The Grothendieck Construction in Enriched, Internal and ∞-Category Theory

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Abstract

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The Grothendieck construction takes a prestack (or pseudofunctor) $B^{\mathsf{op}} \to \mathbf{Cat}$ and returns a cartesian fibration over B. Classically, this construction works for categories with sets of morphisms. Enriched categories have morphisms belonging to another monoidal category $\mathcal{V}$, while internal categories require the objects to also belong to $\mathcal{V}$. Many concepts from ordinary (i.e. **Set**-based) category theory generalize well to enriched and internal category theory, but fibrations and the Grothendieck construction are not one of them. This is especially true if the monoidal product on $\mathcal{V}$ is not given by the cartesian product, such as when $\mathcal{V} = \mathbf{Vect}_k$. In this thesis, we generalize prestacks to $\mathcal{V}$-enriched and $\mathcal{V}$-internal categories, where $\mathcal{V}$ is non-cartesian, and develop a Grothendieck construction for them. As an application, when $\mathcal{V} = \mathbf{sSet}$, we obtain a version of the ∞-categorical Grothendieck construction and show that it is equivalent to existing ∞-categorical constructions.

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DEDICATION

to Ivana, who has enriched my life

INTRODUCTION

The starting point of this thesis is the following result due to Grothendieck:

Theorem 0.0.1 ([Gro61])**.** *Fix a category B. There is a 2-equivalence*

$$\int : [B, \mathbf{Cat}] \xrightarrow{\cong} \mathbf{coCart}^{\mathbf{spl}}_{/B}$$

sending functors $B \to \mathbf{Cat}$ to split cocartesian fibrations over B, and whose inverse is given by taking fibers.

The category B will be referred to as the base of the fibration. Just as categories are many-object versions of monoids, the Grothendieck construction is the many-object version of the following classical result in group theory:

Lemma 0.0.2 (Splitting Lemma for Groups)**.** *Let G be a group. There is a bijective correspondence between group homomorphisms $\varphi \colon G \to \mathsf{Aut}(N)$ and split surjections $\pi \colon E \twoheadrightarrow G$, sending φ to $E = N \rtimes_{\varphi} G$ in one direction and π to $N = \ker \pi$ in the other.*

In this sense, the Grothendieck construction is a generalization of the semi-direct product from groups to categories. The main theme of this thesis is that the Grothendieck construction further generalizes to enriched and internal categories.

Enriched and internal category theory are based on the observation that the *sets* of objects and/or morphisms in a category may be replaced by objects in a monoidal category $\mathcal{V}$ other than $\mathbf{Set}$. For enriched categories, only the morphisms are required to live in $\mathcal{V}$; for internal categories, both the objects and morphisms belong to $\mathcal{V}$. Naturally, the properties of the monoidal category $\mathcal{V}$ play a big role in determining which aspects of ordinary category theory may be incorporated in the enriched or internal setting.

The two archetypical examples of monoidal categories are $\mathbf{Set}$ and $\mathbf{Vect}_k$. A key difference between these monoidal categories is that $\mathbf{Set}$ with the cartesian product $\times$ is *cartesian* *monoidal*[1], while $\mathbf{Vect}_k$ with the tensor product $\otimes_k$ is not. It turns out that cartesianness is essential for many concepts of ordinary category theory to generalize well to the enriched or internal setting. For instance, in the non-cartesian setting, the Grothendieck construction need not even give rise to a functor! It does however give rise to a coaction by a comonoidal category.

With this in mind, the secondary theme of this thesis is that comonoids and comodules mediate the passage from the cartesian to the non-cartesian world, allowing the transfer of concepts and results from ordinary category theory to non-cartesian enriched or internal categories.

Related work and contributions of this thesis

The use of comonoids and comodules is not new to this thesis. Indeed, in the k-linear setting, coalgebras and their comodules are a basic object of study in representation theory and non-commutative algebra. We give a brief survey of recent developments in this area that are related to the Grothendieck construction.

The papers that are most similar in scope to this these are [CM06], [Low08] and [Tam09]. These papers draw their inspiration from the seminal paper [CM84] on smash products for group algebras. The key result of [CM84] is that semi-direct products of groups may be extended to the k-linear setting. Instead of a group G acting on another group N, we have G acting on a k-algebra A. The semi-direct product is then replaced by the *smash product* $kG\#A$, where kG is the group algebra. Thus 'smash product' may be taken to mean 'the Grothendieck construction' in the k-linear setting. Due to the non-cartesianness of $\mathbf{Vect}_k$, the resulting algebra $kG\#A$ does not have an algebra homomorphism down to kG. However, it does have a G-grading, and a second key result of [CM84] is that G-gradings are equivalent to *kG-coactions*.

A k-algebra may be treated as a k-linear category with one object. It is natural to ask if

[1] The word 'cartesian' is used here in a slightly different sense than in '(co)cartesian fibration'. The two uses are distinct, though related.

smash products may be generalized to the many-object setting. In [CM06], smash products are defined for a group G acting on a k-linear category. In [Low08], the result is extended further, with the group G being replaced by a category C. Note that this involves two levels of generalization: we may pass from one object to many objects, and we may also drop the invertibility requirements on G. The resulting smash product in [Low08] is a C-graded k-linear category. Finally in [Tam09], this C-graded category is shown to be equivalent to a kC-comodule category, in the same way that G-graded algebras are kG-comodule algebras.

Instead of restricting ourselves to the k-linear setting, in this thesis we work over a monoidal category $\mathcal{V}$ possessing certain properties. In Chapter 2, we identify conditions on $\mathcal{V}$ that allow the C-graded categories of [Low08] and the kC-comodule categories of [Tam09] to be instantiated as actual functors down to the free $\mathcal{V}$-category $C_{\mathcal{V}}$. Note that $\mathbf{Vect}_k$ does not satisfy these conditions, so that the result in this chapter parallels, rather than generalizes, those of [Low08] and [Tam09]. A true generalization will have to wait till Chapter 4.

In Chapter 3, we apply the results of the previous chapter to the specific case where $\mathcal{V} = \mathbf{sSet}$. Categories enriched over $\mathbf{sSet}$, or simplicial categories, are models of ∞-categories, and a Grothendieck construction for ∞-categories has been developed in [Lur09] in the language of quasicategories and marked simplicial sets. The key result of Chapter 3 is that our $\mathbf{sSet}$-enriched Grothendieck construction is compatible with the existing ∞-categorical Grothendieck construction. One benefit of our $\mathbf{sSet}$-enriched construction is that it allows for explicit computations. We demonstrate this by factoring the operadic nerve of a monoidal $\mathbf{sSet}$-category as defined in [Lur09], and showing that this operadic nerve commutes with taking opposites.

In Chapter 4, we return to the non-cartesian setting, and generalize the constructions of [Low08] and [Tam09] to the case where the base category C is replaced by a comonoidal $\mathcal{V}$-category. We then take this a step further, and replace this $\mathcal{V}$-category by a comonoidal *internal* category. With the exception of the definition of a non-cartesian internal category from [Agu97], all the results of this chapter are new. We note that internal presheaves and internal discrete fibrations have been explored in [BJ01, §7.1] and [Joh03, §B2.5], but both of these treatments deal with the *discrete* (i.e. presheaves rather than prestacks) and *cartesian* case.

In addition to the above results, this thesis includes an introduction to the Grothendieck construction and fibrations in Chapter 1. With the exception of §1.7, where we define the notion of a *fibration across a 2-functor*, none of this material is new.

List of publications

This thesis is to be read in conjuction with the the first, third and fifth papers in the following list (summarized in Chapters 2, 3 and 4, respectively). None of the material of the remaining papers is covered in this thesis.

Published

1. [BW19] Jonathan Beardsley and Liang Ze Wong. *The enriched Grothendieck construction*. Advances in Mathematics, 344:234261, 2019.
2. [CSW17] Alex Chirvasitu, S Paul Smith and Liang Ze Wong. *Noncommutative geometry of homogenized quantum* $\mathfrak{sl}(2, \mathbb{C})$, Pacific Journal of Mathematics 292 (2017), no. 2, 305354.

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3. [BW18] Jonathan Beardsley and Liang Ze Wong. *The operadic nerve, relative nerve, and the Grothendieck construction*. arXiv:1808.08020, 2018. (to appear in: Theory and Applications of Categories, vol. 34, 2019.)
4. [KLW19] Krzysztof Kapulkin, Zachery Lindsey and Liang Ze Wong. *A co-reflection of cubical sets into simplicial sets with applications to model structures*, 2019. (to appear in: New York Journal of Mathematics)

Preprints

5. [Won19] Liang Ze Wong. *Smash products for Non-cartesian Internal Prestacks*, 2019. (included in full in the Appendix)

In preparation

6. Simon Cho, Cory Knapp, Clive Newstead and Liang Ze Wong. *Weak equivalences between categories of models of type theory.*

Chapter 1

THE GROTHENDIECK CONSTRUCTION AND FIBRATIONS

This thesis is about a certain class of functors, called *fibrations*, and their relation to a certain construction, called the *Grothendieck construction*. In this chapter, we give a gentle introduction to these two concepts.

1.1 Semi-direct products

We start with a very special case of the Grothendieck construction that should be familiar to anyone who has taken a course in group theory. Fix a group G, and suppose that G acts on another group N, via a group homomorphism $\varphi\colon G \to \mathsf{Aut}(N)$. For $g \in G$, let φ_g be the automorphism $\varphi(g)\colon N \to N$.

Then we can form the semi-direct product $N \rtimes_\varphi G$ whose underlying set is $N \times G$, whose identity is (e_N, e_G), but whose multiplication is given by (for all $m, n \in N$ and $f, g \in G$)

$$(n, g) \cdot (m, f) := \big(n\varphi_g(m), gf\big). \tag{1.1}$$

We simply multiply g and f together, but we 'twist' m by the action of φ_g before multiplying it with n. The semi-direct product $N \rtimes_\varphi G$ fits into the short exact sequence of groups

$$N \xhookrightarrow{\iota} N \rtimes_\varphi G \xrightarrow{\pi} G \tag{1.2}$$

where ι includes n as (n, e_G) and π projects (n, g) down to g. This is in fact a *split* short exact sequence: π has a section $\sigma\colon G \to N \rtimes_\varphi G$ which sends g to (e_N, g).

Further, every split short exact sequence of groups arises in this manner, and there is a bijective correspondence between split short exact sequences $N \hookrightarrow E \twoheadrightarrow G$ and homomorphisms $G \to \mathsf{Aut}(N)$. Since a short exact sequence $N \hookrightarrow E \twoheadrightarrow G$ is equivalently a surjection $\pi\colon E \twoheadrightarrow G$ with $N = \ker \pi$, we thus have:

Lemma 1.1.1 (Splitting Lemma for Groups)**.** *There is a bijective correspondence between split surjections $\pi\colon E \twoheadrightarrow G$ and group homomorphisms $\varphi\colon G \to \mathsf{Aut}(N)$.*

The premise of this thesis is the following observation: G and E **need not be groups**. They can be monoids, categories, ∞-categories and even k-algebras or algebroids! The rest of this thesis is dedicated to making this claim precise.

1.2 The Grothendieck construction

Let C be a category, and $\varphi\colon C \to \mathbf{Cat}$ a functor. For each $c \in C$, let N_c denote the category $\varphi(c)$, and for each $c \xrightarrow{g} d$ in C, let

$$\varphi_g\colon N_c \to N_d$$

denote the functor $\varphi(g)$. We may define a new category $N_\bullet \rtimes_\varphi C$ as follows:

- The objects of $N_\bullet \rtimes_\varphi C$ are pairs (x, c) where $c \in C$ and $x \in N_c$
- The set of arrows from (x, c) to (y, d) is the set of pairs (n, g) where

$$\varphi_g(x) \xrightarrow{n} y \ \in N_d, \quad \text{and} \quad c \xrightarrow{g} d \ \in C$$

- The identity on (x, c) is the pair $(1_x, 1_c)$
- Given composable arrows $(w, b) \xrightarrow{(m,f)} (x, c) \xrightarrow{(n,g)} (y, d)$, their composite is

$$(n, g) \circ (m, f) := \big(n\varphi_g(m), gf\big). \tag{1.3}$$

Further, this has a functor $p\colon N_\bullet \rtimes_\varphi C \to C$ which projects (x, c) to c and (n, g) to g.

Unpacking the definition, the objects and arrows of $N_\bullet \rtimes_\varphi C$ are pairs of objects and arrows from the various N_c's and C. Given composable arrows (m, f) and (n, g), we may simply compose their C-components f and g, but since m lives in N_c while n lives in N_d, m and n cannot be composed. We need to use the action of $\varphi_g\colon N_c \to N_d$ to transport m to N_d, where it can then be composed with n.

Note the similarity between the formula for multiplication in $N \rtimes_\varphi G$ (1.1) and for composition in $N_\bullet \rtimes_\varphi C$ (1.3). The difference is that while G acts on a single group N, the category C acts on a collection of categories N_c, one for each of its objects.

Example 1.2.1 (Semi-direct products)**.** Semi-direct products are a special case of the Grothendieck construction where C is a category with a single object $*$ and $C(*,*) = G$, and N_* is the category with a single object $*$ and $N_*(*,*) = N$.

Example 1.2.2 (The codomain functor)**.** For any category C, we have a functor $C \to \mathbf{Cat}$ sending each c to the *slice* or *comma* category $C_{/c}$ of arrows into c.

Applying the Grothendieck construction to $C_{/\bullet}$ gives the *arrow category*

$$C_{/\bullet} \rtimes C = \mathbf{Arr}(C),$$

and the resulting cocartesian fibration is the *codomain functor* $\mathsf{cod}\colon \mathbf{Arr}(C) \to C$ sending $f\colon x \to y$ to y.

The notation $N_\bullet \rtimes_\varphi C$ is non-standard, and has been used only to highlight its similarity with semi-direct products. From now on, we will use $\int \varphi$ to denote $N_\bullet \rtimes_\varphi C$. The construction that takes $\varphi\colon C \to \mathbf{Cat}$ and produces $\int \varphi$ is known as the **Grothendieck construction**, and in fact extends to a functor

$$\int : [C, \mathbf{Cat}] \to \mathbf{Cat}_{/C},$$

where $[C, \mathbf{Cat}]$ is the category of functors $C \to \mathbf{Cat}$ and natural transformations between them, and $\mathbf{Cat}_{/C}$ is the category of functors $X \to C$ and commuting triangles over C.

The Grothendieck construction $\int$ is faithful (but not full!), hence restricts to an equivalence between $[C, \mathbf{Cat}]$ and the image of $\int$ (on objects and morphisms) in $\mathbf{Cat}_{/C}$. A **split cocartesian fibration** is then precisely a functor $p\colon E \to C$ in the image of $\int$, thanks to the following analogue of Lemma 1.1.1:

Theorem 1.2.3 (Split Grothendieck Correspondence)**.** *The Grothendieck construction*

$$\int : [C, \mathbf{Cat}] \to \mathbf{coCart}^{\mathbf{spl}}_{/C}$$

is an equivalence of categories.

Given a split cocartesian fibration $p\colon E \to C$, the functor $F\colon C \to \mathbf{Cat}$ that gives rise to p sends $c \in C$ to the fiber E_c over c. Thus, split cocartesian fibrations are functors over C whose fibers vary functorially.

In order to give more examples of the Grothendieck construction, we introduce its dual version: Let C be a category, and $\varphi\colon C^{\mathsf{op}} \to \mathbf{Cat}$ a functor. For each $c \in C$, let N_c denote the category $\varphi(c)$, and for each $c \xrightarrow{g} d$ in C, let

$$\varphi_g\colon N_d \to N_c$$

denote the functor $\varphi(g)$. We may define a new category $N_\bullet \rtimes_\varphi C$ as follows:

- The objects of $N_\bullet \rtimes_\varphi C$ are pairs (x, c) where $c \in C$ and $x \in N_c$
- The set of arrows from (x, c) to (y, d) is the set of pairs (n, g) where

$$x \xrightarrow{n} \varphi_g(y) \ \in N_c, \quad \text{and} \quad c \xrightarrow{g} d \ \in C$$

- The identity on (x, c) is the pair $(1_x, 1_c)$
- Given composable arrows $(w, b) \xrightarrow{(m,f)} (x, c) \xrightarrow{(n,g)} (y, d)$, their composite is

$$(n, g) \circ (m, f) := \big(\varphi_g(n)\, m, gf\big).$$

Further, this has a functor $p\colon N_\bullet \rtimes_\varphi C \to C$ which projects (x, c) to c and (n, g) to g.

Again, we will henceforth use $\int \varphi$ to denote $N_\bullet \rtimes_\varphi C$. The functors in the image of $\int$ are precisely the **split cartesian fibrations** over C, thanks to the dual version of Theorem 1.2.3:

Theorem 1.2.4 (Split Grothendieck Correspondence)**.** *The Grothendieck construction*

$$\int : [C^{op}, \mathbf{Cat}] \to \mathbf{Cart}^{\mathbf{spl}}_{/C}$$

is an equivalence of categories.

Split cartesian fibrations are thus functors over C whose fibers vary functorially, in a contravariant manner.

1.3 Pullbacks and fibrations

We have seen that split (co)cartesian fibrations are functors whose fibers vary functorially. We will now see the archetypical example of a fibration that is *not* split.

Let C be a category with pullbacks. For each cospan $x \xrightarrow{g} z \xleftarrow{f} y$ we use the following notation for its pullback:

$$\begin{tikzcd} f^*x \arrow[r] \arrow[d, "f^*g"'] \arrow[dr, phantom, "\lrcorner", very near start] & x \arrow[d, "g"] \\ y \arrow[r, "f"] & z \end{tikzcd}$$

The universal property of pullbacks ensures that we have a functor $\varphi_f\colon C_{/z} \to C_{/y}$ sending $g\colon x \to z$ to $f^*g\colon f^*x \to y$, so we may attempt to give $\mathsf{cod}\colon \mathbf{Arr}(C) \to C$ from Example 1.2.2 the structure of a split cartesian fibration.

Unfortunately, pullbacks are not strictly functorial: given another map $h\colon w \to y$, we only have an isomorphism $(fh)^*x \cong h^*f^*x$ instead of an equality, so we cannot obtain a functor $F\colon C^{\mathsf{op}} \to \mathbf{Cat}$ which sends $c \in C$ to $C_{/c}$. Thus, the codomain functor cod is not necessarily a split fibration. It turns out, however, that F is a *pseudo*functor, which means that it is functorial *up to isomorphism*. Further, we can carry out a version of the Grothendieck construction for pseudofunctors $F\colon C^{\mathsf{op}} \to \mathbf{Cat}$, and have non-split versions of Theorems 1.2.3 and 1.2.4:

Theorem 1.3.1 (Grothendieck Correspondence)**.** *The Grothendieck construction*

$$\int\colon [C, \mathbf{Cat}]_{\mathsf{ps}} \to \mathbf{coCart}_{/C}$$

is an equivalence of categories.

Theorem 1.3.2 (Grothendieck Correspondence)**.** *The Grothendieck construction*

$$\int\colon [C^{\mathsf{op}}, \mathbf{Cat}]_{\mathsf{ps}} \to \mathbf{Cart}_{/C}$$

is an equivalence of categories.

Thus, fibrations are functors over C whose fibers vary pseudofunctorially, in a covariant or contravariant fashion. Keeping this slogan in mind, we turn to the proper definition of a fibration in the next section, starting with the notion of a cartesian arrow.

1.4 Cartesian fibrations and functors

We now give a proper definition of a fibration, starting with the notion of a cartesian arrow.

Cartesian arrows generalize the notion of pullbacks in a category C. Recall that the square on the left is a pullback in C iff for all $x \in C$, the square on the right is a pullback in **Set**:

$$\begin{tikzcd} d \arrow[r, "\chi_f"] \arrow[d, "\chi_p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & e \arrow[d, "p"] \\ b \arrow[r, "f"] & c \end{tikzcd} \iff \begin{tikzcd} C(x,d) \arrow[r, "\chi_f \circ -"] \arrow[d, "\chi_p \circ -"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C(x,e) \arrow[d, "p \circ -"] \\ C(x,b) \arrow[r, "f \circ -"] & C(x,c) \end{tikzcd}$$

This means that for all x and g, h such that $ph = fg$, we have a unique (g, h) making everything commute:

$$\begin{tikzcd} x \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "g"'] \arrow[dr, dashed, "\exists!\,(g,h)"] & & \\ & d \arrow[r, "\chi_f"] \arrow[d, "\chi_p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & e \arrow[d, "p"] \\ & b \arrow[r, "f"'] & c \end{tikzcd} \tag{1.4}$$

While it is common to refer to the object d as the pullback, it is really the whole square that has this universal property. With that in mind, we turn to the definition of a p-cartesian arrow.

Definition 1.4.1 (Cartesian arrow)**.** Let $p\colon E \to B$ be a functor. An arrow $\chi\colon d \to e$ in E

is p-**cartesian** if the following square is a pullback in **Set** for all $x \in E$:

$$\begin{tikzcd} E(x,d) \arrow[r, "\chi\circ -"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E(x,e) \arrow[d, "p"] \\ B(px,pd) \arrow[r, "p\chi\circ -"] & B(px,pe) \end{tikzcd}$$

Letting $b = pd, c = pe$ and $f = p\chi$, this says that for all x and g, h in E such that $ph = fg$, there exists a unique (g,h) making everything commute (where the dotted arrows indicate the action of p):

$$\begin{tikzcd} x \arrow[rr, bend left, "h"] \arrow[r, dashed, "\exists!\,(g,h)"'] \arrow[d, dotted] & d \arrow[r, "\chi"'] \arrow[d, dotted] & e \arrow[d, dotted] \\ px \arrow[r, "g"'] \arrow[rr, bend left, "ph"] & b \arrow[r, "f"'] & c \end{tikzcd} \tag{1.5}$$

Definition 1.4.2 (Fibration)**.** A functor $p\colon E \to B$ is a **(cartesian) fibration** if for every $e \in E$ and $f\colon b \to pe$ in B, there exists a p-cartesian lift χ_f of f with codomain e.

$$\begin{tikzcd} f^*e \arrow[r, "\chi_f"] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & e \arrow[d, dotted] \\ b \arrow[r, "f"] & pe \end{tikzcd}$$

The domain of χ_f will often be denoted f^*e as above, and we sometimes use the pullback symbol $\lrcorner$ to indicate the cartesianness of χ_f.

Definition 1.4.3 (Cartesian functor)**.** Let $p\colon D \to A$ and $q\colon E \to B$ be fibrations. A **cartesian functor** from p to q is a commuting square $F = (F^\top, F^\perp)$

$$\begin{tikzcd} D \arrow[r, "F^\top"] \arrow[d, "p"'] & E \arrow[d, "q"] \\ A \arrow[r, "F^\perp"] & B \end{tikzcd}$$

such that $F^\top$ sends p-cartesian arrows to q-cartesian arrows.

Definition 1.4.4 (Category of fibrations)**.** The 2-category $\mathbf{Cart}(\mathbf{Cat})$ is the subcategory of $\mathbf{Arr}(\mathbf{Cat})$ consisting of cartesian fibrations and cartesian functors between them, and 2-cells inherited from $\mathbf{Arr}(\mathbf{Cat})$.

For $B \in \mathbf{Cat}$, the 2-category $\mathbf{Cart}(\mathbf{Cat})_{/B} \subset \mathbf{Cat}_{/B}$ is the category of cartesian fibrations over B and cartesian functors where $F^{\perp} = 1_B$, and 2-cells inherited from $\mathbf{Cat}_{/B}$.

1.5 Further properties

In this section, we review a few more properties of cartesian arrows and fibrations. We start by investigating the properties of cartesian arrows in a fibration.

Proposition 1.5.1. *Let $p\colon E \to B$ be a fibration. Then:*

i) if $\chi = \omega\psi$ in E and ψ and ω are p-cartesian, then so is χ (i.e. the composite of p-cartesian arrows is p-cartesian);

$$\begin{tikzcd} \cdot \arrow[rr, "\chi"] \arrow[dr, "\psi"'] & & \cdot \\ & \cdot \arrow[ur, "\omega"'] & \end{tikzcd}$$

ii) if $\chi = \omega\psi$ in E and χ and ω are p-cartesian, then so is ψ;

iii) if $\chi = \omega\psi$ in E, χ and ω are p-cartesian, and $p\psi$ is an isomorphism in B, then ψ is an isomorphism;

iv) isomorphisms in E are p-cartesian;

*v) every arrow in E factors as $\cdot \xrightarrow{v} \cdot \xrightarrow{\chi} \cdot$ where χ is p-cartesian and v is **vertical** (i.e. pv is an identity);*

Proof. (i) is well-known, and can be found in [Bor94, Lemma 8.1.4], for instance. (i-iii) are also found in [RV17a, Lemma 3.2.10], with proofs in [RV17b, 5.1.8, 5.1.9, 4.1.3], respectively. (iv-v) are well-known, but will be proved here for lack of a citable reference.

(iv) Keeping the notation of (1.5), if χ is an isomorphism in E, then so is $f = p\chi$ in B. Given g and h, let $(g, h) := \chi^{-1}h$. Since $ph = fg$, we have $p(g, h) = p(\chi^{-1}h) = f^{-1}ph = g$, so that (p, h) is indeed a lift of g. To show uniqueness, if $(g, h)'$ is another filler such that $\chi(g, h)' = h$, then $(g, h)' = \chi^{-1}h = (g, h)$.

(v) Let h be an arrow in E, and let $f = ph$ in (1.5), so that g is an identity. Then $h = \chi v$ where χ is a p-cartesian lift of f and $v = (g, h)$. □

Lemma 1.5.2. *Every isomorphism of categories is a fibration.*

Proof. If $p\colon E \to B$ is an isomorphism of categories, we may take $f^*e := p^{-1}b$ and $\chi_f := p^{-1}f$ (in the notation of Definition 1.4.2). One can then use fullness and faithfulness of p to check that χ_f is indeed p-cartesian. □

Remark 1.5.3. *As the previous Lemma hints, fibrations are 'evil' in the sense that they respect isomorphisms of categories, not equivalences. To rectify this, one would have to work with* Street fibrations [Str80], *where f^*e in Definition 1.4.2 is only required to lie over an object* isomorphic *(rather than equal) to b.*

Proposition 1.5.4 ([RV17b, 5.2.1])**.** *Fibrations are closed under composition and pullback. Moreover, if we have a pullback square*

$$\begin{tikzcd} D \arrow[r, "G^\top"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "q"] \\ A \arrow[r, "G^\perp"] & B \end{tikzcd}$$

where q (and hence p) is a fibration, then $(G^\top, G^\perp)$ is a cartesian functor and an arrow χ in D is p-cartesian iff $G^\top\chi$ is q-cartesian.

The previous proposition says that pullbacks of fibrations are fibrations. We can similarly show that pullbacks of cartesian functors are cartesian functors.

Corollary 1.5.5. *Suppose we have the commuting diagram on the left, where p and p' are fibrations. Pulling back p and p' along $G^\perp$ and $G'^\perp$, resp., we obtain fibrations q and q', and* *an induced map* $H^\top$*.*

$$
\begin{tikzcd}
 & & E \arrow[dr, "F^\top"] \arrow[dd, "p"'] & \\
 & & & E' \arrow[dd, "p'"] \\
A \arrow[rr, "G^\perp"] \arrow[dr, "H^\perp"'] & & B \arrow[dr, "F^\perp"] & \\
 & A' \arrow[rr, "G'^\perp"'] & & B'
\end{tikzcd}
\qquad
\begin{tikzcd}
D \arrow[rr, "G^\top"] \arrow[dr, dashed, "H^\top"] \arrow[dd, "q"'] & & E \arrow[dr, "F^\top"] \arrow[dd] & \\
 & D' \arrow[rr, "G'^\top"] \arrow[dd, "q'"'] \arrow[dr, phantom, "\lrcorner", very near start] & & E' \arrow[dd, "p'"] \\
A \arrow[rr] \arrow[dr, "H^\perp"'] & & B \arrow[dr] & \\
 & A' \arrow[rr, "G'^\perp"'] & & B'
\end{tikzcd}
$$

If $F = (F^\top, F^\perp)$ *is a cartesian functor, then so is* $H = (H^\top, H^\perp)$*.*

Proof. Suppose χ is a q-cartesian arrow in D. We need to show that $H^\top\chi$ is q'-cartesian. By the previous proposition, this is so iff $G'^\top H^\top\chi$ is p'-cartesian. But $G'^\top H^\top\chi = F^\top G^\top\chi$, which is p'-cartesian because both $G = (G^\top, G^\perp)$ and F are cartesian functors. □

1.6 Fibrations in a 2-category

In fact, fibrations may be defined in any 2-category. Just as we may use pullbacks in **Set** to define pullbacks representably in any category, we may use fibrations in **Cat** to define fibrations in a 2-category. The fibrations we have seen are fibrations in the 2-category **Cat**.

The contents of this section may be found in [Rie10, §3].

Definition 1.6.1 (Fibration)**.** Let $\mathcal{C}$ be a 2-category. A 1-cell $p\colon E \to B$ in $\mathcal{C}$ is a **fibration** if for every $X \in \mathcal{C}$, the functor $p \circ -\colon \mathcal{C}(X, E) \to \mathcal{C}(X, B)$ is a fibration in **Cat**, and for every $f\colon X \to Y$, the square

$$
\begin{tikzcd}
\mathcal{C}(Y, E) \arrow[r, "- \circ f"] \arrow[d, "p \circ -"'] & \mathcal{C}(X, E) \arrow[d, "p \circ -"] \\
\mathcal{C}(Y, B) \arrow[r, "- \circ f"] & \mathcal{C}(X, B)
\end{tikzcd}
$$

is a cartesian functor of fibrations.

Definition 1.6.2 (Cartesian functor)**.** Let $p\colon D \to A$ and $q\colon E \to B$ be fibrations in $\mathcal{C}$. A **cartesian functor** from p to q is a pair $F = (F^\top, F^\perp)$ such that for all $X \in \mathcal{C}$, the induced square in **Cat** is a cartesian functor.

$$
\begin{tikzcd}
D \arrow[r, "F^{\top}"] \arrow[d, "p"'] & E \arrow[d, "q"] \\
A \arrow[r, "F^{\perp}"] & B
\end{tikzcd}
\xrightarrow{\mathcal{C}(X,-)}
\begin{tikzcd}
\mathcal{C}(X,D) \arrow[r, "F^{\top}\circ -"] \arrow[d, "p\circ -"'] & \mathcal{C}(X,E) \arrow[d, "q\circ -"] \\
\mathcal{C}(X,A) \arrow[r, "F^{\perp}\circ -"] & \mathcal{C}(X,B)
\end{tikzcd}
$$

While these definitions are succinct, it will be helpful to characterize fibrations and cartesian functors in a 2-category more explicitly. This will be particularly useful when the 2-category in question admits comma objects.

Definition 1.6.3 ([Str74])**.** Given 1-cells $A \xrightarrow{F} C \xleftarrow{G} B$ in a 2-category $\mathcal{K}$, the *comma category* is a 0-cell $F{\downarrow}G$ equipped with 1-cells $B \xleftarrow{H} F{\downarrow}G \xrightarrow{K} A$ and a 2-cell $\varphi\colon FK \Rightarrow GH$

$$
\begin{tikzcd}
F{\downarrow}G \arrow[r, "K"] \arrow[d, "H"'] \arrow[dr, phantom, "\swArrow\varphi"] & A \arrow[d, "F"] \\
B \arrow[r, "G"'] & \mathcal{C}
\end{tikzcd}
$$

that has the following universal property:

1. Given any other diagram

$$
\begin{tikzcd}
D \arrow[r, "K'"] \arrow[d, "H'"'] \arrow[dr, phantom, "\swArrow\psi"] & A \arrow[d, "F"] \\
B \arrow[r, "G"'] & \mathcal{C}
\end{tikzcd}
$$

   there exists a *unique* $J\colon D \to F{\downarrow}G$ such that: $KJ = K'$, $HJ = H'$ and

$$
\begin{tikzcd}
D \arrow[dr, dashed, "J"] \arrow[drr, bend left, "K'"] \arrow[ddr, bend right, "H'"'] & & \\
& F{\downarrow}G \arrow[r, "K"] \arrow[d, "H"'] \arrow[dr, phantom, "\swArrow\varphi"] & A \arrow[d, "F"] \\
& B \arrow[r, "G"'] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
D \arrow[r, "K'"] \arrow[d, "H'"'] \arrow[dr, phantom, "\swArrow\psi"] & B \arrow[d, "F"] \\
A \arrow[r, "G"'] & \mathcal{C}
\end{tikzcd}
.
$$

2. Given $J, J' \colon D \to F{\downarrow}G$, $\xi\colon HJ \Rightarrow HJ'$ and $\eta\colon KJ \to KJ'$ such that

$$
\begin{tikzcd}
D \arrow[dr, "J"] \arrow[drr, bend left, "KJ"] \arrow[ddr, bend right, "HJ'"'] & & \\
 & F{\downarrow}G \arrow[r, "K"] \arrow[d, "H"'] \arrow[l, phantom, "\overset{\xi}{\Leftarrow}"] \arrow[dr, phantom, "\swarrow\varphi"] & B \arrow[d, "F"] \\
& A \arrow[r, "G"'] & \mathcal{C}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
D \arrow[dr, "J'"] \arrow[drr, bend left, "KJ"] \arrow[ddr, bend right, "HJ'"'] \arrow[dr, phantom, shift left=3, "\Downarrow\eta"] & & \\
& F{\downarrow}G \arrow[r, "K"] \arrow[d, "H"'] \arrow[dr, phantom, "\swarrow\varphi"] & B \arrow[d, "F"] \\
& A \arrow[r, "G"'] & \mathcal{C}
\end{tikzcd}
,
$$

there exists a unique $\rho\colon J \Rightarrow J'$ such that $\xi = H\rho$ and $\eta = K\rho$, so that both diagrams above are equal to

$$
\begin{tikzcd}
D \arrow[dr, bend left, "J"] \arrow[dr, bend right, "J'"'] \arrow[dr, phantom, "\swarrow\rho"] \arrow[drr, bend left=40, "KJ"] \arrow[ddr, bend right=50, "HJ'"'] & & \\
& F{\downarrow}G \arrow[r, "K"] \arrow[d, "H"'] \arrow[dr, phantom, "\swarrow\varphi"] & B \arrow[d, "F"] \\
& A \arrow[r, "G"'] & \mathcal{C}
\end{tikzcd}
$$

A more succinct way of expressing the universal property of $F{\downarrow}G$ is that there is an 2-natural *isomorphism* of categories

$$
\mathcal{K}(D, F{\downarrow}G) \cong \mathcal{K}(D,F){\downarrow}\mathcal{K}(D,G),
$$

where on the right we have the usual comma category in **Cat**. These are sometimes called *strict* comma categories. However, as these are the only kinds of comma categories we consider, we will omit 'strict'.

Theorem 1.6.4 ([Rie10, 3.1.3])**.** *Let $\mathcal{K}$ be a finitely complete 2-category. A 1-cell $p\colon E \to B$ is a fibration if and only if the canonical inclusion $i\colon E \to B{\downarrow}p$ has a right adjoint over B:*

$$
\begin{tikzcd}
E \arrow[rr, bend left, "i"] \arrow[rr, phantom, "\perp"] \arrow[dr, "p"'] & & B/p \arrow[ll, bend left, "r"] \arrow[dl, "\pi_B"] \\
& B &
\end{tikzcd}
$$

1.7 *Fibrations across a* 2*-functor*

In 1.3, we saw that pullback squares in a category $\mathcal{C}$ are special cases of cod-cartesian arrows for the functor $\mathsf{cod}\colon \mathbf{Arr}(C) \to C$. In this section, we will see that fibrations in a 2-category $\mathcal{C}$ are special cases of cod-fibrations (where $\mathsf{cod}\colon \mathbf{Arr}(\mathcal{C}) \to \mathcal{C}$), which we now define.

Definition 1.7.1 (Φ-Fibration)**.** Let $\Phi\colon \mathcal{T} \to \mathcal{C}$ be a 2-functor.

An object $p \in \mathcal{T}$ is a **Φ-fibration** if for every $u \in \mathcal{T}$, the functor $\Phi_{u,p}\colon \mathcal{T}(u,p) \to \mathcal{C}(\Phi u, \Phi p)$ is a fibration in **Cat**, and for every $f\colon u \to v$ in $\mathcal{T}$, the square

$$\begin{tikzcd}
\mathcal{T}(v,p) \arrow[r, "-\circ f"] \arrow[d, "\Phi"'] & \mathcal{T}(u,p) \arrow[d, "\Phi"] \\
\mathcal{C}(\Phi v, \Phi p) \arrow[r, "-\circ \Phi f"] & \mathcal{C}(\Phi u, \Phi p)
\end{tikzcd}$$

is a cartesian functor.

Definition 1.7.2 (Φ-cartesian functor)**.** Let p and q be Φ-fibrations. A 1-cell $F\colon p \to q$ in $\mathcal{T}$ is a **Φ-cartesian functor** if for all $u \in \mathcal{C}$, the square

$$\begin{tikzcd}
\mathcal{T}(u,p) \arrow[r, "F\circ -"] \arrow[d, "\Phi"'] & \mathcal{T}(u,q) \arrow[d, "\Phi"] \\
\mathcal{C}(\Phi u, \Phi p) \arrow[r, "\Phi F\circ -"] & \mathcal{C}(\Phi u, \Phi q)
\end{tikzcd}$$

is a cartesian functor.

Proposition 1.7.3. *Let $\mathcal{C}$ be a 2-category, and $\mathsf{cod}\colon \mathbf{Arr}(\mathcal{C}) \to \mathcal{C}$ its codomain functor. An arrow $p\colon E \to B$ in $\mathcal{C}$ is a fibration iff it is a cod-fibration, and a commuting square between fibrations p and q*

$$\begin{tikzcd}
D \arrow[r, "F^{\top}"] \arrow[d, "p"'] & E \arrow[d, "q"] \\
A \arrow[r, "F^{\perp}"] & B
\end{tikzcd}$$

is a cartesian functor iff it is cod-cartesian.

Proof. We first note that for $u\colon X \to Y$ in $\mathcal{C}$, $\mathbf{Arr}(\mathcal{C})(u,p)$ fits into a pullback square

$$
\begin{tikzcd}
\mathbf{Arr}(\mathcal{C})(u,p) \arrow[r, "\mathsf{dom}"] \arrow[d, "\mathsf{cod}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}(X,E) \arrow[d, "p\circ -"] \\
\mathcal{C}(Y,B) \arrow[r, "-\circ u"] & \mathcal{C}(X,B)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r] \arrow[d, "u"'] & E \arrow[d, "p"] \\
Y \arrow[r] & B
\end{tikzcd}
\tag{1.6}
$$

where $\mathsf{dom}\colon \mathbf{Arr}(\mathcal{C}) \to \mathcal{C}$ is the domain functor. If $p \circ -$ is a fibration, then so is cod, being a pullback of a fibration. Conversely, if cod is a fibration for all u, setting $u = 1_X$ yields an isomorphism $\mathbf{Arr}(\mathcal{C})(1_X, p) \cong \mathcal{C}(X,E)$ over $\mathcal{C}(Y,B) = \mathcal{C}(X,B)$, so $p \circ -$ is the composite of an isomorphism with a fibration, hence is a fibration as well.

Next, suppose we have a map $f\colon u \to v$ in $\mathbf{Arr}(\mathcal{C})$, where $f = (f^{\top}, f^{\perp})$. Then we obtain a commuting cube whose front and back faces are the pullbacks squares of (1.6), and whose right face is the square in Definition 1.6.1.

$$
\begin{tikzcd}
\mathbf{Arr}(\mathcal{C})(v,p) \arrow[rr, "\mathsf{dom}_{v,p}"] \arrow[dr, "-\circ f"'] \arrow[dd, "\mathsf{cod}_{v,p}"'] & & \mathcal{C}(W,E) \arrow[dr, "-\circ f^{\top}"] \arrow[dd] & \\
& \mathbf{Arr}(\mathcal{C})(u,p) \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathcal{C}(X,E) \arrow[dd, "p\circ -"] \\
\mathcal{C}(Z,B) \arrow[rr] \arrow[dr, "-\circ f^{\perp}"'] & & \mathcal{C}(W,B) \arrow[dr] & \\
& \mathcal{C}(Y,B) \arrow[rr, "-\circ u"'] & & \mathcal{C}(X,B)
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f^{\top}"] \arrow[d, "u"'] & W \arrow[r] \arrow[d, "v"'] & E \arrow[d, "p"] \\
Y \arrow[r, "f^{\perp}"] & Z \arrow[r] & B
\end{tikzcd}
$$

If the left face is a cartesian functor of fibrations for all u and v, then setting $u = 1_X$ and $v = 1_W$, the right face is a cartesian functor as well. Conversely, if the right face is a cartesian functor, Corollary 1.5.5 shows that the left face is a cartesian functor, too.

We have thus shown that fibrations in $\mathcal{C}$ are precisely the cod-fibrations. $\square$

Chapter 2

ENRICHED FIBRATIONS

In this chapter, we provide an overview of [BW19], which is joint work with Jonathan Beardsley, and which gives an enriched version of the Grothendieck correspondence (Theorem 1.3.2). In order to do this, we first introduce enriched categories.

2.1 Enriched category theory

We briefly recall some notions from enriched category theory. For a more detailed account, all of which can be found in [Rie14, Chapter 3] or [Kel82].

An ordinary category C has a *set* of arrows $C(x, y)$ for any two objects $x, y \in C$. The starting point of enriched category theory is that much of category theory can be done even when $C(x, y)$ is not a set, but instead an object of another monoidal category $(\mathcal{V}, \otimes, \mathbf{1})$.

A $\mathcal{V}$*-enriched category* C, or simply a $\mathcal{V}$*-category*, consists of a set of objects $\mathbf{Ob}(\mathcal{C})$ (or sometimes simply $\mathcal{C}$), such that for all $c, d \in \mathcal{C}$ we have a hom-object $\mathcal{C}(c, d) \in \mathcal{V}$, for all $c \in C$ we have an 'identity map' $1_c \colon \mathbf{1} \to \mathcal{C}(c, c)$ in $\mathcal{V}$, and for all $c, d, e \in \mathcal{V}$ we have a 'composition map'

$$\circ_{c,d,e} \colon \mathcal{C}(d, e) \otimes \mathcal{C}(c, d) \to \mathcal{C}(c, e)$$

in $\mathcal{V}$, all of which are required to satisfy associativity and unitality conditions. Henceforth, $\mathcal{V}$-categories will be denote $\mathcal{C}, \mathcal{D}, \mathcal{E} \dots$, while ordinary categories will be denoted C, D, E and so on.

A $\mathcal{V}$*-functor* $F \colon \mathcal{C} \to \mathcal{D}$ consists of a function on objects $F \colon \mathbf{Ob}(\mathcal{C}) \to \mathbf{Ob}(\mathcal{D})$ and for all $c, d \in \mathcal{C}$, a $\mathcal{V}$-morphism on hom-objects

$$F_{c,d} \colon \mathcal{C}(c, d) \to \mathcal{D}(Fc, Fd)$$

respecting the identity and composition maps in $\mathcal{C}$ and $\mathcal{D}$. We will abuse notation slightly, and use F for the functor $\mathcal{C} \to \mathcal{D}$, the function $Ob(\mathcal{C}) \to Ob(\mathcal{D})$ and the $\mathcal{V}$-morphism $\mathcal{C}(c, d) \to \mathcal{D}(Fc, Fd)$.

Finally, given $\mathcal{V}$-functors $F, G\colon \mathcal{C} \to \mathcal{D}$, a *natural transformation* of $\mathcal{V}$-functors $\alpha\colon F \Rightarrow G$ is a family of $\mathcal{V}$-morphisms $\alpha_b\colon \mathbf{1} \to \mathcal{D}(Fc, Gc)$ for each $c \in \mathcal{C}$ such that the following diagram commutes in $\mathcal{V}$,

$$\begin{tikzcd}
\mathcal{C}(c,d) \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{D}(Fc,Fd) \arrow[d, "\alpha_d\circ -"] \\
\mathcal{D}(Gc,Gd) \arrow[r, "-\circ\alpha_b"] & \mathcal{D}(Fc,Gd)
\end{tikzcd}$$

where we leave it to the reader to infer the meaning of $\alpha_d \circ -$ and $- \circ \alpha_b$, as well as how to compose natural transformations horizontally and vertically (or see the Appendix of [BW19]).

All of this allows us to define a 2-category of $\mathcal{V}$-categories, $\mathcal{V}$-functors, and natural transformations, which we call $\mathcal{V}$-**Cat**.

Let $\mathbb{1}$ denote the $\mathcal{V}$-category with a single object $*$ and

$$\mathbb{1}(*, *) := \mathbf{1}.$$

This is a terminal object in $\mathcal{V}$-**Cat**. Since $\mathcal{V}$-**Cat** is a 2-category, its hom-objects are categories. Given a $\mathcal{V}$-category $\mathcal{C}$, we thus have an ordinary category

$$\mathcal{C}_0 := \mathcal{V}\text{-}\mathbf{Cat}(\mathbb{1}, \mathcal{C})$$

which we call the *underlying category* of $\mathcal{C}$. Explicitly, $\mathcal{C}_0$ is the category with the same objects as $\mathcal{C}$ and morphisms $f\colon \mathbf{1} \to \mathcal{C}(b, c)$.

Going the other direction, when $\mathcal{V}$ has coproducts which are preserved by $\otimes$, the *free $\mathcal{V}$-category* on $C \in \mathbf{Cat}$ is the $\mathcal{V}$-category $C_{\mathcal{V}}$ with the same objects as C and hom-objects

$$C_{\mathcal{V}}(b, c) := \coprod_{f \in C(b,c)} \mathbf{1}.$$

2.2 Properties of $\mathcal{V}$

In order to have a good theory of fibrations for $\mathcal{V}$-categories, we require $\mathcal{V}$ to have the following properties:

1. $\mathcal{V}$ has all pullbacks and coproducts, and the monoidal unit $\mathbf{1}$ is terminal.

2. The monoidal product $\otimes$ *preserves coproducts* in the sense that we have a canonical isomorphism
$$\left(\coprod_{i\in I} A_i\right) \otimes \left(\coprod_{j\in J} B_j\right) \cong \coprod_{i\in I}\coprod_{j\in J} A_i \otimes B_j.$$

3. $\mathcal{V}$ is *extensive*, which means that pullbacks interact well with coproducts in the following senses:

   (i) *Pullbacks preserve coproduct injections*: For any set I and family of maps $f_i\colon Y_i \to X_i$ in $\mathcal{V}$, the following square is a pullback:
$$\begin{tikzcd} Y_i \arrow[r, hook] \arrow[d, "f_i"'] & \coprod_{i\in I} Y_i \arrow[d, "\coprod_i f_i"] \\ X_i \arrow[r, hook] & \coprod_{i\in I} X_i \end{tikzcd}$$

   (ii) *Pullbacks preserve coproduct decompositions*: For any set I and family of maps $f_i\colon X_i \to Z$ and $g\colon Y \to Z$ in $\mathcal{V}$, we have a canonical isomorphism
$$Y \times_Z \left(\coprod_i X_i\right) \cong \coprod_i \left(Y \times_Z X_i\right),$$
   where these fibered products are given by the following pullback diagrams:
$$\begin{tikzcd} Y \times_Z (\coprod_i X_i) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ \coprod_{i\in I} X_i \arrow[r, "\coprod_i f_i"] & Z \end{tikzcd} \qquad \begin{tikzcd} Y \times_Z X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X_i \arrow[r, "f_i"] & Z \end{tikzcd}$$

4. The monoidal unit $\mathbf{1}$ is *connected*, which means that

$$\mathcal{V}(\mathbf{1}, -)\colon \mathcal{V} \to \mathbf{Set}$$

preserves coproducts. If $\mathbf{1}$ is terminal, then $\mathcal{V}(\mathbf{1},\mathbf{1}) \cong \{*\}$, so for any set X we have a canonical isomorphism

$$\mathcal{V}\left(\mathbf{1}, \coprod_{x\in X} \mathbf{1}\right) \cong \coprod_{x\in X} \{*\} \cong X. \tag{2.1}$$

2.3 Enriched fibrations

In this section, we develop the theory of enriched cocartesian fibrations. All of this dualizes to cartesian fibrations as well.

Definition 2.3.1. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a $\mathcal{V}$-functor. A map $\chi\colon \mathbf{1} \to \mathcal{E}(e, e')$ is p**-cocartesian** if the following square is a pullback in $\mathcal{V}$ for all $d \in \mathcal{E}$:

$$\begin{tikzcd}
\mathcal{E}(e', d) \arrow[r, "-\circ\chi"] \arrow[d, "p"'] & \mathcal{E}(e, d) \arrow[d, "p"] \\
\mathcal{B}(pe', pd) \arrow[r, "-\circ p\chi"] & \mathcal{B}(pe, pd)
\end{tikzcd} \tag{2.2}$$

Definition 2.3.2. An **cocartesian fibration** is a $\mathcal{V}$-functor $p\colon \mathcal{E} \to \mathcal{B}$ along with, for every $e \in \mathcal{E}$, $b \in \mathcal{B}$ and $f\colon \mathbf{1} \to \mathcal{B}(pe, b)$, an object $f_! e \in \mathcal{E}$ over b and a p-cocartesian map $\chi(f, e)\colon \mathbf{1} \to \mathcal{E}(e, f_! e)$ over f.

Definition 2.3.3. An **cocartesian functor** from $p\colon \mathcal{E} \to \mathcal{B}$ to $q\colon \mathcal{F} \to \mathcal{B}$ is a functor $k\colon \mathcal{E} \to \mathcal{F}$ that satisfies $qk = p$ and sends p-cocartesian maps to q-cocartesian maps.

Definition 2.3.4. Let $\mathbf{coCart}(\mathcal{B})$ denote the 2-category whose objects are cocartesian fibrations over $\mathcal{B}$, morphisms are cocartesian functors, and 2-morphisms are natural transformations over $\mathcal{B}$.

2.4 Enriched comma categories

The 2-category $\mathcal{V}$-**Cat** is finitely complete if $\mathcal{V}$ is finitely complete. In such a situation, we verify that fibrations as defined in Definition 2.3.2 are precisely fibrations in the 2-category $\mathcal{V}$-**Cat** as characterized in Theorem 1.6.4.

We are interested in comma categories of the form $p{\downarrow}\mathcal{B} := p{\downarrow}1_{\mathcal{B}}$, with universal natural transformation φ_p:

$$\begin{tikzcd}
p{\downarrow}\mathcal{B} \arrow[r, "\pi_{\mathcal{E}}"] \arrow[d, "\pi_{\mathcal{B}}"'] \arrow[dr, phantom, "\swarrow\varphi_p"] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{B} \arrow[r, equal] & \mathcal{B}
\end{tikzcd}$$

The objects of $p{\downarrow}\mathcal{B}$ are tuples (e, f, b) where $e \in \mathcal{E}, b \in \mathcal{B}$ and $f \colon \mathbf{1} \to \mathcal{B}(pe, b)$ (so f is an element of $\mathcal{B}_0(pe, b)$), and the hom-objects are pullbacks:

$$\begin{tikzcd}
p{\downarrow}\mathcal{B}\big((e,f,b),(e',f',b')\big) \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\lrcorner", very near start] & \mathcal{E}(e,e') \arrow[d, "p_{e,e'}"] \\
 & \mathcal{B}(pe,pe') \arrow[d, "f'\circ -"] \\
\mathcal{B}(b,b') \arrow[r, "-\circ f"] & \mathcal{B}(pe,b')
\end{tikzcd} \tag{2.3}$$

By the universal property of $p{\downarrow}\mathcal{B}$, the functors $\mathcal{B} \xleftarrow{p} \mathcal{E} \xrightarrow{1} \mathcal{E}$ induce a canonical functor $i \colon \mathcal{E} \to p{\downarrow}\mathcal{B}$.

$$\begin{tikzcd}
\mathcal{E} \arrow[dr, dashed, "i"] \arrow[drr, equal, bend left] \arrow[ddr, bend right, "p"'] & & \\
 & p{\downarrow}\mathcal{B} \arrow[r, "\pi_{\mathcal{E}}"] \arrow[d, "\pi_{\mathcal{B}}"'] \arrow[dr, phantom, "\swarrow\varphi"] & \mathcal{E} \arrow[d, "p"] \\
 & \mathcal{B} \arrow[r, equal] & \mathcal{B}
\end{tikzcd}$$

On objects, we have $ie = (e, 1_{pe}, pe)$, while on morphisms, i is given by the universal property of the pullback:

$$
\begin{tikzcd}
\mathcal{E}(e,e') \arrow[dr, dashed, "i_{e,e'}"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, "p_{e,e'}"'] & & \\
& p{\downarrow}\mathcal{B}(ie,ie') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}(e,e') \arrow[d, "p_{e,e'}"] \\
& \mathcal{B}(pe,pe') \arrow[r, equal] & \mathcal{B}(pe,pe')
\end{tikzcd}
$$

In fact, $i_{e,e'}$ is an isomorphism, so i is full and faithful. Thus $\mathcal{E}$ may be treated as the full subcategory of $p{\downarrow}\mathcal{B}$ on objects of the form $(e, 1_{pe}, pe)$.

Proposition 2.4.1. *A $\mathcal{V}$-functor $p\colon \mathcal{E} \to \mathcal{B}$ is an cocartesian fibration if and only if it is a cocartesian fibration in the* 2*-category* $\mathcal{V}$-**Cat**.

Proof. By Theorem 1.6.4, it suffices to show that p is a cocartesian fibration iff the canonical inclusion $i\colon \mathcal{E} \to p{\downarrow}\mathcal{B}$ has a left adjoint ℓ over $\mathcal{B}$:

$$
\begin{tikzcd}
p{\downarrow}\mathcal{B} \arrow[rr, bend left, "\ell"] \arrow[rr, phantom, "\perp"] \arrow[dr, "\pi_{\mathcal{B}}"'] & & \mathcal{E} \arrow[ll, bend left, hook', "i"] \arrow[dl, "p"] \\
& \mathcal{B} &
\end{tikzcd}
$$

The functor $i\colon \mathcal{E} \hookrightarrow p{\downarrow}\mathcal{B}$ has a left adjoint if and only if for all $(e,f,b) \in p{\downarrow}\mathcal{B}$, there exists $\ell(e,f,b) \in \mathcal{E}$ and a map

$$\eta_{(e,f,b)}\colon \mathbf{1} \to p{\downarrow}\mathcal{B}\big((e,f,b), i\ell(e,f,b)\big)$$

such that the composite

$$\mathcal{E}\big(\ell(e,f,b), d\big) \xrightarrow{i} p{\downarrow}\mathcal{B}\big(i\ell(e,f,b), id\big) \xrightarrow{-\circ\eta_{(e,f,b)}} p{\downarrow}\mathcal{B}\big((e,f,b), id\big) \tag{2.4}$$

is an isomorphism in $\mathcal{V}$ for all $d \in \mathcal{E}$. Further, this adjunction lies over $\mathcal{B}$ if and only if $p\,\ell(e,f,b) = b$ and $\pi_{\mathcal{B}}\,\eta_{(e,f,b)} = 1_b$.

The result follows by observing that the data of $\ell(e,f,b)$ and $\eta_{(e,f,b)}$ is precisely the data of $f_!e$ and $\chi(f,e)$,

$$\ell(e,f,b) = f_!e \qquad\qquad \eta_{(e,f,b)} = \big(\chi(f,e), 1_b\big) \tag{2.5}$$

and that (2.4) is an isomorphism precisely when $\mathcal{E}\big(\ell(e,f,b),d\big)$ is also a pullback of the cospan defining $p{\downarrow}\mathcal{B}\big((e,f,b),id\big)$,

$$\begin{tikzcd} & \mathcal{E}(e,d) \arrow[d, "p"] \\ \mathcal{B}(b,pd) \arrow[r, "-\circ f"] & \mathcal{B}(pe,pd) \end{tikzcd} \tag{2.6}$$

in which case $\chi(f,e)$ is p-cocartesian. □

2.5 The enriched Grothendieck construction and its inverse

In this section, we assume that $\mathcal{V}$ satisfies all the assumptions of §2.2.

Definition 2.5.1 ([BW19, §4.2])**.** Let B be an ordinary (i.e. **Set**-enriched) category treated as a 2-category, and let $F\colon B\to\mathcal{V}\text{-}\mathbf{Cat}$ be a pseudofunctor. Let $F\rtimes B$ denote the $\mathcal{V}$-category

$$\begin{aligned} Ob(F\rtimes B) &:= \coprod_{b\in B} Ob(F_b)\times\{b\}, \\ F\rtimes B\big((x,b),(y,c)\big) &:= \coprod_{f\colon b\to c} F_c(F_f x,y). \end{aligned}$$

Identity morphisms are given by

$$1_{(x,b)} := \xi_x\colon \mathbf{1}\to F_b(F_{1_b}x,x)\subset \coprod_{f\colon b\to b} F_b(F_f x,x) = F\rtimes B\big((x,b),(x,b)\big) \tag{2.7}$$

while composition is induced by the composite

$$\begin{tikzcd} F_b(F_f x,y)\otimes F_d(F_g y,z) \arrow[r, "F_g\otimes 1"] \arrow[d, dotted] & F_d(F_gF_f x,F_g y)\otimes F_d(F_g y,z) \arrow[d, "(-\circ\theta_x)\otimes 1"', "\cong"] \\ F_d(F_{gf}x,z) & F_d(F_{gf}x,F_g y)\otimes F_d(F_g y,z) \arrow[l, "\circ"] \end{tikzcd}$$

where $b\xrightarrow{f}c\xrightarrow{g}d$. This extends to a functor out of $F\rtimes B\big((x,b),(y,c)\big)\otimes F\rtimes B\big((y,c),(z,d)\big)$ because $\otimes$ preserves coproducts.

The **Grothendieck construction of** F is the functor $\int F\colon F\rtimes B\to B_{\mathcal{V}}$ which sends (x,b) to b and whose action on hom-objects is given by the unique map $F_c(F_f x,y)\to\mathbf{1}$.

Proposition 2.5.2 ([BW19, Proposition 4.5])**.** *For $F\colon B \to \mathcal{V}$-**Cat** a pseudofunctor, $\int F$ is a cocartesian fibration.*

Theorem 2.5.3 ([BW19, Theorem 4.7])**.** *The Grothendieck construction extends to a 2-functor*

$$\int_{\mathcal{V}}\colon [B, \mathcal{V}\text{-}\mathbf{Cat}] \to \mathbf{coCart}_{/B_{\mathcal{V}}}.$$

To show that $\int_{\mathcal{V}}$ is a 2-equivalence, we construct its inverse.

Definition 2.5.4. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a $\mathcal{V}$-functor. For each $b \in \mathcal{B}$, treated as a functor $b\colon \mathbb{1} \to \mathcal{B}$, the **fiber** of p over b is the category $\mathcal{E}_b$ given by the pullback:

$$\begin{tikzcd} \mathcal{E}_b \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\ \mathbb{1} \arrow[r, "b"'] & \mathcal{B} \end{tikzcd}$$

Proposition 2.5.5 ([BW19, Proposition 3.13])**.** *If $p\colon \mathcal{E} \to \mathcal{B}$ is a cocartesian fibration, there is a pseudofunctor $\mathcal{E}_\bullet\colon \mathcal{B}_0 \to \mathcal{V}$-**Cat** sending each b to the fiber $\mathcal{E}_b$.*

Proposition 2.5.6 ([BW19, Proposition 3.14])**.** *The construction that sends an cocartesian fibration $p\colon \mathcal{E} \to \mathcal{B}$ to the pseudofunctor $\mathcal{E}_\bullet$ extends to a 2-functor*

$$(-)_\bullet\colon \mathbf{coCart}_{/\mathcal{B}} \to [\mathcal{B}_0, \mathcal{V}\text{-}\mathbf{Cat}].$$

Note that while $(-)_\bullet$ takes cocartesian fibrations over an arbitrary enriched $\mathcal{V}$-category $\mathcal{B}$, it only returns pseudofunctors from an unenriched $\mathcal{B}_0$. It is thus generally not possible to recover an cocartesian fibration $p\colon \mathcal{E} \to \mathcal{B}$ over an arbitrary base $\mathcal{B}$ from its corresponding pseudofunctor $\mathcal{E}_\bullet\colon \mathcal{B}_0 \to \mathcal{V}$-**Cat**. However, when $\mathcal{B}$ is of the form $B_{\mathcal{V}}$ to begin with, our assumptions on $\mathcal{V}$ imply that $\mathcal{B}_0 = (B_{\mathcal{V}})_0 \cong B$, so $(-)_\bullet$ is a 2-functor $\mathbf{coCart}_{/B_{\mathcal{V}}} \to [B, \mathcal{V}\text{-}\mathbf{Cat}]$. The following result shows that in this situation, $\int_{\mathcal{V}}$ and $(-)_\bullet$ are mutual inverses:

Theorem 2.5.7 ([BW19, Theorem 5.9])**.** *There is a 2-equivalence:*

$$\int_{\mathcal{V}}\colon [B, \mathcal{V}\text{-}\mathbf{Cat}] \cong \mathbf{coCart}_{/B_{\mathcal{V}}} \colon (-)_\bullet$$

Chapter 3

∞-CATEGORICAL GROTHENDIECK CONSTRUCTIONS

In this section, we give an application of the enriched Grothendieck construction when $\mathcal{V} = \mathbf{sSet}$, the category of simplicial sets. Categories enriched in $\mathbf{sSet}$, also known as *simplicial categories*, are one of the many models of ∞-categories. In [Lur09], an ∞-categorical Grothendieck construction is given, where the ∞-categories are modelled using both *quasicategories* and *marked simplicial sets*. In [BW18], we compare our $\mathbf{sSet}$-enriched Grothendieck construction with the ∞-categorical one given in [Lur09]. This chapter is a summary of that comparison.

3.1 Unstraightening and the relative nerve

The ∞-categorical Grothendieck construction is the following equivalence

$$\int_{\infty} : [S, \mathbf{Cat}_{\infty}] \xrightarrow{\simeq} \mathbf{coCart}_{/S},$$

where S is a simplicial set, $[S, \mathbf{Cat}_{\infty}]$ (also denoted $(\mathbf{Cat}_{\infty})^S$) is quasicategory of simplicial maps from S to the *quasicategory of ∞-categories*, and $\mathbf{coCart}_{/S}$ is the quasicategory of *cocartesian fibrations over S* (these 'large quasicategories' are defined as nerves of certain simplicial categories. See [Lur09, Ch. 3], or [BW18, Appendix 1 and 2] for details). This equivalence should be interpreted as the ∞-categorical analogue of 1.3.1.

It is not easy to explicitly describe $\int_{\infty} \varphi$ for an arbitrary $\varphi \colon S \to \mathbf{Cat}_{\infty}$. The functor $\int_{\infty}$ is the nerve of the *marked unstraightening functor*, which is in turn given as the right adjoint of the marked straightening functor [Lur09, 3.2.1.6]. However, when S is the nerve of a small category D, and φ is the nerve of a functor $f \colon D \to \mathbf{sSet}$ such that each fd is a quasicategory, the *relative nerve* $\mathrm{N}_f(D)$ of [Lur09, 3.2.5.2] yields a cocartesian fibration equivalent to $\int_{\infty} \mathrm{N}(f)$.

Definition 3.1.1 ([Lur09, 3.2.5.2])**.** Let D be a category, and $f\colon D \to \mathbf{sSet}$ a functor. The **nerve of** D **relative to** f is the simplicial set $\mathrm{N}_f(D)$ whose n-simplices are sets of:

(i) a functor $d\colon [n] \to D$; write d_i for $d(i)$ and $d_{ij}\colon d_i \to d_j$ for the image of the unique map $i \leq j$ in $[n]$,

(ii) for every nonempty subposet $J \subseteq [n]$ with maximal element j, a map $s^J\colon \Delta^J \to fd_j$,

(iii) such that for nonempty subsets $I \subseteq J \subseteq [n]$ with respective maximal elements $i \leq j$, the following diagram commutes:

$$\begin{tikzcd} \Delta^I \arrow[r, "s^I"] \arrow[d, hook] & fd_i \arrow[d, "fd_{ij}"] \\ \Delta^J \arrow[r, "s^J"] & fd_j \end{tikzcd} \tag{3.1}$$

Proposition 3.1.2 ([Lur09, 3.2.5.21])**.** *Let $f\colon D \to \mathbf{sSet}$ be a functor such that each fd is a quasicategory. There is an equivalence of cocartesian fibrations:*

$$\mathrm{N}_f(D) \simeq \int_\infty \mathrm{N}(f).$$

If f further factors as $D \xrightarrow{F} \mathbf{sCat} \xrightarrow{\mathrm{N}} \mathbf{sSet}$, where each Fd is a locally Kan simplicial category, we may instead form the simplicially-enriched Grothendieck construction $\int_{\mathbf{sSet}} F$ and take its nerve. The main contribution of [BW18] is the following result:

Theorem 3.1.3 ([BW18, 2.3.1])**.** *Let $F\colon D \to \mathbf{sCat}$ be a functor, and $f = \mathrm{N}F$. Then there is an isomorphism of coCartesian fibrations*

$$\mathrm{N}(\int_{\mathbf{sSet}} F) \cong \mathrm{N}_f(D).$$

From this we may conclude that the $\mathbf{sSet}$-enriched Grothendieck construction gives an alternative description of the ∞-categorical Grothendieck construction:

Corollary 3.1.4. *Let $F\colon D \to \mathbf{sCat}$ be a functor such that each Fd is a quasicategory, and $f = \mathrm{N}F$. Then there is an equivalence of coCartesian fibrations:*

$$\mathrm{N}(\int_{\mathbf{sSet}} F) \simeq \mathrm{N}_f(D) \simeq \int_{\infty} \mathrm{N}(f).$$

Remark 3.1.5. *Recall that $\int_{\mathbf{sSet}}$ works for* pseudo*functors $F\colon D \to \mathbf{sCat}$ as well. However, we require strict functors in the above results because the relative nerve $\mathrm{N}_f(D)$ is only defined for strict functors $f\colon D \to \mathbf{sSet}$.*

3.2 Monoidal ∞-categories and opposites

In this section, we give an example of a functor $F\colon D \to \mathbf{sCat}$.

Definition 3.2.1. Let $\mathcal{C}$ be a monoidal simplicial category.

For $f\colon [m] \to [n]$ in Δ, let $\mathcal{C}^f\colon \mathcal{C}^n \to \mathcal{C}^m$ be the functor that sends $(x_1, \dots, x_n) \in \mathcal{C}^n$ to $(y_1, \dots, y_m)$, and $(\varphi_1\colon x_1 \to x_1', \dots, \varphi_n)$ to $(\psi_1, \dots, \psi_m)$, where

$$y_i = x_{f(i-1)+1} \otimes \cdots \otimes x_{f(i)},$$
$$\psi_i = \varphi_{f(i-1)+1} \otimes \cdots \otimes \varphi_{f(i)}.$$

Then let $\mathcal{C}^\bullet\colon \Delta^{\mathsf{op}} \to \mathbf{sCat}$ denote the pseudofunctor sending $[n]$ to $\mathcal{C}^n$ and f to $\mathcal{C}^f$.

Remark 3.2.2. *This holds in a more general setting. Let $\mathcal{C}$ be a monoidal $\mathcal{V}$-category. Then we may similarly define a pseudofunctor $\mathcal{C}^\bullet\colon \Delta^{op} \to \mathcal{V}$-$\mathbf{Cat}$ sending $[n]$ to $\mathcal{C}^n$ and f to $\mathcal{C}^f$. The pseudofunctoriality of $\mathcal{C}^\bullet$ witness the fact that the monoidal structure is associative and unital only up to coherent isomorphism, and $\mathcal{C}^\bullet$ is a functor if and only if $\mathcal{C}$ is strict monoidal.*

Applying the Grothendieck construction to $\mathcal{C}^\bullet$ gives the *category of operators* of $\mathcal{C}$:

Definition 3.2.3 ([Lur07, 1.1.1])**.** Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a monoidal simplicial category. Then we define a new category $\mathcal{C}^\otimes$ as follows:

1. An object of $\mathcal{C}^\otimes$ is a finite, possibly empty, sequence of objects of $\mathcal{C}$, denoted $[x_1, \dots, x_n]$.

2. The simplicial set of morphisms from $[x_1, \dots, x_n]$ to $[y_1, \dots, y_m]$ in $\mathcal{C}^{\otimes}$ is defined to be

$$\coprod_{f \in \Delta([m],[n])} \prod_{1 \le i \le m} \mathcal{C}\big(x_{f(i-1)+1} \otimes x_{f(i-1)+2} \otimes \cdots \otimes x_{f(i)} \ , \quad y_i\big)$$

where $x_{f(i-1)+1} \otimes \cdots \otimes x_{f(i)}$ is taken to be $\mathbf{1}$ if $f(i-1) = f(i)$.

A morphism will be denoted $[f; f_1, \dots, f_m]$, where

$$x_{f(i-1)+1} \otimes \cdots \otimes x_{f(i)} \xrightarrow{f_i} y_i.$$

3. Composition in $\mathcal{C}^{\otimes}$ is determined by composition in Δ and $\mathcal{C}$:

$$[g; g_1, \dots g_\ell] \circ [f; f_1, \dots, f_m] = [f \circ g \ ; \quad h_1, \dots, h_\ell],$$
$$\text{where} \quad h_i = g_i \circ (f_{g(i-1)+1} \otimes \cdots \otimes f_{g(i)}).$$

This is associative and unital due to the associativity and unit constraints of $\otimes$.

Lemma 3.2.4 ([BW18, 3.1.6])**.** *For a strict monoidal simplicial category $\mathcal{C}$, there is an isomorphism of simplicial categories*

$$\mathcal{C}^{\otimes} \cong \int_{\mathbf{sSet}} \mathcal{C}^{\bullet}.$$

Suppose now that $\mathcal{C}$ is a strict monoidal *fibrant* (i.e. locally Kan) simplicial category. Then $\mathcal{C}^{\otimes}$ is a fibrant simplicial category as well, so the simplicial nerves of $\mathcal{C}$ and $\mathcal{C}^{\otimes}$ are both quasicategories.

Definition 3.2.5 ([Lur07])**.** Let $(\mathcal{C}, \otimes)$ be a strict monoidal fibrant simplicial category. The **operadic nerve of $\mathcal{C}$ with respect to $\otimes$** is the quasicategory

$$\mathrm{N}^{\otimes}(\mathcal{C}) := \mathrm{N}(\mathcal{C}^{\otimes}).$$

One would expect the nerve of a monoidal simplicial category to be a monoidal ∞-category in some sense. This is indeed the case:

Definition 3.2.6 ([Lur07, 1.1.2])**.** A **monoidal quasicategory** is a coCartesian fibration of simplicial sets $p : X \to N(\Delta^{\mathsf{op}})$ such that for each $n \geq 0$, the functors $X_{[n]} \to X_{\{i,i+1\}}$ induced by $\{i, i+1\} \hookrightarrow [n]$ determine an equivalence of quasicategories

$$X_{[n]} \xrightarrow{\simeq} X_{\{0,1\}} \times \cdots \times X_{\{n-1,n\}} \cong (X_{[1]})^n,$$

where $X_{[n]}$ denotes the fiber of p over $[n]$. In this case, we say that p defines a **monoidal structure on** $X_{[1]}$.

Proposition 3.2.7 ([Lur07, Proposition 1.6.3])**.** *If $\mathcal{C}$ is a strict monoidal fibrant simplicial category then $p\colon \mathrm{N}^{\otimes}(\mathcal{C}) \to \mathrm{N}(\Delta^{op})$ defines a monoidal structure on the quasicategory $\mathrm{N}(\mathcal{C}) \cong (\mathrm{N}^{\otimes}(\mathcal{C}))_{[1]}$.*

Remark 3.2.8. *The definition of a monoidal ∞-category may seem a little strange at first sight. The 'category' that has a monoidal product and unit is not the whole of X, but only the fiber $X_{[1]}$ over $[1]$. Perhaps a better way of interpreting the definition of a monoidal ∞-category is that it defines a monoidal structure on the quasicategory $X_{[1]}$.*

The definition of a monoidal ∞-category is akin to defining a monoidal simplicial category to be the cocartesian fibration $\mathcal{C}^{\otimes} \to \Delta^{op}$, rather the simplicial category $\mathcal{C}$ with a monoidal product and unit. These two ways of interpreting a monoidal category are related by the Grothendieck construction and its inverse: given a monoidal $\mathcal{C}$, we can form $\mathcal{C}^{\bullet}$ and then take $\mathcal{C}^{\otimes} = \int \mathcal{C}^{\bullet}$; conversely given $\mathcal{C}^{\otimes}$, we recover $\mathcal{C}$ by taking the fiber $\mathcal{C}_{[1]}$ over $[1]$.

In light of the results of this section, we obtain the following equivalent characterizations of the monoidal ∞-category (i.e. the operadic nerve) induced by a strict monoidal simplicial category:

Corollary 3.2.9. *Let $\mathcal{C}$ be a strict monoidal fibrant simplicial category, and let f be the composite $\Delta^{op} \xrightarrow{\mathcal{C}^{\bullet}} \mathbf{sCat} \xrightarrow{\mathrm{N}} \mathbf{sSet}$. Then we have the following string of equivalences:*

$$\mathrm{N}^{\otimes}(\mathcal{C}) \simeq \mathrm{N}\Big(\int_{\mathbf{sSet}} \mathcal{C}^{\bullet}\Big) \simeq \mathrm{N}_f(\Delta^{op}) \simeq \int_{\infty} \mathrm{N}(f). \tag{3.2}$$

Remark 3.2.10. *Again, most of the above equivalences hold even if $\mathcal{C}$ is not strict monoidal. The only part of the link that breaks down is* $\mathrm{N}_f(\Delta^{op})$*, simply because the relative nerve has not been defined for pseudofunctors* $f\colon \Delta^{op} \to \mathbf{sSet}$.

As an application of these alternative descriptions of the operadic nerve, we show that the operadic nerve interacts well with opposites. Recall that for a monoidal category $\mathcal{C}$, its opposite is a monoidal category as well. Given a monoidal simplicial category $\mathcal{C}$, we may first form the opposite monoidal category $\mathcal{C}^{\mathsf{op}}$ before taking its operadic nerve to yield $\mathrm{N}^{\otimes}(\mathcal{C}^{\mathsf{op}})$. Alternatively, we may form the monoidal ∞-category $\mathrm{N}^{\otimes}(\mathcal{C})$, then take *fiberwise opposites* by applying $\int_{\infty}^{-1}$ to obtain a map $\mathrm{N}(\Delta^{\mathsf{op}}) \to \mathbf{Cat}_{\infty}$, taking opposites pointwise in $\mathbf{Cat}_{\infty}$, then applying $\int_{\infty}$ to obtain another monoidal ∞-category which we denote $\mathrm{N}^{\otimes}(\mathcal{C})_{\mathsf{op}}$.

The following result shows that these two methods of forming the 'opposite monoidal ∞-category' are the same:

Theorem 3.2.11 ([BW18, 4.3.5])**.** *Let $\mathcal{C}$ be a strict monoidal fibrant simplicial category and equip $\mathcal{C}^{op}$ with its canonical monoidal structure. Then* $\mathrm{N}^{\otimes}(\mathcal{C}^{op})$ *and* $\mathrm{N}^{\otimes}(\mathcal{C})_{op}$ *define equivalent monoidal structures on* $\mathrm{N}(\mathcal{C}^{op}) \simeq \mathrm{N}(\mathcal{C})^{op}$.

The proof of this theorem proceeds by verifying that taking opposites commutes at each of the stages of the equivalence in Corollary 3.2.9.

Chapter 4

INTERNALIZING ENRICHED FIBRATIONS

The enriched Grothendieck construction $\int_{\mathcal{V}}$ in Chapter 2 takes a prestack $F\colon B^{\mathsf{op}} \to \mathcal{V}\text{-}\mathbf{Cat}$ and returns a fibration over $B_{\mathcal{V}}$. Although each Fb for $b \in B$ is a $\mathcal{V}$-category, the category B is an ordinary category, and the base of the fibration $B_{\mathcal{V}}$ is a free $\mathcal{V}$-category rather than an arbitrary $\mathcal{V}$-category.

For a fully enriched Grothendieck construction, we would like the base of the fibration to be an arbitrary $\mathcal{V}$-category $\mathcal{B}$. However, it does not make sense to ask for a $\mathcal{V}$-functor $\mathcal{B}^{\mathsf{op}} \to \mathcal{V}\text{-}\mathbf{Cat}$ because $\mathcal{V}\text{-}\mathbf{Cat}$ itself is not a $\mathcal{V}$-category[1]!

With a view towards defining prestacks over an arbitrary $\mathcal{V}$-category $\mathcal{B}$, let us first revisit the information encapsulated in an ordinary functor $F\colon B^{\mathsf{op}} \to \mathbf{Cat}$. We have:

1. A function $\bar{F}\colon D \to \mathbf{Ob}(\mathbf{Cat})$ sending each b to Fb, where $D = \mathbf{Ob}(B)$. Treating D as a discrete category and applying the Grothendieck construction to $\bar{F}$, we obtain a category
$$E := \coprod_{b \in B} Fb \tag{4.1}$$
along with a functor $p\colon E \to D$ that sends each Fb to the identity on b. In this way, $\bar{F}$ may be encoded as a functor $p\colon E \to D$ whose fibers are precisely the Fb's. Let $X(b) := \mathbf{Ob}(Fb)$ and $X := \coprod_b X(b) = \coprod_b \mathbf{Ob}(Fb) = \mathbf{Ob}(E)$, and write $p\colon X \to D$ for the induced function between sets of objects.

2. Since F is a functor, for each $f\colon b \to c$ in B, we have a functor $f^*\colon Fc \to Fb$. This includes the data of a map on objects $\bar{f}^*\colon X(c) \to X(b)$ for each $f \in B(b,c)$, i.e. a

[1] Even if $\mathcal{V}$ is a complete symmetric monoidal closed category, $\mathcal{V}\text{-}\mathbf{Cat}$ is only enriched over $\mathcal{V}\text{-}\mathbf{Cat}$, not $\mathcal{V}$.

function

$$B(b,c) \to \mathbf{Set}\big(X(c), X(b)\big)$$

which we may uncurry[2] to obtain a map

$$B(b,c) \times X(c) \to X(b).$$

The functoriality of F then implies that we have a 'B-action on X'.

3. The functor $f^*\colon Fc \to Fb$ also includes maps $f^*_{x,y}\colon Fc(x,y) \to Fb(\bar{f}^*x, \bar{f}^*y)$ which together induce a function

$$B(b,c) \times Fc(x,y) \to \coprod_{f \in B(b,c)} Fb(\bar{f}^*x, \bar{f}^*y) \to \coprod_{u,v \in X(b)} Fb(u,v),$$

or in terms of E, a function

$$B(b,c) \times E(x,y) \to \coprod_{u,v \in X(b)} E(u,v).$$

The functoriality of F then implies that we have a 'B-action on E'.

Note that the functor F has to respect sources and targets in E: if an arrow φ in E has source x and target y, then $f^*\varphi$ has to have source $\bar{f}^*x$ and target $\bar{f}^*y$. So the B-actions on X and E have to respect the sources and targets of E.

4. Finally, each $f^*\colon Fc \to Fb$ has to be a functor, which means it has to respect the identities and composition in Fc and Fb. So the B-action on E has to respect identities and composition in E.

In this manner, we may rephrase the definition of a functor $F\colon B^{\mathsf{op}} \to \mathbf{Cat}$ in terms of a B action on another category E. Remarkably, the notion of B acting on E generalizes well to the enriched setting. A prestack over $\mathcal{B}$ will thus be a $\mathcal{B}$ action on another $\mathcal{V}$-category $\mathcal{E}$, rather than a $\mathcal{V}$-functor $\mathcal{B}^{\mathsf{op}} \to \mathcal{V}\text{-}\mathbf{Cat}$.

[2] i.e. apply the Hom-Tensor adjunction.

One final modification needs to be made: in the first step, we defined a functor $p\colon E \to D$ sending each Fb to a point $\{*\}$. This was only possible because **Set** is cartesian, so the unit $\{*\}$ is a terminal object. When $\mathcal{V}$ is non-cartesian, there are no canonical maps down to $\mathbf{1}$, but we do have canonical coactions $V \cong V \otimes \mathbf{1}$ for each $V \in \mathcal{V}$. The final modification we need is that when $\mathcal{V}$ is non-cartesian, the functor $E \to D$ is replaced by a D-*coaction* on E. Of course, this only makes sense if D itself was a comonoid of some sort (which it is when $\mathcal{V} = \mathbf{Set}$).

Thus, a non-cartesian enriched prestack involves a *comonoidal* $\mathcal{V}$-category $\mathcal{B}$ whose objects form another comonoidal $\mathcal{V}$-category $\mathbb{1}_D$, and a $\mathbb{1}_D$-*comodule category* $\mathcal{E}$ equipped with a $\mathcal{B}$-action. In the rest of this chapter, we first make this definition precise, then give a Grothendieck construction that takes enriched prestacks over $\mathcal{B}$ and returns a $\mathcal{B}$-comodule category.

In order to show that the Grothendieck construction does indeed give a $\mathcal{V}$-enriched category, we pass to $\mathcal{V}$-*internal* categories, give an internal Grothendieck construction, show that it gives rise to a $\mathcal{V}$-internal category, then finally show that the enriched construction is a special case of the internal construction. All the material regarding internal categories may be found in [Won19].

By the end of the chapter, we can conclude that the Grothendieck construction generalizes to non-cartesian fully enriched and internal prestacks.

4.1 Preliminaries

Throughout, let $(\mathcal{V}, \otimes, \mathbf{1})$ be a cocomplete symmetric monoidal category. Let $\mathbf{0}$ denote the initial object of $\mathcal{V}$. Assume that $\otimes$ preserves coproducts, and $\mathbf{0} \otimes V \cong \mathbf{0} \cong V \otimes \mathbf{0}$ for all V. Assume also that equalizers preserve coproduct injections and coproduct decompositions, analogously to §2.2. Finally, assume that $\mathcal{V}$ is *regular* in the sense of [Agu97, Definition 2.1.1] (or see Definition 4.4.1 below).

Definition 4.1.1. For any set X, let $\mathbf{1}_X$ denote the free $\mathcal{V}$-object $\coprod_{x \in X} \mathbf{1}$. Let $\mathbb{1}_X$ denote the $\mathcal{V}$-category whose set of objects is X, and whose morphisms are given by

$$\mathbb{1}_X(x,y) = \begin{cases} \mathbf{1} & \text{if } x = y; \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Let $\mathbb{1}$ denote $\mathbb{1}_*$, the monoidal unit in $\mathcal{V}$-**Cat**.

Definition 4.1.2. Let $\mathcal{E}$ be a $\mathcal{V}$-category with set of objects X. Then there is a $\mathcal{V}$-functor

$$i\colon \mathbb{1}_X \to \mathcal{E}$$

which is the identity on objects, and whose action on hom-objects is

$$i_{x,y} = \begin{cases} 1_x & \text{if } x = y; \\ ! & \text{otherwise.} \end{cases}$$

where $1_x\colon \mathbf{1} \to \mathcal{E}(x,x)$ are the identities and ! is the unique map out of the initial object $\mathbf{0}$.

4.2 Enriched prestacks

Definition 4.2.1. A **comonoidal $\mathcal{V}$-category** is a comonoid in $\mathcal{V}$-**Cat**, or equivalently a **Comon**$(\mathcal{V})$-category.

Lemma 4.2.2. *For any set D, $\mathbb{1}_D$ is a comonoidal $\mathcal{V}$-category. If $\mathcal{B}$ is a comonoidal $\mathcal{V}$-category with object set D, then $i\colon \mathbb{1}_D \to \mathcal{B}$ is a comonoidal $\mathcal{V}$-functor.*

Definition 4.2.3. Let $\mathcal{B}$ be a comonoidal $\mathcal{V}$-category. A **right $\mathcal{B}$-comodule category** is a right $\mathcal{B}$-comodule in $\mathcal{V}$-**Cat**. Equivalently, a $\mathcal{V}$-category $\mathcal{E}$ along with a coaction $p\colon \mathcal{E} \to \mathcal{E} \otimes \mathcal{B}$.

Lemma 4.2.4. *A right $\mathcal{B}$-comodule category is equivalently a $\mathcal{V}$-category $\mathcal{E}$ along with:*

1. *a function $p\colon X \to D$ where $X = \mathbf{Ob}(\mathcal{E})$ and $D = \mathbf{Ob}(\mathcal{B})$*

2. *for all $x, y \in X$, a coaction $p_{x,y}\colon \mathcal{E}(x,y) \to \mathcal{E}(x,y) \otimes \mathcal{B}(px, py)$*

such that the following diagrams commute:

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "1_x"] \arrow[d, "\cong"] & \mathcal{E}(x,x) \arrow[d, "p_{x,x}"] \\
\mathbf{1} \otimes \mathbf{1} \arrow[r, "1_x \otimes 1_{px}"] & \mathcal{E}(x,x) \otimes \mathcal{B}(px,px)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{E}(x,y) \otimes \mathcal{E}(y,z) \arrow[rr, "\circ"] \arrow[d, "p_{x,y} \otimes p_{y,z}"] & & \mathcal{E}(x,z) \arrow[dd, "p_{x,z}"] \\
\mathcal{E}(x,y) \otimes \mathcal{B}(px,py) \otimes \mathcal{E}(y,z) \otimes \mathcal{B}(py,pz) \arrow[d, "1 \otimes \mathbb{x} 1"] & & \\
\mathcal{E}(x,y) \otimes \mathcal{E}(y,z) \otimes \mathcal{B}(px,py) \otimes \mathcal{B}(py,pz) \arrow[rr, "\circ \otimes \circ"] & & \mathcal{E}(x,z) \otimes \mathcal{B}(px,pz)
\end{tikzcd}$$

Lemma 4.2.5. *A right $\mathbb{1}_D$-comodule category is precisely a function $p\colon X \to D$ and a $\mathcal{V}$-category $\mathcal{E}$ with objects X such that $\mathcal{E}(x,y) = \mathbf{0}$ if $px \neq py$.*

Definition 4.2.6. Let $(p\colon X \to D, \mathcal{E})$ be a right $\mathbb{1}_D$-comodule category. For each $b \in D$, let $X(b) := p^{-1}(b)$ and let

$$\mathcal{E}_{X(b)} := \coprod_{u,v \in X(b)} \mathcal{E}(u,v).$$

More generally, for any $Y \subseteq X$, let

$$\mathcal{E}_Y := \coprod_{u,v \in Y} \mathcal{E}(u,v).$$

Remark 4.2.7. *Each $\mathbb{1}_D$ is a right comodule category over $\mathbb{1}$ via the unique map $!\colon D \to *$. Note that in this case, $D(*) = D$ and $(\mathbb{1}_D)_{D(*)} = (\mathbb{1}_D)_D = \mathbf{1}_D$.*

Definition 4.2.8. Let $\mathcal{B}$ be a comonoidal $\mathcal{V}$-category with object set D. A **(split) prestack over $\mathcal{B}$** consists of the following data:

1. A right $\mathbb{1}_D$-comodule category $(p\colon X \to D, \mathcal{E})$;

2. For each $b, c \in D$ and $y \in X(c)$, a comonoid map $f_{b;y}\colon \mathcal{B}(b,c) \to \mathbf{1}_{X(b)}$ inducing

$$f_{b,c}\colon \mathcal{B}(b,c) \otimes \mathbf{1}_{X(c)} \to \mathbf{1}_{X(b)}$$

such that the following diagrams commute:

$$
\begin{tikzcd}
\mathbf{1} \arrow[r, "1_b"] \arrow[d, "\cong"'] & \mathcal{B}(b,b) \arrow[d] \\
\mathbf{1}_y \arrow[r, hook] & \mathbf{1}_{X(b)}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B}(a,b) \otimes \mathcal{B}(b,c) \arrow[r] \arrow[d] & \mathcal{B}(a,b) \otimes \mathbf{1}_{X(b)} \arrow[d] \\
\mathcal{B}(a,c) \arrow[r] & \mathbf{1}_{X(a)}
\end{tikzcd}
$$

3. For each $b, c \in \mathcal{D}$ and $x, y \in X(c)$, a map $F_{b;x,y} \colon \mathcal{B}(b,c) \otimes \mathcal{E}(x,y) \to \mathcal{E}_{X(b)}$ inducing

$$F_{b,c} \colon \mathcal{B}(b,c) \otimes \mathcal{E}_{X(c)} \to \mathcal{E}_{X(b)}$$

such that the following diagrams commute:

$$
\begin{tikzcd}
\mathbf{1} \otimes \mathcal{E}(x,y) \arrow[r] \arrow[d, "\cong"'] & \mathcal{B}(b,b) \otimes \mathcal{E}(x,y) \arrow[d] \\
\mathcal{E}(x,y) \arrow[r, hook] & \mathcal{E}_{X(b)}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B}(a,b) \otimes \mathcal{B}(b,c) \otimes \mathcal{E}(x,y) \arrow[r] \arrow[d] & \mathcal{B}(a,b) \otimes \mathcal{E}_{X(b)} \arrow[d] \\
\mathcal{B}(a,c) \otimes \mathcal{E}(x,y) \arrow[r] & \mathcal{E}_{X(a)}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{B}(b,c) \otimes \mathcal{E}(x,y) \arrow[r, "F_{b;x,y}"] \arrow[d] & \mathcal{E}_{X(b)} \arrow[d, "s"] \\
\mathcal{B}(b,c) \otimes \mathcal{E}(x,y) \otimes \mathcal{B}(b,c) \arrow[r, "F_{b;x,y} \otimes f_{b;x}"] & \mathcal{E}_{X(b)} \otimes \mathbf{1}_{X(b)}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{B}(b,c) \otimes \mathcal{E}(x,y) \arrow[r, "F_{b;x,y}"] \arrow[d] & \mathcal{E}_{X(b)} \arrow[d, "t"] \\
\mathcal{B}(b,c) \otimes \mathcal{E}(x,y) \otimes \mathcal{B}(b,c) \arrow[r, "F_{b;x,y} \otimes f_{b;y}"] & \mathcal{E}_{X(b)} \otimes \mathbf{1}_{X(b)}
\end{tikzcd}
$$

The last two diagrams say that $f_{b,c}$ and $F_{b,c}$ are compatible with σ and τ.

4. The maps $f_{b,c}$ and $F_{b,c}$ are compatible with the unit and multiplication of $\mathcal{E}$:

$$
\begin{tikzcd}
\mathcal{B}(b,c) \otimes \mathbf{1} \arrow[r] \arrow[d] & \mathcal{B}(b,c) \otimes \mathcal{E}(x,x) \arrow[d] \\
\mathbf{1}_{X(b)} \arrow[r] & \mathcal{E}_{X(b)}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B}(b,c) \otimes \mathcal{E}(x,y) \otimes \mathcal{E}(y,z) \arrow[r] \arrow[d] & \mathcal{B}(b,c) \otimes \mathcal{E}(x,z) \arrow[d] \\
\mathcal{E}_{X(b)} \otimes_{\mathbf{1}_{X(b)}} \mathcal{E}_{X(b)} \arrow[r] & \mathcal{E}_{X(b)}
\end{tikzcd}
$$

where the last map is

$$\mathcal{E}_{X(b)} \otimes_{\mathbf{1}_{X(b)}} \mathcal{E}_{X(b)} \cong \coprod_{u,v,w \in X(b)} \mathcal{E}(u,v) \otimes \mathcal{E}(v,w) \to \mathcal{E}(u,v) \hookrightarrow \mathcal{E}_{X(b)}.$$

Remark 4.2.9. *Since all prestacks that we consider will be split, we will simply refer to them as prestacks. The definition of a prestack might seem to require a lot of data, but this is no more than the amount of data needed to define a functor* $F\colon B^{op} \to \mathbf{Cat}$ *that we gave at the start of this chapter.*

Remark 4.2.10. *Part of the complication in the above definition is that we have to consider individual hom-objects (e.g.* $\mathcal{E}(x,y)$*) as well as coproducts of these hom-objects (e.g.* $\mathcal{E}_{X(b)}$*). A simpler approach would be to just express everything in terms of* $\mathcal{E}_{X(b)}$ *rather than individual hom-objects. This amounts to defining prestacks* internally*, which we do in Definition 4.4.15.*

4.3 The Grothendieck construction

In this section, we describe the construction of a new category $\mathcal{E} \rtimes \mathcal{B}$ given a prestack $\mathcal{E}$ over $\mathcal{B}$. Recall that we have comonoid maps $f_{b;y}\colon \mathcal{B}(b,c) \to \mathbf{1}_{X(b)}$ for all $y \in X(c)$, which induce coactions $\mathcal{B}(b,c) \to \mathbf{1}_{X(b)} \otimes \mathcal{B}(b,c)$. Each $\mathcal{E}(x,u)$ with $x,u \in X(b)$ also has a coaction $\mathcal{E}(x,u) \xrightarrow{t} \mathcal{E}(x,u) \otimes \mathbf{1}_{X(b)}$.

Definition 4.3.1. For $\mathcal{E}$ a prestack over $\mathcal{B}$, let $\mathcal{E} \rtimes \mathcal{B}$ denote the $\mathcal{V}$-category with the same objects as $\mathcal{E}$, and hom-objects

$$\mathcal{E} \rtimes \mathcal{B}\,(x,y) := \left(\coprod_{u \in X(px)} \mathcal{E}(x,u) \right) \otimes_{\mathbf{1}_{X(px)}} \mathcal{B}(px,py).$$

Identities are given by

$$\mathbf{1} \cong \mathbf{1} \otimes \mathbf{1} \xrightarrow{1_x \otimes 1_{px}} \mathcal{E}(x,x) \otimes \mathcal{B}(px,px) \hookrightarrow \mathcal{E} \rtimes \mathcal{B}\,(x,x),$$

while multiplication is given by the composite in Figure 4.1.

Theorem 4.3.2. *The above definition does indeed give a* $\mathcal{V}$*-category* $\mathcal{E} \rtimes \mathcal{B}$*, which moreover is a right* $\mathcal{B}$*-comodule category.*

We will not prove this theorem directly – the notation is too cumbersome for that. Instead, we will develop a Grothendieck construction for *internal categories*, of which the

$$\begin{gathered}
\left(\coprod_{u\in X(px)}\mathcal{E}(x,u)\right)\otimes_{\mathbf{1}_{X(px)}}\mathcal{B}(px,py)\otimes\left(\coprod_{v\in X(py)}\mathcal{E}(y,v)\right)\otimes_{\mathbf{1}_{X(py)}}\mathcal{B}(py,pz)\\
\downarrow\\
\left(\coprod_{u\in X(px)}\mathcal{E}(x,u)\right)\otimes\mathcal{B}(px,py)\otimes\left(\coprod_{v\in X(py)}\mathcal{E}(y,v)\right)\otimes\mathcal{B}(py,pz)\\
\downarrow\\
\left(\coprod_{u\in X(px)}\mathcal{E}(x,u)\right)\otimes\mathcal{B}(px,py)\otimes\left(\coprod_{v\in X(py)}\mathcal{E}(y,v)\right)\otimes\mathcal{B}(px,py)\otimes\mathcal{B}(py,pz)\\
\downarrow\\
\left(\coprod_{u,w\in X(px)}\mathcal{E}(x,u)\otimes\mathcal{E}(u,w)\right)\otimes\mathcal{B}(px,py)\otimes\mathcal{B}(py,pz)\\
\downarrow\\
\left(\coprod_{w\in X(px)}\mathcal{E}(x,w)\right)\otimes_{\mathbf{1}_{X(px)}}\mathcal{B}(px,pz)
\end{gathered}$$

Figure 4.1: Composition in the enriched category $\mathcal{E}\rtimes\mathcal{B}$

above construction will be a special case. We will then show that a certain *coinvariant category* allows us to recover $\mathcal{E}$ from $\mathcal{E}\rtimes\mathcal{B}$.

4.4 Internal prestacks and the internal Grothendieck construction

Recall that the objects of an enriched category are sets, while each $\mathcal{C}(x,y)$ lives in some other monoidal category $\mathcal{V}$. An internal category takes this a step further: the objects and homomorphisms are both objects of $\mathcal{V}$, in a suitable sense. We will see in §4.5 that under suitable assumptions on $\mathcal{V}$, a $\mathcal{V}$-enriched category gives rise to an internal category in $\mathcal{V}$.

We start by recalling some definitions, leading up to the definition of a category internal to a regular monoidal category $(\mathcal{V},\otimes,\mathbf{1})$ from [Agu97]. We then define internal prestacks and a Grothendieck construction for them, all of which may be found in [Won19] (which has been included as the Appendix to this thesis).

Throughout, we assume that $(\mathcal{V},\otimes,\mathbf{1},\mathbb{x})$ is a symmetric monoidal category, where $\mathbb{x}$ denotes the symmetry. Importantly, $\mathcal{V}$ is not required to be cartesian i.e. the monoidal product $\otimes$ is not necessarily the cartesian product $\times$. We will further assume that $\mathcal{V}$ is *regular* in the following sense:

Definition 4.4.1 ([Agu97, Definition 2.1.1])**.** A monoidal category $(\mathcal{V}, \otimes, \mathbf{1})$ is **regular** if it has all equalizers, and $\otimes$ preserves them (in both variables). In other words, if $E \overset{\mathsf{eq}}{\rightarrowtail} X$ is the equalizer of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, then $A \otimes E \otimes B \xrightarrow{A \otimes \mathsf{eq} \otimes B} A \otimes X \otimes B$ is the equalizer of

$$A \otimes X \otimes B \underset{A \otimes g \otimes B}{\overset{A \otimes f \otimes B}{\rightrightarrows}} A \otimes Y \otimes B.$$

For C, D comonoids in $\mathcal{V}$, let ${}_C\mathbf{Comod}_D$ denote the category of (C, D)-comodules. We write ${}_C\mathbf{Comod} := {}_C\mathbf{Comod}_\mathbf{1}$ and $\mathbf{Comod}_D := {}_\mathbf{1}\mathbf{Comod}_D$. The maps in ${}_C\mathbf{Comod}_D$ are comodule maps respecting both the C and D coactions. More generally, we have:

Definition 4.4.2. Let $f\colon C \to D$ be a comonoid map, $M \in \mathbf{Comod}_C$ and $N \in \mathbf{Comod}_D$.

A **(comodule) map over** f is a map $\varphi\colon M \to N$ such that the diagram on the left commutes, where ρ denotes the respective right coactions.

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\rho"'] & N \arrow[d, "\rho"] \\ M \otimes C \arrow[r, "\varphi \otimes f"] & N \otimes D \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, dotted, "\rho"'] & N \arrow[d, dotted, "\rho"] \\ C \arrow[r, "f"] & D \end{tikzcd}$$

We use the diagram on the right as an abbreviation of the diagram on the left. If $C = D$ and $f = 1_C$, we say that this is a **map over** C. We may similarly define maps over f for left comodules. For bicomodules, we may define maps over (f, g), or simply maps over f if $g = f$. Thus, maps in $\mathbf{Comod}_C$, ${}_C\mathbf{Comod}$ and ${}_C\mathbf{Comod}_C$ are maps over C.

Remark 4.4.3. *A map $\varphi\colon M \to N$ over $f\colon C \to D$ is equivalently a D-comodule map $f_*M \to N$, where f_* is the corestriction along f.*

Definition 4.4.4. Let B, C, D be comonoids, and let $M \in {}_B\mathbf{Comod}_C$ and $N \in {}_C\mathbf{Comod}_D$. The **cotensor over** C of M and N is the equalizer:

$$M \underset{C}{\boxtimes} N \rightarrowtail M \otimes N \underset{M \otimes \lambda_N}{\overset{\rho_M \otimes N}{\rightrightarrows}} M \otimes C \otimes N$$

Theorem 4.4.5 ([Agu97, Theorem 2.2.1])**.** *For C a comonoid in $\mathcal{V}$, $({}_C\mathbf{Comod}_C, \underset{C}{\boxtimes}, C)$ is a monoidal category.*

Definition 4.4.6 ([Agu97, Definition 2.3.1])**.** A $\mathcal{V}$**-internal category** consists of a comonoid C in $\mathcal{V}$ and a monoid A in the category of (C, C)-bicomodules $({}_C\mathbf{Comod}_C, \underset{C}{\boxtimes}, C)$.

In detail, an internal category is a tuple $\mathcal{A} = (C, A, d, e, \sigma, \tau, u, m)$ with

1. a *comonoid of objects* $C \in \mathbf{Comon}(\mathcal{V})$, with comultiplication $d\colon C \to C \otimes C$ and counit $e\colon C \to \mathbf{1}$;

2. a *comodule of maps* $A \in {}_C\mathbf{Comod}_C$, with coactions[3] $\sigma\colon A \to C \otimes A$ and $\tau\colon A \to A \otimes C$;

3. and *identity* and *composition* comodule maps

$$\begin{tikzcd} & C \arrow[dl, dotted, "d"'] \arrow[dr, dotted, "d"] \arrow[dd, "u"] & \\ C & & C \\ & A \arrow[ul, dotted, "\sigma"] \arrow[ur, dotted, "\tau"'] & \end{tikzcd} \qquad \begin{tikzcd} & A \underset{C}{\boxtimes} A \arrow[dl, dotted, "\sigma"'] \arrow[dr, dotted, "\tau"] \arrow[dd, "m"] & \\ C & & C \\ & A \arrow[ul, dotted, "\sigma"] \arrow[ur, dotted, "\tau"'] & \end{tikzcd}$$

   satisfying associativity and unitality.

For brevity, we will sometimes refer to an internal category $\mathcal{A}$ using subtuples such as (C, A).

Remark 4.4.7. *The definition of an internal category does not require the comonoid of objects C to be cocommutative. However, without cocommutativity, it is not possible to define internal prestacks or the internal Grothendieck construction.*

Definition 4.4.8 ([Agu97, Definition 4.1.1])**.** Let $\mathcal{A} = (C, A)$ and $\mathcal{B} = (D, B)$ be internal categories in $\mathcal{V}$. An **internal functor** from $\mathcal{A}$ to $\mathcal{B}$ is a tuple (f, φ) where $f\colon C \to D$ is a

[3] σ for ‘source’ and τ for ‘target’.

comonoid map and $\varphi\colon A \to B$ is a map such that the following diagrams commute:

$$
\begin{tikzcd}
C \arrow[d, "f"'] & A \arrow[l, dotted, "\sigma"'] \arrow[r, dotted, "\tau"] \arrow[d, "\varphi"] & C \arrow[d, "f"] \\
D & B \arrow[l, dotted, "\sigma"'] \arrow[r, dotted, "\tau"] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "u"'] & D \arrow[d, "u"] \\
A \arrow[r, "\varphi"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \underset{C}{\boxtimes} A \arrow[r, "\varphi \underset{f}{\boxtimes} \varphi"] \arrow[d, "m"'] & B \underset{D}{\boxtimes} B \arrow[d, "m"] \\
A \arrow[r, "\varphi"] & B
\end{tikzcd}
$$

Definition 4.4.9. Let $\mathbf{Cat}(\mathcal{V})$ denote the category of internal categories and functors.

Remark 4.4.10. *It is also possible to define internal transformations between internal functors, making $\mathbf{Cat}(\mathcal{V})$ a 2-category, but we will not need the 2-category structure in this paper.*

Recall that if $\mathcal{V}$ is a symmetric monoidal category, its category of comonoids $\mathbf{Comon}(\mathcal{V})$ is also symmetric monoidal, with the same braiding and monoidal product. A similar result holds for internal categories.

Proposition 4.4.11 ([Agu97, §7.1])**.** *$\mathbf{Cat}(\mathcal{V})$ is a monoidal category, with product*

$$(C, A) \otimes (D, B) := (C \otimes D, A \otimes B)$$

and unit $\mathbb{1} := (\mathbf{1}, \mathbf{1})$.

Definition 4.4.12. A **comonoidal internal category** is a comonoid in $\mathbf{Cat}(\mathcal{V})$.

Proposition 4.4.13 (A.3.9)**.** *Let $\mathcal{B} = (D, B, d, e, \sigma, \tau, u, m)$ be a comonoidal internal category. Then:*

1. *D is cocommutative (i.e. d and e are comonoid maps);*

2. *B is a comonoid, and σ and τ are comonoid maps (hence are induced by comonoid maps $s\colon B \to D$ and $t\colon B \to D$);*

3. *σ, τ and δ (the comultiplication of B) are maps over d:*

$$
\begin{tikzcd}
D \arrow[d, "d"'] & B \arrow[l, dotted, "\sigma"'] \arrow[r, dotted, "\tau"] \arrow[d, "\delta"] & D \arrow[d, "d"] \\
D \otimes D & B \otimes B \arrow[l, dotted, "\sigma\otimes\sigma"'] \arrow[r, dotted, "\tau\otimes\tau"] & D \otimes D
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \arrow[r, "\sigma"] \arrow[d, dotted, "\sigma"'] & D \otimes B \arrow[d, dotted, "d\otimes\sigma"] \\
D \arrow[r, "d"] & D \otimes D
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\tau"] \arrow[d, dotted, "\tau"'] & B \otimes D \arrow[d, dotted, "\tau\otimes d"] \\
D \arrow[r, "d"] & D \otimes D
\end{tikzcd}
$$

4. *$B \underset{D}{\boxtimes} B$ is a comonoid, and u and m are comonoid maps.*

Definition 4.4.14. Let $\mathcal{B} = (D, B)$ be a comonoidal internal category. A **right $\mathcal{B}$-comodule category** is a right $\mathcal{B}$-comodule in $\mathbf{Cat}(\mathcal{V})$.

In detail, this is the data of an internal category $\mathcal{A} = (C, A)$ along with:

1. a D-coaction $p\colon C \to C \otimes D$ that is also a comonoid map (by Lemma A.2.12, this coaction is thus induced by a comonoid map $q\colon C \to D$);

2. a B-coaction $\pi\colon A \to A \otimes B$ that is also a map over p;

3. such that $(p, \pi)\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$ is an internal functor.

We henceforth refer to these as simply $\mathcal{B}$-comodule categories or $\mathcal{B}$-comodules.

Recall that if $\mathcal{B} = (D, B)$ is comonoidal, then so is the discrete category $\mathcal{D} = (D, D)$.

Definition 4.4.15. Let $\mathcal{B} = (D, B)$ be a comonoidal internal category and $\mathcal{D} = (D, D)$ its subcategory of objects. A **prestack over $\mathcal{B}$** (or a **$\mathcal{B}$-module category**) consists of:

0. An internal category $\mathcal{A} = (C, A)$ with C cocommutative;

1. A coaction $(p, \pi)\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{D}$ (so $\mathcal{A}$ is a right $\mathcal{D}$-comodule category);

2. A comonoid map $f\colon B \underset{D}{\boxtimes} C \to C$ satisfying:

$$
\begin{tikzcd}
B \underset{D}{\boxtimes} C \arrow[r, "f"] \arrow[d, dotted, "\sigma"'] & C \arrow[d, dotted, "\mathbb{x}p"] \\
D \arrow[r, equal] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
D \underset{D}{\boxtimes} C \arrow[r, "u \underset{D}{\boxtimes} C"] \arrow[d, "\cong"'] & B \underset{D}{\boxtimes} C \arrow[d, "f"] \\
C \arrow[r, equal] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
B \underset{D}{\boxtimes} B \underset{D}{\boxtimes} C \arrow[r, "B \underset{D}{\boxtimes} f"] \arrow[d, "m \underset{D}{\boxtimes} C"'] & B \underset{D}{\boxtimes} C \arrow[d, "f"] \\
B \underset{D}{\boxtimes} C \arrow[r, "f"] & C
\end{tikzcd}
$$

3. A map $\varphi\colon B \underset{D}{\boxtimes} A \to A$ satisfying:

$$\begin{tikzcd}
D \underset{D}{\boxtimes} A \arrow[r, "u \underset{D}{\boxtimes} A"] \arrow[d, "\cong"'] & B \underset{D}{\boxtimes} A \arrow[d, "\varphi"] \\
A \arrow[r, equal] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \underset{D}{\boxtimes} B \underset{D}{\boxtimes} A \arrow[r, "B \underset{D}{\boxtimes} \varphi"] \arrow[d, "m \underset{D}{\boxtimes} A"'] & B \underset{D}{\boxtimes} A \arrow[d, "\varphi"] \\
B \underset{D}{\boxtimes} A \arrow[r, "\varphi"] & A
\end{tikzcd}$$

$$\begin{tikzcd}
B \underset{D}{\boxtimes} C \arrow[d, "f"'] & B \underset{D}{\boxtimes} A \arrow[l, dotted, "\delta \underset{d}{\boxtimes} \sigma"'] \arrow[r, dotted, "\delta \underset{d}{\boxtimes} \tau"] \arrow[d, "\varphi"] & B \underset{D}{\boxtimes} C \arrow[d, "f"] \\
C & A \arrow[l, dotted, "\sigma"'] \arrow[r, dotted, "\tau"] & C
\end{tikzcd}$$

$$\begin{tikzcd}
B \underset{D}{\boxtimes} C \arrow[d, "f"'] & B \underset{D}{\boxtimes} A \arrow[l, dotted, "\delta \underset{d}{\boxtimes} \sigma"'] \arrow[r, dotted, "\delta \underset{d}{\boxtimes} \tau"] \arrow[d, "\varphi"] & B \underset{D}{\boxtimes} C \arrow[d, "f"] \\
C & A \arrow[l, dotted, "\sigma"'] \arrow[r, dotted, "\tau"] & C
\end{tikzcd}$$

4. f and φ further satisfy:

$$\begin{tikzcd}
B \underset{D}{\boxtimes} C \arrow[r, "B \underset{D}{\boxtimes} e"] \arrow[d, "f"'] & B \underset{D}{\boxtimes} A \arrow[d, "\varphi"] \\
C \arrow[r, "e"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \underset{D}{\boxtimes} (A \underset{C}{\boxtimes} A) \arrow[r, "B \underset{D}{\boxtimes} m"] \arrow[d, "\varphi_2"'] & B \underset{D}{\boxtimes} A \arrow[d, "\varphi"] \\
A \underset{C}{\boxtimes} A \arrow[r] & A
\end{tikzcd}$$

Remark 4.4.16. *Compare the above Definition item-wise with Definition 4.2.8.*

Let $\mathcal{A} = (C, A)$ be a prestack over $\mathcal{B} = (D, B)$, with actions f and φ as above.

We make $B \underset{D}{\boxtimes} C$ an object of ${}_C\mathbf{Comod}_C$, with left coaction induced by the comonoid map $f\colon B \underset{D}{\boxtimes} C \to C$, and right coaction induced by the comonoid map $t\colon B \to D$ (which induces the coaction τ),

$$f_*\Delta\colon B \underset{D}{\boxtimes} C \longrightarrow (B \underset{D}{\boxtimes} C) \otimes (B \underset{D}{\boxtimes} C) \xrightarrow{f \otimes (B \underset{D}{\boxtimes} C)} C \otimes (B \underset{D}{\boxtimes} C)$$

$$t_*\Delta\colon B \underset{D}{\boxtimes} C \longrightarrow (B \underset{D}{\boxtimes} C) \otimes (B \underset{D}{\boxtimes} C) \xrightarrow{(B \underset{D}{\boxtimes} C) \otimes (t \underset{D}{\boxtimes} C)} (B \underset{D}{\boxtimes} C) \otimes (D \underset{D}{\boxtimes} C) \cong (B \underset{D}{\boxtimes} C) \otimes C$$

where $\Delta = \delta \underset{d_D}{\boxtimes} d_C$ is the comultiplication of $B \underset{D}{\boxtimes} C$. We also have a right B-coaction induced by the comonoid map $q\colon C \to D$ (which induces the coaction p):

$$q_*\Delta\colon B \underset{D}{\boxtimes} C \longrightarrow (B \underset{D}{\boxtimes} C) \otimes (B \underset{D}{\boxtimes} C) \xrightarrow{(B \underset{D}{\boxtimes} C)\otimes(B \underset{D}{\boxtimes} q)} (B \underset{D}{\boxtimes} C) \otimes (B \underset{D}{\boxtimes} D) \cong (B \underset{D}{\boxtimes} C) \otimes B$$

We are now in a position to define smash products – or the Grothendieck construction – for internal prestacks.

Theorem 4.4.17 (A.5.2)**.** *Let $(f, \varphi)\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$ be an internal prestack. There is an internal category*

$$\mathcal{A} \rtimes \mathcal{B} := \big(C, A \underset{C}{\boxtimes} (B \underset{D}{\boxtimes} C)\big),$$

which we call the ***smash product*** *of $\mathcal{A}$ with $\mathcal{B}$. Further, $\mathcal{A} \rtimes \mathcal{B}$ has the structure of a $\mathcal{B}$-comodule category, with coaction induced by $q_*\Delta$.*

Although we have not defined what a 'cartesian fibered right $\mathcal{B}$-comodule category' should be, we can still verify that the fibers of $\mathcal{A} \rtimes \mathcal{B}$ allow us to recover our original prestack $\mathcal{A}$. We first begin by defining the fibers of any right $\mathcal{B}$-comodule category.

Definition 4.4.18. Let $\mathcal{A}$ be a right $\mathcal{B}$-comodule category, with coaction functor $p\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$. The **coinvariant category** is the coinduction $\mathcal{A}$ along $(D, u)\colon \mathcal{D} \to \mathcal{B}$:

$$\begin{tikzcd}
\mathcal{A} \underset{\mathcal{B}}{\boxtimes} \mathcal{D} \arrow[r] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A} \arrow[d, dotted] \\
\mathcal{D} \arrow[r, "{(D,u)}"] & \mathcal{B}
\end{tikzcd}$$

Equivalently, $\mathcal{A} \underset{\mathcal{B}}{\boxtimes} \mathcal{D}$ is given by the equalizer:

$$\begin{tikzcd}
\mathcal{A} \underset{\mathcal{B}}{\boxtimes} \mathcal{D} \arrow[r, tail] & \mathcal{A} \otimes \mathcal{D} \arrow[r, shift left, "(p,\pi)\otimes\mathcal{D}"] \arrow[r, shift right, "\mathcal{A}\otimes\left((D,u)\otimes\mathcal{D}\right)(d,\delta)"'] & \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{D}
\end{tikzcd}$$

It is clear from the definition that the coinvariant category $\mathcal{A} \underset{\mathcal{B}}{\boxtimes} \mathcal{D}$ is a $\mathcal{D}$-comodule category with objects $C \cong C \underset{D}{\boxtimes} D$ and morphisms $A \underset{B}{\boxtimes} D$.

Given an arbitrary $\mathcal{B}$-comodule category, it is unlikely that its coinvariant category has the structure of a prestack over $\mathcal{D}$. However, when the $\mathcal{B}$-comodule category is of the form $\mathcal{A} \rtimes \mathcal{B}$ for a prestack $\mathcal{A}$, we have:

Theorem 4.4.19 (A.6.4)**.** *Let $\mathcal{A}$ be a prestack over $\mathcal{B}$ and let $\mathcal{A} \rtimes \mathcal{B}$ be the corresponding right $\mathcal{B}$-comodule category. Then the coinvariant category $(\mathcal{A} \rtimes \mathcal{B}) \underset{\mathcal{B}}{\boxtimes} \mathcal{D}$ is a prestack over $\mathcal{B}$, which is moreover isomorphic to $\mathcal{A}$.*

4.5 Internalization

Finally, we return to the topic at the start of this chapter: the internalization of enriched comodule categories and enriched prestacks.

Definition 4.5.1. Let $\mathcal{E}$ be a $\mathcal{V}$-category with set of objects X. The **internalization** of $\mathcal{E}$ is the internal category $\mathcal{E}^{\in} := (\mathbb{1}_X, \mathcal{E}_X)$.

Remark 4.5.2. *The process of internalizing an enriched category, and the properties of $\mathcal{V}$ that are required for this to work, are given in* [CFP17]*. However, a major assumption there is that $\mathcal{V}$ is cartesian monoidal. The methods of this section are inspired by that paper, but are suitable for non-cartesian $\mathcal{V}$. The requirement that $\mathcal{V}$ be extensive is replaced by the requirement that equalizers preserve coproducts.*

The following result is straightforward:

Proposition 4.5.3. *Let $\mathcal{E}, \mathcal{B}$ be a $\mathcal{V}$-categories, with sets of objects X, D, respectively. Then:*

1. *If $\mathcal{B}$ is comonoidal, so is $\mathcal{B}^{\in}$.*

2. *If $\mathcal{E}$ is a $\mathcal{B}$-comodule $\mathcal{V}$-category, then $\mathcal{E}^{\in}$ is a $\mathcal{B}^{\in}$-comodule internal category. Moreoever, the coaction on objects is given by a comonoid map $\mathbf{1}_X \to \mathbf{1}_D$.*

3. *$\mathbb{1}_D^{\in}$ is the discrete internal category $\mathcal{D} = (\mathbf{1}_D, \mathbf{1}_D)$.*

In light of the previous section, we then have:

Theorem 4.5.4. *If $\mathcal{E}$ is an enriched prestack over $\mathcal{B}$, then:*

1. *$\mathcal{E}^{\in}$ is an internal prestack over $\mathcal{B}^{\in}$;*

2. *$(\mathcal{E} \rtimes \mathcal{B})^{\in} = \mathcal{E}^{\in} \rtimes \mathcal{B}^{\in}$.*

Proof. For Item 1, compare Definitions 4.2.8 and 4.4.15, in light of the previous proposition. For Item 2, compare Definitions 4.3.1 and 4.4.17. In both cases, the descriptions match up precisely because equalizers are assumed to preserve coproducts. □

As we have remarked, this correspondence allows us to prove Theorem 4.3.2. Further, Theorem 4.4.17 shows that the enriched Grothendieck construction on prestacks has an inverse given by taking coinvariants.

EPILOGUE

We end with some questions that the author has not had time to consider in this thesis.

Fibrations across a 2-functor

The $\mathcal{V}$-enriched and $\mathcal{V}$-internal prestacks in the last chapter are in fact *split* prestacks. An obvious generalization would be to define non-split prestacks over $\mathcal{B}$, and extend the Grothendieck construction to them. One can also define 1- and 2-cells between these prestacks, to obtain a 2-category of prestacks $\mathbf{Pst}(\mathcal{B})$ over $\mathcal{B}$. Similary, we can define 1- and 2-cells between $\mathcal{B}$-comodule categories. We can then extend the Grothendieck construction to a 2-functor from the category of prestacks over $\mathcal{B}$ to the category of $\mathcal{B}$-comodule categories. When $\mathcal{V} = \mathbf{Set}$, this would recover the classical result that the Grothendieck construction is a 2-functor from the category of prestacks $[B^{\mathsf{op}}, \mathbf{Cat}]$ to the *slice* category $\mathbf{Cat}_{/B}$.

In the classical setting, we know that this 2-functor restricts to a 2-equivalence between $[B^{\mathsf{op}}, \mathbf{Cat}]$ and the category of *fibrations* $\mathbf{Fib}_{/B}$ over B. We expect the analogous result to also hold in the non-cartesian setting. However, we need a replacement for 'fibrations over $\mathcal{B}$', since the image of a prestack under the Grothendieck construction is not even a functor, but a $\mathcal{B}$-comodule category!

By §1.7, an ordinary fibration is in fact a cod-fibration, where $\mathsf{cod}\colon \mathbf{Arr}(\mathbf{Cat}) \to \mathbf{Cat}$. This motivates the following definition:

Definition. Let $\mathbf{Comod}(\mathbf{Cat}(\mathcal{V}))$ denote the 2-category whose objects are pairs $(\mathcal{E}, \mathcal{B})$ where $\mathcal{B} \in \mathbf{Comon}(\mathbf{Cat}(\mathcal{V}))$ and $\mathcal{E}$ is a $\mathcal{B}$-comodule category.

Let $\Phi\colon \mathbf{Comod}(\mathbf{Cat}(\mathcal{V})) \to \mathbf{Comon}(\mathbf{Cat}(\mathcal{V}))$ be the functor which sends $(\mathcal{E}, \mathcal{B})$ to $\mathcal{B}$.

A **fibered $\mathcal{B}$-comodule category** is a Φ-fibration.

We may similarly define cartesian functors between fibered $\mathcal{B}$-comodule categories, and 2-cells between them, yielding a 2-category $\mathbf{Fib}_{/\mathcal{B}}$ of fibered $\mathcal{B}$-comodule categories.

Conjecture. *The Grothendieck construction extends to a 2-equivalence:*

$$\mathbf{Pst}(\mathcal{B}) \cong \mathbf{Fib}_{/\mathcal{B}}.$$

We conjecture also that fibered B-graded $\mathcal{V}$-categories are precisely Φ-fibrations, where $\Phi\colon \mathcal{V}\text{-}\mathbf{Cat}_{\mathbf{Cat}} \to \mathbf{Cat}$, where $\mathcal{V}\text{-}\mathbf{Cat}_{\mathbf{Cat}}$ is the category of $\mathcal{V}$-categories graded by ordinary categories, and Φ projects to the grading category. A similar conjecture may be made for B-parametrized $\mathcal{V}$-categories. We expect that the Grothendieck construction is a 2-equivalence in all these settings.

Smash products for quantum categories

In the last chapter, we saw that the comonoids of objects of both the comonoidal internal category $\mathcal{B}$ and a prestack $\mathcal{E}$ are cocommutative. The cocommutativity was crucial for the Grothendieck construction, so it seems unlikely that we can do away with it entirely.

But there is hope: in [Nik00], a smash product for weak Hopf algebras was given. Instead of assuming that the comonoid of objects is cocommutative, they assume that it is *separable Frobenius*[4]. One would like to compare our smash product construction with theirs.

The theory of duoidal categories provides an abstract setting in which to make this comparison, since our comonoidal internal categories and their weak bialgebras and both examples of bimonoids in a duoidal category. This motivates the following:

Question. *Is there a smash product for B-comodule monoids, where B is a bimonoid in a duoidal category?*

Weak bialgebras are themselves special cases of the *quantum categories* characterized in [Chi11]. In a quantum category, the (co)monoid of objects can be an arbitrary (co)monoid that is neither (co)commutative nor separable Frobenius. We may thus ask:

[4] In fact, they work in a dual framework, with separable Frobenius *algebras* of objects. But separable Frobenius k-algebras are equivalently separable Frobenius k-*co*algebras

Question. *Is there a smash product construction that works for quantum categories?*

Finally, this thesis demonstrates that the Grothendieck construction may be generalized to the non-cartesian setting via the use of comonoids and comodules. It is natural to ask:

Question. *What other categorical constructions may be generalized to the non-cartesian setting, through the use of comonoids and comodules?*

Here is a possible approach to obtaining such generalizations:

1. Identify a categorical notion that may be expressed in terms of the codomain functor $\mathsf{cod}\colon \mathbf{Arr}(\mathbf{Cat}) \to \mathbf{Cat}$.
2. Express this in terms of an arbitrary 2-functor.
3. Specialize to the case of the 2-functor $\Phi\colon \mathbf{Comod}(\mathbf{Cat}(\mathcal{V})) \to \mathbf{Comon}(\mathbf{Cat}(\mathcal{V}))$.

Appendix A

SMASH PRODUCTS FOR NON-CARTESIAN INTERNAL PRESTACKS

A.1 Introduction

Given a group G acting on another group A via a homomorphism $\varphi\colon G \to \mathsf{Aut}(A)$, we may form the semi-direct product $A \rtimes_\varphi G$, or simply $A \rtimes G$. There is also a projection $\pi\colon A \rtimes G \twoheadrightarrow G$, and taking the kernel of π allows us to recover A. This paper synthesizes two classical generalizations of the semi-direct product.

The first is the *Grothendieck construction* [Gro61]. Instead of a group G acting on another group N, we now have a category $\mathcal{B}$ acting on a family of other categories $\{\mathcal{A}_b\}_{b\in\mathcal{B}}$ via a functor $\varphi\colon \mathcal{B}^{\mathsf{op}} \to \mathbf{Cat}$ sending b to $\mathcal{A}_b$. Such functors are also known as *(split) prestacks.* The Grothendieck construction then takes a split prestack and returns a *fibration* $\pi\colon \mathcal{A} \rtimes \mathcal{B} \to \mathcal{B}$ whose *fibers* allow us to recover the categories $\mathcal{A}_b$ that we started with.

The second generalization is the *smash product* construction [CM84]. This time, instead of a group acting on another group, we start with a group G acting on a k-algebra A. We may then form the smash product $A \rtimes G$ (or $A\#G$), which is another k-algebra. Instead of an algebra homomorphism $A \rtimes G \to G$, we have a G-grading on $A \rtimes G$ whose identity component is the original algebra A; equivalently, we have a kG-comodule algebra $A \rtimes G$ whose *coinvariant* subalgebra is A. More generally, given a Hopf algebra H acting on another algebra A (i.e. a H-module algebra), we may form the smash product $A \rtimes H$ which is a H-comodule algebra, and taking the coinvariant subalgebra of $A \rtimes H$ allows us to recover A [BM85, VdB84]. Although the antipode of the Hopf algebra H is used in the definition of the smash product, it is not actually *required*: we may in fact form the smash product $A \rtimes B$ for a bialgebra B acting on A, which coincides with the usual smash product if B is a Hopf algebra.

The starting point of this paper is the observation that categories $\mathcal{B}$ and bialgebras B are both examples of *internal categories* [Agu97]. In fact, they are *comonoidal* internal categories (which we define in §A.3), and we may thus define *comodule categories* and *prestacks* over them (§A.4). In §A.5, we define *smash products* of prestacks, and in §A.6 we show that taking coinvariants allows us to recover the original prestack. Some necessary lemmas regarding comonoids and comodules will be provided in §A.2.

The reader might find many of the statements and proofs in this paper rather technical and unmotivated. This is because they were developed in the following manner:

1. Identify a notion for ordinary categories (i.e. categories internal to **Set**);

2. Define this notion for categories internal to an arbitrary monoidal category $\mathcal{V}$, in the language of comonoids and comodules;

3. Prove the necessary statements using *string diagrams*;

4. Transfer this proof into commutative diagrams.

Consequently, the results and proofs that end up in this paper are already one step removed from the original *method* of proof (string diagrams), and three steps removed from the original *motivation* (ordinary category theory)! Future versions of this paper might attempt to better motivate the results, and present them using string diagrams. For now, we encourage the reader to keep the original categorical constructions in mind and work out the statements and proofs for themselves in string diagrams.

A.2 Comonoids and comodules

In this section, we give a quick overview of comonoids and comodules. Throughout, we assume that $(\mathcal{V}, \otimes, \mathbf{1}, \mathbb{x})$ is a symmetric monoidal category, where $\mathbb{x}$ denotes the symmetry. We will further assume that $\mathcal{V}$ is *regular* in the following sense:

Definition A.2.1 ([Agu97, Definition 2.1.1])**.** A monoidal category $(\mathcal{V}, \otimes, \mathbf{1})$ is **regular** if it has all equalizers, and $\otimes$ preserves them (in both variables). In other words, if $E \overset{\mathsf{eq}}{\rightarrowtail} X$ is the equalizer of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, then $A \otimes E \otimes B \xrightarrow{A \otimes \mathsf{eq} \otimes B} A \otimes X \otimes B$ is the equalizer of

$$A \otimes X \otimes B \underset{A \otimes g \otimes B}{\overset{A \otimes f \otimes B}{\rightrightarrows}} A \otimes Y \otimes B.$$

For C, D comonoids in $\mathcal{V}$, let ${}_C\mathbf{Comod}_D$ denote the category of left C-, right D-bicomodules, or (C, D)-comodules. When either C or D is the monoidal unit $\mathbf{1}$, we write ${}_C\mathbf{Comod} := {}_C\mathbf{Comod}_\mathbf{1}$ and $\mathbf{Comod}_D := {}_1\mathbf{Comod}_D$. The maps in ${}_C\mathbf{Comod}_D$ are comodule maps respecting both the C and D coactions. More generally, we have:

Definition A.2.2. Let $f \colon C \to D$ be a comonoid map, $M \in \mathbf{Comod}_C$ and $N \in \mathbf{Comod}_D$.

A **(comodule) map over** f is a map $\varphi \colon M \to N$ such that the diagram on the left commutes, where ρ denotes the respective right coactions.

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\rho"'] & N \arrow[d, "\rho"] \\ M \otimes C \arrow[r, "\varphi \otimes f"] & N \otimes D \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, dotted, "\rho"'] & N \arrow[d, dotted, "\rho"] \\ C \arrow[r, "f"] & D \end{tikzcd}$$

We use the diagram on the right as an abbreviation of the diagram on the left. In particular, the dotted arrows indicate that M has a C-coaction and N has a D-coaction. In the special case where $C = D$ and $f = 1_C$, we say that φ is a **map over** C. We may similarly define maps over f for left comodules. For bicomodules, we may define maps over (f, g), or simply maps over f if $g = f$. Thus, maps in $\mathbf{Comod}_C$, ${}_C\mathbf{Comod}$ and ${}_C\mathbf{Comod}_C$ are maps over C.

Lemma A.2.3. *Let $f \colon C \to D$ be a comonoid map, $M \in \mathbf{Comod}_C$ and $N \in \mathbf{Comod}_D$.*

A map $\varphi \colon M \to N$ over $f \colon C \to D$ is equivalently a D-comodule map $f_* M \to N$, where f_* is the corestriction along f.

Definition A.2.4. Let B, C, D be comonoids, and let $M \in {}_B\mathbf{Comod}_C$ and $N \in {}_C\mathbf{Comod}_D$. The **cotensor over** C of M and N is the equalizer:

$$M \underset{C}{\boxtimes} N \rightarrowtail M \otimes N \underset{M \otimes \lambda_N}{\overset{\rho_M \otimes N}{\rightrightarrows}} M \otimes C \otimes N$$

Proposition A.2.5 ([Agu97, Proposition 2.2.1]). *When $\mathcal{V}$ is a regular, $M \underset{C}{\boxtimes} N$ has a right D-coaction induced by the coaction on N:*

$$\begin{tikzcd}
M \underset{C}{\boxtimes} N \arrow[r, tail] \arrow[d, dashed, "M\boxtimes\rho_N"'] & M \otimes N \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "M\otimes\rho_N"'] & M \otimes C \otimes N \arrow[d, "M\otimes C\otimes\rho_N"] \\
M \underset{C}{\boxtimes} N \otimes D \arrow[r, tail] & M \otimes N \otimes D \arrow[r, shift left] \arrow[r, shift right] & M \otimes C \otimes N \otimes D
\end{tikzcd}$$

Similarly, $M \underset{C}{\boxtimes} N$ has a left B-coaction making $M \underset{C}{\boxtimes} N$ an object of ${}_B\mathbf{Comod}_D$.

Lemma A.2.6 ([Agu97, Lemma 7.1.1]). *Let $M \in {}_B\mathbf{Comod}_C, N \in {}_C\mathbf{Comod}_D, M' \in {}_{B'}\mathbf{Comod}_{C'}$ and $N' \in {}_{C'}\mathbf{Comod}_{D'}$.*

$$\begin{tikzcd}[column sep=small]
& M \arrow[dl, dotted] \arrow[dr, dotted] & & N \arrow[dl, dotted] \arrow[dr, dotted] & & & M' \arrow[dl, dotted] \arrow[dr, dotted] & & N' \arrow[dl, dotted] \arrow[dr, dotted] & \\
B & & C & & D & B' & & C' & & D'
\end{tikzcd}$$

Then there is a canonical isomorphism in ${}_{B\otimes B'}\mathbf{Comod}_{D\otimes D'}$

$$(M \underset{C}{\boxtimes} N) \otimes (M' \underset{C'}{\boxtimes} N') \cong (M \otimes M') \underset{C\otimes C'}{\boxtimes} (N \otimes N')$$

natural in M, N, M' and N'.

It is further shown in [Agu97, §2.2] that cotensoring extends to a functor

$$- \underset{C}{\boxtimes} -\colon {}_B\mathbf{Comod}_C \times {}_C\mathbf{Comod}_D \to {}_B\mathbf{Comod}_D.$$

In particular, if $\varphi\colon M \to M'$ and $\psi\colon N \to N'$ are maps in ${}_B\mathbf{Comod}_C$ and ${}_C\mathbf{Comod}_D$, there is a ${}_B\mathbf{Comod}_D$-map $\varphi \underset{C}{\boxtimes} \psi\colon M \underset{C}{\boxtimes} N \to M' \underset{C}{\boxtimes} N'$. More generally, we have:

Proposition A.2.7. *Let $f\colon B \to B', g\colon C \to C'$ and $h\colon D \to D'$ be comonoid maps, and let $M \in {}_B\mathbf{Comod}_C, N \in {}_C\mathbf{Comod}_D, M' \in {}_{B'}\mathbf{Comod}_{C'}$ and $N' \in {}_{C'}\mathbf{Comod}_{D'}$, and suppose we have $\varphi\colon M \to M'$ over (f, g) and $\psi\colon N \to N'$ over (g, h).*

$$\begin{tikzcd}[column sep=small]
& M \arrow[dl, dotted] \arrow[dr, dotted] \arrow[dd, "\varphi"'] & & N \arrow[dl, dotted] \arrow[dr, dotted] \arrow[dd, "\psi"] & \\
B \arrow[dd, "f"'] & & C \arrow[dd, "g"] & & D \arrow[dd, "h"] \\
& M' \arrow[dl, dotted] \arrow[dr, dotted] & & N' \arrow[dl, dotted] \arrow[dr, dotted] & \\
B' & & C' & & D'
\end{tikzcd}$$

Then there is a map $\varphi \underset{g}{\boxtimes} \psi\colon M \underset{C}{\boxtimes} N \to M' \underset{C'}{\boxtimes} N'$ *over* (f, h)*.*

Proof. The map $\varphi \underset{g}{\boxtimes} \psi$ is induced by:

$$\begin{tikzcd}
M \underset{C}{\boxtimes} N \arrow[r, tail] \arrow[d, dashed, "\varphi \underset{g}{\boxtimes} \psi"'] & M \otimes N \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\varphi \otimes \psi"'] & M \otimes C \otimes N \arrow[d, "\varphi \otimes g \otimes \psi"] \\
M' \underset{C'}{\boxtimes} N' \arrow[r, tail] & M' \otimes N' \arrow[r, shift left] \arrow[r, shift right] & M' \otimes C' \otimes N'
\end{tikzcd}$$

This is a comodule map over h if the left-most face of the following diagram commutes,

$$\begin{tikzcd}
M_C N \arrow[rr, tail] \arrow[dd, dashed] \arrow[dr, dashed] & & MN \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dd] \arrow[dr] & & MCN \arrow[dd] \arrow[dr] & \\
& M'_{C'} N' \arrow[rr, tail, crossing over] & & M'N' \arrow[rr, shift left, crossing over] \arrow[rr, shift right, crossing over] & & M'C'N' \arrow[dd] \\
M_C N D \arrow[rr, tail] \arrow[dr, dashed] & & MND \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr] & & MCND \arrow[dr] & \\
& M'_{C'} N' D' \arrow[rr, tail] \arrow[from=uu, dashed, crossing over] & & M'N'D' \arrow[rr, shift left] \arrow[rr, shift right] \arrow[from=uu, crossing over] & & M'C'N'D'
\end{tikzcd}$$

where we have omitted $\otimes$ and $\boxtimes$ for brevity. But both composites that make up the left-most face are maps uniquely induced by the diagonal map $M \underset{C}{\boxtimes} N \to M' \otimes N' \otimes D'$, hence are equal. Similarly, $\varphi \underset{g}{\boxtimes} \psi$ is a comodule map over f. $\square$

Theorem A.2.8 ([Agu97, Theorem 2.2.1])**.** *There is a bicategory whose objects are comonoids in* $\mathcal{V}$*, and whose category of arrows from* C *to* D *is* ${}_C\mathbf{Comod}_D$*.*

Corollary A.2.9. *For* C *a comonoid in* $\mathcal{V}$*,* $({}_C\mathbf{Comod}_C, \underset{C}{\boxtimes}, C)$ *is a monoidal category.*

We conclude this section with some useful lemmas.

Lemma A.2.10. *Let* $(D, d, e), (M_1, \delta_1, \epsilon_1)$ *and* $(M_2, \delta_2, \epsilon_2)$ *be comonoids in* $\mathcal{V}$*. If each* M_i *is in* ${}_D\mathbf{Comod}_D$*, and* δ_i *and* ϵ_i *are maps over* d *and* e*, then* $M_1 \underset{D}{\boxtimes} M_2$ *is also a comonoid.*

Proof. By PropositionA.2.7, since each δ_i is a map over d, we have a map $\delta_1 \underset{d}{\boxtimes} \delta_2$, which we may compose with the isomorphism from Lemma A.2.6 to obtain a comultiplication:

$$M_1 \underset{D}{\boxtimes} M_2 \xrightarrow{\delta_1 \underset{d}{\boxtimes} \delta_2} (M_1 \otimes M_1) \underset{D\otimes D}{\boxtimes} (M_2 \otimes M_2) \xrightarrow{\cong} (M_1 \underset{D}{\boxtimes} M_2) \otimes (M_1 \underset{D}{\boxtimes} M_2)$$

Similarly, since each ϵ_i is a comodule map over e, we have a counit

$$M_1 \underset{D}{\boxtimes} M_2 \xrightarrow{\epsilon_1 \underset{e}{\boxtimes} \epsilon_2} \mathbf{1} \underset{\mathbf{1}}{\boxtimes} \mathbf{1} \cong \mathbf{1}.$$

The reader may verify that these maps make $M_1 \underset{D}{\boxtimes} M_2$ a comonoid. $\square$

Lemma A.2.11. *Let (C, δ, ϵ) and (D, d, e) be comonoids, and suppose that C is a D-comodule with coaction $p\colon C \to C \otimes D$. Then p is a comonoid map if and only if δ is a map over d:*

$$\begin{tikzcd}
C \arrow[r, "\delta"] \arrow[d, dotted, "p"'] & C \otimes C \arrow[d, dotted, "p\otimes p"] \\
D \arrow[r, "d"] & D \otimes D
\end{tikzcd}$$

Proof. Note that p always preserves counits, so p is a comonoid map if and only if it also preserves comultiplication.

The diagram in the lemma commutes precisely when the left pentagon in the following diagram commutes:

$$\begin{tikzcd}
C \arrow[rr, "p"] \arrow[d, "\delta"'] & & C \otimes D \arrow[d, "d\otimes\delta"] \\
 & & C \otimes C \otimes D \otimes D \arrow[d, "C\otimes\mathtt{x}\otimes D"] \\
C \otimes C \arrow[r, "p\otimes p"] & C \otimes D \otimes C \otimes D \arrow[r, equal] \arrow[ur, "C\otimes\mathtt{x}\otimes D"] & C \otimes D \otimes C \otimes D
\end{tikzcd}$$

The outer square then says that p is a comonoid map[1].

[1]Note that we need $\mathcal{V}$ to be *symmetric*, not just braided, for the bottom-right corner to commute!

Conversely, if p is a comonoid map, the left pentagon in the following diagram commutes:

$$\begin{tikzcd}
C \arrow[rr, "\delta"] \arrow[dd, "p"'] & & C\otimes C \arrow[d, "p\otimes p"] \\
 & & C\otimes D\otimes C\otimes D \arrow[d, "C\otimes \mathrm{x}\otimes D"] \\
C\otimes D \arrow[r, "\delta\otimes d"] & C\otimes C\otimes D\otimes D \arrow[ur, "C\otimes \mathrm{x}\otimes D"] \arrow[r, equal] & C\otimes C\otimes D\otimes D
\end{tikzcd}$$

The outer square then says that δ is a map over d. $\square$

Lemma A.2.12. *Let C, D be comonoids, and $p\colon C \to C\otimes D$ be a D-coaction that is also a comonoid map. Then p is induced by a comonoid map $q\colon C \to D$.*

Proof. The counit $e\colon C \to \mathbf{1}$ is a comonoid map, so the composite

$$q\colon C \xrightarrow{p} C\otimes D \xrightarrow{e\otimes D} D$$

is a comonoid map. The left square of the following diagram commutes because p is a comonoid map; the upper-right square commutes because p is a coaction.

$$\begin{tikzcd}
C \arrow[r, "p"] \arrow[dd, "d"'] & C\otimes D \arrow[r, equal] \arrow[d, "d\otimes d"'] & C\otimes D \arrow[d, equal] \\
 & C\otimes C\otimes D\otimes D \arrow[r, "C\otimes e\otimes e\otimes D"] \arrow[d, "C\otimes \mathrm{x}\otimes D"'] & C\otimes D \arrow[d, equal] \\
C\otimes C \arrow[r, "p\otimes p"] \arrow[rr, bend right, "C\otimes q"'] & C\otimes D\otimes C\otimes D \arrow[r, "C\otimes e\otimes e\otimes D"] & C\otimes D
\end{tikzcd}$$

The outer diagram then says that q induces p. $\square$

Remark A.2.13. *The converse of Lemma A.2.12 does not hold: given an arbitrary comonoid map $q\colon C \to D$, the coaction*

$$C \xrightarrow{\delta} C\otimes C \xrightarrow{C\otimes q} C\otimes D$$

need not be a comonoid map, because $\delta\colon C \to C\otimes C$ is not a comonoid map (unless C is cocommutative). Thus the two equivalent conditions in Lemma A.2.11 are stronger than the condition in Lemma A.2.12.

Remark A.2.14. *Note that for any comonoid map $q\colon C \to D$ inducing a coaction $p\colon C \to C \otimes D$, the following diagram always commutes:*

$$\begin{tikzcd}
C \arrow[rr, "q"] \arrow[dr, dotted, "p"'] & & D \arrow[dl, dotted, "d"] \\
& D &
\end{tikzcd}$$

Lemma A.2.15. *Let C be a cocommutative comonoid, and $M \in \mathbf{Comod}_C$ with coaction $\rho\colon M \to M \otimes C$. Then ρ is a map over δ:*

$$\begin{tikzcd}
M \arrow[r, "\rho"] \arrow[d, dotted, "\rho"'] & M \otimes C \arrow[d, dotted, "\rho\otimes\delta"] \\
C \arrow[r, "\delta"] & C \otimes C
\end{tikzcd}$$

Proof. We need the following diagram to commute:

$$\begin{tikzcd}
M \arrow[r, "\rho"] \arrow[d, "\rho"'] & M \otimes C \arrow[r, "\rho\otimes C"] & M \otimes C \otimes C \arrow[d, "M\otimes C\otimes\delta"] \\
M \otimes C \arrow[d, "\rho\otimes C"'] & & M \otimes C \otimes C \otimes C \arrow[d, "M\otimes \mathrm{x}\otimes C"] \\
M \otimes C \otimes C \arrow[rr, "M\otimes C\otimes\delta"] & & M \otimes C \otimes C \otimes C
\end{tikzcd}$$

Since $(\rho \otimes C)\rho = (M \otimes \delta)\rho$, this is equivalent to the following diagram commuting,

$$\begin{tikzcd}
M \arrow[r, "\rho"] \arrow[d, "\rho"'] & M \otimes C \arrow[r, "M\otimes\delta"] \arrow[d, equal] & M \otimes C \otimes C \arrow[d, "M\otimes C\otimes\delta"] \\
M \otimes C \arrow[r, equal] \arrow[d, "M\otimes\delta"'] & M \otimes C \arrow[r, "M\otimes(\delta\otimes C)\delta"] \arrow[d, "M\otimes(\delta\otimes C)\delta"] & M \otimes C \otimes C \otimes C \arrow[d, "M\otimes \mathrm{x}\otimes C"] \\
M \otimes C \otimes C \arrow[r, "M\otimes C\otimes\delta"] & M \otimes C \otimes C \otimes C \arrow[r, equal] & M \otimes C \otimes C \otimes C
\end{tikzcd}$$

whose bottom-right square commutes because C is cocommutative. $\square$

A.3 Comonoidal internal categories

We take our definition of a category internal to a regular monoidal category $(\mathcal{V}, \otimes, \mathbf{1})$ from [Agu97]. Importantly, $\mathcal{V}$ is not required to be cartesian i.e. the monoidal product $\otimes$ is not necessarily the cartesian product $\times$.

Definition A.3.1 ([Agu97, Definition 2.3.1])**.** A $\mathcal{V}$**-internal category** consists of a comonoid C in $\mathcal{V}$ and a monoid A in ${}_C\mathbf{Comod}_C$.

In detail, an internal category is a tuple $\mathcal{A} = (C, A, d, e, \sigma, \tau, u, m)$ with

1. a *comonoid of objects* $C \in \mathbf{Comon}(\mathcal{V})$, with comultiplication $d\colon C \to C \otimes C$ and counit $e\colon C \to \mathbf{1}$;

2. a *comodule of maps* $A \in {}_C\mathbf{Comod}_C$, with coactions[2] $\sigma\colon A \to C \otimes A$ and $\tau\colon A \to A \otimes C$;

3. and *identity* and *composition* comodule maps

$$\begin{tikzcd} & C \arrow[dl, dotted, "d"'] \arrow[dr, dotted, "d"] \arrow[dd, "u"] & \\ C & & C \\ & A \arrow[ul, dotted, "\sigma"] \arrow[ur, dotted, "\tau"'] & \end{tikzcd} \qquad \begin{tikzcd} & A \underset{C}{\boxtimes} A \arrow[dl, dotted, "\sigma"'] \arrow[dr, dotted, "\tau"] \arrow[dd, "m"] & \\ C & & C \\ & A \arrow[ul, dotted, "\sigma"] \arrow[ur, dotted, "\tau"'] & \end{tikzcd}$$

   satisfying associativity and unitality.

For brevity, we will sometimes refer to an internal category $\mathcal{A}$ using subtuples such as (C, A).

Remark A.3.2. *The definition of an internal category does not require the comonoid of objects C to be cocommutative. However, it does not seem possible to define internal prestacks or the internal Grothendieck construction without cocommutativity of objects. The internal categories that we subsequently consider will all have cocommutative comonoids of objects. In such a situation, the left coaction σ induces a right coaction $\mathbb{x}\sigma$. Similarly, the right coaction τ induces a left coaction $\mathbb{x}\tau$.*

Example A.3.3. Any monoid A in $\mathcal{V}$ gives rise to the 'one-object' internal category $(\mathbf{1}, A)$. Any comonoid C in $\mathcal{V}$ gives rise to the 'discrete' internal category (C, C). (see [Agu97, Example 2.4.1].)

---

[2] σ for 'source' and τ for 'target'.

Definition A.3.4 ([Agu97, Definition 4.1.1])**.** Let $\mathcal{A} = (C, A)$ and $\mathcal{B} = (D, B)$ be internal categories in $\mathcal{V}$. An **internal functor** from $\mathcal{A}$ to $\mathcal{B}$ is a tuple (f, φ) where $f \colon C \to D$ is a comonoid map and $\varphi \colon A \to B$ is a map such that the following diagrams commute:

$$\begin{tikzcd}
C \arrow[d, "f"'] & A \arrow[l, dotted, "\sigma"'] \arrow[r, dotted, "\tau"] \arrow[d, "\varphi"] & C \arrow[d, "f"] \\
D & B \arrow[l, dotted, "\sigma"'] \arrow[r, dotted, "\tau"] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "u"'] & D \arrow[d, "u"] \\
A \arrow[r, "\varphi"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \boxtimes_C A \arrow[r, "\varphi \boxtimes_f \varphi"] \arrow[d, "m"'] & B \boxtimes_D B \arrow[d, "m"] \\
A \arrow[r, "\varphi"] & B
\end{tikzcd}$$

Definition A.3.5. Let $\mathbf{Cat}(\mathcal{V})$ denote the category of internal categories and functors.

Remark A.3.6. *It is also possible to define internal transformations between internal functors, making* $\mathbf{Cat}(\mathcal{V})$ *a* 2*-category, but we will not need the* 2*-category structure in this paper.*

Recall that if $\mathcal{V}$ is a symmetric monoidal category, its category of comonoids $\mathbf{Comon}(\mathcal{V})$ is also symmetric monoidal, with the same braiding and monoidal product. A similar result holds for internal categories.

Proposition A.3.7 ([Agu97, §7.1])**.** $\mathbf{Cat}(\mathcal{V})$ *is a monoidal category, with product*

$$(C, A) \otimes (D, B) := (C \otimes D, A \otimes B)$$

and unit $\mathbb{1} := (\mathbf{1}, \mathbf{1})$.

Definition A.3.8. A **comonoidal internal category** $\mathcal{B} = (D, B)$ is a comonoid in $\mathbf{Cat}(\mathcal{V})$.

Proposition A.3.9. *Let* $\mathcal{B} = (D, B, d, e, \sigma, \tau, u, m)$ *be a comonoidal internal category. Then:*

1. D *is cocommutative (i.e.* d *and* e *are comonoid maps);*

2. B *is a comonoid, and* σ *and* τ *are comonoid maps (hence are induced by comonoid maps* $s \colon B \to D$ *and* $t \colon B \to D$*);*

3. σ, τ *and* δ *(the comultiplication of* B*) are maps over* d*:*

$$\begin{tikzcd}
D \arrow[d, "d"'] & B \arrow[l, dotted, "\sigma"'] \arrow[r, dotted, "\tau"] \arrow[d, "\delta"] & D \arrow[d, "d"] \\
D \otimes D & B \otimes B \arrow[l, dotted, "\sigma \otimes \sigma"'] \arrow[r, dotted, "\tau \otimes \tau"] & D \otimes D
\end{tikzcd}$$

$$\begin{tikzcd}
B \arrow[r, "\sigma"] \arrow[d, dotted, "\sigma"'] & D \otimes B \arrow[d, dotted, "d\otimes\sigma"] \\
D \arrow[r, "d"] & D \otimes D
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\tau"] \arrow[d, dotted, "\tau"'] & B \otimes D \arrow[d, dotted, "\tau\otimes d"] \\
D \arrow[r, "d"] & D \otimes D
\end{tikzcd}$$

4. $B \underset{D}{\boxtimes} B$ is a comonoid, and u and m are comonoid maps.

Proof. Let $(d', \delta)\colon (D, B) \to (D \otimes D, B \otimes B)$ and $(e', \epsilon)\colon (\mathbf{1}, \mathbf{1}) \to (D, B)$ be internal functors making $\mathcal{B} = (D, B)$ comonoidal. Then (d, e) and (d', e') are both counital comonoidal structures on D such that d' is a comonoid map with respect to d. By the Eckmann-Hilton argument, we have $e' = e, d' = d$, and D is cocommutative.

The maps (δ, ϵ) make B a comonoid, and δ is a map over d by definition of an internal functor. By Lemma A.2.11, both σ and τ are comonoid maps, and by Lemma A.2.15, they are comodule maps over d.

Since δ and ϵ are comodule maps over d and e, Lemma A.2.10 shows that $B \underset{D}{\boxtimes} B$ is a comonoid. Finally, since (d, δ) and (e, ϵ) are internal functors, δ and ϵ are required to make the following diagrams commute:

$$\begin{tikzcd}
D \arrow[r, "d"] \arrow[d, "u"'] & D \otimes D \arrow[d, "u\otimes u"] \\
B \arrow[r, "\delta"] & B \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
D \arrow[r, "e"] \arrow[d, "u"'] & \mathbf{1} \arrow[d, equal] \\
B \arrow[r, "\epsilon"] & \mathbf{1}
\end{tikzcd}
\qquad
\begin{tikzcd}
B \underset{D}{\boxtimes} B \arrow[r, "\epsilon \underset{e}{\boxtimes} \epsilon"] \arrow[d, "m"] & \mathbf{1} \arrow[d, equal] \\
B \arrow[r, "\epsilon"] & \mathbf{1}
\end{tikzcd}$$

$$\begin{tikzcd}
B \underset{D}{\boxtimes} B \arrow[r, "\delta \underset{d}{\boxtimes} \delta"] \arrow[d, "m"'] & (B \underset{D}{\boxtimes} B) \otimes (B \underset{D}{\boxtimes} B) \arrow[d, "m\otimes m"] \\
B \arrow[r, "\delta"] & B \otimes B
\end{tikzcd}$$

But these are precisely the diagrams that make u and m comonoid maps. □

Remark A.3.10. *The previous proposition effectively says that a comonoidal internal category is a 'category internal to* **Comon**$(\mathcal{V})$*'. We write the latter statement in quotes because our definition of internal category requires the ambient monoidal category to be regular, which* **Comon**$(\mathcal{V})$ *need not be.*

Corollary A.3.11. *The following diagram commutes:*

$$\begin{tikzcd}
B \arrow[r, "\delta"] \arrow[d, "\delta"'] & B \otimes B \arrow[d, "(\mathbb{x} \otimes B)(B \otimes \sigma)"] \\
B \otimes B \arrow[r, "\sigma \otimes B"] & D \otimes B \otimes B
\end{tikzcd}$$

Proof. Follows from σ being a map over d. □

Thus, although the comultiplicands of B need not be the same (i.e. B is not cocommutative), their sources are. The analogous statement for targets also holds.

Example A.3.12. If B is a bimonoid, its one-object category $(\mathbf{1}, B)$ is comonoidal.

If D is a cocommutative comonoid, its discrete category (D, D) is comonoidal.

A.4 Internal prestacks

Definition A.4.1. Let $\mathcal{B} = (D, B)$ be a comonoidal internal category. A **right $\mathcal{B}$-comodule category** is a right $\mathcal{B}$-comodule in $\mathbf{Cat}(\mathcal{V})$.

In detail, this is the data of an internal category $\mathcal{A} = (C, A)$ along with:

1. a D-coaction $p\colon C \to C \otimes D$ that is also a comonoid map (hence is induced by a comonoid map $q\colon C \to D$);

2. a B-coaction $\pi\colon A \to A \otimes B$ that is also a map over p;

3. such that $(p, \pi)\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$ is an internal functor.

We henceforth refer to these as simply $\mathcal{B}$-comodule categories or $\mathcal{B}$-comodules.

Recall that if $\mathcal{B} = (D, B)$ is comonoidal, then so is the discrete category $\mathcal{D} = (D, D)$.

Lemma A.4.2. *Let $\mathcal{B} = (D, B)$ be a comonoidal internal category and $\mathcal{D} = (D, D)$ its subcategory of objects. Let $\mathcal{A} = (A, C, \sigma, \tau)$ be a $\mathcal{D}$-comodule category with coaction*

$$(p\colon C \to C \otimes D, \pi\colon A \to A \otimes D),$$

and let $q\colon C \to D$ be the comonoid map that induces p. Then:

1. *$B \underset{D}{\boxtimes} C$ is a comonoid, with comultiplication $\Delta := \delta \underset{d_D}{\boxtimes} d_C$;*

2. *The D-coactions $q_*\sigma$ and $q_*\tau$ on A coincide with π;*

3. *σ and τ are maps over d:*

$$
\begin{tikzcd}
C \otimes A \arrow[d, dotted, "p\otimes\pi"'] & A \arrow[l, "\sigma"'] \arrow[r, "\tau"] \arrow[d, dotted, "\pi"] & A \otimes C \arrow[d, dotted, "\pi\otimes p"] \\
D \otimes D & D \arrow[l, "d"'] \arrow[r, "d"] & D \otimes D
\end{tikzcd}
$$

4. *The coactions σ and τ induce $B \underset{D}{\boxtimes} C$-coactions on $B \underset{D}{\boxtimes} A$;*

Proof. By Lemma A.2.11, since p is a comonoid map, the comultiplication d_C is a comodule map over d_D, and the counit e_C is a comodule map over e_D. By Lemma A.2.10, $B \underset{D}{\boxtimes} C$ is a comonoid.

By Lemma A.2.12, p induces a comonoid map $q\colon C \to D$. Corestricting along q makes A a (D, D)-bicomodule. Since π is a map over p, the left square in the following diagram commutes:

$$
\begin{tikzcd}
A \arrow[r, "\pi"] \arrow[dd, "\sigma"'] & A \otimes D \arrow[d, "\sigma\otimes d"] \arrow[ddr, bend left, "\mathbb{x}", "\cong"'] & \\
& C \otimes A \otimes D \otimes D \arrow[d, "A\otimes\mathbb{x}\otimes D"] & \\
C \otimes A \arrow[r, "p\otimes\pi"] \arrow[rr, bend right, "q\otimes A"'] & C \otimes D \otimes A \otimes D \arrow[r, "e\otimes D\otimes A\otimes e"] & D \otimes A
\end{tikzcd}
$$

The outer diagram then says that $\mathbb{x}\pi$ and $q_*\sigma$ coincide. A similar diagram (with an identity instead of $\mathbb{x}$) shows that π and $q_*\tau$ coincide.

Again, the following diagram commutes, so τ is a map over d:

$$
\begin{tikzcd}
A \arrow[r, "\pi"] \arrow[dd, "\tau"'] & A \otimes D \arrow[d, "\tau\otimes d"] & \\
& A \otimes C \otimes D \otimes D \arrow[d, "A\otimes\mathbb{x}\otimes D"] \arrow[dr, equals, bend left] & \\
A \otimes C \arrow[r, "\pi\otimes p"] & A \otimes D \otimes C \otimes D \arrow[r, "A\otimes\mathbb{x}\otimes D"] & A \otimes C \otimes D \otimes D
\end{tikzcd}
$$

Similarly, σ is a map over d. We may thus form the composites,

$$B \underset{D}{\boxtimes} A \xrightarrow{\delta \underset{d}{\boxtimes} \tau} (B \otimes B) \underset{D \otimes D}{\boxtimes} (A \otimes C) \cong (B \underset{D}{\boxtimes} A) \otimes (B \underset{D}{\boxtimes} C)$$

$$B \underset{D}{\boxtimes} A \xrightarrow{\delta \underset{d}{\boxtimes} \sigma} (B \otimes B) \underset{D \otimes D}{\boxtimes} (C \otimes A) \cong (B \underset{D}{\boxtimes} C) \otimes (B \underset{D}{\boxtimes} A)$$

which are seen to be coactions. □

Definition A.4.3. Let $\mathcal{B} = (D, B)$ be a comonoidal internal category and $\mathcal{D} = (D, D)$ its subcategory of objects. A **prestack over $\mathcal{B}$** (or a **$\mathcal{B}$-module category**) consists of:

0. An internal category $\mathcal{A} = (C, A)$ with C cocommutative;

1. A coaction $(p, \pi)\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{D}$;

2. A comonoid map $f\colon B \underset{D}{\boxtimes} C \to C$ satisfying:

$$\begin{tikzcd}
B \underset{D}{\boxtimes} C \arrow[r, "f"] \arrow[d, dotted, "\sigma"'] & C \arrow[d, dotted, "\mathbb{x} p"] \\
D \arrow[r, equal] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
D \underset{D}{\boxtimes} C \arrow[r, "u \underset{D}{\boxtimes} C"] \arrow[d, "\cong"'] & B \underset{D}{\boxtimes} C \arrow[d, "f"] \\
C \arrow[r, equal] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
B \underset{D}{\boxtimes} B \underset{D}{\boxtimes} C \arrow[r, "B \underset{D}{\boxtimes} f"] \arrow[d, "m \underset{D}{\boxtimes} C"'] & B \underset{D}{\boxtimes} C \arrow[d, "f"] \\
B \underset{D}{\boxtimes} C \arrow[r, "f"] & C
\end{tikzcd}$$

3. A map $\varphi\colon B \underset{D}{\boxtimes} A \to A$ satisfying:

$$\begin{tikzcd}
B \underset{D}{\boxtimes} C \arrow[d, "f"'] & B \underset{D}{\boxtimes} A \arrow[l, dotted, "\delta \underset{d}{\boxtimes} \sigma"'] \arrow[r, dotted, "\delta \underset{d}{\boxtimes} \tau"] \arrow[d, "\varphi"] & B \underset{D}{\boxtimes} C \arrow[d, "f"] \\
C & A \arrow[l, dotted, "\sigma"'] \arrow[r, dotted, "\tau"] & C
\end{tikzcd}$$

$$\begin{tikzcd}
D \underset{D}{\boxtimes} A \arrow[r, "u \underset{D}{\boxtimes} A"] \arrow[d, "\cong"'] & B \underset{D}{\boxtimes} A \arrow[d, "\varphi"] \\
A \arrow[r, equal] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \underset{D}{\boxtimes} B \underset{D}{\boxtimes} A \arrow[r, "B \underset{D}{\boxtimes} \varphi"] \arrow[d, "m \underset{D}{\boxtimes} A"'] & B \underset{D}{\boxtimes} A \arrow[d, "\varphi"] \\
B \underset{D}{\boxtimes} A \arrow[r, "\varphi"] & A
\end{tikzcd}$$

4. f and φ further satisfy:

$$\begin{tikzcd}
B \underset{D}{\boxtimes} C \arrow[r, "B \underset{D}{\boxtimes} e"] \arrow[d, "f"'] & B \underset{D}{\boxtimes} A \arrow[d, "\varphi"] \\
C \arrow[r, "e"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \underset{D}{\boxtimes} (A \underset{C}{\boxtimes} A) \arrow[r, "B \underset{D}{\boxtimes} m"] \arrow[d, "\varphi_2"'] & B \underset{D}{\boxtimes} A \arrow[d, "\varphi"] \\
A \underset{C}{\boxtimes} A \arrow[r] & A
\end{tikzcd}$$

The map φ_2 is given by the following lemma:

Lemma A.4.4. *There is an action* $\varphi_2 \colon B \underset{D}{\boxtimes} (A \underset{C}{\boxtimes} A) \to A \underset{C}{\boxtimes} A$.

Proof. We first observe that we have a map $A \underset{C}{\boxtimes} A \to A \underset{D}{\boxtimes} A$ induced by:

$$\begin{tikzcd}
A \underset{C}{\boxtimes} A \arrow[r, tail] \arrow[d, dashed] & A \otimes A \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & A \otimes C \otimes A \arrow[d, "A \otimes q \otimes A"] \\
A \underset{D}{\boxtimes} A \arrow[r, tail] & A \otimes A \arrow[r, shift left] \arrow[r, shift right] & A \otimes D \otimes A
\end{tikzcd}$$

Next, since the left and right D-coactions on A coincide, the following diagram commutes,

$$\begin{tikzcd}
A \underset{D}{\boxtimes} A \arrow[r, tail] \arrow[d] & A \otimes A \arrow[r, "q_*\sigma \otimes q_*\tau"] \arrow[d, "q_*\sigma"'] & D \otimes A \otimes D \otimes A \arrow[d, "D \otimes \mathbb{x} \otimes A"] \\
D \otimes (A \underset{D}{\boxtimes} A) \arrow[r, tail] & D \otimes A \otimes A \arrow[r, "d \otimes A \otimes A"] & D \otimes D \otimes A \otimes A
\end{tikzcd}$$

so the map $\iota \colon A \underset{C}{\boxtimes} A \longrightarrow A \underset{D}{\boxtimes} A \rightarrowtail A \otimes A$ is a comodule map over $d \colon D \to D \otimes D$. The comultiplication $\delta \colon B \to B \otimes B$ is also a comodule map over d, which we may combine with the above map to obtain a map $B \underset{D}{\boxtimes} (A \underset{C}{\boxtimes} A) \to A \otimes A$:

$$\begin{tikzcd}
B \underset{D}{\boxtimes} (A \underset{C}{\boxtimes} A) \arrow[r, "\delta \underset{D}{\boxtimes} \iota"] \arrow[dd, dashed, "?"'] & (B \otimes B) \underset{D \otimes D}{\boxtimes} (A \otimes A) \arrow[d, "\cong"] \\
& (B \underset{D}{\boxtimes} A) \otimes (B \underset{D}{\boxtimes} A) \arrow[d, "\varphi \otimes \varphi"] \\
A \underset{C}{\boxtimes} A \arrow[r, tail] & A \otimes A
\end{tikzcd}$$

Finally, a routine diagram chase, repeatedly invoking the naturality of $\otimes$, allows us to verify that this map does indeed factor through $A \underset{C}{\boxtimes} A$, giving the desired map. $\square$

A.5 Smash products

Let $\mathcal{A} = (C, A)$ be a prestack over $\mathcal{B} = (D, B)$, with actions f and φ as above.

We make $B \underset{D}{\boxtimes} C$ an object of ${}_C\mathbf{Comod}_C$, with left coaction induced by the comonoid map $f\colon B \underset{D}{\boxtimes} C \to C$, and right coaction induced by the comonoid map $t\colon B \to D$,

$$f_*\Delta\colon B \underset{D}{\boxtimes} C \longrightarrow (B \underset{D}{\boxtimes} C) \otimes (B \underset{D}{\boxtimes} C) \xrightarrow{f \otimes (B \underset{D}{\boxtimes} C)} C \otimes (B \underset{D}{\boxtimes} C)$$

$$t_*\Delta\colon B \underset{D}{\boxtimes} C \longrightarrow (B \underset{D}{\boxtimes} C) \otimes (B \underset{D}{\boxtimes} C) \xrightarrow{(B \underset{D}{\boxtimes} C) \otimes (t \underset{D}{\boxtimes} C)} (B \underset{D}{\boxtimes} C) \otimes (D \underset{D}{\boxtimes} C) \cong (B \underset{D}{\boxtimes} C) \otimes C$$

where $\Delta = \delta \underset{d_D}{\boxtimes} d_C$ is the comultiplication of $B \underset{D}{\boxtimes} C$. We also have a right B-coaction induced by the comonoid map $q\colon C \to D$:

$$q_*\Delta\colon B \underset{D}{\boxtimes} C \longrightarrow (B \underset{D}{\boxtimes} C) \otimes (B \underset{D}{\boxtimes} C) \xrightarrow{(B \underset{D}{\boxtimes} C) \otimes (B \underset{D}{\boxtimes} q)} (B \underset{D}{\boxtimes} C) \otimes (B \underset{D}{\boxtimes} D) \cong (B \underset{D}{\boxtimes} C) \otimes B$$

Lemma A.5.1. *Let $\mathcal{A}$ be an internal prestack over $\mathcal{B}$. Then:*

1. *The coaction π is a bicomodule map over p:*

$$\begin{tikzcd}
C \arrow[d, "p"'] & A \arrow[l, dotted, "\sigma"'] \arrow[r, dotted, "\tau"] \arrow[d, "\pi"] & C \arrow[d, "p"] \\
C \otimes D & A \otimes D \arrow[l, dotted, "\sigma \otimes d"'] \arrow[r, dotted, "\tau \otimes d"] & C \otimes D
\end{tikzcd}$$

2. *The coaction $q_*\Delta$ is a bicomodule map over p:*

$$\begin{tikzcd}
C \arrow[d, "p"'] & B \underset{D}{\boxtimes} C \arrow[l, dotted, "f_*\Delta"'] \arrow[r, dotted, "t_*\Delta"] \arrow[d, "q_*\Delta"] & C \arrow[d, "p"] \\
C \otimes D & (B \underset{D}{\boxtimes} C) \otimes B \arrow[l, dotted, "f_*\Delta \otimes \sigma"'] \arrow[r, dotted, "t_*\Delta \otimes \tau"] & C \otimes D
\end{tikzcd}$$

3. *The coaction $f_*\Delta$ is a comodule map over p:*

$$\begin{tikzcd}
C \arrow[d, "p"'] & B \underset{D}{\boxtimes} C \arrow[l, dotted, "f_*\Delta"'] \arrow[d, "f_*\Delta"] \\
C \otimes D & C \otimes (B \underset{D}{\boxtimes} C) \arrow[l, dotted, "d \otimes \sigma"']
\end{tikzcd}$$

Proof. By Lemma A.2.15, the top squares of the following diagrams commute:

$$\begin{tikzcd}
C & A \arrow[l, dotted, "\sigma"'] \arrow[d, "\sigma"] \\
C \otimes C & C \otimes A \arrow[l, dotted, "d\otimes\sigma"'] \arrow[d, "q\otimes A"] \\
D \otimes C & D \otimes A \arrow[l, dotted, "d\otimes\sigma"'] \arrow[d, "\mathbb{x}"] \\
C \otimes D & A \otimes D \arrow[l, dotted, "\sigma\otimes d"']
\arrow[from=1-1, to=2-1, "d"']
\arrow[from=2-1, to=3-1, "q\otimes C"']
\arrow[from=3-1, to=4-1, "\mathbb{x}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, dotted, "\tau"] \arrow[d, "\tau"'] & C \arrow[d, "d"] \\
A \otimes C \arrow[r, dotted, "\tau\otimes d"] \arrow[d, "A\otimes q"'] & C \otimes C \arrow[d, "C\otimes q"] \\
A \otimes D \arrow[r, dotted, "\tau\otimes d"] & C \otimes D
\end{tikzcd}$$

The remaining squares obviously commute. For the left square, since C is cocommutative, the left vertical composite is p. The right vertical composite is π because $q_*\sigma = \mathbb{x}\pi$ by Lemma A.4.2. This proves the first item.

For the second item, the left square commutes because

$$\begin{aligned}
(p \otimes q_*\Delta) \circ (f_*\Delta) &= (C \otimes D \otimes q_*\Delta) \circ (p \otimes B \underset{D}{\boxtimes} C) \circ (f \otimes B \underset{D}{\boxtimes} C) \circ \Delta \\
\text{\scriptsize f is a map over D} \quad &= (C \otimes D \otimes q_*\Delta) \circ (\mathbb{x} \otimes B \underset{D}{\boxtimes} C) \circ (D \otimes f \otimes B \underset{D}{\boxtimes} C) \circ (\sigma \otimes B \underset{D}{\boxtimes} C) \circ \Delta \\
&= \Big(\big(\mathbb{x} \circ (D \otimes f) \circ \sigma\big) \otimes q_*\Delta \Big) \circ \Delta \\
&= \Big(\big(\mathbb{x} \circ (D \otimes f) \circ \sigma\big) \otimes \Big(\big(B \underset{D}{\boxtimes} C \otimes B \underset{D}{\boxtimes} q\big) \circ \Delta \Big) \Big) \circ \Delta \\
\text{\scriptsize associativity of Δ} \quad &= \Big(\Big(\big(\big(\mathbb{x} \circ (D \otimes f) \circ \sigma\big) \otimes B \underset{D}{\boxtimes} C\big) \circ \Delta \Big) \otimes B \underset{D}{\boxtimes} q \Big) \circ \Delta \\
&= \Big(\big(\big((\mathbb{x} \circ (D \otimes f)) \otimes B \underset{D}{\boxtimes} C\big) \circ \big(\sigma \otimes B \underset{D}{\boxtimes} C\big) \circ \Delta\big) \otimes B \underset{D}{\boxtimes} q \Big) \circ \Delta \\
\text{\scriptsize Cor A.3.11} \quad &= \Big(\big(\big((\mathbb{x} \circ (D \otimes f)) \otimes B \underset{D}{\boxtimes} C\big) \circ \big(\mathbb{x} \otimes B \underset{D}{\boxtimes} C\big) \circ \big(B \underset{D}{\boxtimes} C \otimes \sigma\big) \circ \Delta\big) \otimes B \underset{D}{\boxtimes} q \Big) \circ \Delta \\
&= \Big(\big((f \otimes \sigma) \circ \Delta\big) \otimes B \underset{D}{\boxtimes} q \Big) \circ \Delta \\
&= (f_*\Delta \otimes \sigma) \circ q_*\Delta.
\end{aligned}$$

The right square of the second item and the square in the third item commute by similar arguments. $\square$

We are now in a position to define smash products of internal prestack.

Theorem A.5.2. *Let $(f,\varphi)\colon \mathcal{A}\to\mathcal{A}\otimes\mathcal{B}$ be an internal prestack. There is an internal category*

$$\mathcal{A}\rtimes\mathcal{B} := \big(C, A\underset{C}{\boxtimes}(B\underset{D}{\boxtimes}C)\big),$$

which we call the ***smash product*** *of $\mathcal{A}$ with $\mathcal{B}$. Further, $\mathcal{A}\rtimes\mathcal{B}$ has the structure of a $\mathcal{B}$-comodule category.*

Proof. By Lemma A.5.1, $\pi, q_*\Delta$ and $f_*\Delta$ are all maps over p, allowing us to define the composite in (A.1). In fact, this composite factors through $A\underset{C}{\boxtimes}(B\underset{D}{\boxtimes}C)$, giving the multiplication on $\mathcal{A}\rtimes\mathcal{B}$.

The unit of $\mathcal{A}\rtimes\mathcal{B}$ is given by the composite:

$$C \xrightarrow{\cong} C\underset{C}{\boxtimes}(D\underset{D}{\boxtimes}C) \xrightarrow{u\underset{C}{\boxtimes}(u\underset{D}{\boxtimes}C)} A\underset{C}{\boxtimes}(B\underset{D}{\boxtimes}C)$$

These maps are unital and associative (because of Items 3 and 4 in Definition A.4.3, and the fact that $\mathcal{A}$ and $\mathcal{B}$ are internal categories), so $\mathcal{A}\rtimes\mathcal{B}$ is an internal category.

To see that $\mathcal{A}\rtimes\mathcal{B}$ has the structure of a $\mathcal{B}$-comodule category, note that C already has a D-coaction p. We then take the B-coaction on $A\underset{C}{\boxtimes}(B\underset{D}{\boxtimes}C)$ to be the composite in (A.2). $\square$

A.6 Coinvariants of comodule categories

Although we have not defined what a 'cartesian fibered right $\mathcal{B}$-comodule category' should be, we can still verify that the fibers of $\mathcal{A}\rtimes\mathcal{B}$ allow us to recover our original prestack $\mathcal{A}$. We first begin by defining the fibers of any right $\mathcal{B}$-comodule category.

Definition A.6.1. Let $\mathcal{A}$ be a right $\mathcal{B}$-comodule category, with coaction functor $p\colon\mathcal{A}\to\mathcal{A}\otimes\mathcal{B}$. The **coinvariant category** is the coinduction $\mathcal{A}$ along $(D,u)\colon\mathcal{D}\to\mathcal{B}$:

$$\begin{tikzcd}
\mathcal{A}\underset{\mathcal{B}}{\boxtimes}\mathcal{D} \arrow[r] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner"] & \mathcal{A} \arrow[d, dotted] \\
\mathcal{D} \arrow[r, "{(D,u)}"] & \mathcal{B}
\end{tikzcd}$$

$$
\begin{tikzcd}
A \underset{C}{\boxtimes} (B \underset{D}{\boxtimes} C) \underset{C}{\boxtimes} A \underset{C}{\boxtimes} (B \underset{D}{\boxtimes} C) \arrow[d, "\pi \underset{p}{\boxtimes} (q_* \Delta) \underset{p}{\boxtimes} \pi \underset{p}{\boxtimes} (f_* \Delta)"] \\
\left(A \otimes D \right) \underset{C \otimes D}{\boxtimes} \left((B \underset{D}{\boxtimes} C) \otimes B \right) \underset{C \otimes D}{\boxtimes} \left(A \otimes D \right) \underset{C \otimes D}{\boxtimes} \left(C \otimes (B \underset{D}{\boxtimes} C) \right) \arrow[d, "\cong"] \\
\left(A \underset{C}{\boxtimes} (B \underset{D}{\boxtimes} C) \underset{C}{\boxtimes} A \underset{C}{\boxtimes} C \right) \otimes \left(D \underset{D}{\boxtimes} B \underset{D}{\boxtimes} D \underset{D}{\boxtimes} (B \underset{D}{\boxtimes} C) \right) \arrow[d, "\cong"] \\
\left(A \underset{C}{\boxtimes} B \underset{D}{\boxtimes} A \right) \otimes \left(B \underset{D}{\boxtimes} B \underset{D}{\boxtimes} C \right) \arrow[d, "(A \underset{C}{\boxtimes} \varphi) \otimes (m \underset{D}{\boxtimes} C)"] \\
\left(A \underset{C}{\boxtimes} A \right) \otimes \left(B \underset{D}{\boxtimes} C \right) \arrow[d, "m \otimes (B \underset{D}{\boxtimes} C)"] \\
A \otimes (B \underset{D}{\boxtimes} C)
\end{tikzcd}
$$

Figure A.1: Composition in the internal category $\mathcal{A} \rtimes \mathcal{B}$

$$\begin{tikzcd}
A \underset{C}{\boxtimes} (B \underset{D}{\boxtimes} C) \arrow[d, "\pi \underset{C}{\boxtimes} (q_* \Delta)"] \\
\Big(A \otimes D\Big) \underset{C \otimes D}{\boxtimes} \Big((B \underset{D}{\boxtimes} C) \otimes B\Big) \arrow[d, "\cong"] \\
\Big(A \underset{C}{\boxtimes} (B \underset{D}{\boxtimes} C)\Big) \otimes \Big(D \underset{D}{\boxtimes} B\Big) \arrow[d, "\cong"] \\
\Big(A \underset{C}{\boxtimes} (B \underset{D}{\boxtimes} C)\Big) \otimes B
\end{tikzcd}$$

Figure A.2: $\mathcal{B}$-coaction on $\mathcal{A} \rtimes \mathcal{B}$

Equivalently, $\mathcal{A} \underset{\mathcal{B}}{\boxtimes} \mathcal{D}$ is given by the equalizer:

$$\begin{tikzcd}
\mathcal{A} \underset{\mathcal{B}}{\boxtimes} \mathcal{D} \arrow[r, tail] & \mathcal{A} \otimes \mathcal{D} \arrow[r, shift left, "(p,\pi)\otimes\mathcal{D}"] \arrow[r, shift right, "\mathcal{A}\otimes\big((D,u)\otimes\mathcal{D}\big)(d,\delta)"'] & \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{D}
\end{tikzcd}$$

The following lemmas follow almost by definition:

Lemma A.6.2. *The coinvariant category $\mathcal{A} \underset{\mathcal{B}}{\boxtimes} \mathcal{D}$ is a $\mathcal{D}$-comodule category.*

Lemma A.6.3. *The coinvariant category is given by $\mathcal{A} \underset{\mathcal{B}}{\boxtimes} \mathcal{D} = (C \cong C \underset{D}{\boxtimes} D,\ A \underset{B}{\boxtimes} D)$.*

Given an arbitrary $\mathcal{B}$-comodule category, it is unlikely that its coinvariant category has the structure of a prestack over $\mathcal{D}$. However, when the $\mathcal{B}$-comodule category is of the form $\mathcal{A} \rtimes \mathcal{B}$ for a prestack $\mathcal{A}$, we have:

Theorem A.6.4. *Let $\mathcal{A}$ be a prestack over $\mathcal{B}$ and let $\mathcal{A} \rtimes \mathcal{B}$ be the corresponding right $\mathcal{B}$-comodule category. Then the coinvariant category $(\mathcal{A} \rtimes \mathcal{B}) \underset{\mathcal{B}}{\boxtimes} \mathcal{D}$ is a prestack over $\mathcal{B}$, which is moreover isomorphic to $\mathcal{A}$.*

Proof. First observe that the comonoid of objects for $\mathcal{A}, \mathcal{A} \rtimes \mathcal{B}$ and $(\mathcal{A} \rtimes \mathcal{B}) \underset{\mathcal{B}}{\boxtimes} \mathcal{D}$ are all C. On morphisms, recall that the B-coaction on $A \underset{C}{\boxtimes} (B \underset{D}{\boxtimes} C)$ is given by the copy of B sitting inside $B \underset{D}{\boxtimes} C$. Thus

$$\begin{aligned} A \underset{C}{\boxtimes} (B \underset{D}{\boxtimes} C) \underset{B}{\boxtimes} D &\cong A \underset{C}{\boxtimes} (D \underset{D}{\boxtimes} C) \\ &\cong A \underset{C}{\boxtimes} C \\ &\cong A. \end{aligned}$$

So $\mathcal{A}$ and $(\mathcal{A} \rtimes \mathcal{B}) \underset{\mathcal{B}}{\boxtimes} \mathcal{D}$ are isomorphic categories. We may then transfer the prestack structure of $\mathcal{A}$ over to $(\mathcal{A} \rtimes \mathcal{B}) \underset{\mathcal{B}}{\boxtimes} \mathcal{D}$. $\square$

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