My time as Mike Boardman's student and our work on infinite loop spaces

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Dedicated to my thesis advisor Michael Boardman on the occasion of his 60^{th} *birthday in gratitude*

I have the honor of having been Mike Boardman's first student, but I came as a surprise for him.

In 1965 I graduated from the Johann Wolfgang Goethe-Universität in Frankfurt, Germany, with a scholarship in my pocket for graduate studies in England. The advisor for my diploma thesis was Wolfgang Franz, known for his work on Reidemeister-Franz torsion, but at that time Franz had moved into administrative positions at the university: he was dean of the faculty of sciences, then president and later vice-president of the university. So I asked Heiner Zieschang for advice on where to go and he recommended the University of Warwick in Coventry, England, which opened in 1965 (the graduate school in mathematics started one year earlier in 1964 with students whom the new staff had brought with them).

The University of Warwick started off in a spectacular fashion with a symposium in topology running the whole academic year. I wrote to David Epstein for permission to attend the symposium and to the university authorities for acceptance to the graduate program.

About two weeks into the fall term I realized that all my fellow graduate students had supervisors. When I asked about my supervisor I realized that the Mathematics Institute considered me a visitor of the symposium and had not realized that I was also a graduate student. Since every staff member had his share of students, this meant trouble. Finally, they decided to ask Mike Boardman to take me on. Mike had a postdoctoral fellowship from the Science Research Counsel and was not obliged to take care of a student. Moreover, he was very busy with a number of projects he wanted to complete. Here is the list of the ones I know of:

M. Boardman's projects during 1965/66

1. Work in progress with B. Steer on Hopf invariants (resulting in the publication of [5])
2. Completion of the Warwick notes on his stable category of CW-spectra [1], [2], [3].
3. A revision of his work on singularities (now called Boardman-Thom singularities [4]).
4. The investigation of the algebraic structure of classifying spaces arising from geometry (first results were presented in a talk "Pre-Hilbert spaces and H-spaces" in the spring of 1966).

After a day or two of considerations Mike agreed to take care of me. It took us some time to get used to each other. We all know that Mike is not much of a talker. The following incident was quite typical: during a colloquium talk the speaker made some blunder at the blackboard. The more advanced graduate students in the back row, among them notably Hugh Morton and Elmer Rees, became restless, David Epstein in the first row interrupted the talk and a discussion started about the mathematics on the black board without arriving at a conclusion. Finally, David turned to Mike and asked whether the mathematics was correct or not. The answer was short and precise "It is false!". Looking over into Mike's notes I realized that he had corrected the talk all along for quite some time without saying a word. No wonder that I had some trouble discussing mathematics with him at the very beginning.

The time in Warwick was very exciting, quite different from my undergraduate days in Frankfurt. Most of the world's leading topologists visited the symposium at some point during the year. Moreover, there was a score of advanced lecture courses read by young mathematicians of international reputation. Among these courses was one on homotopy theory given by Mike. It was one of the most excellent courses I ever attended, delivered in very condensed form; and it is the only course during my time as a student to which I still refer for short and precise proofs. (Mike told me later that it was based on a course by Adams in Cambridge, but it definitely had Mike's style and precision).

Some time into spring I learnt that Mike had been invited as a Visiting Lecturer to the University of Chicago for the academic year 1966/67. David Epstein was so kind as to write a letter of recommendation to Kaplanski and I was offered a research assistantship at Chicago. The research environment there matched the one at Warwick. To cut things short, I will not go into details. Here are some of the highlights as far as I was concerned:

- Mike Boardman's course on Stable Homotopy Theory (I should point out that in Section 9 he constructed an associative, commutative, and unital smash product functor up to coherent homotopies using spectra indexed by finite dimensional subspaces of the real inner product space $\mathbb{R}^\infty$, i.e. a Boardman type version of what are now called "coordinate-free spectra").
- Mike suggested as a problem for my Ph.D. thesis some questions connected with exact triangles in the stable homotopy category.
- A seminar on "Iterated homotopies and the bar construction", officially run by Adams and MacLane (since Frank Adams could only come for a month it was mainly organized by Saunders MacLane).

The audience of the seminar "Iterated homotopies and the bar construction" is worth mentioning. Practically all staff members and graduate students of the University of Chicago who were interested in topology, homological algebra or category theory attended it. Besides them, Brayton Gray and Jim Milgram from Chicago Circle and Jim Stasheff from Notre Dame were regular visitors. Frank Adams came for a month, and people from Northwestern came to some of the lectures. Since this seminar influenced our work on infinite loop spaces considerably, let me sketch its program.

Program of the seminar "Iterated homotopies and the bar construction"

1. MacLane (January and February 1967): MacLane developed the theory of $PROP$s and $PACT$s in a series of lectures. A $PROP$ is a category of operators with tensor PROducts and Permutations and $PACT$ stands for Permutations, Addition, Composition, and Tensor product. A $PACT$ is a $PROP$ based on the category of chain complexes rather than the category of sets. $PACT$s codify higher homotopies on DGAs and hence are connected with cohomology operations.

2. J. Milgram (March 6th, 1967): "The bar construction in geometry".
   The talk was about the material of his paper "The bar construction and abelian H-spaces" (Ill. J. Math. 11 (1967)).

3. J. Stasheff (March 9th, 1967): "Higher homotopy associativity". Stasheff used an algebraic analogue of his theory of A_n-spaces to study the following question: if a certain $PACT$ operates on the bar construction $B(A)$ of a DGA A, which kind of $PACT$ operates on A itself? Most important for us was a number of questions he raised at the end of his lecture:
   (1) Can one go back from algebra to topology?
   (2) If the dual of the Steenrod $PACT$ acts on a DGA A, does it also act on $B(A)$?
   Finally he remarked that a topological version of (2) would give infinite loop spaces.

4. F. Adams (April 13th, 1967): "Operations of the Steenrod $PACT$ on the cobar construction and loop spaces".
   Adams addressed the problem of defining a diagonal on the cobar construction $F(C_*(X))$ of a chain complex of a space X which corresponds to the Alexander-Whitney diagonal under the equivalence $F(C_*(X)) \to C_*(\Omega X)$.

5. M. Boardman (May 11th, 1967): " Homotopy everything H-spaces".
   Mike presented the first **topological** $PROP$, namely the "linear isometry" $PROP$ (now called "linear isometry operad") and showed that it acts on the various classifying spaces arising from the geometry of manifolds.

By the end of the academic year Mike and I realized that my research problem resulted in setting up a big machinery to prove a result which could be obtained directly with less effort. Some day in May 1967 during lunch we decided to scrap it. Mike suggested that I should work on infinite loop spaces taking up some of the problems mentioned in MacLane's seminar. Considering the audience of excellent mathematicians my reaction was: "This is a pretty HOT problem". It must have been my accent which made Mike misunderstand the word "hot", because his answer was: "I do not have any EASY ones".

At the beginning our approach was trial and error. We stuck too closely to the algebraic set-up using simplicial methods. The breakthrough came in February 1968 when we both independently came up with the same model for a homotopy-universal A_∞-structure. Modulo minor technicalities the development of the theory was practically straightforward from then on. Let me summarize

Our work on infinite loop spaces

(announced in 1968 in [6])

(1) It suffices to consider *PROP*s in "standard form". A *PROP* in standard form is the same as an "operad", a catchphrase introduced four years later by May. It is a PROP $\mathcal{B}$ which is determined by the morphism spaces $\mathcal{B}(n,1)$, the action of the symmetric group Σ_n on $\mathcal{B}(n,1)$, and composition. So $\mathcal{B}$ is uniquely determined by the operad consisting of the $\mathcal{B}(n,1)$. Each *PROP* $\mathcal{B}$ functorially defines a *PROP* in standard from $\mathcal{B}'$ together with a functor $\mathcal{B}' \to \mathcal{B}$.

(2) We introduced the "endomorphism *PROP*" and "endomorphism operad" of a space, the "Little cubes operad" and the "Linear isometries operad". Moreover, we used the well-known operads $\mathcal{A}$ and $\mathcal{S}$, which define monoids and commutative monoids and introduced what is now called an E_∞-structure.

(3) We defined the "W-construction" which replaces an operad by a cofibrant one (in the sense of homotopy-invariance). It has the universal properties one expects from a cofibrant gadget.

(4) Extending Stasheff's notion of an A_∞-map between monoids we defined homotopy homomorphisms between $\mathcal{B}$-spaces, $\mathcal{B}$ an arbitrary

operad, i.e. maps which preserve the structure up to coherent homotopies.

(5) We showed that $W(\mathcal{B})$-structures are homotopy invariant.

(6) Using interchange and homotopy invariance we clarified the connection between a monoid with additional structure and the structure inherited by its classifying space (this solved the topological version of Stasheff's questions).

(7) Using (3) we constructed homotopy universal operads which characterize spaces homotopy equivalent to a loop space or an infinite loop space, respectively. From (6) we got infinite deloopings of "group-like" E_∞-spaces.

(8) Mike's earlier work then implied that O, SO, F, U, TOP, etc. and their coset spaces F/O etc., and all their iterated classifying spaces are infinite loop spaces, and that the natural maps between them are infinite loop maps.

Detailed proofs were given in our Lecture Notes [7]. We had a similar result for PL, its coset spaces such as PL/O, and BPL which we did not include in our account for two reasons. It would have required a PL substitute of the linear isometry operad. Secondly, at the time of our work, Sullivan made the statement that the canonical map $BPL \to BTOP$ was not an infinite loop map, which contradicted our results.

To this day I am grateful that Mike did not leave me a mathematical orphan although he had other plans at that time. He was a very good thesis advisor, his courses were excellent, I learnt a lot from him, and it was a joy to watch him develop mathematical concepts. In short I owe him much; my academic career most certainly would have been quite different without him.

References

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