Khovanov Homology And Gauge Theory

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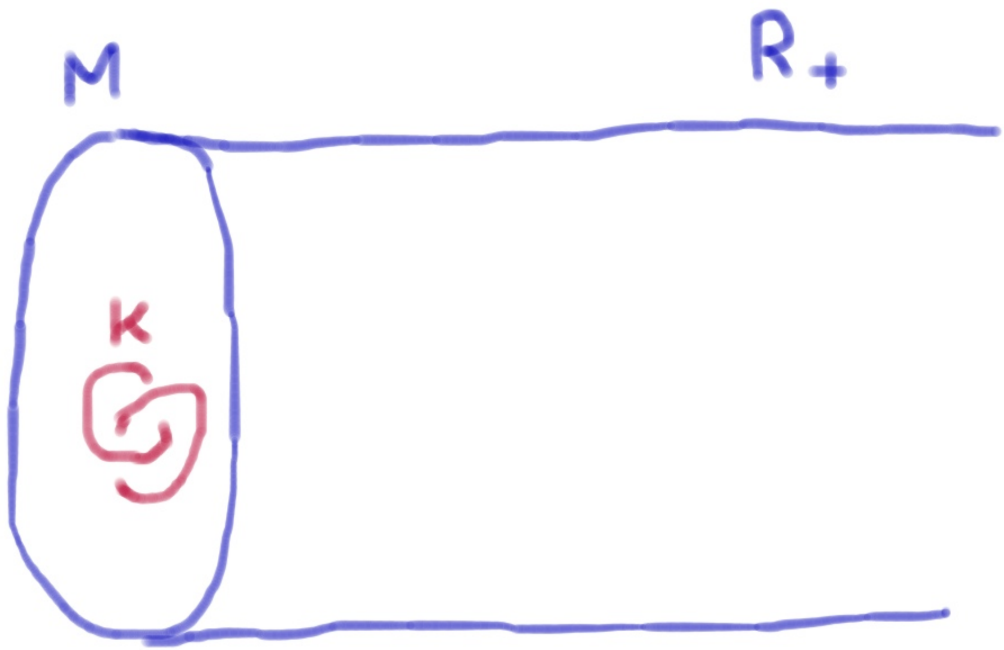
As we discussed yesterday, quantum invariants of a simple Lie group G on a three-manifold M can be computed by counting solutions of a certain system of nonlinear PDE's with gauge group $G^\vee$ on the four-manifold $X = M \times \mathbb{R}_+$:



Here $G^\vee$ is the Langlands or GNO dual group of G.

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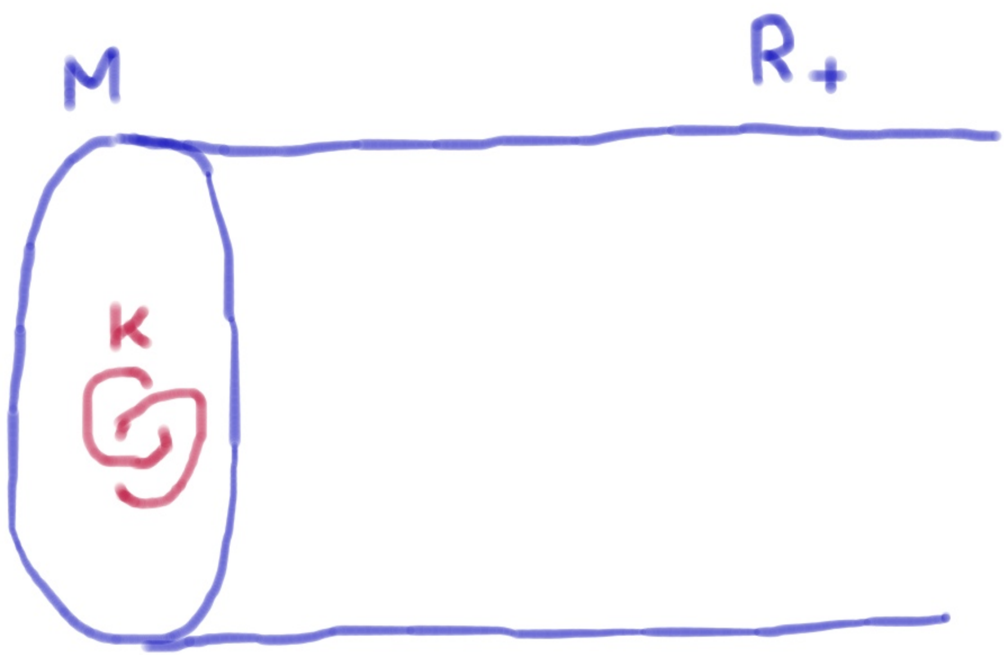
where $\star$ is the Hodge star. The boundary conditions on these equations at the finite end of $X = M \times \mathbb{R}_+$ depend on the knot, as I've tried to suggest in the picture:



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For $M = \mathbb{R}^3$, the difference between G and $G^\vee$ is important primarily when they have different Lie algebras, since for example there is no second Stieffel-Whithey class to distinguish $SU(2)$ from $SO(3)$.

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$$n = \frac{1}{8\pi^2} \int_X \mathrm{Tr}\, F \wedge F,$$

where $F = \mathrm{d}A + A \wedge A$ is the curvature. Here Tr is an invariant quadratic form on $\mathfrak{g}^\vee$ (the Lie algebra of $G^\vee$), which we normalize so that if X has no boundary and $G^\vee$ is simply-connected, then n is $\mathbb{Z}$-valued.

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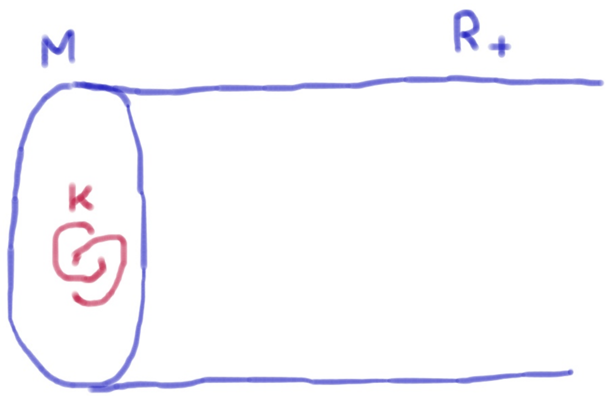
In all of this, the knot is encoded entirely in the boundary condition:



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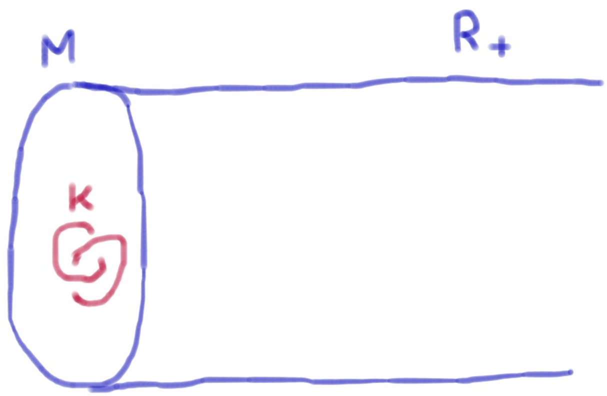

The instanton number

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is an integer if X is compact and without boundary, but our $X = M \times \mathbb{R}_+$ does not have that property and to make n into a topological invariant, we require a trivialization of $E^\vee$ at both the finite and infinite ends of X.

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Let $\mathcal{S}$ be the set of solutions of the KW equation. (It is expected that for a generic embedding of a knot or link in $\mathbb{R}^3$, the KW equations have only finitely many solutions and these are nondegenerate: the linearized operator has trivial kernel and cokernel.) We define a vector space $\mathcal{V}$ by declaring that for every $i \in \mathcal{S}$, there is a corresponding basis vector $|i\rangle$. On $\mathcal{V}$, we will have two "conserved quantum numbers" namely "instanton number," which I will call P and "fermion number," which I will call F. I have already defined the instanton number; it takes values in $\mathbb{Z} + c$ where c is a fixed constant that depends only on the representations. The fermion number F is another integer-valued quantity. We consider $|i\rangle$ to be "bosonic" or "fermionic" depending on whether it has an even or odd eigenvalue of F; the operator that distinguishes bosonic from fermionic states is $(-1)^F$. F will be defined so that, if the solution i contributed $+1$ to the counting of KW solutions, then $|i\rangle$ has even F, and if it contributed -1, then $|i\rangle$ has odd F.

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$$Z(q) = \sum_{i \in \mathcal{S}} (-1)^{f_i} q^{n_i} = \mathrm{Tr}_{\mathcal{V}} \, (-1)^F q^P.$$

Here P is the instanton number operator

$$P|i\rangle = n_i |i\rangle.$$

So far, we have not really done anything except to shift things around. However, on $\mathcal{V}$ we will also have a "differential" Q, which is an operator that commutes with the instanton number P but increases the fermion number F by one unit, and also obeys $Q^2 = 0$.

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This cohomology will be (conjecturally) the Khovanov homology, for those groups and representations where the latter has been defined. It is automatically $\mathbb{Z} \times \mathbb{Z}$-graded, with the two gradings defined by P and F.

The importance of passing from $\mathcal{V}$ to $\mathcal{H}$ is that $\mathcal{H}$ is a topological invariant while $\mathcal{V}$ is not. If one deforms a knot embedded in $\mathbb{R}^3$, solutions of the KW equations on $X = \mathbb{R}^3 \times \mathbb{R}_+$ will appear and disappear, so $\mathcal{V}$ will change. But $\mathcal{H}$ does not change.

Instead of defining the Jones polynomial and analogous invariants in terms of $\mathcal{V}$ by the formula of a couple of slides ago

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The ability to do all this rests on the following facts about the KW equations. (These facts were discovered independently by A. Haydys.) I will just state these facts as facts – which one can verify by a short calculation – rather than trying to explain the full quantum field theory picture which made me look for these facts. We consider the KW equations on $X = M \times \mathcal{I}$ where M is a three-manifold with local coordinates x_i, $i = 1, 2, 3$ and $\mathcal{I}$ is a one-manifold parametrized by y. (In our application, $\mathcal{I} = \mathbb{R}_+$.) We write $\phi = \sum_i \phi_i \mathrm{d}x_i + \phi_y \mathrm{d}y$. Now we replace X by $Y = \mathbb{R} \times X$ where $\mathbb{R}$ is a new "time" direction, parametrized by a time coordinate t, and we replace ϕ_y by $\frac{D}{Dt}$ everywhere that it appears in the KW equations. To be explicit about this, here

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + [A_t \cdot]$$

where ϕ_y is reinterpreted as A_t.

If one makes this replacement in a random differential equation that involves ϕ_y, one would not even get a differential equation, but a differential operator. In the case of the KW equations, ϕ_y appears only inside commutators $[\phi_i, \phi_y]$ and covariant derivatives $D_\mu \phi_y$ and the substitutions proceed by

$$[\phi_i, \phi_y] \to [\phi_i, D_t] = -D_t \phi_i, \quad D_\mu \phi_y = [D_\mu, \phi_y] \to [D_\mu, D_t] = F_{\mu t}.$$

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This five-dimensional equation has a four-dimensional symmetry that isn't obvious from what I've said so far. We started on $X = M \times \mathcal{I}$ with M a three-manifold, and then via $\phi_y \to D/Dt$, we replaced X with $Y = \mathbb{R} \times M \times \mathcal{I}$. It turns out that here $\mathbb{R} \times M$ can be replaced by any oriented four-manifold Z, and our five-dimensional equation can be naturally defined on $Y = Z \times \mathcal{I}$. At a certain point, we will make use of this four-dimensional symmetry.

Another crucial fact is that the five-dimensional equation that we get this way can be formulated as a gradient flow equation

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for a certain functional $\Gamma(\Phi)$. (I have just schematically combined all fields A, ϕ into Φ.) This means that we are in the situation explored by Floer when he formulated Floer cohomology in the mid-1980's: we can define (modulo analytic subtleties) an infinite-dimensional version of Morse theory, with Γ as a middle-dimensional Morse function.

In Morse theory, we define a complex (or more simply a vector space) $\mathcal{V}$ with a basis vector $|i\rangle$ for each critical point of Γ and then we define a "differential" $Q : \mathcal{V} \to \mathcal{V}$ by

$$Q|i\rangle = \sum_j n_{ij}|j\rangle$$

where for each pair of critical points i, j, n_{ij} is the "number" of solutions of the gradient flow equation

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that start at i in the past and end at j in the future. (In the counting, one factors out by the time-translation symmetry and one includes a sign ± 1 given by the sign of the fermion determinant, that is, of the determinant of the linearization of the flow equation. This is the procedure explained somewhat imperfectly in my paper "Supersymmetry and Morse Theory" (1982) and much developed later by others.) Q commutes with the instanton number P and increases the fermion number F by 1.

When we do this in the present context, the time-independent solutions in five dimensions are just the solutions of the KW equation in four-dimensions (with A_t reinterpreted as ϕ_y), since when we ask for a solution to be time-independent, we undo what we did to go from four to five dimensions. So the space $\mathcal{V}$ on which the differential of Morse theory acts is the same space we introduced before in writing the Jones polynomial as a trace.

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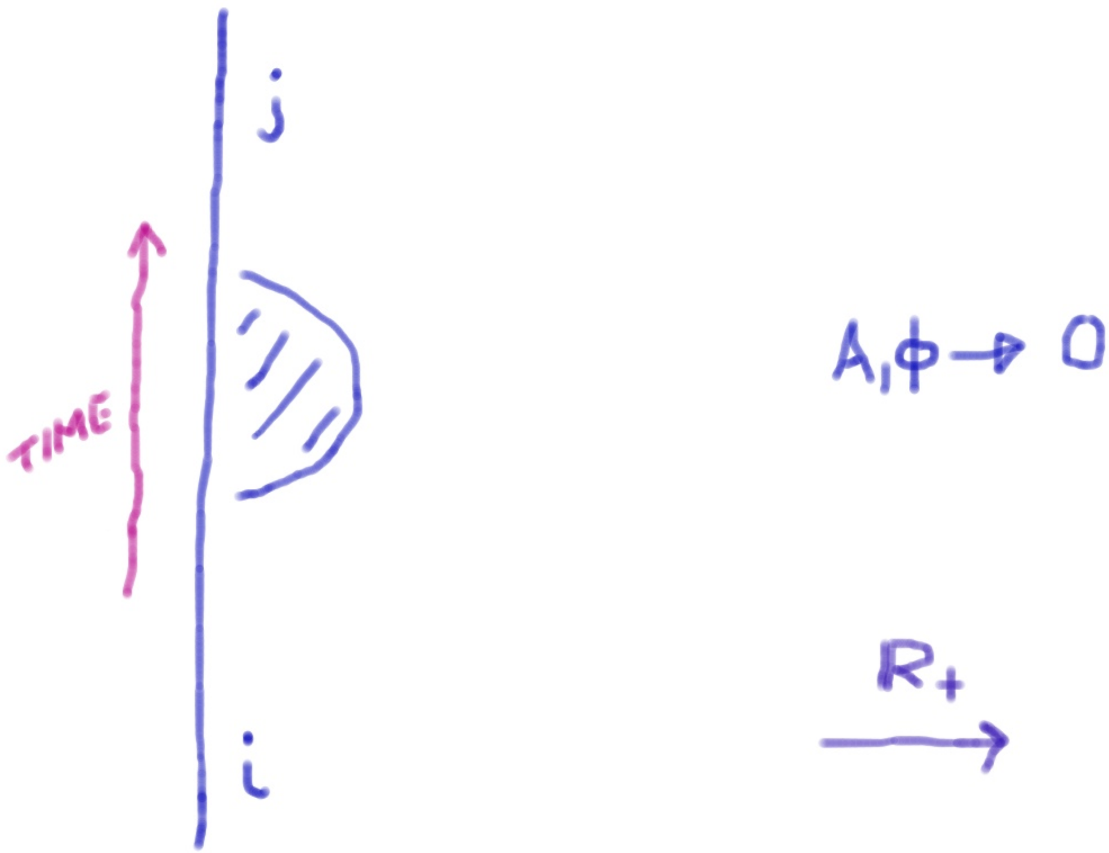

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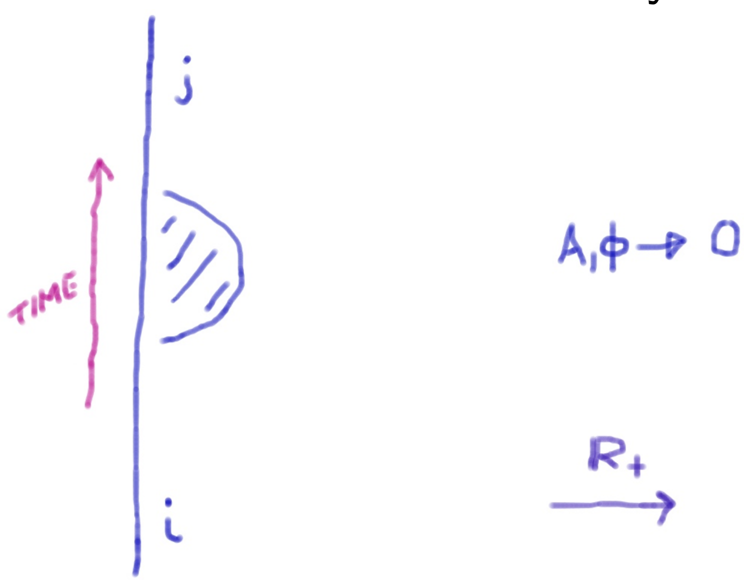
In general, the cohomology of a manifold B can be twisted by a flat line bundle $\mathcal{L} \to B$. Instead of the ordinary cohomology $H^i(B, \mathbb{Z})$, we can consider the twisted cohomology with values in $\mathcal{L}$, $H^i(B, \mathcal{L})$. The possible $\mathcal{L}$'s are classified by $\mathrm{Hom}(\pi_1(B), \mathbb{C}^*)$. In the present case, B is a function space, consisting of pairs (A, ϕ) on $X = M \times \mathbb{R}_+$ (which define initial data for "time"-dependent fields on $Y = \mathbb{R} \times X$ where $\mathbb{R}$ is parametrized by "time"). We only care about the pairs (A, ϕ) up to $G^\vee$-valued gauge transformations (which because of the boundary conditions are trivial on the boundaries of X) and for $M = \mathbb{R}^3$, this means that $\pi_1(B) = \pi_4(G^\vee)$, where π_4 comes in because X is four-dimensional. We have for a simple Lie group $G^\vee$

$$\pi_4(G^\vee) = \begin{cases} \mathbb{Z}_2 & G^\vee = \mathrm{Sp}(2n) \text{ or } \mathrm{Sp}(2n)/\mathbb{Z}_2, \quad n \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

So Khovanov homology is unique unless $G = \mathrm{Spin}(2n+1)$, $G^\vee = \mathrm{Sp}(2n)/\mathbb{Z}_2$ (or $G = SO(2n+1)$, $G^\vee = \mathrm{Sp}(2n)$), in which case there are two versions of Khovanov homology. Concretely, an $\mathrm{Sp}(2n)$ bundle on a five-manifold Y (with a trivialization on ∂Y) has a $\mathbb{Z}_2$-valued invariant η derived from $\pi_4(\mathrm{Sp}(2n)) = \mathbb{Z}_2$. When we define the differential by counting five-dimensional solutions



we have the option to modify the differential by weighting each solution with a factor of $(-1)^\eta$. If we do this, we get a second differential Q' that still obeys $(Q')^2 = 0$ and is congruent mod 2 to the original Q (defined without mentioning the factor $(-1)^\eta$).

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As a preliminary to describing the boundary condition, I need to tell you about an important equation in gauge theory, which is Nahm's equation. Nahm's equation is a system of ordinary differential equations for a triple X_1, X_2, X_3 valued in $\mathfrak{g}^3$, where $\mathfrak{g}$ is the Lie algebra of G. Nahm's equation reads

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and cyclic permutations. On a half-line $y \geq 0$, Nahm's equations have the special solution

$$X_i = \frac{t_i}{y},$$

where the t_i are elements of $\mathfrak{g}$ that obey the $\mathfrak{su}(2)$ commutation relations $[t_1, t_2] = t_3$, etc. We are mainly interested in the case that the t_i define a "principal $\mathfrak{su}_2$ subalgebra" of $\mathfrak{g}$, in the sense of Kostant.

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This sort of singular solution of Nahm's equations was important in the work of Nahm on monopoles, and in later work of Kronheimer and others. We will use it to define an exotic but elliptic boundary condition for our equations. (Ellipticity has been proved in recent work, to appear soon, of R. Mazzeo and EW.)

In fact, Nahm's equations can be embedded in the KW equations

$$F - \phi \wedge \phi + \star \mathrm{d}_A \phi = 0, \quad \mathrm{d}_\star \phi = 0$$

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on $\mathbb{R}^3 \times \mathbb{R}_+$. If we look for a solution that is (i) invariant under translations of $\mathbb{R}^3$, (ii) has the connection $A = 0$, (iii) has $\phi = \sum_{i=1}^3 \phi_i \, \mathrm{d}x_i + 0 \cdot \mathrm{d}y$ (where x_1, x_2, x_3 are coordinates on $\mathbb{R}^3$ and y is the normal coordinate) then our four-dimensional equations reduce to Nahm's equations

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We define an elliptic boundary condition by declaring that we will allow only solutions that are asymptotic to this one for $y \to 0$.

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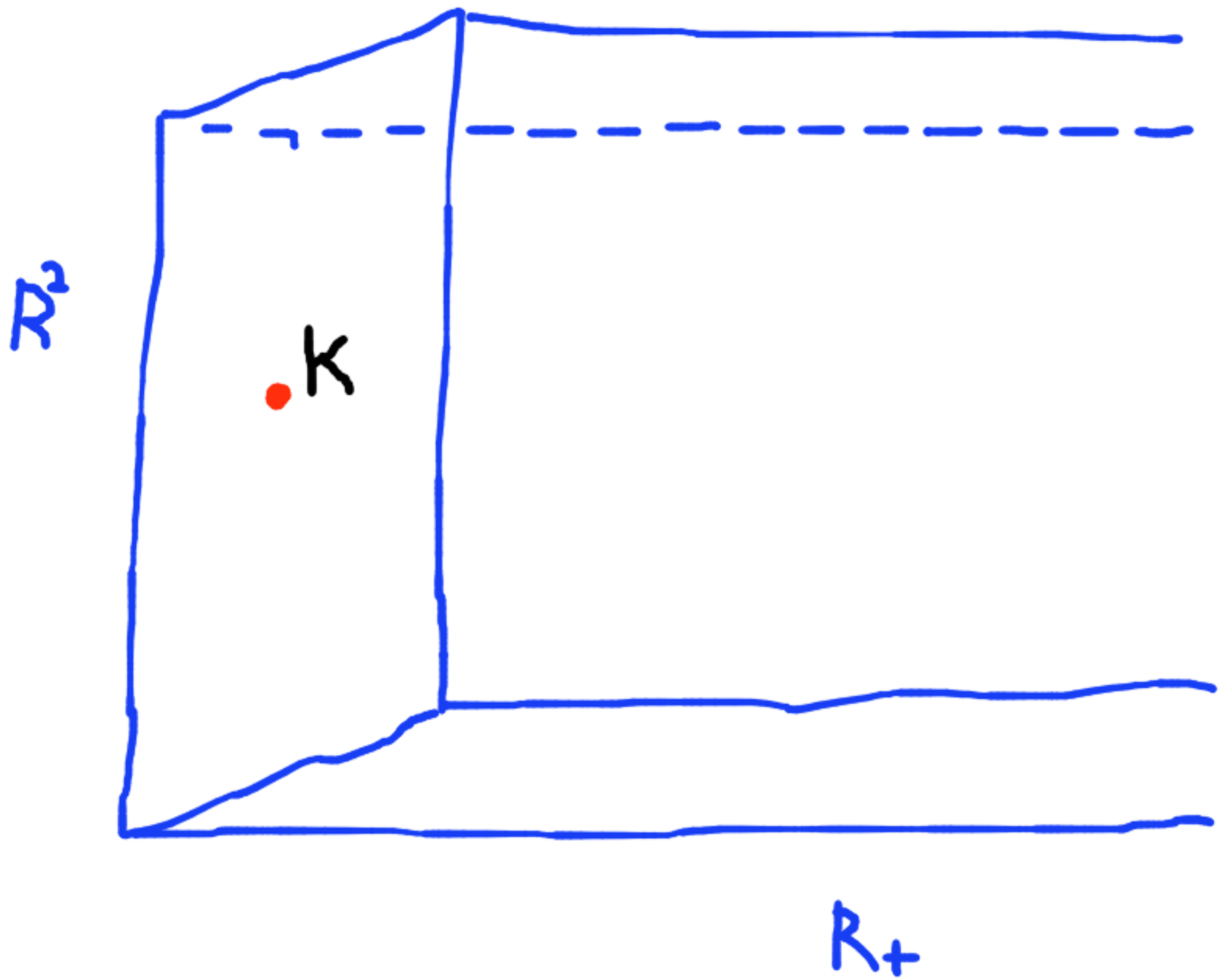

$\mathbb{R}^2$
K
$\mathbb{R}_+$

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The model solution has a singularity that, in the boundary, is of codimension 2. When we go to five dimensions, the singularity remains of codimension 2 so now (since the boundary dimension is 4) the singularity is supported on a 2-surface, not on a knot.

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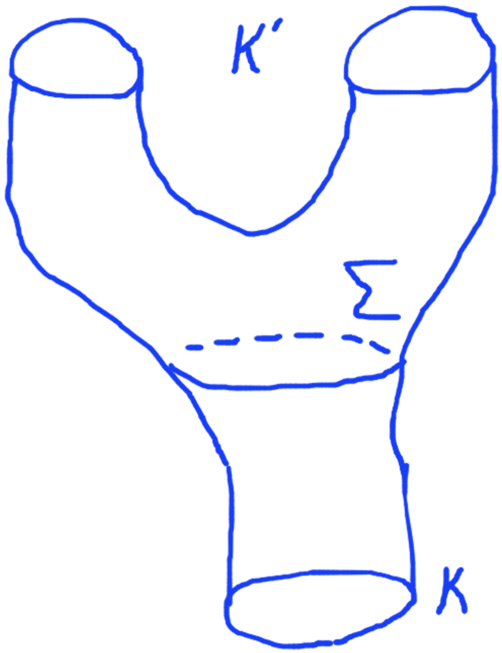

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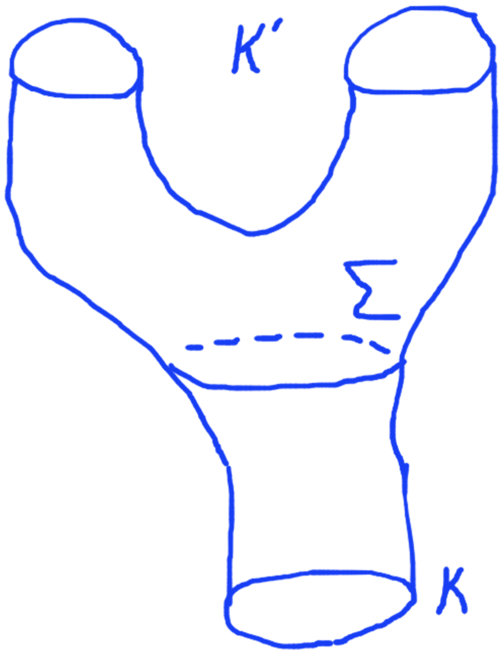

In other words, counting solutions with the boundary conditions described in the picture gives a time-dependent transition from a physical state in the presence of K in the past to a physical state in the presence of K' in the future. (In the simple example shown, K is an unknot and K' consists of two unlinked unknots.)

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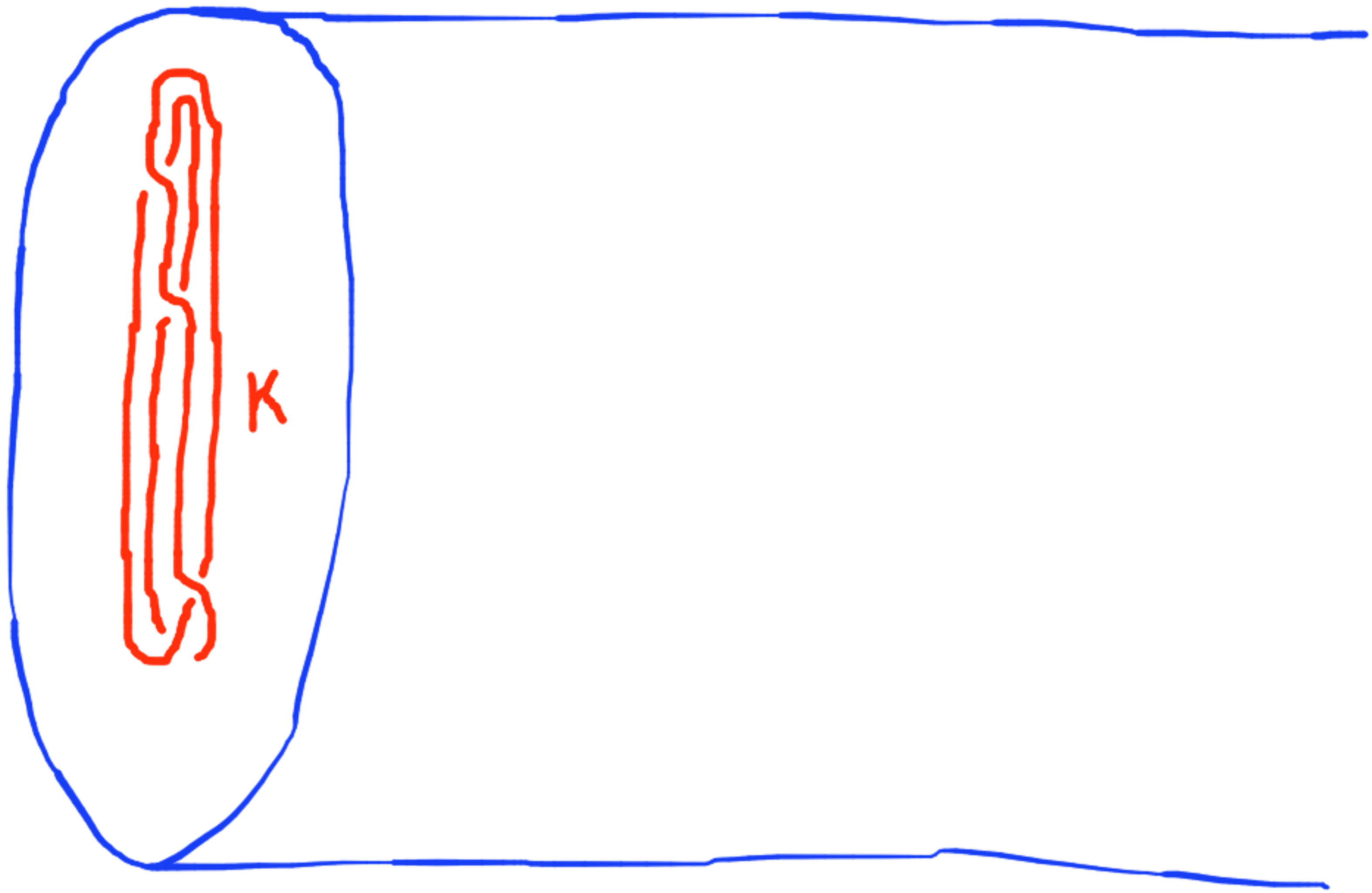
K

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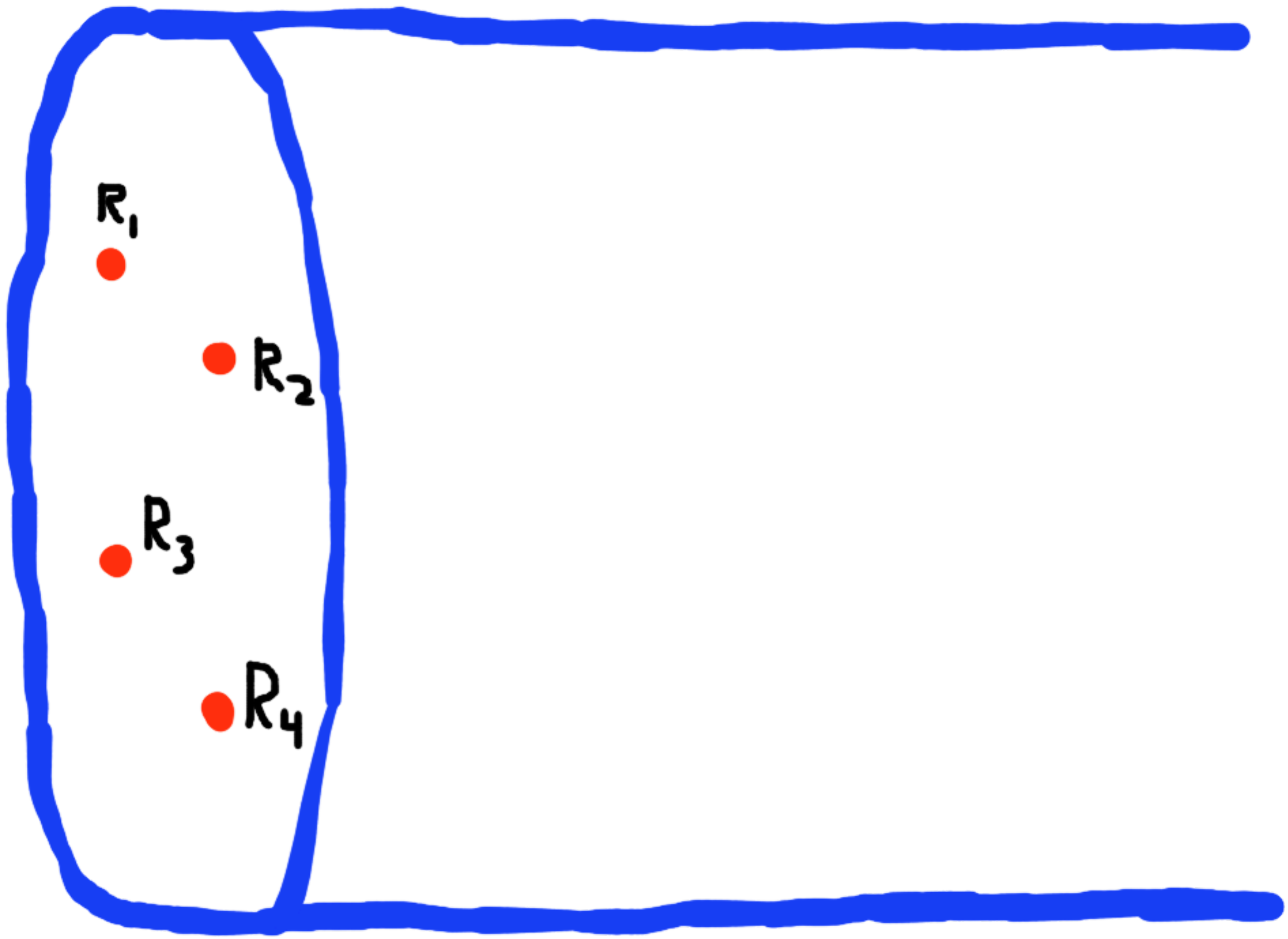
In the present approach, this category should be the *A*-model category of the moduli space of solutions of the reduced three-dimensional equations in the appropriate geometry, sketched in the next picture. (There is also a mirror approach that we haven't had time for today that involves a *B*-model category of almost the same space rather than an *A*-model.)


R_1
R_2
R_3
R_4

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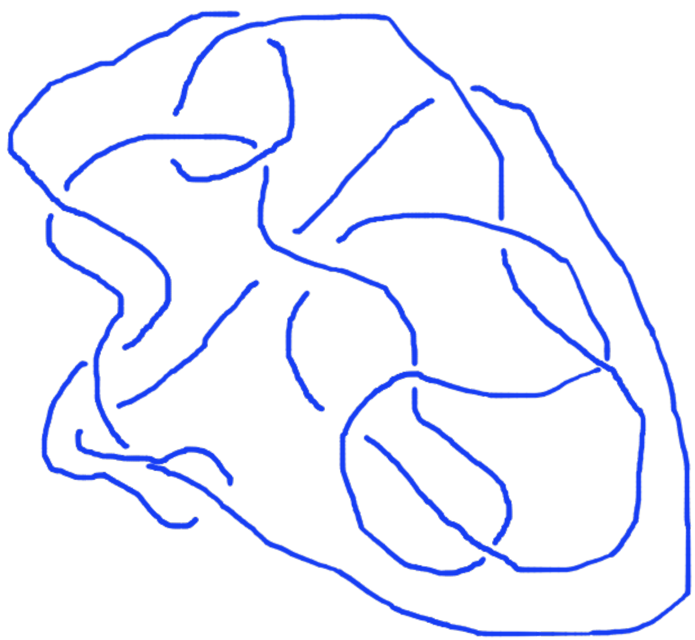

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$$\phi \to \sum_i c_i \cdot \mathrm{d}x^i$$

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for $y \to \infty$. (x^1, x^2, x^3 are Euclidean coordinates on $\mathbb{R}^3$.) We use the fact that the equations have an exact solution for $A = 0$ and ϕ of the form I indicated.

The counting of solutions of an elliptic equation is constant under continuous variations (provided certain conditions are obeyed) so one expects that the Jones polynomial can be computed with this more general asymptotic condition, for an arbitrary choice of $\vec{c} = (c_1, c_2, c_3)$.

If $G = SU(2)$, then $\mathfrak{t}$ is one-dimensional. So if $\vec{c}$ is non-zero, it has the form $\vec{c} = c \cdot \vec{a}$ where c is a fixed (nonzero) element of $\mathfrak{t}$ and $\vec{a}$ is a vector in three-space. So picking $\vec{c}$ essentially means picking a vector $\vec{a}$ pointing in some direction in three-space.

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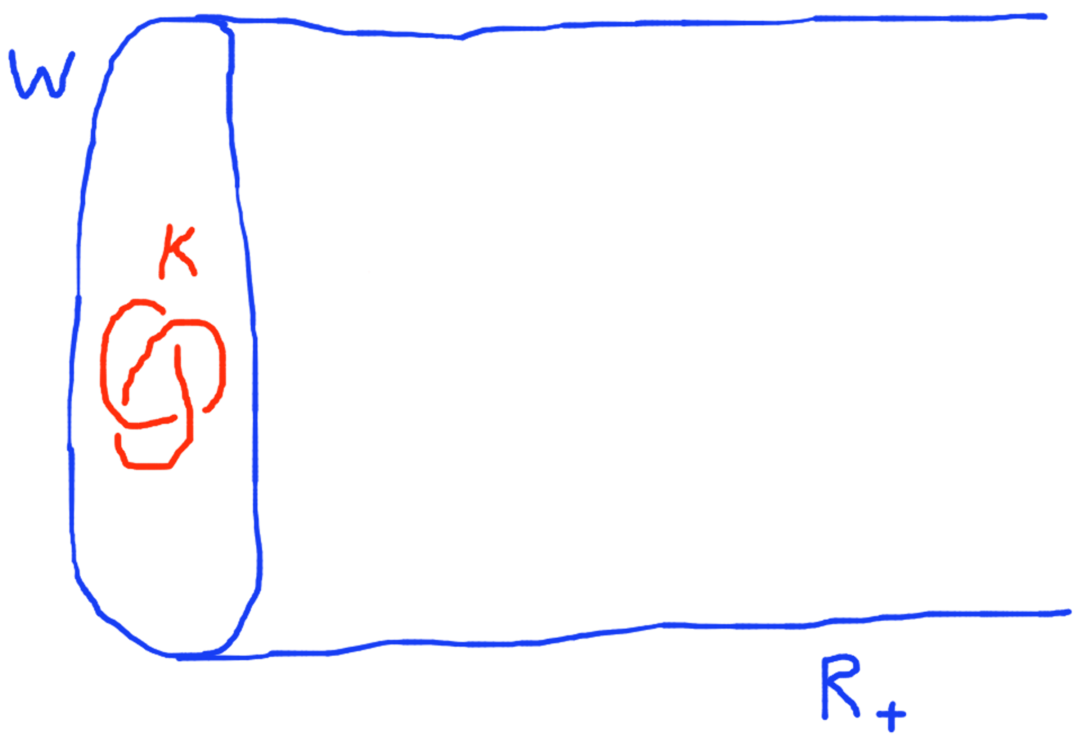

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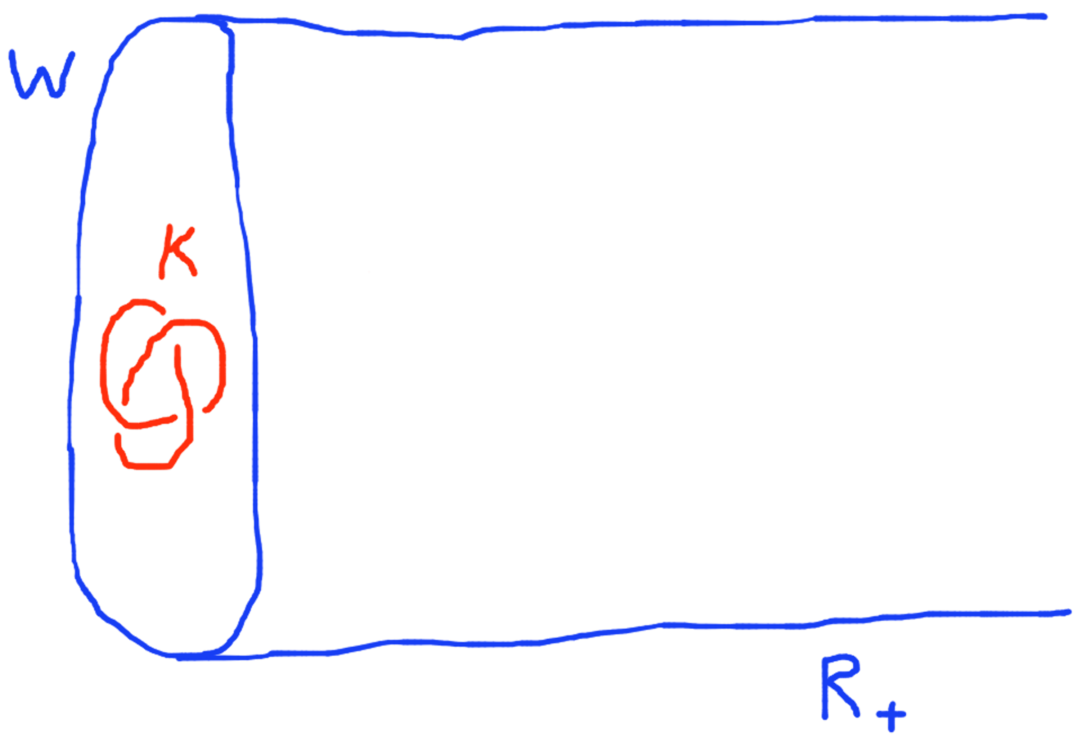

Gaiotto and I were able to understand from this picture the origin of the "vertex model," with which I began yesterday's lecture.