Equivariant bordism and applications in Differential Geometry

Michael Wiemeler

Universität Augsburg

michael.wiemeler@math.uni-augsburg.de

Mathematisches Kolloquium
Augsburg, January 2016

Universität Augsburg
Institut für Mathematik

Outline

Outline

Manifolds

Definition

A n-dimensional manifold M is a second countable Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$.

Manifolds

Definition

A n-dimensional manifold M is a second countable Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$.

- Preimages of regular values of smooth maps $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$ are manifolds.

Manifolds

Definition

A n-dimensional manifold M is a second countable Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$.

- Preimages of regular values of smooth maps $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$ are manifolds.
- In particular, $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1};\ \sum_{i=1}^{n+1} x_i^2 = 1\}$ is a manifold.

Manifolds

Definition

A n-dimensional manifold M is a second countable Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$.

- Preimages of regular values of smooth maps $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$ are manifolds.
- In particular, $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1};\ \sum_{i=1}^{n+1} x_i^2 = 1\}$ is a manifold.
- One can also construct manifolds by patching together open subsets of $\mathbb{R}^n$.

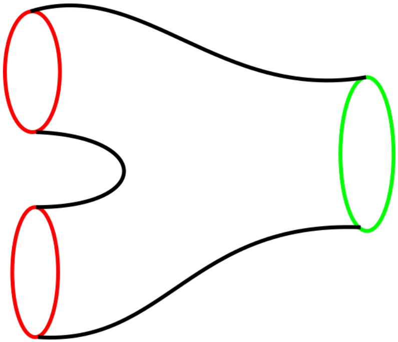
- Goal: Classify manifolds up to some equivalence relation.

- Goal: Classify manifolds up to some equivalence relation.
- Classification up to diffeomorphism or homeomorphism to hard or even impossible.

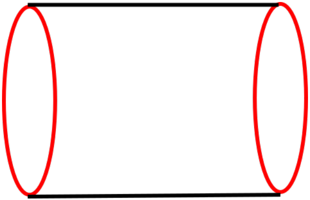
- Goal: Classify manifolds up to some equivalence relation.
- Classification up to diffeomorphism or homeomorphism to hard or even impossible.
- Therefore classification up to bordism.

- Two closed n-manifolds M_1, M_2 are called bordant if there is an compact $n+1$-manifold W with boundary $\partial W = M_1 \amalg M_2$.

- Two closed n-manifolds M_1, M_2 are called bordant if there is an compact $n+1$-manifold W with boundary $\partial W = M_1 \amalg M_2$.

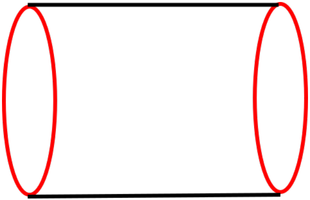


Bordism is an equivalence relation – reflexivity and symmetry

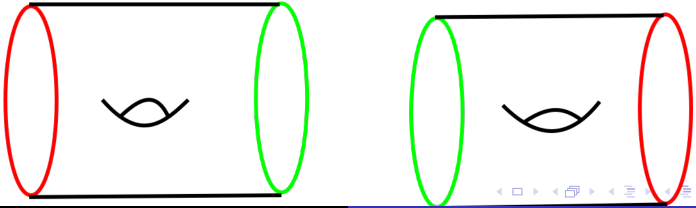


$M \times I$

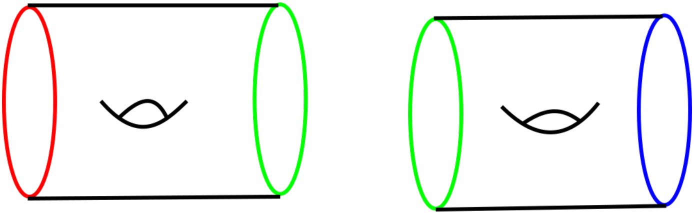
Bordism is an equivalence relation – reflexivity and symmetry



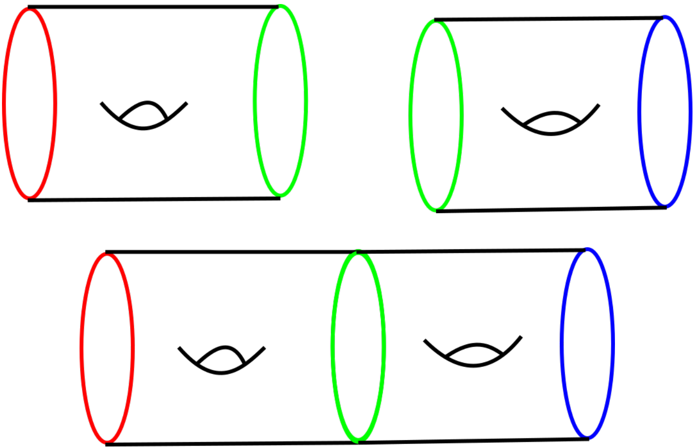
$M \times I$



Bordism is an equivalence relation – transitivity



Bordism is an equivalence relation – transitivity



The unoriented bordism ring

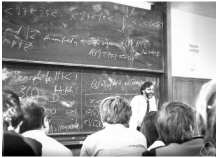
The set $\mathfrak{N}_*$ of all bordism classes of all manifolds forms a graded ring with:

The unoriented bordism ring

The set $\mathfrak{N}_*$ of all bordism classes of all manifolds forms a graded ring with:

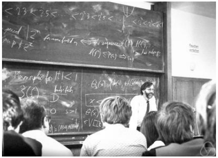
- addition induced by disjoint union
- multiplication induced by cartesian product
- grading by dimension

The oriented bordism ring



- $\Omega_*^{SO} \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}P^{2i};\ i \in \mathbb{N}]$ (Thom 1954)

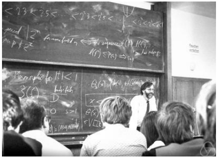
The oriented bordism ring



- $\Omega_*^{SO} \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}P^{2i};\ i \in \mathbb{N}]$ (Thom 1954)
- All non-trivial torsion elements in Ω_*^{SO} are of order two. (Milnor, Averbuh, Wall 1958/1959)

Universität Augsburg
Institut für Mathematik

The oriented bordism ring



- $\Omega_*^{SO} \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}P^{2i};\ i \in \mathbb{N}]$ (Thom 1954)
- All non-trivial torsion elements in Ω_*^{SO} are of order two. (Milnor, Averbuh, Wall 1958/1959)
- Ω_*^{SO} is generated by
  - Milnor hypersurfaces
  - Dold manifolds
  - bundles with fibers products of Dold manifolds over tori

Oriented bordism in low dimensions

n	Ω_n^{SO}	generators
0	$\mathbb{Z}$	$\{pt\}$
1	0	
2	0	
3	0	
4	$\mathbb{Z}$	$\mathbb{C}P^2$
5	$\mathbb{Z}/2$	$P(1,2)$
6	0	
7	0	
8	$\mathbb{Z}^2$	$\mathbb{C}P^2 \times \mathbb{C}P^2, \mathbb{C}P^4$

Tea and Coffee

Assume you have a tea cup like this ...

Tea and Coffee

Assume you have a tea cup like this ...

...but you want to drink coffee.

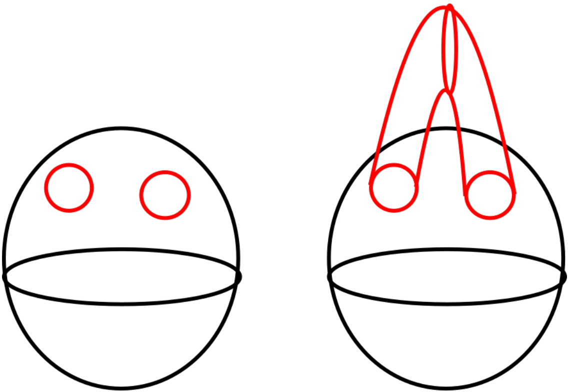
Tea and Coffee

Assume you have a tea cup like this ...

...but you want to drink coffee.

What can you do?

Surgery



Surgery

More formally, surgery is the following cutting and pasting process:

Surgery

More formally, surgery is the following cutting and pasting process:

- Let $\Phi : S^k \times D^{n-k} \hookrightarrow N^n$ be an embedding.

Surgery

More formally, surgery is the following cutting and pasting process:

- Let $\Phi : S^k \times D^{n-k} \hookrightarrow N^n$ be an embedding.
- Cut $\operatorname{im}(\Phi)$ out of N^n and glue in $D^{k+1} \times S^{n-k-1}$ instead.

Surgery

More formally, surgery is the following cutting and pasting process:

- Let $\Phi : S^k \times D^{n-k} \hookrightarrow N^n$ be an embedding.
- Cut $\operatorname{im}(\Phi)$ out of N^n and glue in $D^{k+1} \times S^{n-k-1}$ instead.

Surgery and bordism

Theorem

Two manifolds M and N are bordant if and only if M can be constructed by surgery from N.

Universität Augsburg
Institut für Mathematik

Outlook: Topological Quantum Field Theories

- The bordism category $\mathcal{B}_n$, is the category with objects compact oriented n-dimensional manifolds, and morphisms bordisms between these manifolds

Outlook: Topological Quantum Field Theories

- The bordism category $\mathcal{B}_n$, is the category with objects compact oriented n-dimensional manifolds, and morphisms bordisms between these manifolds
- A TQFT is a functor $F : \mathcal{B}_n \to Vect$ such that
  1. $F(M_1) \cong F(M_2)$ if M_1 and M_2 are orientation preserving diffeomorphic
  2. $F(M_1 \amalg M_2) \cong F(M_1) \otimes F(M_2)$,

Outlook: Topological Quantum Field Theories

- The bordism category $\mathcal{B}_n$, is the category with objects compact oriented n-dimensional manifolds, and morphisms bordisms between these manifolds
- A TQFT is a functor $F : \mathcal{B}_n \to Vect$ such that
  1. $F(M_1) \cong F(M_2)$ if M_1 and M_2 are orientation preserving diffeomorphic
  2. $F(M_1 \amalg M_2) \cong F(M_1) \otimes F(M_2)$,
- Physically
  1. is related to relativistic invariance
  2. is induced by the quantum nature of the theory

Universität Augsburg
Institut für Mathematik

Outlook: Topological Quantum Field Theories

- The bordism category $\mathcal{B}_n$, is the category with objects compact oriented n-dimensional manifolds, and morphisms bordisms between these manifolds
- A TQFT is a functor $F : \mathcal{B}_n \to Vect$ such that
  1. $F(M_1) \cong F(M_2)$ if M_1 and M_2 are orientation preserving diffeomorphic
  2. $F(M_1 \amalg M_2) \cong F(M_1) \otimes F(M_2)$,
- Physically
  1. is related to relativistic invariance
  2. is induced by the quantum nature of the theory
- TQFT's have applications in Seiberg–Witten theory, topological string theory and knot theory.

Universität Augsburg
Institut für Mathematik

Scalar curvature

- Let (M, g) be a Riemannian manifold.

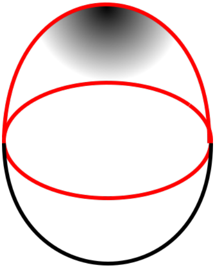
Scalar curvature

- Let (M, g) be a Riemannian manifold.
- The scalar curvature of M is a function $scal : M \to \mathbb{R}$

Scalar curvature

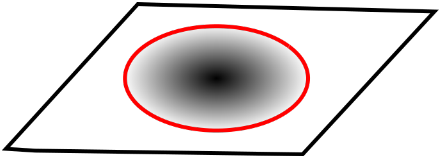
- Let (M, g) be a Riemannian manifold.
- The scalar curvature of M is a function $scal : M \to \mathbb{R}$
- For small $r > 0$ and $x \in M$ we have :

$$\operatorname{vol}(B_r(x)) = \operatorname{vol}_{euclid}(B_r(0))(1 - \frac{scal(x)}{6(n+2)} r^2 + O(r^4))$$



$$scal(x) = 2, \quad \mathrm{vol}(B_{\pi/2}(x)) = 2\pi$$

$$\mathrm{vol}_{euclid}(B_{\pi/2}(0)) = \pi \cdot \pi^2/4$$



What functions are the scalar curvature of a metric on a manifold?

Theorem (Kazdan and Warner 1975)

Let M be a manifold with $\dim M \geq 3$. Then:

- *Every C^∞-function on M which is somewhere negative is the scalar curvature of some metric on M.*

What functions are the scalar curvature of a metric on a manifold?

Theorem (Kazdan and Warner 1975)

Let M be a manifold with $\dim M \geq 3$. Then:

- *Every C^∞-function on M which is somewhere negative is the scalar curvature of some metric on M.*
- *Every C^∞-function on M is the scalar curvature of some metric on M if and only if there is a metric of positive scalar curvature on M.*

A basic question

Question

Let M be a closed connected manifold.
Does there exist a metric of positive scalar curvature on M?

Dimension two

Theorem (Gauss–Bonnet)

For a two-dimensional orientable manifold M, we have

$$\int_M scal(x)\, dvol = 4\pi\chi(M)$$

Dimension two

Theorem (Gauss–Bonnet)

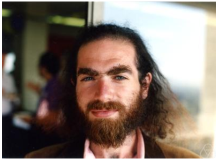
For a two-dimensional orientable manifold M, we have

$$\int_M scal(x)\, dvol = 4\pi\chi(M)$$

- Hence, the only surfaces which admit metrics of positive scalar curvature are S^2 and $\mathbb{R}P^2$.

Universität Augsburg
Institut für Mathematik

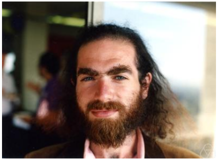
Dimension three and four



Theorem (Perelman 2003)

If M is a manifold of dimension three, then M admits a metric of positive scalar curvature if and only if M is diffeomorphic to a connected sum of several copies of $S^1 \times S^2$ *and spherical space forms.*

Dimension three and four

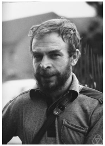


Theorem (Perelman 2003)

If M is a manifold of dimension three, then M admits a metric of positive scalar curvature if and only if M is diffeomorphic to a connected sum of several copies of $S^1 \times S^2$ and spherical space forms.

- Dimension four is open.

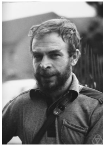
Surgery and positive scalar curvature




Theorem (Gromov and Lawson / Schoen and Yau)

If M is constructed from N by a surgery of codimension at least three and N admits a metric of positive scalar curvature, then the same holds for M.

Surgery and positive scalar curvature




Theorem (Gromov and Lawson / Schoen and Yau)

If M is constructed from N by a surgery of codimension at least three and N admits a metric of positive scalar curvature, then the same holds for M.

Corollary

A manifold M with $\dim M \geq 5$ admits a metric of positive scalar curvature, if and only if its class in a certain bordism ring can be represented by a manifold with such a metric.

Bordism classes of manifolds of positive scalar curvature

Lemma

Let Ω_* be a bordism ring and $I \subset \Omega_*$ the set of bordism classes which can be represented by manifolds with positive scalar curvature.
Then I is an ideal.

Bordism classes of manifolds of positive scalar curvature

Lemma

Let Ω_* be a bordism ring and $I \subset \Omega_*$ the set of bordism classes which can be represented by manifolds with positive scalar curvature.
Then I is an ideal.

- Let M be a compact manifold and N a compact manifold with metric of positive scalar curvature.

Bordism classes of manifolds of positive scalar curvature

Lemma

Let Ω_* be a bordism ring and $I \subset \Omega_*$ the set of bordism classes which can be represented by manifolds with positive scalar curvature.
Then I is an ideal.

- Let M be a compact manifold and N a compact manifold with metric of positive scalar curvature.
- We have: $scal_{M\times N}(x,y) = scal_M(x) + scal_N(y)$.

Bordism classes of manifolds of positive scalar curvature

Lemma

Let Ω_* be a bordism ring and $I \subset \Omega_*$ the set of bordism classes which can be represented by manifolds with positive scalar curvature.
Then I is an ideal.

- Let M be a compact manifold and N a compact manifold with metric of positive scalar curvature.
- We have: $scal_{M\times N}(x, y) = scal_M(x) + scal_N(y)$.
- By shrinking N we get $scal_N \to +\infty$

Theorem (Gromov and Lawson 1980)

Assume that $\pi_1(M) = 0$, $\dim M \geq 5$ *and M does not admit a spin-structure.*
Then M admits a metric of positive scalar curvature.

Theorem (Gromov and Lawson 1980)

Assume that $\pi_1(M) = 0$, $\dim M \geq 5$ *and M does not admit a spin-structure.*
Then M admits a metric of positive scalar curvature.

- The relevant bordism group for M is Ω_n^{SO}.

Universität Augsburg
Institut für Mathematik

Theorem (Gromov and Lawson 1980)

Assume that $\pi_1(M) = 0$, $\dim M \geq 5$ and M does not admit a spin-structure.
Then M admits a metric of positive scalar curvature.

- The relevant bordism group for M is Ω_n^{SO}.
- Ω_*^{SO} is generated by fiber bundles with fibers manifolds with positive scalar curvature.

Theorem (Gromov and Lawson 1980)

Assume that $\pi_1(M) = 0$, $\dim M \geq 5$ *and* M *does not admit a spin-structure.*
Then M *admits a metric of positive scalar curvature.*

- The relevant bordism group for M is Ω_n^{SO}.
- Ω_*^{SO} is generated by fiber bundles with fibers manifolds with positive scalar curvature.
- The total spaces of these fiber bundles therefore admit metrics of positive scalar curvature.

psc-metrics and Spin-structures

If M is spin and admits a metric of positive scalar curvature, then

- the Dirac-operator D on M is invertible (Lichnerowicz 1963).

psc-metrics and Spin-structures

If M is spin and admits a metric of positive scalar curvature, then

- the Dirac-operator D on M is invertible (Lichnerowicz 1963).
- Hence its index vanishes.

psc-metrics and Spin-structures

If M is spin and admits a metric of positive scalar curvature, then

- the Dirac-operator D on M is invertible (Lichnerowicz 1963).
- Hence its index vanishes.
- $\operatorname{ind} D = \widehat{A}(M)$ is an invariant of the spin-bordism type of M (Atiyah-Singer 1968).

Spin bordism

Main results on spin bordism rings are due to Anderson, Brown, Peterson 1966/1967

- $\Omega_*^{\text{Spin}} \otimes \mathbb{Q} \cong \Omega_*^{SO} \otimes \mathbb{Q}$

Spin bordism

Main results on spin bordism rings are due to Anderson, Brown, Peterson 1966/1967

- $\Omega_*^{\text{Spin}} \otimes \mathbb{Q} \cong \Omega_*^{SO} \otimes \mathbb{Q}$
- All non-trivial torsion elements in Ω_*^{Spin} are of order two

Theorem (Stolz 1992)

Assume that $\pi_1(M) = 0$, $\dim M \geq 5$ *and M admits a spin structure.*
Then M admits a metric of positive scalar curvature if and only if $\alpha(M) = 0$.

Theorem (Stolz 1992)

Assume that $\pi_1(M) = 0$, $\dim M \geq 5$ and M admits a spin structure.
Then M admits a metric of positive scalar curvature if and only if $\alpha(M) = 0$.

- The relevant bordism group for M is Ω_n^{Spin}.

Theorem (Stolz 1992)

Assume that $\pi_1(M) = 0$, $\dim M \geq 5$ and M admits a spin structure.
Then M admits a metric of positive scalar curvature if and only if $\alpha(M) = 0$.

- The relevant bordism group for M is Ω_n^{Spin}.
- Stolz shows that $\ker \alpha$ is generated by $\mathbb{H}P^2$-bundles over spin manifolds.

Theorem (Stolz 1992)

Assume that $\pi_1(M) = 0$, $\dim M \geq 5$ *and M admits a spin structure.*
Then M admits a metric of positive scalar curvature if and only if $\alpha(M) = 0$.

- The relevant bordism group for M is Ω_n^{Spin}.
- Stolz shows that $\ker \alpha$ is generated by $\mathbb{H}P^2$-bundles over spin manifolds.
- As in the non-spin-case these admit psc-metrics.

Outlook: Scalar curvature in General Relativity

- The vacuum Einstein field equation

$$Ric_g - \frac{scal_g}{2} g \qquad = 0$$

Outlook: Scalar curvature in General Relativity

- The vacuum Einstein field equation

$$Ric_g - \frac{scal_g}{2} g = 0$$

is the Euler equation for the variational problem for the total scalar curvature functional

$$g \mapsto \int_M scal_g \, dvol_g$$

Outlook: Scalar curvature in General Relativity

- The vacuum Einstein field equation

$$Ric_g - \frac{scal_g}{2} g + \lambda g = 0$$

is the Euler equation for the variational problem for the total scalar curvature functional

$$g \mapsto \int_M scal_g \, dvol_g$$

- positive scalar curvature corresponds to positive mass density or positive cosmological constant λ.

Outlook: Scalar curvature in General Relativity

- The vacuum Einstein field equation

$$Ric_g - \frac{scal_g}{2} g + \lambda g = 0$$

is the Euler equation for the variational problem for the total scalar curvature functional

$$g \mapsto \int_M scal_g \, dvol_g$$

- positive scalar curvature corresponds to positive mass density or positive cosmological constant λ.
- Beginning in the 1990s, measurements suggest that λ is small but positive.

Outline

- Let G be a compact Lie-group and M_1 and M_2 closed G-manifolds. M_1 and M_2 are called G-equivariantly bordant if there is a G-manifold with boundary W such that $\partial W = M_1 \amalg -M_2$.
- The set of all equivariant bordism classes $\Omega_*^{SO,G}$ is an algebra over Ω_*^{SO}.

Computations of equivariant bordism rings

Theorem (Uchida / Hattori and Taniguchi 1970-1972)

As a module over Ω_*^{SO}, Ω_*^{SO,S^1} is generated by twisted $\mathbb{C}P^n$-bundles.

Computations of equivariant bordism rings

Theorem (Uchida / Hattori and Taniguchi 1970-1972)

As a module over Ω_*^{SO}, Ω_*^{SO,S^1} is generated by twisted $\mathbb{C}P^n$-bundles.

- Results on the module structure of the unitary S^1-equivariant bordism ring by Kosniowski and Yahia (1982).
- Sinha (2005) gives generators and relations for the semi-free unitary S^1-equivariant bordism ring.

Computations of equivariant bordism rings II

Theorem (2015)

As a module over $\Omega_*^{Spin}[\frac{1}{2}]$, $\Omega_*^{Spin,S^1}[\frac{1}{2}]$ *is generated by:*

- *semi-free* S^1*-manifolds,*
- *generalized Bott manifolds*

Generalized Bott manifolds

A $2n$-dimensional manifold is called generalized Bott manifold if there is a sequence of fibration

$$M = N_k \to N_{k-1} \to \cdots \to N_1 \to N_0 = \{pt\}$$

such that:

- each N_i is the projectivization of a sum of $n_i + 1$ complex line bundles over N_{i-1}.

Universität Augsburg
Institut für Mathematik

Generalized Bott manifolds

A $2n$-dimensional manifold is called generalized Bott manifold if there is a sequence of fibration

$$M = N_k \to N_{k-1} \to \cdots \to N_1 \to N_0 = \{pt\}$$

such that:

- each N_i is the projectivization of a sum of $n_i + 1$ complex line bundles over N_{i-1}.

Then we have:

- There is an effective action of a torus T of dimension $n = \sum_i n_i$ on M.

Generalized Bott manifolds

A $2n$-dimensional manifold is called generalized Bott manifold if there is a sequence of fibration

$$M = N_k \to N_{k-1} \to \cdots \to N_1 \to N_0 = \{pt\}$$

such that:

- each N_i is the projectivization of a sum of $n_i + 1$ complex line bundles over N_{i-1}.

Then we have:

- There is an effective action of a torus T of dimension $n = \sum_i n_i$ on M.
- This action is induced by multiplication on the line bundles from above.

Generalized Bott manifolds

A $2n$-dimensional manifold is called generalized Bott manifold if there is a sequence of fibration

$$M = N_k \to N_{k-1} \to \cdots \to N_1 \to N_0 = \{pt\}$$

such that:

- each N_i is the projectivization of a sum of $n_i + 1$ complex line bundles over N_{i-1}.

Then we have:

- There is an effective action of a torus T of dimension $n = \sum_i n_i$ on M.
- This action is induced by multiplication on the line bundles from above.
- The S^1-action on M is given by restriction of the T-action to some circle subgroup.

Outline

Universität Augsburg
Institut für Mathematik

A basic question

Question

Assume that a compact connected Lie group G acts effectively on a closed connected manifold M.
Does there exist an G-invariant metric of positive scalar curvature on M?

First existence theorem

Theorem (2013)

Let M be a connected $(G \times S^1)$-manifold such that $\operatorname{codim} M^{S^1} = 2$.
Then M admits a $(G \times S^1)$-invariant metric of positive scalar curvature.

From now on assume that M is an S^1-manifold such that:

- codim $M^{S^1} \geq 4$
- $\pi_1(M_{max}) = 0$
- All singular strata in M are orientable.
  This is always satisfied if M is spin.

The bordism principle for invariant metrics

Theorem

If $\dim M \geq 6$ and M_{max} is not spin, then M admits a normally symmetric metric of positive scalar curvature if and only if its class in $\Omega^{SO,S^1}_{\geq 4,n}$ can be represented by a manifold which admits such a metric.

The bordism principle for invariant metrics

Theorem

If $\dim M \geq 6$ *and* M_{max} *is not spin, then* M *admits a normally symmetric metric of positive scalar curvature if and only if its class in* $\Omega^{SO,S^1}_{\geq 4,n}$ *can be represented by a manifold which admits such a metric.*

Theorem

If $\dim M \geq 6$ *and* M *is spin, then* M *admits a normally symmetric metric of positive scalar curvature if and only if its class in* $\Omega^{Spin,S^1}_{\geq 4,n}$ *can be represented by a manifold which admits such a metric.*

Existence results

Theorem (2015)

If $\dim M \geq 6$ *and*

- M_{max} *is not spin, or*
- M *is spin and the* S^1*-action of odd type,*

Existence results

Theorem (2015)

If $\dim M \geq 6$ *and*

- M_{max} *is not spin, or*
- M *is spin and the* S^1*-action of odd type,*

then there is an $\ell \in \mathbb{N}$ *such that the equivariant connected sum of* 2^ℓ *copies of* M *admits an invariant metric of positive scalar curvature.*

Existence results

Theorem (2015)

If $\dim M \geq 6$ *and*

- M_{max} *is not spin, or*
- M *is spin and the* S^1*-action of odd type,*

then there is an $\ell \in \mathbb{N}$ *such that the equivariant connected sum of* 2^ℓ *copies of* M *admits an invariant metric of positive scalar curvature.*

- In the first case ℓ can be taken to be 1.
- If the action is semi-free, ℓ can be taken to be 1.

Existence results II

Theorem (2015)

If $\dim M \geq 6$, M is spin and the S^1-action of even type, then $\widehat{A}_{S^1}(M/S^1) = 0$ if and only if there is an $\ell \in \mathbb{N}$ such that the equivariant connected sum of 2^ℓ copies of M admits an invariant metric of positive scalar curvature.

Existence results II

Theorem (2015)

If $\dim M \geq 6$, M is spin and the S^1-action of even type, then $\widehat{A}_{S^1}(M/S^1) = 0$ if and only if there is an $\ell \in \mathbb{N}$ such that the equivariant connected sum of 2^ℓ copies of M admits an invariant metric of positive scalar curvature.

- $\widehat{A}_{S^1}(M/S^1)$ is a $\mathbb{Z}[\frac{1}{2}]$-valued equivariant bordism invariant of M.
- For free actions it is the $\widehat{A}$-genus of the orbit space.
- For semi-free actions it was defined by Lott (2000).

$\hat{A}$-genus and S^1-actions

Theorem (Atiyah and Hirzebruch 1970)

Let M be a spin-manifold with $\dim M \geq 6$ which admits a non-trivial S^1-action.
Then $\hat{A}(M) = 0$.

$\hat{A}$-genus and S^1-actions

Theorem (Atiyah and Hirzebruch 1970)

Let M be a spin-manifold with $\dim M \geq 6$ which admits a non-trivial S^1-action.
Then $\hat{A}(M) = 0$.

- The original proof uses the Lefschetz fixed point formula and complex analysis.

$\widehat{A}$-genus and S^1-actions

Theorem (Atiyah and Hirzebruch 1970)

Let M be a spin-manifold with $\dim M \geq 6$ which admits a non-trivial S^1-action.
Then $\widehat{A}(M) = 0$.

- The original proof uses the Lefschetz fixed point formula and complex analysis.
- From the original proof no relation to positive scalar curvature follows.

$\widehat{A}$-genus and S^1-actions

Theorem (Atiyah and Hirzebruch 1970)

Let M be a spin-manifold with $\dim M \geq 6$ which admits a non-trivial S^1-action.
Then $\widehat{A}(M) = 0$.

- The original proof uses the Lefschetz fixed point formula and complex analysis.
- From the original proof no relation to positive scalar curvature follows.
- Such a relation can be deduced from our existence results for positive scalar curvature metrics on S^1-manifolds.

$\widehat{A}$-genus and S^1-actions

Theorem (Atiyah and Hirzebruch 1970)

Let M be a spin-manifold with $\dim M \geq 6$ which admits a non-trivial S^1-action.
Then $\widehat{A}(M) = 0$.

$\widehat{A}$-genus and S^1-actions

Theorem (Atiyah and Hirzebruch 1970)

Let M be a spin-manifold with $\dim M \geq 6$ which admits a non-trivial S^1-action. Then $\widehat{A}(M) = 0$.

- We may assume that $\dim M = 4k$.

$\widehat{A}$-genus and S^1-actions

Theorem (Atiyah and Hirzebruch 1970)

Let M be a spin-manifold with $\dim M \geq 6$ which admits a non-trivial S^1-action.
Then $\widehat{A}(M) = 0$.

- We may assume that $\dim M = 4k$. $\Rightarrow \widehat{A}_{S^1}(M/S^1) = 0$.

$\hat{A}$-genus and S^1-actions

Theorem (Atiyah and Hirzebruch 1970)

Let M be a spin-manifold with $\dim M \geq 6$ which admits a non-trivial S^1-action.
Then $\hat{A}(M) = 0$.

- We may assume that $\dim M = 4k$. $\Rightarrow \hat{A}_{S^1}(M/S^1) = 0$.
- Therefore $2^\ell M$ is equivariantly spin-bordant to an S^1-manifold N with an invariant metric of positive scalar curvature.

$\widehat{A}$-genus and S^1-actions

Theorem (Atiyah and Hirzebruch 1970)

Let M be a spin-manifold with $\dim M \geq 6$ which admits a non-trivial S^1-action.
Then $\widehat{A}(M) = 0$.

- We may assume that $\dim M = 4k$. $\Rightarrow \widehat{A}_{S^1}(M/S^1) = 0$.
- Therefore $2^\ell M$ is equivariantly spin-bordant to an S^1-manifold N with an invariant metric of positive scalar curvature.
- Hence, $2^\ell \widehat{A}(M) = \widehat{A}(N) = 0$.

$\hat{A}$-genus and S^1-actions

Theorem (Atiyah and Hirzebruch 1970)

Let M be a spin-manifold with $\dim M \geq 6$ which admits a non-trivial S^1-action.
Then $\hat{A}(M) = 0$.

- We may assume that $\dim M = 4k$. $\Rightarrow \hat{A}_{S^1}(M/S^1) = 0$.
- Therefore $2^\ell M$ is equivariantly spin-bordant to an S^1-manifold N with an invariant metric of positive scalar curvature.
- Hence, $2^\ell \hat{A}(M) = \hat{A}(N) = 0$. $\Rightarrow \hat{A}(M) = 0$.

Genera

- Let Λ be a $\mathbb{Q}$-algebra. A Λ-genus is a ring homomorphism $\varphi : \Omega_*^{SO} \to \Lambda$.

Genera

- Let Λ be a $\mathbb{Q}$-algebra. A Λ-genus is a ring homomorphism $\varphi : \Omega_*^{SO} \to \Lambda$.
- Examples:
  - The Signature and the $\widehat{A}$-genus are genera.

Elliptic genera

- A genus φ is called elliptic if there are $\delta, \epsilon \in \Lambda$ such that

$$\sum_{i\geq 0} \frac{\varphi([\mathbb{C}P^{2i}])}{2i+1} u^{2i+1} = \int_0^u \frac{1}{\sqrt{1 - 2\delta t^2 + \epsilon t^4}} dt$$

Elliptic genera

- A genus φ is called elliptic if there are $\delta, \epsilon \in \Lambda$ such that

$$\sum_{i\geq 0} \frac{\varphi([\mathbb{C}P^{2i}])}{2i+1} u^{2i+1} = \int_0^u \frac{1}{\sqrt{1-2\delta t^2+\epsilon t^4}} dt$$

Theorem (Ochanine 1987)

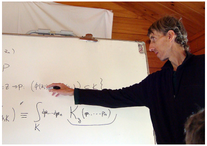
A genus φ is elliptic if and only if $\varphi(E) = 0$ for all total spaces E of fiber bundles with fiber $\mathbb{C}P^{2i+1}$, $i \geq 0$, and simply connected base manifold.

Equivariant genera

For every Λ-genus $\varphi : \Omega_*^{SO} \to \Lambda$ there exists an S^1-equivariant version

$$\varphi_{S^1} : \Omega_*^{SO,S^1} \to H^{**}(BS^1; \Lambda) = \Lambda[[u]].$$

Equivariant genera



For every Λ-genus $\varphi : \Omega_*^{SO} \to \Lambda$ there exists an S^1-equivariant version

$$\varphi_{S^1} : \Omega_*^{SO,S^1} \to H^{**}(BS^1; \Lambda) = \Lambda[[u]].$$

Theorem (Bott and Taubes 1989)

A Λ-genus is elliptic if and only if for every spin S^1-manifold M, the power series $\varphi_{S^1}(M)$ is constant in u.

Summary

- We have generators of the S^1-equivariant Spin-bordism ring
- These can be used to prove
  - the rigidity of elliptic genera
  - existence of S^1-invariant metrics of positive scalar curvature.

Copyright information

- The photos of mathematicians are taken from the MFO Photo Collection or from Wikipedia.
- The photo of the tea cup is taken from artdentity.de.