A NON-ELEMENTARY PROOF OF THE SNAKE LEMMA

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Abstract. Every student of homological algebra has proved the snake lemma.

Well, every student of homological algebra has at least proved the snake lemma in the category of R-modules and then mumbled something about the Freyd–Mitchell Embedding Theorem.

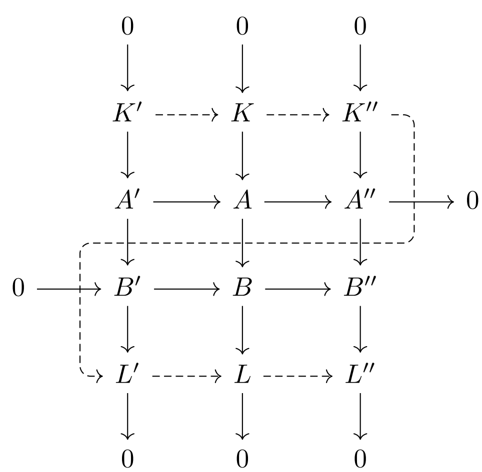
Okay, every student of homological algebra has at least made all of the constructions in a proof of the snake lemma in the category of R-modules, done some of the tedious verifications, and then gotten tired and done something else.

We will give a proof that is valid in any abelian category and avoids all of the unpleasant verifications. We also give a proof of Bergman's salamander lemma.

1. The snake lemma

The snake lemma is best stated with a picture:

(S)



Theorem 1. *In any abelian category, any diagram (S) of solid lines with exact rows and columns can be completed by dashed arrows making the sequence of dashed arrows exact.*

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2. Abelian categories

Among the many axiomatizations of an abelian category, we will use the following one:

Definition 1. A category $\mathscr{C}$ is abelian if it possesses the following properties:

AB0 finite products and finite coproducts exist and coincide;
AB1 all morphisms have kernels and cokernels;
AB2 images and coimages coincide.

The precise meaning of **AB0** is that, for any finite set I and any family of objects A_i indexed by i, the canonical map $\coprod_{i \in I} A_i \to \prod_{i \in I} A_i$, induced by the identity maps on all A_i, is an isomorphism.

The axioms **AB1** and **AB2** were given by Grothendieck [Gro57, §1.4]. Grothendieck used a stronger assumption than **AB0**, but the conjunction of the axioms yields the same notion of an abelian category.

2.1. The additive structure on morphisms.

By itself, **AB0** implies that the set $\mathrm{Hom}(A, B)$ has the structure of a commutative monoid with unit for any A and B in $\mathscr{C}$. First we'll construct the zero element of $\mathrm{Hom}(A, B)$. Let $0 \in \mathscr{C}$ denote the empty product, which by **AB0** is also the empty coproduct. The empty product is the final object of the category, so there is a unique morphism $A \to 0$; likewise, the empty coproduct is an initial object, so there is a unique morphism $0 \to B$. Composing these gives a morphism $A \to B$ that is also denoted 0.

Note first that there is a canonical identification

$$\mathrm{Hom}(A \sqcup B, C \times D) \simeq \mathrm{Hom}(A, C) \times \mathrm{Hom}(A, D) \times \mathrm{Hom}(B, C) \times \mathrm{Hom}(B, D) \tag{1}$$

from the universal properties of product and coproduct. We therefore write elements of $\mathrm{Hom}(A \sqcup B, C \times D)$ as 2×2 matrices. In particular, there is a map

$$\begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix} : A \sqcup A \to A \times A.$$

Since products and coproducts coincide, this map is always an isomorphism.

We can now construct an addition law on $\mathrm{Hom}(A, B)$. Consider a pair of maps $f, g : A \to B$. These induce a map $(f, g) : A \to B \times B$. We obtain

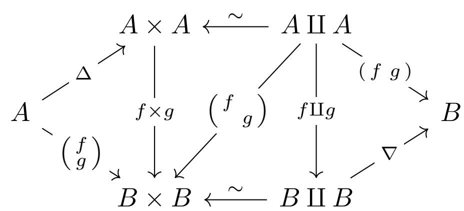
$$A \xrightarrow{\binom{f}{g}} B \times B \xleftarrow{\sim} B \amalg B \xrightarrow{\nabla} B.$$

The sum could also have been constructed as the composition

$$A \xrightarrow{\Delta} A \times A \xleftarrow{\sim} A \amalg A \xrightarrow{(f\ g)} B.$$

Fortunately, diagram (2) is commutative so these definitions agree!

(2)



We leave it as an exercise to verify that the addition law is commutative (use the automorphism $A \times A \simeq A \times A$ exchanging the factors) and associative (use the isomorphism $A \times (A \times A) \simeq (A \times A) \times A$).

We will employ the following standard notation for cokernels and nonstandard notation for kernels:

$$B/A = \operatorname{coker}(A \to B)$$
$$B : A = \ker(A \to B)$$

Lemma 1. *In an abelian category $\mathscr{C}$, a morphism with trivial kernel and cokernel is an isomorphism.*

Proof. Consider $f : A \to B$ with trivial kernel and cokernel. Then we can factor f as

$$A \to \operatorname{coim} f \xrightarrow{\sim} \operatorname{im} f \to B.$$

But $\operatorname{coim} f = A/\ker(f) = A/0 = A$ and $\operatorname{im} f = (B : \operatorname{coker}(f)) = (B : 0) = B$. □

From now on, we will write products and coproducts with the same symbol: $\oplus$.

Assuming **AB1**, we can construct differences in $\operatorname{Hom}(A, B)$. Let $i : K \to A \oplus A$ be the kernel of $\nabla : A \oplus A \to A$. Composing with the two projections $p_1, p_2 : A \oplus A \to A$ gives two maps $p_1 i, p_2 i : K \to A$.

Lemma 2. *The maps $p_1 i$ and $p_2 i$ are isomorphisms and $p_1 i + p_2 i = 0$ in $\operatorname{Hom}(K, A)$.*

Proof. Let's consider the cokernel:

$$\operatorname{coker}(p_1 i) = A/p_1 iK = A \oplus A/(0 \oplus A + iK) = A/(\nabla(0 \oplus A)) = A/A = 0$$

Now let's consider the kernel:

$$\ker(p_1 i) \subset \ker(p_1) \cap \ker(\nabla).$$

But $\ker(p_1) = 0 \oplus A$ and ∇ restricts to the isomorphism $0 \oplus A \simeq A$ on $0 \oplus A$. Therefore $\ker(p_1) \cap \ker(\nabla) = 0$.

Thus $p_1 i$ has zero kernel and zero cokernel. By Lemma 1, it must be an isomorphism. The proof for $p_2 i$ is similar and is omitted.

Now, we compute $p_1 i + p_2 i$. By definition,

$$p_1 i + p_2 i = \nabla \begin{pmatrix} p_1 i & p_2 i \end{pmatrix} = \nabla i = 0,$$

as desired. □

Now, $p_1 i \circ (p_2 i)^{-1}$ gives a map $A \to A$ called $-\mathrm{id}$. As $p_1 i + p_2 i = 0$, it follows that $\mathrm{id} + (-\mathrm{id}) = (p_1 i + p_2 i) \circ (p_2 i)^{-1} = 0$. Composing with $-\mathrm{id}$ allows us to define $-f \in \mathrm{Hom}(A, B)$ for any B. Thus $\mathrm{Hom}(A, B)$ has the structure of an abelian group.

2.2. **Exact sequences.**

Lemma 3. *Suppose that* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *are morphisms in an abelian category and* $gf = 0$*. Then the natural map* $(Z : Y)/X \to Z : (Y/X)$ *is an isomorphism.*

Proof. We may identify $(Z : Y)/X$ with $\mathrm{coim}(D : C \to C/X)$ and $Z : (Y/X)$ with $\mathrm{im}(D : C \to C/X)$. □

In the situation of the lemma, the notation $Z : Y/X$ is unambiguous, so we omit the parentheses in the future.

Definition 2. A sequence of morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in an abelian category is said to be *exact* if any of the following equivalent conditions hold:

(i) $\mathrm{im}(f) = \ker(g)$
(ii) $\mathrm{coker}(f) = \mathrm{im}(g)$
(iii) $gf = 0$ and $C : B/A = 0$

Lemma 4. *Suppose that*

$$A \to B \to C \to D \to E \tag{3}$$

is an exact sequence in an abelian category and $X \to B$ *is any morphism. Then the sequence*

$$A \to B/X \to C/X \to D \to E \tag{4}$$

is also exact.

Proof. The exactness of (3) gives an isomorphism $B/A \to D : C$. Dividing both sides by X gives an isomorphism $(B/X)/A \to D : C/X$, which proves the exactness of (4) at B/X and C/X. For exactness at D, we observe that $E : D/(C/X) = E : D/C = 0$. □

3. Proof of the snake lemma

The proof proceeds by updating the diagram by taking a series of quotients and kernels. Let's begin with the sequence

$$L' \to L \to L'', \tag{5}$$

which is easier to construct. Begin with the diagram

$$
\begin{array}{ccccccc}
 & & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & L' & \dashrightarrow & L & \dashrightarrow & L'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{6}
$$

Divide first by A'.

$$
\begin{array}{ccccccc}
0 & \longrightarrow & A/A' & \longrightarrow & A'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
B'/A' & \longrightarrow & B & \longrightarrow & B'' & & \\
\downarrow\wr & & \downarrow & & \downarrow & & \\
L' & \dashrightarrow & L & \dashrightarrow & L'' & & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

Then divide by A/A'.

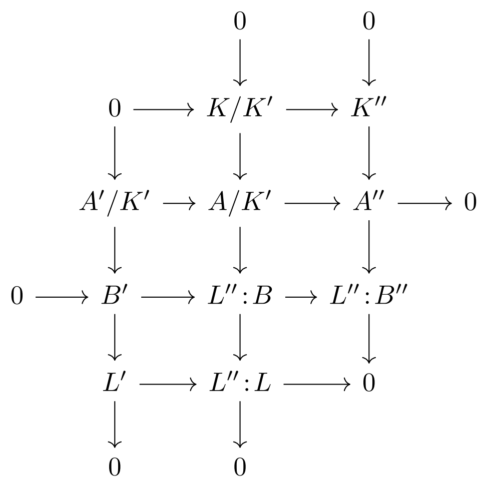
$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
B'/A' & \longrightarrow & B/A & \longrightarrow & B''/A'' \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
L' & \dashrightarrow & L & \dashrightarrow & L'' \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

This gives the exact sequence (5). A similar argument using kernels instead of cokernels gives the exact sequence

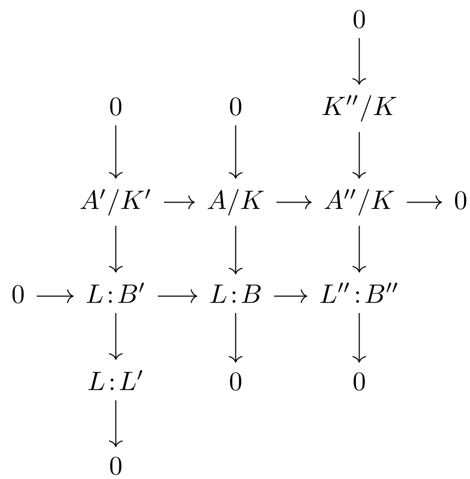
$$
K' \to K \to K'' \tag{7}
$$

If we fill these arrows into diagram (S), we see that the only thing left to do is produce a map $K'' \to L'$ and show that it induces an isomorphism $K''/K \simeq L{:}L'$.

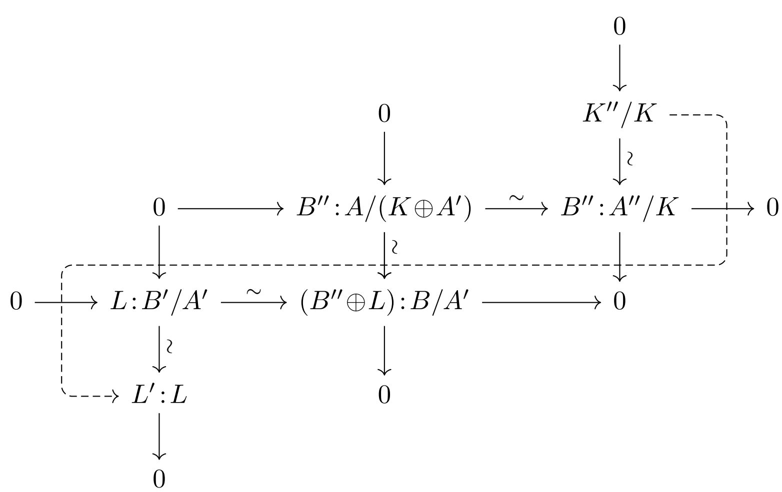
First take quotients by K' and kernels into L''.



Now divide by K/K' and take kernels into $L\!:\!L''$.



Finally, divide by A'/K' and take kernels into $L''\!:\!B''$. Here is the result:



Now we can follow the chain of isomorphisms and the snake lemma is proved!

4. The salamander lemma

Theorem 2. *Let K be an object of an abelian category, equipped with a morphism $d : K \to K$ such that $d^3 = 0$. Then the following sequence is exact:*

$$\frac{\ker d^2}{\operatorname{im} d} \overset{d}{\rightleftarrows} \frac{\ker d}{\operatorname{im} d^2} \tag{8}$$

The morphism $\frac{\ker d}{\operatorname{im} d^2} \to \frac{\ker d^2}{\operatorname{im} d}$ is induced by the inclusions $\ker d \subset \ker d^2$ and $\operatorname{im} d^2 \subset \operatorname{im} d$.

Proof. It is equivalent to show that the map

$$\frac{\ker d^2}{\operatorname{im} d} \Big/ \frac{\ker d}{\operatorname{im} d^2} \xrightarrow{\ d\ } \frac{\ker d^2}{\operatorname{im} d} : \frac{\ker d}{\operatorname{im} d^2}$$

is an isomorphism. We can identify this with

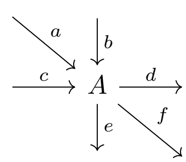
$$\frac{\ker d^2}{\operatorname{im} d + \ker d} \xrightarrow{\ d\ } \frac{\ker d \cap \operatorname{im} d}{\operatorname{im} d^2} \tag{9}$$

Under the map

$$d : \ker d^2 \to \ker d \cap \operatorname{im} d$$

we have $d^{-1}(\operatorname{im} d^2) = \operatorname{im} d + \ker d$ and $d(\ker d^2) = \ker d \cap \operatorname{im} d$. Thus (9) is an isomorphism. ☐

The Salamander Lemma concerns a double complex and is due to Bergman [Ber]. We will follow the presentation of [Ger]. Consider a position in a double complex:



One introduces notation:

$$\begin{aligned} {}_{=}A &= \frac{\ker d}{\operatorname{im} c} \\ A^{\|} &= \frac{\ker e}{\operatorname{im} b} \\ {}^{\square}A &= \frac{\ker d \cap \ker e}{\operatorname{im} a} \\ A_{\square} &= \frac{\ker f}{\operatorname{im} b + \operatorname{im} c} \end{aligned}$$

Theorem 3 (Salamander Lemma)**.** *In a double complex containing* (10)*, the sequence* (11) *is exact.*

$$\begin{array}{ccccc} & \downarrow r & & & \\ \xrightarrow{\alpha} & A & & & \\ & \downarrow \beta & & & \\ \xrightarrow{s} & B & \xrightarrow{\gamma} & C & \xrightarrow{t} \\ & & & \downarrow \delta & \\ & & & D & \xrightarrow{\epsilon} \\ & & & \downarrow u & \end{array} \tag{10}$$

$$A_\square \to_= B \to B_\square \to {}^\square C \to_= C \to {}^\square D \tag{11}$$

First proof. Nothing will be changed in the sequence (11) if we replace diagram (10) with

$$\begin{array}{ccccc} \xrightarrow{\alpha} & \operatorname{coker}(r) & & & \\ & \downarrow \beta & & & \\ & \operatorname{coker}(s) & \xrightarrow{\gamma} & \ker(t) & \\ & & & \downarrow \delta & \\ & & & \ker(u) & \xrightarrow{\epsilon} \end{array}$$

We can therefore assume $r = s = t = u = 0$ without any loss of generality. We can rearrange the diagram linearly:

$$\cdots \xrightarrow{\alpha} A \xrightarrow{\beta} B \xrightarrow{\gamma} C \xrightarrow{\delta} D \xrightarrow{\epsilon} \cdots$$

Let K be the direct sum of all the entries, with 'differential' $d : K \to K$. Note that $d^3 = 0$ so we can apply Theorem 2 to get an exact sequence:

$$\frac{\ker \gamma\beta}{\operatorname{im} \alpha} \xrightarrow{\beta} \frac{\ker \gamma}{\operatorname{im} \beta\alpha} \to \frac{\ker \delta\gamma}{\operatorname{im} \beta} \xrightarrow{\gamma} \frac{\ker \delta}{\operatorname{im} \gamma\beta} \to \frac{\ker \epsilon\delta}{\operatorname{im} \gamma} \xrightarrow{\delta} \frac{\ker \epsilon}{\operatorname{im} \delta\gamma}$$

This is exactly the sequence we require. □

Second proof. We can also prove the Salamander lemma as a corollary of the snake lemma. In Diagram (10), we can make the following replacements without changing the sequence (11):

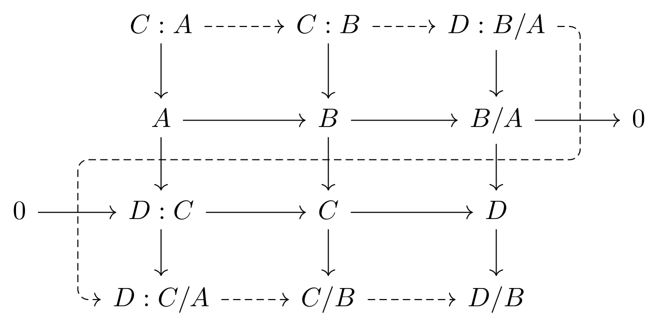
$$\begin{aligned} A &\rightsquigarrow A/\operatorname{im}(r) + \operatorname{im}(\alpha) \\ B &\rightsquigarrow B/\operatorname{im}(s) \\ C &\rightsquigarrow \ker(t) \\ D &\rightsquigarrow \ker(\epsilon) \cap \ker(u) \end{aligned}$$

Then the sequence (11) becomes

$$C : A \to C : B \to D : B/A \to D : C/A \to C/B \to D/B \tag{12}$$

This is the snake in

(13)



□

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