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Twists of the Iwasawa-Tate Zeta Function

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Introduction

The intent of this paper is to develop the theory of a simple variation of the adelic zeta function first described by Tate [14] and Iwasawa [9] in independent efforts to simplify Hecke's analysis of his L-functions attached to Größencharakters. Our "twists" of the Iwasawa-Tate zeta function arise in the framework of a more general theory of zeta functions associated to representations of algebraic groups. Many of the basic problems of this general theory have already been solved by Yukie and the author (see [20] and [21]) but still await appearance in final written form. Thus, it seems worthwhile to give here a general statement of these problems before proceeding to the very special case considered in this paper.

Let k be a global field (i.e. algebraic number field or function field of a curve defined over a finite field). Let $\mathbb{A}$ and $\mathbb{A}^\times$ denote the ring of adeles and the group of ideles of k, respectively. Let ω be a quasicharacter of the idele group trivial on the embedded group $k^\times$ of nonzero elements of k. The group $\Omega = \Omega_k$ of all such quasicharacters ω has a natural Riemann surface structure. Let $|\cdot|_{\mathbb{A}}$ denote the usual idele norm on $\mathbb{A}^\times$, and for $s \in \mathbb{C}$ let ω_s be the principal quasicharacter $\omega_s(t) = |t|_{\mathbb{A}}^s$. Let $\operatorname{Re}(\omega)$ be the real part of $\omega \in \Omega$, defined as the unique real number σ such that $|\omega| = \omega_\sigma$.

For any reductive linear algebraic group G defined over k, let $G_{\mathbb{A}}$ denote the "adelization" of G over k as defined in [16], and let G_k be the group of k-rational points of G embedded as a discrete group in $G_{\mathbb{A}}$. For simplicity, we shall fix G as Gl_n now, although the theory to be described below can also be effected for any reductive algebraic group with at least one nontrivial k-rational character. Let $\rho: G \to \mathrm{Gl}(V)$ be an irreducible k-rational representation of G in the finite-dimensional vector space V of dimension m. Let I be the identity matrix in G. By Schur's lemma, there is an integer d such that $\rho(tI) = t^d \rho(I)$ for all scalars t. We shall always assume that $d > 0$. Let dg be any convenient left-invariant measure on the quotient $G_{\mathbb{A}}/G_k$. Let $\mathscr{S}(V_{\mathbb{A}})$ be the Schwartz-Bruhat space of functions on $V_{\mathbb{A}}$. For the time being, let V_k' be any G_k-invariant subset of V_k. The zeta function associated to the representation ρ may now be defined:

$$Z_\rho(\omega, \Phi) = \int_{G_{\mathbb{A}}/G_k} \omega(\det g) \sum_{x \in V_k'} \Phi(\rho(g)x)\, dg \tag{I.1}$$

for ω in Ω and Φ in $\mathscr{S}(V_{\mathbb{A}})$. Unless necessary, we shall omit the subscript ρ.

There are several major phases to the investigation of this zeta function. The first objective is to establish the absolute and locally uniform convergence of $Z(\omega, \Phi)$ for all Φ in $\mathscr{S}(V_{\mathbb{A}})$ and all ω with sufficiently large real part. The question of convergence amounts to determining the largest G_k-invariant subset V_k' of V_k such that this happens. This question can be settled by means of geometric invariant theory and reduction theory.

The second main problem is to obtain the analytic continuation of the zeta function to all of Ω. The design of the zeta function envisioned the application of the Poisson summation formula for this purpose. Let $\hat{\Phi}$ denote the Fourier transform of $\Phi \in \mathscr{S}(V_{\mathbb{A}})$ defined with respect to a suitable k-rational inner product on V. Let ρ^* be the contragredient representation to ρ with respect to this inner product. Define $\kappa = \frac{dm}{n}$. The main theorem should assert that under certain assumptions $Z(\omega, \Phi)$ has an analytic continuation to all of Ω which is holomorphic everywhere except possibly for poles at $\omega = \chi\omega_s$ with

$$s = 0, 1, \ldots, n-1, \kappa-(n-1), \ldots, \kappa-1, \kappa$$

and χ of finite order. Moreover, the following functional equation should hold

$$Z_\rho(\omega, \hat{\Phi}) = Z_{\rho^*}(\omega_\kappa \omega^{-1}, \Phi).$$

The third major problem is an extension of the previous one, namely, to determine precisely the nature of the poles of the zeta function, including a calculation of the leading terms of the Laurent expansions. The singularities of the zeta function are largely determined by the nature of the "missing" lattice points in its definition, that is, $S_k = V_k - V_k'$. The answer to this problem should be expressed in terms of special values of zeta functions associated to smaller representations.

After these basic analytic questions have been answered about the zeta function, there remains the vague problem of interpretation of these results. Often this will involve a great deal of analysis over local fields. The general idea is that the answers to the above questions will reveal interesting information about the distribution of the G_k-orbits in V_k'.

The above program was first delineated in something approaching this generality in the work of Sato and Shintani on zeta functions associated with prehomogeneous vector spaces (see [10, 12, 13]). For reasons connected with the local analysis, they chose to deal exclusively with representations for which there is a G-orbit of dimension equal to that of V, the so-called prehomogeneous representations. For all the questions listed above there is no reason why such an assumption must be made.

In the papers [18, 1, 2, 3], Datskovsky and the author presented an adelic formulation of the above type for Shintani's work on the natural representation of Gl_2 in the space of binary cubic forms. The end product was a set of theorems on the distribution of discriminants of cubic extensions of k. It is worth noting an important difference between the representation in these papers and that in Shintani's paper. Vectors $x=(x_1, x_2, x_3, x_4)$ are identified with binary cubic forms as follows

$$F_x(u, v)=x_1 u^3+x_2 u^2 v+x_3 u v^2+x_4 v^3.$$

In Shintani's work, the representation $\rho(g)$ of $g\in \mathrm{Gl}_2$ is defined by the functional equation

$$F_{\rho(g)x}(u, v)=F_x((u, v) g),$$

where we simply multiply the variable vector (u, v) on the right by g. In the papers of Datskovsky and the author, a slightly twisted representation ρ' defined by

$$\rho'(g)=\frac{1}{\det g}\,\rho(g)$$

is used instead. For the representation ρ, the degree d turns out to be 3, whereas for ρ' we have $d=1$. For the twisted representation ρ', the G_k-orbits in V_k' are in precise correspondence with the extensions of degree at most three of k. This is not true for Shintani's representation. One also has to deal with the difference between $k^\times$ and the subgroup k^3 of cubes in $k^\times$. While the residues of the zeta function for ρ' may be evaluated in terms of the Iwasawa-Tate zeta function, in the case of ρ the analogous analysis leads to a twisted version of this zeta function. This observation is what led to the subject of the present paper.

We shall deal here only with the case $G=\mathrm{Gl}_1$ and $\dim(V)=1$. All the representations are given by $\rho_n(t)x=t^n x$, for integral n. Let Φ be a Schwartz-Bruhat function on the locally compact abelian group $\mathbb{A}$. Let $|d^\times t|_{\mathbb{A}}$ be a multiplicative Haar measure on $\mathbb{A}^\times$. The Iwasawa-Tate zeta function is a distribution defined as follows

$$\zeta(\omega, \Phi)=\int_{\mathbb{A}^\times} \omega(t)\,\Phi(t)\,|d^\times t|_{\mathbb{A}}.$$

Special choices of ω and Φ lead to all possible Hecke L-functions. The basic references for this theory are [14] and [17]. A summary is given in Sect. 1 of [18]. Using the principle of telescoping series, we may rewrite the zeta function as

$$\zeta(\omega, \Phi)=\int_{\mathbb{A}^\times/k^\times} \omega(t) \sum_{x\in k^\times} \Phi(tx)\,|d^\times t|_{\mathbb{A}}.$$

It is in this guise that we see the natural generalization stated above. For any positive integer n, define the n-th twist of the Iwasawa-Tate function by

$$\zeta^{(n)}(\omega, \Phi)=\int_{\mathbb{A}^\times/k^\times} \omega(t) \sum_{x\in k^\times} \Phi(t^n x)\,|d^\times t|_{\mathbb{A}}. \tag{I.2}$$

This zeta function has no apparent Euler product decomposition for $n>1$. Indeed, if we mimic the telescoping of the Iwasawa-Tate zeta function, we arrive at the formula

$$\zeta^{(n)}(\omega, \Phi)=\frac{1}{w_n} \sum_{x\in k^\times/k^n} \int_{\mathbb{A}^\times} \omega(t)\, \Phi(t^n x)\, |d^\times t|_{\mathbb{A}}, \tag{I.3}$$

where w_n denotes the number of n-th roots of unity contained in k and k^n is the subgroup of n-th powers in $k^\times$. Nonetheless, the analytic continuation of this twisted zeta function may be carried out exactly as in the untwisted case. A summary of results is given in Sect. 1, in particular, Theorem 1.1.

The integrals that appear here in (I.3) are "orbital integrals" attached to the orbit of x under multiplication by k^n. If the characteristic of k is prime to n, there is a finite-to-one correspondence between elements of $k^\times/k^n$ and Kummer extensions $k(\sqrt[n]{x})$. Thus, the properties of the twisted zeta function are closely connected with the distribution of such extensions. In Sect. 2, we shall briefly study the local versions of these orbital zeta functions. In Sect. 3, we shall consider the product of these local zeta functions over all the places of k. Along the way, we produce some interesting Dirichlet series as well as their function-theoretic properties, assuming the characteristic of k does not divide n.

Classically, the structure of $k^\times/k^n$ is studied by decomposing this group into three parts, the ideal class group modulo the subgroup of n-th powers, the group of units modulo n-th powers, and finally the full group of fractional ideals of k again modulo the subgroup of n-th powers. Following such traditional paths leads one to suspect that the Dirichlet series appearing in the twisted zeta function are not so different from Hecke L-series after all. The main result of Sect. 4 and the paper brings the study of the twisted zeta function back to the original zeta function of Tate and Iwasawa. Let S be a finite nonempty set of places of k including all infinite places. Let $C_n(S)$ be the finite group of characters $\chi\in\Omega$ which are unramified outside S and which are trivial on the subgroup $\mathbb{A}^n$ of n-th powers in $\mathbb{A}^\times$.

Theorem I.1. *Given any $\omega\in\Omega$ and any Schwartz-Bruhat function Φ on $\mathbb{A}$, there is a finite nonempty set S_0 of places of k containing all infinite places so that for any finite set $S\supset S_0$ we have*

$$\zeta^{(n)}(\omega^n, \Phi)=\frac{1}{n} \sum_{\chi\in C_n(S)} \zeta(\omega\chi, \Phi). \tag{I.4}$$

It is proved in Sect. 1 (Prop. 1.2) that $\zeta^{(n)}(\omega, \Phi)=0$ unless ω is an n-th power of a quasicharacter. The set S_0 depends heavily on ω and Φ, although remarkably the formula does not change by enlarging S_0 to S. The formula in this theorem resembles an integral of some sort over Ω. A direct derivation of this formula via a change-of-variables theorem seems difficult because the group $\mathbb{A}^n$ of n-th powers of ideles is not open in $\mathbb{A}^\times$. In fact, $\mathbb{A}^n$ is a closed subset of measure zero. It is important to note that the quasicharacter ω in (I.4) is completely arbitrary.

Much of the work of Sects. 1 through 3 is performed simply to illustrate the analogue with the theory of the original Iwasawa-Tate zeta function and also that of general zeta functions. Clearly, the short combinatorial proof of the above theorem removes any essential need for those sections, since all the results in those sections can be quickly deduced from (I.4). For arbitrary representations, the properties of the zeta function are blindly pursued along the lines of Sects. 1, 2, and 3. For instance, for the space of binary cubic forms, the orbital decomposition analogous to (I.3) is

$$Z_\rho(\omega, \Phi) = \sum_{x \in G_k \backslash V'_k} \frac{1}{\gamma_x} \int_{G_{\mathbb{A}}} \omega(\det g)\, \Phi(\rho(g)x)\, dg$$

where γ_x is the order of the stabilizer of x in G_k. Once again, there is no obvious Euler product structure for the zeta function. In the case of the twisted zeta function, Eq. (I.4) reveals all the associated Dirichlet series to be finite linear combinations of Euler products. An analogue of (I.4) for the case of the space of binary cubic forms is currently unknown; although, such a formula would be immensely interesting. Part of the reason for presenting this paper is that these twists of the Iwasawa-Tate zeta function are examples where the zeta function is not itself an Euler product but is at least a finite linear combination of Euler products.

The last section of this paper presents an application of this zeta function theory to the analysis of discriminants of quadratic extensions of number fields. We prove the following theorem (Theorem 5.2 and subsequent remarks).

Theorem I.2. *Let k be a global field of characteristic not equal to 2. Choose $\omega \in \Omega$ with $\operatorname{Re}(\omega) > 1$ and the property that ω^2 is unramified at all finite places. Let S be any finite set of places of k containing all infinite places, all places lying over 2, and all places at which ω is ramified. Let $C_2(S)$ be the group of all $\chi \in \Omega$ satisfying $\chi^2 = 1$ and which are unramified outside S. Then*

$$\sum_{[k':k] \leq 2} \omega(\Delta_{\mathbb{A}}(k'/k)) = \frac{2^{-|S|}}{L_{k,S}(\omega^2)} \sum_{\chi \in C_2(S)} \Lambda_S(\chi\omega)\, L_{k,S}(\chi\omega)$$

where $L_{k,S}(\omega)$ are Hecke L-series (defined in Sect. 3), $\Delta_{\mathbb{A}}(k'/k)$ is Fröhlich's relative idelic discriminant (see Sect. 5), and Λ_S is a finite sum defined by

$$\Lambda_S(\omega) = \prod_{v \in S} \sum_{[K:k_v] \leq 2} \omega(\Delta(K/k_v)).$$

This sum ranges over the quadratic extensions K of the local field k_v, with $\Delta(K/k_v)$ being the relative discriminant.

A corollary to this theorem is the elegant identity

$$\sum_{[k':k] \leq 2} |\Delta_{\mathbb{A}}(k'/k)|_{\mathbb{A}}^s = \prod_v \Big\{ \sum_{[K:k_v] \leq 2} \tfrac{1}{2} |\Delta(K/k_v)|_v^s \Big\}$$

which holds if and only if the absolute ray class group modulo 4 of k has no elements of order 2. If the 2-primary part of this ray class group is nontrivial, then in general the global discriminant series is a sum of twists of this Euler product by the characters of order at most 2 of the same ray class group.

The conductor-discriminant formula of class field theory may also be used to obtain similar results (see [3]). Our methods should be considered only as a slightly more precise and elementary alternative.

There are a few conventions in our notation worth describing at the outset of this paper. The fields of rational, real, and complex numbers are denoted by $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. Given any ring R, the group of units is denoted by $R^\times$, and the subgroup of n-th powers of units is written R^n. If T is a subset of the set S, the complement of T in S is denoted $S-T$.

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Section 1. Global Theory

First, we shall establish some notational conventions that will be in force throughout the remainder of this paper. Our notation will largely be drawn from Sect. 1 of [18] and Sect. 6 of [2], which in turn is derived primarily from [17]. Let k denote a global field (i.e. algebraic number field or function field of a curve over a finite field). $\mathbb{A}$ and $\mathbb{A}^\times$ shall denote the ring of adeles and group of ideles of k, respectively. Let M_k be the set of all places of k. The set of all infinite places of k shall be denoted by M_∞. For any place v, let k_v denote the completion of k with respect to v.

The absolute value $|\ |_v$ on k_v is normalized to be the modulus function with respect to any additive Haar measure on k_v. The idele norm $\#_{\mathbb{A}}$ is similarly defined as the modulus with respect to any additive Haar measure on $\mathbb{A}$. The usual product formula for this idele norm is then valid. Let $\mathbb{A}_1$ denote the group of all ideles with norm 1.

The space of Schwartz-Bruhat functions for a locally compact abelian group X shall be denoted $\mathscr{S}(X)$ (see [6], Ch. II and Ch. IV for details). Next, we fix a choice $\langle\ \rangle$ of a nontrivial additive character on $\mathbb{A}$ which is trivial on k, and define the Fourier transform of a function Φ in $\mathscr{S}(\mathbb{A})$ by

$$\hat{\Phi}(y) = \int_{\mathbb{A}} \Phi(x)\langle xy\rangle\, |dx|_{\mathbb{A}}$$

where $|dx|_{\mathbb{A}}$ is the self-dual measure on $\mathbb{A}$ with respect to $\langle\ \rangle$. Then $\hat{\Phi}$ is an element of $\mathscr{S}(\mathbb{A})$, and we have

$$\Phi(x) = \int_{\mathbb{A}} \hat{\Phi}(y)\langle -xy\rangle\, |dy|_{\mathbb{A}}.$$

Moreover, the Poisson summation formula

$$\sum_{x\in k}\Phi(x)=\sum_{x\in k}\hat{\Phi}(x)$$

is valid for all $\Phi\in\mathscr{S}(\mathbb{A})$.

The multiplicative Haar measure $|d^\times x|_{\mathbb{A}}^1$ on $\mathbb{A}_1$ shall be chosen so that the compact quotient $\mathbb{A}_1/k^\times$ has induced measure 1. If k is a number field of degree m over $\mathbb{Q}$, for any positive real number λ define $\underline{\lambda}=(a_v)\in\mathbb{A}^\times$ such that $a_v=\lambda^{1/m}$ for all $v\in M_\infty$ and $a_v=1$ for all finite v. Then $|\underline{\lambda}|_{\mathbb{A}}=\lambda$. The Haar measure $|d^\times x|_{\mathbb{A}}$ on $\mathbb{A}^\times$ is then defined by

$$\int_{\mathbb{A}^\times}\Phi(x)\,|d^\times x|_{\mathbb{A}}=\int_0^\infty\frac{d\lambda}{\lambda}\int_{\mathbb{A}_1}\Phi(\underline{\lambda}x)\,|d^\times x|_{\mathbb{A}}^1.$$

If k is a function field with field of constants of order q, we may choose a fixed element $\pi\in\mathbb{A}^\times$ such that $|\pi|_{\mathbb{A}}=q$. Then the Haar measure $|d^\times x|_{\mathbb{A}}$ on $\mathbb{A}^\times$ is defined by

$$\int_{\mathbb{A}^\times}\Phi(x)\,|d^\times x|_{\mathbb{A}}=\sum_{l=-\infty}^{\infty}\int_{\mathbb{A}_1}\Phi(\pi^l x)\,|d^\times x|_{\mathbb{A}}^1.$$

Let Ω denote the group of complex quasicharacters of $\mathbb{A}^\times$ that are trivial on $k^\times$. Ω has a natural Riemann surface structure. The principal quasicharacters are those of the form $\omega_s(x)=|x|_{\mathbb{A}}^s$, for any $s\in\mathbb{C}$. All quasicharacters that are trivial on $\mathbb{A}_1$ are principal. The "real part" $\mathrm{Re}(\omega)$ of ω is defined to be the unique real number σ such that $|\omega|=\omega_\sigma$. Let Ω_0 be the subgroup of quasicharacters ω such that $\omega(\underline{\lambda})=1$ for all positive λ in the number field case and $\omega(\pi)=1$ in the function field case. Ω_0 is a discrete topological group. Every quasicharacter ω is of the form $\omega=\tilde{\omega}\omega_s$ for some $s\in\mathbb{C}$ and a unique $\tilde{\omega}\in\Omega_0$.

Let n be a fixed positive integer. The n-th twist of the Iwasawa-Tate zeta function is defined by

$$\zeta^{(n)}(\omega,\Phi)=\int_{\mathbb{A}^\times/k^\times}\omega(t)\sum_{x\in k^\times}\Phi(t^n x)\,|d^\times t|_{\mathbb{A}}$$

for $\omega\in\Omega$ and $\Phi\in\mathscr{S}(\mathbb{A})$. For $n=1$, this is precisely the Iwasawa-Tate zeta function

$$\zeta(\omega,\Phi)=\int_{\mathbb{A}^\times}\omega(t)\Phi(t)\,|d^\times t|_{\mathbb{A}}.$$

The relation between $\zeta^{(n)}$ and ζ is not clear at this point. We shall present the precise relation in Theorem 4.4, when the characteristic of k does not divide n. That theorem makes much of the analysis in this and the following two sections unnecessary, since the properties of the twisted zeta function may then be easily derived from those of the original Iwasawa-Tate zeta function. Skimping somewhat on the details (on account of Theorem 4.4), we shall develop the theory of $\zeta^{(n)}$ in a way entirely analogous to that used for ζ. First, an estimate modelled after that of Lemma 1.1 of [18] may be used to show that $\zeta^{(n)}$ converges for all Φ and all $\mathrm{Re}(\omega)>n$. To obtain the analytic continuation of $\zeta^{(n)}$, one would then use the Poisson summation formula in the following form:

$$\sum_{x\in k} \Phi(t^n x)=|t|_{\mathbb{A}}^{-n} \sum_{x\in k} \hat{\Phi}(t^{-n} x).$$

With this formula, simply follow the proof for $n=1$ given in [17]. Further details of the proof of the next theorem are very clearly indicated in the references cited above.

Theorem 1.1. *For all $\Phi\in\mathscr{S}(\mathbb{A})$, $\zeta^{(n)}(\omega, \Phi)$ converges absolutely and locally uniformly for $\mathrm{Re}(\omega)>n$, and has an analytic continuation to all of Ω, which is holomorphic everywhere except possibly for simple poles at $\omega=\omega_n$ and ω_0. The residues there are given by*

$$\operatorname*{Res}_{\omega=\omega_n} \zeta^{(n)}(\omega, \Phi)=c_k \hat{\Phi}(0),$$

$$\operatorname*{Res}_{\omega=\omega_0} \zeta^{(n)}(\omega, \Phi)=-c_k \Phi(0),$$

where $c_k=1$ for number fields k and $c_k=(\log q)^{-1}$ for function fields k. Moreover, $\zeta^{(n)}(\omega, \Phi)$ satisfies the functional equation

$$\zeta^{(n)}(\omega_n \omega^{-1}, \hat{\Phi})=\zeta^{(n)}(\omega, \Phi).$$

Finally, if k is a function field and $\chi\in\Omega_0$, $\chi\neq 1$, then

$$\zeta^{(n)}(\chi\omega_s, \Phi)=q^{Ns} P(\chi, \Phi; q^{-s})$$

for some integer N and some polynomial $P(\chi, \Phi; T)$. Also,

$$\zeta^{(n)}(\omega_s, \Phi)=\frac{q^{Ms} P(\Phi; q^{-s})}{(1-q^{-s})(1-q^{n-s})}$$

for some integer M and some polynomial $P(\Phi; T)$.

Let u be any idele such that $u^n=1$. The change of variables $t\to tu$ establishes the relation

$$\zeta^{(n)}(\omega, \Phi)=\omega(u)\, \zeta^{(n)}(\omega, \Phi).$$

Thus, the twisted zeta function vanishes unless $\omega(u)=1$ for all such ideles u. The Pontryagin duality theorem for locally compact abelian groups implies that any quasicharacter satisfying this triviality condition is in fact an n-th power of a quasicharacter. This proves the following.

Proposition 1.2. *$\zeta^{(n)}(\omega, \Phi)=0$ unless $\omega=\psi^n$ for some $\psi\in\Omega$.*

The final observations of this section concern the decomposition of $\zeta^{(n)}(\omega, \Phi)$ into orbital integrals. Let $w_n=w_n(k)$ denote the number of n-th roots of unity contained in $k^\times$. A straightforward rearrangement proves

Theorem 1.3. *For* $\mathrm{Re}(\omega)>n$ *and* $\Phi\in\mathscr{S}(\mathbb{A})$,

$$\zeta^{(n)}(\omega,\Phi)=\frac{1}{w_n}\sum_{x\in k^\times/k^n}\zeta_x^{(n)}(\omega,\Phi),$$

where the sum ranges over any complete set of coset representatives and

$$\zeta_x^{(n)}(\omega,\Phi)=\int_{\mathbb{A}^\times}\omega(t)\,\Phi(t^n x)\,|d^\times t|_{\mathbb{A}}.$$

This theorem indicates the connection between $\zeta^{(n)}$ and the structure of $k^\times/k^n$. In turn, to each coset $x\in k^\times/k^n$ are associated the extensions $k(\sqrt[n]{x})$ generated by n-th roots of x. To exploit this connection, we need to assume that the characteristic of k does not divide n. It is precisely under this assumption when given such a Kummer extension there are only finitely many corresponding cosets in $k^\times/k^n$. Thus, with this assumption, Theorem 1.3 exposes $\zeta^{(n)}$ to be a generating function counting the Kummer extensions of k of degree not greater than n. However, it is worth emphasizing that all the preceding theory is true independent of the characteristic of k.

We shall see that for Schwartz-Bruhat functions of "product form" the orbital zeta functions appearing in Theorem 1.3 decompose into a product over all places of k of local orbital zeta functions. To the study of these local zeta functions we now turn.

Section 2. Local Theory

Let K be a local field. Recall that these fields are divided into two categories, that of $\mathbb{R}$-fields, consisting of $\mathbb{R}$ and $\mathbb{C}$, and that of p-fields, those with residue class field of characteristic p, for some prime p. Let $|x|$ denote the modulus of multiplication by $x\in K$ with respect to any additive Haar measure on K. For any p-field K, the maximal compact subring shall be denoted by R, the unique maximal ideal by P, and a fixed generator of P by π. Then $|\pi|=q^{-1}$ where q is the order of the residue class field R/P.

Let dx be any additive Haar measure on K. Let $d^\times t$ be a multiplicative Haar measure on $K^\times$, and let the positive constant a be such that $d^\times t=a\dfrac{dt}{|t|}$. If K is a p-field, we shall always assume that a is chosen so that the measure of $R^\times$ is 1. Let Ω denote the group of quasicharacters of $K^\times$. For any $\omega\in\Omega$, let $\mathrm{Re}(\omega)$ denote its real part as usual.

For any $\alpha\in K^\times$, $\omega\in\Omega$, and $\Phi\in\mathscr{S}(K)$, define

$$\zeta_\alpha^{(n)}(\omega,\Phi)=\int_{K^\times}\omega(t)\,\Phi(t^n\alpha)\,d^\times t.$$

When $n=1$, we write simply

$$\zeta(\omega, \Phi) = \int_{K^\times} \omega(t)\,\Phi(t)\,d^\times t.$$

First, we shall discuss the convergence and analytic continuation of these local orbital zeta functions. It is a simple matter to verify that, for any $\alpha \in K^\times$, integer $n \geqq 1$, and $\Phi \in \mathscr{S}(K)$, the function $\Psi(x) = \Phi(x^n \alpha)$ is also a Schwartz-Bruhat function on K. Thus, all the properties of the twisted local zeta function may be deduced from those of the untwisted ($n=1$) version and the equation

$$\zeta_\alpha^{(n)}(\omega, \Phi) = \zeta(\omega, \Psi).$$

A list of these properties is given in Proposition 1.1 of [2]. At this point, we limit our observations to the following.

Lemma 2.1. *$\zeta_\alpha^{(n)}(\omega, \Phi)$ converges absolutely and locally uniformly for all $\Phi \in \mathscr{S}(K)$ and* $\mathrm{Re}(\omega) > 0$. *If Φ vanishes at* 0, *$\zeta_\alpha^{(n)}(\omega, \Phi)$ converges absolutely and locally uniformly to an entire function of ω. $\zeta_\alpha^{(n)}(\omega, \Phi)$ has a meromorphic continuation to all of Ω.*

In addition, for reasons analogous to those behind Proposition 1.2 in the present paper, $\zeta_\alpha^{(n)}(\omega, \Phi)$ vanishes unless ω is trivial on the group of n-th roots of unity contained in K. This in turn implies that ω is an n-th power of a quasicharacter of $K^\times$. Thus, it suffices to analyze $\zeta_\alpha^{(n)}(\omega^n, \Phi)$.

We must now attend to a more detailed description of the orbital structure of $K^\times$ modulo multiplication by n-th powers. For brevity, we shall denote the quotient group $K^\times/K^n$ by $\mathscr{A} = \mathscr{A}_K$. When necessary, we shall also use $\mathscr{A}$ to refer to a specific set of coset representatives always chosen of the following form. In the event that K is $\mathbb{R}$, we take $\mathscr{A} = \{\pm 1\}$ if n is even, and $\mathscr{A} = \{1\}$ if n is odd. If K is $\mathbb{C}$, we take $\mathscr{A} = \{1\}$. When K is a p-field, we shall assume

$$\mathscr{A} = \{\pi^l \varepsilon \mid 0 \leqq l < n,\ \varepsilon \in R^\times/R^n\},$$

where ε ranges over any complete set of coset representatives of $R^\times/R^n$.

If K is a p-field, let $\Phi_0 \in \mathscr{S}(K)$ be the characteristic function of R. Then, for any $\alpha \in \mathscr{A}$, $\Phi_0(t^n \alpha) = \Phi_0(t)$ by a simple consideration of the absolute value of $t^n \alpha$. Thus

$$\zeta_\alpha^{(n)}(\omega, \Phi_0) = \int_R \omega(t)\,d^\times t = \begin{cases} 0, & \text{if } \omega \text{ is ramified,} \\ (1-\omega(\pi))^{-1}, & \text{otherwise}\end{cases}. \tag{2.1}$$

This formula is crucial to the adelic calculations in Sect. 3.

To progress further, we must know something more about the structure of $R^\times/R^n$. For the remainder of this section, we shall assume that the characteristic of K does not divide n. It is precisely this assumption that guarantees that R^n is an open subgroup of $R^\times$. In fact, Hensel's lemma implies that $R^n \supset 1 + \pi n^2 R$. Thus, when the characteristic is prime to n, $R^\times/R^n$ is a finite set.

Integrals over K may be dissected into a sum of orbital integrals. Indeed, for any locally integrable function Φ on K, we have

$$\int_K \Phi(x)\,dx = \sum_{\alpha\in\mathscr{A}} \int_{\alpha(K^\times)^n} \Phi(x)\,dx.$$

The map $x\mapsto\alpha x^n$ is a continuous map of $K^\times$ into $K^\times$ with jacobian $\alpha n x^{n-1}$. The degree of this map is the number $w_n = w_n(K)$ of n-th roots of unity contained in K. Thus, by the change of variables theorem for a local field

$$\int_K \Phi(x)\,dx = \frac{|n|}{w_n} \sum_{\alpha\in\mathscr{A}} |\alpha| \int_K |x|^{n-1} \Phi(\alpha x^n)\,dx.$$

In terms of the multiplicative Haar measure, this formula becomes

$$\int_K |t|\,\Phi(t)\,d^\times t = \frac{|n|}{w_n} \sum_{\alpha\in\mathscr{A}} |\alpha| \int_K |t|^n \Phi(\alpha t^n)\,d^\times t. \tag{2.2}$$

In particular, if we apply this formula to the characteristic function of $R^\times$, when K is a p-field, we find that

$$\mathrm{Card}(R^\times/R^n) = \frac{w_n}{|n|}.$$

By separate consideration of $\mathbb{R}$ and $\mathbb{C}$, we may show that in complete generality

$$\mathrm{Card}(K^\times/K^n) = \frac{n\,w_n}{|n|}. \tag{2.3}$$

Applying (2.2) to the Iwasawa-Tate zeta function yields

$$\zeta(\omega, \Phi) = \frac{|n|}{w_n} \sum_{\alpha\in\mathscr{A}} \omega(\alpha)\,\zeta_\alpha^{(n)}(\omega^n, \Phi). \tag{2.4}$$

We can invert this equation by means of orthogonality of characters of finite abelian groups. Let $C_n = C_n(K)$ denote the group of $\chi\in\Omega$ satisfying $\chi^n = 1$. Then C_n may be naturally interpreted as the group of characters of the group $\mathscr{A}$. By duality of finite abelian groups, we have $\mathrm{Card}(C_n) = \mathrm{Card}(\mathscr{A})$. Then

$$\begin{aligned}\sum_{\chi\in C_n} \bar\chi(\alpha)\,\zeta(\omega\chi, \Phi) &= \frac{|n|}{w_n} \sum_{\chi\in C_n} \sum_{\beta\in\mathscr{A}} \omega\chi(\beta)\,\bar\chi(\alpha)\,\zeta_\beta^{(n)}(\omega^n, \Phi)\\ &= n\,\omega(\alpha)\,\zeta_\alpha^{(n)}(\omega^n, \Phi).\end{aligned} \tag{2.5}$$

The functional equation for $\zeta_\alpha^{(n)}$ may be derived from the above formula together with that for the untwisted case, presented in Sect. 1 of [2], among other places. Let $\Gamma(\omega)$ and τ_ω be the local gamma factor and the local gauss sum, respectively, as defined in Table 1.1 of [2]. (Be aware that $\Gamma(\omega)$ must strictly speaking be considered as a branched meromorphic function on Ω.) Then the local Iwasawa-Tate zeta function satisfies

$$\Gamma(\omega_1\,\omega^{-1})\,\zeta(\omega,\hat{\Phi})=\tau_\omega\,\Gamma(\omega)\,\zeta(\omega_1\,\omega^{-1},\Phi).$$

Applying this functional equation to (2.5) produces

$$\omega(\alpha)\,\zeta_\alpha^{(n)}(\omega^n,\hat{\Phi})=\frac{|n|}{n\,w_n}\sum_{\beta\in\mathscr{A}}\left\{\sum_{\chi\in C_n}\bar{\chi}(\beta)\,\frac{\tau_{\omega\chi}\,\Gamma(\omega\chi)}{\Gamma(\omega_1\,\omega^{-1}\,\bar{\chi})}\right\}\omega_1\,\omega^{-1}(\beta)\,\zeta_\beta^{(n)}(\omega_n\,\omega^{-n},\Phi).$$

This functional equation is interesting because it gives an example of the kind of explicit formula for the functional equation of the local zeta function of a prehomogeneous vector space that would be most desirable (see [7] and [8]).

Section 3. Adelic Synthesis

In this section, we shall apply the local analysis of Sect. 2 to the decomposition of the adelic zeta function presented in Theorem 1.3. All test functions $\Phi\in\mathscr{S}(\mathbb{A})$ considered in this section shall be of "product form", i.e. of the form

$$\Phi(x)=\prod_v \Phi_v(x_v),$$

for functions $\Phi_v\in\mathscr{S}(k_v)$. (Unless explicitly stated otherwise, all products written as above shall be assumed to extend over all places v of k.) For almost all finite places v, Φ_v is the characteristic function $\Phi_{0,v}$ of the maximal compact subring o_v of k_v.

The idelic measure $|d^\times t|_{\mathbb{A}}$ defined in Sect. 1 may be decomposed as a product of local measures. For infinite places v, let $|dt_v|_v$ be the additive Haar measure on k_v for which the set of all $t\in k_v$ with absolute value less than one has measure equal to 1, if v is real, and 2π, if v is complex. The multiplicative Haar measure $|d^\times t_v|_v$ on $k_v^\times$ is chosen to be $\dfrac{|dt_v|_v}{|t_v|_v}$ if v is infinite, and, if v is finite, the measure for which the compact subgroup $o_v^\times$ of units has measure 1. Then the idelic measure $|d^\times t|_{\mathbb{A}}$ selected in Sect. 1 is a constant multiple of the product of all these local measures, a relation we shall write as

$$|d^\times t|_{\mathbb{A}}=\rho_k^{-1}\bigotimes_v |d^\times t_v|_v.$$

The constant ρ_k has a well-known evaluation in terms of the basic number-theoretical constants associated with k. This evaluation is presented in Sect. 6 of [2].

The adelic orbital zeta functions may now be unveiled as products of local zeta functions

$$\zeta_x^{(n)}(\omega^n,\Phi)=\rho_k^{-1}\prod_v \int_{k_v} \omega_v^n(t_v)\,\Phi_v(t_v^n x)\,|d^\times t_v|_v.$$

For every place v, let $\mathscr{A}_v$ be the set of representatives for $k_v^\times/k_v^n$ selected in Sect. 2. For every $x \in k^\times$ and every place v, we may choose $u_{x,v} \in k_v^\times$ and a unique $\alpha_{x,v} \in \mathscr{A}_v$ such that

$$x = u_{x,v}^n \alpha_{x,v}. \tag{3.1}$$

Then $u_x = (u_{x,v})_{v \in M_k}$ and $\alpha_x = (\alpha_{x,v})_{v \in M_k}$ define elements of $\mathbb{A}^\times$. Also, since $\omega(x) = 1$, we have $\omega^{-n}(u_x) = \omega(\alpha_x)$. Then

$$\zeta_x^{(n)}(\omega^n, \Phi) = \rho_k^{-1} \omega(\alpha_x) \prod_v \int_{k_v} \omega_v^n(t_v) \Phi_v(t_v^n \alpha_v) |d^\times t_v|_v.$$

The "tail" of this product may be simplified in the following sense. There is a finite nonempty set S of places of k including all infinite places such that, for all places $v \notin S$, $\Phi = \Phi_{0,v}$ and ω_v is unramified. For each finite place v, let π_v be any generator of the unique maximal ideal of o_v. Then, according to Eq. (2.1),

$$\zeta_x^{(n)}(\omega^n, \Phi) = \rho_k^{-1} \omega(\alpha_x) \{\prod_{v \in S} \int_{k_v} \omega_v(t_v) \Phi_v(t_v^n \alpha_v) |d^\times t_v|_v\} L_{k,S}(\omega^n),$$

where

$$L_{k,S}(\omega) = \prod_{v \notin S} (1 - \omega_v(\pi_v))^{-1}.$$

This last Euler product is the most general form of a Hecke L-function. The function-theoretic properties of the Hecke L-functions are well-known; indeed, that was the purpose of the original treatment of $\zeta(\omega, \Phi)$. Thus, the analytic continuation of the adelic orbital zeta function is now complete. Except for a finite number of Euler factors, it is nothing other than a Hecke L-function.

We shall introduce an abbreviation for the finite product over the places in S appearing in the above formula. Let $\mathscr{A}_S$ be the cartesian product of all $\mathscr{A}_v$ with $v \in S$, also viewed with the natural group structure. For any $\alpha = (\alpha_v) \in \mathscr{A}_S$ set

$$\zeta_{\alpha,S}^{(n)}(\omega, \Phi) = \prod_{v \in S} \int_{k_v} \omega_v(t_v) \Phi_v(t_v^n \alpha_v) |d^\times t_v|_v.$$

Given $x \in k^\times$, let α denote the corresponding representative in $\mathscr{A}_S$. Then the previous formula may be written

$$\zeta_x^{(n)}(\omega^n, \Phi) = \rho_k^{-1} \omega(\alpha_x) L_{k,S}(\omega^n) \zeta_{\alpha,S}^{(n)}(\omega^n, \Phi).$$

Any representative $\alpha \in \mathscr{A}_S$ defines a subset of $\mathbb{A}^\times$, namely, the set of all ideles x such that $x_v \in \alpha_v k_v^n$ for all $v \in S$. In subsequent notation, we shall use α to denote the set of all cosets in $k^\times/k^n$ that belong to this corresponding subset of $\mathbb{A}^\times$. With that in mind, we define the Dirichlet series

$$\xi_{\alpha,S}^{(n)}(\omega) = \sum_{x \in \alpha} \omega(\alpha_x).$$

Remember that when $n = 1$ this series collapses to be identically 1. The decomposition in Theorem 1.3 may now be written

$$\zeta^{(n)}(\omega^n, \Phi) = \frac{L_{k,S}(\omega^n)}{\rho_k w_n} \sum_{\alpha \in \mathscr{A}_S} \xi_{\alpha,S}^{(n)}(\omega) \zeta_{\alpha,S}^{(n)}(\omega^n, \Phi). \tag{3.2}$$

Equation (3.2) is valid under the assumptions stated above for ω and Φ with regard to the choice of S.

To continue our analysis of these Dirichlet series, we must resume our supposition that the characteristic of k is prime to n. Under this assumption, the subset corresponding to any $\alpha\in\mathscr{A}_S$ is open in $\mathbb{A}^\times$. Then for any $\omega\in\Omega$ and any $\alpha_v\in\mathscr{A}_v$, we may choose $\Phi_v\in\mathscr{S}(k_v)$ such that Φ_v has compact support contained in $\alpha_v k_v^n$ and also so that $\zeta_{\alpha_v}^{(n)}(\omega_v^n, \Phi_v)$ is nonzero. Thus, the properties of $\xi_{\alpha,S}^{(n)}$ follow quickly from those of $\zeta^{(n)}$ and $\zeta_{\alpha_v}^{(n)}$ stated in Sects. 1 and 2. We shall refrain from a statement of these properties because we shall shortly see that these Dirichlet series may be written as simple combinations of Hecke L-functions.

Section 4. S-idelic Decompositions

We may derive another series expansion for $\xi_{\alpha,S}^{(n)}$ by applying the S-idelic decomposition to $\zeta^{(n)}$. Throughout most of this section, we shall assume that S is nonempty and contains all infinite places. Let $\mathbb{A}(S)$ and $\mathbb{A}^\times(S)$ denote the ring of S-adeles and group of S-ideles of k, respectively. Once again, this standard terminology and notation is drawn from Sect. 1 of [18] and Sect. 6 of [2]. For the benefit of the reader, we state here the definitions of $\mathbb{A}(S)$ and $\mathbb{A}^\times(S)$ as direct products.

$$\mathbb{A}(S)=\{\prod_{v\in S} k_v\}\times\{\prod_{v\notin S} o_v\}; \qquad \mathbb{A}^\times(S)=\{\prod_{v\in S} k_v^\times\}\times\{\prod_{v\notin S} o_v^\times\}.$$

Define $o_S=k\cap\mathbb{A}(S)$ and $o_S^\times=k^\times\cap\mathbb{A}^\times(S)$ to be the ring of S-integers and the group of S-units in k, respectively. Let o'_S be the set of nonzero S-integers. Let $k_S=\prod_{v\in S} k_v$ and $k_S^\times=\prod_{v\in S} k_v^\times$. For any Schwartz-Bruhat function Φ of product form, let Φ_S be the restricted product $\bigotimes_{v\in S}\Phi_v$. Similarly, for any $\omega\in\Omega$, let $\omega_S(x)=\prod_{v\in S}\omega_v(x_v)$ for all $x\in k_S^\times$. We shall continue to suppose that ω_v is unramified and $\Phi_v=\Phi_{0,v}$ for all $v\notin S$. For convenience, we shall also assume S is chosen so large that the S-ideal class number of k is 1. That is, we shall assume that $\mathbb{A}^\times=\mathbb{A}^\times(S)k^\times$. This may always be achieved by including finitely many more places in S. Note that $\mathbb{A}^\times(S)$ is an open subgroup of $\mathbb{A}^\times$. Thus, we may rewrite the zeta function as follows:

$$\begin{aligned}\zeta^{(n)}(\omega,\Phi)&=\int_{\mathbb{A}^\times(S)k^\times/k^\times}\omega(t)\sum_{x\in k^\times}\Phi(t^n x)\,|d^\times t|_{\mathbb{A}}\\ &=\int_{\mathbb{A}^\times(S)/o_S^\times}\omega(t)\sum_{x\in k^\times}\Phi(t^n x)\,|d^\times t|_{\mathbb{A}}\\ &=\rho_k^{-1}\cdot\int_{k_S^\times/o_S^\times}\omega_S(t)\sum_{x\in o'_S}\Phi_S(t^n x)\,|d^\times t|_S.\end{aligned}$$

Here $|d^\times t|_S$ is the restricted product measure $\bigotimes_{v\in S}|d^\times t_v|_v$. The last equality follows from the assumptions $\Phi_v=\Phi_{0,v}$ and ω_v is unramified outside S. In addition, bear in mind that the measures are chosen so that $o_v^\times$ has measure 1.

To this last formula, we may apply the same orbital decomposition argument underlying the proof of Theorem 1.3. We obtain

$$\begin{aligned}\zeta^{(n)}(\omega,\Phi)&=\frac{1}{\rho_k w_n}\sum_{x\in o_S'/(o_S^\times)^n}\int_{k_S^\times}\omega_S(t)\,\Phi_S(t^n x)\,|d^\times t|_S\\&=\frac{1}{\rho_k w_n}\sum_{\alpha\in\mathscr{A}_S}\Big\{\sum_{x\in o_\alpha/(o_S^\times)^n}\omega_S^{-1}(u_x)\Big\}\,\zeta_\alpha^{(n)}(\omega,\Phi),\end{aligned}$$

where $o_\alpha=o_S\cap\alpha$, considering α as a subset of $\mathbb{A}^\times$. Comparison of this equation with (3.2) proves

$$L_{k,S}(\omega^n)\,\xi_{\alpha,S}^{(n)}(\omega)=\sum_{x\in o_\alpha/o_S^n}w_S^{-n}(u_x),$$

once again substituting ω^n for ω. In light of (3.1), we have

$$L_{k,S}(\omega^n)\,\xi_{\alpha,S}^{(n)}(\omega)=\omega(\alpha)\sum_{x\in o_\alpha/o_S^n}\omega_S^{-1}(x). \tag{4.1}$$

Here, $\omega(\alpha)$ is to be calculated by realizing α as an idele which is 1 at all places outside S.

Specializing (4.1) to the case $n=1$, we obtain simply

$$L_{k,S}(\omega)=\sum_{x\in o_S'/o_S^\times}\omega_S^{-1}(x).$$

This suggests using character sums to identify the right side of (4.1) in terms of Hecke L-series. Any character χ of the group $\mathscr{A}_S$ corresponds to a character on $k_S^\times=\prod_{v\in S}k_v^\times$ which is trivial on all n-th powers. This may be extended to a character in Ω unramified outside S if and only if χ is trivial on the group $o_S^\times$ of S-units in $k^\times$. Summing over $\mathscr{A}_S$, we obtain

$$\begin{aligned}\sum_{\alpha\in\mathscr{A}_S}\bar\chi(\alpha)\sum_{x\in o_\alpha/o_S^n}\omega_S^{-1}(x)&=\sum_{x\in o_S'/o_S^n}\bar\chi(x)\,\omega_S^{-1}(x)\\&=\sum_{x\in o_S'/o_S^\times}\ \sum_{y\in o_S^\times/o_S^n}\bar\chi(xy)\,\omega_S^{-1}(xy),\end{aligned}$$

after grouping into cosets modulo $o_S^\times$. Since ω is unramified outside S, ω_S is trivial on S-units. Thus, the preceding sum may be written as

$$\sum_{x\in o_S'/o_S^\times}\Big\{\sum_{y\in o_S^\times/o_S^n}\bar\chi(y)\Big\}\,\bar\chi(x)\,\omega_S^{-1}(x).$$

The character sum inside the braces vanishes unless χ is trivial on $o_S^\times$. In the latter case, $\chi\in\Omega$, and the above sum simplifies to

$$[o_S^\times : o_S^n]\, L_{k,S}(\chi\,\omega).$$

In addition, Dirichlet's unit theorem implies that

$$[o_S^\times : o_S^n] = w_n\, n^{|S|-1}.$$

At last, we have the identities of our series $\xi^{(n)}_{\alpha,S}$.

Proposition 4.1. *If the character χ of $\mathscr{A}_S$ is trivial on the image of the group $o_S^\times$ of S-units in k, then*

$$\sum_{\alpha\in\mathscr{A}_S} \omega^{-1}\,\bar{\chi}(\alpha)\,\xi^{(n)}_{\alpha,S}(\omega) = w_n\, n^{|S|-1}\,\frac{L_{k,S}(\omega\,\chi)}{L_{k,S}(\omega^n)}.$$

If χ is nontrivial on $o_S^\times$, then the sum on the left side above vanishes.

Orthogonality of characters enables us to write the series $\xi^{(n)}_{\alpha,S}$ in terms of Hecke L-series. Let $w_{n,v}$ denote the number of n-th roots of unity contained in k_v. Define $|n|_S=\prod_{v\in S}|n|_v$ and $w_{n,S}=\prod_{v\in S} w_{n,v}$. The order of the group $\mathscr{A}_S$ is then

$$|\mathscr{A}_S| = \frac{n^{|S|}\, w_{n,S}}{|n|_S}.$$

Let $C_n(S)$ denote the subgroup of characters of Ω which are unramified outside S and such that $\chi^n=1$.

Proposition 4.2. *We have*

$$\xi^{(n)}_{\alpha,S}(\omega) = \left\{\frac{w_n\,|n|_S}{n\, w_{n,S}}\right\} \frac{\sum_{\chi\in C_n(S)} \omega\,\chi(\alpha)\, L_{k,S}(\omega\,\chi)}{L_{k,S}(\omega^n)}.$$

Both these theorems assume that ω is unramified outside S and that the S-ideal class number of k is 1. Inserting this formula back into (3.2) leads us to a formula for $\zeta^{(n)}$.

$$\begin{aligned}\zeta^{(n)}(\omega^n, \Phi) &= \frac{1}{n\rho_k}\left\{\frac{|n|_S}{w_{n,S}}\right\} \sum_{\alpha\in\mathscr{A}_S}\sum_{\chi\in C_n(S)} \omega\,\chi(\alpha)\, L_{k,S}(\omega\,\chi)\,\zeta^{(n)}_{\alpha,S}(\omega^n, \Phi)\\ &= \frac{1}{n\rho_k}\left\{\frac{|n|_S}{w_{n,S}}\right\} \sum_{\chi\in C_n(S)} L_{k,S}(\omega\,\chi) \sum_{\alpha\in\mathscr{A}_S} \omega\,\chi(\alpha)\,\zeta^{(n)}_{\alpha,S}(\omega^n, \Phi).\end{aligned}$$

The last sum may be evaluated in terms of the local Iwasawa-Tate zeta function according to Eq. (2.5). Then, using (3.2) for the case $n=1$, we establish the following.

Proposition 4.3. *Given $\omega \in \Omega$ and $\Phi \in \mathscr{S}(\mathbb{A})$, let S be large enough so that the S-ideal class number of k is 1, ω_v is unramified and $\Phi_v = \Phi_{0,v}$ for $v \notin S$. Then*

$$\zeta^{(n)}(\omega^n, \Phi) = \frac{1}{n} \sum_{\chi \in C_n(S)} \zeta(\omega\chi, \Phi).$$

For any global field, S may be enlarged to the point that the S-class number is 1, as has already been mentioned. Thus, Propositions 4.2 and 4.3 may be easily generalized to any finite set S including all infinite places and all finite places v where ω_v is ramified or where Φ_v is not $\Phi_{0,v}$ (this last inclusion being only necessary for the situation of Proposition 4.3). Indeed, let S be such a set, and let T be any set of places containing S and for which the T-class number of k is 1. Given $\beta \in \mathscr{A}_T$, let $\beta|_S$ denote the natural restriction of β to an element of $\mathscr{A}_S$. Then

$$\xi^{(n)}_{\alpha,S}(\omega) = \sum_{\beta|_S = \alpha} \xi^{(n)}_{\beta,T}(\omega)$$

the sum ranging over all $\beta \in \mathscr{A}_T$ the restriction of which to $\mathscr{A}_S$ is α. Applying Proposition 4.2 to the inner Dirichlet series, we obtain

$$\xi^{(n)}_{\alpha,S}(\omega) = \left\{\frac{w_n |n|_T}{n\, w_{n,T}}\right\} \frac{1}{L_{k,T}(\omega^n)} \sum_{\chi \in C_n(T)} \left\{ \sum_{\beta|_S = \alpha} \omega\chi(\beta) \right\} L_{k,T}(\omega\chi). \tag{4.2}$$

The sum inside the braces may be evaluated as

$$\sum_{\beta|_S = \alpha} \omega\chi(\beta) = \omega\chi(\alpha) \prod_{v \in T-S} \left\{ \sum_{\beta_v \in \mathscr{A}_v} \omega_v \chi_v(\beta_v) \right\}.$$

Assuming that ω is unramified outside S, the sum on the right side of the preceding equation is

$$\left(\sum_{\beta_v \in o_v^\times / o_v^n} \chi_v(\beta_v) \right) \left(1 + \omega_v \chi_v(\pi_v) + \dots + \omega_v \chi_v(\pi_v^{n-1})\right)$$
$$= \begin{cases} \dfrac{w_{n,v}}{|n|_v} \dfrac{(1-\omega_v^n(\pi_v))}{(1-\omega_v \chi_v(\pi_v))}, & \text{if } \chi_v(o_v^\times) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Inserting this result into (4.2) proves Proposition 4.2 for the set S. Repeating the argument prior to Proposition 4.3, we have at last

Theorem 4.4. *Given $\omega \in \Omega$ and $\Phi \in \mathscr{S}(\mathbb{A})$, let S be any finite set of places of k containing all infinite places and all places v at which ω_v is ramified or $\Phi_v \neq \Phi_{0,v}$. Then*

$$\zeta^{(n)}(\omega^n, \Phi) = \frac{1}{n} \sum_{\chi \in C_n(S)} \zeta(\omega\chi, \Phi).$$

Section 5. Generating Series of Quadratic Discriminants

Equality of generating series is a powerful manner in which to describe relations between arithmetic objects. The series $\xi^{(n)}_{\alpha,S}$ count cosets in the quotient $k^\times/k^n$, which correspond, as has already been remarked, to extensions of k of the form $k(\sqrt[n]{x})$. Thus, the equality furnished by Proposition 4.2 may be used to study the distribution of these extensions. We shall give a brief summary of this method in this section in the case $n=2$, where the results are most elegant.

We must assume that the characteristic of k is not 2. In that case, there is a one-to-one correspondence between the cosets of $k^\times/k^2$ and the extensions of k of degree at most 2. Given $x\in k^\times$, let α_x be the idele of representatives modulo squares defined in Sect. 3. The quantity α_x is closely related to the idelic discriminant $\Delta_{\mathbb{A}}(k(\sqrt{x})/k)$ defined in [5]. We shall briefly review the definition of Fröhlich's discriminant. Let k'/k be a finite separable extension of global fields. For any place v of k, we have

$$k'\otimes_k k_v = \bigoplus_{w|v} k'_w$$

where the direct sum ranges over all places w of k' lying over v. The relative discriminant $\Delta(k'_w/k_v)$ is defined in the usual way. For infinite places, $\Delta(\mathbb{R}/\mathbb{R}) = \Delta(\mathbb{C}/\mathbb{C})=1$ and $\Delta(\mathbb{C}/\mathbb{R})=-1$. For finite places, $\Delta(k'_w/k_v)$ is defined as $\det(\theta_j^{(i)})^2$ where $\{\theta_j\}$ is an o_v-module basis of the maximal compact subring of k'_w and $\theta_j^{(i)}$ ranges over the conjugates of θ_j over k_v. The discriminant is in this case well-defined up to multiplication by the square of a unit. The v-part of the discriminant of k'/k is defined to be

$$\Delta_v(k'/k) = \prod_{w|v} \Delta(k'_w/k_v).$$

The idelic discriminant $\Delta_{\mathbb{A}}(k'/k)$ is the idele whose v-component is $\Delta_v(k'/k)$.

To describe the relation between discriminants and α_x, first consider local extensions. For the moment, let n be arbitrary again. Let K be a local field of characteristic not dividing n (see Sect. 2 for the notation used). For convenience, we make the additional assumption that K contains all the n-th roots of unity. To each $\alpha\in\mathscr{A}=K^\times/K^n$ we then associate the extension $K'_\alpha=K(\sqrt[n]{\alpha})$, which is independent of the choice of n-th root. Then K'_α is of degree m over K where m is the order of α in the group $\mathscr{A}$, i.e. the smallest integer such that $\alpha^m\in K^n$. The calculation of the relative discriminant of K'_α/K is in general a delicate matter. Straightforward computation shows that the relative discriminant is a divisor of $n^n\alpha^{n-1}$. If K is a p-field and p does not divide n, then the extension K'_α/K is at worst tamely ramified, and the discriminant is easier to describe. Let e be the smallest positive integer such that $\alpha^e\in R^\times K^n$. It follows that e is a divisor of m. Moreover, without changing the extension K'_α, we can arrange that $\alpha=\pi^{n/e}\varepsilon^{n/m}$ for some $\varepsilon\in R^\times$. Then

$$\Delta(K'_\alpha/K) = m^m \pi^{\frac{m}{e}(e-1)} \varepsilon^{(m-1)}$$

again up to multiplication by squares of units. When $n=2$, we have the following complete statement, taken from [5].

Lemma 5.1. *Suppose K is a p-field of characteristic not equal to 2. Choose $\alpha \in R^\times \cup \pi R^\times$.*

(i) *If $p \neq 2$, then $\Delta(K(\sqrt{\alpha})/K) = \alpha$.*

(ii) *If $p=2$, let e be the positive integer such that $2 \in \pi^e R^\times$. For $\alpha \in \pi R^\times$, we have $\Delta(K(\sqrt{\alpha})/K) = 4\alpha$. If $\alpha \in R^\times$, then*

$$\Delta(K(\sqrt{\alpha}/K) = \frac{4\alpha}{\pi^{2l}}$$

where l is the largest integer less than or equal to e such that $\alpha \in (1+\pi^{2l}R)R^2$. All these equalities hold modulo squares of units.

Returning to the global extensions $k' = k(\sqrt{x})$, we have as a consequence of Lemma 5.1 that $\Delta_v(k'/k) = \alpha_{x,v}$ for all finite places v of k not lying over 2. We are now prepared to use Proposition 4.2 to identify the generating series of quadratic discriminants

$$\xi(\omega) = \sum_{[k':k] \leqq 2} \omega(\Delta_{\mathbb{A}}(k'/k)).$$

Having fixed a set $\mathscr{A}_v$ of representatives of $k_v^\times/k_v^2$ for all places v, we shall suppose that each $\Delta_v(k'/k)$ belongs to $\mathscr{A}_v$. This frees us to select any quasicharacter ω whatsoever as the argument of $\xi(\omega)$.

Choose S to be a finite nonempty set of places of k, containing all the infinite places, all places v at which ω_v is ramified, and all places lying over 2. We may now reinterpret the series $\xi^{(2)}_{\alpha,S}(\omega)$ in terms of quadratic discriminants. Each $\alpha \in \mathscr{A}_S$ determines a collection of extensions k'/k, namely, all those of the form $k(\sqrt{x})$ for some $x \in \alpha$. Abusing our notation slightly, we shall write $k' \in \alpha$. The condition that $k' \in \alpha$ amounts to specifying $k' \otimes k_v$ for each $v \in S$. Then

$$\begin{aligned}\xi^{(2)}_{\alpha,S}(\omega) &= \sum_{x \in \alpha} \omega(\alpha_x)\\ &= \omega(\alpha) \sum_{k' \in \alpha} \prod_{v \notin S} \omega_v(\Delta_v(k'/k)).\end{aligned}$$

From Proposition 4.2 (extended by the remarks at the end of Sect. 4), we obtain

$$\sum_{k' \in \alpha} \omega(\Delta_{\mathbb{A}}(k'/k)) = \frac{2^{-|S|}\omega(\Delta_\alpha)}{L_{k,S}(\omega^2)} \sum_{\chi \in C_2(S)} \chi(\alpha) L_{k,S}(\omega\chi),$$

using the abbreviation

$$\omega(\Delta_\alpha) = \prod_{v \in S} \omega_v(\Delta(k_v(\sqrt{\alpha_v})/k_v)).$$

In simplifying the coefficient in Proposition 4.2, we have made use of the facts that all fields in question contain ± 1 and that $|2|_S=1$ provided S contains all infinite places and all places over 2. Summing over all $\alpha\in\mathscr{A}_S$, we see that

$$\xi(\omega)=\frac{2^{-|S|}}{L_{k,S}(w^2)}\sum_{\chi\in C_2(S)} A_S(\omega;\chi)\,L_{k,S}(\omega\chi), \tag{5.1}$$

where

$$A_S(\omega;\chi)=\sum_{\alpha\in\mathscr{A}_S}\chi(\alpha)\,\omega(\Delta_\alpha).$$

Reviewing the statement of Lemma 5.1, we see that in all cases α and $\Delta(k_v(\sqrt{\alpha})/k_v)$ differ only by multiplication by a square in $k_v^\times$. Thus, since χ is trivial on squares, we may rewrite $A_S(\omega;\chi)$ as $A_S(\omega\chi)$ by means of the notation

$$A_S(\omega)=\sum_{\alpha\in\mathscr{A}_S}\omega(\Delta_\alpha). \tag{5.2}$$

These finite sums A_S have a product decomposition

$$A_S(\omega)=\prod_{v\in S}A_v(\omega) \tag{5.3}$$

with

$$A_v(\omega)=\sum_{\alpha\in\mathscr{A}_v}\omega_v(\Delta_{\alpha_v}).$$

To make (5.1) as explicit as possible, we must evaluate these finite sums. If v is an infinite place, the possible values of this character sum are easily determined:

$$A_v(\omega)=\begin{cases}1, & \text{if } k_v=\mathbb{C},\\ 2, & \text{if } k_v=\mathbb{R} \text{ and } \omega_v(-1)=1,\\ 0, & \text{if } k_v=\mathbb{R} \text{ and } \omega_v(-1)=-1.\end{cases} \tag{5.4}$$

If v is a finite place, the evaluation of $A_v(\omega)$ depends on the choice of representatives for $o_v^\times/o_v^2$, unless we assume that ω_v is trivial on o_v^2. In the ensuing calculations, we shall labor under this assumption. Using Lemma 5.1 (i) and the principle of orthogonality of characters, we find, for finite v not lying over 2,

$$A_v(\omega)=\begin{cases}2(1+\omega_v(\pi_v)), & \text{if } \omega_v(o_v^\times)=1,\\ 0, & \text{otherwise.}\end{cases} \tag{5.5}$$

When v lies over 2, we may use the description of local quadratic discriminants given in Lemma 5.1 (ii). First of all, define the nonnegative integer e_v by $|2|_v=q_v^{-e_v}$. Introduce the subgroup Q_l of $o_v^\times/o_v^2$ consisting of all elements congruent to a square modulo π_v^{2l} for $0\leq l\leq e_v$. That is,

$$Q_l=o_v^2(1+\pi_v^{2l}o_v)/o_v^2,$$

interpreting $Q_0 = o_v^\times / o_v^2$. The cardinality of Q_l is

$$|Q_l| = 2\,q_v^{e_v - l},$$

as may be most easily shown by applying (2.2) to the characteristic function of $(1+\pi_v^{2l} o_v)$. Then, according to Lemma 5.1,

$$\Lambda_v(\omega) = \{\sum_{\varepsilon \in o_v^\times / o_v^2} \omega_v(4\pi_v \varepsilon)\} + \left\{\sum_{l=0}^{e_v} \sum_{\varepsilon \in Q_l - Q_{l+1}} \omega_v\left(\frac{4\varepsilon}{\pi_v^{2l}}\right)\right\}.$$

Here, interpret Q_{e_v+1} as the empty set. Given ω_v, let f be the largest integer such that $f \leqq e_v + 1$ and ω_v is trivial on Q_f. Then

$$\sum_{\varepsilon \in Q_l} \omega_v(\varepsilon) = \begin{cases} |Q_l|, & \text{if } l \geqq f, \\ 0, & \text{if } l < f, \end{cases}$$

by orthogonality of characters of finite abelian groups.

For simplicity, we introduce the abbreviation $\omega_v(\pi_v) = q_v^{-s_v}$ for some complex number s_v. Since $2 \in \pi_v^{e_v} o_v^\times$, we have $4 \in \pi_v^{2e_v} o_v^2$. Consequently, $\omega_v(4) = q_v^{-2e_v s_v}$, since ω_v is trivial on o_v^2. Assume first that $f = 0$.

$$\begin{aligned} \Lambda_v(\omega) &= q_v^{-2e_v s_v}\left\{q_v^{-s_v}\,2\,q_v^{e_v} + \sum_{l=0}^{e_v} q_v^{2ls_v}\,2\,q_v^{e_v - l} - \sum_{l=0}^{e_v - 1} q_v^{2ls_v}\,2\,q_v^{e_v - l - 1}\right\} \\ &= 2\,q_v^{e_v(1-2s_v)}\left\{q_v^{-s_v} + q_v^{e_v(2s_v - 1)} + (1 - q_v^{-1})\,\frac{1 - q_v^{e_v(2s_v - 1)}}{1 - q_v^{2s_v - 1}}\right\}. \end{aligned} \tag{5.6}$$

This formula does not simplify much further. However, it is worth noting that, if $s_v = 1$, then $\Lambda_v(\omega)$ simplifies to $2(1 + q_v^{-1})$.

Finally, if $f > 0$, then a similar calculation shows

$$\Lambda_v(\omega) = 2\,q_v^{e_v(1-2s_v)}(1 - q_v^{-2s_v})\,\frac{q_v^{f(2s_v-1)} - q_v^{(e_v+1)(2s_v-1)}}{1 - q_v^{2s_v - 1}}. \tag{5.7}$$

Although this formula does not simplify further, it is plain that this character sum does not vanish as a function of ω_v unless $f = e_c + 1$. That is, $\Lambda_v(\omega) \neq 0$ unless ω_v is nontrivial on Q_{e_v}.

We summarize all these calculations in the theorem below.

Theorem 5.2. *Let k be a global field of characteristic not equal to 2. Choose $\omega \in \Omega$ with $\mathrm{Re}(\omega) > 1$. Let S be any finite nonempty set of places of k containing all infinite places, all places lying over 2, and all places at which ω is unramified. The generating series of relative discriminants of quadratic extensions of k may be expressed in terms of Hecke L-functions as follows:*

$$\sum_{[k':k] \leqq 2} \omega(\Delta_{\mathbb{A}}(k'/k)) = \frac{2^{-|S|}}{L_{k,S}(\omega^2)} \sum_{\chi \in C_2(S)} \Lambda_S(\chi\omega)\,L_{k,S}(\chi\omega)$$

where the finite character sums Λ_S are defined in (5.2). *If we assume that ω_v is trivial on o_v^2 for all finite places $v \in S$, then Λ_S is explicitly evaluated in* (5.3) *through* (5.7).

This identification of the discriminant series is as precise as possible. A result of this nature can be derived from the conductor-discriminant formula of class field theory (see [19]). From knowledge of the analytic continuation of $L_{k,S}(\omega)$ to all of Ω and from standard Tauberian theorems, we may deduce theorems about the distribution of quadratic discriminants. Some of these theorems are stated in Theorem 4.2 of [3].

The final question to be considered in this paper concerns any possible further simplification in the formula established in Theorem 5.2. We shall suppose that $\omega = \omega_s$ is principal. Then the smallest set S we may choose consists precisely of all infinite places and all the places lying over 2. (This will be the empty set when k is a function field, assumed of characteristic not equal to 2.) Given $\chi \in C_2(S)$, $\Lambda_S(\chi\omega_s)$ is nonzero as a function of s if and only if $\chi_v(-1) = 1$ for all real places v and χ_v is trivial on $Q_{e_v} = 1 + 4o_v$ for all places $v|2$. These are precisely the characters $\chi \in \Omega$ satisfying $\chi^2 = 1$ and which are trivial on

$$U_4 = \prod_{v|\infty} k_v^\times \times \prod_{v|2} (1 + 4o_v) \times \prod_{v \nmid 2,\infty} o_v^\times.$$

These characters correspond to characters of $\mathbb{A}^\times / k^\times U_4$, which is the absolute ray class group modulo 4 of k. If this ray class group has no elements of order 2, then the only χ contributing a nonzero term to the expression for $\xi(\omega)$ in Theorem 5.2 is the trivial character. This establishes the following.

Corollary 5.3. *Assume that* $\mathrm{char}(k) \neq 2$. *The following equality holds if and only if the absolute ray class group modulo* 4 *of k has no elements of order* 2.

$$\sum_{[k':k] \leq 2} |\Delta_{\mathbb{A}}(k'/k)|_{\mathbb{A}}^s = \prod_{v \in M_k} \{\tfrac{1}{2} \sum_{[K:k_v] \leq 2} |\Delta(K/k_v)|_v^s\}.$$

In general, the discriminant series is a sum of twists of the stated Euler product by Dirichlet characters, as stated in the introduction. This is established by the same considerations preceding the corollary.

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