the following Corollary.

Corollary 18. *Let A be a universal algebra whose congruence lattice is distributive. Put $CA = \mathcal{L}$. Then we have Swamy-Wolf's sheaf representation theorem* [9],[16].

Corollary 19. . *Let A be an universal algebra of some type with the following properties:*

(i) *the congruence lattice CA is a normal unital quantale with a compact top element.*

(ii) *the virginal elements of CA permute.*

Then A is isomorphic to the algebra of global sections of a sheaf on the compact Hausdorff space $Max(CA)$.

The proof of Corollary 19 follows from Theorem 9 and Example 16. Once again the result may be extended to 1-sided quantales.

This Corollary unifies the following representation theorems: Georgescu and Voiculescu's representation of normal unital quantales [4], Keimel's sheaf representations for F-rings [6], Koh-Simmons's representations for strongly harmonic rings [7], including Mulvey's representations for Gelfand rings [10], and Cornish's sheaf representations for normal lattices [3].

Finally, if we restrict our attention to representations of rings (rather than general universal algebras) we can eliminate property (*).

Let R be a ring (with an identity). Write $Id\,R$ for the set of all 2-sided ideals of R. A subset $\mathcal{L} \subset Id\,R$ is said to be a *subframe* of $Id\,R$ if for any $I, J \in \mathcal{L}$, $IJ = I \cap J$ and $\mathcal{L}$ is closed under arbitrary sums and finite intersections. A subframe $\mathcal{L}$ of $Id\,R$ is called *small* if for each maximal ideal M of R, $\mathcal{L}(M) = \sum\{I \in \mathcal{L} \mid I \subset M\} \subset \sum\{LAnn\,\langle a\rangle \mid a \notin M\}$, where $\langle a\rangle$ is the principal ideal generated by a and $LAnn\,\langle a\rangle$ denotes the left annihilator ideal of $\langle a\rangle$. Then we have the following:

Theorem 20. *Let R be any ring with identity and $\mathcal{L}$ be a small subframe of $Id\,R$. Then R is isomorphic to the ring of global sections of a sheaf $\mathcal{F}$ whose base space is $Pt(\mathcal{L})$.*

Proof. The sheaf is built by taking as each stalk a quotient R/π, where $\pi \in Pt(\mathcal{L})$. The smallness condition ensures that the canonical projection is a local homeomorphism. The injectivity follows from Lemma 1. Now we show the surjectivity: Let σ be a section of $\mathcal{F}$. For each $\pi \in Pt(\mathcal{L})$, there is an $a_\pi \in A$ with $\sigma(\pi) = \widehat{a}_\pi(\pi)$. Then $\sigma^{-1}(\widehat{a}_\pi(Pt(\mathcal{L}))$ is a neighbourhood of π which contains some basic neighbourhood $d(I_\pi)$ of π, where $I_\pi \in \mathcal{L}$, so that $\sigma|d(I_\pi) = \widehat{a}_\pi|d(I_\pi)$ since they are sections. Thus $\{d(I_\pi) \mid \pi \in Pt(\mathcal{L})\}$ is an open cover of $Pt(\mathcal{L})$ and hence there are $I_1, I_2, \ldots, I_n \in \mathcal{L}$ whose join is R, and elements $a_1, a_2, \ldots, a_n \in R$ such that

$$\sigma|d(I_i) = \widehat{a}_i|d(I_i)$$

for $i = 1, 2, \ldots, n$. Furthermore, there exist $e_i \in I_i$ satisfying $e_1 + e_2 + \cdots + e_n = 1$.

Let $a = \sum_1^n a_i e_i$. We claim $\widehat{a} = \sigma$. In fact, for each $i \leq n$ and each $\pi \in Pt(\mathcal{L})$, if $\pi \notin d(I_i)$, i.e., $I_i \subset \pi$, then

$$[(\sigma - \widehat{a}_i)\widehat{e}_i](\pi) \subset \widehat{e}_i(\pi) \in I_i/\pi \subset \pi/\pi = 0;$$

if $\pi \in d(I_i)$, then $\sigma(\pi) = \widehat{a}_i(\pi)$ which implies

$$[(\sigma - \widehat{a}_i)\widehat{e}_i](\pi) = 0.$$

Thus we have shown that

$$\sum_i^n [(\sigma - \widehat{a}_i)\widehat{e}_i](\pi) = 0,$$

that is, $\widehat{a} = \sigma$. □

Generalizations and applications of Theorem 20 will be taken up elsewhere.

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