ACTION PRINCIPLES AND GLOBAL GEOMETRY

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ABSTRACT

A new universal conserved current, depending on a field and two variations of the field, is defined and shown to provide a general approach to the covariant description of the phase space of a Lagrangian field theory. The global symplectic -- and in certain cases Kählerian -- geometry of phase space is examined for particular models. The moduli space of Riemann surfaces makes an unexpected appearance in three dimensional field theory; in particular, gravity. A brief discussion of quantization concludes a mainly classical treatment of field theory and action principles.

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INTRODUCTION

Modern physics has achieved a high degree of mathematical coherence through its virtually universal application of certain abstract principles. Hamilton's principle of least action is basic throughout mechanics, field theory, particle physics, and, lately, string theory. Noether's principle -- a continuous symmetry of the action leads to a conservation law -- is equally basic. Time translation invariance, for example, yields the law of conservation of energy.

We have discovered a new universal conservation law that holds for any system governed by a least action principle. We have investigated and exploited this new law in a wide variety of physical models and related mathematical problems. Our conservation law endows the phase space of the physical system with a mathematical structure generalizing the canonical structure of Hamiltonian mechanics[1]. Moreover, this generalized canonical structure becomes, for various field theories, the starting point for a quantum mechanical investigation. Finally, our approach to phase space preserves all the symmetries of the original variational problem. Hence, whenever the action is relativistically invariant, both the classical and quantum structures are explicitly compatible with Einstein's special and/or general principles of relativity.

E. Noether's famous 1918 paper, "Invariant Variational Problems"[2], crystallized essential mathematical relationships among symmetries, conservation laws, and identities for the variational or "action" principles of physics. Modern accounts of Noether's paper appear in 3], 4], 5]. Noether herself was motivated by the then new theory of general relativity[6]. In the last decade, physicists have been attempting to unify general relativity with quantum particle physics. Supergravity theory[7]] and superstring theory[8]] are related approaches to unified physics (see 9] for an introductory account for mathematicians.) String theory has spawned string field theory, which, unlike previous models in physics, involves infinite component fields on space-time (see 10]). Virtually all physical models begin with the formulation of an action principle invariant with respect to a Lie group and/or Lie algebra, usually infinite dimensional. Thus, Noether's abstract analysis in 2] continues to be relevant to contemporary physics (as well as to applied mathematics (see 5])).

A basic notion in Noether's theory is that of a conserved current: if the space-time X is n-dimensional, then such a current is an (n-1)-form $J = J(\psi)$ which is "local" in the field ψ (see below), and which is closed whenever ψ is an extremal for the action principle. Extremal fields are the solutions to the Euler-Lagrange field equations derived from the action. Linearization of the field equations about a given extremal leads to the Jacobi equations and their solutions, the Jacobi fields. One can generalize the notion of a conserved current as follows: consider an (n-1)-form $J = J(\psi, \delta_1\psi, \ldots, \delta_m\psi)$ which is local in ψ and a finite number $\delta_1\psi, \ldots, \delta_m\psi$ of infinitesimal variations of ψ; assume further that J is closed whenever ψ is extremal and the variations $\delta_1\psi, \ldots, \delta_m\psi$ are Jacobi fields for ψ. Our universal conservation law, alluded to above, amounts to a completely general construction of a universal conserved current $U = U(\psi, \delta_1\psi, \delta_2\psi)$ which is alternating bilinear in $\delta_1\psi$ and $\delta_2\psi$. Particular examples of the universal current U appear in 11], 12], 13], 14]. In the next section, we formulate a theory of conserved currents (alternating multilinear in the variations) in terms of differential forms on the product $X \times \mathcal{S}$ of space-time, X, with the infinite dimensional manifold $\mathcal{S}$ of field configurations over X.

1. LOCAL DIFFERENTIAL FORMS

Let $\pi: Y \to X$ be a smooth fibration of a manifold Y over the space-time manifold X (incidentally, we are using the term space-time in a rather vague way; we are <u>not</u> assuming X has a semi-Riemannian metric, as in general relativity). Let $\mathcal{S}$ be the manifold of smooth sections of Y over X. The de Rham complex $\Omega(X \times \mathcal{S})$ of smooth differential forms on $X \times \mathcal{S}$ has a number of nontrivial special features:

1) $\Omega(X \times \mathcal{S})$ is bigraded according to the product structure of $X \times \mathcal{S}$. We will write

$$\Omega(X \times \mathcal{S}) = \coprod_{p,q} \Omega^{p,q}(X \times \mathcal{S}).$$

Corresponding to this bigradation, the exterior derivative d on $X \times \mathcal{S}$ breaks into two operators: D, of type $(1,0)$, and ∂, of type $(0,1)$. We have $d = D + \partial$, $d^2 = D^2 = \partial^2 = D\partial + \partial D = 0$.

2) If $J \in \Omega^{p,0}(X \times \mathcal{S})$ and $\psi \in \mathcal{S}$, define a p-form $J(\psi)$ on X by $J(\psi)(x) = J(x,\psi)$. DJ will be in $\Omega^{p+1,0}(X \times \mathcal{S})$, and we will have

$$(DJ)(\psi) = d(J(\psi)),$$

where the d on the right is the exterior derivative on X.

3) More generally, if $J \in \Omega^{p,q}(X \times \mathcal{S})$, $\psi \in \mathcal{S}$,

and $\delta_1\psi,\ldots,\delta_q\psi$ are vectors in $T_\psi\mathcal{S}$ -- the tangent space of $\mathcal{S}$ at ψ --, then we can define a p-form $J(\psi,\delta_1\psi,\ldots,\delta_q\psi)$ on X by

$$J(\psi,\delta_1\psi,\ldots,\delta_q\psi)(x) = \{i(\delta_1\psi)\cdots i(\delta_q\psi)J\}(x,\psi)$$

where $i(\delta\psi)$ is the operation of interior multiplication (contraction) of a tangent vector against a form. We will have

$$(DJ)(\psi,\delta_1\psi,\ldots,\delta_q\psi) = d(J(\psi,\delta_1\psi,\ldots,\delta_q\psi))$$

where again d is the exterior derivative on X.

4) $\Omega(X \times \mathcal{S})$ has a canonical sub-bicomplex $\Omega_{loc}(X \times \mathcal{S})$ defined as follows: let $J^\infty Y$ be the manifold of infinite jets of sections of Y at points of X (see 15]). Let ε_∞ be the evaluation map from $X \times \mathcal{S}$ to $J^\infty Y$: $\varepsilon_\infty(x,\psi)$ = the ∞-jet of ψ at x. We have the induced map ε_∞^* from $\Omega(J^\infty Y)$ to $\Omega(X \times \mathcal{S})$. The image, $\varepsilon_\infty^*\Omega(J^\infty Y)$, is stable under both D and ∂, and hence is a sub-bicomplex, which we call $\Omega_{loc}(X \times \mathcal{S})$. We will write

$$\Omega_{loc}(X \times \mathcal{S}) = \coprod_{p,q} \Omega_{loc}^{p,q}(X \times \mathcal{S}).$$

Takens[5] defines a canonical bicomplex structure on $\Omega(J^\infty Y)$. It turns out that the map ε_∞^* yields an isomorphism of bicomplexes between $\Omega(J^\infty Y)$ and $\Omega_{loc}(X \times \mathcal{S})$.

We call a form K on $X \times \mathcal{S}$ <u>local</u> if K lies in $\Omega_{loc}(X \times \mathcal{S})$. Thus, if $K \in \Omega_{loc}^{p,q}(X \times \mathcal{S})$, $\psi \in \mathcal{S}$, and $\delta_1\psi,\ldots,\delta_q\psi \in T_\psi\mathcal{S}$, the space-time p-form $K(\psi,\delta_1\psi,\ldots,\delta_q\psi)$ (see 3) above) depends on $\psi,\delta_1\psi,\ldots,\delta_q\psi$ in a local fashion, i.e. $K(\psi,\delta_1\psi,\ldots,\delta_q\psi)(x)$ depends only on (finite) jets of $\psi,\delta_1\psi,\ldots,\delta_q\psi$ at x. (Note: we can regard $T_\psi\mathcal{S}$ as the sections of an appropriate vector bundle over X; hence, we can speak of a "jet at x" of an element $\delta\psi$ of $T_\psi\mathcal{S}$).

5) The space $\Omega^{n,1}_{loc}$ of local, type $(n,1)$, forms on $X \times \mathcal{S}$ has a distinguished subspace[15]: we call $K \in \Omega^{n,1}_{loc}$ a <u>source form</u> if for $\psi \in \mathcal{S}$ and $\delta\psi \in T_\psi\mathcal{S}$, the n-form $K(\psi,\delta\psi)(x)$ depends only on a finite jet of ψ and the <u>zero-jet</u> of $\delta\psi$ at x.

We write $\Omega^{n,1}_{source}$ for the space of source forms. $\Omega^{n,1}_{loc}$ also contains the subspace $D\Omega^{n-1,1}_{loc}$.

LEMMA 1[15]:

$$\Omega^{n,1}_{loc} = \Omega^{n,1}_{source} \oplus D\Omega^{n-1,1}_{loc},$$

i.e. if K is a local $(n,1)$ form, there exists a <u>unique</u> source form G and a local $(n-1,1)$ form H such that

$$K = G + DH.$$

6) We have a useful cohomological result for the D operator on Ω_{loc}:

THEOREM 2[15]: If $K \in \Omega^{p,q}_{loc}$ with $p < n$ and $q \geq 1$, then $DK = 0$ implies $K = DF$ for a local form in $\Omega^{p-1,q}_{loc}$. However, D-closed $(p,0)$-forms need <u>not</u> be D-exact; moreover, (n,q)-forms, which are always D-closed, need not be D-exact. For example, a nonzero source form (type $(n,1)$) is never D-exact.

7) The entire theory of local forms on $X \times \mathcal{S}$ can be written in suitable local coordinates: let Z denote the fiber of Y over X -- thus, X can be covered by open coordinate charts $\mathcal{U}$ such that $\pi^{-1}(\mathcal{U})$ is diffeomorphic to $\mathcal{U} \times Z$; moreover, Y can be covered by coordinate charts of the form $\mathcal{U} \times \mathcal{V}$, with

$\mathcal{U}$ as above and with $\mathcal{V}$ an open coordinate chart in Z. We label by $x_1,\ldots,x_n$ the coordinates on $\mathcal{U}$ and by $u_1,\ldots,u_m$ the coordinates on $\mathcal{V}$. Let $\mathcal{W}(\mathcal{U}\times\mathcal{V})$ be the set of pairs (x,ψ) such that $\psi(x)$ is in $\mathcal{U}\times\mathcal{V}$ -- then $\mathcal{W}(\mathcal{U}\times\mathcal{V})$ is an open subset of $X\times\mathcal{S}$, and $X\times\mathcal{S}$ is covered by these special open subsets. For any such $\mathcal{W}=\mathcal{W}(\mathcal{U}\times\mathcal{V})$ we can explicitly describe $\Omega_{loc}(\mathcal{W})$ as follows: define functions x_i on $\mathcal{W}$ by $x_i(x,\psi)=x_i(x)$; then define functions $u_{j,I}$ on $\mathcal{W}$ by

$$u_{j,I}(x,\psi)=D_I[u_j(\psi(x))],$$

where D_I is the partial derivative with respect to the x_i's and corresponding to the multi-index $I=(i_1,\ldots,i_n)$, $i_1,\ldots,i_n\geq 0$. The functions x_i and $u_{j,I}$ are type $(0,0)$ local forms; dx_i and $\partial u_{j,I}$ are local forms of type $(1,0)$ and $(0,1)$ respectively. Any local $(0,0)$ form F on $\mathcal{W}$ is smooth function of finitely many of the variables x_i, u_{jI}. Any local (p,q) form is expressible as a finite sum

$$\sum F_{i_1,\ldots,i_p,(j_1,I_1),\ldots,(j_q,I_q)}\wedge$$

$$dx_{i_1}\wedge\cdots\wedge dx_{i_p}\wedge\partial u_{j_1,I_1}\wedge\cdots\wedge\partial u_{j_q,I_q}$$

where each $F_{(\ldots)}$ is a local $(0,0)$ form.

For the differentials D and ∂ we have the formulae:

$$Dx_i=dx_i,\ \partial x_i=0;$$

$$Du_{j,I}=\sum u_{j,I\cup\{i\}}dx_i;$$

$$\partial u_{j,I}=\partial u_{j,I};$$

$$DF = \sum \frac{\partial F}{\partial x_i} dx_i + \sum \frac{\partial F}{\partial u_{j,I}} Du_{j,I};$$

$$\partial F = \sum \frac{\partial F}{\partial u_{j,I}} \partial u_{j,I};$$

where F is a local $(0,0)$ form.

A variation $\delta\psi$ of ψ, with (x,ψ) in $\mathfrak{U}$, determines a function $\overline{\delta\psi}$ from a neighborhood of x into $\mathbb{R}^m$. We find that

$$\partial u_{j,I}(x,\psi,\delta\psi) = D_I[u_j(\overline{\delta\psi}(x))].$$

Thus, for any $K \in \Omega^{p,q}_{loc}(\mathfrak{U})$, $(x,\psi) \in \mathfrak{U}$, and $\delta_1\psi,\ldots,\delta_q\psi \in T_\psi\mathcal{S}$, we can in principle write an explicit expression for the evaluation at x of the p-form $K(\psi,\delta_1\psi,\ldots,\delta_q\psi)$.

2. THE UNIVERSAL CONSERVED CURRENT

Suppose we now fix L in $\Omega^{n,0}_{loc}(X \times \mathcal{S})$. Then ∂L is in $\Omega^{n,1}_{loc}(X \times \mathcal{S})$, and by 1,5) we can write $\partial L = E + DM$, where E is a source form in $\Omega^{n,1}_{loc}(X \times \mathcal{S})$ and M is a form in $\Omega^{n-1,1}_{loc}(X \times \mathcal{S})$. We can then define U in $\Omega^{n-1,2}_{loc}(X \times \mathcal{S})$ by setting $U = \partial M$. We will have $\partial U = 0$ and $DU = D\partial M = -\partial DM = -\partial(\partial L - E) = +\partial E$. We summarize our construction below:

THEOREM 3 (The Fundamental Formulae):*

1) To any $L \in \Omega^{n,0}_{loc}(X \times \mathcal{S})$ there is a system of forms $\{E,M,U\}$ satisfying

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*During the preparation of this paper we learned of a July, 1986 letter from P. Deligne to D. Kazhdan in which Deligne sketches a result essentially equivalent to our Theorem 3, Part 1).

a) $E \in \Omega^{n,1}_{source}(X \times \mathcal{S})$

b) $M \in \Omega^{n-1,1}_{loc}(X \times \mathcal{S})$

c) $U \in \Omega^{n-1,2}_{loc}(X \times \mathcal{S})$

d) $\partial L = E + DM$

e) $U = \partial M$

f) $\partial U = 0$

g) $DU = \partial E$

2) a) E is determined uniquely by L; moreover, if we replace L by $L + DK$, with $K \in \Omega^{n-1,0}_{loc}(X \times \mathcal{S})$, E does not change.

b) M is not uniiquely determined by L, however, the class of M modulo addition of forms DN, $N \in \Omega^{n-2,1}_{loc}(X \times \mathcal{S})$, <u>is</u> uniquely determined by L.

c) U is uniquely determined by L, modulo the addition to U of a term DV, where V is in $\Omega^{n-2,2}_{loc}(X \times \mathcal{S})$; moreover, if we replace L by $L + DK$, $K \in \Omega^{n-1,0}_{loc}(X \times \mathcal{S})$, then U modulo D-exact forms does not change. In other words, the association of U to L defines a linear map $\tilde{U}$:

$$\tilde{U} : \Omega^{n,0}_{loc}/D\Omega^{n-1,0}_{loc} \to \Omega^{n-1,2}_{loc}/D\Omega^{n-2,2}_{loc}$$

PROOF: We have explained part 1); 2 a) follows from Lemma 1; 2 b) and c) follow from (Taken's) Theorem 2.

Theorem 3 above is quite abstract and needs to be interpreted in the more concrete terms of the calculus of variations and Lagrangian field theory:

1) The form $L \in \Omega^{n,0}_{loc}$ leads to an n-form $L(\psi)$

on X. $L(\psi)$ is called the Lagrangian density (strictly speaking, $L(\psi)/$(fixed volume form on X) is often referred to as the Lagrangian density; but we have no specific choice of volume form in general, so we work here with $L(\psi)$ instead). $L(\psi)$ is local in ψ -- almost all Lagrangians in physics are local. The action of ψ in a space-time domain $\mathcal{U} \subseteq X$ is defined as the integral

$$\int_{\mathcal{U}} L(\psi).$$

2) Suppose $\psi \in \mathcal{S}$, $\mathcal{U}$ is a domain in X with smooth boundary $\partial\mathcal{U}$, and $\delta\psi \in T_\psi\mathcal{S}$ is a variation that vanishes along $\partial\mathcal{U}$. We consider the variation of the action:

$$\delta\int_{\mathcal{U}} L(\psi) = \int_{\mathcal{U}} \partial L(\psi, \delta\psi)$$

Applying our abstract formula $\partial L = E + DM$, we obtain

$$\partial L(\psi,\delta\psi) = E(\psi,\delta\psi) + d(M(\psi,\delta\psi)),$$

and hence, by Stokes Theorem,

$$\delta\int_{\mathcal{U}} L(\psi) = \int_{\mathcal{U}} E(\psi, \delta\psi)$$

DEFINITION 4: $\psi \in \mathcal{S}$ is an extremal for the Lagrangian field theory determined by L if $\delta\int_{\mathcal{U}} L(\psi) = 0$ for all pairs $\mathcal{U}$ and $\delta\psi$ such that $\mathcal{U}$ is a (relatively compact) domain in X and $\delta\psi$ is a variation of ψ that vanishes on the boundary of $\mathcal{U}$.

PROPOSITION 5: ψ is an extremal for L if and only if ψ satisfies the equations $E(\psi,\delta\psi) = 0$, where $\delta\psi$ runs over all variations of ψ. In local coordinates, the system $E(\psi,\delta\psi) = 0$ is equivalent to the standard Euler-Lagrange equations (see 5]).

DEFINITION 6: The solution variety $\mathcal{S}_L$ associated to L is the set of extremals for L.

$\mathcal{S}_L$ is not in general a submanifold of $\mathcal{S}$. However, we can still define, for each $\psi \in \mathcal{S}_L$, a tangent space to ψ in $\mathcal{S}_L$, which we denote by $T_\psi \mathcal{S}_L$:

DEFINITION 7: The subspace $T_\psi \mathcal{S}_L$ of $T_\psi \mathcal{S}$, for $\psi \in \mathcal{S}_L$, consists of those $\delta\psi$ that satisfy the Jacobi equations -- the linearization about ψ of the Euler-Lagrange equations.

It is possible to write the Jacobi equations in our abstract global formalism. We skip the details here, but we do mention an important property of Jacobi fields:

LEMMA 8: If $\psi \in \mathcal{S}_L$ and $\delta_1\psi$ and $\delta_2\psi$ are in $T_\psi \mathcal{S}_L$, then

$$\partial E(\psi, \delta_1\psi, \delta_2\psi) = 0.$$

3) Apparently the form M of Theorem 3 has disappeared from our discussion of Lagrangian field theory. It was E. Noether[2] who observed (in local coordinates) that M is essential to the construction of conservation laws associated to L.

DEFINITION 9: Fix L: a form $J \in \Omega^{n-1,0}_{loc}$ is called a conserved current for L if whenever $\psi \in \mathcal{S}_L$, $J(\psi)$ is a closed $(n-1)$-form on X; more generally, a form $K \in \Omega^{n-1,q}_{loc}$ is a conserved current for L if whenever $\psi \in \mathcal{S}_L$ and $\delta_1\psi, \ldots, \delta_q\psi \in T_\psi \mathcal{S}_L$, then $K(\psi, \delta_1\psi, \ldots, \delta_q\psi)$ is a closed $(n-1)$-form on X.

Following the ideas of Sophus Lie, Noether introduced what are now called (see 5]) generalized symmetries of a Lagrangian L. Moreover, she proved in 2] her celebrated theorem that to every generalized symmetry of L there is an associated conserved current J in $\Omega^{n-1,0}_{loc}$. The form M is an essential ingredient in Noether's construction of J. We give some details of Noether's theory in part 5) of this section.

4) Of course, since U is defined as ∂M, the form M is essential to our own work. Our main result is the following:

THEOREM 10: a) For any Lagrangian L, the associated form $U \in \Omega^{n-1,2}_{loc}(X \times \mathfrak{S})$ is a conserved current.

b) The restriction of U to the manifold portion (smooth locus) of $X \times \mathfrak{S}_L$ defines a closed (n+1)-form.

PROOF; Combine the equations $\partial U = 0$, $DU = \partial E$, and $dU = \partial U + DU$ with Lemma 8.

DEFINITION 11: We call U the universal conserved current associated to L.

Suppose now that C is a compact oriented (n-1)-dimensional submanifold of X. Define a 1-form θ_C and a 2-form ω^0_C on $\mathfrak{S}$ via

$$\theta_C = \int_C M, \quad \omega^0_C = \int_C U;$$

more explicitly, if $\psi \in \mathfrak{S}$ and $\delta_1\psi, \delta_2\psi \in T_\psi\mathfrak{S}$, then

$$\theta_C(\psi, \delta_1\psi) = \int_C M(\psi, \delta_1\psi), \quad \text{and}$$

$$\omega_C^0(\psi, \delta_1\psi, \delta_2\psi) = \int_C U(\psi, \delta_1\psi, \delta_2\psi).$$

Then θ_C and ω_C^0 are smooth on $\mathbb{S}$, and $\omega_C^0 = d\theta_C$. In particular, $d\omega_C^0 = 0$.

COROLLARY OF THEOREM 10: The restriction of ω_C^0 to the smooth part of $\mathbb{S}_L$ is a closed 2-form, denoted by $\omega_{[C]}$, which depends only on the homology class of C in X.

QUESTION: Given X, $Y \xrightarrow{\pi} X$, L, and C as above: when is $\omega_{[C]}$ symplectic -- nondegenerate at every point of the smooth part of $\mathbb{S}_L$?

Of course, the manifold X may not contain any homologically nontrivial submanifolds C of codimension one. We can try in that case to integrate $U(\psi, \delta_1\psi, \delta_2\psi)$ over a noncompact submanifold -- but then we need appropriate boundary conditions to guarantee convergence of the integral as well as ensure the invariance of the integral under a deformation of the noncompact manifold -- contour -- inside X. In reference 12] we discuss linear field equations on an anti-de Sitter space-time X, which is diffeomorphic to $D^3 \times \mathbb{R}$. The noncompactness of the 3-ball D^3 leads to nontrivial difficulties with the two form $\omega_{D^3} = \int_{D^3} U$.

If we choose not to integrate at all, then Theorem 10 b) tells us that the restriction of U to $X \times \mathbb{S}_L'$, where $\mathbb{S}_L'$ is the smooth locus of $\mathbb{S}_L$, defines a cohomology class in $H^{n+1}(X \times \mathbb{S}_L', \mathbb{R})$. We can ask under what conditions this "universal" cohomology class is nonzero.

5) Suppose next that X <u>does</u> contain a homologically nontrivial (n-1)-cycle, C -- a compact

oriented (n-1)-submanifold without boundary. Even for such a space X, the nondegeneracy of $\omega_{[C]}$ depends nontrivially on symmetry properties of L.

DEFINITION 12: a) A smooth vector field ξ on $\mathcal{S}$ is <u>local</u> if when we regard $T_\psi\mathcal{S}$ as sections of a vector bundle over X depending on $\psi \in \mathcal{S}$, then $\xi(\psi)(x)$ is a function of some jet of ψ at x.

b) A local vector field ξ is a generalized symmetry of a Lagrangian L if for some R in $\Omega^{n-1,0}_{loc}$, we have

$$(\partial L)(\psi, \xi(\psi)) = (DR)(\psi)$$

for all ψ in $\mathcal{S}$. In other words, the variation of L by ξ does not change the class of L modulo D-exact local forms.

c) A family of local vector fields $\xi(\varepsilon)$ which depend locally and linearly on an arbitrary smooth section ε of some fixed vector bundle over X -- $\xi(\varepsilon)(\psi)(x)$ depends only on some jets of ψ and ε at x -- is a generalized gauge symmetry of a Lagrangian L if for some family $R(\varepsilon)$ in $\Omega^{n-1,0}_{loc}$ that depends locally on ε, we have

$$\partial L(\psi, \xi(\varepsilon)(\psi)) = DR(\varepsilon)(\psi)$$

for all ψ in $\mathcal{S}$ and all sections ε.

THEOREM 13: a) Suppose a Lagrangian L admits a generalized symmetry ξ: then for any ψ in $\mathcal{S}_L$, $\xi(\psi)$ is in $T_\psi\mathcal{S}_L$ -- $\xi(\psi)$ is a Jacobi field for ψ; moreover, the Lie derivative of $\omega_{[C]}$ along ξ vanishes on $\mathcal{S}_L$.

b) Suppose a Lagrangian L admits a generalized gauge symmetry $\xi(\varepsilon)$: then for any extremal ψ in $\mathcal{S}_L$, the Jacobi field $\xi(\varepsilon)(\psi)$ is in the radical of the two-form $\omega_{[C]}(\Psi)$ -- for any Jacobi field $\delta\Psi$ in $T_\Psi\mathcal{S}_L$, we have

$$\omega_{[C]}(\Psi, \xi(\varepsilon)(\Psi), \delta\Psi) = 0$$

for all sections ε.

PROOF: The details will appear in a forthcoming preprint[16]. A sketch of the proof follows:

We remark here that our Definition 12 is a globalization of concepts that play a key role in Noether's paper[2]. In fact, it is easy at this point to state Noether's main results.

DEFINITION 14: Suppose ξ is a generalized symmetry of a Lagrangian L. The Noether current J_ξ associated to L and ξ is given by

$$J_\xi(\Psi) = R(\Psi) - M(\Psi, \xi(\Psi))$$

for Ψ in $\mathcal{S}$. (Recall that the form R occurs in the condition that ξ be a generalized symmetry.)

THEOREM 15[2]: a) Suppose ξ is a generalized symmetry of L: then the Noether current J_ξ is conserved -- $dJ_\xi(\Psi) = 0$ for any extremal Ψ.

b) Suppose $\xi(\varepsilon)$ is a generalized gauge symmetry of L: then the Noether current $J_{\xi(\varepsilon)}$ has the following triviality property: for any extremal Ψ, $J_{\xi(\varepsilon)}(\Psi)$ is an exact (n-1)-form on X.

The relationship between Noether's theorem and our Theorem 13 is clear from the following:

PROPOSITION 16: Suppose ξ is a generalized symmetry of L: then for any ψ in $\mathcal{S}_L$ and $\delta\psi$ in $T_\psi\mathcal{S}_L$ we have

$$(dQ_\xi)(\psi,\delta\psi) = \omega_{[C]}(\psi,\xi(\psi),\delta\psi),$$

where

$$Q_\xi(\psi) = \int_C K_\xi(\psi) \text{ --}$$

the Noether charge associated to ψ, ξ, L, and C.

Thus, the argument for Theorem 13 b) is simple: If $\xi(\varepsilon)$ is a generalized gauge symmetry of L, and ψ is an extremal, then $J_\xi(\psi)$ is exact on X, by (Noether's) Theorem 15 (part b)). Therefore, the Noether charge $Q_{\xi(\varepsilon)}(\psi)$ vanishes by Stoke's theorem, since C is compact without boundary. Finally, by Proposition 16 we conclude that $\omega_{[C]}(\psi,\xi(\varepsilon)(\psi),\ \delta\psi)$ is zero for any Jacobi field $\delta\psi$ in $T_\psi\mathcal{S}_L$.

Part a) of Theorem 13 is a consequence of the theory of Jacobi fields, which will be discussed in [6].

6) Suppose G is a connected Lie group that operates smoothly on the fiber bundle $Y \xrightarrow{\pi} X$: thus G operates on both Y and X, and for any $y \in Y$, $g \in G$ we have $\pi(g\cdot y) = g\cdot\pi(y)$. We have induced actions of G on $\mathcal{S}$, $X \times \mathcal{S}$, and the bicomplex of local forms $\Omega_{loc}(X \times \mathcal{S})$.

DEFINITION 17: G is a symmetry group of a Lagrangian L if for any $g \in G$, then $g\cdot L - L = DK$ for some $K \in \Omega_{loc}^{n-1,0}$.

Suppose C is a compact (n-1)-cycle in X. Because G is connected, the homology class [C] of C will be G-invariant. Because G is a symmetry group of L mod D-exact local forms, the Euler-Lagrange equations of L will be left invariant by the action of G. Thus, G operates on $\mathcal{S}_L$ and on the differential forms on $\mathcal{S}_L$.

THEOREM 18: Given L, G, and C as above:

a) the closed 2-form $\omega_{[C]}$ on $\mathcal{S}_L$ is G-invariant;

b) Suppose $\psi \in \mathcal{S}_L$ and we denote by G_ψ the stability group of ψ for the action of G on $\mathcal{S}_L$. Then the tangent space $T_\psi \mathcal{S}_L$ carries a linear representation of G_ψ; moreover, the alternating bilinear form $\omega_{[C]}(\psi, \delta_1\psi, \delta_2\psi)$ is invariant for this G_ψ representation; finally, $T_\psi \mathcal{S}_L$/radical $\omega_{[C]}(\psi)$ carries a symplectic representation of G_ψ.

REMARK: In the above Theorem 18, G and G_ψ may both be noncompact, and the various representations of G_ψ may be infinite dimensional (see for example [2]).

3. EXAMPLES

We give now some examples of variational principles:

1) Suppose $X = \mathbb{R}$ with coordinate t, $Y = \mathbb{R} \times Z$, and π is projection onto the first factor. A section $\psi \in \mathcal{S}$ is just a path in Z. Suppose at first that $L(\psi)$ is first order in ψ. Working in local coordinates u_i in a patch on Z, we have

$$\begin{aligned}
\partial L &= \left[\frac{\partial \tilde{L}}{\partial u_{i,t}}\,\partial u_{i,t} + \frac{\partial \tilde{L}}{\partial u_i}\partial u_i\right] \wedge dt \\
&= \left[\frac{\partial \tilde{L}}{\partial u_i} - \frac{d}{dt}\left[\frac{\partial \tilde{L}}{\partial u_{i,t}}\right]\right] \partial u_i \wedge dt \\
&\quad + D\left[-\frac{\partial \tilde{L}}{\partial u_{i,t}}\,\partial u_i\right] ,
\end{aligned}$$

where $\tilde{L} = L/dt$. If we define p_i as $\partial\tilde{L}/\partial u_{i,t}$, we see that $M = -p_i \partial u_i$, and $U = -\partial p_i \wedge \partial u_i$. An (n-1)-cycle C in X is just a point, t_0, so that we have $\omega_{[C]} = (-\partial p_i \wedge \partial u_i)(t_0)$; this form is nondegenerate when we can solve for $u_{i,t}$ in terms of the "canonical variables" $\{u_k, p_k\}$. Relabeling u_i as q_i we see that our general theory reproduces the symplectic structure of classical mechanics.

If Z is a Riemannian or more generally, semi-Riemannian manifold, the arclength action provides a nontrivial case of a first order action with gauge invariance -- in this case reparameterization invariance. Let $\mathcal{E}$ be the variety of complete parameterized geodesics: $\mathcal{E}$ is a subvariety of $\mathcal{S}_L$. The set $\tilde{\mathcal{E}}$ of unparameterized geodesics in Z can be constructed as the quotient of the unit tangent bundle of Z by the action of the geodesic flow. By Theorem 13, our 2-form $\omega_{[t_0]}$ drops down to a 2-form on $\tilde{\mathcal{E}}$. However, $\tilde{\mathcal{E}}$ may not be a manifold at <u>any</u> point: consider the case when Z is a compact surface with negative curvature: the geodesic flow is ergodic, and $\tilde{\mathcal{E}}$ is a nasty object (perhaps approachable by Connes' noncommutative differential geometry[17]). If Z is a sphere, $\tilde{\mathcal{E}}$ is quite nice: it is a homogeneous compact Kähler variety, for which the Kähler form is our form $\omega_{[t_0]}$, pushed forward from $\mathcal{E}$ to $\tilde{\mathcal{E}}$.

Specific higher-order "mechanical" actions arise in elasticity[18] and in the theory of the KdV equation[11]. Our universal current leads again to the known symplectic structures.

2) Suppose X is a three-manifold M_3 and G is a Lie group, regarded as a real matrix group, with matrix Lie algebra $\mathfrak{G}$. Let $\mathbb{Q}$ be the space of

connections: $\mathfrak{G}$-valued 1-forms on X. Define the Chern-Simons[19] action of $A \in \mathfrak{A}$ by

$$\int_{\mathcal{U}} \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$

$\mathcal{U}$ a domain in M_3. The Euler-Lagrange equation is $F = 0$, where F is the curvature of A. The gauge group is enormous: the semi-direct product of $\mathrm{Diff}(M_3)$ with $C^{\infty}(M_3, G)$. Consequently, $\mathfrak{S}_L$ modulo gauge transformations is small: it is the space $\mathfrak{F}$ of geometrical classes of flat G-connections over M_3. If $\pi_1(M_3)$ is finitely presented, $\mathfrak{F}$ is the quotient of a finite dimensional real algebraic variety by the discrete group Γ of components of $\mathrm{Diff}(M_3)$. When $G = SL(2,\mathbb{C})$ we are studying Kleinian groups, i.e. homomorphisms of $\pi_1(M_3)$ to $SL(2,\mathbb{C})$.

If $G = SL(2,\mathbb{R})$ and we assume X is diffeomorphic to $S \times \mathbb{R}$, with S a compact oriented surface, then an open subset of $\mathfrak{F}$ is identifiable with the moduli space of Riemann surface structures on S.

The 2-form $\omega_{[S]}$, pushed forward to $\mathfrak{F}$ via Theorem 13, yields the Weil-Petersson symplectic form on moduli space. This symplectic form is in turn a multiple of the Kähler form for the known Kählerlian structure on moduli space[20].

3) Suppose X is a two-dimensional Riemannian or Lorentzian manifold and M is a given background space-time manifold. The energy of a map from X to M defines an "action" familiar in mathematics. If we regard the energy as a functional in both the map <u>and</u> the metric on X we obtain instead the Polyakov action for bosonic string theory[8]. The symplectic structure for this physical model is studied in the physics literature by means of conformal gauge fixing[8].

4) If we consider the Yang-Mills action on an arbitrary space-time, our universal current U specializes to the simple form

$$U(A,\delta_1 A,\delta_2 A) == \mathrm{Tr}(\delta_1 A \wedge *\delta_2 F - \delta_2 A \wedge *\delta_1 F);$$

here A is a Yang-Mills potential with variations $\delta_1 A$ and $\delta_2 A$, and corresponding curvature variations $\delta_1 F$ and $\delta_2 F$; "*" is the Hodge * operator induced by the background space-time metric. Our formula agrees with the formula in Crnkovic-Witten[13]].

For instantons on a compact closed four-manifold X, all periods of $U(A,\delta_1 A,\delta_2 A)$ vanish, for in this case the universal form is exact. It is unclear what happens when X is a compact manifold with boundary, as in 21].

5) Suppose X is an arbitrary manifold and we consider the Einstein action with cosmological constant for (semi-)Riemannian metrics on X. In this case our universal current U specializes to the Crnkovic-Witten current[13]] and the current in Ashtekar et. al[14]]. If the space-time X is three-dimensional, the extremal gravitational metrics have constant curvature. In the case of negative curvature we are led back to flat $SL(2,\mathbb{C})$ connections over three manifolds, as in 2) above.

In 16] we will compare our "covariant" symplectic geometry for Yang-Mills and Einstein theory with the more standard non-covariant "3 + 1" formalism of Fischer and Marsden[22]].

6) If X is $\mathbb{R}^{26}$ ($\mathbb{R}^{10}$), Witten's open string field (open superstring field) action[10](21]) offers the most sophisticated and mathematically involved variational principle ever considered. The string (superstring) field has infinitely many components.

The Witten action is a kind of noncommutative differential geometry[17] analog of the Chern-Simons action discussed in 2) above. Witten proposes in [21] a covariant symplectic structure for the solution variety modulo string gauge transformations. We plan to investigate the relationship between our universal current U and Witten's symplectic 2-form.

4. CONCLUDING REMARKS

We have limited our discussion to classical bosonic field theory. Fermionic fields may be introduced by extending our formalism to fibrations $Y \xrightarrow{\pi} X$ of a supermanifold Y over a superspace-time X. It should then be possible to incorporate superparticles[24], superstrings[8], and supergravity fields[7],[25] into our collection of physical examples.

Quantum fields -- in a covariant operator formalism -- would be the ultimate goal of our covariant approach to Lagrangian field theory. Under reasonable conditions, the form $\omega_{[C]}$ should push-forward to a symplectic form on the quotient of $\mathcal{S}_L$ by gauge transformations. (When the gauge algebra has field-dependent structure coefficients, as in supergravity theory[25], we have a new difficulty with taking such a quotient.) We would then regard this quotient as our covariant phase space, $\mathbb{P}$. Under ideal conditions (see Example 2) in Section 3) our phase space $\mathbb{P}$ would be Kählerian with the Kähler form equal to the push-forward of our $\omega_{[C]}$. Suppose this Kähler form represented an integral cohomology class of $\mathbb{P}$: then, following the ideas of geometric quantization[26],[27], there would be a holomorphic Hermitian line bundle $\mathfrak{L}$ with connection such that the curvature 2-form would equal the Kähler form. The Hilbert space $\mathfrak{H}$ of square-integrable global holomorphic sections of $\mathfrak{L}$ would serve as the quantum state space for our quantum field theory. There would be two main difficulties: defining "square-integrable" when $\mathbb{P}$ is infinite dimensional; and identifying the classical observables -- functions on $\mathbb{P}$ --, the quantum observables -- operators on $\mathfrak{H}$ --, as well as the connection between these two sets of observables.

An ideal testing ground for geometric quantization is gravitational theory on topologically nontrivial three-manifolds, X. If X is diffeomorphic to $S \times \mathbb{R}$ with S a compact oriented surface, and we choose the cosmological constant to have the proper sign, then our phase space $\mathbb{P}$ will be the space of hyperbolic structures on $X = S \times \mathbb{R}$. If we let $\mathfrak{M}$ denote the moduli space of Riemann surface structures on S, then it is known[20] that $\mathbb{P}' = \mathfrak{M} \times \overline{\mathfrak{M}}$, where $\mathbb{P}'$ is the subset of complete hyperbolic metrics on X, and $\overline{\mathfrak{M}}$ is the space of Riemann surface structures on S^{opp}, i.e. S with the opposite orientation.

Our form $\omega_{[S]}$ becomes a Kähler form for the complex structure[20] on $\mathfrak{M} \times \overline{\mathfrak{M}}$. The theory of Teichmüller modular forms should imply that our quantum state space $\mathfrak{H}$ for three-dimensional gravity is finite dimensional!

One last remark: the appearance of Teichmüller theory above has nothing obvious to do with the current applications of that theory to strings and superstrings[28]. In the latter theories, the moduli space $\mathfrak{M}$ shows up in the quantum theory -- the Feynman path integral reduces to an integral over $\mathfrak{M}$. Clearly, the correspondence between deep mathematical objects and deep physical models is not one-to-one!

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