LECTURE 11: CARTAN'S CLOSED SUBGROUP THEOREM

1. Cartan's closed subgroup theorem

Suppose G is a Lie group and H a closed subgroup of G, i.e. H is subgroup of G which is also a closed subset of G. Let

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \exp(tX) \in H \text{ for all } t \in \mathbb{R}\}.$$

In what follows we will prove the closed subgroup theorem due to E. Cartan. We will need the following lemmas:

Lemma 1.1. *$\mathfrak{h}$ is a linear subspace of $\mathfrak{g}$.*

Proof. Clearly $\mathfrak{h}$ is closed under scalar multiplication. It is closed under vector addition because for any $t \in \mathbb{R}$,

$$H \ni \lim_{n\to\infty}\left(\exp(\frac{tX}{n})\exp(\frac{tY}{n})\right)^n = \lim_{n\to\infty}\left(\exp\left(\frac{t(X+Y)}{n}+O(\frac{1}{n^2})\right)\right)^n = \exp(t(X+Y)).$$

□

Lemma 1.2. *Suppose $X_1, X_2, \cdots$ be a sequence of nonzero elements in $\mathfrak{g}$ so that*

1. *$X_i \to 0$ as $i \to \infty$.*
2. *$\exp(X_i) \in H$ for all i.*
3. *$\lim_{i\to\infty} \frac{X_i}{|X_i|} = X \in \mathfrak{g}$.*

Then $X \in \mathfrak{h}$.

Proof. For any fixed $t \neq 0$, we take $n_i = [\frac{t}{|X_i|}]$ be the integer part of $\frac{t}{|X_i|}$. Then

$$\exp(tX) = \lim_{i\to\infty}\exp(n_iX_i) = \lim_{i\to\infty}\exp(X_i)^{n_i} \in H.$$

□

Lemma 1.3. *The exponential map $\exp : \mathfrak{g} \to G$ maps a neighborhood of 0 in $\mathfrak{h}$ bijectively to a neighborhood of e in H.*

Proof. Take a vector subspace $\mathfrak{h}'$ of $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$. Let $\Phi : \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}' \to G$ be the map

$$\Phi(X+Y) = \exp(X)\exp(Y).$$

Then as we have seen, $d\Phi_0(X+Y) = X+Y$. So Φ is a local diffeomorphism from $\mathfrak{g}$ to G. Since $\exp|_{\mathfrak{h}} = \Phi|_{\mathfrak{h}}$, to prove the lemma, it is enough to prove that Φ maps a neighborhood of 0 in $\mathfrak{h}$ bijectively to a neighborhood of e in H.

Suppose the lemma is false, then we can find a sequence of vectors $X_i + Y_i \in \mathfrak{h} \oplus \mathfrak{h}'$ with $Y_i \neq 0$ so that $X_i + Y_i \to 0$ and $\Phi(X_i + Y_i) \in H$. Since $\exp(X_i) \in H$, we must have $\exp(Y_i) \in H$ for all i. We let Y be a limit point of $\frac{Y_i}{|Y_i|}{}'s$. Then according to the previous lemma, $Y \in \mathfrak{h}$. Since $\mathfrak{h}'$ is a subspace and thus a closed subset, $Y \in \mathfrak{h}'$. So we must have $Y = 0$, which is a contradiction since by construction, $|Y| = 1$. □

Now we are ready to prove

Theorem 1.4 (E. Cartan's closed subgroup theorem)**.** *Any closed subgroup H of a Lie group G is a Lie subgroup (and thus a submanifold) of G.*

Proof. According to the previous lemma, one can find a neighborhood U of e in G and a neighborhood V of 0 in $\mathfrak{g}$ so that $\exp^{-1} : U \to V$ is a diffeomorphism, and so that $\exp^{-1}(U \cap H) = V \cap \mathfrak{h}$. It follows that $(\exp^{-1}, U, V)$ is a chart on G which makes H a submanifold near e. For any other point $h \in H$, we can use left translation to get such a chart. □

As an immediate consequence, we get

Corollary 1.5. *If $\varphi : G \to H$ is Lie group homomorphism, then* $\ker(\varphi)$ *is a closed Lie subgroup of G whose Lie algebra is* $\ker(d\varphi)$*.*

Proof. It is easy to see that $\ker(\varphi)$ is a subgroup of G which is also a closed subset. So according to Cartan's theorem, $\ker(\varphi)$ is a Lie subgroup. It follows that the Lie algebra of $\ker(\varphi)$ is given by

$$\mathrm{Lie}(\ker(\varphi)) = \{X \in \mathfrak{g} \mid \exp(tX) \in \ker(\varphi), \forall t\}.$$

The theorem follows since

$$\begin{aligned} \exp(tX) \in \ker(\varphi), \forall t &\iff \varphi(\exp(tX)) = e, \forall t \\ &\iff \exp(td\varphi(X)) = e, \forall t \\ &\iff d\varphi(X) = 0. \end{aligned}$$

□

As an application, we have

Theorem 1.6. *Any connect abelian Lie group is of the form $\mathbb{T}^r \times \mathbb{R}^k$.*

Proof. Let G be a connect abelian Lie group. Then we have seen that $\exp : \mathfrak{g} \to G$ is a surjective Lie group homomorphism, so G is isomorphic to $\mathfrak{g}/\ker(\exp)$.

On the other hand side, $\ker(\exp)$ is a Lie subgroup of $(\mathfrak{g}, +)$, and it is discrete since $\exp$ is a local diffeomorphism near e. By using induction one can show that $\ker(\exp)$ is a lattice in $(\mathfrak{g}, +)$, i.e. there exists linearly independent vectors $v_1, \cdots, v_r \in \mathfrak{g}$ so that

$$\ker(\exp) = \{n_1 v_1 + \cdots + n_r v_r \mid n_i \in \mathbb{Z}\}.$$

Let $V_1 = \text{span}(v_1, \cdots, v_r)$ and V_2 be a linear subspace of $\mathfrak{g}$ so that $\mathfrak{g} = V_1 \times V_2$. Then

$$G \simeq \mathfrak{g}/\ker(\exp) = V_1/\ker(\exp) \times V_2 \simeq T^r \times \mathbb{R}^k.$$

□

Another important consequence of Cartan's theorem is

Corollary 1.7. *Every continuous homomorphism of Lie groups is smooth.*

Proof. Let $\phi : G \to H$ be a continuous homomorphism, then

$$\Gamma_\phi = \{(g, \phi(g)) \mid g \in G\}$$

is a closed subgroup, and thus a Lie subgroup of $G \times H$. The projection

$$p : \Gamma_\phi \stackrel{i}{\to} G \times H \stackrel{pr_1}{\to} G$$

is bijective, smooth and is a Lie group homomorphism. It follows that dp is a constant rank map, and thus has to be bijective at each point.So p is local diffeomorphism everywhere. Since it is globally invertible, p is also a global diffeomorphism. Thus $\phi = pr_2 \circ p^{-1}$ is smooth. □

As a consequence, for any topological group G, there is at most one smooth structure on G to make it a Lie group. (However, it is possible that one group admits two different topologies and thus have different Lie group structures.)

2. Simply Connected Lie Groups

Recall that a *path* in M is a continuous map $f : [0, 1] \to M$. It is *closed* if $f(0) = f(1)$.

Definition 2.1. Let M be a connected Hausdorff topological space.

(1) Two paths $f, g : [0, 1] \to M$ with the same end points (i.e. $f(0) = g(0), f(1) = g(1)$) are *homotopic* if there is a continuous map $h : [0, 1] \times [0, 1] \to M$ such that

$$h(s, 0) = f(s), h(s, 1) = g(s)$$

for all s, and

$$h(0, t) = f(0), h(1, t) = f(1)$$

for all t.

(2) M is *simply connected* if any two paths with the same ends are homotopic.

(3) A continuous surjection $\pi : X \to M$ is called a *covering* if each $p \in M$ has a neighborhood V whose inverse image under π is a disjoint union of open sets in X each homeomorphic with V under π.

(4) A simply connected covering space is called the *universal cover*.

For example, $\mathbb{R}^n$ is simply connected, $\mathbb{T}^n$ is not simply connected. The map

$$\mathbb{R}^n \to \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n, x \mapsto x + \mathbb{Z}^n$$

is a covering map. The following results are well known:

Facts from topology:

- Let $\pi : X \to M$ is a covering, Z a simply connected space. Suppose $\alpha : Z \to M$ be a continuous map, such that $\alpha(z_0) = m_0$. Then for any $x_0 \in \pi^{-1}(m_0)$, there is a unique "lifting" $\tilde{\alpha} : Z \to X$ such that $\pi \circ \tilde{\alpha} = \alpha$ and $\tilde{\alpha}(z_0) = x_0$.
- Any connected manifold has a simply connected covering space.
- If M is simply connected, any covering map $\pi : X \to M$ is a homeomorphism.

Theorem 2.2. *The universal covering space of a connected Lie group admits a Lie group structure such that the covering map is a Lie group homomorphism.*

Proof. Since G is connected, it has a universal covering $\pi : \widetilde{G} \to G$. One can use the charts on G and the lifting map to define charts on $\widetilde{G}$ so that $\widetilde{G}$ becomes a smooth manifold. Moreover, one can check that under this smooth structure, the lifting of a smooth map is also smooth.

To define a group structure on $\widetilde{G}$, and show π is a Lie group homomorphism, we consider the map

$$\alpha : \widetilde{G} \times \widetilde{G} \to G, \quad (\tilde{g}_1, \tilde{g}_2) \mapsto \pi(\tilde{g}_1)\pi(\tilde{g}_2)^{-1}.$$

Choose any $\tilde{e} \in \pi^{-1}(e)$. Since $\widetilde{G} \times \widetilde{G}$ is simply connected, there is a lifting map $\tilde{\alpha} : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ such that $\pi \circ \tilde{\alpha} = \alpha$ and such that $\tilde{\alpha}(\tilde{e}, \tilde{e}) = \tilde{e}$. Now for any $\tilde{g}_1, \tilde{g}_2 \in \tilde{G}$ we define

$$\tilde{g}^{-1} := \tilde{\alpha}(\tilde{e}, \tilde{g}), \quad \tilde{g}_1 \cdot \tilde{g}_2 = \tilde{\alpha}(\tilde{g}_1, \tilde{g}_2^{-1}).$$

By uniqueness of lifting, we have $\tilde{g}\tilde{e} = \tilde{e}\tilde{g} = \tilde{g}$ for all $\tilde{g} \in \tilde{G}$, since the maps

$$\tilde{g} \mapsto \tilde{g}\tilde{e}, \quad \tilde{g} \mapsto \tilde{e}\tilde{g}, \quad \tilde{g} \mapsto \tilde{g}$$

are all lifting of the map $\tilde{g} \mapsto \pi(\tilde{g})$. Similarly $\tilde{g}\tilde{g}^{-1} = \tilde{g}^{-1}\tilde{g} = \tilde{e}$, and $(\tilde{g}_1\tilde{g}_2)\tilde{g}_3 = \tilde{g}_1(\tilde{g}_2\tilde{g}_3)$. So $\widetilde{G}$ is a group. One can check that the group operations are smooth under the smooth structure chosen above. So $\widetilde{G}$ is actually a Lie group.

Finally by definition $\pi(\tilde{g}^{-1}) = \pi(\tilde{g})^{-1}$ and $\pi(\tilde{g}_1\tilde{g}_2) = \pi(\tilde{g}_1)\pi(\tilde{g}_2)$. So π is a continuous group homomorphism between Lie groups, and thus a Lie group homomorphism. $\square$